RESEARCH STATEMENT

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I study number theory and representation theory, specifically automorphic representations, functoriality in the Langlands program, and the cohomology of arithmetic groups. I am especially motivated by problems in which automorphic forms interact with other areas of mathematics, for example:

Topology: In my thesis [GG21] and subsequent works [DGG22, EGGG], we bound the dimensions of the i^{th} cohomology of arithmetic manifolds, which roughly count the number if i-dimensional holes in the manifold, as they vary in sequences of covering spaces known as towers. A sample theorem is:

Theorem 1 ([GG21]). *Let $\mathfrak{p}$ be a large enough prime in a totally real number field, N be odd, and $X(\mathfrak{p}^n)$ be a tower of full-level compact arithmetic manifolds associated to $U(N-a,a)$. In the smallest nontrivial degree of cohomology a, the following bound holds:*

$$\dim H^a(X(\mathfrak{p}^n), \mathbf{Q}) \ll Vol(X(\mathfrak{p}^n))^{\frac{N}{N^2-1}}.$$

We derive such results from asymptotic bounds on automorphic representations, thereby proving several instances of the 30-year old Conjecture 1 quantifying the failure of the naïve Ramanujan conjecture.

Fourier analysis: In the collaboration [GGV] with Akshay Venkatesh, we give a new proof, using the cohomology of arithmetic groups, of the following Fourier interpolation formula of Radchenko-Viazovska:

Theorem 2 ([RV19]). *Let f be an even Schwartz function on the real line, and $\hat{f}$ its Fourier transform. The function f is entirely determined by the values $(f(\sqrt{n}), \hat{f}(\sqrt{n}))_{n\in\mathbf{N}}$.*

The starting point for this strategy is the observation that the space $\mathcal{S}$ of Schwartz functions carries the Weil representation of the metaplectic group. We deduce Theorem 2 from the vanishing of a cohomology group $H^1(\Gamma, \mathcal{S}^*)$ for a certain Γ, which we prove using the spectral theory of automorphic forms.

Outline. This research statement has three parts: Section 1, discusses the relation between the cohomology of arithmetic groups and automorphic forms, and lays out recent progress and future directions in the program to use Langlands functoriality to understand their asymptotics. Section 2 concerns cohomology or arithmetic groups with infinite-dimensional coefficients, and applications to interpolation theorems. Finally, Section 3 discusses some recent collaborations on mod ℓ gamma factors, and on statistics of Fourier coefficients.

1. Growth of Cohomology and Non-Tempered Representations

A classical measure of the topological complexity of a space X are the cohomology groups $H^i(X, \mathbf{C})$, roughly measuring the number of i-dimensional holes in the space X. These groups are studied geometrically, but also algebraically as the *group cohomology* $H^i(\Gamma, \mathbf{C})$ of the so-called fundamental group Γ of the space X.

A particularly rich class of spaces are *arithmetic manifolds* X_Γ, whose fundamental groups Γ are arithmetic groups, i.e. the integer entries $G(\mathbf{Z})$ of a (reductive) matrix group G such as GL_N. Some key properties are:

- The spaces X_Γ often parameterize objects of interest, such as quadratic forms or abelian varieties.
- The fundamental groups Γ have natural sequences of *congruence subgroups* $\Gamma(n)$, consisting of matrices congruent to the identity modulo n. The spaces $X_{\Gamma(n)}$ are *covering spaces* of X_Γ.
- Natural families of operators T_p act on $H^i(\Gamma, \mathbf{C})$ and their eigenvalues are expected to be connected to Galois representations. As such, the cohomology of the generalization of *modular forms*.

The cohomology of arithmetic manifolds, though well-studied, is in general very hard to compute. A more tractable problem is to study *tower* of covering spaces.

Question 1. In a tower $X_{\Gamma(n)}$ of arithmetic manifolds, how does $H^i(\Gamma(n), \mathbf{C})$ grow as a function of n?

Arithmetic manifolds are Riemannian: as such, their cohomology can be computed using *harmonic* differential forms. Even more is true: these spaces of differential forms naturally carry representations of the Lie group $G(\mathbf{R})$. If X_Γ is compact, Matsushima [Mat67] leveraged this to relate cohomology and representations:

$$H^*(\Gamma, \mathbf{C}) = \oplus_\pi m(\pi, \Gamma) H^*(\mathfrak{g}, K; \pi). \tag{1}$$

This formula expresses the cohomology in terms of irreducible unitary representations π of $G(\mathbf{R})$:

- $m(\pi, \Gamma)$ is the multiplicity of π in the right-regular representation of $G(\mathbf{R})$ on $L^2(\Gamma\backslash G(\mathbf{R}))$.
- The representations contributing to $H^*(\Gamma, \mathbf{C})$ are those whose so called $(\mathfrak{g}, K)$-cohomology $H^*(\mathfrak{g}, K; \pi)$ is nonzero. These *cohomological representations* look like spaces of differential forms in a suitable sense, and are completely classified [VZ84].

This allows us to rephrase Question 1 in terms of representation theory:

Question 2. Given a representation π of $G(\mathbf{R})$, how does $m(\pi, n) := m(\pi, \Gamma(n))$ grow as a function of n?

First progress towards this question was made by deGeorge-Wallach [dGW78], whose work implies that

$$\lim_{|n\to\infty} \frac{m(\pi, n)}{\mathrm{Vol}(X_{\Gamma(n)})} = \begin{cases} k > 0 & \text{if } \pi \hookrightarrow L^2(G) \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Remark 1. (A) The volume of $X_{\Gamma(n)}$, which grows like $n^{\dim G}$, is an upper bound on the multiplicity.
(B) Integrability properties of π determine the rate of growth of $m(\pi, \Gamma(n))$.

Building on observation (B), Sarnak-Xue [SX91] conjectured growth rates for representations $\pi \not\hookrightarrow L^2(G)$:

Conjecture 1 (Sarnak-Xue)**.** Let π be a unitary representation, and $p(\pi) = \inf\{p \mid \pi \hookrightarrow L^p(G)\}$. Then

$$m(\pi, \Gamma(n)) \ll_\epsilon Vol(\Gamma(n)\backslash G/K)^{2/p(\pi)+\epsilon}.$$

Representations such that $p(\pi) > 2$ are said to be *non-tempered.* To discuss them, we adopt a global perspective and consider automorphic representations, i.e. irreducible $G(\mathbf{A})$-subrepresentations of

$$L^2(G(\mathbf{Q})\backslash G(\mathbf{A})) = \oplus_\pi m(\pi)\pi \tag{3}$$

where $\mathbf{A}$ are the adèles of $\mathbf{Q}$. Automorphic representations factor as tensor products $\pi = \otimes'_v \pi_v$ of representations of the local groups $G(\mathbf{Q}_v)$; to underscore our shift in perspective, representations of the Lie group $G(\mathbf{R})$ are now denoted π_∞. The global and classical perspectives are related since up to finite index,

$$\Gamma(n)\backslash G(\mathbf{R}) = G(\mathbf{Q})\backslash G(\mathbf{A})/K(n)$$

for suitable subgroups $K(n) \subset \mathcal{G}(\mathbf{A})$. The multiplicity counts under consideration in Question 2 can then be rewritten as $m(\pi^0_\infty, n) = \sum_{\pi=\pi^0_\infty\otimes\pi_f} m(\pi) \dim \pi_f^{K(n)}$ for π^0_∞ a fixed $G(\mathbf{R})$-representation.

Arithmetically, the richest part of the decomposition (3) and of generalizations to non-compact quotients is the *cuspidal* part, which is the subject a central conjecture in the analytic theory of automorphic forms:

Conjecture 2 (Ramanujan-Petersson)**.** If G is split and π is cuspidal, the constituents π_v of π are tempered.

This conjecture has analytic implications in number theory and beyond [Li20]. Yet for general G, it fails: the first counterexamples appear in [Kur78]. Conjecture 1 quantifies their rarity: it can be rephrased as

The Ramanujan-Petersson Conjecture holds asymptotically as $n \to \infty$.

1.1. **Results.** My research program aims to prove Conjecture 1 for cohomological representations using results in the Langlands Program. The main idea is:

Multiplicities of non-tempered representations grow slowly because these representations come from smaller groups.

To be more precise, we outline the web of conjecture known as the *Langlands Program.* To any group G, it attaches a dual group $\hat{G}$. The *Reciprocity Conjecture* then predicts a correspondence

$$\{\text{Automorphic representations } \pi_\rho \text{ of } G\} \leftrightarrow \left\{\text{Representations } \rho_\pi \text{ of } \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \text{ into } \hat{G}.\right\} \tag{4}$$

If a Galois representation ρ_π factors through a subgroup $\hat{H} \subset \hat{G}$, then (4) predicts a corresponding π^H_ρ of H, and according to the Langlands *Functoriality Conjecture* the characters of π^G_ρ and π^H_ρ should be related.

A well-understood instance of these relations is the Endoscopic Classification of Representation (ECR), proved by Arthur [Art13] for SO_N and Sp_N, and extended to unitary groups in [Mok15] and [KMSW14]. It describes the decomposition (3) in terms of the contributions of smaller, *endoscopic* groups.

In the footsteps of [MS18] who use endoscopy to prove Conjecture 1 for $U(N-1,1)$, I prove in my thesis:

Theorem 3 ([GG19],[GG21])**.** *Let $U(N-a,a)$ be a unitary group of odd rank defined with respect to a CM extension E/F of number fields, let $\mathfrak{p}$ be a prime of F, and let $\Gamma(\mathfrak{p}^n)$ be a sequence of cocompact congruence subgroups. Then for the smallest degree a whose cohomology does not identically vanish,*

$$\dim H^a(\Gamma(\mathfrak{p}^n), \mathbf{C}) \ll |\mathfrak{p}^n|^N,$$

where $|\cdot| : F \to \mathbf{Q}$ is the norm. In particular, Conjecture 1 holds for the corresponding representations.

The idea of the theorem is to not only appeal to the ECR, but also repurpose its tools, namely the *stabilization of the trace formula*. This is an expression for the multiplicity $m(\pi_\infty, \Gamma(\mathfrak{p}^n))$ in terms of a so-called stable contribution of G and of error terms coming from the endoscopic groups.

$$m(\pi_\infty, \mathfrak{p}^n) = St^G_{\pi_\infty}(\mathfrak{p}^n) + \sum_{H \text{endoscopic}} St^H_{\pi^H_\infty}(\mathfrak{p}^n). \tag{5}$$

The main observation powering Theorem 3 is:

Under good conditions, of the "error terms" in (5) is in fact the dominant term as n grows.

This, combined with observation (A) from Remark 1, allows me to bound the multiplicities in terms of the smaller rate of growth $|\mathfrak{p}^n|^{\dim H}$. In subsequent joint work with Rahul Dalal, we vastly extend the results:

Theorem 4 ([DGG22])**.** *Let G be a unitary group defined with respect to a CM extension E/F. For any cohomological π_∞ on $G(\mathbf{R})$, there is an explicit constant $R(\pi_\infty)$ such that for any sequence of ideals $\mathfrak{n}$ split in E:*

(i) $m(\pi_\infty, \mathfrak{n}) \ll |\mathfrak{n}|^{R(\pi_\infty)}$ and the exponent $R(\pi_\infty)$ proves Conjecture 1 if $N > 4$.
(ii) Under some conditions on E, and π_∞, $R(\pi_\infty)$ is sharp.
For example, if $G = U(N-a,a)$ with N odd and a the smallest degree whose cohomology does not identically vanish, then there exist sequences of cocompact lattices $\Gamma(\mathfrak{n})$ such that

$$H^a(\Gamma(\mathfrak{n}), \mathbf{C}) \asymp Vol(X(\mathfrak{n}))^{\frac{N}{N^2-1}}.$$

In fact, we prove more refined statistics in [DGG22]: we weigh multiplicities by unramified functions and show that under the assumptions of (ii), the finite factors π_v representations with $\pi_\infty = \pi^0_\infty$ satisfy a Sato-Tate theorem on average with respect to a measure predicted by Langlands functoriality.

The proof of Theorem 4 builds on that of Theorem 3, but tackles the much more delicate cases where the G-term in (5) dominates. Inspired by [Taï17], we then analyze this term by taking seriously the intuition:

Endoscopy is the echo for classical groups of parabolic induction happening on GL_N.

In practice, this means that the main work of Theorem 4 is controlling traces of representations obtained by *twisted parabolic induction.* We give upper bounds on these in general, and exact identities in some cases.

Can these methods go beyond unitary groups? We begin to tackle this question with Shai Evra and Henrik Gustaffson:

Theorem 5 ([EGGG])**.** *Let F be a totally real number field, and let $G = SO_5/F$ or an inner form unramified at the finite places.*

(i) Conjecture 1 holds for cohomological representations of G. In particular, if $\Gamma(\mathfrak{n})$ is a sequence of cocompact lattices in $G(\mathbf{R})$, then for the unique degree d of interest we have:

$$\dim H^d(\Gamma(\mathfrak{n}), \mathbf{C}) \ll |\mathfrak{n}|^5, \qquad d = \begin{cases} 2 & \text{if } G(\mathbf{R}) = SO(3,2) \\ 1 & \text{if } G(\mathbf{R}) = SO(4,1). \end{cases}$$

(ii) If the group G is compact at the infinite places, the analogue of Conjecture 1 for finite places v also holds: the multiplicity of non-tempered π_v are negligible as the level grows.

The generalization of Conjecture 1 in (ii) is the *Sarnak-Xue Density Hypothesis*: proving it opens new applications in which arithmetic manifolds are replaced by discrete analogues such as simplicial complexes. As a corollary of Theorem 5, we exhibit infinite families of so-called Density Ramanujan graphs: regular graphs with optimal mixing properties, which are sought after for applications to computer science.

Theorem 5 is also proved in the framework of functoriality, but adopts a mix approach in which the multiplicity counts are split into a *global* and a *local* components, and estimates both independently. Additionally, it relies on an explicit descriptions of the automorphic representations of SO_5 proved by Schmidt [Sch20].

1.2. **Future Work.** I plan on building on the above and give sharp asymptotics on the multiplicities of cohomological representations for all U_N, SO_N, and Sp_N. The current obstacles to fulfilling this program involve foundational questions of interest in their own right, which I plan on tackling in the coming years:

Asymptotics of root numbers in higher rank. The arithmetic of the ECR is encoded in Arthur's so-called ϵ-factors: certain sign characters obtained from root numbers of local L-functions associated to the automorphic representations. Even coarse control on their distribution as the level $\mathfrak{n}$ grows would extend the sharp results of Theorem 4 to a much wider class of representations. With Rahul Dalal, our current strategy to compute these statistics uses the trace formula, and exploits the fact that the sign of the functional equation is equal to the eigenvalue of an Atkin-Lehner type operator on the space of newforms.

Character expansion of local representations. To generalize the local-global methods of Theorem 5 and to obtain sharp asymptotics the key is to control the dimensions of fixed vectors in local representations. The rate of growth as the level grows is the well-understood *Gelfand-Kirrilov* dimension of the representation, but the constants in this asymptotic growth are mysterious. Bounding these constants is a tractable problem for induced representations by Mackey theory, so the work lies in the study of cuspidal representations. For this, a promising strategy is to rely on the recent developments in which all cuspidal representations are constructed as compact-inductions of representations of finite groups.

Twisted parabolic induction. The restrictions in part (ii) of Theorem 4, as well as the obstacle to extending it to SO_N and S_N, come from difficulties in comparing twisted traces on GL_N and some of its Levi subgroups. The type of identities come in two categories: *spectral stable transfer*, in which one compares representations of GL_N to those of a product $GL_{N_1} \times ... \times GL_{N_r}$ for $\sum N_r = N$, and *Speh transfer*, which compares non-tempered representations on GL_{Nd} to tempered representations on $(GL_N)^d$. Once again, we need only the coarsest asymptotic information to derive results. Work in this direction will begin by understanding of the decomposition representations realizes as twisted parabolic induction.

2. Cohomology of Arithmetic Groups and Interpolation

Recently, as part of the series of works leading up to the sphere-packing results for the E_8 and Leech lattices, Radchenko-Viazovska proved the following interpolation theorem.

Theorem 6 ([RV19]). *Let f be an even Schwartz function on the real line, and $\hat{f}$ its Fourier transform. The function f is entirely determined by the values $(f(\sqrt{n}), \hat{f}(\sqrt{n}))_{n\in\mathbf{N}}$, and the only relation comes from Poisson summation formula.*

In joint work with Akshay Venkatesh, we give a new proof of this result in the language of the cohomology of arithmetic groups. The main insight driving this new proof is:

The space $\mathcal{S}$ of Schwartz functions carries the Weil representation of the metaplectic group.

The metaplectic group is the degree 2 cover of SL_2, and the Weil representation is defined by:

- $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ multiplies $f(x)$ by $e^{t\pi i x^2}$ and $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$ multiplies $\hat{f}(x)$ by $e^{t\pi i x^2}$, and
- the lift of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ acts by the Fourier transform.

The extension of these formulas to the dual $\mathcal{S}^*$ of the Weil representation, together with a Mayer-Vietoris argument, allow us to reduce Theorem 6 to

$$H^0(\Gamma, \mathcal{S}^*) = \mathbf{C}, \quad \text{and} \quad H^1(\Gamma, \mathcal{S}^*) = 0, \qquad \text{for} \quad \Gamma = \langle \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \rangle \subset SL_2(\mathbf{Z}). \tag{6}$$

In order to prove this, we extend the framework of Section 1 relating group cohomology and automorphic representations: the main difference is that the groups

$$H^*(\Gamma, \mathbf{C}) \qquad \text{are replaced by} \qquad H^*(\Gamma, V)$$

where the coefficient system is no longer trivial, or finite dimensional, but is an infinite-dimensional topological vector space V. For these representations and when the ambient group G is a cover of SL_2, we extend a result of Bunke-Olbrich [BO98] and prove a Matsushima's formula (1) for infinite-dimensional coefficients.

Theorem 7 ([GGV]). *Let Γ be a torsion-free lattice in a covering group of $SL_2(\mathbf{R})$, and let W be an infinite-dimensional irreducible smooth Fréchet representation of with G infinitesimal character λ. Then there is a unique complementary representation W^{cl} such that*

$$H^0(\Gamma, W^*) = m(\Gamma, W), \quad H^1(\Gamma, W^*) = m_{\text{cusp}}(\Gamma, W^{cl}),$$

where $m_{cusp}(\Gamma, V)$ denotes the multiplicity of V in the cuspidal spectrum at level Γ.

We give two proofs of this result: one based on [BO98] using standard resolutions, and one, in the spirit of Matsushima's formula, using the Casselman-Wallach globalization theorem and $(\mathfrak{g}, K)$ cohomology. The crux in both cases is the surjectivity of a certain differential operator on the space of automorphic forms, which we prove using the spectral decomposition of the space of automorphic forms and a truncation argument.

Specialization of this theorem to $W = \mathcal{S}$ gives the desired dimensions from (6) and allows us to deduce Theorem 6. Moreover, changing the coefficients to other spaces of functions gives new interpolation theorems:

Theorem 8 ([GGV]). *Let $\mathbf{P}^1(\mathbf{R})$, be the real projective line, with homogenous coordinates $[x : y]$. Then any $\Phi \in C^\infty(\mathbf{P}^1(\mathbf{R}))$ is entirely determined the following family of integrals:*

$$a_n(\Phi) = \int_{\mathbf{R}} \Phi(x, 1) e^{\pi i n x} dx, \qquad b_n(\Phi) = \int_{\mathbf{R}} \Phi(1, y) e^{\pi i n y} dy.$$

2.1. **Future Work.** This work has natural extensions in several directions: we would like to prove the technical Theorem 7 for more degrees of cohomology, and more general class of groups. Additionally, it would be interesting to systematize the relation between cohomology vanishing and interpolation theorems.

Another possible line of work would move away from the archimedean place towards the p-adic places: can one deduce p-adic interpolation theorems from computing cohomology with values spaces of functions on p-adic manifolds? In this context, it is likely that the relevant lattices would be finite covolume subgroups of $SL_2(\mathbf{R}) \times SL_2(\mathbf{Q}_p)$.

3. Gamma Factors and L-functions

To conclude, I discuss two collaborative projects [BGGG+23] and [FGGHT23] on invariants of automorphic representations, namely gamma factors and L-functions. Though very different in nature, both projects are very concrete: in the first, we compute local gamma factors as Gauss sums, and in the second, we relax assumptions of a theorem from GRH to a count of zeros in an explicit rectangle in the critical strip.

3.1. **Gamma factors and a converse theorem mod ℓ.** We consider the classical problem of *classifying the irreducible representations of $GL_n(\mathbf{F}_q)$*. We study *gamma factors* $\gamma(\rho, \pi)$, invariants introduced by Piatetski-Shapiro for complex representations ρ of $GL_2(\mathbf{F}_q)$, with π a character of $GL_1(\mathbf{F}_q)$. The construction of $\gamma(\rho, \pi)$ relies on the realization of ρ as a space of functions, its *Whittaker Model*, and Piatetski-Shapiro proves

Theorem 9 ([PS83]). *A $\mathbf{C}$-valued cuspidal representation of $GL_2(\mathbf{F}_q)$ is determined by its gamma factors.*

This statement is a *converse theorem*: such result are motivated by local Langlands correspondences, which are characterized bye equalities of gamma factors and L-functions. Macdonald [Mac80] showed that $GL_n(\mathbf{F}_q)$ admits such a correspondence, in which they correspond to equivalence classes of representations of the Weil group. Moreover, representations of $GL_n(\mathbf{F}_q)$ give rise to depth-zero representations of the corresponding p-adic GL_n, and [Mac80] is compatible with the usual local Langlands for depth zero representations.

3.1.1. *Results.* In [BGGG$^+$23], we consider coefficients in any characteristic $\ell \nmid q$; in particular, ℓ may divide $|GL_n(\mathbf{F}_q)|$. In this situation, Vigneras has extended many results from the complex case [Vig88]. Using these, we first show that one can construct mod ℓ gamma factors. Then, exploiting their explicit description as Gauss sums in the case of $GL_2(\mathbf{F}_q)$, we compute counterexamples to the converse theorem on SAGE.

Example 10. When $(\ell, q) = (2, 5)$, $(3, 7)$, $(11, 23)$, or $(29, 59)$, there are non-isomorphic mod ℓ cuspidal representations of $GL_2(\mathbf{F}_q)$ whose gamma factors agree.

This suggests that for mod ℓ representations, one needs to modify the naïve construction. We do this by replacing the Whittaker model of ρ valued in the constants $\bar{\mathbf{F}}_\ell$ by its analogue consisting of functions valued in a certain Artinian algebra $R(\pi)$ associated to π. With this new gamma factor, we find:

Theorem 11 ([BGGG$^+$23])**.** *Let ρ be a mod ℓ cuspidal representation of $GL_n(\mathbf{F}_q)$. If $n \geq 2$, a cuspidal representation ρ is entirely determined by its $R(\pi)$-valued γ factors $\gamma(\rho, \pi)$, as π runs over irreducible representations of $GL_{n-1}(\mathbf{F}_q)$.*

The crux of the proof is establishing a completeness of $R(\pi)$-valued Whittaker models: we do this via an analysis of ρ and π via their Bernstein-Zelevinsky derivatives.

3.1.2. *Future Work.*
Jacquet's conjecture mod ℓ. We show that mod ℓ representations of $GL_n(\mathbf{F}_q)$ are determined by their gamma factors $\gamma(\rho, \pi)$ against representations of $GL_{n-1}(\mathbf{F}_q)$. However, it was conjectured by Jacquet, and proved by Nien [Nie14] for $\mathbf{C}$-valued representations, that $\gamma(\rho, \pi)$ determine ρ for representations π of $GL_m(\mathbf{F}_q)$ with $m \leq \lfloor \frac{n}{2} \rfloor$. Such a statement mod ℓ is likely within reach using our Bernstein–Zelevinsky-type methods.

Characterization of local Langlands-type correspondances. The correspondence [Mac80] preserves so-called Godement-Jacquet gamma factors, a different harmonic-analytic construction. Vigneras [Vig94] constructed a correspondence mod ℓ, and recently, Li-Shotton [LS23] proved a version in families. We would like to assert whether these correspondences are characterized by equality of our (Rankin-Selberg) gamma factors: we will start by showing that the $\gamma(\rho, \pi)$ are multiplicative over parabolic induction.

3.2. **Signs in Fourier coefficients of modular forms and zeros of the L-function.** This project concerns classical modular Hecke eigenforms f of weight k, given by a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n(f) n^{\frac{k-1}{2}} e^{\pi i n z},$$

with z in the complex upper half-plane. The Fourier coefficients $a_n(f)$ are multiplicative and the Ramanujan conjecture for SL_2, proved by Deligne [Del74] for modular forms of weight $k \geq 2$, states that

$$|a_n(f)| \leq \tau(n), \tag{7}$$

for $\tau(n)$ the number of positive divisors of n. We study cancellation between the $a_n(f)$, i.e. of the growth of

$$S(f, x) := \sum_{n \leq x} a_n(f).$$

The best bounds independent of f were proved by Lamzouri [Lam19]: under the Generalized Riemann Hypothesis (GRH) for the L-function $L(s, f)$ of f, he showed that $S(f, x) = o(x \log x)$ as $\frac{\log x}{\log \log k} \to \infty$.

3.2.1. *Results.* The work [FGGHT23] with Claire Frechette, Alia Hamieh, and Naomi Tanabe, proves that the sums $S(f, x)$ are $o(x \log x)$ in the range $x \geq k^\epsilon$, under assumptions both weaker and more explicit than GRH. We explicitly relate a too rapid growth of $S(f, x)$ to the presence of many unexpected zeros of $L(s, f)$.

Theorem 12 ([FGGHT23])**.** *Let $f \in S_k(N)$ be a primitive cusp form. Let ϵ and T be real numbers with $\epsilon \geq (\log k)^{-1/8}$ and $1 \leq T \leq (\log k)^{1/200}$. Suppose that the region*

$$R(T) = \left\{ s \,:\, \Re(s) \geq \frac{3}{4},\, |\Im(s)| \leq T + \frac{1}{4} \right\}$$

contains no more than $\epsilon^2 \log k / 5000$ zeroes of $L(s, f)$. Then for all $x \geq k^\epsilon$, we have $|S(f, x)| \ll (x \log x)/T$.

Our proof relies on recent developments in analytic number theory pioneered by Granville–Soundararajn, whose analogous result [GS18] on character sums was the inspiration for the project. The framework, known as *pretentious number theory*, applies naturally to bounded functions. Recently, the key input, known as Halazs' Theorem was extended by Mangerel [Man23] to multiplicative functions satisfying (7).

The strategy is to compare $a_f(n)$ and to n^{it} for $t \in \mathbf{R}$: such functions see very little cancellation, and for good t the twisted sums $S(f,x,t) = \sum_{n \leq x} a_f(n) n^{it}$ remain close enough to $S(f,x)$. This is useful because:

$S(f,x,t)$ and t-translates of the L-function $L(f,s)$ are related by the Fourier transform.

This allows us to use Plancherel's formula to compare $S(f,x)$ and a suitable shift of $L(s,t)$. By assuming for contradiction that $S(f,x)$ is large, we first produce a lower bound involving $L(s+it,f)$ and then express $L(s+it,f)$ to in terms of its zeros using Hadamard's formula to obtain our result.

3.2.2. *Future Work.* A long-term goal using the methods of our theorem would be to extend the strategy to the cuspidal representations of $GL_n/\mathbf{Q}$ which satisfy the Ramanujan conjecture. A first project to test the robustness of the method is to bound sums of Rankin-Selberg products of modular forms.

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