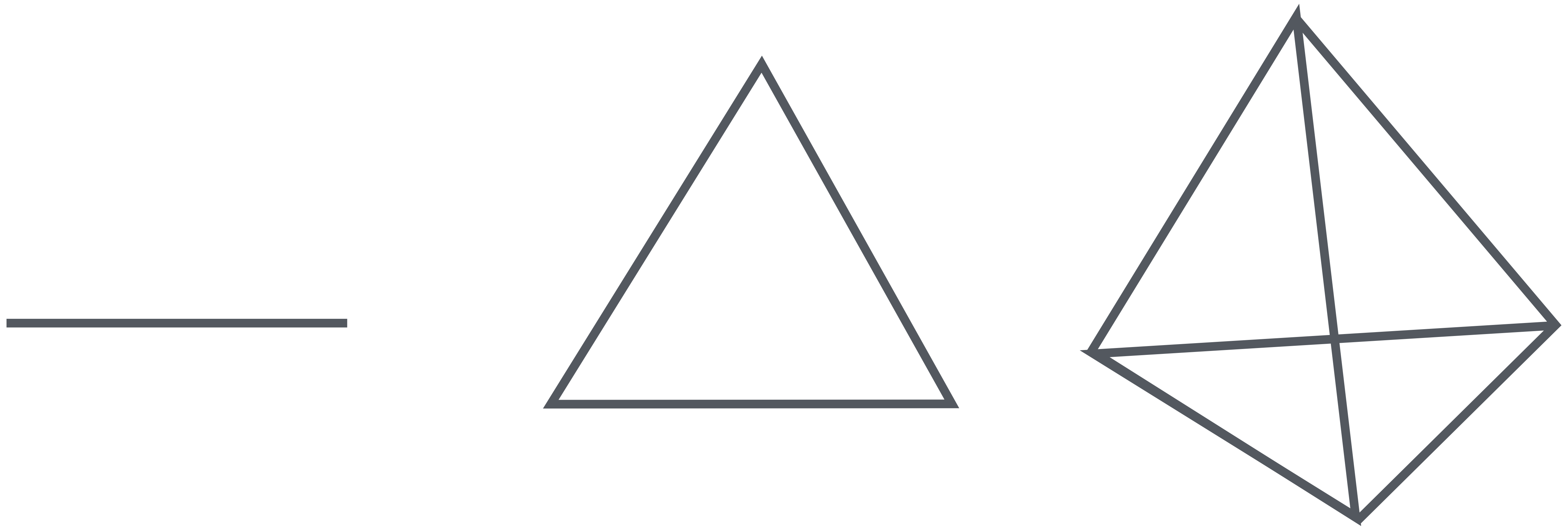
The mathematics of VoteKit

Voting Methods Modeling Workshop
Day 1 – Wednesday June 18, 2025 – Technical

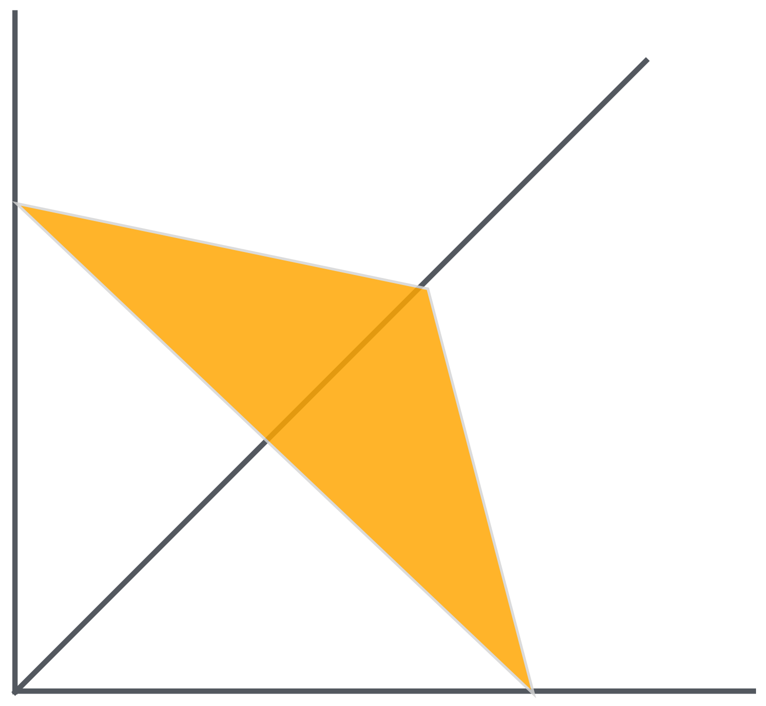
Simplices, coordinates, and measures

Simplex: polytope formed as convex hull of n points



Can form this in Euclidean space as the collection of convex combinations:

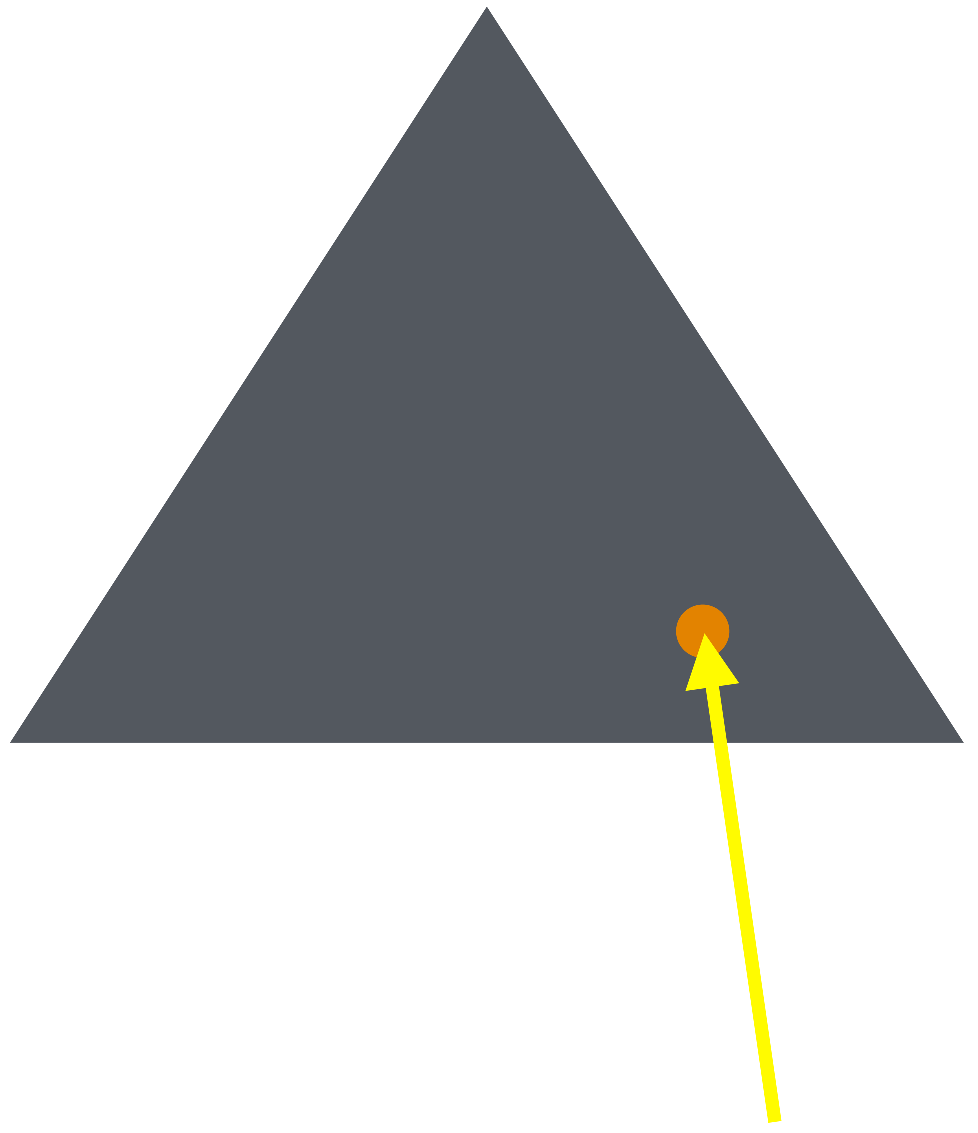
$$\left\{ a_1 v_1 + \ldots + a_n v_n : \sum a_i = 1 \right\}$$



$x+y+z=1,\ x,y,z \geq 0$

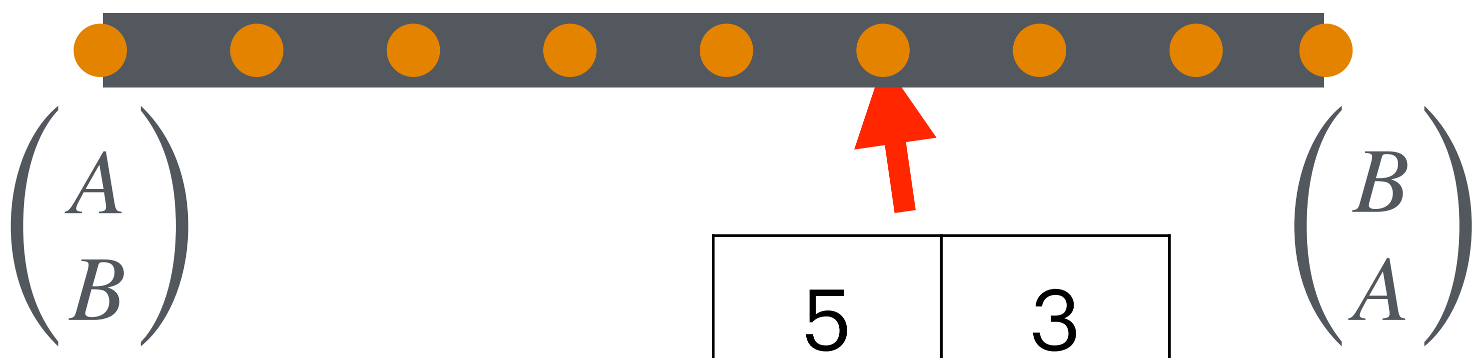
You can think of this as weighted averages of the points.

Standard (regular) n-dim simplex sits in $\mathbb{R}^{n+1}$ as slice of the first orthant, taking v_i as the standard (0,...,0,1,0,...,0)



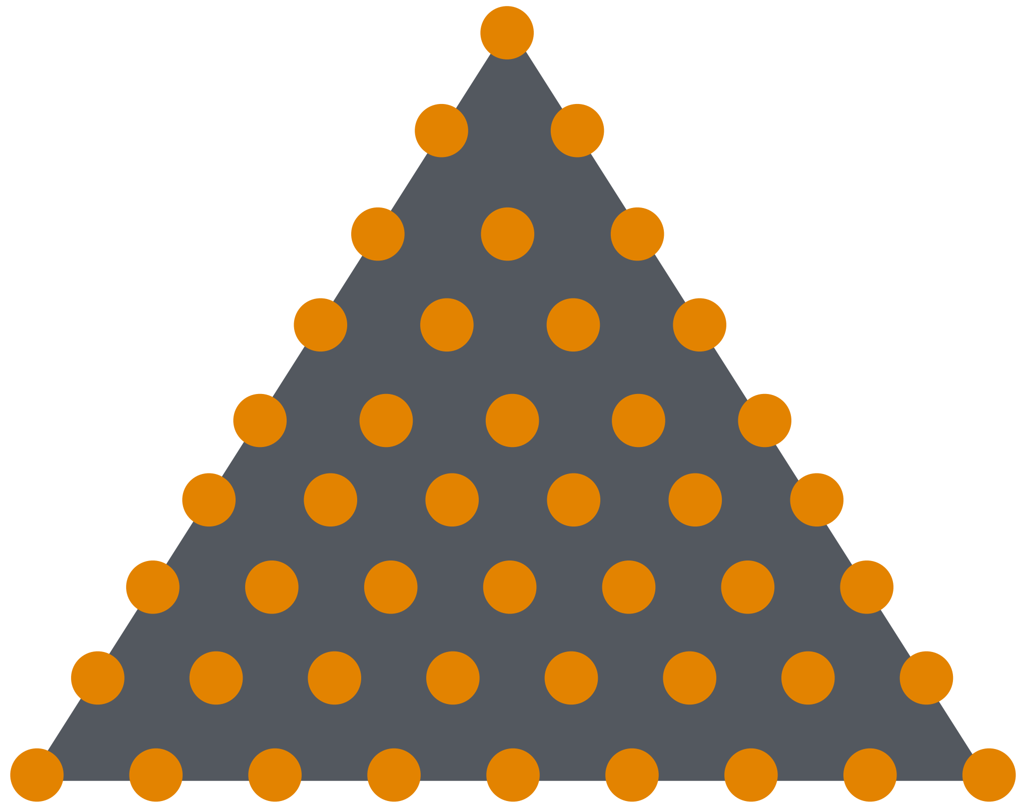
The coordinate construction should make it clear that every point in a simplex is a convex combination of the extreme vertices in a **unique** way, so there's a 1-1 correspondence between points in the simplex and probability vectors on the extreme points.

(vectors with coords summing to 1)

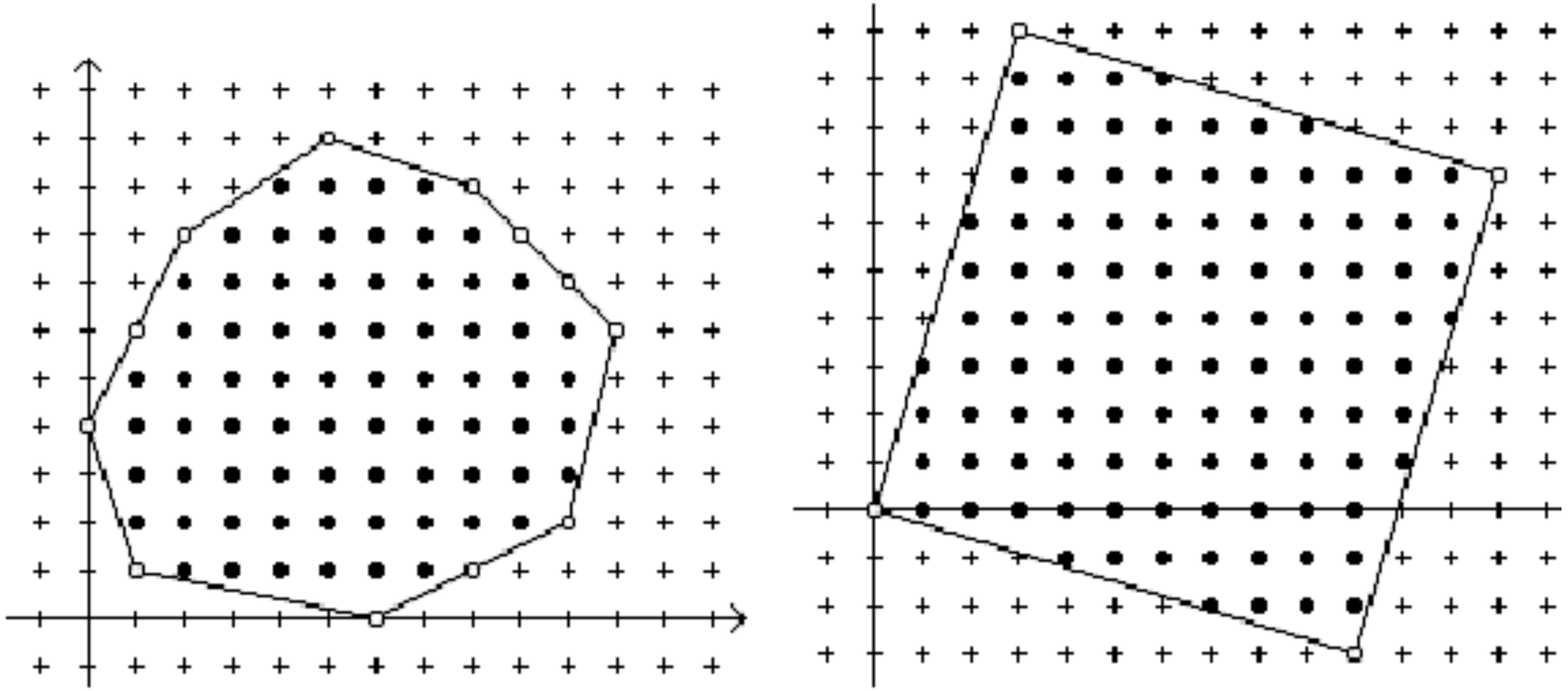


5	3
B	A
A	B

Next, observe that for a fixed number of voters, the set of possible profiles sits as a lattice in the simplex on the possible ballots. For example, this image shows 8 voters and two ballot types.



8 voters and
3 ballot types

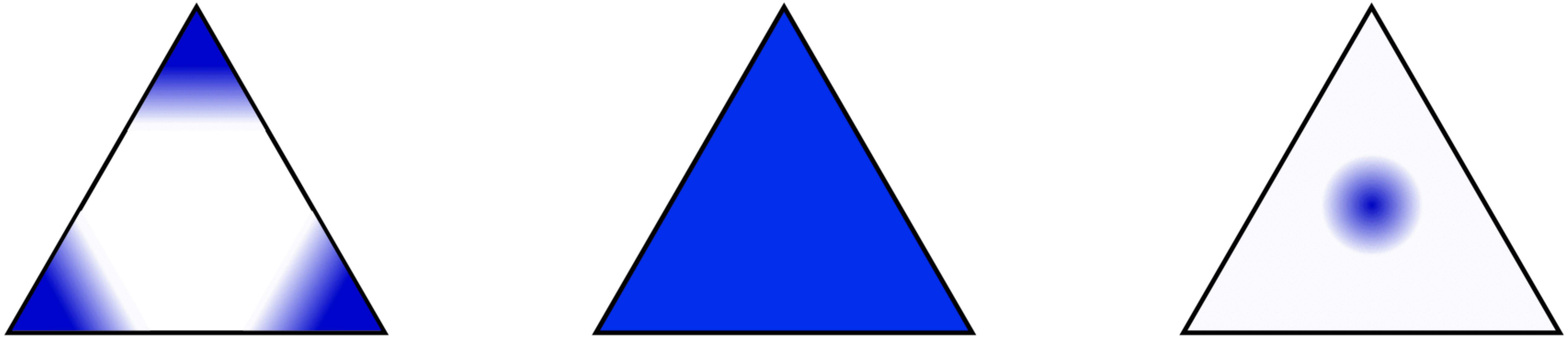


Classical fact (well studied by Gauss, Riemann, Sierpinski, ...)

When a lattice gets finer and finer, the count of lattice points in a region is proportional to its area/volume.

Dirichlet distributions

a family of measures on the simplex



$\alpha \to 0$ $\alpha = 1$ $\alpha \to \infty$

We'll use two ways:

- Dirichlet on **ballot** simplex will get us a probability on profiles (IC, IAC are special cases)
- Dirichlet on **candidate** smiplex will get us preference weights on candidates

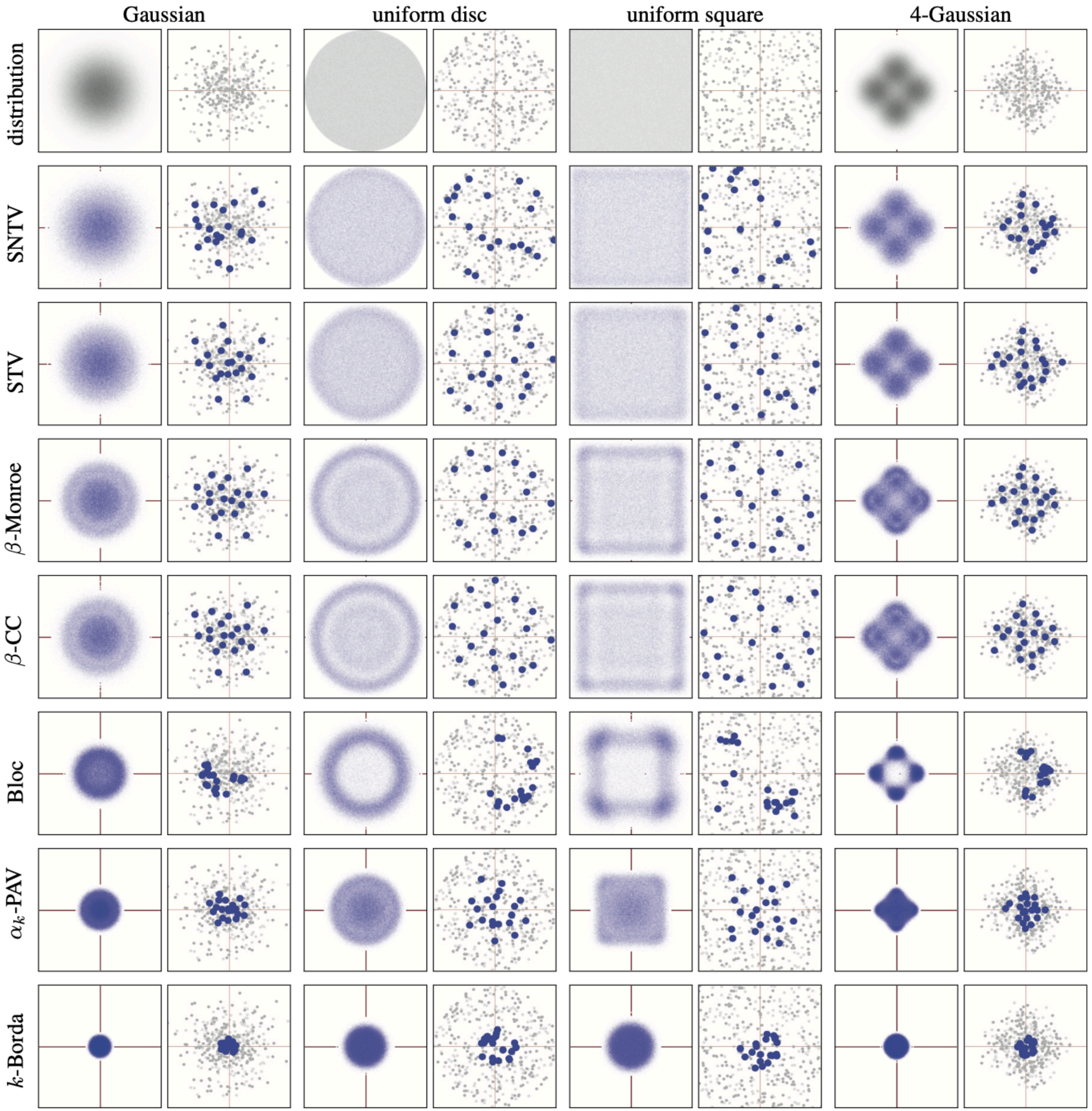
generative models

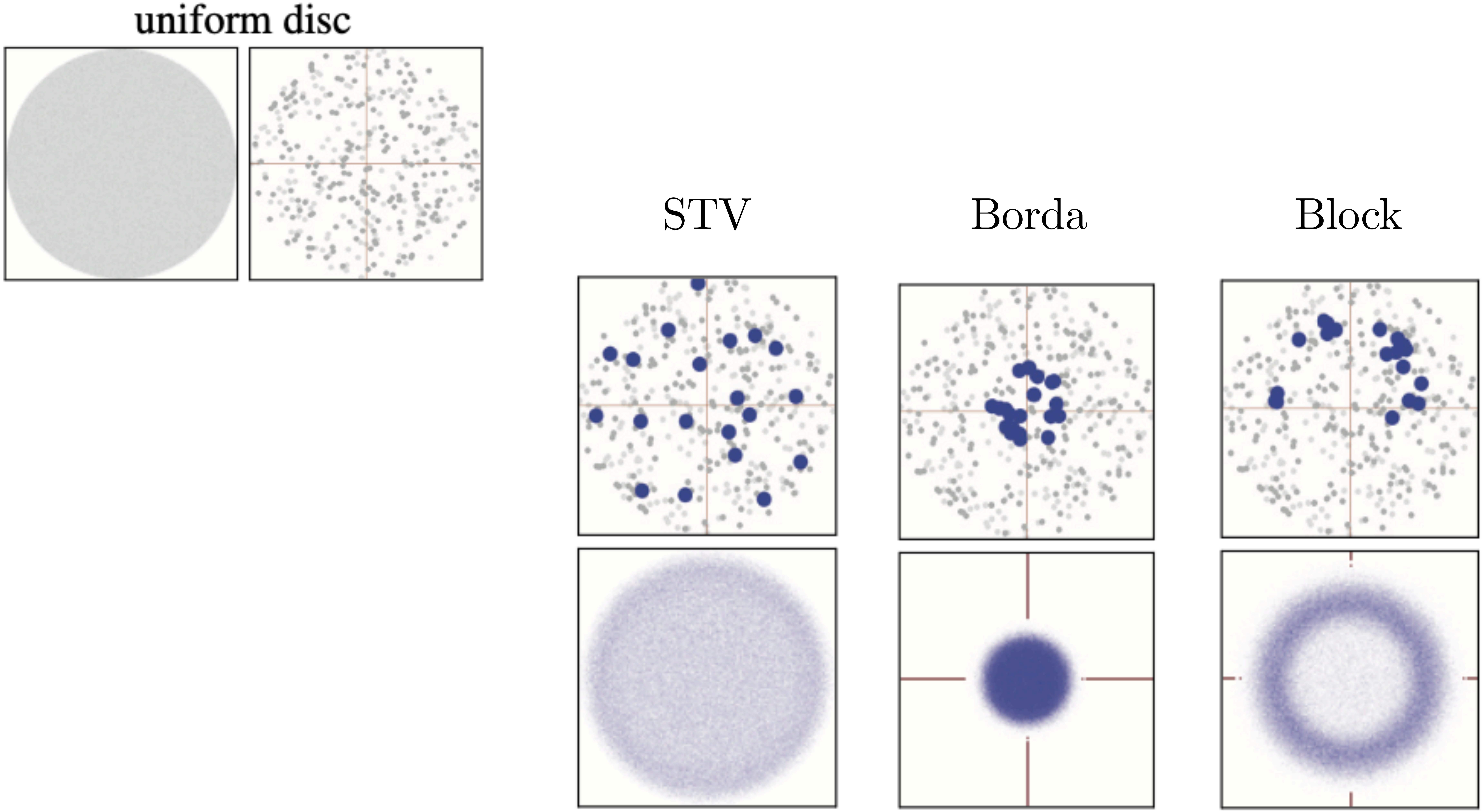
spatial models

imagine that the plane is an "issue space"

rankings are by proximity

Elkind et al.




uniform disc
STV
Borda
Block

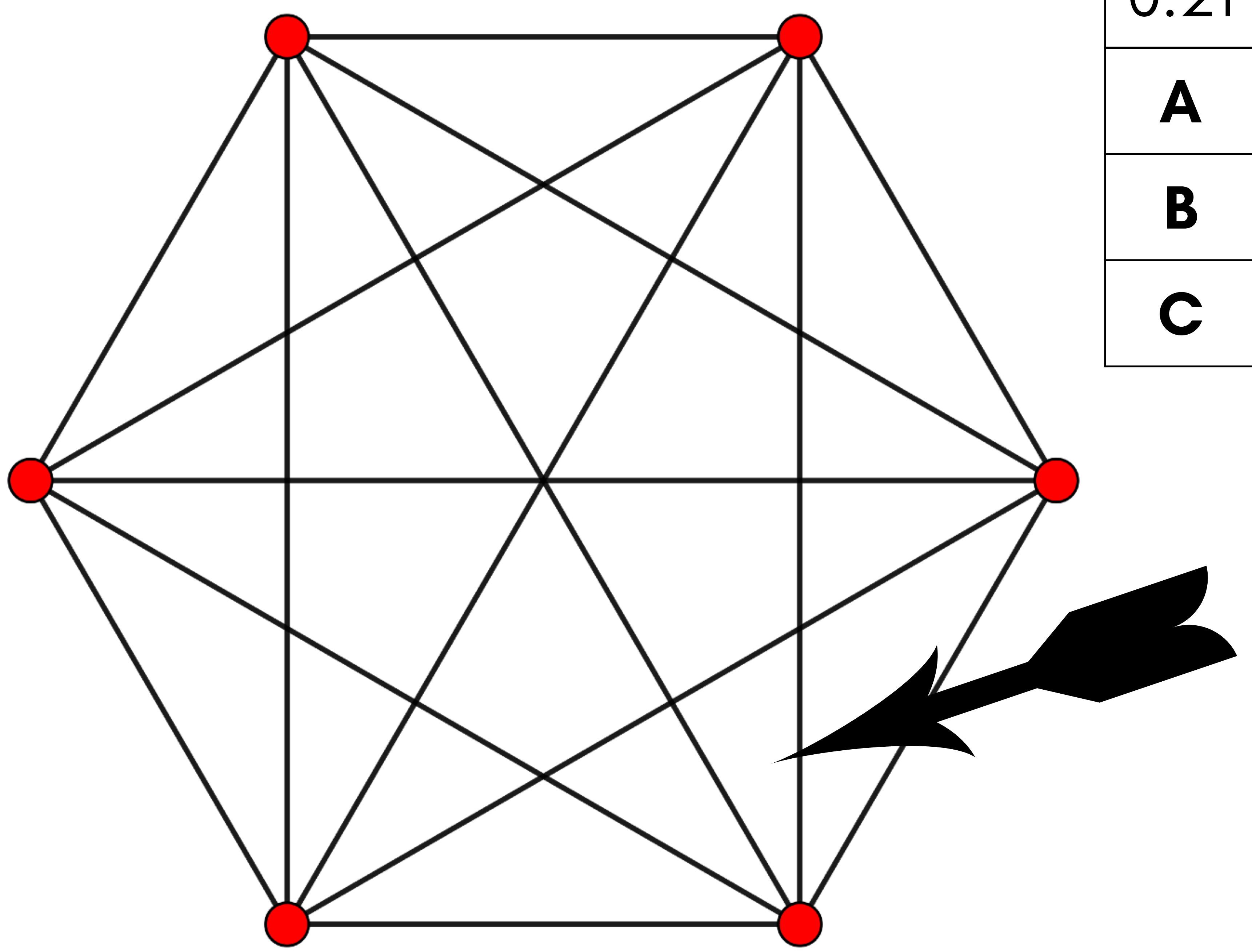
“Impartial Culture”

all permutations equally likely

0.16	0.17	0.16	0.17	0.17	0.16
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

“Impartial Anonymous Culture”

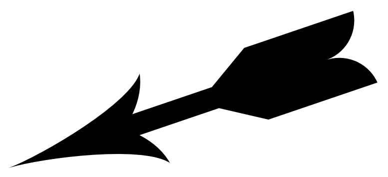
elections drawn by throwing darts



0.21	0.04	0.12	0.40	0.07	0.16
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

Tideman–Plassmann, *The Structure of the Election-Generating Universe*

None of the 11 models discussed so far are based on the belief that the associated distributions of P might actually describe rankings in actual elections. IAC, IC, UUP, DC, and UP assume that various components of p are equally likely, for the sake of algebraic tractability. $\mathrm{IAC}_b(k_b)$, $\mathrm{IAC}_t(k_t)$, $\mathrm{IAC}_c(k_c)$ and SSP seek to describe rankings that have meaningful interpretations for the problem of defining probabilities of observing Condorcet's paradox. The Borda and Condorcet models are rationalizations of claims about how one ought to determine the winner in an election.



By contrast, their realistic model is… spatial voting!

new models,
using classic ranking ideas

rankings from preference intervals

Plackett-Luce: voter fills i… …without replacement according to pref…

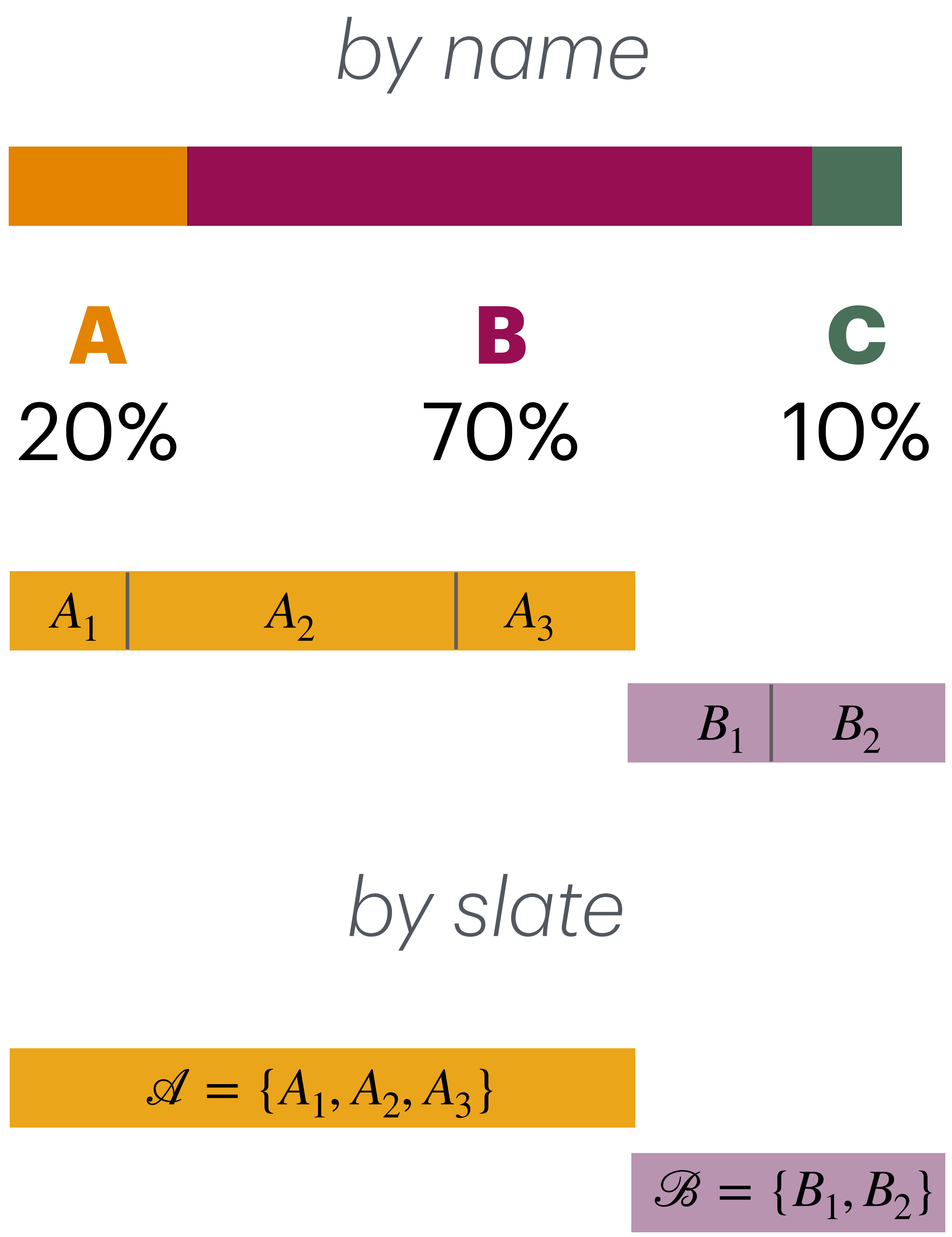
impulsive voter

Bradley-Terry: probability of ABC ballot i… …ional to

$$\left(\frac{\ell(A)}{\ell(A)+\ell(B)}\right)\left(\frac{\ell(A)}{\ell(A)+\ell(C)}\right)\left(\frac{\ell(B)}{\ell(B)+\ell(C)}\right)$$

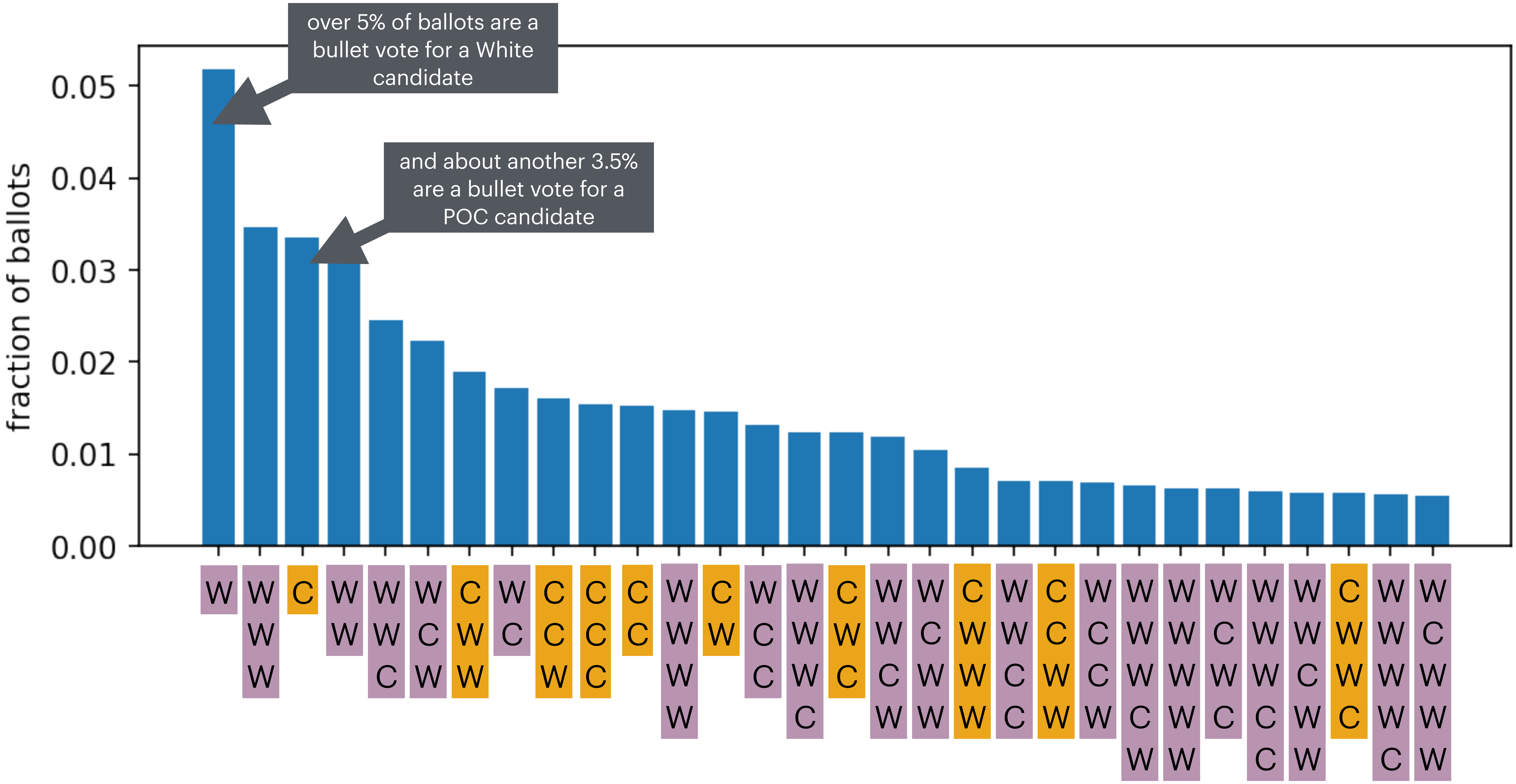
deliberative voter

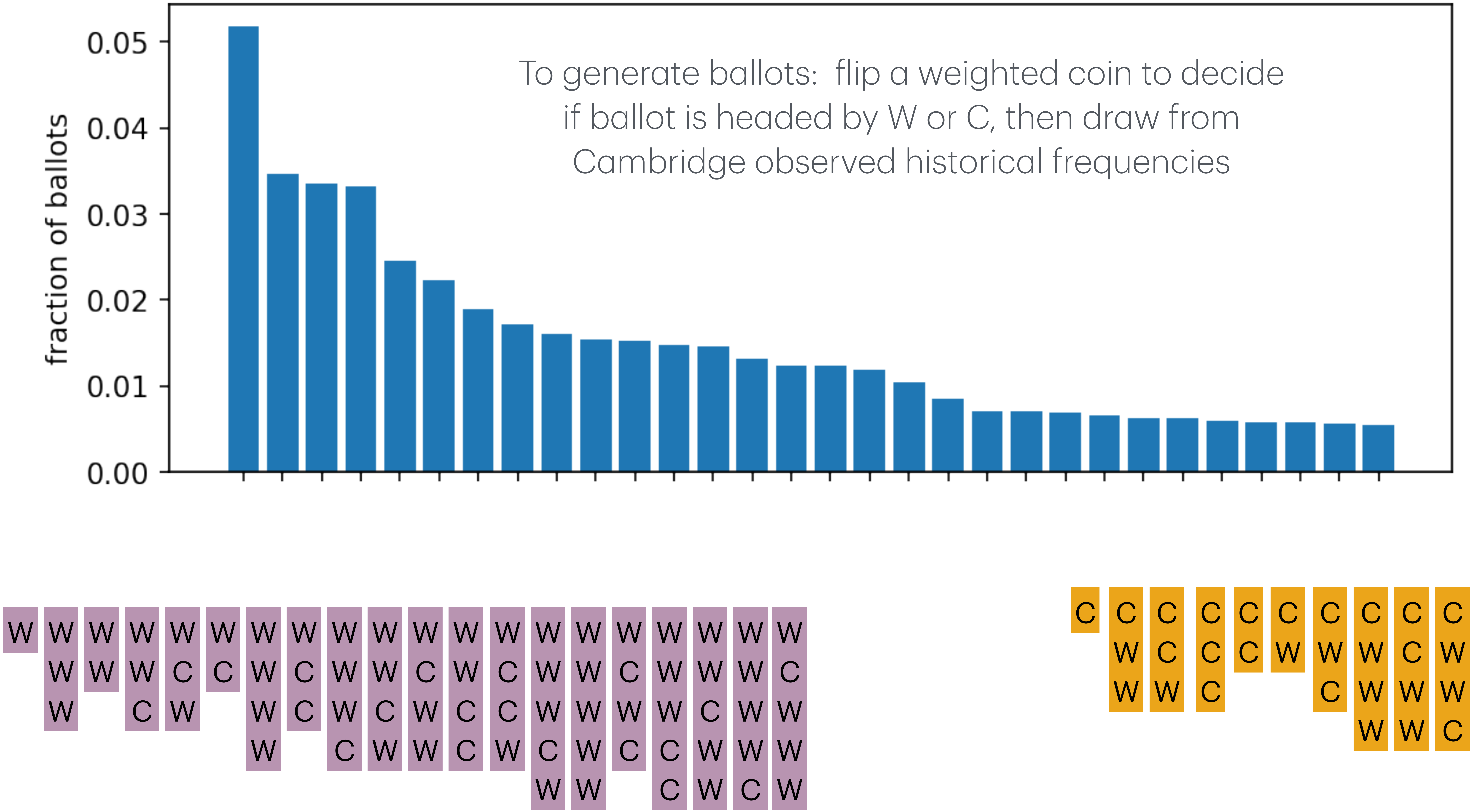
(product of prob… …pairwise comparisons A>B, A>C, B>C)



Cambridge Sampler

based on actual Cambridge, MA ballots




To generate ballots: flip a weighted coin to decide if ballot is headed by W or C, then draw from Cambridge observed historical frequencies
fraction of ballots
0.05
0.04
0.03
0.02
0.01
0.00
W
W W W
W W
W W C
W C W
W C
W W W W
W C C
W W W C
W W C W
W C W W
W W C C
W C C W
W W W C W
W W W W W
W C W C
W W W C C
W W C W W
W W W W C
W C W W W
C
C W W
C C W
C C C
C C
C W
C W C
C W W W
C C W W
C W W C

extras

Toward model validation