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Analysis of degenerate Burgers' equations involving small perturbation and large wave amplitude

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We only focus on the shock waves to Burgers system by considering the porous medium diffusion and singularity in energy estimates. In this paper, the conditions $u_+ = 0$ and $D \geq \mu$ are considered. Moreover, we establish the existence of shock waves through the phase plane. The weighted function is employed to handle the singularity in the weighted energy estimates under small perturbations and arbitrary wave amplitudes. Finally, the weighted energy estimates are then used to prove the stability of shock waves.

KEYWORDS

arbitrary wave amplitude, Burgers' system, porous medium diffusion, singularity, small perturbation, stability

MSC CLASSIFICATION

35A01, 35B40

1 | INTRODUCTION

We consider the following one-dimensional degenerate Burgers' system,

$$\begin{cases} u_t + (uv)_x = D(u^m)_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ v_t + \frac{1}{2}\left(u^2 + v^2\right)_x = \mu(v^m)_{xx}, & x \in \mathbb{R}, \quad t > 0, \end{cases} \tag{1.1}$$

where $(u, v) = (u, v)(x, t)$, $m > 0$, and the initial data

$$(u, v)(0, x) = (u_0, v_0)(x) \to \begin{cases} (u_+, v_+) & \text{as } x \to +\infty, \\ (u_-, v_-) & \text{as } x \to -\infty. \end{cases} \tag{1.2}$$

Moreover, $u(x, t)$ and $v(x, t)$ represent the magnetic field and velocity field of the fluid, respectively, and D and μ are positive diffusivity.

System (1.1) is continuation from the following one-dimensional magnetohydrodynamic Burgers' system,

$$\begin{cases} u_t + (uv)_x = Du_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ v_t + \frac{1}{2}\left(u^2 + v^2\right)_x = \mu v_{xx}, & x \in \mathbb{R}, \quad t > 0, \end{cases} \tag{1.3}$$

which was derived in earlier studies [1, 2] to study the small-scale structure of magnetohydrodynamic turbulence. System (1.3) is the simplest possible system which allows to interchange between magnetic and fluid energies, where an ensemble of Alfvenic shock waves on a homogeneous density background represents the magnetohydrodynamic turbulence. Moreover, the opposite limit of a fluid-dominated (i.e., unmagnetized) system can be represented by system (1.3), where density variations react to an adiabatic pressure [3]. We refer the readers to other works [4–6] for more applications of magnetohydrodynamic Burgers' system (1.3).

Other studies of shock wave solutions to Burgers' system were extensively investigated in previous research [7–11] by considering the diffusionless and square root Fisher,

$$\begin{aligned} & u_t + \alpha_1 \sqrt{u} u_x = D_1 u_{xx}, \\ & u_t + \alpha_2 \sqrt{u} u_x = \beta_1 \sqrt{u} - \beta_2 u, \end{aligned} \tag{1.4}$$

where $\alpha_1 > 0, \alpha_2 > 0, D_1 > 0, \beta_1 > 0, \beta_2 > 0$.

In the current paper, we establish the existence and the stability of shock wave solutions (1.1) with porous medium diffusion. Moreover, we also consider the singularity for the current paper and then establish the weighted energy estimates of shock waves solutions (1.1) by dealing with large wave amplitude and small perturbation through the strategies studied in other studies [12, 13]. The elementary energy method can also be used to establish the stability problem of shock wave solutions for the following coupled Burger's equation without the porous medium diffusion [14], where the smallness assumption on the wave amplitude and the coefficients was not required.

Moreover, Li and Wang [15] studied the stability of shock waves to the following chemotaxis model under small perturbation and large wave strength,

$$\begin{aligned} & u_t - (uv)_x = Du_{xx}, \\ & v_t + (\varepsilon v^2 - u)_x = \varepsilon v_{xx}, \end{aligned} \tag{1.5}$$

which was derived from Keller-Segel and the smallness of coefficients ε was considered. The research studied by Li and Wang [15] was distinct with the study in Hu [14] which does not require the smallness of coefficients. Moreover, we suggest the readers to other studies [16–18] which study the stability analysis by the elementary energy method. Some references were related to (1.5); we refer to earlier studies [19, 20].

The main issues of this paper are $u_+ = 0$ and porous medium diffusion of $D(u^m)_{xx}$ and $\mu(v^m)_{xx}$ which are the challenge to study. The purpose of this paper is to provide a first attempt to break down these barriers and prove the existence and stability of shock waves to system (1.1). Moreover, we employ the same strategy to study the porous medium diffusion as in Ghani et al. [21]. The porous medium problem was also studied in Choi and Kim [22] for the existence of chemotactic traveling waves with compact support

$$\begin{aligned} & r_t = \left(\gamma(s,r)\left(r_x - \frac{r}{s}s_x\right)\right), \\ & s_t = -\omega(s,r)r, \end{aligned} \tag{1.6}$$

where r represents population density of bacteria, s represents resource (food) density that bacteria consume. Moreover, $\gamma(s,r)$ and $\omega(s,r)$ represent diffusivity and consumption rate, respectively, which are defined for both as follows:

$$\begin{aligned} & \gamma(s,r) = \frac{p}{s^2}\left(\frac{r}{s}\right)^{p-1}, \quad p > \delta \\ & \omega(s,r) = \alpha(s) r^{-\delta}, \quad 0 < \delta < 1. \end{aligned} \tag{1.7}$$

This paper is organized as follows. In Section 2, we present the existence of shock wave solution of U, V to the Burgers' system (1.1) and derive the appropriate perturbations. Section 3 establishes the a priori estimate by using the weighted energy estimates. In Section 4, we prove the stability of shock waves to Burgers' system (1.1).

Remark 1.1. In this paper, we consider the nonlinear diffusion $D(u^m)_{xx}$ and $\mu(v^m)_{xx}$ for $m > 0$ and $D \geq \mu$. This paper is different with the previous study in Jin et al.[3] which consider $m = 1$. The challenge of this paper is the presence of degeneracies if $m > 1$ and singularities if $0 < m < 1$. The singularities come from the presence of the term $\frac{1}{U(z)}$ for $u_+ = 0$ as $z \to +\infty$, where this case can be handled by introducing the weighted function in Remark 1.2 to provide the weighted energy estimates as in [13]. Moreover, the strategy in (3.11) is also introduced to handle the higher-order term estimates.

In this paper, the norms of Sobolev space $H^r(\mathbb{R})$ are written as $\|p\|_r := \sum_{k=0}^{r} \|\partial_x^k p\|$ and $\|p\| := \|p\|_{L^2(\mathbb{R})}$. Furthermore, the norms of weighted Sobolev space $H_w^r(\mathbb{R})$ are given as $\|p\|_{r,w} := \sum_{k=0}^{r} \|\sqrt{w(x)}\partial_x^k p\|$ and $\|p\|_w := \|p\|_{L_w^2(\mathbb{R})}$.

2 | PRELIMINARIES AND MAIN RESULTS

2.1 | Existence of shock waves

We first establish the shock wave $(U,V)(x-st)$ of the Burgers' system (1.1) by substituting the following shock wave ansatz

$$(u,v)(x,t) = (U,V)(z), \quad z = x - st, \tag{2.1}$$

into (1.1), where s and z are wave speed and moving coordinate, respectively, then one has

$$\begin{cases} -sU_z + (UV)_z = D(u^m)_{zz}, \\ -sV_z + \frac{1}{2}\left(U^2 + V^2\right)_z = \mu(v^m)_{zz} \end{cases} \tag{2.2}$$

and the boundary conditions

$$(U,V)(z) \to (u_\pm, v_\pm) \text{ as } z \to \pm\infty. \tag{2.3}$$

Then, integrating (2.2) with respect to z, one has

$$\begin{cases} -sU + (UV) + F_1 = D(u^m)_z, \\ -sV + \frac{1}{2}\left(U^2 + V^2\right) + F_2 = \mu(v^m)_z, \end{cases} \tag{2.4}$$

where F_1 and F_2 are constants satisfying

$$\begin{aligned} F_1 &= su_- - u_-v_- = su_+ - u_+v_+, \\ F_2 &= sv_- - \frac{1}{2}\left(u_-^2 + v_-^2\right) = sv_+ - \frac{1}{2}\left(u_+^2 + v_+^2\right). \end{aligned} \tag{2.5}$$

By dealing with the fact $(U_z, V_z) \to 0$ as $z \to \pm\infty$, then Rankine–Hugoniot condition is obtained as follows:

$$\begin{cases} s(u_+ - u_-) = u_+v_+ - u_-v_-, \\ s(v_+ - v_-) = \frac{1}{2}\left(u_+^2 - u_-^2\right) + \frac{1}{2}\left(v_+^2 - v_-^2\right), \end{cases} \tag{2.6}$$

and (2.6) under the fact $u_+ = 0$ gives the following quadratic equation of s,

$$s^2 - sv_- = 0. \tag{2.7}$$

Without loss of generality, we only consider

$$s = v_-, \tag{2.8}$$

and the following conditions are satisfied

$$0 = u_+ < u_-, \;\; v_+ < v_-. \tag{2.9}$$

Then, employing the classical phase-plane analysis, one can prove the following existence of shock wave solutions to system (1.1).

Proposition 2.1. *Let $u_\pm$ and $v_\pm$ satisfy (2.6). Then, one has a monotone shock wave solution $(U,V)(x-st)$ to (2.2) satisfying $U_z<0$ and $V_z<0$, where the wave speed s is given by (2.8). Moreover, (U,V) decays exponentially as $z\to\pm\infty$ with rates*

$$U-u_\pm\sim e^{\lambda_\pm z},\ \ as\ \ z\to\pm\infty,\ V-v_\pm\sim e^{\lambda_\pm z},\ \ as\ \ z\to\pm\infty, \tag{2.10}$$

where

$$\lambda_+=\frac{v_+^{1-m}(v_+-s)}{\mu m},\quad \lambda_-=\frac{1}{m}\sqrt{\frac{u_-^{3-m}v_-^{1-m}}{D\mu}}. \tag{2.11}$$

Proof. We first establish the existence of shock waves $(U,V)(x-st)$ of system (1.1). Then, it follows from (2.4), one has

$$\begin{aligned} U_z&=\frac{U^{1-m}}{Dm}(-sU+(UV)+F_1),\\ V_z&=\frac{V^{1-m}}{\mu m}\left(-sV+\frac{1}{2}\left(U^2+V^2\right)+F_2\right). \end{aligned} \tag{2.12}$$

Noting that system (2.12) has only two stationary points (u_+,v_+) and (u_-,v_-). Then, system (2.12) at $(u_\pm,v_\pm)$ is linearized to gain

$$\begin{bmatrix} U-u_\pm\\ V-v_\pm\end{bmatrix}_z=\left[J(U,V)\right]\begin{bmatrix} U-u_\pm\\ V-v_\pm\end{bmatrix}. \tag{2.13}$$

$[J(U,V)]$ is the Jacobian matrix which consists of some entries: $a_{11}=(1-m)\frac{U^{-m}}{Dm}(-sU+UV+F_1)+\frac{U^{1-m}}{Dm}(-s+V)$, $a_{12}=\frac{U^{2-m}}{Dm}$, $a_{21}=\frac{UV^{1-m}}{\mu m}$, and $a_{22}=(1-m)\frac{V^{-m}}{\mu m}\left(-sV+\frac{1}{2}\left(U^2+V^2\right)+F_2\right)+\frac{V^{1-m}}{\mu m}(-s+V)$.

By substituting points $(u_\pm,v_\pm)$ and employing $F_1=su_\pm-u_\pm v_\pm$ and $F_2=sv_\pm-\frac{1}{2}(u_\pm^2+v_\pm^2)$, then one has

$$\left[J(U,V)\right]_{(u_\pm,v_\pm)}=\begin{bmatrix} \frac{u_\pm^{1-m}}{Dm}(-s+v_\pm) & \frac{u_\pm^{2-m}}{Dm}\\ \frac{u_\pm v_\pm^{1-m}}{\mu m} & \frac{v_\pm^{1-m}}{\mu m}(-s+v_\pm)\end{bmatrix}, \tag{2.14}$$

which has the characteristic equation as follows:

$$\lambda^2-\frac{\mu u_\pm^{1-m}+Dv_\pm^{1-m}}{D\mu m}(-s+v_\pm)\lambda+\frac{u_\pm^{1-m}v_\pm^{1-m}\left((-s+v_\pm)^2-u_\pm^2\right)}{D\mu m^2}=0, \tag{2.15}$$

for two real roots λ_1 and λ_2. Then, it follows from (2.15), we have

$$\begin{aligned} \lambda_1+\lambda_2|_{(u_\pm,v_\pm)}&=\frac{\mu u_\pm^{1-m}+Dv_\pm^{1-m}}{D\mu m}(-s+v_\pm)<0,\\ \pm\lambda_1\lambda_2|_{(u_\pm,v_\pm)}&=\frac{u_\pm^{1-m}v_\pm^{1-m}\left((-s+v_\pm)^2-u_\pm^2\right)}{D\mu m^2}<0, \end{aligned} \tag{2.16}$$

which imply that (u_+,v_+) is a stable point and (u_-,v_-) is a saddle point.

Next, we claim the following region Ω as the solution starting from (u_-,v_-) in the plane (U,V),

$$\Omega:\begin{cases} N_1(U,V):=\frac{U^{1-m}}{Dm}(-sU+UV+F_1)<0,\\ N_2(U,V):=\frac{V^{1-m}}{\mu m}\left(-sV+\frac{1}{2}\left(U^2+V^2\right)+F_2\right)<0,\\ u_+<U<u_-,v_+<V<v_-. \end{cases} \tag{2.17}$$

Now, we differentiate $N_1(U,V)$ and $N_2(U,V)$ with respect to U and V to get

$$\begin{aligned}
\frac{\partial N_1}{\partial U} &= (1-m)\frac{U^{-m}}{Dm}(-sU+UV+F_1)+\frac{U^{1-m}}{Dm}(-s+V),\\
\frac{\partial N_1}{\partial V} &= \frac{U^{2-m}}{Dm},\\
\frac{\partial N_2}{\partial U} &= \frac{UV^{1-m}}{\mu m},\\
\frac{\partial N_2}{\partial V} &= (1-m)\frac{V^{-m}}{\mu m}\left(-sV+\frac{1}{2}\left(U^2+V^2\right)+F_2\right)+\frac{V^{1-m}}{\mu m}(-s+V),
\end{aligned}$$

which gives the tangent directions of nullclines $N_i(U,V)=0, \quad i=1,2$

$$\left(\frac{dV}{dU}\right)^{N_1}_{(u_-,v_-)}=\frac{s-v_-}{u_-},\quad \left(\frac{dV}{dU}\right)^{N_2}_{(u_-,v_-)}=\frac{u_-}{s-v_-}. \tag{2.18}$$

Next, we need to show that the following inequality holds

$$\frac{s-v_-}{u_-}<\left(\frac{dV}{dU}\right)_{(u_-,v_-)}<\frac{u_-}{s-v_-}. \tag{2.19}$$

Obviously, it suffices to consider the positive eigenvalue of (2.15) for (u_-,v_-)

$$\lambda=\frac{\mu u_-^{1-m}+Dv_-^{1-m}}{D\mu m}(v_- - s)+\frac{1}{2}\sqrt{\frac{(v_- - s)^2(\mu u_-^{1-m}-Dv_-^{1-m})^2}{D^2\mu^2m^2}+\frac{4u_-^{3-m}v_-^{1-m}}{D\mu m^2}},$$

which has the eigenvector

$$\left(\lambda-\frac{u_-^{1-m}}{Dm}(v_- - s),\frac{u_-^{2-m}}{Dm}\right)\quad\text{or}\quad\left(\frac{u_-v_-^{1-m}}{\mu m},\lambda-\frac{v_-^{1-m}}{\mu m}(v_- - s)\right).$$

It can easily be checked that

$$\frac{s-v_-}{u_-}<\frac{\frac{u_-^{2-m}}{Dm}}{\lambda-\frac{u_-^{1-m}(v_- - s)}{Dm}}=\frac{\lambda-\frac{v_-^{1-m}(v_- - s)}{\mu m}}{\frac{u_-v_-^{1-m}}{\mu m}}<\frac{u_-}{s-v_-},$$

which completes the claim stated before. Figure 1 represents the phase portrait of (2.17) in the region Ω, where the blue one is the point $(u_-,v_-)=(1.63,1.63)$ and the red one is the point $(u_+,v_+)=(0.37,0.37)$. By employing the simple algebra, one can check that $N_1(U,V)<0$ (represents the direction of nullclines goes left towards the region Ω) on $N_2(U,V)=0$ and $N_2(U,V)<0$ (represents the direction of nullclines goes downward towards the region Ω) on $N_1(U,V)=0$ as in Figure 1. Then, those imply that $U_z<0, V_z=0$ on $N_2(U,V)=0$ and $V_z<0, U_z=0$ on $N_1(U,V)=0$. Moreover, the asymptotic decay rate can be easily proved by solving (2.15) directly at the points $(u_\pm,v_\pm)$. □

Remark 1.2. Since the singularity is considered in this paper, we cannot apply the technique used in Li and Wang [16] directly. We introduce the weighted function to handle the singular term and establish the weighted energy estimate as in Li et al. [13]. We further present the following weighted function

$$w(z)=1+e^{\eta z}\quad\text{where}\quad \eta=\frac{1}{m}\sqrt{\frac{\left(u_-^2v_-\right)^{1-m}}{D\mu}}\quad\text{for all }\ z\in\mathbb{R}. \tag{2.20}$$

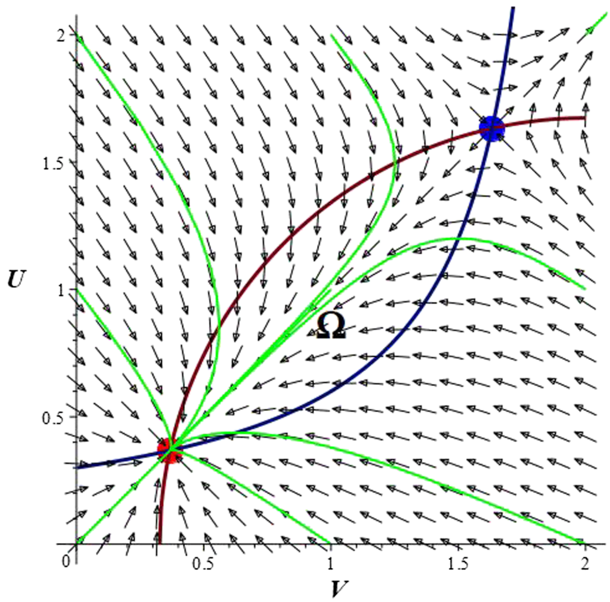

FIGURE 1 Phase portrait of $N_1(U,V)=0$ and $N_2(U,V)=0$ with the parameters $s=2, c_1=c_2=0.6$. [Colour figure can be viewed at wileyonlinelibrary.com]

Based on the weighted function defined in (2.20), we can find two constants $\beta > \alpha > 0$ such that

$$\alpha w(z) \leq \frac{1}{U(z)} \leq \beta w(z) \ \text{ for all } \ z \in \mathbb{R}. \tag{2.21}$$

2.2 | Reformulation of the problems

Next, we define the following perturbation for the Burgers' system (1.1)

$$(\zeta_0, \psi_0)(z) = \int_{-\infty}^{z} (u_0 - U, v_0 - V)(y)dy,$$

which is the zero mass perturbation (see [23, 24]). Then we have the following stability result.

Theorem 2.1. *Let $(U,V)(x-st)$ be the shock wave obtained in Proposition 2.1. Then there exists a constant $\varepsilon_0 > 0$ such that if $\|u_0 - U\|_{1,w} + \|v_0 - V\|_{1,w} + \|(\zeta_0, \psi_0)\|_w \leq \varepsilon_0$, then the Cauchy problems (1.1) and (1.2) have a unique global solution $(u,v)(x,t)$ satisfying*

$$(u-U, v-V) \in C\left([0,\infty); H_w^1\right) \cap L^2\left([0,\infty); H_w^1\right)$$

and

$$\sup_{x\in\mathbb{R}} |(u,v)(x,t) - (U,V)(x-st)| \to 0 \ \ \textit{as} \ \ t \to +\infty.$$

Changing the variables $(x,t) \to (z = x - st, t)$, the Burgers' system (1.1) becomes

$$\begin{cases} u_t - su_z + (uv)_z = D(u^m)_{zz}, \\ v_t - sv_z + \frac{1}{2}\left(u^2 + v^2\right)_z = \mu(v^m)_{zz}. \end{cases} \tag{2.22}$$

By constructing the solution (u,v) of (2.22),

$$(u,v)(z,t) = (U,V)(z) + (\zeta_z, \psi_z)(z,t). \tag{2.23}$$

We further have

$$(\zeta,\psi)(z,t)=\int_{-\infty}^{z}(u(y,t)-U(y),v(y,t)-V(y))dy. \tag{2.24}$$

By substituting (2.23) into (2.22) and integrating the resultant equation with respect to z, one has

$$\begin{cases}\zeta_t=(DmU^{m-1}\zeta_z)_z+G_1+(s-V)\zeta_z-U\psi_z-\zeta_z\psi_z,\\ \psi_t=(\mu mU^{m-1}\psi_z)_z+G_2+(s-V)\psi_z-U\zeta_z-\frac{1}{2}\left(\zeta_z^2+\psi_z^2\right),\end{cases} \tag{2.25}$$

where $G_1=\left(D(\zeta_z+U)^m-U^m-DmU^{m-1}\zeta_z\right)_z$, $G_2=\left(\mu(\psi_z+V)^m-V^m-\mu mU^{m-1}\psi_z\right)_z$ and initial values of (ζ,ψ)

$$(\zeta,\psi)(z,0)=(\zeta_0,\psi_0)(z)=\int_{-\infty}^{z}(u_0-U,v_0-V)dy \tag{2.26}$$

with $(\zeta_0,\psi_0)(\pm\infty)=0$. We further find the solution of reformulated problems (2.25) and (2.26) in the space

$$X(0,T):=\left\{(\zeta,\psi)(z,t)\in C([0,T),H_w^2):\zeta_z\in L^2\left((0,T);H_w^2\right),\psi_z\in L^2\left((0,T);H_w^2\right)\right\}$$

with $0<T\leq+\infty$. Let

$$N(t):=\sup_{0\leq\tau\leq t}\left\{\|\zeta(.,\tau)\|_{2,w}+\|\psi(.,\tau)\|_{2,w}\right\}.$$

From the Sobolev inequality $\|f\|_{L^\infty}\leq\sqrt{2}\|f\|_{L_w^2}^{\frac{1}{2}}\|f_x\|_{L_w^2}^{\frac{1}{2}}$, it follows that

$$\sup_{\tau\in[0,t]}\{\|\zeta(\cdot,\tau)\|_{L^\infty},\|\zeta_z(\cdot,\tau)\|_{L^\infty},\|\psi(\cdot,\tau)\|_{L^\infty},\|\psi_z(\cdot,\tau)\|_{L^\infty}\}\leq N(t).$$

For (2.25) and (2.26), we have the following global well-posedness.

Theorem 2.2. *Under the assumptions of Theorem 2.1, there exists a constant $\delta_1>0$ such that if $N(0)\leq\delta_1$, then the Cauchy problems (2.25) and (2.26) have a unique global solution $(\zeta,\psi)\in X(0,+\infty)$ such that*

$$\|\zeta(.,t)\|_{2,w}^2+\|\psi(.,t)\|_{2,w}^2+\int_0^t\left(\|\zeta_z(.,\tau)\|_{2,w}^2+\|\psi_z(.,\tau)\|_{2,w}^2\right)d\tau\leq C\left(\|\zeta_0\|_{2,w}^2+\|\psi_0\|_{2,w}^2\right)\leq CN^2(0). \tag{2.27}$$

Moreover, it holds that

$$\sup_{z\in R}|(\zeta_z,\psi_z)(z,t)|\to 0\ \text{ as } t\to+\infty \tag{2.28}$$

Based on the solutions (ζ,ψ) that resulted in Theorem 2.2 and (U,V) in Proposition (2.1), we can establish the desired solutions of systems (1.1) and (1.2) through the relation (2.23). The global existence of (ζ,ψ) stated in Theorem 2.2 follows from the local existence theorem and the a priori estimates which are given below.

Proposition 2.2 (Local existence). *For any $\varepsilon_1>0$, there exists a positive constant T depending on ε_1 such that if $(\zeta,\psi)\in H_w^2$ with $N(0)\leq\varepsilon_1/2$, then problems (2.25) and (2.26) have a unique solution $(\zeta,\psi)\in X(0,T)$ satisfying $N(t)\leq 2N(0)$ for any $0\leq t\leq T$.*

Moreover, we can prove the local existence in a standard procedure (e.g., see Nishida [25]).

Proof. We differentiate (2.25) in z twice that

$$\begin{aligned}
\zeta_{zzt} &= \left(DmU^{m-1}\zeta_{zzz}\right)_z + Dm\left(\left(U^{m-1}\right)_{zz}\zeta_z + 2\left(U^{m-1}\right)_z\zeta_{zz}\right)_z \\
&\quad + G_{1zz} + ((s-V)\zeta_z)_{zz} - U_{zz}\psi_z - 2U_z\psi_{zz} - U\psi_{zzz} - (\zeta_z\psi_z)_{zz}, \\
\psi_{zzt} &= \left(\mu mU^{m-1}\psi_{zzz}\right)_z + \mu m\left(\left(U^{m-1}\right)_{zz}\psi_z + 2\left(U^{m-1}\right)_z\psi_{zz}\right)_z \\
&\quad + G_{2zz} + ((s-V)\psi_z)_{zz} - U_{zz}\zeta_z - 2U_z\zeta_{zz} - U\zeta_{zzz} - \frac{1}{2}\left(\zeta_z^2 + \psi_z^2\right)_{zz}.
\end{aligned}$$

We multiply two above equations by ζ_{zz}/U and ψ_{zz}/U, respectively, integrate the results in t, apply $N(t) = min\{D_m(w(z)+u_-), \mu_m(w(z)+u_-)\}$ and $\frac{1}{U} \leq Cw(z)$, then for some $C > 0$, one yields

$$\int \zeta_{zz}^2 w + \psi_{zz}^2 w + \int_0^t \int \zeta_{zzz}^2 w + \int_0^t \int \psi_{zzz}^2 w \leq \int \zeta_{0zz}^2 w + \psi_{0zz}^2 w + \int_0^t \int CN(t)\left(\zeta_z^2 + \zeta_{zz}^2 + \psi_z^2 + \psi_{zz}^2\right) w.$$

By then employing the theorem of Sobolev embedding, we can bound all the terms of right-hand side

$$\int_0^t \int CN(t)(\zeta_{zzz}^2 w + \psi_{zzz}^2 w),$$

then one gets

$$N^2(t) + \int_0^t \int \left(\zeta_{zzz}^2 w + \psi_{zzz}^2 w\right) \leq CN^2(0) + \int_0^t \int CN(t)\left(\zeta_{zzz}^2 w + \psi_{zzz}^2 w\right).$$

By dealing with $N(t) \leq 1/C$, then we derive

$$N(t) \leq \frac{N(0)}{\sqrt{C}} < 2N(0).$$

Hence, we complete the proof of Proposition 2.2. □

Proposition 2.3. *For some time $T > 0$, we consider that $(\zeta, \psi) \in X(0,T)$ be solution of (2.25) and (2.26). Then there is a constant $\varepsilon_1 > 0$, independent of T, such that if $N(T) < \varepsilon_1$, then (ζ, ψ) satisfies (2.27) for any $0 \leq t \leq T$.*

Proposition 2.3 is proved through the energy estimates given as follows.

Lemma 2.1. *Let $(\zeta, \psi) \in H_w^2(\mathbb{R})$ and (ζ, ψ) be solution of (2.25) and (2.26), then there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\|\zeta(.,t)\|_{2,w}^2 + \|\psi(.,t)\|_{2,w}^2 + \int_0^t \|\zeta_z(.,\tau)\|_{2,w}^2 d\tau + \int_0^t \|\psi_z(.,\tau)\|_{2,w}^2 d\tau \\
&\leq C\left(\|\zeta_0\|_{2,w}^2 + \|\psi_0\|_{2,w}^2\right) + CN(t)\int_0^t \int \left(\zeta_z^2 + \zeta_{zz}^2 + \psi_z^2 + \psi_{zz}^2\right) w.
\end{aligned} \tag{2.29}$$

Proposition 2.3 can be then proved through the Lemma 2.1.

Proof. To prove the desired result, the a priori estimate (2.27) is sufficiently proved. For some $C > 0$, by dealing with the theorem of Sobolev embedding, we can bound all the nonlinear terms of right-hand side (2.29)

$$\int_0^t CN(t)\|(\zeta_z, \psi_z)(.,\tau)\|_{2,w}^2 d\tau.$$

Through Lemma 2.1, for some $C > 0$, one can derive

$$N^2(t) + \int_0^t \|(\zeta_z, \psi_z)(.,\tau)\|_{2,w}^2 d\tau \leq CN^2(0) + \int_0^t CN(t)\|(\zeta_z, \psi_z)(.,\tau)\|_{2,w}^2 d\tau,$$

where $0 \leq t \leq T$. Then, by choosing $N(t) \leq 1/2C$ for any $t \in [0, T]$, we can derive

$$N^2(t) + \int_0^t \|(\zeta_z, \psi_z)(.,\tau)\|_{2,w}^2 d\tau \leq CN^2(0),$$

which gives the result (2.27). □

3 | WEIGHTED ENERGY ESTIMATES

In this section, we establish the a priori estimates for solution (ζ, ψ) of (2.25) and (2.26), and hence, Proposition 2.3 is proved. Since we are concerned with a singularity in the energy estimates if $u_+ = 0$, then we consider $C_1 w(z) \leq \frac{1}{U(z)} \leq C_2 w(z)$ for all $z \in \mathbb{R}$.

3.1 | L^2 estimate of (ζ, ψ)

Lemma 3.1. *Under the same assumptions of Proposition 2.3, if $N(t) \ll 1$, then*

$$\|\zeta(.,t)\|_w^2 + \|\psi(.,t)\|_w^2 + D\int_0^t \|\zeta_z(.,\tau)\|_w^2 d\tau + \mu\int_0^t \|\psi_z(.,\tau)\|_w^2 d\tau \leq C\left(\|\zeta_0\|_w^2 + \|\psi_0\|_w^2\right) + \int_0^t \int CN(t)\left(\frac{\zeta_{zz}^2 + \psi_{zz}^2}{U}\right). \tag{3.1}$$

Proof. We multiply Equation $(2.25)_1$ by ζ/U, $(2.25)_2$ by ψ/U, and integrate the results in z to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(\frac{\zeta^2+\psi^2}{U}\right) + Dm\int U^{m-2}\zeta_z^2 + \mu m\int U^{m-2}\psi_z^2 \\
&= -\int\frac{\zeta^2}{2}\left[\left(\frac{s-V}{U}\right)_z - \left(DmU^{m-1}\left(\frac{1}{U}\right)_z\right)_z\right] + \int\frac{G_1\zeta}{U} - \int\frac{\zeta\zeta_z\psi_z}{U} \\
&\quad -\int\frac{\psi^2}{2}\left[\left(\frac{s-V}{U}\right)_z - \left(\mu mU^{m-1}\left(\frac{1}{U}\right)_z\right)_z\right] + \int\frac{G_2\psi}{U} - \int\frac{\psi\zeta_z^2 + \psi\psi_z^2}{2U}.
\end{aligned} \tag{3.2}$$

It follows from the first equation of (2.4), $U_z < 0$, $V_z < 0$, and the fact $u_+ = 0$, one has

$$\begin{aligned}
\left(\frac{s-V}{U}\right)_z - \left(DmU^{m-1}\left(\frac{1}{U}\right)_z\right)_z &= \left[\frac{s-V}{U} - DmU^{m-1}\left(\frac{1}{U}\right)_z\right]_z \\
&= \left[\frac{u_+(s-v_+)}{U^2}\right]_z \\
&= \frac{-2u_+(s-v_+)U_z}{U^3} = 0, \\
\left(\frac{s-V}{U}\right)_z - \left(\mu mU^{m-1}\left(\frac{1}{U}\right)_z\right)_z &= \left[\frac{s-V}{U} - \mu mU^{m-1}\left(\frac{1}{U}\right)_z\right]_z \\
&= \frac{2F_1}{U^3}(-U_z) + \left[(D-\mu)mU^{m-1}\left(\frac{1}{U}\right)_z\right]_z \\
&= \frac{\mu}{D}\frac{2F_1}{U^3}(-U_z) + \frac{(D-\mu)}{DU}\left[(-V_z) + \frac{s-V}{U}(-U_z)\right] \\
&= \frac{(D-\mu)}{DU}\left[(-V_z) + \frac{s-V}{U}(-U_z)\right] \geq 0,
\end{aligned} \tag{3.3}$$

where this holds for $D \geq \mu$. Noting that

$$
\begin{aligned}
|G_1| &\leq Dm(\zeta_z + U)^{m-1}(\zeta_{zz} + U_z) + mU_zA_m + Dm\left((m-1)U^{m-2}\zeta_z U_z + U^{m-1}\zeta_{zz}\right), \\
|G_2| &\leq \mu m(\psi_z + V)^{m-1}\left(\psi_{zz} + V_z\right) + mV_zB_m + \mu m\left((m-1)U^{m-2}\psi_z U_z + U^{m-1}\psi_{zz}\right),
\end{aligned}
\tag{3.4}
$$

where $A_m = u_-^{m-1}$, $B_m = v_-^{m-1}$ if $m \geq 1$ and $A_m = C(1+e^{\eta z})^{m-1}$, $B_m = v_+^{m-1}$ if $0 < m < 1$ for $\eta = \frac{u_-^{1-m}}{Dm}(s + \chi v_+)$, and some constants $C > 0$. We further estimate the terms $(\zeta_z + U)^{m-1}$ and $(\psi_z + V)^{m-1}$ through the following estimations

$$
\begin{aligned}
(\zeta_z + U)^m &\leq (\zeta_z + u_-)^m = u_-^m\left(\frac{\zeta_z}{u_-} + 1\right)^m = \sum_{k=0}^{m} u_-^m \frac{Q_k^m}{k!}\left(\frac{\zeta_z}{u_-}\right)^k, \\
(\psi_z + V)^m &\leq (\psi_z + v_-)^m = v_-^m\left(\frac{\psi_z}{v_-} + 1\right)^m = \sum_{k=0}^{m} v_-^m \frac{Q_k^m}{k!}\left(\frac{\psi_z}{v_-}\right)^k,
\end{aligned}
\tag{3.5}
$$

where $Q_k^m = \frac{m!}{(m-k)!}$. By dealing with the fact $N(t) \ll 1$, it implies that $\|\zeta_z(\cdot,t)\|_{L^\infty} \leq 1$ and $\|\psi_z(\cdot,t)\|_{L^\infty} \leq 1$. Equation (3.5) then becomes

$$
\begin{aligned}
(\zeta_z + U)^m &\leq u_-^m(m!)^2\zeta_z^2 \sum_{k=0}^{m} \frac{1}{k!}\left(\frac{1}{u_-}\right)^k = u_-^m(m!)^2\zeta_z^2 e^{1/u_-} \leq C\zeta_z^2, \\
(\psi_z + V)^m &\leq v_-^m(m!)^2\psi_z^2 \sum_{k=0}^{m} \frac{1}{k!}\left(\frac{1}{v_-}\right)^k = v_-^m(m!)^2\psi_z^2 e^{1/v_-} \leq C\psi_z^2.
\end{aligned}
\tag{3.6}
$$

By the similar ways as in (3.6), the estimate of terms $(\zeta_z + U)^{m-1}$ and $(\psi_z + V)^{m-1}$ can be provided, where the term $(\zeta_z + U)^{m-1}$ has two conditions: For $m \geq 1$, one has $(\zeta_z + U)^{m-1} \leq (\zeta_z + u_-)^{m-1}$ and for $0 < m < 1$, one has $(\zeta_z + U)^{m-1} \leq (\zeta_z + Cw(z))^{m-1}$, where $w(z)$ is a weighted function.

Noting that $0 = u_+ < U < u_-$, $\|\zeta_z(\cdot,t)\|_{L^\infty} \leq N(t) \ll 1$ and $\|\psi_z(\cdot,t)\|_{L^\infty} \leq N(t) \ll 1$, then it holds

$$
\begin{aligned}
|G_1| &\leq C\left(|\zeta_{zz}||\zeta_z| + |\zeta_z|^2 + |\zeta_{zz}|^2\right), \\
|G_2| &\leq C\left(|\psi_{zz}||\psi_z| + |\psi_z|^2 + |\psi_{zz}|^2\right).
\end{aligned}
\tag{3.7}
$$

By Young's inequality, we have

$$
\begin{aligned}
\left|\int \frac{G_1\zeta}{U}\right| &\leq CN(t)\int\left(\frac{|\zeta_z|^2 + |\zeta_{zz}|^2}{U}\right), \\
\left|\int \frac{G_2\psi}{U}\right| &\leq CN(t)\int\left(\frac{|\psi_z|^2 + |\psi_{zz}|^2}{U}\right),
\end{aligned}
\tag{3.8}
$$

where $\|\zeta(\cdot,t)\|_{L^\infty} \leq N(t)$ and $\|\psi(\cdot,t)\|_{L^\infty} \leq N(t)$ have been employed. Similarly,

$$
\begin{aligned}
\left|\int \frac{\zeta\zeta_z\psi_z}{U}\right| &\leq CN(t)\int\left(\frac{|\zeta_z|^2 + |\psi_z|^2}{U}\right), \\
\left|\int \frac{\psi\zeta_z^2 + \psi\psi_z^2}{2U}\right| &\leq CN(t)\int\left(\frac{|\zeta_z|^2 + |\psi_z|^2}{U}\right).
\end{aligned}
\tag{3.9}
$$

Substituting (3.3), (3.8), and (3.9) into (3.2), one has

$$\begin{aligned}
&\int\left(\frac{\zeta^2+\psi^2}{U}\right)+\int_0^t\int\left(D_mU^{m-2}-\frac{CN(t)}{U}\right)\zeta_z^2+\int_0^t\int\left(\mu_mU^{m-2}-\frac{CN(t)}{U}\right)\psi_z^2\\
&\leq\int\left(\frac{\zeta_0^2+\psi_0^2}{U}\right)+CN(t)\int_0^t\int\left(\frac{\zeta_{zz}^2+\psi_{zz}^2}{U}\right).
\end{aligned}\tag{3.10}$$

To estimate the second and third terms in (3.10), we can split the term U^{m-2} into two conditions as stated as follows.

$$\begin{aligned}
&\text{If } 0<m<2, \text{ then } U^{m-2}=\left(\frac{1}{U}\right)^{2-m}\leq C^*w(z)\leq\frac{Cw(z)}{U}.\\
&\text{If } m\geq 2, \text{ then } U^{m-2}\leq C^{**}u_-\leq\frac{Cu_-}{U}.
\end{aligned}\tag{3.11}$$

In this case, we have used the weighted function $w(z)$ defined in Remark 1.2 and $C = max\{C^*, C^{**}\} = max\left\{\frac{\alpha}{m-\alpha}, (m+\alpha)^m\right\}$ for $\alpha>0$ and $m>\alpha$. Then, it follows from (3.11), Equation (3.10) becomes

$$\begin{aligned}
&\int\left(\frac{\zeta^2+\psi^2}{U}\right)+\int_0^t\int C\left(D_m(w(z)+u_-)-N(t)\right)\frac{\zeta_z^2}{U}+\int_0^t\int C\left(\mu_m(w(z)+u_-)-N(t)\right)\frac{\psi_z^2}{U}\\
&\leq\int\left(\frac{\zeta_0^2+\psi_0^2}{U}\right)+CN(t)\int_0^t\int\left(\frac{\zeta_{zz}^2+\psi_{zz}^2}{U}\right).
\end{aligned}\tag{3.12}$$

By employing $N(t)=min\{D_m(w(z)+u_-), \mu_m(w(z)+u_-)\}$ and $\frac{1}{U}\leq Cw(z)$, we can prove (3.1) from (3.12). □

3.2 | H^1 estimate of (ζ,ψ)

Lemma 3.2. *Under the same assumptions of Proposition 2.3, if $N(t)\ll 1$, then*

$$\|\zeta(.,t)\|_{1,w}^2+\|\psi(.,t)\|_{1,w}^2+D\int_0^t\|\zeta_z(.,\tau)\|_{1,w}^2d\tau+\mu\int_0^t\|\psi_z(.,\tau)\|_{1,w}^2d\tau\leq C\left(\|\zeta_0\|_{1,w}^2+\|\psi_0\|_{1,w}^2\right).\tag{3.13}$$

Proof. We differentiate (2.25) in z to get

$$\begin{cases}
\zeta_{zt}=\left(DmU^{m-1}\zeta_{zz}\right)_z+\left(Dm\left(U^{m-1}\right)_z\zeta_z\right)_z+G_{1z}+((s-V)\zeta_z)_z\\
\quad-U_z\psi_z-U\psi_{zz}-(\zeta_z\psi_z)_z,\\
\psi_{zt}=\left(\mu mU^{m-1}\psi_{zz}\right)_z+\left(\mu m\left(U^{m-1}\right)_z\psi_z\right)_z+G_{2z}+((s-V)\psi_z)_z\\
\quad-U_z\zeta_z-U\zeta_{zz}-\frac{1}{2}\left(\zeta_z^2+\psi_z^2\right)_z.
\end{cases}\tag{3.14}$$

Now, multiplying $(3.14)_1$ by ζ_z/U and $(3.14)_2$ by ψ_z/U, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(\frac{\zeta_z^2+\psi_z^2}{U}\right)+Dm\int U^{m-2}\zeta_{zz}^2+\mu m\int U^{m-2}\psi_{zz}^2\\
&=-\int\frac{\zeta_z^2}{2}\left[\left(\frac{s-V}{U}\right)_z-\left(DmU^{m-1}\left(\frac{1}{U}\right)_z\right)_z\right]-2\int\frac{U_z\zeta_z\psi_z}{U}-\int\frac{\left(\zeta_z^2+\psi_z^2\right)V_z}{U}\\
&\quad-\int\frac{\psi_z^2}{2}\left[\left(\frac{s-V}{U}\right)_z-\left(\mu mU^{m-1}\left(\frac{1}{U}\right)_z\right)_z\right]-\int(G_1-\zeta_z\psi_z)\left(\frac{\zeta_z}{U}\right)_z\\
&\quad-\int\left(G_2-\frac{\zeta_z^2+\psi_z^2}{2}\right)\left(\frac{\psi_z}{U}\right)_z+m\int\left(D\zeta_z^2+\mu\psi_z^2\right)\left[\frac{\left(U^{m-1}\right)_{zz}}{U}-\frac{1}{2}\left(\frac{\left(U^{m-1}\right)_z}{U}\right)_z\right].
\end{aligned}\tag{3.15}$$

Using (2.4), $s = v_-$, and the facts $0 = u_+ < U < u_-$, $v_+ < V < v_-$, we have

$$\begin{aligned} \left|\frac{U_z}{U}\right| &\leq \left|\frac{F_1 - (s-V)U}{DmU^m}\right| \leq C, \\ |V_z| &\leq \left|-\frac{s}{\mu m}V^{2-m} + \frac{U^2+V^2}{2\mu m}V^{1-m} + \frac{F_2}{\mu m}V^{1-m}\right| \leq C. \end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.15), using the fact (3.3) and integrating the results with respect to t, we have

$$\begin{aligned} &\int\left(\frac{\zeta_z^2+\psi_z^2}{U}\right) + Dm\int_0^t\int U^{m-2}\zeta_{zz}^2 + \mu m\int_0^t\int U^{m-2}\psi_{zz}^2 \\ &\leq \int\left(\frac{\zeta_{0z}^2+\psi_{0z}^2}{U}\right) + \int_0^t\int\frac{\zeta_z^2}{U} + \int_0^t\int\frac{\psi_z^2}{U} + C\int_0^t\int |G_1 - \zeta_z\psi_z|\left(|\zeta_z| + \frac{|\zeta_{zz}|}{U}\right) \\ &+ C\int_0^t\int\left|G_2 - \frac{\zeta_z^2+\psi_z^2}{2}\right|\left(|\psi_z| + \frac{|\psi_{zz}|}{U}\right). \end{aligned} \tag{3.17}$$

Noting that

$$\begin{aligned} &\int_0^t\int |G_1 - \zeta_z\psi_z|\left(|\zeta_z| + \frac{|\zeta_{zz}|}{U}\right) \leq CN(t)\int_0^t\int\left(\frac{|\zeta_z|^2+|\psi_z|^2+|\zeta_{zz}|^2}{U}\right), \\ &\int_0^t\int\left|G_2 - \frac{\zeta_z^2+\psi_z^2}{2}\right|\left(|\psi_z| + \frac{|\psi_{zz}|}{U}\right) \leq CN(t)\int_0^t\int\left(\frac{|\zeta_z|^2+|\psi_z|^2+|\psi_{zz}|^2}{U}\right), \end{aligned} \tag{3.18}$$

where we have used $\|(\zeta,\psi)(\cdot,t)\|_{L^\infty} \leq N(t)$. Substituting (3.18) into (3.17), and combining the results with (3.1), one has

$$\int\left(\frac{\zeta_z^2+\psi_z^2}{U}\right) + \int_0^t\int\left(D_m U^{m-2} - \frac{CN(t)}{U}\right)\zeta_{zz}^2 + \int_0^t\int\left(\mu_m U^{m-2} - \frac{CN(t)}{U}\right)\psi_{zz}^2 \leq \int\left(\frac{\zeta_{0z}^2+\psi_{0z}^2}{U}\right). \tag{3.19}$$

Then, we apply the same strategies as in Lemma 3.1 to estimate the second and third terms in (3.19) and one has

$$\int\left(\frac{\zeta_z^2+\psi_z^2}{U}\right) + \int_0^t\int C\left(D_m(w(z)+u_-) - N(t)\right)\frac{\zeta_{zz}^2}{U} + \int_0^t\int C\left(\mu_m(w(z)+u_-) - N(t)\right)\frac{\psi_{zz}^2}{U} \leq \int\left(\frac{\zeta_{0z}^2+\psi_{0z}^2}{U}\right). \tag{3.20}$$

By employing $N(t) = min\{D_m(w(z)+u_-), \mu_m(w(z)+u_-)\}$ and $\frac{1}{U} \leq Cw(z)$, then the proof of (3.13) is completed. □

3.3 | H^2 estimate of (ζ, ψ)

Lemma 3.3. *Under the same assumptions of Proposition 2.3, if $N(t) \ll 1$, then*

$$\|\zeta(.,t)\|_{2,w}^2 + \|\psi(.,t)\|_{2,w}^2 + D\int_0^t \|\zeta_z(.,\tau)\|_{2,w}^2 d\tau + \mu\int_0^t \|\psi_z(.,\tau)\|_{2,w}^2 d\tau \leq C\left(\|\zeta_0\|_{2,w}^2 + \|\psi_0\|_{2,w}^2\right). \tag{3.21}$$

Proof. We differentiate (3.14) in z to get

$$\begin{cases} \zeta_{zzt} = \left(DmU^{m-1}\zeta_{zzz}\right)_z + Dm\Big(\left(U^{m-1}\right)_{zz}\zeta_z + 2\left(U^{m-1}\right)_z\zeta_{zz}\Big)_z \\ \qquad + G_{1zz} + ((s-V)\zeta_z)_{zz} - U_{zz}\psi_z - 2U_z\psi_{zz} - U\psi_{zzz} - (\zeta_z\psi_z)_{zz}, \\ \psi_{zzt} = \left(\mu mU^{m-1}\psi_{zzz}\right)_z + \mu m\Big(\left(U^{m-1}\right)_{zz}\psi_z + 2\left(U^{m-1}\right)_z\psi_{zz}\Big)_z \\ \qquad + G_{2zz} + ((s-V)\psi_z)_{zz} - U_{zz}\zeta_z - 2U_z\zeta_{zz} - U\zeta_{zzz} - \frac{1}{2}\left(\zeta_z^2 + \psi_z^2\right)_{zz}. \end{cases} \tag{3.22}$$

Now, multiplying $(3.22)_1$ by ζ_{zz}/U and $(3.22)_2$ by ψ_{zz}/U, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(\frac{\zeta_{zz}^2+\psi_{zz}^2}{U}\right)+Dm\int U^{m-2}\zeta_{zzz}^2+\mu m\int U^{m-2}\psi_{zzz}^2\\
&=-\int\frac{\zeta_{zz}^2}{2}\left[\left(\frac{s-V}{U}\right)_z-\left(DmU^{m-1}\left(\frac{1}{U}\right)_z\right)_z\right]-4\int\frac{U_z\zeta_{zz}\psi_{zz}}{U}-\int\frac{2\left(\zeta_{zz}^2+\psi_{zz}^2\right)V_z}{U}\\
&\quad-\int\frac{\psi_{zz}^2}{2}\left[\left(\frac{s-V}{U}\right)_z-\left(\mu mU^{m-1}\left(\frac{1}{U}\right)_z\right)_z\right]-\int\frac{(\zeta_z\zeta_{zz}+\psi_z\psi_{zz})V_{zz}}{U}\\
&\quad-\frac{U_{zz}\left(\psi_z\zeta_{zz}+\zeta_z\psi_{zz}\right)}{U}-\int(G_1-\zeta_z\psi_z)_z\left(\frac{\zeta_{zz}}{U}\right)_z-\int\left(G_2-\frac{\zeta_z^2+\psi_z^2}{2}\right)_z\left(\frac{\psi_{zz}}{U}\right)_z\\
&\quad+m\int\left(D\zeta_{zz}^2+\mu\psi_{zz}^2\right)\left[\frac{3\left(U^{m-1}\right)_{zz}}{U}-\left(\frac{\left(U^{m-1}\right)_z}{U}\right)_z\right]+m\int\left(D\zeta_z^2+\mu\psi_z^2\right)\left(\frac{\left(U^{m-1}\right)_{zzz}}{2U}\right)_z.
\end{aligned}
\tag{3.23}
$$

Using (2.2), (3.16), the facts $0=u_+<U<u_-$, $v_+<V<v_-$, and $s=v_-$, one can derive

$$
\begin{aligned}
|U_{zz}|&=\left|\frac{(V-s)U_z+UV_z-Dm(m-1)U^{m-2}U_z^2}{DmU^{m-1}}\right|\leq C,\\
|V_{zz}|&=\left|\frac{(V-s)V_z+UU_z-\mu m(m-1)V^{m-2}V_z^2}{\mu mV^{m-1}}\right|\leq C.
\end{aligned}
\tag{3.24}
$$

Then, we employ (3.24) and Cauchy–Schwarz inequality to get the following estimates

$$
\begin{aligned}
-\int\frac{V_{zz}(\zeta_z\zeta_{zz}+\psi_z\psi_{zz})}{U}&\leq C\int\frac{|\zeta_z\zeta_{zz}+\psi_z\psi_{zz}|}{U}\\
&\leq C\int\frac{\zeta_z^2+\psi_z^2+\zeta_{zz}^2+\psi_{zz}^2}{U},\\
-\int\frac{2(\zeta_{zz}^2+\psi_{zz}^2)V_z}{U}&\leq C\int\frac{\zeta_{zz}^2+\psi_{zz}^2}{U},\\
-\int\frac{U_{zz}(\psi_z\zeta_{zz}+\zeta_z\psi_{zz})}{U}&\leq C\int\frac{|\psi_z\zeta_{zz}+\zeta_z\psi_{zz}|}{U}\\
&\leq C\int\frac{\psi_z^2+\zeta_{zz}^2+\zeta_z^2+\psi_{zz}^2}{U},\\
-4\int\frac{U_z\zeta_{zz}\psi_{zz}}{U}&\leq C\int\frac{\zeta_{zz}^2+\psi_{zz}^2}{U}.
\end{aligned}
\tag{3.25}
$$

Noting that

$$
\begin{aligned}
G_{1z}&=\left(Dm(\zeta_z+U)^{m-1}(\zeta_{zz}+U_z)-Dm(m-1)U^{m-2}U_z\zeta_z-DmU^{m-1}\zeta_{zz}\right)_z,\\
&=Dm(m-1)(\zeta_z+U)^{m-2}(\zeta_{zz}+U_z)^2-Dm(m-1)(m-2)U^{m-3}U_z^2\zeta_z\\
&\quad-Dm(m-1)U^{m-2}\left(U_{zz}\zeta_z+2U_z\zeta_{zz}\right)+Dm(\zeta_z+U)^{m-1}U_{zz}\\
&\quad+Dm\left((\zeta_z+U)^{m-1}-U^{m-1}\right)\zeta_{zzz}\\
G_{2z}&=\left(\mu m(\psi_z+V)^{m-1}(\psi_{zz}+V_z)-\mu m(m-1)U^{m-2}U_z\psi_z-\mu mU^{m-1}\psi_{zz}\right)_z,\\
&=\mu m(m-1)(\psi_z+V)^{m-2}(\psi_{zz}+V_z)^2-\mu m(m-1)(m-2)U^{m-3}U_z^2\psi_z\\
&\quad-\mu m(m-1)U^{m-2}\left(U_{zz}\psi_z+2U_z\psi_{zz}\right)+\mu m(\psi_z+V)^{m-1}V_{zz}\\
&\quad+\mu m((\psi_z+V)^{m-1}-U^{m-1})\psi_{zzz},
\end{aligned}
\tag{3.26}
$$

which gives

$$\begin{aligned}&\int (G_1-\zeta_z\psi_z)_z\left(\frac{\zeta_{zz}}{U}\right)_z \le CN(t)\int \left(\frac{\zeta_{zz}^2+\zeta_{zzz}^2+\psi_{zz}^2}{U}\right),\\&\int \left(G_2-\frac{\zeta_z^2+\psi_z^2}{2}\right)\left(\frac{\psi_{zz}}{U}\right)_z \le CN(t)\int \left(\frac{\zeta_{zz}^2+\psi_{zz}^2+\psi_{zzz}^2}{U}\right),\end{aligned} \tag{3.27}$$

where we have employed $\|(\zeta_z,\psi_z)(\cdot,t)\|_{L^\infty}\le N(t)$ and the term U^{m-1} can be estimated by the similar ways as in (3.11) for two conditions: $0<m<1$ and $m\ge 1$. Substituting (3.25) and (3.27) into (3.23), using the fact (3.3) and integrating the results in t, one has

$$\begin{aligned}&\int \left(\frac{\zeta_{zz}^2+\psi_{zz}^2}{U}\right)+\int_0^t\int \left(D_mU^{m-2}-\frac{CN(t)}{U}\right)\zeta_{zzz}^2+\int_0^t\int \left(\mu_mU^{m-2}-\frac{CN(t)}{U}\right)\psi_{zzz}^2\\&\le \int \left(\frac{\zeta_{0zz}^2+\psi_{0zz}^2}{U}\right)+CN(t)\int_0^t\int \left(\frac{\zeta_z^2+\zeta_{zz}^2+\psi_z^2+\psi_{zz}^2}{U}\right).\end{aligned} \tag{3.28}$$

Finally, we can use the similar ways as in Lemmas 3.1 and 3.2 for the second and third terms to complete the proof of (3.21). □

4 | STABILITY

Proof of Theorem 2.2. We are now concerned with the main theorem of this paper. By focusing to Equation (2.23), the proof of Theorem 2.1 is based on Theorem 2.2. The a priori estimate (2.27) states that small enough $N(0)$ gives small $N(t)$. Thus, applying the procedure in a standard way, the global well-posedness of (2.25) and (2.26) in $X(0,+\infty)$ is established. Then, the convergence (2.28) is proved. Since $w\ge 1$, it makes sense that the $\zeta\in H_w^2$ gives $\zeta\in H^2$. By dealing with the global estimate (2.27), one can derive

$$\int_0^t\int_{-\infty}^{\infty}\zeta_z^2(z,\tau)dzd\tau\le C\left(\|\zeta_0\|_{2,w}^2+\|\psi_0\|_{2,w}^2\right)\le CN^2(0). \tag{4.1}$$

In view of the first equation of (2.25), by Young's inequality,

$$\begin{aligned}&\frac{d}{dt}\int_{-\infty}^{\infty}\zeta_z^2(z,t)dz\\&=-2\int_{-\infty}^{\infty}\zeta_t\zeta_{zz}dz\\&=-2\int_{-\infty}^{\infty}\zeta_{zz}\left(Dm(U^{m-1}\zeta_z)_z+G_1+(s-V)\zeta_z-U\psi_z-\zeta_z\psi_z\right)\\&\le C\int_{-\infty}^{\infty}(\zeta_{zz}^2+\zeta_z^2+\psi_z^2).\end{aligned}$$

By referring to the global estimate (2.27), one has

$$\int_0^\infty\left|\frac{d}{dt}\int_{-\infty}^{\infty}\zeta_z^2(z,t)dz\right|\le C\int_0^\infty\int_{-\infty}^{\infty}\left(\zeta_{zz}^2+\zeta_z^2+\psi_z^2\right)\le C\left(\|\zeta_0\|_2^2+\|\psi_0\|_2^2\right)\le CN^2(0). \tag{4.2}$$

From (4.1) and (4.2), we get

$$\int_{-\infty}^{\infty}\zeta_z^2(z,t)dz\to 0\,\text{as}\,t\to+\infty.$$

Moreover, by the dealing with Cauchy–Schwarz inequality, we obtained

$$\begin{aligned}
\zeta_z^2(z,t) &= 2\int_{-\infty}^{z} \zeta_z\zeta_{zz}(y,t)dy \\
&\leq 2\left(\int_{-\infty}^{+\infty} \zeta_z^2(y,t)dy\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty} \zeta_{zz}^2(y,t)dy\right)^{\frac{1}{2}} \\
&\leq C\left(\int_{-\infty}^{+\infty} \zeta_z^2(y,t)dy\right)^{\frac{1}{2}} \to 0 \,\text{as}\, t \to +\infty.
\end{aligned}$$

Applying the same argument to ψ_z yields

$$\sup_{z\in\mathbb{R}} |\psi_z(z,t)| \to 0 \quad \text{as} \quad t \to +\infty. \tag{4.3}$$

Hence, (2.28) is proved. □

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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