RESEARCH STATEMENT

MATHILDE GERBELLI-GAUTHIER

I study number theory and representation theory, specifically automorphic representations, functoriality in the Langlands program, and the cohomology of arithmetic groups. I am motivated by problems in which automorphic forms interact with other areas of mathematics, for example:

- *Topology*: my main research program, see Section 1, uses functoriality in the Langlands program to study the growth of cohomology in towers covering spaces.
- *Fourier analysis*: by studying the versions of the above cohomology groups with infinite-dimensional coefficients, the work of Section 2 (re-)proves interpolation theorems for classical spaces of functions.

I also like working on problems that give concrete incarnations to general principles. In the aforementioned program 1, the phenomenon of endoscopy manifests statistically by the dearth of cohomology in certain degrees. The project of Section 3 constructs Langlands-inspired invariants for finite groups, which in this context are explicitly-computable Gauss sums. Finally, in Section 4, we reprove a conditional result on sums of Fourier coefficients of cuspforms: instead of assuming the Generalized Riemann Hypothesis for the corresponding L-function, we exhibit a rectangle which should not contain too many zeros.

1. Automorphic Statistics, the Sarnak-Xue conjecture, and applications.

My central research program concerns asymptotic counts of automorphic representations, with applications to the growth of cohomology in towers of arithmetic manifolds and to the construction of Ramanujan-type complexes. In my thesis [GG21], and the works [DGG22] and [EGGG], we leverage the theory of endoscopy to compute limit multiplicities and other statistics of non-tempered automorphic representations.

1.1. **Background.** Is this section, G is a reductive Lie group and Γ is a cocompact lattice in G: a discrete subgroup of G such that $\Gamma\backslash G$ is a compact manifold. The space $L^2(\Gamma\backslash G)$ is naturally equipped with the regular representation of G induced by multiplication. It decomposes into a direct sum

$$L^2(\Gamma\backslash G) = \oplus_\pi m(\pi,\Gamma)\pi \tag{1}$$

of irreducible representations π, each appearing with multiplicity $m(\pi,\Gamma)$. A very general question is:

Question 1. What is the multiplicity $m(\pi,\Gamma)$ of any given representation π in the decomposition?

This question can be specialized in several directions, of which we discuss two:

- **Choice of lattice Γ.** If $G = \mathcal{G}(\mathbf{R})$ for a group $\mathcal{G}$ defined over a number field F, we can let Γ be *arithmetic*, which amounts to being a finite-index subgroup of $\mathcal{G}(\mathbf{Z})$. Arithmetic groups have rich connections to number theory: the representations π appearing in the decomposition of $L^2(\Gamma\backslash G)$ are generalization of modular forms. Additionally, arithmetic groups admit a natural filtration by *congruence subgroups* given by the kernel of reduction modulo $\mathfrak{n}$ for each ideal $\mathfrak{n}$ of F.
- **Choice of representation π.** Let K be a maximal compact subgroup of G. Some representations π occurring in the decomposition (1) capture information about the topology of the space $\Gamma\backslash G/K$. Indeed, their multiplicities dictate the singular cohomology following Matsushima's formula [Mat67]:
$$H^*(\Gamma\backslash G/K,\mathbf{C}) = \oplus_\pi m(\pi,\Gamma)H^*(\mathfrak{g},K;\pi). \tag{2}$$
By definition, the representations π whose so-called $(\mathfrak{g},K)$ cohomology $H^*(\mathfrak{g},K;\pi)$ is nonzero are *cohomological*. Vogan-Zuckerman [VZ84] have characterized the finite list of cohomological representations for each Lie group.

Even in the well-studied case of Γ arithmetic and π cohomological, Question 1 is extremely delicate; a more tractable version of the problem asks the question asymptotically.

Question 2. Let $|\mathfrak{n}|$ be the norm of the ideal $\mathfrak{n}$. How does the multiplicity $m(\pi,\Gamma(\mathfrak{n}))$ grow as $|\mathfrak{n}|\to\infty$?

First progress towards this question was made by deGeorge-Wallach [dGW78], whose work implies that

$$\lim_{|\mathfrak{n}|\to\infty} \frac{m(\pi,\Gamma(\mathfrak{n}))}{\mathrm{Vol}(\Gamma(\mathfrak{n})\backslash G/K)} = \begin{cases} k>0 & \text{if } \pi \hookrightarrow L^2(G) \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

In particular, analytic properties of π determine the rate of growth of $m(\pi,\Gamma(\mathfrak{n}))$. One can ask for more precise rates of growth for representations such that $\pi \not\hookrightarrow L^2(G)$. This led Sarnak-Xue [SX91] to conjecture:

Conjecture 1 (Sarnak-Xue)**.** Let π be a unitary representation, and $p(\pi) = \inf\{p \mid \pi \hookrightarrow L^p(G)\}$. Then

$$m(\pi,\Gamma(\mathfrak{n})) \ll_\epsilon Vol(\Gamma(\mathfrak{n})\backslash G/K)^{\frac{2}{p(\pi)}+\epsilon}.$$

Representations such that $p(\pi)>2$ are *non-tempered*. Roughly speaking, less tempered representations (with larger values of $p(\pi)$) contribute to cohomology in lower degrees in (2).

1.2. **First results.** My thesis was concerned with the real points of a unitary group $\mathcal{G}$ preserving a Hermitian form of signature $(N-a,a)$ and a prime ideal $\mathfrak{p}$ large enough. For cohomology, it implies

Theorem 1 ([GG21])**.** *Let N be odd and a be the smallest degree i such that $H^i(\Gamma(\mathfrak{p}^n)\backslash G/K,\mathbf{C}) \neq 0$. Then*

$$\dim H^a(\Gamma(\mathfrak{p}^n)\backslash G/K) \ll Vol(\Gamma(\mathfrak{p}^n)\backslash G/K)^{\frac{N}{N^2-1}}.$$

In view of DeGeorge-Wallach's result (3), the main intuition powering my proof is:

The growth of non-tempered representations is slow because it tracks the growth of tempered representations in towers associated to groups of smaller dimension.

This idea is already present in work of Marshall [Mar14] and Marshall-Shin [MS18]. Theorem 1 recovers some of their results when $a=1$. The groups of smaller dimension occurring in this context are called *endoscopic groups of $\mathcal{G}$*, and the mechanism by which properties of their representations are transferred to $\mathcal{G}$ is *endoscopy*: one the best-understood instance of functoriality in the Langlands program. Consequently, we shift our focus to automorphic representation: these are irreducible summands in the decomposition

$$L^2(\mathcal{G}(F)\backslash\mathcal{G}(\mathbf{A})) = \oplus_\pi m(\pi)\pi \tag{4}$$

where $\mathbf{A}$ is the ring of adèles of F. They factor as $\pi = \otimes'_v \pi_v$; to underscore our shift in perspective, representations of the Lie group G are now denoted π_∞. Finally, the groups $\Gamma(\mathfrak{n})$ are replaced by compact-open subgroups $K(\mathfrak{n}) \subset \mathcal{G}(\mathbf{A})$ such that $\Gamma(\mathfrak{n}) = \mathcal{G}(F)\cap K(\mathfrak{n})$. Question 2 then becomes:

Question 3. For a given π_∞, what is the rate of growth of $m(\pi_\infty,\mathfrak{n}) = \sum_{\pi=\pi_\infty\otimes\pi_f} m(\pi)\dim \pi_f^{K(\mathfrak{n})}$?

The result I prove relies on the endoscopic classification of representations: pioneering work of Arthur [Art13], extended to unitary groups by Mok [Mok15] and Kaletha-Minguez-Shin-White [KMSW14]. In this classification, automorphic representations are sorted into so-called Arthur packets Π_ψ attached to A-parameters ψ. In the case of unitary groups, the data of ψ includes an N-dimensional representation of SL_2, or, by the Jacobson-Morozov theorem, a partition of N. The main technical result of my thesis is:

Theorem 2 ([GG21])**.** *Let ψ_∞ be a local archimedean parameter of $\mathcal{G}$ with regular infinitesimal character, and partition $2k+1+...+1$ of N, and let $\Psi(\psi_\infty)$ be the set of global parameters who localize to ψ_∞. For a prime $\mathfrak{p}$ of F, denote $Nm(\mathfrak{p})$ its norm and p its residue characteristic. If $p \geq N^2[F:\mathbf{Q}]+1$, then*

$$\sum_{\psi\in\Psi(\psi_\infty)} \sum_{\pi\in\Pi_\psi} m(\pi)\pi_f^{K(\mathfrak{p}^n)} \ll |\mathfrak{p}|^{n(N-2k)}. \tag{5}$$

Moreover, if π_∞ occurs only in packets associated to such ψ_∞, then π_∞ satisfies conjecture 1.

I obtain the bound by re-purposing the main tool of the classification: the stabilization of the trace formula. This is an expression for the trace $R(\mathfrak{p}^n)_{\psi_\infty}$ of the operator on $L^2(\mathcal{G}(F)\backslash G(\mathbf{A}_F))_{\psi_\infty}$ given by convolution with the indicator function $1_{K(\mathfrak{p}^n)}$. By construction, $R(\mathfrak{p}^n)_{\psi_\infty}$ computes the left-hand side of (5). The stabilization of the trace formula then writes $R(\mathfrak{p}^n)_{\psi_\infty}$ as a sum of so-called stable traces: a main term corresponding to $\mathcal{G}$, and error terms coming from each of its endoscopic groups $\mathcal{H}$:

$$R(\mathfrak{p}^n)_{\psi_\infty} = St_{\mathcal{G}}(\mathfrak{p}^n)_{\psi_\infty} + \sum_{\mathcal{H}} St_{\mathcal{H}}(\mathfrak{p}^n)_{\psi_\infty}. \tag{6}$$

These endoscopic groups $\mathcal{H}$ are products of unitary groups of smaler rank; the stabilization (6) is mechanism to pass information between $\mathcal{G}$ and $\mathcal{H}$. Crucial to this decomposition are Ngô's Fundamental Lemma [Ngo10], extended to congruence subgroups by Ferrari [Fer07]. The key observation powering my proof is

For ψ_∞ associated to non-tempered representations, one of the "error terms" $St_{\mathcal{H}}(\mathfrak{p}^n)$ dominates $n \to \infty$.

This allows me to inductively control the rates of growth of $m(\pi_\infty, \mathfrak{p}^n)$ on $\mathcal{G}$ in terms of an analogous count on $\mathcal{H}$, making precise the intuition laid out at the outset of this project.

1.3. **Exact multiplicities at split level for unitary groups.** In light of the above results, one can ask:

- Are the bounds sharp? Can we obtain finer asymptotics?
- Can we give bounds for more representations, and thus more degrees of cohomology?

My next project [DGG22], with Rahul Dalal, makes substantial progress in both those directions. The endoscopic classification again plays a prominent role, but we engage more seriously with the intuition:

Endoscopy for subgroups of GL_N is an echo of parabolic induction on GL_N.

Our study of asymptotics of automorphic representations again start from the stable trace formula:

$$R_\Sigma(\mathfrak{n}) = St_{\mathcal{G}}(\mathfrak{n})_\Sigma + \sum_{\mathcal{H}} St_{\mathcal{H}}(\mathfrak{n})_\Sigma.$$

The subscript Σ denotes a *shape*: the set of all cohomological A-parameters with a common combinatorial structure and properties at the real places, and a systematization of the partitions appearing in Theorem 2. We then adopt an inductive strategy:

(i) Identify the group $\mathcal{H}$ whose contribution dominates asymptotically as in the strategy of Theorem 2.
(ii) Using *twisted endoscopy*, identify $St_{\mathcal{H}}(\mathfrak{n})_\Sigma$ with a twisted trace on a general linear group GL_N
(iii) The shape Σ determines the parabolic induction structure of representations contributing to the twisted trace. Using this, compare the twisted trace of (ii) to a twisted trace of tempered representations on products of smaller GL_{N_i}.
(iv) Compute asymptotics of these tempered representations using the work of Shin-Templier [ST16].

We prove:

Theorem 3 ([DGG22])**.** *Let E/F be a CM extension defining a unitary group $\mathcal{G}/F$, π_∞ a cohomological representation, and $\mathfrak{n}$ a sequence of ideals in $\mathcal{O}_F$ which split in E. There exists an explicitly computable $R(\pi_\infty)$ such that:*

(i)

$$\sum_{\pi=\pi_\infty\otimes\pi_f} m(\pi)\pi_f^{K(\mathfrak{n})} \ll_\epsilon |\mathfrak{n}|^{R(\pi_\infty)+\epsilon}.$$

(ii) If E/F is unramified at all finite places and π_∞ is a Genralized Saito-Kurokawa representation*, then for any unramified test function f_S at a set of finite places coprime to $\mathfrak{n}$, there exist constants A, B and C, and explicit quantities $L_{\pi_0}(\mathfrak{n})$ and $M(\pi_0)$ such that*

$$|\mathfrak{n}|^{-R(\pi_\infty)} L_{\pi_0}(\mathfrak{n})^{-1} \sum_{\pi=\pi_\infty\otimes\pi_f} m(\pi)\pi_f^{K(\mathfrak{n})} \operatorname{tr} \pi_S f_S = M(\pi_\infty)\mu_S^{pl(\pi_\infty)}(f_S) + O(|\mathfrak{n}|^{-C} q_S^{A+B\kappa(f_S)}).$$

In particular, for such representations, the upper bounds of (i) are sharp.

From part (i), we derive for all representation of unitary groups, save a single family on $U(2,2)$:

Corollary 4 ([DGG22])**.** *The Sarnak-Xue Conjecture 1 holds at split level for cohomological representations.*

Our bounds *beat* Sarnak-Xue in general, and we predict a sharp bound computed from A-parameters.

Conjecture 2 ([DGG22])**.** Let $\Sigma = \oplus\tau_i[d_i]$ be the shape of the maximal Arthur paremeter of π. Then Conjecture 1 is sharp if $2/p(\pi_\infty)$ is replaced by $R(\pi)/2\dim G^{\mathrm{der}}$, where

$$R(\pi) = N^2 - \sum_i T_i(T_i - d_i^2 - 1).$$

1.3.1. *Applications.* Theorem 3 can be reformulated in terms of the cohomology of Shimura varieties:

Corollary 5. *Let E/F, G, and $\mathfrak{n}$ be as in Theorem 3. Assume that G_∞ has at least one compact factor, so that the lattices $\Gamma(\mathfrak{n}) = G(F) \cap K(\mathfrak{n})$ are cocompact. Then for every degree i of cohomology, there is an explicitly computable constant $R(i)$ such that*

$$\dim H^i(\Gamma(\mathfrak{n}); \mathbf{C}) \ll |\mathfrak{n}|^{R(i)}.$$

Moreover, if E/F is unramified and i is a so-called GSK degree, the exponent $R(i)$ is sharp.

The GSK degrees can be explicitly computed: if the $\Gamma(\mathfrak{n})$ are lattices in $U(N-1,1)$, they are exactly those congruent to $N-1$ mod 2. As a consequence, the bounds of [MS18] are sharp in half the degrees.

As a second application, we weigh the count of automorphic representation by unramified test functions. We then prove an average Sato-Tate-type result for the Satake parameters of GSK representations.

Corollary 6. *Let E/F, G, π_∞, and $\mathfrak{n}_i$ be as in part (ii) Theorem 3. Fix a sequence of finite places v_i of F which (i) share the same splitting behavior in E (ii) are coprime to $\mathfrak{n}_i$ (iii) grow slowly relatively to $|\mathfrak{n}_i|$.*

Then there exists an explictly computable group $\mathcal{H}/F$ such that as v_i, $\mathfrak{n}_i \to \infty$, the Satake parameters at v_i of automorphic forms appearing at level $\mathfrak{n}_i$ equidistribute with respect to the pushforward of the Sato-Tate measure on $\mathcal{H}$.

As in Conjecture 2, the group $\mathcal{H}$ depends only on the A-parameters of π_∞. It should be the smallest group from which the representations with component π_∞ at infinity transfer under Langlands functoriality.

1.4. **Sarnak-Xue for orthogonal groups of rank 5.** In [EGGG], together with Shai Evra and Henrik Gustafsson, we tackle a generalization of the Sarnak-Xue conjecture for orthogonal groups..

1.4.1. *The Sarnak-Xue Density Hypothesis.* Let F be a number field and G/F be a reductive group. We first give the construction of a family of automorphic representations in the level aspect. Let:

- ξ be an infinitesimal character at the infinite places;
- S be a finite set of places of F;
- $K_N \subset K_{N-1}$ be a sequence of of compact-open subgroups of $G(\mathbf{A}^S)$, with $[K_N : K_{N-1}]$ finite.

Then the *family* $\mathcal{F}_N$ is the set of automorphic representations $\pi = \pi_S \otimes \pi^S$ with infinitesimal character ξ and such that $\dim \pi^{K_N} \neq 0$. At each place $v \in S$, fix a representation π_v^0, and let $\mathcal{F}_N^0 = \{\pi \in \mathcal{F}_N \mid \pi_v = \pi_v^0 \text{ for } v \in S\}$. Let $p(\pi_S^0) = \max_{v\in S} \inf\{p \geq 2 \mid \pi_v^0 \hookrightarrow L^p(G_v)\}$. Let

$$m(N) = \sum_{\pi \in \mathcal{F}_N} m(\pi) \dim(\pi^S)^{K_N}, \quad m^0(N) = \sum_{\pi = \pi_S^0 \otimes \pi^S in \mathcal{F}_N} m(\pi) \dim(\pi^S)^{K_N}.$$

Conjecture 3 (Sarnak-Xue Density Hypothesis, level aspect)**.** As $N \to \infty$, we have $m^0(N) \ll_\epsilon m(N)^{\frac{2}{p(\pi_S^0)}+\epsilon}$.

Conjecture 1 specializes to Conjecture 3 when S is the set of archimedean places. We prove:

Theorem 7 ([EGGG])**.** *Let F be totally real field and G/F be either SO_5 or an inner form split at all finite places. Conjecture 3 holds at full level for G.*

The proof uses Arthur's endoscopic classification, an extension to inner forms by Taïbi [Taï17], and Schmidt's explicit description of non-tempered packets of SO_5 [Sch20]. The strategy of Theorem 3 is unavailable since SO_5 is not a twist of GL_N. Instead, we adopt a local-global strategy: we bound the number of automorphic representations at a given level, *and* the dimensions of fixed vectors in the local representations.

1.4.2. *Applications.* Our results have applications to the cohomology of arithmetic lattices in SO_5, as well as to the construction quasi-Ramanujan graphs and optimal strong approximation. First, by choosing an inner form compact at only one non-archimedean place, we give upper bounds on cohomology growth for the full range of degrees. The completely new result is:

Theorem 8. *Let $\Gamma(\mathfrak{n})$ be a cocompact lattice in the noncompact group $SO(a,b)$ coming from a Gross inner form. Then*

$$\dim H^{\frac{ab}{2}-1}(\Gamma(\mathfrak{n}), \mathbf{C}) \ll_\epsilon |\mathfrak{n}|^5 + \epsilon.$$

Next, by choosing an inner form of SO_5 which is compact at the archimedean places, and letting $S = \{v_{\mathfrak{q}}\}$ for a prime $\mathfrak{q}$, we get:

Corollary 9. *Let B be the Bruhat-Tits building of $SO_5(F_{v_{\mathfrak{q}}})$, and let $\Gamma_{\mathfrak{n}} = SO_5(F) \cap K^{v_{\mathfrak{q}}}(\mathfrak{n})$. Then the quotients $X_N = B/\Gamma_{\mathfrak{n}}$ are almost Ramanujan, in the sense that they exhibit the cutoff property.*

Cutoff is a mixing property of the complex: roughly, it states that the distance between the probability distribution of a random walk on the complex and the uniform distribution abruptly approaches zero in a short interval of time called the cutoff window, see [LLP20]. This property is currently only known for Ramanujan graphs and complexes, which are optimal expanders from a spectral perspective, and we give the first non-Ramanujan complexes which display it.

1.5. Future directions.

In the following years, I plan on building on the work described above and:

- give sharp asymptotics of cohomological automorphic representations for classical groups;
- prove Conjecture 1 and the stronger Conjecture 2 for cohomological representations.

Some of the current obstacles to the fulfillment of this program involve foundational technical questions which are of interest in their own right, which I plan on tackling in the coming years:

Asymptotics of root numbers in higher rank. The restriction on the parity of the good degrees in Corollary 5 comes from Arthur's so-called ϵ-factors. These are characters of a 2-group $\mathcal{S}_\psi$; they are computed in terms of root numbers of local L-functions associated to the parameters ψ. Even the coarsest control on their asymptotic distribution as the ramification of the parameters grows would extend our results to a much wider class of representations. Rahul Dalal and I are considering various avenues to study this: our current project uses the the trace formula to compute statistics, exploiting the fact that the sign of the functional equation is equal to the eigenvalue of an Atkin-Lehner type operator on the space of newforms.

Twisted parabolic induction. The restriction on the ramification, level, and shape in Theorem 3, as well as the difficulty to extend the strategy to symplectic and orthogonal groups, come from technical obstructions in step (iii) of the proof strategy, namely relating the trace of a twisted parabolic induction of GL_N to the twisted trace of the inducing data. The type of identities come in two categories: *Stable transfer*, which refers to breaking a composite shape into its simple constituents, and *Speh transfer*, which removes the Arthur SL_2 from a simple shape to access the underlying tempered parameter. Yet, once again, we need only the coarsest asymptotic information to derive results. Work in this direction will begin by developing an understanding of the decomposition representations obtained by twisted parabolic induction.

Character expansion of local representations. In order to generalize the local-global strategy of to higher rank groups and to prove Conjecture , the first step is a finer understanding of the spaces of fixed vectors of local representations. The asymptotic behavior as the level grows is well-understood and encapsulated in the notion of *Gelfand-Kirrilov* dimension, but the constants in this asymptotic growth remain mysterious. This question can be framed as either an explicit Bernstein uniform admissibility at full level, or as a question about the constants in the Howe–Harish-Chandra character expansion of the representation. The problem is tractable for induced representations by Mackey theory, and so the work lies in the study of cuspidal representations, for which the first step will be the understanding of the proof of the character expansion.

2. Fourier Interpolation and Cohomology of Arithmetic Groups

In this joint work with Akshay Venkatsh, we use automorphic forms and their relation to the cohomology of arithmetic groups to reprove a theorem of Radchenko-Viazovska.

2.1. Background.

Schwartz functions on $\mathbf{R}$ are smooth functions whose derivatives of all orders are rapidly decreasing. They appear naturally in harmonic analysis, since the space of Schwartz functions if closed under the Fourier transform. In the project [GGV], we give a new proof of the Fourier interpolation theorem:

Theorem 10 ([RV19])**.** *Let f be an even Schwartz function on the real line, and $\hat{f}$ its Fourier transform. The function f is entirely determined by the values $(f(\sqrt{n}), \hat{f}(\sqrt{n}))_{n\in\mathbf{N}}$, and the only relation comes from Poisson summation formula.*

Radchenko-Viazovska do more: they provide an explicit interpolation basis. Their result and its generalizations are used in the proof of optimal universality of sphere packing for the E_8 and Leech lattices.

2.2. **Results.** In [GGV], we give a new proof of Theorem 10 based in the following fact:

The space of Schwartz functions carries the Weil representation W of the metaplectic group.

The metaplectic group G is the connected cover of degree 2 of $SL_2(\mathbf{R})$, and its action on the infinite-dimensional space of Schwartz functions is given by the following defining properties:

- the lift to G of $U_t = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ acts on Schwartz functions by multiplying a function $f(x)$ by $e^{t\pi i x^2}$,
- the lift of $L_t = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$ multiplies the Fourier transform $\hat{f}(x)$ by $e^{t\pi i x^2}$,
- the element which conjugates U_t to L_t acts by the Fourier transform.

We work with the dual space W^* of distributions, equipped with the dual of the Weil representation. The main observation powering our argument is:

The distributions fixed by U_2 are exactly $f \mapsto f(\sqrt{n})$, and similarly for L_2 and $f \mapsto \hat{f}(\sqrt{n})$.

Let $\Gamma = \langle U_2, L_2 \rangle \subset G$; it is a free group on two generators. From the above observation and a Mayer-Vietoris argument, we deduce Theorem 10 from the the following:

Theorem 11 ([GGV])**.** *Let W^* be the continuous dual of W. Then*

$$\dim H^0(\Gamma, W^*) = 1 \quad \textit{and} \quad H^1(\Gamma, W^*) = 0.$$

The computation of $H^0(\Gamma, W^*)$ is classical and uses weight $\frac{1}{2}$ theta series for Γ. For $H^1(\Gamma, W^*)$, we generalize arguments of Bunke-Ohlrich [BO98] to finite covers of $SL_2(\mathbf{R})$ and prove:

Theorem 12. *Let Γ be a torsion-free cocompact lattice in a covering group of $SL_2(\mathbf{R})$, and let W be an infinite-dimensional irreducible smooth Fréchet representation of G infinitesimal character λ. If there are no cuspforms for Γ with Laplace eigenvalue λ, then $H^1(\Gamma, W^*) = 0$.*

This is a version, for infinite-dimensional coefficients, of the classical Eichler-Shimura isomorphism. Bunke-Ohlrich construct acyclic resolutions to compute the cohomology. We had initially devised a different strategy using $(\mathfrak{g}, K)$-cohomology, reminiscent of Matsushima's formula (2):

- Relate $H^i(\Gamma, W^*)$ to $\mathcal{A} = \mathcal{A}(\Gamma\backslash G)$, a suitable space of automorphic forms on $\Gamma\backslash G$. Specifically, rewrite the first cohomology as $\mathrm{Ext}^1(W, \mathcal{A})$ and ultimately as the $(\mathfrak{g}, K)$-cohomology $\mathrm{Ext}^1_{\mathfrak{g},K}(W_K, \mathcal{A}_K)$, replacing representations by their spaces of smooth K-finite vectors. In this analogue of Shapiro's lemma for group cohomology, the main issues are topological, and the argument relies on reduction theory and the Casselman-Wallach globalization theorem.
- Compute the dimensions of $\mathrm{Ext}^1_{\mathfrak{g},K}(W_K, \mathcal{A}_K)$. The key result is the surjectivity of differential operator on a suitable subspace of $\mathcal{A}$, which enables us to restrict to automorphic forms with generalized eigenvalue λ, and also to show that the Eisenstein series do not contribute to cohomology. The proof relies on a truncation argument, the spectral theory of automorphic forms, and an explicit description of the categoryof $(\mathfrak{g}, K)$-modules of covers of $SL_2(\mathbf{R})$.

Our main technical result is general, and we derive interpolation theorems for new spaces of functions. For example, the following is obtained by considering the so-called weight 2 discrete series representation:

Theorem 13 ([GGV])**.** *Let $\mathbf{P}^1(\mathbf{R})$, be the real projective line, with homogenous coordinates $[x : y]$. Then any $\Phi \in C^\infty(\mathbf{P}^1(\mathbf{R}))$ is entirely determined the following family of integrals:*

$$a_n(\Phi) = \int_{\mathbf{R}} \Phi(x,1) e^{\pi i n x} dx, \qquad b_n(\Phi) = \int_{\mathbf{R}} \Phi(1,y) e^{\pi i n y} dy.$$

2.3. **Future work.** Some natural directions of extension of this work include:

Extension to other groups and degrees of cohomology. Our initial strategy constructs an isomorphism reducing the (a priori analytic) computation of the cohomology of arithmetic groups to an (algebraic) computation in $(\mathfrak{g}, K)$ cohomology. Such a comparison theorem was proved for cocompact lattices by Olbrich-Bunke, for which it is harder to deduce interpolation theorems. Our current extension to non-cocompact lattices is very hands-on and does not extend beyond the first degree of cohomology. A goal would be to provide a general theorem giving comparisons of cohomologies, in all degrees, between group cohomology and $(\mathfrak{g}, K)$-cohomology.

Interpolation theorem for spaces of p-adic functions. A second direction would consider interpolation theorem for spaces of functions on p-adic manifolds, starting with Schwartz functions on the p-adic line $\mathbf{Q}_p$ which also carry a Weil representation, but also considering other naturally occurring spaces of functions. This project is exploratory – is likely that the relevant lattices would be finite covolume subgroups of $SL_2(\mathbf{R}) \times SL_2(\mathbf{Q}_p)$.

3. Gamma factors and a converse theorem mod ℓ

This work is motivated by the classical problem of *classifying the irreducible representations of* $GL_n(\mathbf{F}_q)$. We construct Langlands-inspired invariants and prove that they characterize mod ℓ cuspidal representations.

3.1. Background.

For any finite group G, the best-know complete invariants of irreducibls representations ρ are the characters χ_ρ, which are functions on its conjugacy classes. For $GL_n(\mathbf{F}_q)$, different invariants are available, which not only characterize the representation more efficiently than characters, but also situate the classification of irreducible representations of $GL_n(\mathbf{F}_q)$ in the framework of the Langlands program.

These invariants are the *gamma factors* $\gamma(\rho, \pi)$, introduced by Piatetski-Shapiro [PS83] for complex generic representations ρ of $GL_2(\mathbf{F}_q)$, with π a character of $GL_1(\mathbf{F}_q)$. The construction of $\gamma(\rho, \pi)$ is inspired by the representation theory of $GL_2(\mathbf{Q}_p)$: It is harmonic analytic in nature, and relies on the realization of ρ as a space of functions, its *Whittaker Model.* Piatetski-Shapiro then proved a *converse theorem* for cuspidal representations, the atomic building blocks from which other irreducible representations are built.

Theorem 14 ([PS83])**.** *A $\mathbf{C}$-valued cuspidal representation of $GL_2(\mathbf{F}_q)$ is determined by its gamma factors.*

Nien [Nie14] has defined gamma factors for cuspidal representations $GL_n(\mathbf{F}_q)$ and proved the best expected converse theorem, namely that a cuspidal representation of $GL_n(\mathbf{F}_q)$ is determined by $\gamma(\rho, \pi)$ for π running over representations of $GL_m(\mathbf{F}_q)$ for $m \leq n/2$. In particular, cuspidal representations are determined by $O(q^{n/2})$ data points instead of the $O(q^n)$ one gets with characters.

Beyond this economical characterization, gamma factors position representations of $GL_2(\mathbf{F}_q)$ in a broader picture. Indeed, they are the finite field version of invariants that can also be attached to Galois representations, and that characterize the local Langlands correspondance for $GL_n(\mathbf{Q}_p)$. The finite groups $GL_n(\mathbf{F}_q)$ admit a version of this correspondance, proved by Macdonald [Mac80], in which irreducible representations correspond to so-called tame inertial classes of representations of the Weil group. Representations of $GL_n(\mathbf{F}_q)$ give rise to so-called depth-zero representations of the corresponding p-adic groups, and Macdonald's correspondance is compatible with the usual local Langlands for depth zero representations.

3.2. Results.

In [BGGG+23], we consider coefficients in any characteristic $\ell \nmid q$; in particular, ℓ may divide $|GL_n(\mathbf{F}_q)|$. In this situation, many results from the complex case have been extended by Vigneras [Vig88]. Using these, we first show that one can extend the definitions of gamma factors. Then, exploiting their explicit description as Gauss sums in the case of $GL_2(\mathbf{F}_q)$, we compute using Sage that the converse theorem does not hold in general for $\bar{\mathbf{F}}_\ell$-valued representations.

Example 15. When $(\ell, q) = (2, 5)$, $(3, 7)$, $(11, 23)$, or $(29, 59)$, there are non-isomorphic mod ℓ cuspidal representations of $GL_2(\mathbf{F}_q)$ whose gamma factors agree.

This suggests that for mod ℓ representations, the right construction of gamma factors is not the naïve one. Since the construction of $\gamma(\rho, \pi)$ is harmonic-analytic, the natural place to look for new definitions is in different spaces of functions. We have done this in two ways:

- For GL_2, we define ℓ-regular gamma factors from the ℓ-regular Whittaker model of ρ: this is a space of functions on the subgroup of of $GL_2(\mathbf{F}_q)$ whose determinant in $\mathbf{F}_q^\times$ has order prime to ℓ.
- For GL_n with $n \geq 2$, we replace the Whittaker model of ρ valued in the constants $\bar{\mathbf{F}}_\ell$ by its analogue consisting of functions valued in a certain Artinian algebra $R(\pi)$ associated to π.

With these two modifications, we construct gamma factors of cuspidal representations and prove:

Theorem 16 ([BGGG+23])**.** *Let ρ be a mod ℓ cuspidal representation of $GL_n(\mathbf{F}_q)$.*

(i) If $n = 2$, a cuspidal representation ρ is entirely determined by its ℓ-regular γ-factors $\gamma_\ell(\rho, \pi)$, as π runs over irreducible representations of $GL_1(\mathbf{F}_q)$, i.e. by $O(q)$ invariants.

(ii) If $n \geq 2$, a cuspidal representation ρ is entirely determined by its $R(\pi)$-valued γ factors $\gamma(\rho, \pi)$, as π runs over irreducible representations of $GL_{n-1}(\mathbf{F}_q)$, i.e. by $O(q^{n-1})$ invariants.

The proof of the first statement relies on elementary techniques. For GL_n, the crux of the proof is establishing a completeness of $R(\pi)$-valued Whittaker models, i.e. a statement that there are enough functions in the Whittaker model to tell apart representations; this is reminiscent of linear independence of characters. The proofs are inspired by the p-adic construction: they rely on an analysis of auxiliary representations attached to ρ and π via the notion of Bernstein-Zelevinsky derivatives.

3.3. **Future Work.** Next possible projects in this direction aim at strengthening the parallels between our gamma factor for finite fields and representations of p-adic GL_N.

Jacquet's conjecture mod ℓ. We show that mod ℓ representations of $GL_n(\mathbf{F}_q)$ are determined by their gamma factors $\gamma(\rho, \pi)$ against representations of $GL_{n-1}(\mathbf{F}_q)$. However, it was conjectured by Jacquet, and proved by Nien [Nie14] for $\mathbf{C}$-valued representations, that it should suffice to consider representations π of $GL_m(\mathbf{F}_q)$ with $1 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor$. Obtaining a mod ℓ version of this result will likely rely on an extension of the Bernstein–Zelevinsky-type tools employed in the paper.

Characterization of local Langlands-type correspondances. The Macdonald correspondence [Mac80] described above preserves so-called Godement-Jacquet gamma factors, an different construction also inspired by the p-adic case. Vigneras [Vig94] constructed a correspondence for mod ℓ representations, and recently, Li-Shotton [LS23] proved a version in families, where the correspondence is realized as an isomorphism of algebras. A next phase in our investigation is to assert whether these various correspondences are characterized by equalities of our (Rankin-Selberg) gamma factors. The first step is to show that the gamma factors are multiplicative over parabolic induction.

4. Signs in Fourier coefficients of modular forms

4.1. **Background.** This project concerns classical modular forms of weight k on SL_2, which are the simplest types of automorphic forms. Such modular forms are given by a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n(f) n^{\frac{k-1}{2}} e^{\pi i n z},$$

where z belongs to the complex upper half-plane. Supposing additionally that f is a Hecke eigenform, the Fourier coefficients are multiplicative and encode the arithmetic properties of f, and in particular its connection to Galois representations. Asymptotic properties of the a_n have long been of interest: the Ramanujan conjecture proved by Deligne as the final part of his proof of the Weil conjectures, states that

$$|a_n(f)| \leq \tau(n), \tag{7}$$

where $\tau(n)$ is the number of positive divisors of n. We consider the question of alternating signs, or cancellation, between the Fourier coefficients, i.e. of the growth of the sum

$$S(f, x) = \sum_{n \leq x} a_n(f).$$

Much work has been devoted to bounding this sum, since that of Hecke who showed that $S(f, x) \ll_f x^{1/2}$. Here, the implicit constant depends on the specific modular form f, and the best bounds independent of x were proved by Lamzouri: under the assumption that the Generalized Riemann Hypothesis (GRH) for the L-function $L(s, f)$, he showed that

$$S(f, x) = o(x \log x) \qquad \text{as} \qquad \frac{\log x}{\log \log k} \to \infty.$$

4.2. **Results.** The work [FGGHT23] with Claire Frechette, Alia Hamieh, and Naomi Tanabe, proves that the partial sums $S(f,x)$ are $o(x \log x)$ in the range $x \geq k^\epsilon$, under assumptions both weaker and more explicit than those of Lamzouri. Indeed, instead of assuming the GRH, which states that all nontrivial zeros $L(f,s)$ lie on the line $\Re(s) = 1/2$, it exhibits a rectangle $R(T)$ disjoint from this line, and relates a too-rapid growth of $S(f,x)$ to the presence of many zeros in this region. A succinct form of our result is:

Theorem 17 ([FGGHT23]). *Let $f \in S_k(N)$ be a primitive cusp form. Let ϵ and T be real numbers with $\epsilon \geq (\log k)^{-1/8}$ and $1 \leq T \leq (\log k)^{1/200}$. Suppose that the region*

$$R(T) = \left\{ s \, : \, \Re(s) \geq \frac{3}{4}, \, |\Im(s)| \leq T + \frac{1}{4} \right\}$$

contains no more than $\epsilon^2 \log k/5000$ zeroes of $L(s,f)$. Then for all $x \geq k^\epsilon$, we have $|S(f,x)| \ll (x \log x)/T$.

Our proof relies on recent developments in analytic number theory pioneered by Granville–Soundararajn, whose analogous result [GS18] on character sums was the inspiration for the project. The framework, known as *pretentious number theory*, applies naturally to bounded functions. Recently, the key input, known as Halazs' Theorem was extended by Mangerel [Man23] to multiplicative functions satisfying (7).

The heart of the strategy is to compare $a_f(n)$ with the family of functions n^{it} for $t \in \mathbf{R}$: such functions see a minimal amount of cancellation, and for good values of t the twisted sums $S(f,x,t) = \sum_{n \leq x} a_f(n) n^{it}$ remain close enough to $S(f,x)$. This is useful because of the following principle:

$S(f,x,t)$ and t-translates of the L-function $L(f,s)$ are related by the Fourier transform.

This allows us to use Plancherel's formula to compare $S(f,x)$ and a suitable shift of $L(s,t)$. By assuming for contradiction that $S(f,x)$ is large, we first produce a lower bound involving $L(s+it,f)$ and then express $L(s+it,f)$ to in terms of its zeros using Hadamard's formula. Finally, by bounding the contribution of the zeros away from it, we deduce the presence of many zeros in $R(T)$.

4.3. **Future Work.** The methods utilized in this project are new, powerful, and promise to have applications in a wide range of situations. As an endgoal, we would like to extend the strategy to the cuspidal representations of $GL_n/\mathbf{Q}$ which satisfy the Ramanujan conjecture. A first project to test the robustness of the method is to bound sums of Rankin-Selberg products of modular forms.

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