RESEARCH STATEMENT

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1. Introduction

How do automorphic representations and the Langlands program connect to other areas of mathematics? My background is in number theory and representation theory, more specifically in Langlands functionality and the cohomology of arithmetic groups, but I work on problems motivated by bridges between automorphic representations and other fields. This includes:

Topology: The results of my thesis [GG21] and of a subsequent project [DGG] with Rahul Dalal, give bounds on the dimension of the singular cohomology of certain arithmetic manifolds. These manifolds occur naturally in sequences of covering spaces referred to as towers. A sample theorem is:

Theorem 1 ([GG21])**.** *Let $\mathfrak{p}$ be a large enough prime in a totally real number field and $X(\mathfrak{p}^n)$ be a tower of full-level compact arithmetic manifolds associated to the group $U(N-a,a)$ for N odd. In the first nontrivial degree of cohomology a, the following bound holds:*

$$\dim H^a(X(\mathfrak{p}^n), \mathbf{Q}) \ll Vol(X(\mathfrak{p}^n))^{\frac{N}{N^2-1}}.$$

To obtain this result, I prove a special case of a 30 year-old Conjecture 1 on growth of automorphic forms. In the project [DGG], we develop a framework to prove the sharpness of these upper bounds by refining our asymptotics. This line of research is described in Section 2.

Graph theory: The project in progress [EGGG] with Shai Evra and Henrik Gustafsson establishes a different strategy to bound the multiplicities of automorphic forms for classical groups. We are currently implementing it for the symplectic group Sp_4. As a application, we expect to produce new families of simplicial complexes which are almost Ramanujan, in the sense that they posses some of the key mixing properties of Ramanujan graphs, see Section 3 for more details.

Fourier analysis: As described in Section 4, the collaboration [GGV] with Akshay Venkatesh gives a new proof of the following Fourier interpolation Formula of Radchenko-Viazovska:

Theorem 2 ([RV19])**.** *Let f be an even Schwartz function on the real line, and $\hat{f}$ its Fourier transform. The function f is entirely determined by the values $(f(\sqrt{n}), \hat{f}(\sqrt{n}))_{n\in\mathbf{N}}$.*

We prove this result in the language of the cohomology of arithmetic groups and the spectral theory of automorphic forms: the starting point for this strategy is the observation that Schwartz functions are the underlying space of the Weil representation of the metaplectic group.

Representation theory of finite groups: The project [BGGG$^+$], outlined in Section 5, proves a mod ℓ *converse theorem* for $GL_n(\mathbf{F}_q)$ and $\ell \nmid q$. To do this, we first construct γ-factors of mod ℓ representations: invariants inspired by the theory of automorphic forms and built from harmonic analysis on $GL_n(\mathbf{F}_q)$. They are naturally attached to Langlands-type classifications, and additionally determine a cuspidal representation more effectively than its character. We then show that:

Theorem 3 ([BGGG$^+$])**.** *A mod ℓ cuspidal representation ρ of $GL_n(\mathbf{F}_q)$ is determined by its γ factors $\gamma(\rho, \pi)$ where π runs over generic representations of $GL_{n-1}(\mathbf{F}_q)$.*

In all the above project projects, the core question is the study of a group G and its representations, and a through line is the following paradigm from representation theory:

Paradigm 1. *A fruitful approach to study a group G is to understand the naturally-occurring representations of G on spaces of functions on $\Gamma\backslash G$ for different subgroups Γ.*

2. Cohomology growth and Automorphic representations.

2.1. **Background.** Is this section, G is a reductive Lie group and Γ is a cocompact lattice in G: a discrete subgroup of G such that $\Gamma\backslash G$ is a compact manifold. The space $L^2(\Gamma\backslash G)$ is naturally equipped with the regular representation of G induced by multiplication. It decomposes into a direct sum

$$L^2(\Gamma\backslash G) = \oplus_\pi m(\pi,\Gamma)\pi \tag{1}$$

of irreducible representations π, each appearing with multiplicity $m(\pi,\Gamma)$. A very general question, in the spirit of Paradigm 1, is:

Question 1. What is the multiplicity $m(\pi,\Gamma)$ of any given representation π in the decomposition?

This question can be specialized in several directions, of which we discuss two:

- **Choice of lattice Γ.** If $G = \mathcal{G}(\mathbf{R})$ for a group $\mathcal{G}$ defined over a number field F, we can let Γ be *arithmetic*, which amounts to being a finite-index subgroup of $\mathcal{G}(\mathbf{Z})$. Arithmetic groups have rich connections to number theory: the representations π appearing in the decomposition of $L^2(\Gamma\backslash G)$ are generalization of modular forms. Additionally, arithmetic groups admit a natural filtration by *congruence subgroups* given by the kernel of reduction modulo $\mathfrak{n}$ for each ideal $\mathfrak{n}$ of F.
- **Choice of representation π.** Let K be a maximal compact subgroup of G. Some representations π occurring in the decomposition (1) are related to the topology of the space $\Gamma\backslash G/K$. Indeed, their multiplicities dictate the singular cohomology following Matsushima's formula [Mat67]:

$$H^*(\Gamma\backslash G/K,\mathbf{C}) = \oplus_\pi m(\pi,\Gamma)H^*(\mathfrak{g},K;\pi). \tag{2}$$

  By definition, those representations π whose so-called $(\mathfrak{g},K)$ cohomology $H^*(\mathfrak{g},K;\pi)$ is nonzero are *cohomological*. Vogan-Zuckerman [VZ84] have characterized the finite list of cohomological representations for each Lie group.

Even in the well-studied case of Γ arithmetic and π cohomological, Question 1 is extremely delicate; a more tractable version of the problem asks the question asymptotically:

Question 2. Fix a prime $\mathfrak{p}$. How does $m(\pi,\Gamma(\mathfrak{p}^n))$ grow in a sequence of $\mathfrak{p}$-power congruence subgroups?

First progress towards this question was made by deGeorge-Wallach [dGW78], whose work implies that

$$\lim_{n\to\infty}\frac{m(\pi,\Gamma(\mathfrak{p}^n))}{\mathrm{Vol}(\Gamma(\mathfrak{p}^n)\backslash G/K)} = \begin{cases} k>0 & \text{if } \pi\hookrightarrow L^2(G)\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

In particular, analytic properties of π determine the rate of growth of $m(\pi,\mathfrak{p}^n)$. One can ask for more precise rates of growth for the representations such that $\pi\not\hookrightarrow L^2(G)$. This has led Sarnak-Xue [SX91] to conjecture:

Conjecture 1 (Sarnak-Xue). Let π be a unitary representation, and $p(\pi) = \inf\{p \mid \pi\hookrightarrow L^p(G)\}$. Then

$$m(\pi,\Gamma(\mathfrak{p}^n)) \ll_\epsilon Vol(\Gamma(\mathfrak{p}^n)\backslash G/K)^{\frac{2}{p(\pi)}+\epsilon}.$$

Representations such that $p(\pi)>2$ are *non-tempered*. When they are cohomological, representations with larger values of $p(\pi)$ appear in the lower degrees of cohomology in (2).

2.2. **Thesis work: limit multiplicity and cohomology for unitary groups.** In my thesis, I prove the Sarnak-Xue conjecture for a certain class of representations of $U(N-a,a)$, the real points of a unitary group preserving a Hermitian form of signature $(N-a,a)$ and for $\mathfrak{p}$ large enough. For cohomology, it implies

Theorem 4 ([GG21]). *Let N be odd and a be the smallest degree i such that $H^i(\Gamma(\mathfrak{p}^n)\backslash G/K,\mathbf{C})\neq 0$. Then*

$$\dim H^a(\Gamma(\mathfrak{p}^n)\backslash G/K) \ll Vol(\Gamma(\mathfrak{p}^n)\backslash G/K)^{\frac{N}{N^2-1}}.$$

In view of DeGeorge-Wallach's result (3), the main intuition powering my proof is:

The growth of non-tempered representations is slow because it tracks the growth of tempered representations in towers associated to groups of smaller dimension.

This idea is already present in work of Marshall [Mar14] and Marshall-Shin [MS18]. Theorem 4 recovers some of their results when $a = 1$. The groups of smaller dimension occurring in this context are called *endoscopic groups of* $\mathcal{G}$, and the mechanism by which properties of their representations are transferred to $\mathcal{G}$ is *endoscopy*: one the best-understood instance of functoriality in the Langlands program. Consequently, we shift our focus to automorphic representation: in keeping with Paradigm 1, automorphic representations π are irreducible summands in the decomposition

$$L^2(\mathcal{G}(F)\backslash\mathcal{G}(\mathbf{A})) = \oplus_\pi m(\pi)\pi \tag{4}$$

where $\mathbf{A}$ is the ring of adèles of F. They factor as $\pi = \otimes'_v \pi_v$; to underscore our shift in perspective, representations of the Lie group G are now denoted π_∞. Finally, the groups $\Gamma(\mathfrak{p}^n)$ are replaced by compact-open subgroups $K(\mathfrak{p}^n) \subset \mathcal{G}(\mathbf{A})$ such that $\Gamma(\mathfrak{p}^n) = \mathcal{G}(F) \cap K(\mathfrak{p}^n)$. Question 2 then becomes:

Question 3. For a given π_∞, what is the rate of growth of $m(\pi_\infty, \mathfrak{p}^n) = \sum_{\pi=\pi_\infty\otimes\pi_f} m(\pi) \dim \pi_f^{K(\mathfrak{p}^n)}$?

The result I prove relies on the endoscopic classification of representations: pioneering work of Arthur [Art13], extended to unitary groups by Mok [Mok15] and Kaletha-Minguez-Shin-White [KMSW14]. In this classification, automorphic representations are sorted into so-called Arthur packets Π_ψ attached to A-parameters ψ. In the case of unitary groups, the data of ψ includes an N-dimensional representation of SL_2, or, by the Jacobson-Morozov theorem, a partition of N. The main technical result of my thesis is:

Theorem 5 ([GG21])**.** *Let ψ_∞ be a local archimedean parameter of $\mathcal{G}$ with regular infinitesimal character, and partition $2k+1+...+1$ of N, and let $\Psi(\psi_\infty)$ be the set of global parameters who localize to ψ_∞. For a prime $\mathfrak{p}$ of F, denote $Nm(\mathfrak{p})$ its norm and p its residue characteristic. If $p \geq N^2[F:\mathbf{Q}]+1$, then*

$$\sum_{\psi\in\Psi(\psi_\infty)} \sum_{\pi\in\Pi_\psi} m(\pi)\pi_f^{K(\mathfrak{p}^n)} \ll Nm(\mathfrak{p})^{n(N-2k)}. \tag{5}$$

Moreover, if π_∞ occurs only in packets associated to such ψ_∞, then π_∞ satisfies conjecture 1.

I obtain the bound by re-purposing the main tool of the classification: the stabilization of the trace formula. This is an expression for the trace $R(\mathfrak{p}^n)_{\psi_\infty}$ of the operator on $L^2(\mathcal{G}(F)\backslash G(\mathbf{A}_F))_{\psi_\infty}$ given by convolution with the indicator function $1_{K(\mathfrak{p}^n)}$. By construction, $R(\mathfrak{p}^n)_{\psi_\infty}$ computes the left-hand side of (5). The stabilization of the trace formula then writes $R(\mathfrak{p}^n)_{\psi_\infty}$ as a sum of so-called stable traces: a main term corresponding to $\mathcal{G}$, and error terms coming from each of its endoscopic groups $\mathcal{H}$:

$$R(\mathfrak{p}^n)_{\psi_\infty} = St_{\mathcal{G}}(\mathfrak{p}^n)_{\psi_\infty} + \sum_{\mathcal{H}} St_{\mathcal{H}}(\mathfrak{p}^n)_{\psi_\infty}. \tag{6}$$

These endoscopic groups $\mathcal{H}$ are products of unitary groups of smaler rank; the stabilization (6) is mechanism to pass information between $\mathcal{G}$ and $\mathcal{H}$. Crucial to this decomposition are Ngô's Fundamental Lemma [Ngo10], extended to congruence subgroups by Ferrari [Fer07]. The key observation powering my proof is

For ψ_∞ associated to non-tempered representations, one of the "error terms" $St_{\mathcal{H}}(\mathfrak{p}^n)$ dominates $n \to \infty$.

This allows me to inductively control the rates of growth of $m(\pi_\infty, \mathfrak{p}^n)$ on $\mathcal{G}$ in terms of an analogous count on $\mathcal{H}$, making precise the intuition laid out at the outset of this project.

2.3. **Sharp asymptotics for unitary groups.** In light of the proof of Theorem 5, questions arise naturally:

- Are the bounds sharp? Can we obtain finer asymptotics?
- Can we give bounds for more representations, and thus more degrees of cohomology?

My joint work [DGG] with Rahul Dalal addresses these questions. In contrast with my thesis work, which made a coarse use of the trace formula, this next step in the program entails a much finer spectral study of each term in the decomposition (6). In particular, we engage seriously with the intuition:

Endoscopy for subgroups of GL_N is an echo of parabolic induction on GL_N.

Inspired by work of Taibi [Taï17] who computed exact dimensions of automorphic forms for small levels, we go one step further in re-purposing the trace formula, by writing each of the summands in (6) as a twisted trace on an appropriate general linear group. This trace can be expressed as a twisted trace on a smaller group via twisted parabolic induction. Specifically, for the term $St_G(\mathfrak{p}^n)$, this looks like:

$$St_{\mathcal{G}}(\mathfrak{p}^n) \approx \tilde{R}_{GL_N}(\mathfrak{p}^n) \approx \tilde{R}_{GL_{N_1}}(\mathfrak{p}^n) \times \tilde{R}_{GL_{N_2}}(\mathfrak{p}^n) \approx St_{\mathcal{G}_1}(\mathfrak{p}^n) \times St_{\mathcal{G}_2}(\mathfrak{p}^n), \tag{7}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are unitary groups of smaller rank. In this process, we no longer discern individual parameters ψ_∞, and we work instead with so-called shapes Δ, which encode, roughly, an N-dimensional representation of SL_2 and an infinitesimal character.

In the outcome of the inductive process sketched in (7), the representations of G_i associated to an appropriate choice of Δ are either characters or tempered, and very precise asymptotics for the multiplicities of such representations were computed by Shin-Templier [ST16]. As a consequence, we can give precise results about growth of cohomology. A sample theorem we obtain is:

Theorem 6 ([DGG])**.** *Assume that the CM extension E/F defining the unitary group $U(N-1,1)$ is everywhere unramified. Then, for $\mathfrak{p}$ large enough and $i \equiv N-1 \mod 2$,*

(i) the bounds on $\dim H^i(\Gamma(\mathfrak{p}^n))$ obtained by Marshall-Shin in [MS18] *are sharp,*
(ii) the rates of growth are the same for all weights $(p, i-p)$ in the Hodge decomposition.

Our results produce lower bounds in a range of degrees for unitary groups of arbitrary signature. In view of the work of Bergeron-Millson-Moeglin [BMM16] on A-parameters and Hodge classes, part (ii) implies:

Corollary 7. *Under the assumptions of Theorem 6, and supposing that N is odd and that $\mathfrak{p}$ splits in E, then as $n \to \infty$, a positive proportion of Hodge classes for $X(\mathfrak{p}^n) = \Gamma(\mathfrak{p}^n)\backslash G/K$ satisfy the Hodge conjecture.*

2.4. **Future work.** A medium-term project is to give a full proof of Conjecture 1 for unitary groups coming from Hermitian forms. The restrictions on the results of [DGG] point towards different directions of research, which I aim to pursue in the coming years.

Asymptotics of epsilon-factors. The restriction on the parity of the degrees comes from our lack of control on Arthur's so-called ϵ-factors. These are characters of a 2-group $\mathcal{S}_\psi$; they are computed in terms of root numbers of local L-functions associated to the parameters ψ. Even the coarsest control on their asymptotic distribution as the ramification of the parameters grows could give us control over a much wider class of representations. Dalal and I are considering various avenues to study this, including whether one can vary the ϵ factor by imposing an auxiliary level as in [Mar14], how the ϵ factor varies when twisted by a parameter, and whether results on what representations appear as the image of the theta lift can give us asymptotics.

Twisted parabolic induction. The restriction on the ramifications of E/F, on $\mathfrak{p}$, and on the possible shapes Δ we can give bounds for come from difficulties controlling twisted parabolic induction and twisted transfer for congruence subgroups. In particular, identities relating twisted traces of Langlands quotients to those of the inducing data. Yet, once again, we need only the coarsest asymptotic information to derive results. Work in this direction will begin by developing an understanding of the decomposition representations obtained by twisted parabolic induction.

Further work for upper bounds on cohomology growth. In a joint upcoming project with Simon Marshall, we aim to expand the strategy of [MS18] to give upper bounds for the growth of cohomology for all degrees for groups $U(a,b)$, under the assumption that $\mathfrak{p}$ is split in the extension E/F defining the unitary group.

3. The Sarnak-Xue Density Hypothesis and Almost-Ramanujan Complexes.

This project with Shai Evra and Henrik Gustafsson is motivated by a generalization of Conjecture 1, predicting how rare representation which are non-tempered at an arbitrary place should be. In the spirit of Paradigm 1, it considers a generalization of Question 1 where the real Lie group is replaced by a p-adic one. Below, we begin with an automorphic formulation of the problem and conjecture.

3.1. **Background.** Let F be a number field and G/F be a reductive group. We first give the construction of a family of automorphic representations in the level aspect. Let:

- ξ be an infinitesimal character at the infinite places;

- S be a finite set of places of F;
- $K_N \subset K_{N-1}$ be a sequence of of compact-open subgroups of $G(\mathbf{A}^S)$, with $[K_N : K_{N-1}]$ finite.

Then the *family* $\mathcal{F}_N$ is the set of automorphic representations $\pi = \pi_S \otimes \pi^S$ with infinitesimal character ξ and such that $\dim \pi^{K_N} \neq 0$. At each place $v \in S$, fix a representation π_v^0, and let $\mathcal{F}_N^0 = \{\pi \in \mathcal{F}_N \mid \pi_v = \pi_v^0 \text{ for } v \in S\}$. Let $p(\pi_S^0) = \max_{v\in S} \inf\{p \geq 2 \mid \pi_v^0 \hookrightarrow L^p(G_v)\}$. Let

$$m(N) = \sum_{\pi \in \mathcal{F}_N} m(\pi) \dim(\pi^S)^{K_N}, \quad m^0(N) = \sum_{\pi = \pi_S^0 \otimes \pi^S in \mathcal{F}_N} m(\pi) \dim(\pi^S)^{K_N}.$$

Conjecture 2 (Sarnak-Xue Density Hypothesis, level aspect)**.** As $N \to \infty$, we have $m^0(N) \ll_\epsilon m(N)^{\frac{2}{p(\pi_S^0)}+\epsilon}$.

Conjecture 1 is the specialization of Conjecture 2 to the case that S is the set of archimedean places.

3.2. **Results.** The project [EGGG] lays out a strategy to prove Conjecture 2 for classical groups, and implements it for cohomological representations of the symplectic groups Sp_4 and prime-power level. We hope to prove the following:

Theorem 8 ([EGGG], in progress)**.** *Let F be a number field, G an inner form of Sp_4/F, and $\mathfrak{p}$ a prime of F. Then for, ξ the infinitesimal character of the trivial representation, a place S and representation π_S^0, and $K_N = K^S(\mathfrak{p}^N)$, Conjecture 2 holds.*

By letting v be a unique non-compact infinite place, this will give upper bounds on the L^2 cohomology of full-level lattices in Sp_4. By choosing an inner form of Sp_4 which is compact at the archimedean places, and letting $S = \{v_\mathfrak{q}\}$ for a prime $\mathfrak{q}$, we expect to obtain the following corollary:

Corollary 9. *Let B be the Bruhat-Tits building of $Sp_4(F_{v_\mathfrak{q}})$, and let $\Gamma_N = Sp_4(F) \cap K^{v_\mathfrak{q}}(\mathfrak{p}^N)$. Then the quotients $X_N = B/\Gamma_N$ are almost Ramanujan, in the sense that they exhibit the cutoff property.*

The cutoff property concerns the mixing properties of the complex: roughly, it states that the distance between the probability distribution of a random walk on the complex and the uniform distribution abruptly approaches zero in a short interval of time called the cutoff window, see [LLP20]. This property is currently only known for Ramanujan graphs and complexes, which are optimal expanders from a spectral perspective, and our result will give the first non-Ramanujan complexes which display it.

The key insight in the proof of theorem 8 is that:

The growth of multiplicity can be controlled inductively in terms of lower-rank groups which may not be endoscopic groups of G.

To prove Theorem 8, we use a combination of results in the Langlands program, including the detailed classification of the A-packets of Sp_4 by Schmidt [Sch20], a depth-preservation result of Oi [Oi22], and known asymptotic counts of representations on SL_2. The argument decouples the two terms in the sum $m(N)$:

- The global term $m(\pi)$ is controlled, via an induction process afforded by the endoscopic classification of representations, by counts on smaller groups.
- The local term $(\pi^S)^{K_N}$ is bounded following the announced proof of Shahidi-Liu of a conjecture of Jiang relating the representation of SL_2 attached to an A-packet to rates of growth of $(\pi^S)^{K_N}$ via the Howe–Harish-Chandra character expansion and the notion of Gelfand-Kirillov dimension.

4. Automorphic Representations and Fourier Analysis

4.1. **Background.** Schwartz functions on the real line are smooth functions whose derivatives of all orders are rapidly decreasing. They appear naturally in harmonic analysis, as the space of Schwartz functions if closed under the Fourier transform. In the joint project [GGV] with Akshay Venkatesh, we give a new proof of a Fourier interpolation theorem of Radchenko-Viazovska:

Theorem 10 ([RV19])**.** *Let f be an even Schwartz function on the real line, and $\hat{f}$ its Fourier transform. The function f is entirely determined by the values $(f(\sqrt{n}), \hat{f}(\sqrt{n}))_{n\in\mathbf{N}}$.*

In fact, Radchenko-Viazovska do more: they provide an explicit interpolation basis. Their result and its generalizations are used in the proof of optimal universality of sphere packing for the E_8 and Leech lattices.

4.2. **Results.** In [GGV], we give a new proof of Theorem 10 based in the following observation:

The space of Schwartz functions carries the Weil representation W of the metaplectic group.

The metaplectic group G is the connected cover of degree 2 of $SL_2(\mathbf{R})$. In the representation W, the lift of the element $e = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ to $\widetilde{SL_2(\mathbf{R})}$ acts on a Schwartz function by multiplying it by $e^{2\pi i x^2}$. Similarly, $f = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ acts by multiplying the Fourier transform $\hat{f}$ of f by $e^{2\pi i x^2}$, and the element which conjugates e to f acts by the Fourier transform. Let $\Gamma = \langle \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \rangle \subset G$, and let W^* be the continuous dual of W. Then Γ acts on W^*, and e and f respectively fix the functionals $\varphi \mapsto \varphi(\sqrt{n})$ and $\varphi \mapsto \hat{\varphi}(\sqrt{n})$. This allows us to reformulate Theorem 10 as:

Theorem 11 ([GGV])**.** *Let W^* be the continuous dual of W. Then*

$$\dim H^0(\Gamma, W^*) = 1 \quad \textit{and} \quad H^1(\Gamma, W^*) = 0.$$

We deduce Theorem 11 from the computation of $H^0(\Gamma, W^*)$ using weight $\frac{1}{2}$ theta series for Γ, together with a more general result on group cohomology:

Theorem 12. *Let G be a finite cover of $SL_2(\mathbf{R})$, Γ a lattice in G, and W an irreducible infinite-dimensional $(\mathfrak{g}, K)$-module. Then*

$$\dim H^1(\Gamma, W^*) = \begin{cases} 0, & W \textit{ has a highest or lowest weight,} \\ \textit{multiplicity of } W \textit{ in cusp forms} & \textit{otherwise.} \end{cases}$$

This is an extension to infinite-dimensional $(\mathfrak{g}, K)$-modules of well-known results when W is finite-dimensional. The proof of Theorem 12 proceeds in two parts:

- In the spirit of Paradigm 1, we relate the $H^i(\Gamma, W^*)$ to $\mathcal{A} = \mathcal{A}(\Gamma\backslash G)$, a suitable space of automorphic forms on $\Gamma\backslash\widetilde{SL_2}$. Specifically, we rewrite the cohomology groups as $\mathrm{Ext}^1(W, \mathcal{A})$. This is an analogue of Shapiro's lemma for group cohomology; the main technical issues are topological. Our argument makes use of reduction theory and of a result of Casselman-Wallach on completions of $(\mathfrak{g}, K)$-modules.
- We compute the dimensions of $\mathrm{Ext}^1(W, \mathcal{A})$. The key technical result here is the surjectivity of a translate of the Laplacian on a suitable subspace of $\mathcal{A}$, which enables us to compute the dimensions in a simpler category. Surjectivity is proved by decomposing any given function into a part supported along the cusp, where surjectivity amounts to the solution of an ordinary differential equation, and a part which belongs to $L^2(\Gamma\backslash G)$, for which we make use of the spectral theory of automorphic forms.

4.3. **Future Work.** Our results parallel the work of Franke [FS98] on the computation of cohomology using spaces of automorphic forms, in a simple case where the computations are much more straightforward. A feature of the simplicity of our argument is the presence of a single differential operator, namely the Casimir. This points to the possibility of extending the results, at first, to more general groups of rank one.

5. Automorphic-inspired invariants in the representation theory of finite groups.

5.1. **Background.** In this project, we are motivated by a very classical question, that of the representation theory of $GL_n(\mathbf{F}_q)$ for q a prime power. It is known that for any finite group G, an irreducible representation ρ is determined by its character χ_ρ, a function on the conjugacy classes of G. The group $GL_n(\mathbf{F}_q)$ has $O(q^n)$ conjugacy classes, and one could ask for invariants that more effectively parameterize the representations.

An example of such invariants are the so-called γ factor $\gamma(\rho, \pi)$ introduced by Piatetski-Shapiro [PS83] for $\mathbf{C}$-valued generic representations of $GL_2(\mathbf{F}_q)$. Here π is a representation of $GL_1(\mathbf{F}_q)$. The construction of γ factors is inspired by the representation theory of $GL_2(\mathbf{Q}_p)$, and is harmonic analytic in nature, relying on the realization of ρ as a space of functions, its *Whittaker Model*, in the spirit of Paradigm 1. Piatetski-Shapiro then proved a converse theorem for cuspidal representations, which constitute the building blocks from which the other irreducible representations are built.

Theorem 13 ([PS83])**.** *A $\mathbf{C}$-valued cuspidal representation of $GL_2(\mathbf{F}_q)$ is determined by its γ factors.*

In particular, it is determined by $O(q)$ elements. Nien [Nie14] has defined γ factors for $GL_n(\mathbf{F}_q)$ and proved that cuspidal representations are determined by their γ factors, i.e. by $O(q^{n/2})$ data points.

5.2. **Results.** In the project [BGGG$^+$], we ask similar questions when the coefficient field $\mathbf{C}$ is replaced by $\bar{\mathbf{F}}_\ell$, when ℓ divided $|GL_n(\mathbf{F}_q)|$, but $\ell \nmid q$. In this case, many results from the complex case have been extended by Vigneras [Vig88]. We first show that one can still define γ factors. Then making use of the fact that in practice, γ factors for GL_2 are Gauss sums, we computed on Sage that the corresponding converse theorem does not hold for mod ℓ representations. For example we have:

Example 14 (Counterexamples to the naive converse theorem mod ℓ)**.** When $(\ell, q) = (2, 17)$, $(3, 19)$, or $(5, 11)$ there are non-isomorphic cuspidal representations of $GL_2(\mathbf{F}_q)$ mod ℓ with the same γ factors.

This suggests that for mod ℓ representations, one has to give a different construction of γ factors to obtain a converse theorem. Since the construction of $\gamma(\rho, \pi)$ is harmonic-analytic, the natural place to look for new definitions is in different spaces of functions associated to ρ. We have done this in two ways:

- For GL_2, we define ℓ-regular Γ factors from the ℓ-regular Whittaker model of ρ: this is a space of functions on the subgroup of of $GL_2(\mathbf{F}_q)$ whose determinant in $\mathbf{F}_q^\times$ has order prime to ℓ.
- For GL_n with $n \geq 2$, in order to construct the γ factor $\gamma(\rho, \pi)$, we change the space of functions associated to the pair (ρ, π). Instead of the Whittaker model of ρ valued in the constants $\bar{\mathbf{F}}_\ell$, we consider spaces of functions valued in a certain Artinian algebra $R(\pi)$ associated to π.

With these two modifications, we prove a converse theorem:

Theorem 15 ([BGGG$^+$])**.** *Let ρ be a mod ℓ cuspidal representation of $GL_n(\mathbf{F}_q)$.*

(i) If $n = 2$, a cuspidal representation ρ is entirely determined by its ℓ-regular γ-factors $\gamma_\ell(\rho, \pi)$, as π runs over irreducible representations of $GL_1(\mathbf{F}_q)$, i.e. by $O(q)$ invariants.

(ii) If $n \geq 2$, a cuspidal representation ρ is entirely determined by its $R(\pi)$-valued γ factors $\gamma(\rho, \pi)$, as π runs over irreducible representations of $GL_{n-1}(\mathbf{F}_q)$, i.e. by $O(q^{n-1})$ invariants.

In the case of GL_2, the proof is done by elementary methods. The crux of the proof for GL_n is establishing a completeness of $R(\pi)$-valued Whittaker models, and relies on a finer analysis of auxiliary representations attached to ρ and π via Bernstein-Zelevinsky derivatives.

5.3. **Future Work.** This project is still ongoing – we would like to replace the algebra $R(\pi)$ by $\bar{\mathbf{F}}_\ell$. We are currently studying the interaction between our two constructions of γ factors in the case $n = 2$ where both exist, with the hope that translation from one to the other will shed light on a way to construct $\bar{\mathbf{F}}_\ell$-valued γ factors for $GL_n(\mathbf{F}_q)$. Additional questions that we would like to explore are relations with the depth zero local Langlands correspondence, and the question of what Galois-theoretic phenomenon reflects the congruences mod ℓ that we observe between γ factors.

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