Available online at www.sciencedirect.com
SCIENCE DIRECT
PHYSICS LETTERS A
Physics Letters A 334 (2005) 306–311
www.elsevier.com/locate/pla

Numerical solution of classical kicked rotor and local Lyapunov exponents

Yindong Zheng, Donald H. Kobe *

Physics Department, University of North Texas, Denton, TX 76203-1427, USA

Received 22 June 2004; accepted 20 November 2004

Available online 2 December 2004

Communicated by A.R. Bishop

Abstract

We calculate a local Lyapunov exponent for the kicked rotor that is constant for a duration that depends on the initial separation between two trajectories, but then decreases asymptotically as $O(t^{-1}\ln t)$. This behavior is consistent with an upper bound that is determined analytically.

© 2004 Elsevier B.V. All rights reserved.

Keywords: Kicked rotor; Local Lyapunov exponent; Classical chaos; Asymptotic behavior; Numerical solution

1. Introduction

The classical kicked rotor or "standard map" or Chirikov–Taylor map [1] is often used to test techniques and methods in nonlinear analysis [2–5], since it is a simple model that exhibits complex behavior. This behavior includes both chaotic and nonchaotic regimes. The characterization of classical chaos is the strong sensitivity to initial conditions [3–6]. If two adjacent trajectories with initial values very close to each other depart exponentially in time, the position of a particle cannot be reliably calculated after a sufficiently long time because of the uncertainty in the initial position. The coefficient of time in the exponential separation of adjacent trajectories is called the *global* Lyapunov exponent and is used as a criterion of chaos. Strictly speaking, the global Lyapunov exponent is defined only in the limit as the initial separation distance between the two trajectories goes to zero and time goes to infinity. Since mathematical limits cannot be attained by numerical methods, other definitions have been proposed for a *local* Lyapunov exponent to characterize chaos [7–12].

In this Letter we define a *local* Lyapunov exponent that approaches a global Lyapunov in the limit as the separation distance approaches zero and time approaches infinity. Using our m-file programs in Matlab [13], we solve the classical kicked rotor in both chaotic and nonchaotic regimes. In the chaotic regime the local Lyapunov exponent is calculated as a function of

* Corresponding author.
E-mail address: kobe@unt.edu (D.H. Kobe).

0375-9601/$ – see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.physleta.2004.11.034

separation distance and time. Our computations show that for sufficiently long times two adjacent trajectories in the chaotic regime depart polynomially, instead of exponentially. This behavior is confirmed analytically by showing that an upper bound to the departure of two adjacent trajectories is also a polynomial. The corresponding bound on the local Lyapunov exponent decreases asymptotically as $O(t^{-1}\ln t)$. However, in the beginning stage of time development exponential departure of two adjacent trajectories does occur and extends for a longer time as the initial separation distance decreases.

In Section 2 we define a local Lyapunov exponent. We calculate trajectories for the kicked rotor in Section 3 and obtain their local Lyapunov exponents in the chaotic regime. An upper bound to the local Lyapunov exponent is obtained analytically in Section 4. The conclusion is given in Section 5.

2. Local Lyapunov exponent

We use a local Lyapunov exponent because numerically only finite separation distances and times are possible. The trajectories have initial displacements $\mathbf{x}_1(0) = (x_1(0), p_1(0)) = (x_0, p_0)$ and $\mathbf{x}_2(0) = (x_2(0), p_2(0))$ in phase-space, and their initial separation in phase-space is $\mathbf{d}(0)$. We define a *local* Lyapunov exponent $L(x_0, d(0), t)$ as

$$L\big(\mathbf{x}_0, d(0), t\big) \equiv \frac{1}{t}\ln\left[\frac{d(t)}{d(0)}\right] \to \lambda \quad \text{as}$$
$$d(0) \to 0, \quad t \to \infty, \tag{1}$$

where $L(\mathbf{x}_0, d(0), t)$ depends in general on the initial displacement in phase-space $\mathbf{x}_0 = \mathbf{x}_1(0) = (x_0, p_0)$, the initial separation distance $d(0) = |\mathbf{d}(0)|$, and time t. The modulus of the displacement in phase-space at time t is $d(t) = |\mathbf{d}(t)|$. The double limit of the local Lyapunov exponent as the initial separation distance $d(0)$ approaches zero and time t approaches infinity is defined as the *global* Lyapunov exponent λ [3], which in general does not depend on the initial point $\mathbf{x}_0$ in phase-space because of ergodicity [14,15]. The definition of the global Lyapunov exponent in Eq. (1) may depend on how the double limit is taken.

3. Numerical calculations for trajectories of the kicked rotor

The dynamics of the classical kicked rotor are given by the standard map

$$p(N+1) = p(N) + K\sin\big[x(N)\big],$$
$$x(N+1) = x(N) + p(N+1), \tag{2}$$

where N is the number of kicks. However, in general we measure time t in units of the period T of the kicks, so $t = N$ is a continuous variable. The context shows whether N is an integer or not. The only parameter in Eq. (2) is $K = kT$, the product of the kick strength k and the period T. The displacement $x(N)$ is in radians, while the momentum $p(N)$ is in radians per unit time. For our calculations we chose $p_1(0) = p_2(0) = 0$, so the initial separation distance $d(0)$ in phase space is equal to the initial separation distance in coordinate space $x_2(0) - x_1(0) > 0$.

Fig. 1 shows the numerical solution of Eq. (2) in the chaotic regime with $K = 10$. The other parameters chosen are the initial displacement $x_0 = 3$, the initial separation distance $d(0) = 10^{-6}$, and the initial momenta $p_1(0) = p_2(0) = 0$. The number of kicks N (or time) goes from 0 to 1000. Subgraph (a) shows the displacement of the two trajectories in coordinate space without restricting the displacement to be from 0 to 2π. The coordinate space separation of the two trajectories increases with time, but not exponentially. Subgraph (b) shows the phase-space separation distance ratio $d(t)/d(0)$. Subgraph (c) is the logarithm of the separation distance ratio, $\ln[d(t)/d(0)]$ and subgraph (d) shows the corresponding local Lyapunov exponent in Eq. (1). The local Lyapunov exponent in this loglog plot is obviously approaching zero as the time or number of kicks increases beyond about 10.

Fig. 2 studies the dependence of the local Lyapunov exponent on the initial separation distance $d(0)$ as the separation distance is reduced. The values of the separation distances are $d(0) = 10^{-6}, 10^{-8}, 10^{-10}$, and 10^{-12} for $x_0 = 3$. For a global Lyapunov exponent to exist, Eq. (1) shows that the local Lyapunov exponent must approach a constant in the double limit as $d(0) \to 0$ and time $N \to \infty$. We see in Fig. 2 that the local Lyapunov exponent is essentially constant in a plateau region that widens as $d(0)$ is re-

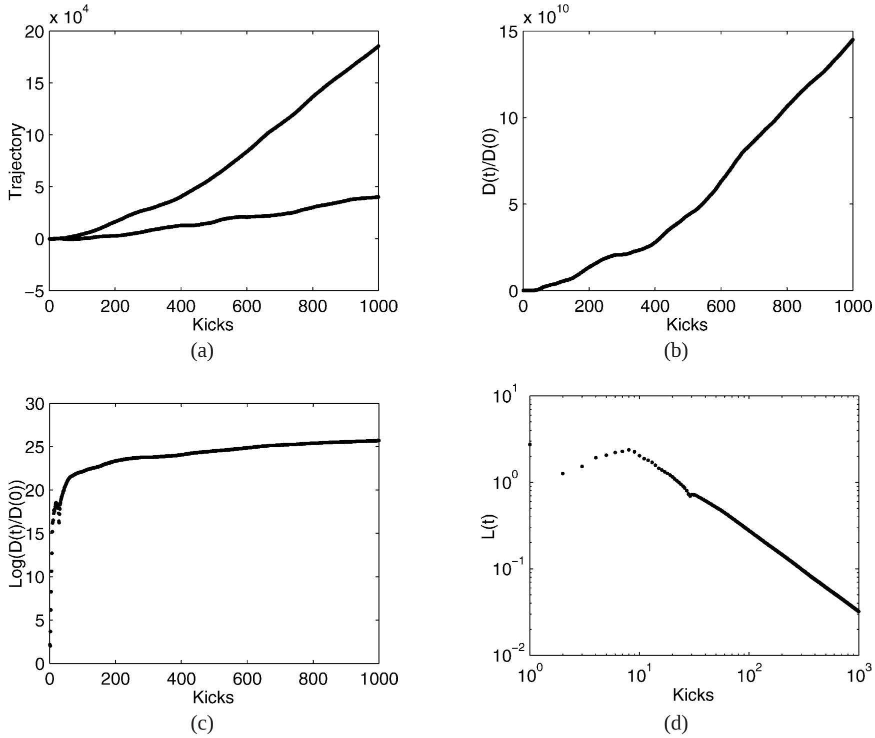

Fig. 1. Trajectory simulation and Lyapunov exponent in the chaotic regime with $K = 10$ as a function of time N. (a) Simulation of two adjacent trajectories in coordinate space with initial separation distance $d(0) = 10^{-6}$ and $x_0 = 3$. (b) Ratio $d(N)/d(0)$ of distances in phase-space as a function of time N. (c) Logarithm of $d(N)/d(0)$. (d) Corresponding local Lyapunov exponent L in Eq. (1).

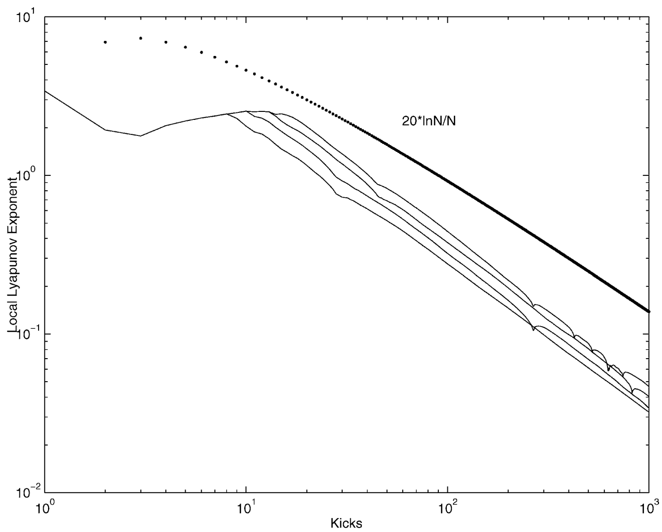

Fig. 2. Local Lyapunov exponent L as a function of time N for $K = 10$, $x_0 = 3$ and different values of $d(0)$. From left to right $d(0) = 10^{-6}$, 10^{-8}, 10^{-10}, and 10^{-12}, respectively.

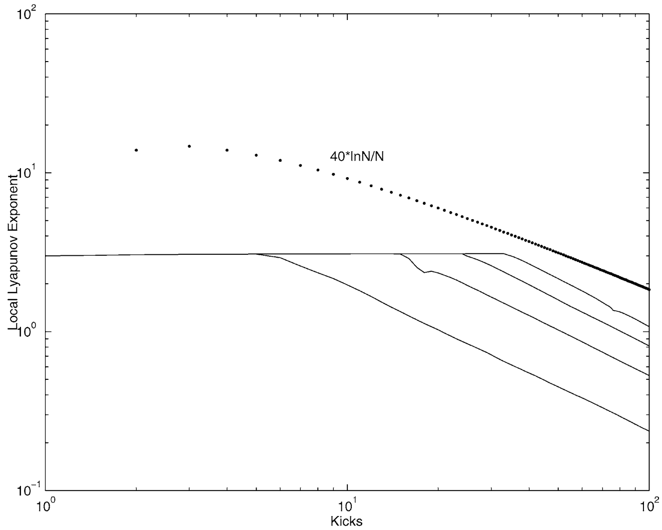

Fig. 3. Local Lyapunov exponent L as a function of time N for $K = 10$, $x_0 = 0$ and different values of $d(0)$. From left to right $d(0) = 10^{-5}, 10^{-15}, 10^{-25}$, and 10^{-35}, respectively.

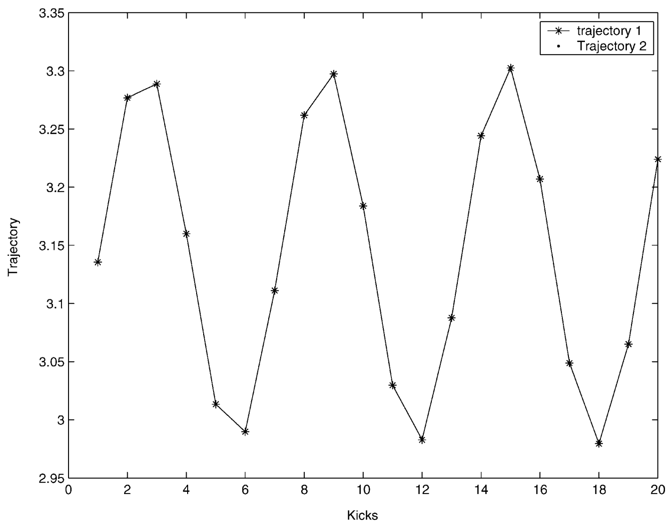

Fig. 4. Trajectory simulation in the nonchaotic regime for $K = 0.96$, $x_0 = 3$ and $d(0) = 10^{-6}$. On the scale shown the two trajectories overlap each other. The dashed line is only a guide for the eye.

duced. For sufficiently large times, however, all the curves show an asymptotic approach to zero. An upper bound to the local Lyapunov exponents is $20N^{-1}\ln N$, the dotted line in Fig. 2. According to our analysis in Section 4 all the curves of local Lyapunov exponents for fixed $d(0)$ approach zero asymptotically faster than $O(N^{-1}\ln N)$. For the parameters and units chosen here the value of the local Lyapunov exponent is about 2.5.

To investigate further the behavior of the local Lyapunov exponent on the separation distance $d(0)$, we used Matlab [13] variable precision arithmetic with 40 significant figures. In Fig. 3 the initial value of one trajectory is $x_1(0) = x_0 = 0$ and the other is $x_2(0) =$

$d(0)$. The initial separation distances are taken to be $d(0) = 10^{-5}, 10^{-15}, 10^{-25}$ and 10^{-35}. The time or number of kick goes from 0 to 100. The plateau in Fig. 3 extends for longer times as $d(0)$ decreases. The dotted line in Fig. 3 is $40N^{-1}\ln N$, which is an upper bound to the local Lyapunov exponents. Here also, all the local Lyapunov exponents approach zero faster than $O(N^{-1}\ln N)$. For the parameters and units chosen here the value of the local Lyapunov exponent is about 3.

Fig. 4 shows two trajectories for a parameter $K = 0.96$ with $x_0 = 3$ and $d(0) = 10^{-6}$ from a time N going from 0 to 20. On the scale used in Fig. 4 the two trajectories have such a small separation that they appear to overlap each other. The trajectories are bounded between about 3.0 and 3.3, and are somewhat periodic. This behavior continues for times as large as $N = 1000$. The line shown is only a guide for the eye. Because there is no significant separation of the trajectories for the parameters chosen, the system is nonchaotic in this case. This behavior is qualitatively different from the trajectories in Fig. 1 for $K = 10$ in the chaotic regime. In the chaotic regime the trajectories and their separation are constantly increasing, while in the nonchaotic regime the trajectories are confined to a very narrow region with little separation.

4. Upper bound for local Lyapunov exponents

From the map in Eq. (2) we can analytically obtain an upper bound for the local Lyapunov exponents for fixed values of $d(0)$. This "worst case scenario" for their asymptotic behavior substantiates the numerical calculations in Figs. 1–3 for large times.

For the upper bound we consider two trajectories initially at $\mathbf{x}_1(0) = (x_1(0), p_1(0))$ and $\mathbf{x}_2(0) = (x_2(0), p_2(0))$, where $d(0)$ is the separation distance in phase-space. Since the sine function in Eq. (2) is no less than -1, the momentum $p_1(1)$ satisfies the inequality

$$p_1(1) = p_1(0) + K\sin[x_1(0)] \geqslant p_1(0) - K. \tag{3}$$

Assuming that

$$p_1(N-1) \geqslant p_1(0) - (N-1)K, \tag{4}$$

we prove by mathematical induction that the momentum for the first trajectory satisfies the inequality

$$p_1(N) \geqslant p_1(0) - NK, \tag{5}$$

for all $N = 0, 1, 2, \ldots$.

Since the sine function in Eq. (2) is no more than $+1$, we can also prove by mathematical induction that the momentum for the second trajectory satisfies the inequality

$$p_2(N) \leqslant p_2(0) + NK, \tag{6}$$

for all $N = 0, 1, 2, \ldots$.

Likewise we can also prove by mathematical induction from Eq. (2) that the coordinate $x_1(N)$ of the first trajectory satisfies the inequality

$$x_1(N) \geqslant x_1(0) + Np_1(0) - \frac{1}{2}N(N+1)K, \tag{7}$$

and that the coordinate $x_2(N)$ of the second trajectory satisfies the inequality

$$x_2(N) \leqslant x_2(0) + Np_2(0) + \frac{1}{2}N(N+1)K, \tag{8}$$

for all $N = 0, 1, 2, \ldots$. From Eqs. (5)–(8) an upper bound on the separation distance $d(N)$ in phase-space after N kicks is

$$\begin{aligned} d(N) \equiv\ & \big\{\big[x_2(N) - x_1(N)\big]^2 \\ & + \big[p_2(N) - p_1(N)\big]^2\big\}^{1/2} \\ \leqslant\ & \big\{\big[x_2(0) - x_1(0) + N\big(p_2(0) - p_1(0)\big) \\ & + N(N+1)K\big]^2 \\ & + \big[p_2(0) - p_1(0) + 2Nk\big]^2\big\}^{1/2} \\ \to\ & KN^2 \quad \text{as } N \to \infty, \end{aligned} \tag{9}$$

so two trajectories initially very close in phase-space depart asymptotically like $O(N^2)$. If $p_2(0) = p_1(0)$ the initial separation distance in phase-space $d(0)$ is also the initial separation distance in coordinate space

$$\begin{aligned} d(0) &\equiv \big\{\big[x_2(0) - x_1(0)\big]^2 + \big[p_2(0) - p_1(0)\big]^2\big\}^{1/2} \\ &= x_2(0) - x_1(0) > 0. \end{aligned} \tag{10}$$

We have chosen the condition $p_2(0) = p_1(0) = 0$ for our calculations in this Letter, but for the sake of generality have not made this choice in this section.

From Eq. (9) the local Lyapunov exponent in Eq. (1) for time N satisfies the inequality

$$
\begin{aligned}
&L\big(\mathbf{x}_0, d(0), N\big) \\
&\quad \leqslant \frac{1}{2N} \ln\Bigg\{ \frac{1}{d(0)^2} \Big\{ \big[x_2(0) - x_1(0) \\
&\qquad + N\big(p_2(0) - p_1(0)\big) + N(N+1)K\big]^2 \\
&\qquad + \big[p_2(0) - p_1(0) + 2NK\big]^2 \Big\} \Bigg\} \\
&\quad \to 2N^{-1} \ln N \quad \text{as } N \to \infty,
\end{aligned} \tag{11}
$$

so asymptotically the local Lyapunov exponent is $O(N^{-1}\ln N)$. That is, it approaches zero no slower than a constant times $N^{-1}\ln N$ as the time N approaches infinity in agreement with our numerical results.

5. Conclusion

In the chaotic region two adjacent trajectories separate exponentially for a duration that depends on the initial separation distance $d(0)$. The smaller the initial separation distance $d(0)$, the longer is the length of time that the local Lyapunov exponent is essentially constant. For sufficiently long times, however, we can see from Figs. 1–3 that all curves decrease asymptotically more rapidly than $O(N^{-1}\ln N)$. This upper bound on the asymptotic behavior of the Lyapunov exponents is obtained analytically. In general, for any polynomial separation distance between adjacent trajectories, the corresponding local Lyapunov exponent decreases asymptotically as $O(N^{-1}\ln N)$.

In the nonchaotic regime, two adjacent trajectories remain extremely close to each other and are limited to a very narrow range of coordinates. This behavior is qualitatively different from the chaotic regime.

We have also used the method of Benettin et al. [8,9] to study this problem. In the chaotic regime for $K = 10$, we chose $x_0 = 3$, $d(0) = 10^{-6}$ and took the time of each segment to be $\tau = 4$. Their "stochastic parameter" analogous to an average of our local Lyapunov exponent was calculated for times N going from 0 to 5000. After a time $N = 1500$ their stochastic parameter had converged to about 2, a value consistent with our Fig. 2 at a time $N = 4$ and $x_0 = 3$. The method of Benettin et al., however, guarantees that different values of x_0 are sampled.

The de Broglie–Bohm approach to quantum mechanics introduces the concept of quantum trajectories [16]. This approach makes it possible to use the same criterion for quantum chaos as for classical chaos [17]. A positive Lyapunov exponent is taken as the signature of both classical and quantum chaos. A thorough understanding of the numerical methods for calculating classical Lyapunov exponents is necessary to study quantum case.

Acknowledgements

We would like to thank Professors Paolo Grigolini, Arkadii Krokhin, Jacek Kowalski, John Neuberger, Wolfgang Schleich, Robert O'Connell and Constantino Tsallis for the helpful discussions. We also thank Dr. Gerardo Aquino for carefully reading the Letter and Prof. Marlan O. Scully for his encouragement. This work was supported by ONR N00014-03-1-0639/TAMU TEES 53494.

References

[1] B.V. Chirikov, Phys. Rep. 52 (1979) 263.
[2] M.C. Gutzwiller, Chaos in Classical and Quantum Mechanics, Springer-Verlag, New York, 1990.
[3] L.E. Reichl, The Transition to Chaos, Springer-Verlag, New York, 1992, p. 44.
[4] E. Ott, Chaos in Dynamical Systems, Cambridge Univ. Press, Cambridge, 1993.
[5] A.L. Lichtenberg, M.A. Lieberman, Regular and Stochastic Motion, Springer-Verlag, New York, 1992.
[6] Y.S. Weinstein, S. Lloyd, C. Tsallis, Phys. Rev. Lett. 89 (2002) 214101.
[7] L.E. Ballentine, Phys. Rev. A 63 (2001) 024101.
[8] G. Benettin, L. Galgani, J.-M. Strelcyn, Phys. Rev. A 14 (1976) 2338.
[9] M. Casartelli, E. Diana, L. Galgani, A. Scotti, Phys. Rev. A 13 (1976) 1921.
[10] H. Shibata, R. Ishizaki, Physica A 197 (1993) 130.
[11] A.K. Pattanayak, P. Brumer, Phys. Rev. E 56 (1997) 5174.
[12] B. Eckhardt, D. Yao, Physica D 65 (1993) 100.
[13] Matlab 6.5, Release 13, The MathWorks Inc., 2002.
[14] D. Braun, Dissipative Quantum Chaos, Springer-Verlag, Berlin, 2001, p. 9.
[15] D. Ruelle, Chaotic Evolution and Strange Attractors, Cambridge Univ. Press, Cambridge, 1989, Chapter 9.1.
[16] P.R. Holland, The Quantum Theory of Motion—an Account of the de Broglie–Bohm Causal Interpretation of Quantum Mechanics, Cambridge Univ. Press, Cambridge, 1993.
[17] U. Schwengelbeck, F.H.M. Faisal, Phys. Lett. A 199 (1995) 281.