QUADRATIC COUNTS OF HIGHLY TANGENT LINES TO HYPERSURFACES

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Abstract.

1. Introduction

1.1. Outline.

`https://byu.zoom.us/my/mckean`

`https://uni-bonn.zoom-x.de/j/6273166727?pwd=PbrnHJs8ZxmuXEuBCrwa88eLInu09q.1&omn=66254591316`

1.2. Code and Data.

Our code and all the data we generated is available online at the following link:

`https://github.com/wgabrielong/highly_tangent`

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2. The parameter space and the bundle

Given a hypersurface X of degree $2n-1$ in $\mathbb{P}^n$, we are interested in counting the lines that meet X to order $2n-1$ at a single point. More generally, one can ask to count the number of r-planes in $\mathbb{P}^n$ that meet a degree d hypersurface to order m at some point. Our restrictions ($r=1$ and $d=m=2n-1$) are to give us particularly nice geometry to analyze in Section 5.

In this section, we will describe the parameter space and vector bundle underlying the more general counting problem. We mostly follow [EH16, §11.1] for this setup, although we will not restrict ourselves to characteristic 0 as is done in *loc. cit.* Instead, we will only assume $\operatorname{char} k \notin [2,d]$. Explain how this implies $(\operatorname{char} k, (m-1)!) = 1$, so we have no issues arising from tangency in positive characteristic or symmetric products of bundles.

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2.1. **Parameter space of pointed r-planes.** The objects we are counting are r-planes in $\mathbb{P}^n$, so the projective Grassmannian $\mathbb{G}\mathrm{r}(r,n)$ is the first scheme one thinks of when looking for our desired parameter space. However, we need a little more data than is carried by $\mathbb{G}\mathrm{r}(r,n)$. Because we want to consider the order to which an r-plane H meets our hypersurface X at $p \in H \cap X$, our parameter space needs to encode both H and p leading us to consider the flag variety of pointed r-planes (add this to signpost better?).

here we consider $\mathrm{Gr}(r+1,n+1)$ because $\mathbb{G}\mathrm{r}(r,n)$ does not carry enough information? Should the $\mathrm{Gr}(r+1,n+1)$ above actually be $\mathbb{G}\mathrm{r}(r,n)$ and suggest thinking about $\mathrm{Gr}(r+1,n+1)$ to define the flag variety as the tautological bundle over $\mathrm{Gr}(r+n,n+1)$?

The $\mathrm{Gr}(r+1,n+1)$ should have been $\mathbb{G}\mathrm{r}(r,n)$. What I meant by "does not carry enough information" is that the Grassmannian (whether affine or projective) does not know about specific points within the planes it parameterizes. We need pointed planes, because our enumerative problem remembers the point at which a given plane meets the hypersurface. Working with the flag variety this problem.

Definition 2.1. Define the flag variety $\Phi_{r,n} := \{(H,p) \in \mathbb{G}\mathrm{r}(r,n) \times \mathbb{P}^n : p \in H\}$, which $\Phi_{r,n}$ parameterizes pointed r-planes in $\mathbb{P}^n$. We will often omit r and n from the notation when these values are contextually clear.

Remark 2.2. Note that $\Phi_{r,n} \cong \mathbb{P}\mathcal{S}$, where $\mathcal{S}$ is the tautological bundle over the affine Grassmannian $\mathrm{Gr}(r+1,n+1) \cong \mathbb{G}\mathrm{r}(r,n)$. Indeed, the fiber in $\Phi_{r,n}$ over $[H] \in \mathrm{Gr}(r+1,n+1)$ is given at the point-set level by the set of points in $\mathbb{P}H$, which is exactly the fiber in $\mathbb{P}\mathcal{S}$ over H.

Notation 2.3. Under the incidence description given in Definition 2.1, projection onto the first or second factor gives us projection morphisms

$$\begin{tikzcd} & \Phi_{r,n} \arrow[dl, "\pi"'] \arrow[dr, "\beta"] & \\ \mathbb{G}\mathrm{r}(r,n) & & \mathbb{P}^n. \end{tikzcd}$$

We will also denote the projective bundle map of Remark 2.2 by $\pi : \mathbb{P}\mathcal{S} \to \mathrm{Gr}(r+1,n+1)$. This convention should not cause confusion, since $\Phi_{r,n} \cong \mathbb{P}\mathcal{S}$ and $\mathbb{G}\mathrm{r}(r,n) \cong \mathrm{Gr}(r+1,n+1)$.

We now make a few simple calculations about $\Phi_{r,n}$ that will be useful to us later.

Lemma 2.4. *We have* $\dim \Phi_{r,n} = r(n-r)+n$.

Proof. The dimension of a projective bundle $\mathbb{P}V \to X$ is given by $\dim \mathbb{P}V = \dim X + \operatorname{rank} V - 1$, so it suffices to recall the computations $\dim \mathrm{Gr}(r+1,n+1) = (r+1)(n-r)$ and $\operatorname{rank} \mathcal{S} = r+1$. □

Lemma 2.5. *Let* $\Phi := \Phi_{r,n}$ *and* $G := \mathrm{Gr}(r+1,n+1)$*. Then* $\omega_\Phi \cong \mathcal{O}_\Phi(-r-1) \otimes \pi^*\mathcal{O}_G(-n)$*, where* ω *denotes the canonical bundle.*

Proof. Let $V \to X$ be a vector bundle of rank v, and let $\pi : \mathbb{P}V \to X$ be the associated projective bundle. Then there is an isomorphism

$$\omega_{\mathbb{P}V} \cong \mathcal{O}_{\mathbb{P}V}(-v) \otimes \pi^* \det V^\vee \otimes \pi^* \omega_X.$$

Add ref/proof. Since $\det \mathcal{S} \cong \mathcal{O}_G(-1)$ and $\omega_G \cong \mathcal{O}_G(-n-1)$, we find that

$$\begin{aligned}\omega_\Phi &\cong \mathcal{O}_\Phi(-r-1) \otimes \pi^*\mathcal{O}_G(1) \otimes \pi^*\mathcal{O}_G(-n-1)\\ &\cong \mathcal{O}_\Phi(-r-1) \otimes \pi^*\mathcal{O}_G(-n).\end{aligned}$$ □

Lemma 2.6. *Assume the same notation as Lemma 2.5. Let $\Omega_{\Phi/G}$ denote the relative cotangent bundle of $\pi : \Phi \to G$. Then $\operatorname{rank} \Omega_{\Phi/G} = r$.*

Proof. Since Φ and G are both smooth, we have $\operatorname{rank} \Omega_{\Phi/G} = \operatorname{rank} \Omega_\Phi - \operatorname{rank} \Omega_G = \dim \Phi - \dim G$ (add ref/proof). We then compute $\dim \Phi = r(n-r) + n$ and $\dim G = (r+1)(n-r)$, so $\operatorname{rank} \Omega_{\Phi/G} = r(n-r) + n - (r+1)(n-r) = r$. □

Lemma 2.7. *Assume the same notation as Lemma 2.6. Then $\det \Omega_{\Phi/G} \cong \mathcal{O}_\Phi(-r-1) \otimes \pi^*\mathcal{O}_G(1)$.*

Proof. By the relative Euler sequence

$$0 \to T_{\Phi/G} \to T_\Phi \to \pi^* T_G \to 0,$$

we have $\det T_{\Phi/G} \cong \det T_\Phi \otimes \pi^* \det T_G^\vee$. Thus $\det \Omega_{\Phi/G} \cong \det T_\Phi^\vee \otimes \pi^* \det T_G$. In Lemma 2.5, we computed $\det T_\Phi^\vee \cong \omega_\Phi \cong \mathcal{O}_\Phi(-r-1) \otimes \pi^*\mathcal{O}_G(-n)$. We also used the fact that $\omega_G \cong \mathcal{O}_G(-n-1)$. Putting these together, we have

$$\begin{aligned}\det \Omega_{\Phi/G} &\cong \mathcal{O}_\Phi(-r-1) \otimes \pi^*\mathcal{O}_G(-n) \otimes \pi^*\mathcal{O}_G(n+1)\\ &\cong \mathcal{O}_\Phi(-r-1) \otimes \pi^*\mathcal{O}_G(1).\end{aligned}$$ □

2.2. **Bundle of relative principal parts.** We have constructed a scheme $\Phi_{r,n}$ that parameterizes pointed r-planes in $\mathbb{P}^n$. We now want a vector bundle $\mathcal{E}_m \to \Phi_{r,n}$ such that, given a hypersurface $X \subset \mathbb{P}^n$ of degree d, we get a section $\sigma_X : \Phi_{r,n} \to \mathcal{E}_m$ that vanishes on a pointed r-plane (H, p) if and only if H meets X to order m at p. Bundles of principal parts give a natural setting in which to study higher order intersections. Because our parameter space is really a projective bundle, we will need to use the bundle of *relative* principal parts.

Let $\pi : X \to Y$ be a smooth proper map of schemes. Let V be a vector bundle on X. The *bundle of relative principal parts* $\mathcal{P}^m_{X/Y}(V)$ is a vector bundle whose stalk at $x \in X$ is the vector space

$$\mathcal{P}^m_{X/Y}(V)_x = \frac{\{\text{germs of sections of } V|_{\pi^{-1}(\pi(x))} \text{ vanishing at } x\}}{\{\text{germs vanishing to order } \geq m+1 \text{ at } x\}}.$$

From this description, it is apparent that we should take $\mathcal{E}_m := \mathcal{P}^{m-1}_{\Phi/G}(\beta^*\mathcal{O}_{\mathbb{P}^n}(d))$. Indeed, a hypersurface $X \subset \mathbb{P}^n$ determines a section $\sigma_X \in H^0(\Phi_{r,n}, \beta^*\mathcal{O}_{\mathbb{P}^n}(d))$, and the germ of σ_X vanishing at (H, p) to order m corresponds to H meeting X to order at least m at p.

Remark 2.8. In positive characteristic, unexpected tangencies can arise due to the power rule. This prevents $\mathcal{P}^{m-1}_{\Phi/G}(\beta^*\mathcal{O}_{\mathbb{P}^n}(d))$ from accounting for all r-planes that are tangent to X to order m at some point. However, we can circumvent this issue by assuming that $\operatorname{char} k > d$ (or $\operatorname{char} k = 0$).

"unexpected tangencies can arise" seems to suggest overcounting, while "prevents from accounting for all..." seems to suggest undercounting. It's supposed to be too many, right?

What I was thinking was that the power rule gives us more tangent planes than the bundle of principal parts can account for. We should actually work out the details and see what goes wrong in positive characteristic.

For a precise definition and detailed discussion of bundles of relative principal parts, see [EH16, §11.1.1]. For our purposes, it will suffice to mention a few properties.

Lemma 2.9. *Let $\pi : X \to Y$ be a smooth proper map of k-schemes. Let V be a vector bundle on X. Then $\mathcal{P}^0_{X/Y}(V) := V$. Moreover, for $m > 0$ (if $\operatorname{char} k = 0$) or $\operatorname{char} k > m > 0$ (if $\operatorname{char} k > 0$), there is a short exact sequence*

$$0 \to V \otimes \operatorname{Sym}^m(\Omega_{X/Y}) \to \mathcal{P}^m_{X/Y}(V) \to \mathcal{P}^{m-1}_{X/Y}(V) \to 0.$$

Proof. See [EH16, Theorem 11.2]. Probably should say something here about positive characteristic? Citing 3264 for the characteristic 0 case should be fine. □

Remark 2.10. If $\operatorname{char} k \neq 0$, then our assumption that $m < \operatorname{char} k$ ensures that $m! \in k^\times$ and hence $\operatorname{Sym}^m(\Omega_{X/Y})$ is smooth. Check if this is necessary, mention 3264. Paired with Remark 2.8, we are assuming $\operatorname{char} k = 0$ or $\operatorname{char} k > \max\{d, m\}$.

Using Lemma 2.9, we can now calculate the rank and determinant of $\mathcal{P}^{m-1}_{\Phi/G}(\beta^*\mathcal{O}_{\mathbb{P}^n}(d))$.

Corollary 2.11. *Let $\Phi := \Phi_{r,n}$ and $G = \mathbb{G}\mathrm{r}(r,n)$. Let $m \geq 1$. Then*

$$\operatorname{rank} \mathcal{P}^{m-1}_{\Phi/G}(\beta^*\mathcal{O}_{\mathbb{P}^n}(d)) = 1 + \sum_{i=1}^{m-1} \binom{i+r-1}{i} = \frac{m}{r}\binom{m+r-1}{m}.$$

$$\operatorname{rank} \mathcal{P}^{m-1}_{\Phi/G}(\beta^*\mathcal{O}_{\mathbb{P}^n}(d)) = \binom{m+r-1}{r}.$$

Proof. Let $\mathcal{E}_i := \mathcal{P}^i_{\Phi/G}(\beta^*\mathcal{O}_{\mathbb{P}^n}(d))$ and $r_i := \operatorname{rank} \mathcal{E}_i$. Because rank is additive along short exact sequences, we can compute inductively. By definition, $\mathcal{E}_0 = \beta^*\mathcal{O}_{\mathbb{P}^n}(d)$, which is a line bundle. Thus $r_0 = 1$. For $i > 0$, Lemma 2.9 implies that

$$r_i = r_{i-1} + \operatorname{rank}(\beta^*\mathcal{O}_{\mathbb{P}^n}(d) \otimes \operatorname{Sym}^i(\Omega_{\Phi/G})).$$

Since $\operatorname{rank} \beta^*\mathcal{O}_{\mathbb{P}^n}(d) = 1$, we have $\operatorname{rank}(\beta^*\mathcal{O}_{\mathbb{P}^n}(d) \otimes \operatorname{Sym}^i(\Omega_{\Phi/G})) = \operatorname{rank} \operatorname{Sym}^i(\Omega_{\Phi/G})$. If V is a vector bundle of rank v, then $\operatorname{rank} \operatorname{Sym}^i(V) = \binom{i+v-1}{i}$. By Lemma 2.6, we thus have $\operatorname{rank} \operatorname{Sym}^i(\Omega_{\Phi/G}) = \binom{i+r-1}{i}$, so $r_i = r_{i-1} + \binom{i+r-1}{i}$. □

Corollary 2.12. *Assume the notation of Corollary 2.11. Then*

$$\det \mathcal{P}^{m-1}_{\Phi/G}(\beta^*\mathcal{O}_{\mathbb{P}^n}(d)) \cong \beta^*\mathcal{O}_{\mathbb{P}^n}(d(1+A)) \otimes \mathcal{O}_\Phi((-r-1)B) \otimes \pi^*\mathcal{O}_G(B),$$

where $A = \sum_{i=1}^{m-1}\binom{i+r-1}{i} = \frac{m}{r}\binom{m+r-1}{m} - 1$ *and* $B = \sum_{i=1}^{m-1}\binom{i+r-1}{i-1} = \frac{m-1}{r+1}\binom{m+r-1}{m-1}$.

Proof. Let $\mathcal{E}_i := \mathcal{P}^i_{\Phi/G}(\beta^*\mathcal{O}_{\mathbb{P}^n}(d))$. Because determinants are multiplicative along short exact sequences, we can compute inductively. As $\mathcal{E}_0 = \beta^*\mathcal{O}_{\mathbb{P}^n}(d)$ is a line bundle, we have $\det \mathcal{E}_0 = \mathcal{E}_0$. For $i > 0$, Lemma 2.9 implies that

$$\det \mathcal{E}_i \cong \det \mathcal{E}_{i-1} \otimes \det(\beta^*\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathrm{Sym}^i(\Omega_{\Phi/G})).$$

Given vector bundles V and W of ranks v and w, respectively, there is an isomorphism $\det(V\otimes W) \cong (\det V)^{\otimes w} \otimes (\det W)^{\otimes v}$. One can show that $\det \mathrm{Sym}^i(V) = (\det V)^{\otimes \binom{i+v-1}{i-1}}$. We have already seen that $\mathrm{rank}\,\mathrm{Sym}^i(\Omega_{\Phi/G}) = \binom{i+r-1}{i}$, so Lemma 2.7 gives us

$$\begin{aligned}
&\det(\beta^*\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathrm{Sym}^i(\Omega_{\Phi/G}))\\
&\quad\cong \beta^*\mathcal{O}_{\mathbb{P}^n}(d)^{\otimes\binom{i+r-1}{i}} \otimes \det \mathrm{Sym}^i(\Omega_{\Phi/G})^{\otimes 1}\\
&\quad\cong \beta^*\mathcal{O}_{\mathbb{P}^n}\left(d\binom{i+r-1}{i}\right) \otimes (\det \Omega_{\Phi/G})^{\otimes\binom{i+r-1}{i-1}}\\
&\quad\cong \beta^*\mathcal{O}_{\mathbb{P}^n}\left(d\binom{i+r-1}{i}\right) \otimes (\mathcal{O}_\Phi(-r-1)\otimes \pi^*\mathcal{O}_G(1))^{\otimes\binom{i+r-1}{i-1}}\\
&\quad\cong \beta^*\mathcal{O}_{\mathbb{P}^n}\left(d\binom{i+r-1}{i}\right) \otimes \mathcal{O}_\Phi\left((-r-1)\binom{i+r-1}{i-1}\right) \otimes \pi^*\mathcal{O}_G\left(\binom{i+r-1}{i-1}\right).
\end{aligned}$$

We now obtain the desired result by applying this computation to $\det \mathcal{E}_{m-1} \cong \beta^*\mathcal{O}_{\mathbb{P}^n}(d) \otimes \bigotimes_{i=1}^{m-1} \det \mathcal{E}_i$. ☐

Corollary 2.13. *Assume the notation of Corollary 2.11. For any non negative integer k we have*

$$\det \mathcal{P}^k_{\Phi/G}(\beta^*\mathcal{O}_{\mathbb{P}^n}(d)) \cong \mathcal{O}_\Phi\left(d\binom{k+r}{k} - (r+1)\binom{r+k}{k-1}\right) \otimes \pi^*\mathcal{O}_G\left(\binom{r+k}{k-1}\right).$$

Proof. We argue by induction on k. If $k=0$ the thesis follows from

$$\beta^*\mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{O}_\Phi(1). \tag{2.1}$$

Suppose the result true up to $k-1$. From

$$\begin{aligned}
&\ker(\det \mathcal{P}^i_{\Phi/G}(\beta^*\mathcal{O}_{\mathbb{P}^n}(d)) \to \det \mathcal{P}^{i-1}_{\Phi/G}(\beta^*\mathcal{O}_{\mathbb{P}^n}(d)))\\
\cong&\det(\beta^*\mathcal{O}_{\mathbb{P}^n}(d)\otimes \mathrm{Sym}^i(\Omega_{\Phi/G}))\\
\cong&\mathcal{O}_\Phi\left(d\binom{i+r-1}{i} - (r+1)\binom{i+r-1}{i-1}\right) \otimes \pi^*\mathcal{O}_G\left(\binom{i+r-1}{i-1}\right),
\end{aligned}$$

and (see [Knu97, (10)])

$$\sum_{i=0}^{k}\binom{r+i}{i} = \binom{r+1+k}{k},$$

we have the result. ☐

In particular, $\det \mathcal{P}^{m-1}_{\Phi/G}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right) \otimes \omega_\Phi$ is isomorphic to

$$\mathcal{O}_\Phi\left(d\binom{r+m-1}{m-1} - (r+1)\left[\binom{r+m-1}{m-2} - 1\right]\right) \otimes \pi^*\mathcal{O}_G\left(\binom{r+m-1}{m-2} - n\right).$$

This implies that $\mathcal{P}^{m-1}_{\Phi/G}$ is relatively orientable if and only if the following equations hold:

$$\begin{aligned} r(n-r)+n &= \binom{m+r-1}{m-1} && \text{rank condition,}\\ d\binom{r+m-1}{m-1} &\equiv (r+1)\left[\binom{r+m-1}{m-2} - 1\right] \bmod 2 && \text{parity of } \mathcal{O}_\Phi,\\ \binom{r+m-1}{m-2} &\equiv n \bmod 2 && \text{parity of } \pi^*\mathcal{O}_G. \end{aligned}$$

These coincide with Corollary 3.3 if we take into account Equation (2.1). This is a good news because it increases the number of tuples (d, r, m, n) such that $\mathcal{P}^{m-1}_{\Phi/G}$ is relatively orientable.

3. Relative orientability

3.1. **General computations.** Orientability hypotheses for vector bundles are often needed when giving quadratic counts for an enumerative problem. We recall the setup from [BW23].

Definition 3.1. Let X be a a smooth proper k-scheme $X \to \operatorname{Spec} k$ of dimension n, a rank n vector bundle V over X is said to be orientable if there exists a line bundle $\mathcal{L}$ and an isomorphism $\rho : \det V \otimes \omega_{X/k} \to \mathcal{L}^{\otimes 2}$.

Given an orientable vector bundle, coherent duality defines a trace map that allows us to construct the Euler number of the vector bundle valued in $\mathrm{GW}(k)$, the Grothendieck-Witt ring of the field k. Herein, we describe a family of orientable problems: tuples $(d, r, m, n) \in \mathbb{N}^4$ such that $\mathcal{P}^{m-1}_{\Phi/G}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)$ is is the tensor square of a line bundle. Sections of this bundle would correspond to r-planes meeting a degree d hypersurface $X \subseteq \mathbb{P}^n_k$ at a point of order of contact m.

Lemma 3.2. *Let Φ be the flag variety of pointed r-planes in $\mathbb{P}^n_k$ and $\mathcal{P}^{m-1}_{\Phi/G}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)$ is the $(m-1)$-th bundle of principal parts of the pullback of $\mathcal{O}_{\mathbb{P}^n_k}(d)$ to Φ via β. Then*

$$\det \mathcal{P}^{m-1}_{\Phi/G}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right) \otimes \omega_\Phi \cong \beta^*\mathcal{O}_{\mathbb{P}^n}(d(1+A)) \otimes \mathcal{O}_\Phi((-r-1)B - r - 1) \otimes \pi^*\mathcal{O}_G(B-n)$$

where $A = \sum_{i=1}^{m-1}\binom{i+r-1}{i} = \frac{m}{r}\binom{m+r-1}{m} - 1$ *and* $B = \sum_{i=1}^{m-1}\binom{i+r-1}{i-1} = \frac{m-1}{r+1}\binom{m+r-1}{m-1}$.

Proof. Combining Lemma 2.5 and Corollary 2.12, we compute the above. ☐

This allows us to determine when bundles of principal parts are orientable. More explicitly, we deduce the following:

Corollary 3.3. *Assume the notation of Lemma 3.2. The bundle of principal parts* $\mathcal{P}^{m-1}_{\Phi/G}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)$ *is orientable if and only d, r, m, n satisfy the following equations:*

$$\begin{aligned} r(n-r)+n &= \binom{m+r-1}{m-1}, \\ d\binom{m+r-1}{m-1} &\equiv 0 \pmod 2, \\ -(r+1)\left(\frac{m-1}{r+1}\binom{m+r-1}{m-1}+1\right) &\equiv 0 \pmod 2, \\ \frac{m-1}{r+1}\binom{m+r-1}{m-1} - n &\equiv 0 \pmod 2. \end{aligned}$$

Table 1 reports several small examples of orientable bundles of principal parts: tuples $(d, r, m, n) \in \mathbb{N}^4$ describing a r-plane meeting a degree d hypersurface in $\mathbb{P}^n_k$ to order m. The tuples in Table 1 were found by brute force computer search. The code and associated files can be found in the `GitHub` repository (see Section 1.2).

(6, 3, 5, 11)	(22, 3, 21, 445)	(38, 3, 37, 2287)
(8, 3, 5, 11)	(24, 3, 21, 445)	(40, 3, 37, 2287)
(10, 3, 5, 11)	(26, 3, 21, 445)	(42, 3, 37, 2287)

Table 1. A list of tuples (d, r, m, n) satisfying the conditions of Corollary 3.3 for $1 \leq d, r, m, n \leq 4000$.

We can actually clean this up a little. None of the orientability computations involve d, so we're searching for r, m, n that give relative orientability. Then we can take $d \geq m$. Right, so we can check triples m, r, n that satisfy the condition and Proposition **??** and for those that $(-r-1)B - r - 1$ and $B - n$ are 0 mod 2. For those m, r, n, either $d = m$ or $d = m + 1$ will cause $d(1 + A)$ to be 0 mod 2. This makes this $O(n^3)$ when searching for solutions. We can parallelize code (see the file in the GitHub), we can make this $O(n^3/c)$ where c is the number of cores. Checking $n \leq 1000$ can be done in in about 2h on a 16 core machine yielding

$$(6, 3, 5, 11), (22, 3, 21, 445)$$

where the computation for 500 previously took the better part of a day. I'm currently running $n \leq 2000$ on the Bowdoin cluster (which is curiously taking longer than a day). Hopefully can get up to 5,000.

Example 3.4. The bundle $\mathcal{P}^4_{\Phi/\mathbb{G}\mathrm{r}(3,11)}\left(\beta^*\mathcal{O}_{\mathbb{P}^{11}_k}(6)\right)$ describing 3-planes meeting a sextic hypersurface in $\mathbb{P}^{11}_k$ is orientable.

Generically, a r-plane in $\mathbb{P}^n_k$ intersects a degree d hypersurface $X \subseteq \mathbb{P}^n_k$ along a subscheme of dimension $r - 1$. This subscheme is a finite number of closed points if and only if it is dimension 0, that is $r = 1$, where the line meets X at exactly d points counted with multiplicity. Note that this implies that if a line ℓ meets a degree d hypersurface in $\mathbb{P}^n_k$ to order d at a point p, we have $X \cap \ell = \{p\}$ which results in especially nice geometry when $d = m$. This motivates us to focus on the cases where $r = 1$ in which case $\dim \Phi_{1,n} = 2n-1$ which we set equal to $\operatorname{rank} \mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(m)\right) = m$ which in turn leads us to consider the orientability of the bundle $\mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(m)\right)$.

Corollary 3.5. *Assume the notation of Lemma 3.2. Then*

$$\begin{aligned}&\det\mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(m)\right)\otimes\omega_\Phi\\ &\qquad\cong \beta^*\mathcal{O}_{\mathbb{P}^n_k}(m^2)\otimes\mathcal{O}_\Phi(-m^2+m-2)\otimes\pi^*\mathcal{O}_{\mathbb{G}\mathrm{r}(1,n)}\left(\frac{m^2-2m-1}{2}\right).\end{aligned}$$

Proof. We specialize to $r = 1, m = d = 2n-1$. From Lemma 3.2, we have $A = m-1, B = \frac{m^2-m}{2}$ and making the appropriate substitutions for r, d, n we yield

$$\begin{aligned}&\cong \beta^*\mathcal{O}_{\mathbb{P}^n_k}(m^2)\otimes\mathcal{O}_\Phi\left(-2\left(\frac{m^2-m}{2}\right)-2\right)\otimes\pi^*\mathcal{O}_{\mathbb{G}\mathrm{r}(1,n)}\left(\frac{m^2-m}{2}-\frac{m+1}{2}\right)\\ &\qquad\cong \beta^*\mathcal{O}_{\mathbb{P}^n_k}(m^2)\otimes\mathcal{O}_\Phi(-m^2+m-2)\otimes\pi^*\mathcal{O}_{\mathbb{G}\mathrm{r}(1,n)}\left(\frac{m^2-2m-1}{2}\right)\end{aligned}$$

as desired. ∎

Note, however, that $\det\mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(m)\right)\otimes\omega_\Phi$ is not the tensor-square of a line bundle since for all $n \in \mathbb{N}$, $2n-1 = m$ is odd which causes both m^2, $\frac{m^2-2m-1}{2}$ to be odd. In this setting, however, one can still compute Euler numbers relative to a divisor. Section on relative orientation relative to a divisor is moved to the following section.

4. Local indices-second attempt

Let $\{e_1, \dots, e_{n+1}\}$ denote a basis for k^{n+1}. A rational point $p \in \mathbb{G}\mathrm{r}(r,n)$ parameterizes a vector subspace

$$H = \left\langle \sum_{i=1}^{n+1}\tilde{x}_{1,i}e_i, \sum_{i=1}^{n+1}\tilde{x}_{2,i}e_i, \dots, \sum_{i=1}^{n+1}\tilde{x}_{r+1,i}e_i \right\rangle. \tag{4.1}$$

Note that H is the space generated by the column vector

$$\begin{pmatrix}\tilde{x}_{1,1} & \cdots & \tilde{x}_{1,n+1}\\ \tilde{x}_{2,1} & \cdots & \tilde{x}_{2,n+1}\\ \vdots & \vdots & \vdots\\ \tilde{x}_{r+1,1} & \cdots & \tilde{x}_{r+1,n+1}\end{pmatrix}\begin{pmatrix}e_1\\ e_2\\ \vdots\\ e_{n+1}\end{pmatrix}.$$

We denote by M the matrix $(\tilde{x}_{t,s})^{n+1}_{t,s=1}$. There esists a standard open cover of $\mathbb{G}\mathrm{r}(r,n)$: Take an ordered set of $r+1$ indeces $I = \{i_\imath\}^{r+1}_{\imath=1}$ such that $1 \le i_1 < \ldots < i_{r+1} \le n+1$, define U_I to be the subset of all points H such that $\det((\tilde{x}_{t,i_s})^{n+1}_1) \neq 0$. Thus $\mathbb{G}\mathrm{r}(r,n)$ is the union of all U_I for any set of indexes I. Consider local coordinates

$$\varphi\colon U_I \longrightarrow \operatorname{Spec} k[x_{1,1}, \dots, x_{r+1,n-r}]$$

such that $(x_{1,1},\dots,x_{r+1,n-r})$ corresponds to the span of the set of vectors $\{\widetilde{e}_i\}_{i\in I}$ where

$$\widetilde{e}_{i_\imath} := e_{i_\imath} + \sum_{j=1}^{n-r} x_{\imath,j} e_{\imath_j} \tag{4.2}$$

where $I^c = \{\imath_1, \imath_2, \dots, \imath_{n-r}\}$. We define the standard atlas $\{(U_I, \varphi_I)\}_I$. If $H \in \mathbb{G}\mathrm{r}(r,n)$ is expressed like in Equation (4.1), we obtain its coordinates $(x_{1,1},\dots)$ in the following way: Let M_I be the square matrix obtained from M by considering only the columns $\tilde{x}_{*,i}$ where $i \in I$. The product $M' = M_I^{-1}M$ is a matrix such that M'_I is the identity matrix. The coefficients $x_{*,*}$ are the coefficients of M' not in M'_I.

The fiber of the vector bundle $\Pi\colon \mathcal{S} \to \mathbb{G}\mathrm{r}(r,n)$ at a point $H \in \mathbb{G}\mathrm{r}(r,n)$ is the vector space parameterized by H. This means that, supposing $H \in U_I$, $\Pi^{-1}(H) \cong \mathbb{A}_k^{r+1}$ where $\mathbb{A}_k^{r+1}$ has coordinates $(y_1,\dots,y_{r+1})$ and parameterizes all linear combinations of the vectors $\{\widetilde{e}_i\}_{i\in I}$ in Equation (5.1). That is, $\Pi^{-1}(p)$ parameterizes vectors

$$v = \sum_{\imath=1}^{r+1} y_\imath \tilde{e}_{i_\imath}. \tag{4.3}$$

Using this language, it is easy to write the transition functions of $\Pi\colon \mathcal{S} \to \mathbb{G}\mathrm{r}(r,n)$ for the atlas $\{(\Pi^{-1}(U_I), \varphi_I \times \mathrm{id})\}_I$. It is enough to consider maps

$$\varphi_J \times \mathrm{id} \circ (\varphi_I \times \mathrm{id})^{-1}\colon \mathbb{A}_k^{(r+1)(n-r)} \times \mathbb{A}_k^{r+1} \longrightarrow \mathbb{A}_k^{(r+1)(n-r)} \times \mathbb{A}_k^{r+1} \tag{4.4}$$

sending coordinates $(x_{1,1},\dots) \times (y_1,\dots,y_{r+1})$ to coordinates $(x'_{1,1},\dots) \times (y'_1,\dots,y'_{r+1})$ where $x'_{*,*}$ are like before and

$$(y'_1,\dots,y'_{r+1}) = (y_1,\dots,y_{r+1}) N_{I,J},$$

where $N_{I,J}$ is the matrix $(M_I^{-1}M)_J$. We deduce that from the simple relation

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_{r+1} \end{pmatrix} \begin{pmatrix} \widetilde{e}_{i_1} \\ \widetilde{e}_{i_2} \\ \vdots \\ \widetilde{e}_{i_{r+1}} \end{pmatrix} = \begin{pmatrix} y_1 & y_2 & \cdots & y_{r+1} \end{pmatrix} N_{I,J} N_{I,J}^{-1} M_I^{-1} M \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n+1} \end{pmatrix}. \tag{4.5}$$

The projective bundle $\pi\colon \Phi \to \mathbb{G}\mathrm{r}(r,n)$ parameterizes pairs $(H, \langle v \rangle)$ where H is like in Equation (4.1) and $\langle v \rangle$ is the projective line generated by the non-zero vector in Equation (4.3). An open cover of Φ is given by subsets $U_{I,p}$ where I is a set of indeces and $p \in \{1,\dots,r+1\}$. Those sets parameterize points $(H, \langle v \rangle)$ where $H \in U_I$ and the coordinate of e_p of v is nonzero.

4.1. **Case of lines.** Let us fix $r = 1$. Let $z_1,\dots,z_{n+1}$ be projective coordinates of $\mathbb{P}^n$. Let $w_{1,2},\dots,w_{a,b},\dots,w_{n,n+1}$ with $1 \le a < b \le n+1$ be projective coordinates of $\mathbb{P}^{n(n+1)/2-1}$. The map $\beta\colon \Phi \to \mathbb{P}^n$ is defined as

$$\langle \sum_{i=1}^{n+1} \tilde{x}_{1,i} e_i, \sum_{i=1}^{n+1} \tilde{x}_{2,i} e_i \rangle, \langle y_1 \sum_{i=1}^{n+1} \tilde{x}_{1,i} e_i + y_2 \sum_{i=1}^{n+1} \tilde{x}_{2,i} e_i \rangle \longmapsto \langle \sum_{i=1}^{n+1} (y_1 \tilde{x}_{1,i} + y_2 \tilde{x}_{2,i}) e_i \rangle,$$

and the Plücker map composed with π is $\Phi \to \mathbb{P}^{(n(n+1)/2)-1}$, that is

$$\langle \sum_{i=1}^{n+1} \tilde{x}_{1,i} e_i, \sum_{i=1}^{n+1} \tilde{x}_{2,i} e_i \rangle, \sum_{i=1}^{n+1} z_i e_i \longmapsto \langle \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} (\tilde{x}_{1,i}\tilde{x}_{2,j} - \tilde{x}_{1,j}\tilde{x}_{2,i}) e_i \wedge e_j \rangle.$$

The projective bundle $\pi\colon \Phi \to \mathbb{G}\mathrm{r}(1,n)$ admits an atlas $\{(U_{I,p}, \varphi_{I,p})\}_{I,p}$ where I is a set of indeces as before, and $p \in \{1,2\}$. The open subsets $U_{I,p}$ parameterizes pairs $(H, \langle v \rangle)$ where

$$\begin{aligned} \tilde{x}_{1,i_1}\tilde{x}_{2,i_2} - \tilde{x}_{1,i_2}\tilde{x}_{2,i_1} &\neq 0 \\ y_1\tilde{x}_{1,i_p} + y_2\tilde{x}_{2,i_p} &\neq 0. \end{aligned}$$

The functions $\varphi_{I,p}\colon U_{I,p} \to \mathbb{A}^{2(n-2)} \times \mathbb{A}^1$ are defined in the following way:

$$\varphi_{I,p}(p, \langle v \rangle) = \left(\varphi_I(p), \frac{y_1\tilde{x}_{1,i_{\bar{p}}} + y_2\tilde{x}_{2,i_{\bar{p}}}}{y_1\tilde{x}_{1,i_p} + y_2\tilde{x}_{2,i_p}} \right),$$

where we denoted by $\bar{p}$ the integer such that $\{p, \bar{p}\} = \{1,2\}$.

4.2. **Transition functions.** In this section, we compute explicitly the transition functions of the atlas $\{(U_{I,p}, \varphi_{I,p})\}_{I,p}$. These are necessary for computing the transition functions of the bundle $\det(T_\Phi)$. Just for simplifying the notation, we compute the transition functions of $\varphi_{J,q} \circ \varphi_{I,p}^{-1}$ where I is fixed to be the pair of indices $\{n, n+1\}$. The general case does not require additional care other than a more complicated notation. Let

$$\varphi_{J,q} \circ \varphi_{I,p}^{-1}((x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2}, \ldots), (y)) = (x'_{1,1}, x'_{2,1}, x'_{1,2}, x'_{2,2}, \ldots), (y'). \tag{4.6}$$

From our previous discussion, the following equality holds:

$$\begin{pmatrix} x_{1,j_1} & x_{1,j_2} \\ x_{2,j_1} & x_{2,j_2} \end{pmatrix}^{-1} \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n-1} & 1 & 0 \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n-1} & 0 & 1 \end{pmatrix} = \begin{pmatrix} x'_{1,1} & \cdots & x'_{1,n-1} \\ x'_{2,1} & \cdots & x'_{2,n-1} \end{pmatrix},$$

where the columns j_1 and j_2 of the matrix on the right form the identity matrix.

Let $\Delta_{I,J} := \det N_{I,J} = x_{1,j_1}x_{2,j_2} - x_{1,j_2}x_{2,j_1}$. An easy computation shows that for any $\alpha \in \{1, 2, \ldots, n-3\}$, denoting by $\bar{\alpha}$ the lowest integer such that $|\{1, 2, \ldots, \bar{\alpha}\} \setminus J| = \alpha$, we have

$$\begin{aligned} x'_{1,\alpha} &= x_{1,\bar{\alpha}} \frac{x_{2,j_2}}{\Delta_{I,J}} + x_{2,\bar{\alpha}} \frac{-x_{1,j_2}}{\Delta_{I,J}} & x'_{1,n-2} &= \frac{x_{2,j_2}}{\Delta_{I,J}} & x'_{1,n-1} &= \frac{-x_{1,j_2}}{\Delta_{I,J}} \\ x'_{2,\alpha} &= x_{1,\bar{\alpha}} \frac{-x_{2,j_1}}{\Delta_{I,J}} + x_{2,\bar{\alpha}} \frac{x_{1,j_1}}{\Delta_{I,J}} & x'_{2,n-2} &= \frac{-x_{2,j_1}}{\Delta_{I,J}} & x'_{2,n-1} &= \frac{x_{1,j_1}}{\Delta_{I,J}}. \end{aligned}$$

In order to compute y', recall that the integer $\bar{q}$ is the integer such that $\{q, \bar{q}\} = \{1,2\}$. Hence

$$y' = \frac{x_{q,i_{\bar{p}}} + y x_{\bar{q},i_{\bar{p}}}}{x_{q,i_p} + y x_{\bar{q},i_p}}.$$

Let us compute the transition function of $\det(T_\Phi)$. We adopt the following trick. In Equation (4.6), we move to the right both the variables (x_{1,j_1}, x_{2,j_1}) of exactly $2(n-1-j_1)$ positions, in such a way that the new coordinates are

$$(x_{1,1}, x_{2,1}, \ldots, x_{1,j_1-1}, x_{2,j_1-1}, x_{1,j_1+1}, x_{2,j_1+1}, x_{1,j_1+2}, x_{2,j_1+2}, \ldots, x_{1,j_1}, x_{2,j_1}), (y).$$

Now we move to the right also (x_{1,j_2}, x_{2,j_2}) of $2(n-1-j_2)$ positions, and call the new set of coordinates "shifted." Those coordinates are

$$(x_{1,1},\dots,x_{2,j_1-1},x_{1,j_1+1},\dots,x_{2,j_2-1},x_{1,j_2+1},\dots,x_{1,n-1},x_{2,n-1},x_{1,j_1},x_{2,j_1},x_{1,j_2},x_{2,j_2}),(y).$$

Let us compute the Jacobian matrix with respect to the shifted coordinates. The derivation with respect to the variables $x_{1,1}$ and $x_{1,2}$ of all the functions $x'_{*,*}$ gives the first and second columns of the matrix. Those variables only appear in $x'_{1,1}$ and $x'_{2,1}$ (supposing that $j_1 > 1$), so we deduce that the first two columns of the Jacobian are

$$\begin{pmatrix} \frac{x_{2,j_2}}{\Delta_{I,J}} & \frac{-x_{1,j_2}}{\Delta_{I,J}} \\ \frac{-x_{2,j_1}}{\Delta_{I,J}} & \frac{x_{1,j_1}}{\Delta_{I,J}} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}. \tag{4.7}$$

We may repeat the same argument for other variables. We deduce that the contribution to the determinant of the functions $(x'_{1,\alpha}, x'_{2,\alpha})$ is given by the non trivial square matrix in Equation (4.7), whose determinant is $\Delta_{I,J}^{-1}$. It remains the contribution of the submatrix of order 5 given by the last 5 variables $x_{1,j_1}, x_{2,j_1}, x_{1,j_2}, x_{2,j_2}, y$, which is

$$W = \begin{pmatrix} -\frac{x_{2,j_2}^2}{\Delta_{I,J}^2} & \frac{x_{1,j_2}x_{2,j_2}}{\Delta_{I,J}^2} & \frac{x_{2,j_1}x_{2,j_2}}{\Delta_{I,J}^2} & -\frac{x_{1,j_2}x_{2,j_1}}{\Delta_{I,J}^2} & 0 \\ \frac{x_{2,j_1}x_{2,j_2}}{\Delta_{I,J}^2} & -\frac{x_{1,j_1}x_{2,j_2}}{\Delta_{I,J}^2} & -\frac{x_{2,j_1}^2}{\Delta_{I,J}^2} & \frac{x_{1,j_1}x_{2,j_1}}{\Delta_{I,J}^2} & 0 \\ \frac{x_{1,j_2}x_{2,j_2}}{\Delta_{I,J}^2} & -\frac{x_{1,j_2}^2}{\Delta_{I,J}^2} & -\frac{x_{1,j_1}x_{2,j_2}}{\Delta_{I,J}^2} & \frac{x_{1,j_1}x_{1,j_2}}{\Delta_{I,J}^2} & 0 \\ -\frac{x_{1,j_2}x_{2,j_1}}{\Delta_{I,J}^2} & \frac{x_{1,j_1}x_{1,j_2}}{\Delta_{I,J}^2} & \frac{x_{1,j_1}x_{2,j_1}}{\Delta_{I,J}^2} & -\frac{x_{1,j_1}^2}{\Delta_{I,J}^2} & 0 \\ \frac{\partial y'}{\partial x_{1,j_1}} & \frac{\partial y'}{\partial x_{2,j_1}} & \frac{\partial y'}{\partial x_{1,j_2}} & \frac{\partial y'}{\partial x_{2,j_2}} & \frac{\partial y'}{\partial y} \end{pmatrix}.$$

Note that

$$\begin{aligned} \det(W) &= \frac{1}{\Delta_{I,J}^4}\frac{\partial y'}{\partial y} \\ &= \frac{1}{\Delta_{I,J}^4}\frac{x_{\bar q,i_{\bar p}}x_{q,i_p} - x_{q,i_{\bar p}}x_{\bar q,i_p}}{(x_{q,i_p}+yx_{\bar q,i_p})^2} \\ &= \frac{1}{\Delta_{I,J}^3}\frac{(-1)^{p+q}}{(x_{q,i_p}+yx_{\bar q,i_p})^2}. \end{aligned}$$

Considering that the first $2n-6$ rows and columns of the matrix $J_{\varphi_{J,q}\circ\varphi_{I,p}^{-1}}$ contribute with $\Delta^{-(n-3)}$ to the determinant, eventually we have

$$\det\left(J_{\varphi_{J,q}\circ\varphi_{I,p}^{-1}}\right) = \frac{(-1)^{p+q}}{\Delta_{I,J}^n}\frac{1}{(x_{q,i_p}+yx_{\bar q,i_p})^2}. \tag{4.8}$$

Note that in order to compute the determinant with respect to the shifted coordinates, we applied an even number of transpositions. Thus the determinant with respect to the shifted coordinates coincides with the same computed with the original coordinates.

4.3. **Trivializations and orientations.** This is part is still work in progress. We conclude this sections by showing some trivializations of the line bundles relevant for our discussion. The proof of the following lemma is standard in the field, see for example Lemma 3.8 and Proposition 5.1 of [Mur24].

Lemma 4.1. *The line bundle $\pi^*\mathcal{O}_G(1)$ is locally trivialized over $U_{I,p}$ by*

$$\frac{w_{i_1,i_2}}{w_{1,2}} = \frac{\tilde{x}_{1,i_1}\tilde{x}_{2,i_2} - \tilde{x}_{1,i_2}\tilde{x}_{2,i_1}}{\tilde{x}_{1,1}\tilde{x}_{2,2} - \tilde{x}_{1,2}\tilde{x}_{2,1}}.$$

The line bundle $\mathcal{O}_\Phi(1)$ is locally trivialized over $U_{I,p}$ by

$$\frac{z_{i_p}}{z_1} = \frac{y_1\tilde{x}_{1,i_p} + y_2\tilde{x}_{2,i_p}}{y_1\tilde{x}_{1,1} + y_2\tilde{x}_{2,1}}.$$

The line bundle $\det(\mathcal{E}_m)$ is locally trivialized on $U_{I,p}$ by the distinguished element

$$s_{I,p} := \left(\frac{z_{i_p}}{z_1}\right)^N \left(\frac{w_{i_1,i_2}}{w_{1,2}}\right)^M,$$

where $N = m(d-m+1)$ and $M = m(m-1)/2$.

The integers N and M are obtained from Lemma **??**, imposing $r = 1$.

Now, we need a system of Nisnevich coordinates for Φ and a relative orientation for $\mathcal{E}_m$ compatible with these coordinates. We claim that this system is $\{(U_{I,p}, \tilde{\varphi}_{I,p})\}_{I,p}$ where

$$\tilde{\varphi}_{I,p}(H, \langle v\rangle) = \left(\varphi_I(H), (-1)^p \frac{y_1\tilde{x}_{1,i_{\bar{p}}} + y_2\tilde{x}_{2,i_{\bar{p}}}}{y_1\tilde{x}_{1,i_p} + y_2\tilde{x}_{2,i_p}}\right).$$

Indeed, the coordinates $(U_{I,p}, \tilde{\varphi}_{I,p})$ determine the basis

$$\partial_{I,p} := (-1)^p \frac{\partial}{\partial x_{1,1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{2,n-1}} \wedge \frac{\partial}{\partial y}$$

of $\Gamma(U_{I,p}, \det T_{U_{I,p}})$.

I will continue from here, as this part has been stagnant for a while.

Let $m = 2n-1$, so that a zero section of $\mathcal{E}_m$ parameterizes highly tangent lines. From the equation

$$\det \mathcal{E}_m \otimes \omega_\Phi \cong \mathcal{O}_\Phi((2n-1)(d-2n+2)-2) \otimes \pi^*\mathcal{O}_G(2n^2-4n+1),$$

we deduce that $\mathcal{E}_m$ is not relatively orientable, for any value of n and d.

From now on we suppose that d is even and $D \subset \Phi$ is a divisor parameterizing pointed lines meeting a codimension 2 linear subspace of $\mathbb{P}^n$. In particular, $\mathcal{O}(D) = \pi^*\mathcal{O}_G(1)$. We may suppose, without loss of generality, that the zero locus of a global section 1_D of D parameterizes those lines as in Equation (**??**) such that

$$w_{1,2} = \tilde{x}_{1,1}\tilde{x}_{2,2} - \tilde{x}_{1,2}\tilde{x}_{2,1} = 0$$

We will show that $\mathcal{E}_m$ is relatively orientable relative to D. In particular, following Definition **??**, there exists an isomorphism

$$\psi\colon \operatorname{Hom}(\det T_\Phi, \det \mathcal{E}_m) \otimes \mathcal{O}(D) \longrightarrow L^{\otimes 2},$$

where

$$L := \mathcal{O}_\Phi\left((2n-1)\frac{(d-2n+2)}{2} - 1\right) \otimes \pi^*\mathcal{O}_G(n^2-2n+1).$$

Let us define $\lambda_{I,p}$ to be the linear map sending $\partial_{I,p}$ to $s_{I,p}$. We define $\psi_{I,p}$ on the open cover given by $U_{I,p}$ as the map sending

$$\lambda_{I,p} \otimes \frac{1}{w_{1,2}} \tag{4.9}$$

to

$$\left(\left(\frac{z_{i_p}}{z_1}\right)^{(2n-1)\frac{(d-2n+2)}{2}-1}\left(\frac{w_{i_1,i_2}}{w_{1,2}}\right)^{n^2-2n+1}\right)^2.$$

We prove that the maps $\psi_{I,p}$ so defined glue together giving an isomorphism ψ, such that the image of (4.9) to a square. Note that, by Equation (4.8),

$$\begin{aligned}
\lambda_{J,q}(\partial_{I,p}) &= \left(\frac{\tilde{x}_{1,i_1}\tilde{x}_{2,i_2} - \tilde{x}_{1,i_2}\tilde{x}_{2,i_1}}{\tilde{x}_{1,j_1}\tilde{x}_{2,j_2} - \tilde{x}_{1,j_2}\tilde{x}_{2,j_1}}\right)^n \left(\frac{y_1\tilde{x}_{1,i_p} + y_2\tilde{x}_{2,i_p}}{y_1\tilde{x}_{1,j_q} + y_2\tilde{x}_{2,j_q}}\right)^2 \lambda_{J,q}(\partial_{J,q}) \\
&= \left(\frac{\tilde{x}_{1,i_1}\tilde{x}_{2,i_2} - \tilde{x}_{1,i_2}\tilde{x}_{2,i_1}}{\tilde{x}_{1,j_1}\tilde{x}_{2,j_2} - \tilde{x}_{1,j_2}\tilde{x}_{2,j_1}}\right)^n \left(\frac{y_1\tilde{x}_{1,i_p} + y_2\tilde{x}_{2,i_p}}{y_1\tilde{x}_{1,j_q} + y_2\tilde{x}_{2,j_q}}\right)^2 s_{J,q} \\
&= \left(\frac{\tilde{x}_{1,i_1}\tilde{x}_{2,i_2} - \tilde{x}_{1,i_2}\tilde{x}_{2,i_1}}{\tilde{x}_{1,j_1}\tilde{x}_{2,j_2} - \tilde{x}_{1,j_2}\tilde{x}_{2,j_1}}\right)^{n-M} \left(\frac{y_1\tilde{x}_{1,i_p} + y_2\tilde{x}_{2,i_p}}{y_1\tilde{x}_{1,j_q} + y_2\tilde{x}_{2,j_q}}\right)^{2-N} s_{I,p}.
\end{aligned}$$

Thus $\psi_{J,q}(\lambda_{I,p} \otimes \frac{1}{w_{1,2}})$ is equal to

$$\left(\frac{\tilde{x}_{1,i_1}\tilde{x}_{2,i_2} - \tilde{x}_{1,i_2}\tilde{x}_{2,i_1}}{\tilde{x}_{1,j_1}\tilde{x}_{2,j_2} - \tilde{x}_{1,j_2}\tilde{x}_{2,j_1}}\right)^{M-n+1} \left(\frac{y_1\tilde{x}_{1,i_p} + y_2\tilde{x}_{2,i_p}}{y_1\tilde{x}_{1,j_q} + y_2\tilde{x}_{2,j_q}}\right)^{N-2} \psi_{J,q}\left(\lambda_{J,q} \otimes \frac{1}{w_{1,2}}\right)$$

that is

$$\left(\frac{\tilde{x}_{1,i_1}\tilde{x}_{2,i_2} - \tilde{x}_{1,i_2}\tilde{x}_{2,i_1}}{\tilde{x}_{1,j_1}\tilde{x}_{2,j_2} - \tilde{x}_{1,j_2}\tilde{x}_{2,j_1}}\right)^{4n^2-4n+2} \left(\frac{y_1\tilde{x}_{1,i_p} + y_2\tilde{x}_{2,i_p}}{y_1\tilde{x}_{1,j_q} + y_2\tilde{x}_{2,j_q}}\right)^{(2n-1)(d-2n+2)-2} \psi_{J,q}\left(\lambda_{J,q} \otimes \frac{1}{w_{1,2}}\right)$$

Using linearity, it is easily seen that the maps $\psi_{I,p}$ glue.

I think this part is mature enough to be merged into Article.tex

4.4. Stephen's approach.

It is not clear to me what you are doing. It must be $m = 2n-1$ as we want that the rank of $\mathcal{E}_m$ equals the dimension of Φ. In particular, $\frac{m^2-m-n}{2}$ does not make any sense as m^2-m-n is not even. Moreover, the equation $L^{\otimes 2} \cong \det \mathcal{E}_m \otimes \omega_\Phi$ is surely wrong as $\mathcal{E}_m$ is not relatively orientable, as we have said multiple times.

Let

$$L := \mathcal{O}_\Phi\left(\tfrac{-m^2+(d+1)m-2}{2}\right) \otimes \pi^*\mathcal{O}_G\left(\tfrac{m^2-m-n}{2}\right),$$

so that $L^{\otimes 2} \cong \det \mathcal{E}_m \otimes \omega_\Phi$. We can construct an isomorphismwho is X?

$$\psi_{I,p} : \mathrm{Hom}(\det T_X|_{U_{I,p}}, \det \mathcal{E}_m|_{U_{I,p}}) \longrightarrow L^{\otimes 2}|_{U_{I,p}}$$

as follows. Since $\det T_X|_{U_{I,p}}$ and $\det \mathcal{E}_m|_{U_{I,p}}$ are both 1-dimensional, we can choose the map $\eta_{I,p} := [\partial_{I,p} \mapsto s_{I,p}]$ as a generator for $\mathrm{Hom}(\det T_X|_{U_{I,p}}, \det \mathcal{E}_m|_{U_{I,p}})$. We then define $\psi_{I,p}$ by specifying $\psi_{I,p}(\eta_{I,p}) = \ell_{I,p} \otimes \ell_{I,p}$, where

$$\ell_{I,p} := \left(\frac{z_{i_p}}{z_1}\right)^{\frac{-m^2+(d+1)m-2}{2}} \otimes \left(\frac{w_{i_1,i_2}}{w_{1,2}}\right)^{\frac{m^2-m-n}{2}}$$

generates $L|_{U_{I,p}}$ (and hence $\ell_{I,p}^{\otimes 2}$ generates $L^{\otimes 2}|_{U_{I,p}}$).

To prove that our Nisnevich coordinates and local trivializations are compatible via $\psi_{I,p}$, it remains to show that the transition functions for $\eta_{I,p}$ agree with those of $\ell_{I,p} \otimes \ell_{I,p}$. The transition functions for $\ell_{I,p}^{\otimes 2}$ are straightforward to calculate. Just take the ratio $\ell_{I,p}^{\otimes 2} \otimes \ell_{J,q}^{-\otimes 2}$. Via the isomorphism

$$\mathrm{Hom}(\det T_X|_{U_{I,p}}, \det \mathcal{E}_m|_{U_{I,p}}) \cong \det \mathcal{E}_m|_{U_{I,p}} \otimes (\det T_X|_{U_{I,p}})^\vee,$$

the transition functions for $\eta_{I,p}$ are given by the transition functions of $s_{I,p} \otimes \partial_{I,p}^\vee$. Again, the transition functions of $s_{I,p}$ are straightforward to compute; $\partial_{I,p}^\vee$ is where the difficulty lies. To do for Stephen: compute transition functions.

5. Local indices

This section needs much more polishing. We should also decide if we want to trim this to the universal line case. Ideally this section and the next will either both do the computations for $\mathbb{G}\mathrm{r}(r,n)$ or both for $\mathbb{G}\mathrm{r}(1,n)$ as opposed to computing local coordinates for $\mathbb{G}\mathrm{r}(r,n)$ here and only doing the Chern class computation for $\mathbb{G}\mathrm{r}(1,n)$ in the next. In this section, we introduce Nisnevich coordinates for Φ and trivializations of the bundle $\mathcal{P}^{m-1}_{\Phi/G}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)$. Note that $\mathbb{P}\mathcal{S} \cong \Phi$ is a $\mathbb{P}^r_k$ bundle over $\mathbb{G}\mathrm{r}(r,n)$ and we thus adapt the standard affine covers of $\mathbb{G}\mathrm{r}(r,n)$ and $\mathbb{P}^r_k$ for our purposes following the setup in [DGGM23, §3.1] and [SW21, §2.3].

Let $\{e_1, \ldots, e_{n+1}\}$ denote a basis for k^{n+1}. The r-dimensional vector subspace $\mathbb{P}H$ for $H = \mathrm{Span}_k\{e_{n-r+1}, \ldots, e_{n+1}\}$ denotes a closed k-rational point of the Grassmannian $\mathbb{G}\mathrm{r}(r,n)$. Consider local coordinates

$$U_{[\mathbb{P}H]} \cong \mathbb{A}^{(r+1)(n-r)} = \mathrm{Spec}\, k[x_{1,1}, \ldots, x_{r+1,n-r}] \to \mathbb{G}\mathrm{r}(r,n)$$

around the closed point $[\mathbb{P}H]$ of $\mathbb{G}\mathrm{r}(r,n)$ such that $(x_{1,1}, \ldots, x_{r+1,n-r})$ corresponds to the span of $\{\widetilde{e}_{n-r+1}, \ldots, \widetilde{e}_{n+1}\}$ where $\{\widetilde{e}_1, \ldots, \widetilde{e}_{n+1}\}$ is the basis of k^{n+1} defined by

$$\begin{cases} e_i & 1 \le i \le n-r \\ \sum_{j=1}^{n-r} e_j x_{1,j} + e_{n-r+1} & i = n-r+1 \\ \vdots & \\ \sum_{j=1}^{n-r} e_j x_{r+1,j} + e_{n+1} & i = n+1. \end{cases}$$

Let $V_l = \{y_l \neq 0\} \subseteq \mathbb{P}^r_k$ and $\psi_l : V_l \to \mathbb{A}^r_k$ by

$$[y_0 : \cdots : y_r] \mapsto (y_{0/l}, \dots, y_{l-1/l}, y_{l+1/l}, \dots, y_{r/l})$$

where $y_{i/l} = \frac{y_i}{y_l}$ for $i \neq l, 1 \leq i, l \leq r$ with inverse given by $\psi_l^{-1} : \mathbb{A}^r_k \to V_l$ taking

$$(y_{0/l}, \dots, y_{l-1/l}, y_{l+1/l}, \dots, y_{r/l}) \mapsto [y_{0/l} : \cdots : y_{l-1/l} : 1 : y_{l+1/l} : \cdots : y_{r/l}].$$

We can construct Nisnevich coordinates on Φ using the coordinates on the product $U_{[\mathbb{P}H]} \times V_l \cong \mathbb{A}^{r(n-r)+n}$. For some point

$$(x_{1,1}, \dots, x_{r+1,n-r}) \times (y_{0/l}, \dots, y_{l-1/l}, y_{l+1/l}, \dots, y_{r/l}) \in U_{[\mathbb{P}H]} \times V_l$$

we can consider the maps $\pi : \Phi \to \mathbb{G}\mathrm{r}(r, n), \beta : \Phi \to \mathbb{P}^n_k$ described in Notation 2.3 where

$$\begin{aligned}\pi\big((x_{1,1}, \dots, x_{r+1,n-r}) \times (y_{0/l}, \dots, y_{l-1/l}, y_{l+1/l}, \dots, y_{r/l})\big) \\ = \big[\mathbb{P}\big(\operatorname{Span}_k\{\widetilde{e}_{n-r+1}, \dots, \widetilde{e}_{n+1}\}\big)\big] \in \mathbb{G}\mathrm{r}(r, n)\end{aligned}$$

and

$$\begin{aligned}\beta\big((x_{1,1}, \dots, x_{r+1,n-r}) \times (y_{0/l}, \dots, y_{l-1/l}, y_{l+1/l}, \dots, y_{r/l})\big) \\ = \mathbb{P}\big(\operatorname{Span}_k\{y_{0/l} \cdot \widetilde{e}_{n-r+1} + \dots 1 \cdot \widetilde{e}_{n-r+l-1} + \cdots + y_{r/l} \cdot \widetilde{e}_{n+1}\}\big) \in \mathbb{P}^n_k.\end{aligned}$$

5.1. **Relative orientation relative to a divisor.** In the case of $\Phi = \Phi_{1,n}$, we consider the relative orientation of our bundle of principal parts with respect to $\mathcal{O}_\Phi(1) \otimes \pi^*\mathcal{O}_G(1)$, the complement of the locus of lines meeting a given $(n-2)$-plane in $\mathbb{P}^n_k$ and point not lying on a hyperplane in $\mathbb{P}^n_k$. Recall that in the Grassmannian $\mathbb{G}\mathrm{r}(1, n)$, the complement of the Schubert class Σ_1 is affine and corresponds to lines in $\mathbb{G}\mathrm{r}(1, n)$ not meeting a given $(n-2)$-plane. Choosing coordinates on the Grassmannian U_{ij} where $1 \leq i < j \leq n+1$, we have the following chart

$$(x_{1,1}, \dots, x_{1,n-1}, x_{2,1}, \dots, x_{2,n-1}) \in \mathbb{A}^{2(n-1)},$$

which equivalently can be represented by a matrix (when $i = n = j - 1$)

$$\begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n-1} & 1 & 0 \\ x_{2,1} & x_{2,2} & \dots & x_{2,n-1} & 0 & 1 \end{bmatrix}$$

isomorphic to $\mathbb{A}^{2n-2}_k$ open in $\mathbb{G}\mathrm{r}(1, n)$ not meeting the hyperplane $V(X_i, X_j) \subseteq \mathbb{P}^n_k$ denoting the coordinates on $\mathbb{P}^n_k$ by $[X_0 : \cdots : X_n]$. Furthermore, we have charts $U_{ij\alpha} \subset U_{ij} \times \mathbb{P}^1$ where $\alpha \in \{0, 1\}$ represents all points of $\pi^{-1}(U_{ij})$ where $Y_\alpha \neq 0$, where the coordinates of $\mathbb{P}^1$ in the fiber are given by $[Y_0 : Y_1]$.

Take $U_{ij\alpha}$ and $U_{qp\beta}$

Furthermore, the charts $V_i = \{y_i \neq 0\}$ for $i \in \{0, 1\}$ correspond to pointed lines with point not meeting $V(X_{n+1})$ and $V(X_n)$, respectively.

Giosuè's idea: set $f_i = \partial(F|_L)/\partial x_i$ and $g_i = \partial(F|_L)/\partial y_i$. Then the local index is the Wronskian determinant of $(f_1, \dots, f_n, g_1, \dots, g_{n-1})$. We now just need to describe how $\mathrm{Wr}(f, g)$ relates to the geometry of $\mathbb{V}(F)$ and (L, p).

5.2. **Rewriting this section from scratch.** In this section, we introduce Nisnevich coordinates for Φ and trivializations of the bundle $\mathcal{P}^{m-1}_{\Phi/G}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)$. Note that $\mathbb{P}\mathcal{S} \cong \Phi$ is a $\mathbb{P}^r_k$ bundle over $\mathbb{G}\mathrm{r}(r,n)$

Let $\{e_1,\ldots,e_{n+1}\}$ denote a basis for k^{n+1}. There esists a standard open cover of $\mathbb{G}\mathrm{r}(r,n)$: Take an ordered set of $r+1$ indeces $I=\{i_\imath\}_{\imath=1}^{r+1}$ such that $1\leq i_1<\ldots<i_{r+1}\leq n+1$. Consider local coordinates

$$U_I \cong \mathbb{A}^{(r+1)(n-r)} = \operatorname{Spec} k[x_{1,1},\ldots,x_{r+1,n-r}] \to \mathbb{G}\mathrm{r}(r,n)$$

such that $(x_{1,1},\ldots,x_{r+1,n-r})$ corresponds to the span of the set of vectors $\{\widetilde{e}_i\}_{i\in I}$ where

$$\widetilde{e}_{i_\imath} := e_{i_\imath} + \sum_{j=1}^{n-r} x_{\imath,j} e_{\imath_j} \tag{5.1}$$

where $I^c=\{\imath_1,\imath_2,\ldots,\imath_{n-r}\}$. The standard cover of $\mathbb{G}\mathrm{r}(r,n)$ is the open cover of the open affine sets U_I, for all possible set of indeces I, with the obvious trivializing functions:

$$\varphi_I\colon U_I \longrightarrow \mathbb{A}_k^{(r+1)(n-r)}.$$

There is a very convenient way to write the relation expressed by Equation (5.1), using matrices. Indeed, we may define the matrix M_I of order $(r+1)\times(n+1)$ defined in the following way: the submatrix spanned by the column indexed by I form the identity matrix, all other columns span $\{x_{*,*}\}$. For example, if $I=\{1,\ldots,r+1\}$ we have

$$M_{\{1,\ldots,r+1\}} = \begin{pmatrix} 1 & 0 & \cdots & 0 & x_{1,1} & \cdots & x_{1,n-r} \\ 0 & 1 & \cdots & 0 & x_{2,1} & \cdots & x_{2,n-r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & x_{r+1,1} & \cdots & x_{r+1,n-r} \end{pmatrix}. \tag{5.2}$$

Thus, Equation 5.1 is equivalent to

$$\begin{pmatrix} \widetilde{e}_{i_1} \\ \widetilde{e}_{i_2} \\ \vdots \\ \widetilde{e}_{i_{r+1}} \end{pmatrix} = M_I \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n+1} \end{pmatrix}. \tag{5.3}$$

The transition functions of the atlas $\{(U_I,\varphi_I)\}_I$ are easily computed: the map

$$\varphi_I \circ \varphi_J^{-1}\colon \mathbb{A}_k^{(r+1)(n-r)} \longrightarrow \mathbb{A}_k^{(r+1)(n-r)}$$

sends the coordinates $(x_{1,1},...)$ to the coordinates $(x'_{1,1},...)$ such that $M'_J = N_{I,J}^{-1} M_I$. where M'_J is M_J where $x_{*,*}$ is replaced by $x'_{*,*}$, and $N_{I,J}$ is the submatrix of M_I formed by the columns indexed by J. For example, if $I=\{n-2r,\ldots,n-r-1,n-r\}$ and $I\cap J=\emptyset$,

$$N_{\{n-2r,\ldots,n-r\},J} = \begin{pmatrix} x_{1,j_1} & x_{1,j_2} & \cdots & x_{1,j_{r+1}} \\ \vdots & \vdots & \vdots & \vdots \\ x_{r+1,j_1} & x_{r+1,j_2} & \cdots & x_{r+1,j_{r+1}} \end{pmatrix}. \tag{5.4}$$

The fiber of the vector bundle $\Pi\colon \mathcal{S} \to \mathbb{G}\mathrm{r}(r,n)$ at a point $p \in \mathbb{G}\mathrm{r}(r,n)$ is the vector space parameterized by p. This means that, supposing $p \in U_I$, $\Pi^{-1}(p) \cong \mathbb{A}_k^{r+1}$ where $\mathbb{A}_k^{r+1}$ has coordinates $(y_1, \ldots, y_{r+1})$ and parameterizes all linear combinations of the vectors $\{\widetilde{e}_i\}_{i\in I}$ in Equation (5.1). That is, $\Pi^{-1}(p)$ parameterizes products

$$(5.5)\qquad \begin{pmatrix} y_1 & y_2 & \cdots & y_{r+1}\end{pmatrix} \begin{pmatrix} \widetilde{e}_{i_1} \\ \widetilde{e}_{i_2} \\ \vdots \\ \widetilde{e}_{i_{r+1}} \end{pmatrix} = \sum_{i\in I}\left(y_i \left(e_i + \sum_{j\in I^c} x_{i+r-n,j} e_j \right)\right).$$

Using this language, it is easy to write the transition functions of $\Pi\colon \mathcal{S} \to \mathbb{G}\mathrm{r}(r,n)$ for the atlas $\{(\Pi^{-1}(U_I), \varphi_I \times \mathrm{id})\}_I$. It is enough to consider maps

$$(5.6)\qquad \varphi_I \times \mathrm{id} \circ (\varphi_J \times \mathrm{id})^{-1}\colon \mathbb{A}_k^{(r+1)(n-r)} \times \mathbb{A}_k^{r+1} \longrightarrow \mathbb{A}_k^{(r+1)(n-r)} \times \mathbb{A}_k^{r+1}$$

sending coordinates $(x_{1,1}, \ldots)\times(y_1, \ldots, y_{r+1})$ to coordinates $(x'_{1,1}, \ldots)\times(y'_1, \ldots, y'_{r+1})$ where $x'_{*,*}$ are like before and

$$(y'_1, \ldots, y'_{r+1}) = (y_1, \ldots, y_{r+1}) N_{I,J}.$$

We deduce that from the simple relation

$$(5.7)\qquad \begin{pmatrix} y_1 & y_2 & \cdots & y_{r+1}\end{pmatrix} \begin{pmatrix} \widetilde{e}_{i_1} \\ \widetilde{e}_{i_2} \\ \vdots \\ \widetilde{e}_{i_{r+1}} \end{pmatrix} = \begin{pmatrix} y_1 & y_2 & \cdots & y_{r+1}\end{pmatrix} N_{I,J} N_{I,J}^{-1} M_I \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n+1} \end{pmatrix}.$$

Finally, considering the projective bundle $\pi\colon \mathbb{P}\mathcal{S} \to \mathbb{G}\mathrm{r}(r,n)$, there is an atlas $\{(U_{I,p}, \varphi_{I,p})\}$ where I is a set of indeces as before, and $p \in \{1, \ldots, n+1\}$. The open subsets $U_{I,p}$ are defined in such a way that

$$\bigcup_{p=1}^{n+1} U_{I,p} = \pi^{-1}(U_I)$$

induces the standard open cover of the projective space $\mathbb{P}_k^r$. Now, let us define rational maps $\sigma_p\colon \mathbb{A}_k^{r+1} \dashrightarrow \mathbb{A}_k^r$ sending coordinates $(y_1, \ldots, y_{r+1})$ to $\left(\frac{y_1}{y_p}, \ldots, \frac{y_{r+1}}{y_p}\right)$. These maps have obvious inverses σ_p^{-1} sending $(Y_1, \ldots, Y_r)$ to $(Y_1, \ldots, Y_{p-1}, 1, Y_{p+1}, \ldots, Y_n)$. The maps $\varphi_{I,p}$ are defined as the composition need accuracy

$$\varphi_{I,p} := (\mathrm{id}_{\mathbb{A}_k^{(r+1)(n-r)}} \times \sigma_p) \circ \varphi_I \circ (\mathrm{id}_{\mathbb{A}_k^{(r+1)(n-r)}} \times \sigma_p)^{-1}.$$

5.3. **Projective lines.** Now we focus only in the case $r = 1$ (sometimes I leave the symbol r, we will correct this in the polished version). Let us write explicitly the transition functions Should we not write here the composite $\varphi_{J,q}^{-1} \circ \varphi_{I,p}$

$$U_{I,p} \to \mathbb{A}_k^{2(n-1)} \times \mathbb{A}_k^1 \to U_{J,q}$$

ie. do $\varphi_{I,p} : U_{I,p} \to \mathbb{A}_k^{2(n-1)} \times \mathbb{A}_k^1$ first then $\varphi_{J,q}^{-1} : \mathbb{A}_k^{2(n-1)} \times \mathbb{A}_k^1 \to U_{J,q}$ after? I think this is a notation thing. I redid your computations up to (4.16) and everything agrees up to this notation change.

Yes you are right. Thank you. $\varphi_{I,p} \circ \varphi_{J,q}^{-1}$, sending

$$(x_{1,1}, \ldots), (y)$$

to

$$(x'_{1,1}, \ldots), (y').$$

Let $\Delta_{I,J} := \det N_{I,J} = x_{1,j_1}x_{2,j_2} - x_{1,j_2}x_{2,j_1}$. In order to compute $x'_{*,*}$, we recall by Equation (5.4) that

$$M'_J = \frac{1}{\Delta_{I,J}} \begin{pmatrix} x_{2,j_2} & -x_{1,j_2} \\ -x_{2,j_1} & x_{1,j_1} \end{pmatrix} M_I \tag{5.8}$$

Let us consider the case $I = \{n-r-1, n-r\}$. We have

$$M'_J = \begin{pmatrix} x_{1,1}\frac{x_{2,j_2}}{\Delta_{I,J}} + x_{2,1}\frac{-x_{1,j_2}}{\Delta_{I,J}} & \cdots & \frac{x_{2,j_2}}{\Delta_{I,J}} & \frac{-x_{1,j_2}}{\Delta_{I,J}} \\ x_{1,1}\frac{-x_{2,j_1}}{\Delta_{I,J}} + x_{2,1}\frac{x_{1,j_1}}{\Delta_{I,J}} & \cdots & \frac{-x_{2,j_1}}{\Delta_{I,J}} & \frac{x_{1,j_1}}{\Delta_{I,J}} \end{pmatrix} \tag{5.9}$$

That is, for any $\alpha \in \{1, 2, \ldots, n-r-2\}$, denoting by $\bar{\alpha}$ the lowest integer such that $|\{1, 2, \ldots, \bar{\alpha}\} \setminus J| = \alpha$, we have

$$x'_{1,\alpha} = x_{1,\bar{\alpha}}\frac{x_{2,j_2}}{\Delta_{I,J}} + x_{2,\bar{\alpha}}\frac{-x_{1,j_2}}{\Delta_{I,J}} \qquad x'_{1,n-r-1} = \frac{x_{2,j_2}}{\Delta_{I,J}} \qquad x'_{1,n-r} = \frac{-x_{1,j_2}}{\Delta_{I,J}} \tag{5.10}$$

$$x'_{2,\alpha} = x_{1,\bar{\alpha}}\frac{-x_{2,j_1}}{\Delta_{I,J}} + x_{2,\bar{\alpha}}\frac{x_{1,j_1}}{\Delta_{I,J}} \qquad x'_{2,n-r-1} = \frac{-x_{2,j_1}}{\Delta_{I,J}} \qquad x'_{2,n-r} = \frac{x_{1,j_1}}{\Delta_{I,J}}. \tag{5.11}$$

Let us compute y'. We will do it case-by-case, depending on $p, q \in \{1, 2\}$.

$$y' = \frac{x_{1,j_2} + yx_{2,j_2}}{x_{1,j_1} + yx_{2,j_1}} \qquad \text{if } p = q = 1, \tag{5.12}$$

$$y' = \frac{x_{1,j_1} + yx_{2,j_1}}{x_{1,j_2} + yx_{2,j_2}} \qquad \text{if } p = 2, q = 1, \tag{5.13}$$

$$y' = \frac{yx_{1,j_1} + x_{2,j_1}}{yx_{1,j_2} + x_{2,j_2}} \qquad \text{if } p = q = 2, \tag{5.14}$$

$$y' = \frac{yx_{1,j_2} + x_{2,j_2}}{yx_{1,j_1} + x_{2,j_1}} \qquad \text{if } p = 1, q = 2. \tag{5.15}$$

Let us compute the transition function of $\det(T_\Phi)$. Consider coordinates

$$(x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2}, \ldots, x_{1,n-r}, x_{2,n-r}), (y) \in \mathbb{A}_k^{2(n-2)} \times \mathbb{A}_k^1. \tag{5.16}$$

We need to compute the determinant of the Jacobian matrix of $\varphi_{(n-r-1,n-r),p} \circ \varphi_{J,q}^{-1}$ for arbitrary J, p, q. First of all, note that

$$\bar{\alpha} = \begin{cases} \alpha & \text{if } 1 \leq \alpha < j_1 \\ \alpha + 1 & \text{if } j_1 \leq \alpha < j_2 \\ \alpha + 2 & \text{if } j_2 \leq \alpha \leq n-r-2 \end{cases} \tag{5.17}$$

So, we adopt the following trick. We move to the right both the variables (x_{1,j_1}, x_{2,j_1}) of exactly $2(n-r-j_1)$ positions, in such a way that the new coordinates are

$$(x_{1,1}, x_{2,1}, \ldots, x_{1,j_1-1}, x_{2,j_1-1}, x_{1,j_1+1}, x_{2,j_1+1}, x_{1,j_1+2}, x_{2,j_1+2}, \ldots, x_{1,j_1}, x_{2,j_1}), (y) \tag{5.18}$$

Now we move to the right also (x_{1,j_2}, x_{2,j_2}) of $2(n-r-j_2)$ positions, and call the new set of coordinates "shifted." Those coordinates are

$$(x_{1,1},\dots,x_{2,j_1-1},x_{1,j_1+1},\dots,x_{2,j_2-1},x_{1,j_2+1},\dots,x_{1,n-r},x_{2,n-r},x_{1,j_1},x_{2,j_1},x_{1,j_2},x_{2,j_2}),(y) \tag{5.19}$$

Let us compute the Jacobian matrix with respect to the shifted coordinates. The derivation with respect to the variables $x_{1,1}$ and $x_{1,2}$ of all the functions $x'_{*,*}$ gives the first and second columns of the matrix. Those variables only appear in $x'_{1,1}$ and $x'_{2,1}$ (supposing that $j_1 > 1$), so we deduce that the first two columns of the Jacobian are

$$\begin{pmatrix} \frac{x_{2,j_2}}{\Delta_{I,J}} & \frac{-x_{1,j_2}}{\Delta_{I,J}} \\ \frac{-x_{2,j_1}}{\Delta_{I,J}} & \frac{x_{1,j_1}}{\Delta_{I,J}} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}. \tag{5.20}$$

We may repeat the same argument for other variables. We deduce that the contribution to the determinant of the functions $(x'_{1,\alpha}, x'_{2,\alpha})$ is given by the non trivial square matrix in Equation (4.7), whose determinant is $\Delta_{I,J}^{-1}$. It remains the contribution of the submatrix of order 5 given by the last 5 variables $x_{1,j_1}, x_{2,j_1}, x_{1,j_2}, x_{2,j_2}, y$. This matrix, in the case $p = q = 1$, is

$$\begin{pmatrix} -\frac{x_{2,j_2}^2}{\Delta_{I,J}^2} & \frac{x_{1,j_2}x_{2,j_2}}{\Delta_{I,J}^2} & \frac{x_{2,j_1}x_{2,j_2}}{\Delta_{I,J}^2} & -\frac{x_{1,j_2}x_{2,j_1}}{\Delta_{I,J}^2} & 0 \\ \frac{x_{2,j_1}x_{2,j_2}}{\Delta_{I,J}^2} & -\frac{x_{1,j_1}x_{2,j_2}}{\Delta_{I,J}^2} & -\frac{x_{2,j_1}^2}{\Delta_{I,J}^2} & \frac{x_{1,j_1}x_{2,j_1}}{\Delta_{I,J}^2} & 0 \\ \frac{x_{1,j_2}x_{2,j_2}}{\Delta_{I,J}^2} & -\frac{x_{1,j_2}^2}{\Delta_{I,J}^2} & -\frac{x_{1,j_1}x_{2,j_2}}{\Delta_{I,J}^2} & \frac{x_{1,j_1}x_{1,j_2}}{\Delta_{I,J}^2} & 0 \\ -\frac{x_{1,j_2}x_{2,j_1}}{\Delta_{I,J}^2} & \frac{x_{1,j_1}x_{1,j_2}}{\Delta_{I,J}^2} & \frac{x_{1,j_1}x_{2,j_1}}{\Delta_{I,J}^2} & -\frac{x_{1,j_1}^2}{\Delta_{I,J}^2} & 0 \\ -\frac{yx_{2,j_2}+x_{1,j_2}}{(yx_{2,j_1}+x_{1,j_1})^2} & -\frac{y(yx_{2,j_2}+x_{1,j_2})}{(yx_{2,j_1}+x_{1,j_1})^2} & \frac{1}{yx_{2,j_1}+x_{1,j_1}} & \frac{y}{yx_{2,j_1}+x_{1,j_1}} & \frac{\Delta_{I,J}}{(yx_{2,j_1}+x_{1,j_1})^2} \end{pmatrix} \tag{5.21}$$

I used Wolfram Math.

Just to be clear here. Are you claiming that the Jacobian matrix is block-diagonal with terms like (4.20) in all but the last diagonal block which is (4.21)? Because while I agree the columns will be of the form (4.20), the rows may have nonzero entries – namely, derivatives with respect to $x_{1,j_1}, x_{2,j_1}, x_{1,j_2}, x_{2,j_2}$. Take for example the transition function for the matrices

$$\begin{bmatrix} x_{1,1} & 1 & 0 & x_{1,4} & x_{1,5} \\ x_{2,1} & 0 & 1 & x_{2,4} & x_{2,5} \end{bmatrix} \to \begin{bmatrix} x'_{1,1} & x'_{1,2} & x'_{1,3} & 1 & 0 \\ x'_{2,1} & x'_{2,2} & x'_{2,3} & 0 & 1 \end{bmatrix}$$

given by the matrix

$$\frac{1}{x_{1,4}x_{2,5} - x_{1,5}x_{2,4}} \begin{bmatrix} x_{2,5} & -x_{1,5} \\ -x_{2,4} & x_{1,4} \end{bmatrix}$$

we have $x'_{1,1} = x_{1,1}\frac{x_{2,5}}{x_{1,4}x_{2,5}-x_{1,5}x_{2,4}} - x_{2,1}\frac{x_{1,5}}{x_{1,4}x_{2,5}-x_{1,5}x_{2,5}}$ where the derivative with respect to $x_{1,4}$ is nonzero so our matrix isn't just block diagonal. But maybe I'm misunderstanding what you're trying to say. I do not claim that the Jacobian matrix is block-diagonal.

I claim that the determiant of the Jacobian will be the product of the blocks in the diagonal. That is, the rows are not zero, but they will have no influence in the calculation of the determinant. This happens also in the case $r = 0$. In which case I've gone through your computations and everything seems fine.

In order to pass from regular coordinates to the shifted coordinates, we used an even number of transpositions. Hence, the determinant of the Jacobian is the product of $\Delta^{-(n-r-2)}$ and the determinant of (4.2), thus

These equations are wrong. I confused signs when copying from Wolfram Math.'s notebook. I am sorry.

$$\det T_\Phi = \frac{\Delta^{-(n-r-2)}}{(-\Delta)^3\,(yx_{2,j_1}+x_{1,j_1})^2} = \frac{-1}{\Delta^n_{I,J}\,(yx_{2,j_1}+x_{1,j_1})^2}.$$

$$\begin{aligned}
\det T_\Phi &= (-1)\frac{1}{\Delta^n_{I,J}\,(yx_{2,j_1}+x_{1,j_1})^2} && \text{if } p=q=1,\\
&= (-1)\frac{1}{\Delta^n_{I,J}\,(x_{1,j_2}+yx_{2,j_2})^2} && \text{if } p=2, q=1,\\
&= \frac{1}{\Delta^n_{I,J}(yx_{1,j_2}+x_{2,j_2})^2} && \text{if } p=q=2,\\
&= \frac{1}{\Delta^n_{I,J}(yx_{1,j_1}+x_{2,j_1})^2} && \text{if } p=1, q=2.
\end{aligned}$$

Or, equivalently,

$$\det T_\Phi = (-1)^q \frac{1}{\Delta^n_{I,J}\left(y^{q-1}x_{1,j_p}+y^{2-q}x_{2,j_p}\right)^2}. \tag{5.22}$$

In order to compute the transition matrix of $\varphi_{I,p}\circ\varphi^{-1}_{J,q}$ for arbitrary I and J, just note that, for any $s\in\{1,2\}$,

$$\varphi_{I,p}\circ\varphi^{-1}_{J,q} = (\varphi_{I,p}\circ\varphi^{-1}_{(n-r-1,n-r),s})\circ(\varphi_{(n-r-1,n-r),s}\circ\varphi^{-1}_{J,q}).$$

Since the Jacobian of the inverse is the reciprocal of the original Jacobian, and

$$(\varphi_{I,p}\circ\varphi^{-1}_{(n-r-1,n-r),s}) = (\varphi_{(n-r-1,n-r),s}\circ\varphi^{-1}_{I,p})^{-1},$$

we have

$$\det\left(J_{\varphi_{I,p}\circ\varphi^{-1}_{(n-r-1,n-r),s}}\right) = \det\left(J_{\varphi_{(n-r-1,n-r),s}\circ\varphi^{-1}_{I,p}}\right)^{-1} = (-1)^p\Delta^n_{\{n-2,n-1\},I}(y^{p-1}x_{1,i_s}+y^{2-p}x_{2,i_s})^2. \tag{5.23}$$

As before,

$$\det\left(J_{\varphi_{(n-r-1,n-r),s}\circ\varphi^{-1}_{J,q}}\right) = (-1)^q\frac{1}{\Delta^n_{\{n-2,n-1\},J}(y^{q-1}x_{1,j_s}+y^{2-q}x_{2,j_s})^2}. \tag{5.24}$$

Finally,

$$\det\left(J_{\varphi_{I,p}\circ\varphi_{J,q}^{-1}}\right) = (-1)^{p+q}\frac{\Delta_I^n\left(y^{p-1}x_{1,i_s}+y^{2-p}x_{2,i_s}\right)^2}{\Delta_{I,J}^n\left(y^{q-1}x_{1,j_s}+y^{2-q}x_{2,j_s}\right)^2}. \tag{5.25}$$

That last expression has to be independent of s. But it seems that it is not. The reason I think it is wrong is that if I take

$$\varphi_{I,p}\circ\varphi_{J,q}^{-1} = (\varphi_{I,p}\circ\varphi_{(n-r-1,n-r),q}^{-1})\circ(\varphi_{(n-r-1,n-r),q}\circ\varphi_{J,q}^{-1}).$$

I expect to obtain the same result, but this is not the case. After we solve this small issue, we can get rid of the $(-1)^{p+q}$ by taking the trivializing functions

$$\tilde{\varphi}_{I,p}\colon U_{I,p}\longrightarrow \mathbb{A}_k^{(r+1)(n-r)}\times\mathbb{A}_k^1$$

sending $(\{\tilde{e}_i\}_{i\in I},\langle\sum_{\imath=1,2}y_\imath\tilde{e}_{i_\imath}\rangle)$ to $(x_{1,1},\ldots)((-1)^p\frac{y_{3-p}}{y_p})$.

That is, a mix of the regular trivializing functions and McKean's trivializing functions. Note that this is very similar to what I did in my article. There I used a mix of the two trivializing functions, although the open cover was different.

5.3.1. *Trivialization of the bundle of principal parts.* Let $z_1,\ldots,z_{n+1}$ be projective coordinates of $\mathbb{P}^n$. Let $w_{1,2},\ldots,w_{a,b},\ldots,w_{n,n+1}$ with $1\leq a<b\leq n+1$ be projective coordinates of $\mathbb{P}^{n(n+1)/2}$. We have a map $\Phi\to\mathbb{P}^n$ defined as

$$\langle\sum_{i=1}^{n+1}\tilde{x}_{1,i}e_i,\sum_{i=1}^{n+1}\tilde{x}_{2,i}e_i\rangle,\langle y_1\sum_{i=1}^{n+1}\tilde{x}_{1,i}e_i+y_2\sum_{i=1}^{n+1}\tilde{x}_{2,i}e_i\rangle\longmapsto\sum_{i=1}^{n+1}(y_1\tilde{x}_{1,i}+y_2\tilde{x}_{2,i})e_i$$

and another map $\Phi\to\mathbb{P}^{n(n+1)/2}$

$$\langle\sum_{i=1}^{n+1}\tilde{x}_{1,i}e_i,\sum_{i=1}^{n+1}\tilde{x}_{2,i}e_i\rangle,\sum_{i=1}^{n+1}z_ie_i\longmapsto\sum_{i=1}^{n}\sum_{j=i+1}^{n+1}(\tilde{x}_{1,i}\tilde{x}_{2,j}-\tilde{x}_{1,j}\tilde{x}_{2,i})e_i\wedge e_j.$$

Lemma 5.1. *The line bundle $\pi^*\mathcal{O}_G(1)$ is locally trivialized over $U_{I,p}$ by*

$$\frac{w_{i_1,i_2}}{w_{1,2}}=\frac{\tilde{x}_{1,i_1}\tilde{x}_{2,i_2}-\tilde{x}_{1,i_2}\tilde{x}_{2,i_1}}{\tilde{x}_{1,1}\tilde{x}_{2,2}-\tilde{x}_{1,2}\tilde{x}_{2,1}}.$$

The line bundle $\mathcal{O}_\Phi(1)$ is locally trivialized over $U_{I,p}$ by

$$\frac{z_{i_p}}{z_1}=\frac{y_1\tilde{x}_{1,i_p}+y_2\tilde{x}_{2,i_p}}{y_1\tilde{x}_{1,1}+y_2\tilde{x}_{2,1}}.$$

Proof. TODO ∎

Proposition 5.2. *The line bundle* $\det P$ *is trivialized on $U_{I,p}$ by the distinguished element*

$$\left(\frac{w_{i_1,i_2}}{w_{1,2}}\right)^M\left(\frac{z_{i_p}}{z_1}\right)^N,$$

where $M=$ and $N=$.

5.4. Some Notes on Relatively Orienting to a Divisor.

Gabriel wrote this part. In order to improve reading, it is better to leave it black and use colours just for small comments.

In our setup for projective lines, we have opens $U_{I,p}$ in Φ with $I = \{i_1, i_2\}$ and $p \in \{1, 2\}$. The open set parametrizes lines

$$\mathbb{P}\mathrm{Span}\left\{e_{i_1} + \sum_{j\in I^c} x_{1,j}e_j, e_{i_2} + \sum_{j\in I^c} x_{2,j}e_j\right\}$$

with e_j the standard basis vectors of k^{n+1} and point

$$e_{i_1} + ye_{i_2} + \sum_{j\in I^c}(x_{1,j} + yx_{2,j})e_j; ye_{i_1} + e_{i_2} + \sum_{j\in I^c}(yx_{1,j} + x_{2,j})e_j$$

for $p = 1, 2$, respectively, which in the coordinates of the projective space are

$$[(x_{1,1} + yx_{2,1}) : \cdots : (x_{1,i_1-1} + yx_{2,i_1-1}) : 1 : (x_{1,i_1} + yx_{2,i_1+1}) : \cdots : (x_{1,i_2-1} + yx_{2,i_2-1}) : y : (x_{1,i_2+1} + yx_{2,i_2+1}) : \cdots : (x_{1,n+1} + yx_{2,n+1})]$$

$$[(yx_{1,1} + x_{2,1}) : \cdots : (yx_{1,i_1-1} + x_{2,i_1-1}) : y : (yx_{1,i_1} + x_{2,i_1+1}) : \cdots : (yx_{1,i_2-1} + x_{2,i_2-1}) : 1 : (yx_{1,i_2+1} + x_{2,i_2+1}) : \cdots : (yx_{1,n+1} + x_{2,n+1})]$$

for $p = 1, 2$, respectively.

By our computation of the orientability conditions of the bundle, we are relatively orienting to the divisor $\mathcal{O}_\Phi(1) \otimes \pi^*\mathcal{O}_{\mathbb{G}\mathrm{r}(1,n)}(1)$ where $\mathcal{O}_\Phi(1) \cong \beta^*\mathcal{O}_{\mathbb{P}^n_k}(1)$. Geometrically, the orienting divisors $\mathcal{O}_\Phi(1)$ and $\pi^*\mathcal{O}_{\mathbb{G}\mathrm{r}(1,n)}(1)$ correspond to the point of our pointed line not lying on a hyperplane in $\mathbb{P}^n_k$ and the line of our pointed line not meeting a codimension 2 subspace of $\mathbb{P}^n_k$, respectively. In particular, we can take $\mathcal{O}_\Phi(1)$ to be the Schubert cycle σ_1 which is a hyperplane section of the Grassmannian in the Plücker embedding, ie. a divisor. In the affine setting, the Schubert cycle σ_1 corresponds to the 2-dimensional linear subspaces of $\mathbb{A}^{n+1}_k$ meeting a $n+1-2 = n-1$ dimensional linear subspace. Projectively, under the correspondence $\mathrm{Gr}(2, n+1) \cong \mathbb{G}\mathrm{r}(1, n)$, the Schubert cycle σ_1 corresponds to lines in $\mathbb{P}^n_k$ meeting a fixed $(n-2)$-dimensional linear projective subspace.

Having written our coordinates in this way, our opens $U_{I,p}$ in Φ, we have natural choices for the divisor since on this chart our lines do not meet the codimension 2 linear subspace $V(X_{i_1}, X_{i_2}) \subseteq \mathbb{P}^n_k$ and our point does not meet the hyperplane $V(X_{i_p}) \subseteq \mathbb{P}^n_k$.

5.5. Cubic curves in $\mathbb{P}^2$.

5.6. Quintic surfaces in $\mathbb{P}^3$.

5.7. Septic threefolds in $\mathbb{P}^4$.

5.8. Degree $2n-1$ hypersurfaces in $\mathbb{P}^n$.

6. Euler numbers

Let V be a relatively orientable vector bundle over a smooth proper k-scheme X of dimension $\dim X = \operatorname{rank} V$ and σ a section of V with only isolated zeroes, one can define an Euler number of V valued in $\mathrm{GW}(k)$ as the sum of local indices $e(V) = \sum_{p\in\sigma^{-1}(0)} \operatorname{ind}_p\sigma$ which is independent of the choice of section σ [KW21, Definition 35, Theorem 3]. In cases where the base scheme X is smooth and proper over $\operatorname{Spec}\mathbb{Z}$, the Euler number of V over an arbitrary field k not of characteristic 2 is completely determined by the real and complex Euler numbers. We thus show that Φ satisfies this hypothesis before further discussing the computation of these Euler numbers.

Proposition 6.1. *The flag variety Φ is smooth and proper over* $\operatorname{Spec}\mathbb{Z}$.

Proof. We first show that $\mathbb{G}\mathrm{r}(r,n)$ is smooth and proper over $\operatorname{Spec}\mathbb{Z}$ and since the compositions of smooth and proper morphisms are smooth and proper, it suffices to show $\pi : \mathbb{P}\mathcal{S} \to \mathbb{G}\mathrm{r}(r,n)$ is smooth and proper.

For $\mathbb{G}\mathrm{r}(r,n)$ being smooth and proper over $\operatorname{Spec}\mathbb{Z}$, we adapt the proof from [Bej20]. By construction, the Grassmannian admits a cover by finitely many finite-dimensional affine spaces and is thus of finite type over $\operatorname{Spec}\mathbb{Z}$. Moreover, since the Grassmannian is reduced, it is endowed with the reduced induced closed subscheme structure of projective space in the sense of [Sta23, 01J4] via the Plücker embedding [GW10, Proposition 8.23] and the morphism $\mathbb{G}\mathrm{r}(r,n) \to \operatorname{Spec}\mathbb{Z}$ is proper by [Sta23, 0CYL]. Smoothness over $\operatorname{Spec}\mathbb{Z}$ is given by [GW10, Corollary 8.15].

The flag variety Φ is a projective bundle, thus π is a projective morphism in the sense of [Sta23, 01W8] and therefore proper [Sta23, 0BCL]. Since π is proper, it is of finite type and noting $\mathbb{G}\mathrm{r}(r,n)$ is Noetherian, π is locally of finite presentation [Sta23, 01TX]. Furthermore, π is surjective and both Φ and $\mathbb{G}\mathrm{r}(r,n)$ are regular schemes so π is flat [Liu06, Remark 4.3.11]. In particular, π is a flat morphism locally of finite presentation with smooth fibers and thus π is a smooth morphism by [Sta23, 01V8]. ∎

We can thus compute the Euler numbers of $\mathcal{P}^{m-1}_{\Phi/G}(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d))$ using [BW23, Theorem 5.11] which we now state.

Lemma 6.2. *Suppose X is smooth and proper over $\mathbb{Z}$, V a relatively oriented vector bundle on X, and V_k denote the base change of V to k for any field k, then*

$$e(V_k) = \frac{n_{\mathbb{C}} + n_{\mathbb{R}}}{2} + \frac{n_{\mathbb{C}} - n_{\mathbb{R}}}{2}\langle -1\rangle$$

where $n_{\mathbb{C}}$ and $n_{\mathbb{R}}$ are the complex and real Euler numbers, respectively.

Of interest to us here is the computation of the Euler numbers of the vector bundles $\mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)$ as well as in the orientable cases we discussed in Section 3.

Proposition 6.3. *If $\mathcal{P}^{m-1}_{\Phi/G}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)$ is orientable, then the Euler number*

$$e\left(\mathcal{P}^{m-1}_{\Phi/G}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)\right)$$

is hyperbolic.

Proof. It suffices to show that in the orientable case, $\dim\Phi$ is odd since Euler numbers of vector bundles over odd-dimensional schemes are hyperbolic [SW21, Proposition 19]. The vector bundle $\mathcal{P}^{m-1}_{\Phi/G}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)$ is orientable only if $(-r-1)B-r-1$ and $B-n$ are both congruent to $0 \pmod 2$ so either $B,n\equiv 0 \pmod 2$ or $B,n\equiv 1 \pmod 2$. Suppose $B,n\equiv 0 \pmod 2$ then $-r-1\equiv 0 \pmod 2$ and r is odd so $\dim\Phi = r(n-r)+n$ is odd for n even. Otherwise $B,n\equiv 1 \pmod 2$ and $\dim\Phi = r(n-r)+n$ is odd regardless of the parity of r. □

Denote $\mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(m)\right)_{\mathbb{C}}, \mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(m)\right)_{\mathbb{R}}$ the base change of the bundle of principal parts to $\mathbb{C}$ and $\mathbb{R}$, respectively. In the case of the complex vector bundle, the Euler number agrees with the top Chern class [BT82, 20.10.6]. Thus, we show the following.

For example, in the case $(5,3,5,11)$, we will have to compute the 4th symmetric power of $\Omega_{\Phi/G}$ tensored with $\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)$, which by Proposition 5.17 of [EH16] is

$$c_{\binom{m+r-1}{m}}\left(\mathrm{Sym}^m\Omega_{\Phi/G}\otimes\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right) = \sum_{l=0}^{\binom{m+r-1}{m}}\left(\binom{r-l}{\binom{m+r-1}{m}-l}(d\zeta)^{\binom{m+r-1}{m}-l}c_l(\mathrm{Sym}^m\Omega_{\Phi/G})\right)$$

Theorem 6.4. *The complex Euler number $n_{\mathbb{C}} = e\left(\mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)_{\mathbb{C}}\right)$ is given by the degree of the top Chern class*

$$\deg\left(c_m\left(\mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)_{\mathbb{C}}\right)\right) = \sum_{j=m-n}^{m-1}(m!\cdot e_{m-j-1})\cdot\deg\left(\sigma_1^j\sigma_{m-j-1}\right).$$

Proof. The bundle of principal parts $\mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)$ admits a filtration

$$\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d), \left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\otimes\Omega_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\right),\ldots,\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\otimes\mathrm{Sym}^{m-1}\Omega_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\right)$$

of line bundles from which one can use Whitney's formula [EH16, Theorem 5.3 (c)] to compute the top Chern class $c_m\left(\mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)\right)$ as the product

$$c_1\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)\times\prod_{j=1}^{m-1}c_1\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\otimes\mathrm{Sym}^j\Omega_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\right). \tag{6.1}$$

We see that $\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)$ is the pullback of the hyperplane class on $\mathbb{P}^n_k$ so $c_1(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)) = d\zeta$ where $\zeta = c_1(\mathcal{O}_\Phi(1))$ and $d\geq m$.

Given that the relative cotangent bundle $\Omega_{\Phi/\mathbb{G}\mathrm{r}(1,n)}$ is dual to the relative tangent bundle $T_{\Phi/\mathbb{G}\mathrm{r}(1,n)}$, we use [EH16, Theorem 11.4] to compute $c_1\left(\Omega_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\right) = \sigma_1 - 2\zeta$ where

$\sigma_1 = c_1(\mathcal{S})$. The total Chern class of the symmetric power of $\Omega_{\Phi/\mathbb{G}\mathrm{r}(1,n)}$ is then given by $c\left(\mathrm{Sym}^j \Omega_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\right) = 1 + j\sigma_1 - 2j$ and the first Chern class of $\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d) \otimes \mathrm{Sym}^j \Omega_{\Phi/\mathbb{G}\mathrm{r}(1,n)}$ is thus

$$c_1\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d) \otimes \mathrm{Sym}^j \Omega_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\right) = j\sigma_1 + (d-2j)\zeta \tag{6.2}$$

by [EH16, Proposition 5.18]. Substituting (6.2) into (6.1) we conclude

$$\begin{aligned} c_m\left(\mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)\right) &= \prod_{j=0}^{m-1} (j\sigma_1 + (d-2j)\zeta) \\ &= m!\zeta \cdot \prod_{j=1}^{m-1}\left(\sigma_1 + \frac{m-2j}{j}\zeta\right). \end{aligned} \tag{6.3}$$

Expanding (6.3) in terms of symmetric polynomials in the roots $-\frac{d-2j}{j}$, let

$$e_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le d-1} \prod_{j \in \{i_1, \dots, i_k\}} \left(\frac{d-2j}{j}\right) \tag{6.4}$$

allowing us to write the Chern class (6.3) as

$$c_m\left(\mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)\right) = \sum_{j=0}^{m-1} \left(d \cdot (m-1)! \cdot e_{m-j-1}\right) \sigma_1^j \zeta^{m-j}.$$

Using Segre classes [EH16, Definition 10.1, Proposition 10.3] and the push-pull formula [EH16, Theorem 1.23 (b)], we see that $\deg(\sigma_1^i\zeta^j) = \deg(\sigma_1^i s_{j-1}(\mathcal{S})) = \deg(\sigma_1^i \sigma_{j-1})$ as $s_{j-1}(\mathcal{S}) = c_{j-1}(\mathcal{Q})$ where $\mathcal{Q}$ is the universal quotient bundle with total Chern class given in [EH16, §5.6.2]. This gives us the degree of the m-th Chern class

$$\deg\left(c_m\left(\mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)\right)\right) = \sum_{j=m-n}^{m-1} \left(d \cdot (m-1)! \cdot e_{m-j-1}\right) \cdot \deg\left(\sigma_1^j \sigma_{m-j-1}\right)$$

as the ζ^{m-j} vanish for $m - j \ge n$. □

We can compute the number of lines highly tangent to a projective hypersurface using `Macaulay2` code we provide in the `GitHub` repository (see Section 1.2). This computation of the complex Euler number allows us to recover two classical results from enumerative geometry. In addition, we provide a count of the number of lines meeting a septic threefold in $\mathbb{P}^4_k$ to order 7, for which we could not find a reference. More explicitly, one can immediately deduce the following.

Corollary 6.5. *Over the complex numbers, there are:*

(i) 9 lines meeting a plane conic to order three,

(ii) 575 lines meeting a cubic surface in $\mathbb{P}^3_k$ to order five,

(iii) and 99,715 lines meeting a septic threefold in $\mathbb{P}^4_k$ to order 7.

Proposition 6.6. *If $W_1 \subseteq \mathbb{P}^n_k$ is a hyperplane, $W_2 \subseteq \mathbb{P}^n_k$ an $(n-2)$-plane corresponding to the divisor σ_1 on $\mathbb{G}\mathrm{r}(1,n)$, and*

$$H^0\left(\Phi, \mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)\right) = \left\{(X, \{T_i\}_{i\in I}, \{p_i\}_{i\in I}) : \begin{array}{r} X \subseteq \mathbb{P}^n_k \text{ such that } T_i \text{ is highly} \\ \text{tangent line to } X \text{ at } p_i \in \mathbb{P}^n_k \end{array}\right\}$$

then the locus

$$Z = \left\{(X, \{T_i\}_{i\in I}, \{p_i\}_{i\in I}) \in H^0\left(\Phi, \mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)\right) : \begin{array}{c} W_1 \cap (\cup_{i\in I} p_i) \neq \emptyset \\ W_2 \cap (\cup_{i\in I} T_i) \neq \emptyset \end{array}\right\}$$

has k-codimension at least 2 in $H^0\left(\Phi, \mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)\right)$ and the bundle of principal parts relatively oriented with respect to W_1 and W_2

$$\mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)_{W_1,W_2} = H^0\left(\Phi, \mathcal{P}^{m-1}_{\Phi/\mathbb{G}\mathrm{r}(1,n)}\left(\beta^*\mathcal{O}_{\mathbb{P}^n_k}(d)\right)\right) \setminus Z$$

is $\mathbb{A}^1$-connected and the $\mathbb{A}^1$-Euler number is independent of section.

Proof. Assume without loss of generality that $W_2 \subseteq W_1 \subseteq \mathbb{P}^n_k$ so it sufficies to show that

$$W_2 \cap (\cup_{i\in I} (T_i \cap W_1)) = \emptyset.$$

Note that there is an $\dim_k(X \cap W_1) = (n-2)$-dimensional space of lines in $\mathbb{P}^n_k$ such that $p_i \notin W_1$ for all $i \in I$ and by a computation of Schubert classes on $\mathbb{G}\mathrm{r}(1,n)$, there is a $(2n-2)$-dimensional space of lines not meeting the $(n-2)$-dimensional plane $W_2 \subseteq \mathbb{P}^n_k$ the complement of the Schubert class σ_1 as discussed in Section **??**. We verify that the inequalities

$$\binom{n+d}{d} - (n-2) \geq 2$$
$$\binom{n+d}{d} - (2n-2) \geq 2$$

hold.

The inequalities follow by applying a standard inequality $\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$ (vis. eg. [CLRS09, p. 1186]) to $\binom{n+d}{d}$ in the cases $n \geq 3$ and can be verified by direct computation for $n = 2$. This shows the statement on codimension, implying that the space of sections is $\mathbb{A}^1$-connected. Harder's theorem [KW19, Lemma 30] thus implies that the $\mathbb{A}^1$-Euler number is independent of section. □

7. Compatible local trivializations

[A couple of words on compatible local trivilalizations]

Let $\Pi\colon \mathbb{G}\mathrm{r}(r,n) \to \mathbb{P}^{\binom{n}{r}-1}$ be the Plücker embedding. Let (U_i, φ) be McKean trivialization. We define a trivialization on $\mathbb{G}\mathrm{r}(r,n)$, still denoted (U_i, φ), given by the pull back of the trivialization on $\mathbb{P}^{\binom{n}{r}-1}$ under Π.

8. Orienting divisors

As seen in the previous section, the bundle $\mathcal{E}_m \to \Phi$ is often not orientable, notably in our cases of interest ($r = 1$). In order to resolve this issue, we will follow the ideas of [LV21] and work relative to a divisor.

Definition 8.1. Let X be a a smooth proper k-scheme $X \to \operatorname{Spec} k$of dimension n. A vector bundle V over X is said to be *orientable relative to a divisor* $D \subset X$ if there exists a line bundle $\mathcal{L}$ and an isomorphism $\rho\colon \det V \otimes \omega_{X/k} \otimes \mathcal{O}(D) \to \mathcal{L}^{\otimes 2}$.

8.1. **Compatible local trivializations.** In this section, $\Phi = \Phi_{1,n}$ is the flag variety of pointed lines in $\mathbb{P}^n$. Let $U_i \subset \mathbb{P}^n$ be the open subsets of points $[x_0 : \ldots : x_n]$ such that $x_i \neq 0$. A point of the Grassmannian $\mathbb{G}\mathrm{r}(1, n)$ can be represented as a $\mathrm{GL}(2, k)$-orbit

$$M = \left[\begin{pmatrix} x_0 & \ldots & x_n \\ y_0 & \ldots & y_n \end{pmatrix}\right]_{\mathrm{GL}(2,k).}$$

where M has rank 2 and represents a line through $[x_0 : \ldots : x_n]$ and $[y_0 : \ldots : y_n]$. The subset $\beta^{-1}(U_i)$ consists of pairs

$$[x_0 : \ldots : x_n] \times \left[\begin{pmatrix} x_0 & \ldots & x_n \\ y_0 & \ldots & y_n \end{pmatrix}\right]_{\mathrm{GL}(2,k).} \tag{8.1}$$

where $x_i \neq 0$. The submatrix of order 1 containing x_i has rank 1, thus M has rank 2 if and only if M contains a non-singular submatrix

$$\begin{pmatrix} x_i & x_\alpha \\ y_i & y_\alpha \end{pmatrix}, \tag{8.2}$$

where α is such that $0 \leq \alpha \leq n$ and $\alpha \neq i$. We denote by $U_{i\alpha} \subset \Phi$ the open set parameterizing points of $\beta^{-1}(U_i)$ such that the minor in Equation (8.2) is non-singular.

I think the submatrix condition preceding (7.2) is a little awkwardly worded. Maybe "thus M has rank 2 if and only if M contains a submatrix ..."?

I changed it.

Definition 8.2. Let $i, \alpha \in \{0, \ldots, n\}$ be integers such that $i \neq \alpha$. The morphism $\tilde{\varphi}_{i\alpha}\colon U_{i\alpha} \to \mathbb{A}^n \times \mathbb{A}^{n-1}$ is defined as the inverse of the following:

$$\begin{gathered}(x_0, \ldots, \widehat{x}_i, \ldots, x_n) \times (y_0, \ldots, \widehat{y}_i, \ldots, \widehat{y}_\alpha, \ldots, y_n) \longmapsto \\ [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n] \times \left[\begin{pmatrix} x_0 & \ldots & 1 & \ldots & x_\alpha & \ldots & x_n \\ (-1)^\alpha y_0 & \ldots & 0 & \ldots & 1 & \ldots & y_n \end{pmatrix}\right]_{\mathrm{GL}(2,k).}\end{gathered}$$

Roughly speaking, $\tilde{\varphi}$ takes a point of $U_{i\alpha}$ like in Equation (8.1), where the $\mathrm{GL}(2, k)$-orbit is chosen such that $y_i = 0$, and send it to

$$\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right) \times \left((-1)^\alpha \frac{y_0}{y_\alpha}, \ldots, \frac{y_{i-1}}{y_\alpha}, \frac{y_{i+1}}{y_\alpha}, \ldots, \frac{y_n}{y_\alpha}\right).$$

Maybe "Roughly speaking, $\widetilde{\varphi}$ takes a matrix-point pair ..." alternatively "takes a point of Φ (or $U_{i\alpha}$) to ..."? Because $\widetilde{\varphi}$ maps the whole pair, not just the matrix.

I changed it.

Proposition 8.3. *The covering maps $\{(U_{i\alpha}, \tilde{\varphi}_{i\alpha})\}$ are Nisnevich coordinates.*

Moreover, $\tilde{\varphi}_{i\alpha}$ determines a distinguished basis element of $\det T_{\Phi|U_i}$ with transition functions $g_{(i\alpha),(j\beta)} \colon T_{\Phi|U_{j\beta}} \to T_{\Phi|U_{i\alpha}}$ such that

$$\det g_{(i\alpha)(j\beta)} = (\frac{x_i}{x_j})^2(\frac{x_i y_\alpha}{x_j y_\beta})^n. \tag{8.3}$$

Proof. See [?,Proposition 3.7] □

Lemma 8.4. *The line bundles $\mathcal{O}_\Phi(1)$ and $\pi^*\mathcal{O}_{\mathbb{G}\mathrm{r}(1,n)}(1)$ are locally trivialized over $U_{i\alpha}$, respectively, by*

$$(\frac{x_i}{x_0}), \qquad (\frac{x_i y_\alpha}{x_0 y_1 - x_1 y_0}).$$

Moreover, the transition functions are, respectively,

$$(\frac{x_i}{x_j}), \qquad (\frac{x_i y_\alpha}{x_j y_\beta}).$$

Proof. See [?,Lemma 3.8]. □

Proposition 8.5. *Let $m = 2n-1$. The line bundle $\det \mathcal{E}_m$ is trivialized on $U_{i\alpha}$ by the distinguished element*

$$(\frac{x_i}{x_0})^{m(d-m+1)}(\frac{x_i y_\alpha}{x_0 y_1 - x_1 y_0})^{m(m-1)/2}, \tag{8.4}$$

with transition functions $\det \mathcal{E}_{m|U_{j\beta}} \to \det \mathcal{E}_{m|U_{i\alpha}}$ being

$$(\frac{x_i}{x_j})^{m(d-m+1)}(\frac{x_i y_\alpha}{x_j y_\beta})^{m(m-1)/2}. \tag{8.5}$$

Proof. Combine Lemma 8.4 with Lemma **??**. □

For each open set $U_{i\alpha}$, let us consider the following map $\lambda_{i\alpha} \colon \det T_{\Phi|U_{i\alpha}} \to \det \mathcal{E}_{m|U_{i\alpha}}$ where

$$\lambda_{i\alpha}((-1)^\alpha \cdot \partial_i \wedge \partial_\alpha) = (\frac{x_i}{x_0})^{m(d-m+1)}(\frac{x_i y_\alpha}{x_0 y_1 - x_1 y_0})^{m(m-1)/2}.$$

Combining Proposition 8.3 and 8.5, we see that

$$\lambda_{i\alpha}((-1)^\alpha \cdot \partial_i \wedge \partial_\alpha) = (\frac{x_j}{x_i})^M(\frac{x_i y_\alpha}{x_j y_\beta})^N \lambda_{j\beta}((-1)^\alpha \cdot \partial_i \wedge \partial_\alpha) \tag{8.6}$$

where we denoted

$$M := m(d-m+1) - 2, \qquad N := m(m-1)/2 - n.$$

Note that $N \equiv 1 \bmod 2$ and $M \equiv d \bmod 2$. Let $D := \pi^*(\{x_0y_1 - x_1y_0 = 0\})$, and d be even. Note that $\mathcal{O}_\Phi(D) = \pi^*\mathcal{O}_{\mathbb{G}\mathrm{r}(1,n)}(1)$. Since M and $N+1$ are even, we have a tensor square

$$\mathrm{Hom}(\det(T_\Phi), \det(\mathcal{E})) \otimes \mathcal{O}_\Phi(D) = \left(\pi^*\mathcal{O}_{\mathbb{G}\mathrm{r}(1,n)}(\frac{N+1}{2}) \otimes \mathcal{O}_\Phi(\frac{M}{2})\right)^{\otimes 2}.$$

As $\mathcal{O}_\Phi(D)$ is locally trivialized by $(\frac{x_iy_\alpha}{x_0y_1-x_1y_0})$, we define on $U_{i\alpha}$

$$\tilde{\psi}_{|U_{i\alpha}}((\frac{x_i}{x_0})^M(\frac{x_iy_\alpha}{x_0y_1 - x_1y_0})^{N+1}) = \lambda_{i\alpha}. \tag{8.7}$$

It follows that there exists a well-defined morphism

$$\tilde{\psi}\colon \pi^*\mathcal{O}_{\mathbb{G}\mathrm{r}(1,n)}(N) \otimes \mathcal{O}_\Phi(M) \otimes \mathcal{O}_\Phi(D) \longrightarrow \mathrm{Hom}(\det(T_\Phi), \det(\mathcal{E})).$$

Lemma 8.6. *Let d be even. The local trivializations*

$$(\frac{x_i}{x_0})^{m(d-m+1)}(\frac{x_iy_\alpha}{x_0y_1 - x_1y_0})^{m(m-1)/2}$$

of $\det \mathcal{E}_m$ are compatible with the Nisnevich coordinates $\{(U_{i\alpha}, \tilde{\varphi}_{i\alpha})\}$ and the relative orientation $(\pi^*\mathcal{O}_{\mathbb{G}\mathrm{r}(1,n)}(\frac{N+1}{2}) \otimes \mathcal{O}_\Phi(M/2), \tilde{\psi})$ relative to the divisor $\pi^*\mathcal{O}_{\mathbb{G}\mathrm{r}(1,n)}(1)$.

Proof. The canonical section 1_D of $\mathcal{O}_\Phi(D)$ is locally given by (the pullback of) $\frac{x_iy_\alpha}{x_0y_1-x_1y_0}$. By construction,

$$\lambda_{i\alpha}((-1)^\alpha \cdot \partial_i \wedge \partial_\alpha) \otimes (\frac{x_iy_\alpha}{x_0y_1 - x_1y_0}) = (\frac{x_i}{x_0})^{m(d-m+1)}(\frac{x_iy_\alpha}{x_0y_1 - x_1y_0})^{m(m-1)/2},$$

so by Equation (8.6),

$$\lambda_{i\alpha}((-1)^\alpha \cdot \partial_i \wedge \partial_\alpha) \otimes (\frac{x_iy_\alpha}{x_0y_1 - x_1y_0}) = (\frac{x_i}{x_j})^M(\frac{x_iy_\alpha}{x_jy_\beta})^N \lambda_{j\beta}((-1)^\alpha \cdot \partial_i \wedge \partial_\alpha) \otimes (\frac{x_jy_\beta}{x_0y_1 - x_1y_0}).$$

Thus the maps $\tilde{\psi}_{|U_{j\beta}}$ and $\tilde{\psi}_{|U_{i\alpha}}$ differ by the transition function

$$\left((\frac{x_i}{x_j})^{M/2}(\frac{x_iy_\alpha}{x_jy_\beta})^{(N+1)/2}\right)^{\otimes 2}.$$

So $\{(U_{i\alpha}, \tilde{\varphi}_{i\alpha})\}$ is well-defined, and the relative orientation is a square by construction. This proves the lemma. □

If d is an odd number, then we may consider $D := \pi^*(\{x_0y_1 - x_1y_0 = 0\})\beta^*(\{x_0 = 0\})$. Following the same steps, we can easily prove the following.

Lemma 8.7. *Let d be odd. The local trivializations*

$$(\frac{x_i}{x_0})^{m(d-m+1)}(\frac{x_iy_\alpha}{x_0y_1 - x_1y_0})^{m(m-1)/2}$$

of $\det \mathcal{E}_m$ are compatible with the Nisnevich coordinates $\{(U_{i\alpha}, \tilde{\varphi}_{i\alpha})\}$ and the relative orientation $(\pi^*\mathcal{O}_{\mathbb{G}\mathrm{r}(1,n)}(\frac{N+1}{2}) \otimes \mathcal{O}_\Phi(\frac{M+1}{2}), \tilde{\psi})$ relative to the divisor $\pi^*\mathcal{O}_{\mathbb{G}\mathrm{r}(1,n)}(1) \otimes \beta^*\mathcal{O}_\Phi(1)$.

9. Relatively orientable case

TO DO: give a list of orientable (n, r, m, d) for $n, m, r, d \leq$ some bound.

Up to 5000, the only families tuples are $(d, r, m, n) \in \{(2, 1, 1, 1), (6, 3, 5, 11), (22, 3, 21, 445), (38, 3, 37, 2287$

I inserted this section in the polished file.

10. Local index

A hypersurface $\mathbb{V}(F) \subset \mathbb{P}^n$ of degree d determines a section

$$\sigma_F \colon \Phi \longrightarrow \mathcal{E}_m.$$

Our next goal is to write out a formula for the composite

$$g_U := \tilde{\varphi}^{-1}|_U \circ \sigma_X \circ \tau|_U \colon \mathbb{A}^{2n-1} \longrightarrow \mathbb{A}^{2n-1}$$

around a zero $p \in U \subset \Phi$, where τ is the local trivialization. After we have done this, we will compute the local degree $\deg_p(g_U)$ and relate it to the geometry of X and its highly tangent line p. If the highly tangent line is geometrically simple (i.e. geometrically reduced), then $\deg_p(g_U) = \mathrm{Tr}_{k(p)/k}\langle \mathrm{Jac}(g_U)|_p \rangle$ (assuming $k(p)/k$ is separable).

Given a pointed line $(L, p) \in \Phi$, we can evaluate the section σ_F by restricting F (viewed as a section of $\mathcal{O}_{\mathbb{P}^n}(d)$) to a section $F|_L$ of $\mathcal{O}_L(d)$ I think this is misleading. We then consider the summands of the m-jet of $F|_L$ at p:

$$\sigma_F(L, p) = \left(F|_L(p), D^{(1)}F|_L(p), \ldots, D^{(m-1)}F|_L(p)\right). \tag{10.1}$$

Here, $D^{(i)}$ denotes the *Hasse derivative*, characterized on polynomials by

$$D^{(i)} z^j = \begin{cases} \binom{j}{i} z^{j-i} & j \geq i, \\ 0 & \text{otherwise,} \end{cases}$$

with respect to which we have Taylor's theorem in the coordinate ring of L (regardless of $\operatorname{char} k$).

Let $[z_0 : \ldots : z_n]$ denote coordinates on $\mathbb{P}^n$. For $i, \alpha \in \{0, \ldots, n\}$ such that $i \neq \alpha$, the pointed line $\tilde{\varphi}^{-1}_{i\alpha}(x_1, \ldots, x_n, y_1, \ldots, y_{n-1})$ is given by

$$(L, p) = \left(\mathbb{V}(\{z_j - (a_j z_i + b_j(z_\alpha - a_\alpha z_i))\}_{j \neq i, \alpha}), [a_0 : \ldots : a_n]\right),$$

Note: this can be written also

$$(L, p) = \left(\mathbb{V}(\{z_j - (a_j z_i + b_j(z_\alpha - a_\alpha z_i))\}_{j \neq i, \alpha}), (a_\alpha)\right),$$

because once you have L, the point $[a_0 : \ldots : a_n]$ is obtained setting $z_i = 1, z_\alpha = a_\alpha$. This points to the fact that a_α is a local coordinate of L

where

$$[a_0 : \ldots : a_n] = [x_1 : \ldots : x_{i-1} : 1 : x_i : \ldots : x_n],$$

$$[b_0 : \ldots : b_n] = \begin{cases} [(-1)^\alpha y_1 : y_2 : \ldots : y_{i-1} : 0 : y_i : \ldots : y_\alpha : 1 : y_{\alpha+1} : \ldots : y_n] & \alpha > i, \\ [(-1)^\alpha y_1 : y_2 : \ldots : y_{\alpha-1} : 1 : y_\alpha : \ldots : y_i : 0 : y_{i+1} : \ldots : y_n] & \alpha < i. \end{cases}$$

The defining equation for L can be obtained by taking the parametric solution $L = \{(s-t)a + tb : [s : t] \in \mathbb{P}^1\}$ for the line through $a = [a_0 : \ldots : a_n]$ and $b = [b_0 : \ldots : b_n]$. Noting that $a_i = 1$, $b_i = 0$, and $b_\alpha = 1$, we can then solve for $z_i = s - t$ and $z_\alpha = (s-t)x_\alpha + t$. Substituting these values into $\{z_j = (s-t)a_j + tb_j\}_{j=0}^n$ yields the $n-1$ linear forms defining L, along with two constant forms ($z_i = z_i$ and $z_\alpha = z_\alpha$).

We can now compute the section

$$F|_L = F(a_0 z_i + b_0(z_\alpha - a_\alpha z_i), \ldots, z_i, \ldots, z_\alpha, \ldots, a_n z_i + b_n(z_\alpha - a_\alpha z_i))$$

in terms of the homogeneous coordinates $[z_i : z_\alpha]$ on L. In order to compute $\sigma_F(L,p)$ as in Equation (10.1), we need a local parameter about p with respect to which we will Hasse differentiate. This local parameter is given by $z_\alpha - a_\alpha z_i$. We may therefore write

$$F|_L = \sum_{\ell \geq 0} c_\ell (z_\alpha - a_\alpha z_i)^\ell z_i^{d-\ell}, \tag{10.2}$$

so that $\sigma_F(L,p) = (c_0, \ldots, c_{m-1})$ more precisely, $\sigma_F(L,p)_{|L} = (c_0, \ldots, c_{m-1})$.

Note: L is canonically identified with a fibre of π. Thus $F|_L$ is the restriction of $\beta^* F$ to the fibre L.

Note: we first considered the restriction $F_{|L}$, and then took jets. The result coincides with $\sigma_F(L,p)_{|L}$ (which is constructed from $\beta^* F$) due to the functorial properties of the bundle of principal parts [Gro67, (16.7.9.1)].

TO DO: trivialization; this should be almost given by coordinate projections from $(c_0, \ldots, c_{2n-2})$, but we have to pay attention to the divisor at ∞, etc. The Jacobian is then given in terms of the partials with respect to x_i and y_i; we need to compute $\partial c_\ell / \partial x_i$, $\partial c_\ell / \partial y_i$. In order to compute these, we need to more explicitly compute $c_\ell = D^{(\ell)} F|_L$, where the Hasse derivative is being taken with respect to $z_\alpha - a_\alpha z_i$. Note that $\partial a_\alpha / \partial x_{\alpha+1} = 1$ if $\alpha < i$, $\partial a_\alpha / \partial x_\alpha = 1$ if $i < \alpha$, and $\partial a_\alpha / \partial x_j = 0$ otherwise.

For example, let $n = 3, i = 1, \alpha = 2$ and

$$F = \sum_{a+b+c=3} t_{abc} z_0^a z_1^b z_2^c = t_{300} z_0^3 + t_{210} z_0^2 z_1 + t_{201} z_0^2 z_2 + t_{120} z_0 z_1^2 + t_{111} z_0 z_1 z_2 + t_{102} z_0 z_2^2$$

$$+ t_{012} z_1 z_2^2 + t_{021} z_1^2 z_2 + t_{030} z_1^3 + t_{003} z_2^3.$$

On the line L given by $z_0 - a_0 z_1 + b_0(z_2 - a_2 z_1)$, we first make the substitution $z_0 = a_0 z_1 + b_0(z_2 - a_2 z_1), z_2 = (a_2 z_1 + (z_2 - a_2 z_1))$ and introduce $T = (z_2 - a_2 z_1)$. Making the appropriate substitutions, we get

$$\begin{aligned}F|_L &= t_{300}(a_0z_1+b_0T)^3+t_{210}(a_0z_1+b_0T)^2z_1+t_{201}(a_0z_1+b_0T)^2(a_2z_1+T)+t_{120}(a_0z_1+b_0T)z_1^2\\&\quad+t_{111}(a_0z_1+b_0T)z_1(a_2z_1+T)+t_{102}(a_0z_1+b_0T)(a_2z_1+T)^2+t_{012}z_1(a_2z_1+T)^2\\&\quad+t_{021}z_1^2(a_2z_1+T)+t_{030}z_1^3+t_{003}(a_2z_1+T)^3\\&=(t_{300}b_0^3+t_{201}b_0^2+t_{102}b_0+t_{003})T^3\\&\quad+(3t_{300}a_0b_0^2+t_{201}b_0^2a_2+2t_{201}a_0b_0+t_{210}b_0^2+2t_{102}b_0a_2+t_{102}a_0+t_{111}b_0+3t_{003}a_2+t_{012})T^2z_1\\&\quad+(3t_{300}a_0^2b_0+2t_{201}a_0b_0a_2+t_{102}b_0a_2^2+t_{201}a_0^2+2t_{210}a_0b_0+2t_{102}a_0a_2+t_{111}b_0a_2+3t_{003}a_2^2\\&\qquad+t_{111}a_0+t_{120}b_0+2t_{012}a_2+t_{021})Tz_1^2\\&\quad+(t_{300}a_0^3+t_{201}a_0^2a_2+t_{102}a_0a_2^2+t_{003}a_2^3+t_{210}a_0^2+t_{111}a_0a_2+t_{012}a_2^2+t_{120}a_0+t_{021}a_2+t_{030})z_1^3\end{aligned}$$

and thus the c_i are given by

$$\begin{aligned}c_2&=3t_{300}a_0b_0^2+t_{201}b_0^2a_2+2t_{201}a_0b_0+t_{210}b_0^2+2t_{102}b_0a_2+t_{102}a_0+t_{111}b_0+3t_{003}a_2+t_{012}\\c_1&=3t_{300}a_0^2b_0+2t_{201}a_0b_0a_2+t_{102}b_0a_2^2+t_{201}a_0^2+2t_{210}a_0b_0+2t_{102}a_0a_2+t_{111}b_0a_2+3t_{003}a_2^2\\&\quad+t_{111}a_0+t_{120}b_0+2t_{012}a_2+t_{021}\\c_0&=t_{300}a_0^3+t_{201}a_0^2a_2+t_{102}a_0a_2^2+t_{003}a_2^3+t_{210}a_0^2+t_{111}a_0a_2+t_{012}a_2^2+t_{120}a_0+t_{021}a_2+t_{030}\end{aligned}$$

This has been amended and agrees with Giosuè's computation.

Example 10.1. Let us take the Fermat cubic $F=z_0^3+z_1^3+z_2^3$, that is $n=2$. Let us take also $i=1,\alpha=2$. Hence we have

$$L: z_0=a_0z_1+b_0(z_2-a_2z_1),$$

hence the notation $F_{|L}$ is misleading

$$\begin{aligned}F|_L&=(a_0z_1+b_0(z_2-a_2z_1))^3+z_1^3+(a_2z_1+(z_2-a_2z_1))^3\\&=(\underbrace{a_0^3+a_2^3+1}_{c_0})z_1^3+(\underbrace{3a_0^2b_0+3a_2^2}_{c_1})z_1^2(z_2-a_2z_1)+(\underbrace{3a_0b_0^2+3a_2}_{c_2})z_1(z_2-a_2z_1)^2\\&\quad+(1+b_0^3)(z_2-a_2z_1)^3.\end{aligned}$$

The Jacobian of $\sigma_F(L,p)=(c_0,c_1,c_2)$ is

$$J_{\sigma_F}=\begin{pmatrix}3a_0^2&3a_2^2&0\\6a_0b_0&6a_2&3a_0^2\\3b_0^2&3&6a_0b_0\end{pmatrix}.$$

The line L is inflexional at the point $p=[a_0:1:a_2]$ if we choose $(L,p)=(a_0,a_2,b_0)$ such that $c_0=c_1=c_2=0$. Let us consider the point $(L,p)=(-1,0,0)\in U_{i\alpha}$, so that

$$J_{\sigma_F|(L,p)}=\begin{pmatrix}3&0&0\\0&0&3\\0&3&0\end{pmatrix}.$$

Is there any geometric interpretation of this matrix? I do not know. There should be. Stephen in §5 of his thesis shows that local indices can be interpreted as an intersection volume in the moduli space. There should be a direct way to interpret this geometrically given that the space Φ is pretty nice. Based on Stephen's thesis, we should say that the geometric interpretation is the volume of the parallelepiped defined by the gradient vectors $\{\nabla c_i^l(p)\}_{i\neq l}$. Note that this volume is not independent of $U_{i\alpha}$. In our last meeting, Sthepen suggested to look for a less obvious interpretation. I guess what I was trying to say was that since Φ is roughly the product of the Grassmannian and a line, does a paralellepiped in this setting have a nice geometric interpretation? I probably won't have time to sit down and think about this until next week, moving, graduation, etc.

In this moment, I do not have any good idea in this direction, sorry.

So in the setup above there are 3 highly tangent lines at the point $p = (-1, 0, 0)$? Because each highly tangent pointed line should contribute a rank 1 form. So I'm not even sure what you are asking for is possible, since the matrix above is not a local index, which should be $\sigma_F|_{(L,p)}$.

I confused a little bit the notation, now it is correct. I denoted the point in $\mathbb{P}^n$ by p, but I also denoted by p what it was supposed to be denoted by (L, p). In our last meeting, Sthepen talked about finding a geometric explanation for the matrix $J_{\sigma_{F|(L,p)}}$. Note that the "number" of maximally tangent lines is the rank of the bilinear form given by $\det(J_{\sigma_{F|(L,p)}})$. In particular, it is one if both L and p are k-rational.

Are Newton Polygons the right interpretation for local indices?

Example 10.2. Consider the following Clebsch-like example

$$F = z_0^3 + z_1^3 + z_2^3 + 3z_0z_1z_2.$$

All solutions are reduced, one of them being $(a_0, a_2, b_0) = (0, -1, 1)$ at which the local $\mathbb{A}^1$-degree is $\langle 27\rangle$. This corresponds to an inflection at the point $[0 : 1 : -1]$.

Nb. Sturmfels and coauthors in their book on metric algebraic geometry have a section on curvature (Ch. 6). Could the local degree be interpreted as some type of fundamental form?

This code compute $F_{|L}$ in Wolfram

```
F = Sum[Subscript[t, a, b, 3 - a - b]*Subscript[z, 0]^a*Subscript[z,
   1]^b*Subscript[z, 2]^(3 - a - b), {a, 0, 3}, {b, 0, 3 - a}]
FL = F /. {Subscript[z, 0] -> Subscript[a, 0]*1 + Subscript[b, 0]*T,
   Subscript[z, 2] -> Subscript[a, 2]*1 + T, Subscript[z, 1] -> 1}
Collect[FL, T]
Coefficient[FL, T, 0]
Coefficient[FL, T, 1]
Coefficient[FL, T, 2]
Coefficient[FL, T, 3]
```

Example 10.3. Gabriel's example. Let us compute the Jacobian matrix of $\sigma_F(L,p)$ where $\mathbb{V}(F)$ is a cubic curve given by

$$F = \sum_{a+b+c=3} t_{abc} z_0^a z_1^b z_2^c.$$

We restrict to the open subset given by $i = 1, \alpha = 2$. This parameterizes lines L passing through a point $p = [a_0 : 1 : a_2]$ given by the equation $z_0 = a_0 z_1 + b_0(z_2 - a_2 z_1)$. Using the change of coordinates $(w_i, w_\alpha) = (z_i, z_\alpha - a_\alpha z_i)$, it is easy to see that around (L,p) the pull back $\beta^* F$ can be written as

$$\beta^* F = \sum_{\ell=0}^{3} c_\ell w_\alpha^\ell \in \Gamma(U_{i\alpha}, \mathcal{O}_{U_{i\alpha}}).$$

The Jacobian matrix is

$$\begin{pmatrix} 3a_0^2 t_{300} + 2a_2 a_0 t_{201} + 2a_0 t_{210} + a_2^2 t_{102} + a_2 t_{111} + t_{120} & a_0^2 t_{201} + 2a_2 a_0 t_{102} + a_0 t_{111} + 3a_2^2 t_{003} + 2a_2 t_{012} + t_{021} & \\ 2a_2 b_0 t_{201} + 6a_0 b_0 t_{300} + 2a_2 t_{102} + 2a_0 t_{201} + 2b_0 t_{210} + t_{111} & 2a_2 b_0 t_{102} + 2a_0 b_0 t_{201} + 6a_2 t_{003} + 2a_0 t_{102} + b_0 t_{111} + 2t_{012} & 3a_0^2 t_{300} + 2a_2 a \\ 3b_0^2 t_{300} + 2b_0 t_{201} + t_{102} & b_0^2 t_{201} + 2b_0 t_{102} + 3t_{003} & 2a_2 b_0 t_{201} + 6a_0 b \end{pmatrix}$$

Note that in $\beta^* F$ we are actually working with both the trivialization with respect to π and β.

Classically, we know that the inflection points of a curve C correspond to the intersection points of the Hessian curve H_C with C. Does this extend to the setting of $\mathbb{A}^1$ homotopy theory? In particular, Stephen gives an interpretation of local data in Bézout's theorem as the signed parallepiped volume of the tangent vectors to the hypersurfaces

Continuing my example, we can consider the point $(a_0, a_2, b_0) = (0, -1, 1)$ which is a point in the preimage of the vanishing of the section

$$\begin{aligned} c_0 &= a_0^3 + a_2^3 + 3a_0 a_2 + 1 \\ c_1 &= 3a_0^2 b_0 + 3a_2^2 + 3a_2 b_0 + a_0 \\ c_2 &= 3a_0 b_0^2 + 3a_2 + 3b_0 \end{aligned}$$

computing the $\mathbb{A}^1$-degree at this point, we get

$$\left\langle \det\left(\begin{bmatrix} -3 & 3 & 0 \\ 1 & -3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \right) \right\rangle = \langle -36 \rangle = \langle -1 \rangle.$$

On the other hand, we can apply the Bézout-McKean theorem to consider this as the local index of an intersection of the curve with the Hessian curve in $\mathbb{P}^2$. We compute the Hessian curve of C which is $-54z_0^3 - 54z_1^3 - 54z_2^3 + 270 z_0 z_1 z_2$. Computing its derivative at $[z_0 : z_1 : z_2] = [a_0 : 1 : a_2] = [0 : 1 : -1]$ we have

$$\begin{aligned} \partial_{z_0} H_C([0:1:-1]) &= -270 \\ \partial_{z_2} H_C([0:1:-1]) &= -162 \end{aligned}$$

and for the curve

$$\begin{aligned} \partial_C([0:1:-1]) &= -3 \\ \partial_C([0:1:-1]) &= 3 \end{aligned}$$

and the signed parallepiped volume is given by the determinant of the matrix

$$\begin{bmatrix} -270 & -3 \\ -162 & 3 \end{bmatrix} = -1296$$

which is equivalent to $\langle -1 \rangle$ up to squares and agrees with the local degree via sections.

From Giosué's computation of the Jacobian above, we have an algebraic expression of the local index in terms of the t_{ijk}s, a_0, a_2, and b_0 (and differently for different choices of $U_{i\alpha}$).

A priori, the computation of the local degree via the determinant of the sections of the Jacobian (ie. the displayed matrix Giosué computed), is a polynomial in t_{ijk}'s, a_0, a_2, and b_0. On the other hand, the intersection volume of the paralellepiped defined by C and H_C in $\mathbb{P}^2$ is a polynomial in the t_{ijk}'s, a_0, and a_2. In particular, the intersection volume of the parallelepiped is independent of b_0.

Recall that b_0 is defines the line between the projective points $[a_0 : 1 : a_2]$ and $[b_0 : 0 : 1]$ which is the tangent line of C at the point $[a_0 : 1 : a_2]$. In particular, we would expect that we can write b_0 in terms of the t_{ijk}'s, a_0, and a_2 using the equation $c_1 = 0$.

$$b_0 = -\frac{t_{201}a_0^2 + 2t_{102}a_0a_2 + 3t_{003}a_2^2}{3t_{300}a_0^2 + 2t_{201}a_0a_2 + t_{102}a_2^2 + 2t_{210}a_0 + t_{111}a_2}$$

but making this substitution into the polynomial defined by the intersection volume of the parallelepiped, the polynomial does not agree with the Jacobian determinant of the sections.

Note that computing the local degree using the Jacobian of (c_0, c_1, c_2) is very different from using the Jacobian of (C, H_C). They are defined on different spaces.

I agree that it could be enough to get a result valid only for plane curves, like this one using the Hessian. Note that this reduces to the Bezout theorem applied to (C, H_C). It seemed to me that Stephen avoided this approach.

Yes, aware that defined on different spaces, and not using Jacobian of H_C at all, just the signed volume of the parallelepiped defined by the gradient vectors, all in $\mathbb{P}^2$. The upshot is that while the universal interpretation as an intersection volume defined by (c_0, c_1, c_2) is taken in the moduli space Φ, this interpretation of intersection volume of the Hessian takes place in $\mathbb{P}^2$, and in this sense is more geometric. I think the higher analogue of the Hessian curve is the second fundamental form and umbillical umbilical points, but I'm not exactly sure how these correspond to higher tangency.

I do not know exactly what you have in mind. Anyway, the following article that deals with the second fundamental form in AG could be useful: arxiv.org/pdf/1509.05435

Actually, the idea of using the second fundamental form is somehow usable. Recall that the second fundamental of $X \subset \mathbb{P}$ is a morphism of $\mathcal{O}_X$-modules:

$$\mathrm{Sym}^2(T_X) \longrightarrow N_{X/\mathbb{P}}.$$

Can be this defined also for varieties over a field k where $k \notin \{\mathbb{C}, \mathbb{R}\}$?

I do not know but an answer with applications in $\mathbb{A}^1$-homotopy is interesting in his own.

Let us suppose that $X \subset \mathbb{P}^2$ is a plane curve of degree d. Thus the second fundamental form is equivalent to a map

$$\mathcal{O}_X \longrightarrow \mathrm{Sym}^2\Omega_X \otimes N_{X/\mathbb{P}^2}.$$

It can be easily seen that $\deg(\mathrm{Sym}^2\Omega_X \otimes N_{X/\mathbb{P}^2}) = 3d(d-2)$. So that, the inflectional points of a plan curve are exactly the points where the second fundamental form fails to be surjective.

If $\dim(X) = 2$, the cokernel of the second fundamental form is a torsion sheaf, thus it is supported in a finite number of points. Are those points the highly tangent points? Is this still true in higher dimension?

11. Wronskian

The idea is to interpret the Jacobian matrix as a Wronskian. This is a step toward an interpretation of the local index that is not a simple triviality. This task is revealing more challenging for me.

Let us consider $F \in \Gamma(\mathbb{P}^3, \mathcal{O}(3))$. If we restrict to a line

$$L = \left(\mathbb{V}(\{z_j - (a_j z_i + b_j(z_\alpha - a_\alpha z_i))\}_{j\neq i,\alpha})\right),$$

then we get $F_{|L} = f(z_1, z_2) \in \Gamma(L, \mathcal{O}(3)_{|L}) = \Gamma(\mathbb{P}^1, \mathcal{O}(3))$.

Stephen manages to write this polynomial in the form

$$F_{|L} = \sum_{\ell=0}^{3} c_\ell (z_2 - a_2 z_1)^\ell z_1^{d-\ell},$$

and claims that $\sigma_F(L, p) = (c_0, c_1, c_2)$. I have some objections, your honor:

(1) Strictly speaking, c_ℓ is a constant, as the scalars a_i, b_i are fixed. Somehow you ignore that and promote c_ℓ to a function on the variables a_i, b_i. That is, $c_\ell \in \Gamma(U_{i\alpha}, \mathcal{O}_{U_{i\alpha}})$. More justifications are needed.

(2) $F_{|L}$ is a global section of the line bundle $\mathcal{O}_L(3)$ of a fixed line L in $\mathbb{P}^3$. I do not understand the steps that lead you to construct σ_F, which is a global section of a vector bundle of a different space.

I do not claim that your result are wrong. But they would benefit from a more formal approach. I think it is time to write things formally accurate, if we want to submit this anytime soon.

Here is my approach. Just in order to keep everything down-to-earth, we fix a section $F = z_0^3 + z_1^3 + z_2^3$. First of all, we pull-back F to where derivations belong, that is to Φ,

$$\beta^*F = \beta^*z_0^3 + \beta^*z_1^3 + \beta^*z_2^3 \in \Gamma(\Phi, , \mathcal{O}_\Phi(3)).$$

Recall that on $U_{i\alpha}$, $\beta(x_1, x_2, y_1) = (x_1, x_2)$, thus

$$\beta^*F_{|U_{i\alpha}} = x_1^3 + 1 + x_2^3 \in \Gamma(U_{i\alpha}, \mathcal{O}_{U_{i\alpha}}).$$

Let A, B, T be coordinates of $U_{i\alpha}$ such that $\pi(A, B, T) = (A, B)$. That is, the pair (A, B) parameterizes lines, while T is a local coordinate of the fiber of π. In order to compute σ_F, we need to compute the Hasse derivatives of $\beta^* F$ with respect to T. In particular, we need to express $\beta^* F$ as a polynomial in T. Using matrix manipulation, we see that

$$\begin{pmatrix} x_1 & 1 & x_2 \\ y_1 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} x_1 - x_2 y_1 & 1 & 0 \\ y_1 & 0 & 1 \end{pmatrix}.$$

Thus $A = x_1 - x_2 y_1, B = y_1, T = x_2$. We deduce that

$$\beta^* F = (A + BT)^3 + 1 + T^3$$

and

$$\begin{aligned} \sigma_F &= (\beta^* F, D_T^1 \beta^* F, D_T^2 \beta^* F) \\ &= ((A + BT)^3 + 1 + T^3, 3(A + BT)^2 B + 3T^2, 3(A + BT)B^2 + 3T) \\ &= 1(x_1 1 \neq 3 + x_2^3 + 1, 3x_1^2 y_1 + 3x_2^2, 3x_1 y_1^2 + 3x_2). \end{aligned}$$

$$\sigma_F = \big((A + BT)^3 + 1 + T^3, 3(A + BT)^2 B + 3T^2, 6(A + BT)B^2 + 6T\big)$$

This is not correct because $D_T^1 D_T^1 \neq D_T^2$.

We get exactly the same result as using Stephen's approach. The Jacobian of this matrix is obtained by combining these three vectors

$$\begin{aligned} \nabla_{x_1 x_2 y_1} \beta^* F &= (3x_1^2, 3x_2^2, 0) \\ \nabla_{x_1 x_2 y_1} D_T^1 \beta^* F &= (6x_1 y_1, 6x_2, 3x_1^2) \\ \nabla_{x_1 x_2 y_1} D_T^2 \beta^* F &= (3y_1^2, 3, 6x_1 y_1) \\ \nabla_{x_1 x_2 y_1} D_T^2 \beta^* F &= (6y_1^2, 6, 12x_1 y_1) \end{aligned}$$

Unfortunately, it seems that $\nabla_{x_1 x_2 y_1}$ and D_T^i do not commute. Note that $\beta^* F, D_T^1 \beta^* F, D_T^2 \beta^* F$ are not independent, so we expect that their divergent are neither.

The following document might have some ideas that we can incorporate to interpret local indices: hal.science/hal-01779785v1/document. In particular, section 3 discusses lines meeting hypersurfaces in $\mathbb{P}^n$ to high order, and Lemma 3.2 talks about writing such highly tangent lines as a map $\mathbb{A}^1 \to \mathbb{P}^n$ and Lemma 3.4 connects osculation with high tangency.

From a conversation with Stephen in Utah it may be helpful to do this computation over the "moving basis" trivialization since that expresses the coordinate on the fiber more naturally than the trivialization we are doing right now.

Recall that Kass-Wickelgren's "moving basis" is defines coordinates on the base by $x_{1,1}, \ldots, x_{1,n-1}, x_{2,1}, \ldots, x_{2,n-1}, t$ where the line is given by the span of

$$\begin{bmatrix} x_{1,1} & \ldots & x_{1,n-1} & 1 & 0 \\ x_{2,1} & \ldots & x_{2,n-1} & 0 & 1 \end{bmatrix},$$

and the local coordinate on the fiber is given by t. That is, the point on the highly tangent line is given by either

$$[tx_{1,1}+x_{2,1}:\cdots:tx_{1,n-1}+x_{2,n-1}:t:1] \text{ or } [x_{1,1}+tx_{2,1}:\cdots:x_{1,n-1}+tx_{2,n-1}:1:t]$$

depending on the chart of $\mathbb{P}^1$.

In the case of the Fermat surface, we and picking the latter chart for $n=2$ we would be considering the line spanned by the vectors

$$\begin{bmatrix} x_{1,1} & 1 & 0 \\ x_{2,1} & 0 & 1 \end{bmatrix}$$

and using the first chart on $\mathbb{P}^1$, the highly tangent point is $[tx_{1,1}+x_{2,1}:t:1]$. The equation of the line is given by $-X_0+x_{1,1}X_1+x_{2,1}X_2$ for $[X_0:X_1:X_2]$ the coordinates on $\mathbb{P}^2$. For the Fermat, the restriction to the line is given by $(x_{1,1}X_1+x_{2,1}X_2)^3+X_1^3+X_2^3$ and substituing $X_2=1$ we have $(x_{1,1}X_1+x_{2,1})^3+X_1^3+1$

$$= x_{1,1}^3X_1^3+3x_{1,1}x_{2,1}X_1^2+3x_{1,1}x_{2,1}^2X_1+x_{2,1}^3+X_1^3+1$$

taking derivatives, we get sections

$$\begin{aligned} c_0 &= x_{1,1}^3t^3+3x_{1,1}x_{2,1}t^2+3x_{1,1}x_{2,1}^2t+x_{2,1}^3+t^3+1 \\ c_1 &= 3x_{1,1}^3t^2+6x_{1,1}x_{2,1}t+3x_{1,1}x_{2,1}^2+3t^2 \\ c_2 &= 3x_{1,1}^3t+3x_{1,1}x_{2,1}+3t \end{aligned}$$

(Continuing here following the discussion Giosuè and I had after the call Wednesday August 7th)

This gives us a Jacobian

$$\begin{bmatrix} 3x_{1,1}^2t^3+3x_{2,1}t^2+3x_{2,1}^2t & 3x_{1,1}t^2+6x_{1,1}x_{2,1}t+3x_{2,1}^2 & 3x_{1,1}^3t^2+6x_{1,1}x_{2,1}t+3x_{1,1}x_{2,1}^2+3t^2 \\ 9x_{1,1}^2t^2+6x_{2,1}t+3x_{2,1}^2 & 6x_{1,1}t+6x_{1,1}x_{2,1} & 6x_{1,1}^3t+6x_{1,1}x_{2,1}+6t \\ 9x_{1,1}^2t+3x_{2,1} & 3x_{1,1} & 3x_{1,1}^3+3 \end{bmatrix}$$

Conversely, we can consider the divergent vector

$$\begin{bmatrix} 3x_{1,1}^2t^3+3x_{2,1}t^2+3x_{2,1}^2t \\ 3x_{1,1}t^2+6x_{1,1}x_{2,1}t+3x_{2,1}^2 \\ 3x_{1,1}^3t^2+6x_{1,1}x_{2,1}t+3x_{1,1}x_{2,1}^2+3t^2 \end{bmatrix}$$

and computing successive Hasse derivatives with respect to t, we get the same result.

Since $D_{x_{1,1}}D_t^i = D_t^iD_{x_{1,1}}$ and $D_{x_{1,2}}D_t^i = D_t^iD_{x_{1,2}}$ for any i=0,1,2, we can write

$$\begin{aligned} \nabla_{tx_{1,1}x_{2,1}}\beta^*F &= \nabla_{tx_{1,1}x_{2,1}}\beta^*F \\ \nabla_{tx_{1,1}x_{2,1}}D_t^1\beta^*F &= D_t^1\nabla_{tx_{1,1}x_{2,1}}\beta^*F \\ \nabla_{tx_{1,1}x_{2,1}}D_t^2\beta^*F &= D_t^2\nabla_{tx_{1,1}x_{2,1}}\beta^*F \end{aligned}$$

Thus, the local index is the Wronskian with respect to t of the divergent vector of the hypersurface β^*F.

12. Second Fundamental form

Question about the second fundamental form: Griffiths and Harris give a whole sequence of higher fundamental forms in "Algebraic geometry and local differential geometry." Landsberg says the second fundamental form measures first-order deviation of X from T_xX at a point x, and that these fundamental forms form a linear series of quadratic forms. See "On second fundamental forms of projective varieties" and `https://arxiv.org/pdf/math/9809184`. Can we sum up the local second fundamental forms over the flex locus to recover the Euler number?

In "Algebraic geometry and local differential geometry" on page 367 of the paper in the paragraph preceding (1.21), they note that the base of the linear system of the second fundamental form corresponds geometrically to lines in the ambient projective space meeting the complex manifold to order three with a specified tangent direction.

See also the following preprint by Ein and Niu `https://arxiv.org/abs/2304.08430`. Here's the setup: let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a quasiprojective variety of dimension $d \geq 1$, $L = \mathcal{O}_X(1)$, and $V = H^0(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1))$. For each $k \geq 0$ there is a "Taylor-series map" $\alpha_k : V \otimes \mathcal{O}_X \to P^k(L)$ where $P^k(L)$ the bundle of kth principal parts with kernel $R_k \otimes L$ and image $P_k \otimes L$. This allows them to define the k-th projective tangent space $\mathbb{T}^k_x(X)$ of $x \in X$ which is a linear space in $\mathbb{P}^n_{\mathbb{C}}$ as $\mathbb{P}(P_k \otimes L \otimes \kappa(x))$ here denoting $\kappa(x)$ the residue field of the point x. These kth projective tangent spaces are also known as osculating spaces. In §2.2 of the preprint by Di Rocco and coauthors `https://arxiv.org/abs/1410.4811` gives a construction of the kth osculating space as the Proj of a global section and its derivatives. These projectivized tangent spaces/osculating spaces are the images of Gauss maps taking an open subset of the variety X to a Grassmannian by $x \mapsto \mathbb{T}^k_x(X)$ which are determined by fundamental forms. See Thm. 3.3 of the Ein-Niu preprint and the surrounding discussion.

Overall, I think that there is probably a way to connect the fundamental form with our geometric problem of the highly tangent lines. I think the pieces are there to, but I'm not sure how to put them together.

Let $X \subset \mathbb{P}^2$ be a plane curve of degree d. The second fundamental form is defined as a certain local section of $E = \mathrm{Sym}^2\Omega_X \otimes N_{X/\mathbb{P}^2}$. By [Lan98], we know that it can be defined in any characteristic.

Proposition 12.1. *The bundle E is relatively orientable if and only if d is even.*

Proof. Let us denote by $\mathcal{O}_X(1)$ the ample line bundle $\mathcal{O}_{\mathbb{P}^2}(1)|_X$. It is known that

$$N_{X/\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(d)|_X = \mathcal{O}_X(d^2).$$

Moreover, from the cotangent exact sequence we have

$$\Omega_X \otimes N_{X/\mathbb{P}^2} = \det(\Omega_{\mathbb{P}^2})|_X = \mathcal{O}_X(-3d).$$

Finally

$$\Omega_X \otimes E = \mathcal{O}_X(4d^2 - 9d).$$

Thus, $\Omega_X \otimes E$ is a square if and only if $4d^2 - 9d$ is even or $\mathcal{O}_X(1)$ is a square. Both conditions are verified if and only if d is even. □

It follows that the $\mathbb{A}^1$-degree of E, if d is even, is $\frac{3d(d-2)}{2}\mathbb{H}$.

Remark 12.2. The reason why E is orientable (for d even), but $\mathcal{E}_m$ is not is that $\mathcal{E}_m$ contains more information that E. So you need more "orienting information." For this reason E is more likely to be relatively orientable.

Remark 12.3. When d is odd, E is not orientable. An alternative approach in the case $d = 3$ is an enriched count of 3-torsion points in an abelian curve. Points of a cubic curve where a maximally tangent curve passes through correspond to 3-torsion points of the Jacobian of the curve.

What are we gonna do with this section? What do we want to contain?

TODO:

Give a characteristic-free construction of the second fundamental form.

Expand Remarks 12.2 and 12.3.

Is it possible to do this relative to a divisor and get $3d(d-2)$ with odd d? Ie. the points where $\mathcal{O}_X \to \mathrm{Sym}^2\Omega_X \otimes N_{X/\mathbb{P}^2}$ is not surjective away from the orienting divisors are the highly tangent points? We should also show that the second fundamental form not being surjective corresponds to the geometric interpretation of the local index. I've been trying to read Griffiths-Harris but can't seem to figure this out just yet.

Some Remarks on the Second Fundamental Form in Griffiths-Harris Following the setup in G-H on 363 (b) ff., we can write out in the case of plane cubic curves Darboux frames "centered" at some $p \in C$ as in the conditions of their (1.12) of vectors $\{A_0; A_1; A_2\}$ where $A_0 \in k^3$ is in the same projective class as p and A_0, A_1 span the projective tangent space $\widetilde{T}_pC \subseteq \mathbb{P}^2_k$. To this Darboux frame we get a system of 1-forms $\omega_0, \omega_1, \omega_2$ to the A's above where ω_1 gives a basis for the dual affine tangent space $T_pC^\vee$ and $\omega_2 = 0$. The second fundamental form consists of quadrics on the line $p = \mathbb{P}^0_k = \mathbb{P}T_pC$ of the form $q_{1,1,2}\omega_1^2$ here following the notation of the equation at the end of G-H's p. 365. How does this glue? The basepoints of this linear system consist of the points of C where all quadrics vanish. Can we connect this to the local index?

Constructing the second fundamental form algebraically. Consider the Fermat plane cubic in $\mathbb{P}^2_k$. We can compute the sheaf of differentials on each affine chart of the curve as

$\Omega_{X/k}|_{X_0\neq 0} = \frac{d\cdot k[x_{1/0},x_{2/0}]\oplus d\cdot k[x_{1/0},x_{2/0}]}{(x_{1/0}^2\cdot dx_{1/0}+x_{2/0}^2\cdot dx_{2/0})}, \Omega_{X/k}|_{X_1\neq 0} = \frac{d\cdot k[x_{0/1},x_{2/1}]\oplus d\cdot k[x_{0/1},x_{2/1}]}{(x_{0/1}^2\cdot dx_{0/1}+x_{2/1}^2\cdot dx_{2/1})}, \Omega_{X/k}|_{X_2\neq 0} = \frac{d\cdot k[x_{0/2},x_{1/2}]\oplus d\cdot k[x_{0/2},x_{1/2}]}{(x_{0/2}^2\cdot dx_{0/2}+x_{1/2}^2\cdot dx_{1/2})}$

The relative conormal sheaf of the embedding is just the sheaf of ideals defining the curve modulo its square and the normal sheaf is its dual.

Torsion Points Kummer in §8 of arxiv.org/pdf/2301.10621 gives a construction of the 2-torsion points of an Abelian variety in terms of $\mathbb{A}^1$-degrees. Under the (non-canonical) isomorphism between the curve and its Picard group, we would expect that the 3-torsion points correspond to 3-torsion line bundles. We want to find a relative orientation of the identity endomorphism on the curve C so for L a 3-torsion line bundle, we want to find a map $L \otimes L \otimes L \to \mathcal{O}_C$. In the 2-torsion case, it is noted that such an isomorphism is given by a rational function f on the Abelian variety such that the divisor of f is $2D$ where D is the divisor of L. In this case, is it true that such a map is induced by a rational function f on the curve of with associated divisor $3D$ and D the divisor of L? The contribution to the count of lines is given as the Jacobian of this f as in §8.1 of the abovementioned paper.

13. A Comment on the Second Fundamental Form

Add Felipe Voloch, Steven Kleiman and Wenbo Niu to Ack

Let T_X be the tangent space of a smooth projective variety $X \subset \mathbb{P}^n$ over k. The second fundamental form of X is a linear system of bilinear forms on the vector spaces $T_{X,x}$ (for $x \in X$) parameterized by the dual of the normal bundle $N_{X/\mathbb{P}^n}$. In complex differential geometry, its properties are exploited in the influential paper [Lan94]. In particular, the second fundamental form is degenerate at a point of $T_{X,x}$ if and only of that point spans a line of $\mathbb{P}^n$ meeting X at x with contact order 3, as explained on [GH79, pag.367].

Hence, we have an alternative way to count highly tangent lines of a plane curve. As those lines meet the curve with contact order 3, it is enough to count the points where the second fundamental form degenerates.

Given a short exact sequence of vector bundles on X

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0, \tag{13.1}$$

in [AK70, I.3] the second fundamental form is constructed as a morphism

$$E' \longrightarrow \Omega_X \otimes E''. \tag{13.2}$$

In the case that the exact sequence (13.1) is

$$0 \longrightarrow N^\vee_{X/\mathbb{P}^n} \longrightarrow \Omega_{\mathbb{P}^n}|_X \longrightarrow \Omega_X \longrightarrow 0, \tag{13.3}$$

we have a map:

$$N^\vee_{X/\mathbb{P}^n} \longrightarrow \Omega_X \otimes \Omega_X. \tag{13.4}$$

If $X \subset \mathbb{P}^2$ is a plane curve of degree d, then Ω_X is a line bundle. Thus the second fundamental form is equivalent to a global section of $M = \mathrm{Sym}^2\Omega_X \otimes N_{X/\mathbb{P}^2}$.

Proposition 13.1. *The bundle M is relatively orientable if and only if d is even.*

Proof. Let us denote by $\mathcal{O}_X(1)$ the ample line bundle $\mathcal{O}_{\mathbb{P}^2}(1)|_X$. It is known that

$$N_{X/\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(d)|_X = \mathcal{O}_X(d^2).$$

Moreover, from the dual of (13.3) we have

$$\Omega_X \otimes N^\vee_{X/\mathbb{P}^2} = \det(\Omega_{\mathbb{P}^2})|_X = \mathcal{O}_X(-3d).$$

Finally

$$\Omega_X \otimes M = \mathrm{Sym}^3\Omega_X \otimes N_{X/\mathbb{P}^2} = \mathcal{O}_X(4d^2 - 9d).$$

Thus, $\Omega_X \otimes M$ is a square if and only if $4d^2 - 9d$ is even or $\mathcal{O}_X(1)$ is a square. Both conditions are verified if and only if d is even. □

It follows that for an even number d, the enriched count of inflectional points of a plane curve of degree d is $\frac{3d(d-2)}{2}\mathbb{H}$.

13.1. **Local indices.** In this subsection, $X \subset \mathbb{P}^n$ denotes a smooth projective variety over $k = \mathbb{C}$. For reader convenience, we sum up the construction of the t^{th}-fundamental forms as in [EN23].

Let $L = \mathcal{O}_X(1)$ and $V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ be the space of linear forms. For each integer $t \geq 0$, the sheaf of t^{th} order principal parts $P^k(L)$ of L is equipped with a Taylor series map

$$\alpha_t \colon V \otimes \mathcal{O}_X \longrightarrow P^t(L). \tag{13.5}$$

We denote by R_t the sheaf such that $R_t \otimes L = \ker(\alpha_t)$, and by i_t the embedding map $\ker(\alpha_t) \hookrightarrow V \otimes \mathcal{O}_X$. Note that $P^t(L)$ satisfies the following short exact sequence

$$0 \longrightarrow \mathrm{Sym}^t\Omega_X \otimes L \xrightarrow{\pi'_{t+1}} P^{t+1}(L) \xrightarrow{\pi_{t+1}} P^t(L) \longrightarrow 0.$$

The composition

$$R_{t-1} \otimes L \xrightarrow{\alpha_t \circ i_{t-1}} P^t(L) \xrightarrow{\pi_t} P^{t-1}(L)$$

is zero because $\pi_t \circ \alpha_t = \alpha_{t-1}$, thus there exists an induced map $R_{t-1} \otimes L \to \mathrm{Sym}^t\Omega_X \otimes L$.

Definition 13.2. The t^{th} fundamental form is the induced map

$$F_t \colon R_{t-1} \longrightarrow \mathrm{Sym}^t\Omega_X.$$

Example 13.3. By [EN23, pag 4], we know that the first and second fundamental forms are, respectively,

$$R_1 = \Omega_{\mathbb{P}^n}|_X \xrightarrow{F_1} \Omega_X \tag{13.6}$$

$$R_2 = N^\vee_{X/\mathbb{P}^n} \xrightarrow{F_2} \mathrm{Sym}^2\Omega_X. \tag{13.7}$$

Note that the first fundamental form is the canonical map of the cotangent sheaves moreove, the first fundamental form may be seen as the map (13.2) when (13.1) is the Euler exact sequence restricted to X..

In the case that X is a plane curve, (13.7) is equivalent to (13.4).

Now, we prove the following result.

Proposition 13.4. *There exists an exact sequence*

$$0 \longrightarrow R_t \otimes L \longrightarrow R_{t-1} \otimes L \xrightarrow{F_t \otimes \mathrm{id}} \mathrm{Sym}^t \Omega_X \otimes L,$$

such that the composition $R_t \otimes L \to R_{t-1} \otimes L \xrightarrow{i_t} V \otimes L$ *coincides with* $R_t \otimes L \xrightarrow{i_t} V \otimes L$.

Proof. The map $R_t \otimes L \to P^{t-1}(L)$ given by $\alpha_{t-1} \circ i_t$ is zero because it factorizes through $\pi_{t-1} \circ \alpha_t$. Thus $R_t \otimes L$ maps to $R_{t-1} \otimes L$ and the composition with $F_t \otimes \mathrm{id}$ is zero. Finally, since $R_t \otimes L \to R_{t-1} \otimes L$ factorizes a monomorphism, it must be a monomorphism itself. $\square$

For the reader's convenience, we provide the following diagram, which includes all the maps defined so far. All squares are commutative, and the dashed arrows are $F_t \otimes \mathrm{id}$.

(13.8)
$$\begin{tikzcd}
\ldots \arrow[r, hook] & R_2 \otimes L \arrow[r, hook] \arrow[d, hook, "i_2"] & R_1 \otimes L \arrow[r, hook] \arrow[d, hook, "i_1"] \arrow[ddl, dotted] & R_0 \otimes L \arrow[d, hook, "i_0"] \arrow[ddl, dotted] \\
\ldots \arrow[r, equal] & V \otimes \mathcal{O}_X \arrow[r, equal] \arrow[d, "\alpha_2"] & V \otimes \mathcal{O}_X \arrow[r, equal] \arrow[d, "\alpha_1"] & V \otimes \mathcal{O}_X \arrow[d, "\alpha_0"] \\
\ldots \arrow[r, two heads] & P^2(L) \arrow[r, two heads, "\pi_1"] & P^1(L) \arrow[r, two heads, "\pi_0"] & P^0(L) \\
\ldots \arrow[r] & \mathrm{Sym}^2 \Omega_X \otimes L \arrow[r, "0"] \arrow[u, hook, "\pi'_2"] & \Omega_X \otimes L \arrow[r, "0"] \arrow[u, hook, "\pi'_1"] & L \arrow[u, hook, "\pi'_0"]
\end{tikzcd}$$

this part need revision.

Finally, let X be a plane curve, and suppose that F_2 is given locally by

$$f|_X \mapsto \frac{d^2}{dx^2} f|_X dx \otimes dx,$$

where f is a homogeneous polynomial in the coordinates of $\mathbb{P}^2$, dx is a local generator of Ω_X, and $f|_X = \frac{d}{dx} f|_X = 0$. If we consider the Jacobian of F_2 in the locus of points where F_2 vanishes, we obtain a map $f|_X \mapsto \frac{d^3}{dx^3} f|_X dx \otimes dx \otimes dx$, which is the third fundamental form. Hence, the local index of the second fundamental form is the third fundamental form.

This is consistent with the description of the fundamental forms of [GH79]. Indeed, there F_{t+1} is defined as a derivative of F_t.

G.M. thinks that such a geometric interpretation still holds for arbitrary fields k.

First of all, we need to be sure that "t^{th} fundamental form" still makes sense for arbitrary k. I think Equation (14.1) and everything after that still makes sense for k arbitrary, so Ein–Niu construction is still possible (I sent an email to Niu asking that. He confirmed that this construction works.)

An alternative construction could be the following: Let $F_t\colon R_{t-1} \to \mathrm{Sym}^t\Omega_X$ be the t^{th} fundamental form. Since the kernal and image of a map of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent (see [Har77, Proposition.II.5.7]), we can apply Altaman–Kleiman construction to the exact sequence

$$0 \longrightarrow \ker(F_t) \longrightarrow R_{t-1} \longrightarrow Im(F_t) \longrightarrow 0.$$

We get a morphism $\ker(F_t) \to \Omega_X \otimes Im(F_t)$, that we define to be the $(t+1)^{\text{th}}$ fundamental form.This approach does not work because at some point the exact sequence is not locally split, so AK construction does not apply. I sent an email to Kleiman, but it was of little help.

The characteristic of the field could give some problem. In that case we take the Hasse derivatives D^2 instead of the usual derivative. Since $\frac{d}{dx}D^2f = 3D^3f$, we have that if in k the element 3 is a square, then the two local indices coincide.

14. Second Fundamental Form

Let T_X be the tangent space of a smooth projective variety $X \subset \mathbb{P}^n$ over k. The second fundamental form of X is a linear system of bilinear forms on the vector spaces $T_{X,x}$ (for $x \in X$) parameterized by the dual of the normal bundle $N_{X/\mathbb{P}^n}$. In complex differential geometry, its properties are exploited in the influential paper [Lan94]. In particular, the second fundamental form is degenerate at a point of $T_{X,x}$ if and only of that point spans a line of $\mathbb{P}^n$ meeting X at x with contact order 3, as explained on [GH79, pag.367].

Hence, we have an alternative way to count highly tangent lines of a plane curve. As those lines meet the curve with contact order 3, it is enough to count the points where the second fundamental form degenerates.

For reader convenience, we sum up the construction of the t^{th}-fundamental forms as in [EN23]. Although the paper assumes $k = \mathbb{C}$, all constructions work in any field.

Let $L = \mathcal{O}_X(1)$ and $V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ be the space of linear forms. For each integer $t \geq 0$, the sheaf of t^{th} order principal parts $P^k(L)$ of L is equipped with a Taylor series map

$$\alpha_t\colon V \otimes \mathcal{O}_X \longrightarrow P^t(L). \tag{14.1}$$

Such map is the composition of the natural map

$$V \otimes \mathcal{O}_X \longrightarrow L,$$

with the derivative map

$$L \longrightarrow P^t(L).$$

We denote by R_t the sheaf such that $R_t \otimes L = \ker(\alpha_t)$, and by i_t the embedding map $\ker(\alpha_t) \hookrightarrow V \otimes \mathcal{O}_X$. Note that $P^t(L)$ satisfies the following short exact sequence

$$0 \longrightarrow \mathrm{Sym}^t\Omega_X \otimes L \xrightarrow{\pi'_{t+1}} P^{t+1}(L) \xrightarrow{\pi_{t+1}} P^t(L) \longrightarrow 0.$$

The composition

$$R_{t-1} \otimes L \xrightarrow{\alpha_t \circ i_{t-1}} P^t(L) \xrightarrow{\pi_t} P^{t-1}(L)$$

is zero because $\pi_t \circ \alpha_t = \alpha_{t-1}$, thus there exists an induced map $R_{t-1} \otimes L \to \mathrm{Sym}^t\Omega_X \otimes L$.

Definition 14.1. The t^{th} fundamental form is the induced map

$$F_t\colon R_{t-1} \longrightarrow \mathrm{Sym}^t\Omega_X.$$

Now, we prove the following result.

Proposition 14.2. *There exists an exact sequence*

$$0 \longrightarrow R_t \otimes L \longrightarrow R_{t-1} \otimes L \xrightarrow{F_t \otimes \mathrm{id}} \mathrm{Sym}^t\Omega_X \otimes L,$$

such that the composition $R_t \otimes L \to R_{t-1} \otimes L \xrightarrow{i_{t-1}} V \otimes L$ *coincides with* $R_t \otimes L \xrightarrow{i_t} V \otimes L$.

Proof. The map $R_t \otimes L \to P^{t-1}(L)$ given by $\alpha_{t-1} \circ i_t$ is zero because it factorizes through $\pi_{t-1} \circ \alpha_t$. Thus $R_t \otimes L$ maps to $R_{t-1} \otimes L$ and the composition with $F_t \otimes \mathrm{id}$ is zero. Finally, since $R_t \otimes L \to R_{t-1} \otimes L$ factorizes a monomorphism, it must be a monomorphism itself. $\square$

For the reader's convenience, we provide the following diagram, which includes all the maps defined so far. All squares are commutative, and the dashed arrows are $F_t \otimes \mathrm{id}_L$.

(14.2)
$$\begin{tikzcd}
\dots \arrow[r, hook] & R_2 \otimes L \arrow[r, hook] \arrow[d, hook, "i_2"] & R_1 \otimes L \arrow[r, hook] \arrow[d, hook, "i_1"] \arrow[dddl, dotted] & R_0 \otimes L \arrow[d, hook, "i_0"] \arrow[dddl, dotted] \\
\dots \arrow[r, equal] & V \otimes \mathcal{O}_X \arrow[r, equal] \arrow[d, "\alpha_2"] & V \otimes \mathcal{O}_X \arrow[r, equal] \arrow[d, "\alpha_1"] & V \otimes \mathcal{O}_X \arrow[d, "\alpha_0"] \\
\dots \arrow[r, two heads] & P^2(L) \arrow[r, two heads, "\pi_1"] & P^1(L) \arrow[r, two heads, "\pi_0"] & P^0(L) \\
\dots \arrow[r] & \mathrm{Sym}^2\Omega_X \otimes L \arrow[r, "0"] \arrow[u, hook, "\pi'_2"] & \Omega_X \otimes L \arrow[r, "0"] \arrow[u, hook, "\pi'_1"] & L \arrow[u, hook, "\pi'_0"]
\end{tikzcd}$$

By [EN23, pag 4], we know that the second fundamental form is a morphism

(14.3)
$$N^\vee_{X/\mathbb{P}^n} \xrightarrow{F_2} \mathrm{Sym}^2\Omega_X.$$

In other words, it is a global section of the vector bundle $\mathrm{Hom}(N^\vee_{X/\mathbb{P}^2}, \mathrm{Sym}^2\Omega_X)$.

Proposition 14.3. *Let $X \subset \mathbb{P}^2$ be a smooth projective plane curve of degree d. Let M be* $\mathrm{Hom}(N^\vee_{X/\mathbb{P}^2}, \mathrm{Sym}^2\Omega_X)$. *The line bundle M is relatively orientable if and only if d is even.*

Proof. Let us denote by $\mathcal{O}_X(1)$ the ample line bundle $\mathcal{O}_{\mathbb{P}^2}(1)|_X$. It is known that

$$N_{X/\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(d)|_X = \mathcal{O}_X(d^2).$$

Moreover, from the cotangent exact sequence we have

$$\Omega_X \otimes N^\vee_{X/\mathbb{P}^2} = \det(\Omega_{\mathbb{P}^2})|_X = \mathcal{O}_X(-3d).$$

Finally,

$$\Omega_X \otimes M = \mathrm{Sym}^3\Omega_X \otimes N_{X/\mathbb{P}^2} = \mathcal{O}_X(4d^2 - 9d).$$

Thus, $\Omega_X \otimes M$ is a square if and only if $4d^2 - 9d$ is even or $\mathcal{O}_X(1)$ is a square. Both conditions are verified if and only if d is even. □

It follows that for an even number d, the enriched count of inflectional points of a plane curve of degree d is $\frac{3d(d-2)}{2}\mathbb{H}$.

Remark 14.4. There is an alternative construction of the map (14.3). Let us consider the exact sequence of locally free sheaves [Har77, Theorem.II.8.17(2)]:

$$0 \longrightarrow N^\vee_{X/\mathbb{P}^n} \longrightarrow \Omega_{\mathbb{P}^n}|_X \longrightarrow \Omega_X \longrightarrow 0.$$

Let us apply the construction devised by [AK70, I.3], so that we have the following map of locally free sheaves, also called second fundamental form:

$$N^\vee_{X/\mathbb{P}^n} \longrightarrow \Omega_X \otimes \Omega_X.$$

Such map coincides with (14.3) after symmetrization.

14.1. **Local indices.** In this section, we give an interpretation of the local indices of the second fundamental form. We assume that X is a smooth affine plane curve given by a polynomial $F(T, S) \in k[T, S]$ of even degree. We denote by F_T (resp., F_S) the derivative of F with respect to T (resp., S). We also denote by t (resp., s) the class of T (resp., S) in $k[T, S]/F$.

The sheaf of differentials of X is the $\mathcal{O}_X$ module freely generated by $\mathrm{d}t$ and $\mathrm{d}s$, where

$$\mathrm{d}s = -\left(\frac{F_T(t, s)}{F_S(t, s)}\right) \mathrm{d}t,$$

see [Liu06, Example 6.1.23]. Note that $F_T(t, s)$ and $F_S(t, s)$ cannot both vanish at any point because X is smooth. The second fundamental form is defined in the following way: It sends each global section $g \in N^\vee_{X/\mathbb{P}^2}$ to $\mathrm{d}(\mathrm{d}g)$. This is a well defined map of $\mathcal{O}_X$ module. Indeed, $N^\vee_{X/\mathbb{P}^2}$ is by definition the quotient $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of $X \subset \mathbb{P}^2$. It means that such global section g must be of the form $g = hF$, where $h \in \mathcal{O}_X$. Hence

$$\begin{aligned}
\mathrm{d}(\mathrm{d}g) &= \mathrm{d}(\underset{=0}{F}\mathrm{d}h + h\mathrm{d}F) \\
&= \mathrm{d}h\mathrm{d}F + h\mathrm{d}(\mathrm{d}F) \\
&= \mathrm{d}h(F_T(t, s)\mathrm{d}t + F_S(t, s)\mathrm{d}s) + h\mathrm{d}(\mathrm{d}F) \\
&= h\mathrm{d}(\mathrm{d}F).
\end{aligned}$$

Thus, the second fundamental form is the map of $\mathcal{O}_X$ modules sending $h \in \mathcal{O}_X$ to the morphism sending $hF \in N^\vee_{X/\mathbb{P}^2}$ to $h\mathrm{d}(\mathrm{d}F)$. Note that

$$\mathrm{d}(\mathrm{d}F) = \left(F_{TT} - 2\left(\frac{F_T}{F_S}\right) F_{TS} + \left(\frac{F_T}{F_S}\right)^2 F_{SS}\right) \mathrm{d}t \otimes \mathrm{d}t.$$

The third fundamental form can be easily described as the map $h \mapsto h\mathrm{d}(\mathrm{d}(\mathrm{d}F))$ where h is such that $h\mathrm{d}(\mathrm{d}F) = 0$. Note that

$$\mathrm{d}(\mathrm{d}(\mathrm{d}F)) = (F_{TTT} + ...)\mathrm{d}t \otimes \mathrm{d}t \otimes \mathrm{d}t.$$

We can suppose that, if p is a simple zero, p is k-rational by [KW21, Proposition 34]. Without loss of generality, we suppose that $p = (0,0)$ and $F_S|_p \neq 0$. By Hensel's lemma, there exists a $G(T) \in k[T]_{(T)}$ such that $G(0) = 0$ and $F(T, G(T)) = 0$. It follows that there exists an isomorphism

$$\frac{k[T,S]_{(T,S)}}{(F(T,S))} \cong \frac{k[T,S]_{(T,S)}}{(G(T)-S)}.$$

Indeed, $F(T,S) = (G(T,S) - S)H(T,S)$ for some power series $H(T,S)$ where $H(0,0) \neq 0$. Thus

$$U = \mathrm{Spec}\left(\frac{k[T,S]_{(T,S)}}{(G(T,S)-S)}\right) \longrightarrow \mathrm{Spec}\,(k[T])$$

is the Nisnevich coordinate around p. With these coordinates, the associated vector bundle (see [Har77, Exercise.II.5.18]) to the second fundamental form becomes

$$T \longmapsto G_{TT}(T).$$

Note that this is consistent with the classical definition of second fundamental form of a plane curve locally given by an equation $S = G(T)$.

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