RESEARCH STATEMENT

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1. Introduction

How do automorphic representations and the Langlands program connect to other areas of mathematics? My background is in number theory and representation theory, more specifically in Langlands functionality and the cohomology of arithmetic groups, but I work on problems motivated by bridges between automorphic representations and other fields. This includes:

Topology: In my thesis [GG21] and a subsequent project [DGG] with Rahul Dalal, we bound the dimension of the singular cohomology of certain classes of arithmetic manifolds. These occur naturally in sequences of covering spaces X_n referred to as towers. A sample theorem is:

Theorem 1 ([GG21])**.** *Let $\mathfrak{p}$ be a large enough prime in a totally real number field and $X(\mathfrak{p}^n)$ be a tower of full-level compact arithmetic manifolds associated to the group $U(N-a,a)$ for N odd. In the first nontrivial degree of cohomology a, the following bound holds:*

$$\dim H^a(X(\mathfrak{p}^n), \mathbf{Q}) \ll Vol(X(\mathfrak{p}^n))^{\frac{N}{N^2-1}}.$$

I deduce this from the proof of a special case of the 30 year-old Conjecture 1 on growth of automorphic forms. In [DGG], we develop a framework to prove the sharpness of these upper bounds by computing much finer asymptotics.

Fourier analysis: The collaboration [GGV] with Akshay Venkatesh gives a new proof of the following Fourier interpolation Formula of Radchenko-Viazovska:

Theorem 2 ([RV19])**.** *Let f be an even Schwarz function on the real line, and $\hat{f}$ its Fourier transform. The function f is entirely determined by the values $(f(\sqrt{n}), \hat{f}(\sqrt{n}))_{n\in\mathbf{N}}$.*

We prove this result in the language of the cohomology of arithmetic groups and the spectral theory of automorphic forms: the starting point for this strategy is that Schwartz functions are the underlying space of the Weil representation.

High-dimensional expanders: The joint project in progress with [EGGG] with Shai Evra and Henrik-Gustaffson establishes a framework to bound the multiplicities of a class of automorphic forms for classical groups. We implement it for the symplectic group Sp_4: as a application, we produce the first family of expanders – complexes with exceptional spectral properties and applications to computer science – attached to groups of type other than A_n.

Representation theory of finite groups: The six-author project [BGGG$^+$] proves a mod ℓ converse theorem for $GL_2(\mathbf{F}_q)$ and $\ell \nmid q$: we construct γ-factors –invariants constructed from Fourier analysis on $GL_n(\mathbf{F}_q)$ – of mod ℓ representations, and show that

Theorem 3. *A mod ℓ cuspidal representation ρ of GL_n is determined by its γ factors. In particular for large q, γ-factors are a more effective set of invariants than characters.*

2. Limit Multiplicity: Cohomology and Expanders

2.1. **Background.** A through-line in all of my research is the following paradigm from representation theory:

Paradigm A:. *A fruitful approach to study a group G is to understand the naturally-occurring representations of G on spaces of functions on $H\backslash G$ for different subgroups H.*

Is this section, G is a reductive Lie group (e.g. the indefinite unitary group $U(N-a,a)$) and Γ is a cocompact lattice in G, i.e. a discrete subgroup of G such that $\Gamma\backslash G$ is a compact manifold, giving the space $L^2(\Gamma\backslash G)$ a spectral decomposition into the sum of irreducible representations. A very general question is:

Question 1. Let G be a Lie group and Γ be a cocompact lattice, so that $L^2(\Gamma\backslash G) = \oplus_\pi m(\pi,\Gamma)$. What is the multiplicity $m(\pi,\Gamma)$ of any given representation π?

If additionally G is the real points of a group $\mathcal{G}$ defined over a number field F, we can let Γ be *arithmetic*, which amounts to Γ being a finite-index subgroup of $\mathcal{G}(\mathbf{Z})$. Arithmetic groups have rich connections to number theory: representations appearing in the decomposition of $L^2(\Gamma\backslash G)$ are generalization of modular forms. In particular, arithmetic groups admit a natural filtration by *congruence subgroups* associated to each prime $\mathfrak{p}$ of F, denoted $\Gamma(\mathfrak{p}^n)$.

Some representations occuring in the decomposition of

$$L^2(\Gamma\backslash G) = \oplus_\pi m(\pi,\Gamma)\pi \tag{1}$$

into a sum of irreducible unitary representations are directly to the topology of the space $\Gamma\backslash G$. These are called cohomological representations, and they are related to cohomology by Matsushima's formula [Mat67]:

$$H^*(\Gamma\backslash G/K,\mathbf{C}) = \oplus_\pi m(\pi,\Gamma)H^*(\mathfrak{g},K;\pi).$$

Here K is a maximal compact subgroup of G, the sum os taken over unitary representations of G, and $H^*(\mathfrak{g},K;\pi)$ is the so-called $(\mathfrak{g},K)$-cohomology of π. Cohomological π are by definition those with nonzero $(\mathfrak{g},K)$ cohomology, and Vogan-Zuckerman [VZ84] have characterized the finite list of cohomological representations for each Lie group.

The question of computing $m(\pi,\Gamma)$ for π cohomological and Γ arithmetic is extremely delicate; a more tractable version of the problem considers the question in towers:

Question 2. How does $m(\pi,\Gamma(\mathfrak{p}^n))$ grow in a sequence of congruence subgroups?

The first progress towards this question was made by DeGeorge-Wallach [dGW78], whose work implies that

$$\lim_{n\to\infty}\frac{m(\pi,\Gamma(\mathfrak{p}^n))}{\mathrm{Vol}(\Gamma(\mathfrak{p}^n)\backslash G/K)} = \begin{cases} k>0 & \text{if } \pi\hookrightarrow L^2(G) \\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

One can ask for more precise rates of growth for representations such that $m(\pi,\Gamma(\mathfrak{p}^n)) = o(Vol(\Gamma(\mathfrak{p}^n)\backslash G/K))$. This has led Sarnak-Xue [SX91] to conjecture:

Conjecture 1 (Sarnak-Xue)**.** Let π be a unitary representation, and $p(\pi) = \inf\{p \mid \pi\hookrightarrow L^p(G)\}$. Then

$$m(\pi,\Gamma(\mathfrak{p}^n)) \ll_\epsilon Vol(\Gamma(\mathfrak{p}^n)\backslash G/K)^{\frac{2}{p(\pi)}+\epsilon}.$$

Representations such that $p(\pi)>2$ are called *non-tempered*: their presence in the decomposition. In particular, analytic properties of π determine the rate of growth of $m(\pi,\mathfrak{p}^n)$.

2.2. **Thesis work on upper bounds.** In my thesis, I prove the Sarnak-Xue conjecture for a certain class of representations of $U(N-a,a)$, the real points of a unitary group preserving a Hermitian form of signature $(N-a,a)$ and for $\mathfrak{p}$ large enough. For cohomology, it implies

Theorem 4. *Let N be odd and a be the smallest degree i such that $H^i(\Gamma(\mathfrak{p}^n)\backslash G/K,\mathbf{C})\neq 0$. Then*

$$\dim H^a(\Gamma(\mathfrak{p}^n)\backslash G/K) \ll Vol(\Gamma(\mathfrak{p}^n)\backslash G/K)^{\frac{N}{N^2-1}}.$$

In view of DeGeorge-Wallach's result (2), the main intuition powering my proof is:

The growth of nontempered representatinos is slow because it tracks the growth of tempered representations in towers associated to smaller groups.

This idea is already present in work of Marshall [Mar14] and Marshall-Shin [MS18], and Theorem 4 recovers some of their work when $a = 1$.

The smaller groups occuring in this context are called *endoscopic groups of* $\mathcal{G}$, and the mechanism by which properties of their representations are transferred to $\mathcal{G}$ is called *endoscopy.* Endoscopy is the best-understood instance of functoriality in the Langlands program. Consequently, we shift our focus to automorphic representation: in keeping with Paradigm A, automorphic representations π are the summands in the decomposition

$$L^2(\mathcal{G}(F)\backslash\mathcal{G}(\mathbf{A})) = \oplus_\pi m(\pi)\pi$$

where $\mathbf{A}$ is the ring of adéles of F. They factor as $\pi = \otimes'_v \pi_v$; to underline our shift in perspective, the representations of the Lie group G are now denoted π_∞. Finally, the groups $\Gamma(\mathfrak{p}^n)$ are replaced by compact-open subgroups $K(\mathfrak{p}^n) \subset \mathcal{G}(\mathbf{A})$ such that $\Gamma(\mathfrak{p}^n) = \mathcal{G}(F) \cap K(\mathfrak{p}^n)$. The reformulation of Question **??** becomes

Question 3. For a given π_∞, what is the rate of growth of $m(\pi_\infty, \mathfrak{p}^n) = \sum_{\pi=\pi_\infty\otimes\pi_f} m(\pi) \dim \pi_f^{K(\mathfrak{p}^n)}$?

The result I prove relies pioneering work of Arthur [Art13], extended to unitary groups by Mok [Mok15] and Kaletha-Minguez-Shin-White [KMSW14], known as the endoscopic classification of representations. It involves the subdivision of automorphic representations into so-called Arthur packets Π_ψ attached to A-parameters ψ. In the case of unitary groups, the data of ψ includes an N-dimensional representation of SL_2, which by the Jacobson-Morozov theorem is equivalent to a partition of N. My main technical theorem is:

Theorem 5. *Let ψ_∞ be a local packet at infinity with regular infinitesimal character, and corresponding partition $2k+1+...+1$ of N, and let $\Psi(\psi_\infty)$ be the set of global packets whose localization is ψ_∞. Let $\mathfrak{p}$ be a prime of F; denote $Nm(\mathfrak{p})$ its norm and p its residue characteristic. If $p \geq N^2[F : \mathbf{Q}] + 1$, we have*

$$\sum_{\psi\in\Psi(\psi_\infty)} \sum_{\pi\in\Pi_\psi} m(\pi)\pi_f^{K(\mathfrak{p}^n)} \ll Nm(\mathfrak{p})^{n(N-2k)}. \tag{3}$$

This theorem proves, and even beats the predictions of, Conjecture 1, for a certain class of non-tempered representations.

I obtain the bound by repurposing the main tool of the stabilization: the stabilization of the trace formula. This is an expression for the trace $R_{\psi_\infty}(\mathfrak{p}^n)$ of the operator on $L^2(\mathcal{G}(F)\backslash G(\mathbf{A}_F))_{\psi_\infty}$ given by convolution with the indicator function of $1_{K(\mathfrak{p}^n)}$. By construction, $R(\mathfrak{p}^n)_{\psi_\infty}$ computes the left-hand side of (3). The stabilization of the trace formula is the mechanism by which information can be passed between groups: it writes $R(\mathfrak{p}^n)$ as a sum of so-called stable traces: a main term coming from $\mathcal{G}$, and error terms coming from each of its endoscopic groups, which are smaller unitary groups:

$$R(\mathfrak{p}^n)_{\psi_\infty} = St_G(\mathfrak{p}^n)_{\psi_\infty} + \sum_H St_H(\mathfrak{p}^n)_{\psi_\infty}. \tag{4}$$

Crucial to this decomposition are Ngo's Fundamental Lemma [Ngo10], extended to congruence subgroups by Ferrari [Fer07]. The key observation powering my proof is

For ψ_∞ associated to non-tempered representations, one of the error terms $St_H(\mathfrak{p}^n)$ grows the fastest as $n \to \infty$.

This allows me to inductively control the rates of growth of $m(\pi_\infty, \mathfrak{p}^n)$ on G in terms of an analoguous count on H, making precise the intuition laid out at the outset of this project. A strategy relying on the same observation was previously implemented by Marshall-Shin, and gane bounds for cohomological representations of $U(N, 1)$ in all degrees.

2.3. **More precise asymptotics for the cohomology of arithmetic groups.** In light of the proof of Theorem 5, many questions arise naturally:

- Are the bounds sharp?
- Can we obtain finer asymptotics?
- Can we give bounds for more representations, and thus more degrees of cohomology?

My joint work [DGG] with Rahul Dalal adresses these questions. In contrast with my thesis work, which made a coarse use of the trace formula, this next step in the program entails a much finer spectral study of each term in the decomposition (4). In particular, we engage seriously with the intuition:

Endoscopy for subgroups of GL_N is an echo of parabolic induction on GL_N.

Inspired by work of Chenevier-Taibi [?] who computed exact dimensions for small levels, we go one step further in repurposing the trace formula, by writing each of the summands in (4) as a twisted trace on an appropriate general linear group. This trace can be expressed as a twisted trace on a smaller group via twisted parabolic induction. Specifically, for the term $St_G(\mathfrak{p}^n)$, this looks like:

$$St_G(\mathfrak{p}^n) \approx \tilde{R}_{GL_N}(\mathfrak{p}^n) \approx \tilde{R}_{GL_{N_1}(\mathfrak{p}^n)} \times \tilde{R}_{GL_{N_2}}(\mathfrak{p}^n) \approx St_{G_1}(\mathfrak{p}^n) St_{G_2}(\mathfrak{p}^n), \tag{5}$$

where the groups G_1 and G_2 are smaller unitary groups, and their product may or may not be an endoscopic groups. In this process, one no longer discerns individual parameters ψ_∞, and we work instead with so-called shapes Δ, which encode, roughly, an N-dimensional representation of SL_2 and an infinitesimal character.

At the end of this process, the representations of G_i associated to Δ for an appropriate choice of Δ are either characters or tempered, and very precise asymptotics for the multiplicities of the representations were computed by Dalal in his thesis [?]. Subtleties of twisted traces and twisted endoscopy mean that we can currently carry out our strategy under some assumptions. A sample theorem is :

Theorem 6 ([DGG])**.** *Assume that the CM extension E/F defining the unitary group $U(N-1,1)$ is everywhere unramified. Then, for $\mathfrak{p}$ large enough and $i \equiv N-1 \mod 2$,*

(i) the bounds on $\dim H^i(\Gamma(\mathfrak{p}^n))$ obtained by Marshall-Shin in [MS18] *are sharp,*

(ii) the rates of growth are the same for all weights $(p, i-p)$ in the Hodge decomposition.

Our results also produce lower bounds in a range of degrees for unitary groups of arbitrary signature. In view of the work of Bergeron-Millson-Moeglin [BMM16] on A-parameters and Hodge cycles, this implies:

Corollary 7. *Under the assumptions of Theorem 6, and supposing additionally that N is odd and that $\mathfrak{p}$ splits in E, then as $n \to \infty$, a positive proportion of Hodge classes for $X(\mathfrak{p}^n) = \Gamma(\mathfrak{p}^n)\backslash G/K$ satisfy the Hodge conjecture.*

2.4. **Future work: unitary groups.** A medium-term project is to give a full proof of Conjecture 1 for unitary groups coming from Hermitian forms. The restrictions on the results of [DGG] point towards different directions of research, which I aim to pursue in the coming years.

Asymptotics of epsilon-factors. The restriction on the parity of the degrees comes from our lack of control on Arthur's so-called ϵ-factors. These are characters of a 2-group $\mathcal{S}_\psi$; they are computed in terms of root numbers of local L-functions associated to the parameters ψ. Even the coarsest control on their asymptotic distribution as the ramification of the parameters grows could give us control over a much wider class of representations. Dalal and I are considering various avenues to study this, including whether one can vary the ϵ factor by imposing an auxiliary level as in [Mar14], how the ϵ factor varies when twisted by a parameter, and whether results on what representations appear as the image of the theta lift can give us asymptotics.

Twisted parabolic induction. The restriction on the ramifications of E/F, on $\mathfrak{p}$, and on the possible shapes Δ we can give bounds for come from difficulties controlling twisted parabolic induction and twisted transfer for congruence subgroups. In particular, identities relating twisted traces of Langlands quotients to those of the inducing data. Yet, once again, we need only the coarsest asymptotic information to derive results. Work in this direction will begin by developing an understanding of the decomposition representations obtained by twisted parabolic induction.

Further work for upper bounds on cohomology growth. In a joint upcoming project with Simon Marshall, we aim to expand the strategy of [?], together with new ideas of Marshall to bound the dimensions of fixed vectors in representations of $GL_N(\mathbf{Q}_p)$, to give upper bounds for the growth of cohomology for all degrees for groups $U(a,b)$, under the assumption that $\mathfrak{p}$ is split in the extension E/F defining the unitary group.

2.5. **Work at an earlier stage: symplectic groups, avoiding the trace formula, and expander graphs.** This project with Shai Evra and Henrik Gustaffson is motivated by a generalization of Conjecture 1, predicting rare non-tempered representations should be.

Let F be a number field and G/F be a reductive group. We first give the construction of a family of automorphic representations in the level aspect. Let:

- infinitesimal character ξ of G at the infinite places;
- S be a finite set of places of F;
- $K_N \subset K_{N-1}$ be a sequence of of compact-open subgroups of $G(\mathbf{A}^S)$, with $[K_N : K_{N-1}]$ finite.

Then the *family* $\mathcal{F}_N$ is the set of automorphic representations $\pi = \pi_S \otimes \pi^S$ with infinitesimal character ξ and such that $\dim \pi^{K_N} \neq 0$. At each place $v \in S$, fix a representation π_v^0, and let $\mathcal{F}_N^0 = \{\pi \in \mathcal{F}_N \mid \pi_v = \pi_v^0 \text{ for } v \in S\}$. Let $p(\pi_S^0) = \max_{v \in S} \inf\{p \geq 2 \mid \pi_v^0 \hookrightarrow L^p(G_v)\}$. Let

$$m(N) = \sum_{\pi in \mathcal{F}_N} m(\pi) \dim(\pi^S)^{K_N}, \quad m^0(N) = \sum_{\pi = \pi_S^0 \otimes \pi^S in \mathcal{F}_N} m(\pi) \dim(\pi^S)^{K_N}.$$

Conjecture 2 (Sarnak-Xue Density Hypothesis, level aspect)**.** As $N \to \infty$, we have $m^0(N) \ll_\epsilon m(N)^{\frac{2}{p(\pi_S^0)}+\epsilon}$.

Conjecture 1 is the specialization of Conjecture 2 to the case that S is the set of archimedean place. The project [EGGG] lays out a strategy to prove Conjecture 2 for classical groups and implements it for cohomological representations of the symplectic groups Sp_4 and prime-power level. We hope to prove the following:

Theorem 8 ([EGGG], in progress)**.** *Let F be a number field, G an inner form of Sp_4/F, and $\mathfrak{p}$ a prime of F. Then for, ξ the infinitesimal character of the trivial representation, $K_N = K^{v_1}(\mathfrak{p}^N)$, and any place $v \neq v_{\mathfrak{p}}$ and π_0^v, Conjecture 2 holds.*

By letting S be the infinite placee, this will give upper bounds on the L^2 cohomology of full-level lattices in Sp_4. By choosing an inner form of Sp_4 which is compact at the archimedean places, and letting $S = \{v_{\mathfrak{q}}\}$ for a prime $\mathfrak{q}$, we obtain the following corollary:

Corollary 9. *Let B be the Bruhat-Tits building of $G(F_{v_{\mathfrak{q}}})$, and let $\Gamma_n = G(F) \cap K^{v_{\mathfrak{q}}}(\mathfrak{p}^N)$. Then the quotients $X_N = B/\Gamma_N$ are higher-dimensional expanders.*

High-dimensional expanders are a generalization of expander graphs, a class of graphs with exceptional spectral properties. The study of high-dimensional expanders started about 15 years ago; they have already found applications to some areas of computer science.

The key insight in the proof of theorem 8

> *The growth of multiplicity can be controlled inductively in terms of lower-rank groups which may not be endoscopic groups of G.*

To prove Theorem 8, we use a combination of results in the Langlands program, including the detailed classification of the A-packets of Sp_4, some depth-preservation results, and known asymptotic counts of representations on SL_2. A feature of this argument is that it decouples the two terms in the sum $m(N)$

- The global term $m(\pi)$ is controlled, via an induction process afforded by the endoscopic classification of representations, by counts on smaller groups.
- The local term $(\pi^S)^{K_N}$ is bounded following results of Shahidi-Liu, who have announced a proof of a conjecture of Jiang relating the representation of SL_2 attached to an A-packet (and thus the invariant $p(\pi_S)$) to rate of growth of $(\pi^S)^{K_N}$ via Howe–Harish-Chandra character expansion, and the notion of Gelfand-Kirrilov dimension.

2.5.1. *Future work: doing this in general for higher dimensions.* We hope to extend the method to other groups – the key input missing for high-rank groups is count of automorphic representations on the inducing groups – in thie case SL_2.

3. Automorphic Representations and Fourier Analysis

Schwartz functions on the real line are smooth functions whose derivatives of all order are rapidly decreasing – it appears naturally in harmonic analysis as the largest class of functions closed under the Fourier transform. In the joint project [GGV] with Akshay Venkatesh, we give a new proof of a Fourier interpolation theorem of Radchenko-Viasovskaya:

Theorem 10 ([RV19])**.** *Let f be an even Schwarz function on the real line, and $\hat{f}$ its Fourier transform. The function f is entirely determined by the values $(f(\sqrt{n}), \hat{f}(\sqrt{n}))_{n \in \mathbf{N}}$.*

In fact, Radchenko-Viasovskaya do more: they provide an explicit interpolation basis. Their result and its generalizations are used in the proof of optimal universality of sphere packing for the E_8 and leech lattices.

In [GGV], we give a new proof of Theorem 10 based in the following observation:

The space of Schwarz functions carries the Weil representation W of the metaplectic group.

The metaplectic group $\widetilde{SL_2(\mathbf{R})}$ is the connected cover of degree 2 of $SL_2(\mathbf{R})$. The lift of the element $e = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ to $\widetilde{SL_2(\mathbf{R})}$ acts on a Schwartz function by multiplying it by $e^{2\pi 2x^2}$. Similarly, $f = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ acts by multiplying $\hat{f}$ by $e^{2\pi i x^2}$, and the element which conjugates e to f acts by the Fourier transform. Let $\Gamma = \langle \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \rangle$, and let W^* be the continuous dual of W. Then $\widetilde{SL_2(\mathbf{R})}$ acts on W^*, and e and f respectively fix the functionals $\varphi \mapsto \varphi(\sqrt{n})$ and $\varphi \mapsto \hat{\varphi}(\sqrt{n})$. This allows us to reformulate Theorem 10 as:

Theorem 11 ([GGV])**.** *Let W^* be the continuous dual of W. Then*

$$\dim H^0(\Gamma, W^*) = 1 \quad \textit{and} \quad H^1(\Gamma, W^*) = 0.$$

We deduce Theorem 11 from a more general cohomology vanishing statement:

Theorem 12. *Let G be $SL_2(\mathbf{R})$ or a finite cover, Γ a lattice in G, and W an irreducible infinite-dimensional $(\mathfrak{g}, K)$-module. Then*

$$\dim H^1(\Gamma, W^*) = \begin{cases} 0, & W \textit{ has a highest or lowest weight.} \\ \textit{multiplicity of } W \textit{ in cusp forms.} & \textit{else} \end{cases}$$

This is an extension to infinite-dimesnional $(\mathfrak{g}, K)$-modules of well-known results when W is finite-dimensional. Proof of Theorem 12, whose core result is the vanishing of H^1 proceeds in two parts:

- Rewrite the cohomology groups as $\mathrm{Ext}^1(W, \mathcal{A}(\Gamma\backslash G))$, where the $\mathcal{A} = \mathcal{A}(\Gamma\backslash G)$ is a suitable space of automorphic forms for the degree 2 cover of SL_2. This is an analogue of Shapiro's lemma, and the proof relies on results of Casselman-Wallach on representations on Frechet spaces, as well as reduction theory for finite covers fo $SL_2(\mathbf{R})$.
- Show that the Ext group vanishes. The key technical result here is the surjectivity of the Laplacian on a suitable subspace of $\mathcal{A}$, which enables us to compute the vanishing in a simpler category. The surjectivity result depends on the spectral theory of automorphic forms.

3.1. **Future Research.** Our work parallels that of Franke on the computation of cohomology using spaces of automorphic forms, but the argument are much more straightforward. One reason our method works well is because we have a single differential operator to deal with, namely the Casimir. This points to the possibility of first extending the results to more general groups of rank 1.

4. Automorphic-inspired invariants in the representation theory of finite groups.

In this project, we are motivated by a very classical question, that of the representation theory of $GL_n(\mathbf{F}_q)$ for q a prime power. It is known that for any finite group G, an irreducible representation ρ is determined by its character χ_ρ, a function on the conjugacy classes of G. The group $GL_n(\mathbf{F}_q)$ has $O(q^n)$ conjugacy classes, and one could ask for invariants that more effectively parameterize the representations.

An example of such invariants are the so-called γ factor $\gamma(\rho, \pi)$ introduced by Piatetskii-Shapiro [PS83] for $\mathbf{C}$-valued generic representations of $GL_2(\mathbf{F}_q)$. Here π is a representation of $GL_1(\mathbf{F}_q)$. The construction of γ factors is inspired by the representation theory of $GL_2(\mathbf{Q}_p)$, and is harmonic analytic in nature, relying on the realization of ρ as a space of functions, its *Whittaker Model*. In practice, γ factors for GL_2 are computed as Gauss sums. Piatetskii-Shapiro then proved a converse theorem, for cuspidal representations, which constitute the building blocks from which the other irreducible representations are built.

Theorem 13 ([PS83)**.** *] A $\mathbf{C}$-valued cuspidal representation of $GL_2(\mathbf{F}_q)$ is determined entirely by its γ factors.*

In particular, it is determined by $O(q)$ elements. Nian [Ni12] has defined γ factors for $GL_n(\mathbf{F}_q)$ and proved an analogous statement: cuspital representations are determined by their Γ factors, i.e. by $O(q^{n-1})$ data points.

In the project [BGGG$^+$], we ask similar questions when the coefficient field $\mathbf{C}$ is replaced by $\bar{F}_\ell$, when ℓ divided $|GL_n(\mathbf{F}_q)|$, but $\ell \nmid q$. In this case, one can still define γ factors, but we have computed on Sage that the corresponding converse theorem does not hold for mod ℓ representations. for example we have:

Example 14 (Counterexamples to the naive converse theorem mod ℓ)**.** When $(\ell, q) = (2, 17)$, $(3, 19)$, or $(5, 11)$ there are non-isomorphic cuspidal representations of $GL_2(\mathbf{F}_q)$ mod ℓ with the same γ factors.

This suggests that for mod ℓ representations, one has to give a different construction of γ factors to obtain a converse theorem. Since the construction of γ facros is harmonic-analytic, the natural place to look for new definitions is in different spaces of functions associated to ρ. We have done this in two ways:

- For GL_2, we define ℓ-regular Γ factors from the ℓ-regular Whittaker model of ρ: this is a space of functions on the subgroup of of $GL_2(\mathbf{F}_q)$ whose determinant in $\mathbf{F}_q^\times$ has order prime to ℓ.
- For GL_n with $n \geq 2$, in order to construct the γ factor $\gamma(\rho, \pi)$, we change the space of functions associated to ρ, π. Instead of the Whittaker model of ρ valued in the constants $\bar{\mathbf{F}}_\ell$, we consider spaces of functions (and thus γ) factors valued in a certain Artinian algebra $R(\pi)$ associated to π.

With these two modifications of the Γ factor, we prove a converse theorem:

Theorem 15 ([BGGG$^+$])**.** *Let ρ be a mod ℓ cuspidal representation of $GL_n(\mathbf{F}_q)$.*

(i) If $n = 2$, a cuspidal representation ρ is entirely determined by its ℓ-regular γ-factors $\gamma_\ell(\rho, \pi)$, as π runs over irreducible representations of $GL_1(\mathbf{F}_q)$, i.e. by $O(q)$ invariants.

(ii) If $n \geq 2$, a cuspidal representation ρ is entirely determined by its $R(\pi)$-valued γ factors $\gamma(\rho, \pi)$, as π runs over irreducible representations of $GL_{n-1}(\mathbf{F}_q)$, i.e. by $O(q^{n-1})$ invariants.

In the case of GL_2, the proof is done by elementary methods. The crux of the proof for GL_n is establishing a "completeness of $R(\pi)$-valued Whittaker models", and relies on a finer analysis of auxiliary representations attached to ρ and π by applying Bernstein-Zelevinsky derivatives.

This project is still ongoing – we would like to replace the algebra $R(\pi)$ by $\bar{\mathbf{F}}_\ell$. We are currently studying the interaction between our two contructions of γ factors inthe case $n = 2$ where both exist, with the hope that translation from one to the other will shed light on a way to construct $\bar{F}_\ell$-valued γ factors for $GL_n(\mathbf{F}_q)$.

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