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ORIGINAL RESEARCH

Traveling fronts of viscous Burgers' equations with the nonlinear degenerate viscosity

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Abstract

In this paper, we continue the study of viscous Burgers' equations by Il'in and Oleinik for single model equation with convex nonlinearity [6] by introducing the nonlinear degenerate viscosity. Since the degeneracy of this paper is considered, then there are two conditions for estimate of the traveling fronts U when $m \geq 1$ and $0 < m < 1$. The following higher-order estimate of traveling fronts U is introduced to overcome the energy estimate

$$U^{m-2} = \left(\frac{1}{U}\right)^{2-m} \leq Ku_+ \leq Cu_+, \text{ if } 0 < m < 2,$$

$$U^{m-2} \leq Lu_- \leq Cu_-, \text{ if } m \geq 2$$

where $C = max\{K, L\} = max\left\{\frac{b}{m-b}, (m+b)^m\right\}$ for $b > 0$ and $m > b$. Moreover, the following Taylor expansion is employed

$$f(U+\pi_z) - f^{'}(U)\pi_z - f(U) = \int_0^1 \left((1-s)f(U+s\pi_z)ds\right)\pi_z^2 = \mathcal{O}(1)\pi_z^2$$

to overcome the estimate of term F_1, where this term is transformation result of nonlinearity for first order derivative in (1). The stability of traveling fronts U is presented to give the information how close the distance between the solution u of (1) and the traveling fronts U is under the small perturbations. This stability result is based on the energy estimates under the condition $N(t) \leq Dm(u_+ + u_-)$. To validate our works and to illustrate the effect of nonlinear degenerate viscosity, the numerical simulations are provided by using the standard finite difference for discretization steps.

Keywords Stability · Degenerate viscous Burgers' equations · Small perturbation · Large wave amplitude

Mathematics Subject Classification 35A01 · 35B40

Introduction

In this paper, we are interested in the study of stability of traveling fronts to the following viscous Burgers' equations with the nonlinear degenerate viscosity

$$u_t + (f(u))_x = D(u^m)_{xx}, \quad (1)$$

where $m > 0$, $u = u(x, t)$, the initial conditions

$$u(x, 0) = u_0(x) \to u_\pm \text{ as } x \to \pm\infty, \quad (2)$$

and the smooth function $f(u)$ satisfies

$$f(u) \geq 0,\ f'(u) > 0,\ f^{''}(u) \geq 0. \quad (3)$$

The viscous Burgers' equations (1) is the modified version of following viscous Burgers' equations involving the convex nonlinearity

$$u_t + f(u)_x = Du_{xx}, \quad (4)$$

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where a constant $D > 0$. Il'in and Oleinik studied the stability of shock profiles to (4) based on the maximum principle [6] and spectral analysis studied by Sattinger in [21].

For $f(u) = \frac{u^2}{2}$, the Eq. (1) becomes the Burger's equation. Moreover, Mickens and Oyedeji [19] and other references [3, 22] investigated the traveling wave solutions to Burgers' equations with square root and non-diffusion Fisher

$$\begin{aligned} &u_t + \psi_1 \sqrt{u} u_x = \psi_2 u_{xx}, \\ &u_t + \psi_3 \sqrt{u} u_x = \psi_4 \sqrt{u} - \psi_5 u, \end{aligned} \tag{5}$$

where $\psi_i > 0, i = 1, 2, 3, 4$.

Buckmire et al [1], Jordan [9], and Mickens [17, 18] for additional research containing the square root term. We are interested in the stability of traveling waves of (1) with nonlinear degenerate viscosity in this paper. Moreover the singularity of energy estimate under small perturbation and arbitrary wave amplitude was studied in [8, 10]. As in [7], the energy method was also used to establish the stability of traveling waves to coupled Burgers' equations

$$\begin{aligned} &u_t + \left(\frac{1}{2}u^2 + \frac{1}{2}v^2\right)_x = D_1 u_{xx}, \\ &\quad v_t + (uv)_x = D_2 v_{xx}, \end{aligned} \tag{6}$$

Moreover, Li and Wang [14] studied the following system

$$\begin{aligned} &u_t - (uv)_x = D v_{xx}, \\ &\quad v_t + (\varepsilon v^2 - u)_x = \varepsilon v_{xx}, \end{aligned} \tag{7}$$

where (7) was derived from the Keller–Segel model, also known as the chemotaxis model, and the coefficients ε were assumed to be small enough. In contrast to the research in [7], the smallness of coefficients was not required in the study of [14]. Other references using the elementary energy method to study the stability problem include [11, 12, 15].

Our concern of this paper is the nonlinear degenerate viscosity which has two conditions for the estimate of traveling fronts U when $0 < m < 1$ and $m \geq 1$. We also present the higher-order estimate to overcome the energy estimate. The nonlinear diffusion was studied in [5] and [4] for the case of chemotaxis model under the condition $u_+ > 0$ and for the case of coupled Burgers' system under the condition $u_+ = 0$, respectively. Moreover, Choi and Kim [2] studied the porous medium problem to provide the existence of chemotaxis equations arising from the food metric under the compact support

$$\begin{aligned} &r_t = \left(\psi(s, r)\left(r_x - \frac{r}{s}s_x\right)\right), \\ &s_t = -\omega(s, r) r. \end{aligned} \tag{8}$$

The parameters r and s, respectively, describe the interactions between the population density of bacteria and resource (food) density. Moreover, the functions of $\psi(s, r)$ and $\omega(s, r)$ give the diffusivity and consumption rate which are, respectively, defined as follows

$$\begin{aligned} &\psi(s, r) = \frac{p}{s^2}\left(\frac{r}{s}\right)^{p-1}, \quad p > \delta \\ &\omega(s, r) = \alpha(s) r^{-\delta}, \quad 0 < \delta < 1. \end{aligned} \tag{9}$$

Other parts of this paper consist of five aspects. The existence of traveling fronts U is presented in the Sect. 2 by using the ansatz traveling fronts. Meanwhile, the appropriate perturbations of traveling fronts U are also presented in the Sect. 2. The a priori estimate by using the higher-order estimate and the stability of traveling fronts U are provided in the Sect. 3. Section 4, we present the numerical simulation to illustrate the effect of nonlinear degenerate viscosity. Finally, the conclusions of this paper are provided in Sect. 5.

Remark 1 In this paper, we consider the nonlinear degenerate viscosity in the viscous Burgers' equations where this problem gives two conditions for the estimate of traveling fronts U when $0 < m < 1$ and $m \geq 1$. The following higher-order estimate is introduced to overcome the energy estimate

$$\begin{aligned} &U^{m-2} = \left(\frac{1}{U}\right)^{2-m} \leq K u_+ \leq C u_+, \text{ if } 0 < m < 2, \\ &\quad U^{m-2} \leq L u_- \leq C u_-, \text{ if } m \geq 2, \end{aligned}$$

where $C = max\{K, L\} = max\left\{\frac{b}{m-b}, (m+b)^m\right\}$ for $b > 0$ and $m > b$. Moreover, the following Taylor expansion is employed

$$\begin{aligned} &f(U + \pi_z) - f'(U)\pi_z - f(U) \\ &\quad = \int_0^1 \left((1-s) f''(U + s\pi_z) ds\right) \pi_z^2 = \mathcal{O}(1)\pi_z^2, \end{aligned}$$

to overcome the estimate of term F_1, where this term is transformation result of nonlinearity for first order derivative in (1).

Notation 1 In this paper, the Sobolev space $H^p(\mathbb{R})$ is introduced by defining the norms as follows $\|v\|_p := \sum_{r=0}^{p} \|\partial_x^r v\|$ and $\|v\| := \|v\|_{L^2(\mathbb{R})}$.

Transformation of the original equation

The viscous Burgers' equations of (1) can be written as the following traveling fronts

$$u(x, t) = U(z), \quad z = x - st, \tag{10}$$

satisfying

$$-sU_z + (f(U))_z = D(U^m)_{zz}, \tag{11}$$

with the boundary conditions

$$U(z) \to u_\pm \ \text{ as } \ z \to \pm\infty. \tag{12}$$

The Eq. (11) then provides the integration in z,

$$D(U^m)_z = -sU + f(U) + K, \tag{13}$$

where $K = su_\pm - f(u_\pm)$. By dealing with $U_z \to 0$ as $z \to \pm\infty$, and (3), then we can derive the Rankine–Hugoniot condition

$$s(u_+ - u_-) = f(u_+) - f(u_-), \tag{14}$$

which yields

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-} > 0. \tag{15}$$

The existence of traveling fronts with the nonlinear degenerate viscosity is given below.

Proposition 1 *Consider that $u_\pm$ satisfy (14). The monotonicity of traveling fronts $U(x-st)$ to (11) be unique with the translation $z = x - st$ and the condition $U_z < 0$ hold. The wave speed s is provided in (15). Then, the traveling fronts U satisfies the monotonicity criteria as follows*

$$U - u_\pm \sim e^{\lambda_\pm z} \text{ as } z \to \pm\infty,$$

where

$$\lambda_\pm = -\frac{f(u_+) - f(u_-)}{u_+ - u_-} \cdot (u_\mp - u_\pm).$$

Proof By assuming $M = (U^m)_z$, then one has

$$\begin{aligned} U_z &= \frac{MU^{1-m}}{m}, \\ M_z &= \frac{MU^{1-m}}{m} \cdot \left(f'(U) - \frac{f(u_+) - f(u_-)}{u_+ - u_-} \right), \end{aligned} \tag{16}$$

which gives

$$\begin{cases} \frac{dM}{dU} = f'(U) - \frac{f(u_+)-f(u_-)}{u_+-u_-}, \\ M(u_\pm) = 0. \end{cases} \tag{17}$$

Let M_s be the solution of (17), then one provides

$$\begin{aligned} M_s(U) \sim m u_\pm^{m-1}(u_\mp - u_\pm)\left(f'(U) - \frac{f(u_+) - f(u_-)}{u_+ - u_-} \right) \\ (U - u_\pm), \ \text{ as } U \to u_\pm. \end{aligned} \tag{18}$$

Employing the first equation of (16), L'Hospital's rule, and (18), one has

$$\begin{aligned} &\lim_{U \to u_\pm} \frac{z}{\ln(U - u_\pm)} \\ &= \lim_{U \to u_\pm} \frac{m u_\pm^{m-1}(U - u_\pm)}{m u_\pm^{m-1}(u_\mp - u_\pm)\left(f'(U) - \frac{f(u_+)-f(u_-)}{u_+-u_-}\right)(U - u_\pm)} \\ &= \lim_{U \to u_\pm} \frac{1}{(u_\mp - u_\pm)\left(f'(U) - \frac{f(u_+)-f(u_-)}{u_+-u_-}\right)} \\ &= -\frac{1}{(u_\mp - u_\pm)\left(\frac{f(u_+)-f(u_-)}{u_+-u_-}\right)}, \end{aligned}$$

which provides

$$U - u_\pm \sim e^{-\left(\frac{f(u_+)-f(u_-)}{u_+-u_-} \cdot (u_\mp - u_\pm)\right)z}, \ \text{ as } z \to \pm\infty.$$

□

We refer to [13, 16] to provide the following zero mass perturbation of π

$$\pi_0(z) = \int_{-\infty}^{z} (u_0 - U)(y)dy.$$

To provide the stability of traveling fronts of viscous Burgers' equations (1), then the following theorems are given.

Theorem 1 *Let $U(x-st)$ be the traveling fronts obtained in Proposition 1. Then there exists a constant $\varepsilon_0 > 0$ such that if $\|u_0 - U\| + \|\pi_0\| \le \varepsilon_0$, then the Cauchy problem (1)-(2) has an unique global solution $u(x,t)$ satisfying*

$$u - U \in C\big([0,\infty); H^1\big) \cap L^2\big([0,\infty); H^1\big),$$

and the asymptotic stability

$$\sup_{x \in \mathbb{R}} |u(x,t) - U(x - st)| \to 0 \ \text{ as } \ t \to +\infty.$$

The Eq. (1) is changed by employing the variables $(x,t) \to (z = x - st, t)$

$$u_t - su_z + (f(u))_z = (u^m)_{zz}. \tag{19}$$

Moreover, the solution u can be stated as follows

$$u(z,t) = U(z) + \pi_z(z,t), \tag{20}$$

which gives

$$\pi(z,t) = \int_{-\infty}^{z} (u(y,t) - U(y))dy. \tag{21}$$

To get the Eq. (22), the Eq. (20) is applied into (19), and the integration in z is then employed,

$$\pi_t = s\pi_z + f'(U)\pi_z + m\big(U^{m-1}\pi_z\big)_z + F_1 + F_2, \tag{22}$$

where $F_1 = f(U+\pi_z) - f'(U)\pi_z - f(U)$, $F_2 = \left((\pi_z + U)^m - U^m - mU^{m-1}\pi_z\right)_z$, and the initial data of π

$$\pi(z,0) = \pi_0(z) = \int_{-\infty}^{z} (u_0 - U)dy \tag{23}$$

with $\pi_0(\pm\infty) = 0$. We further find the solution of transformed problem (22)-(23) in the space

$$X(0,T) := \left\{\pi(z,t) \in C([0,T), H^2) : \pi_z \in L^2((0,T);H^2))\right\}$$

with $0 < T \leq +\infty$. Let

$$N(t) := \sup_{0\leq\tau\leq t} \left\{\|\pi(.,\tau)\|_2\right\}.$$

From the Sobolev inequality $\|f\|_{L^\infty} \leq \sqrt{2}\| \quad f\|_{L^2}^{\frac{1}{2}}\| \quad f_x\|_{L^2}^{\frac{1}{2}}$, it follows that

$$\sup_{\tau\in[0,t]} \{\|\pi(\cdot,\tau)\|_{L^\infty}, \|\pi_z(\cdot,\tau)\|_{L^\infty}\} \leq N(t).$$

According to (22)-(23), the global well-posedness is provided below.

Theorem 2 *There exists a constant $\delta_1 > 0$, such that if $N(0) \leq \delta_1$, then Cauchy problem (22)-(23) has an unique global solution $\pi \in X(0,+\infty)$ such that*

$$\|\pi(.,t)\|_2^2 + \int_0^t \|\pi_z(.,\tau)\|_2^2 \, d\tau \leq C\|\pi_0\|_2^2 \leq CN^2(0), \tag{24}$$

for any $t \in [0,\infty)$ and the asymptotic stability

$$\sup_{z\in R} |\pi_z(z,t)| \to 0 \text{ as } t \to +\infty. \tag{25}$$

The existence of π in global sense in Theorem 2 is based on the existence theorem in local sense and the a priori estimates provided below.

Proposition 2 (Local existence) *For any $\varepsilon_1 > 0$, there exists a positive constant T depending on ε_1 such that if $\pi \in H^2$ with $N(0) \leq \varepsilon_1/2$, then problem (22), (23) has an unique solution $\pi \in X(0,T)$ satisfying $N(t) \leq 2N(0)$ for any $0 \leq t \leq T$.*

Conducting [20], the existence in local sense can be provided in standard strategy.

Proof Firstly, differentiating twice Eq. (22) with respect to z, one can provide

$$\begin{aligned}\pi_{zzt} = s\pi_{zzz} &+ f'''(U)\pi_z U_z^2 + f''(U)(2\pi_{zz}U_z \\ &+ \pi_z U_{zz}) + f'(U)\pi_{zzz} \\ &+ m(U^{m-1}\pi_z)_{zzz} + (F_1)_{zz} + (F_2)_{zz}.\end{aligned}$$

For $C > 0$, multiplying the above equation by π_{zz}/U, calculating the integration of results with respect to t, and conducting $N(t) \leq Dm(u_+ + u_-)$, $f''(U) > 0$, and $f'''(U) > 0$, we provide

$$\int \pi_{zz}^2 + \int_0^t \int \pi_{zzz}^2 \leq C \int \pi_{0zz}^2 + CN(t) \int_0^t \int \pi_{zz}^2.$$

Conducting the Sobolev theorem, the boundedness of all terms at the right hand side is given below

$$CN(t) \int_0^t \int \pi_{zzz}^2,$$

then one gets

$$N^2(t) + \int_0^t \int \pi_{zzz}^2 \leq CN^2(0) + CN(t) \int_0^t \int \pi_{zzz}^2.$$

By employing $N(t) \leq 1/C$, we have

$$N(t) \leq N(0)C^{-1/2} < 2N(0),$$

providing the proof of Proposition 2. $\square$

Proposition 3 (A priori estimates) *Let $\pi \in X(0,T)$ be a solution obtained in Proposition 2 for $T > 0$. Then there exists a constant $\varepsilon_2 > 0$, independent of T such that if $N(t) < \varepsilon_2$, then π satisfies (24) for any $0 \leq t \leq T$.*

The following energy estimates is conducted to provide the proof of Proposition 3.

Lemma 1 *Consider $\pi \in H^2(\mathbb{R})$ and the solution of (22)-(23). Then, for some constant $C > 0$ one can provide*

$$\begin{aligned}\|\pi(.,t)\|_2^2 + \int_0^t \|\pi_z(.,\tau)\|_2^2 d\tau &\leq C\|\pi_0\|_2^2 \\ + CN(t) \int_0^t \int \pi_{zz}^2\end{aligned} \tag{26}$$

The proof Proposition 3 can be provided from the Lemma 1.

Proof The proof of a priori estimate (24) is sufficiently employed to prove the Lemma 1. Conducting the Sobolev theorem at the right-hand side of nonlinear term (26), then one can provide the following boundedness

$$CN(t) \int_0^t \|\pi_z(.,\tau)\|_2^2 d\tau,$$

for some $C > 0$. Following from Lemma 1, and some $C > 0$, we can provide

$$N^2(t) + \int_0^t \|\pi_z(.,\tau)\|_2^2 d\tau \le CN^2(0) + CN(t)\int_0^t \|\pi_z(.,\tau)\|_2^2 d\tau,$$

in any $t \in [0, T]$. Conducting the condition $N(t) \le 1/2C$

$$N^2(t) + \int_0^t \|\pi_z(.,\tau)\|_2^2 d\tau \le CN^2(0),$$

then we can provide the proof of (24) in any $t \in [0, T]$.

□

Energy estimates

The a priori estimates of π in (22)-(23) is now a concern in this section, as a result the Proposition 3 is proved.

L^2-estimate of π

Lemma 2 *Under the assumptions of Lemma 1, if $N(t) \le Dm(u_+ + u_-)$, then*

$$\|\pi(.,t)\|^2 + \int_0^t \|\pi_z(.,\tau)\|^2 d\tau \le C\|\pi_0\|^2 + CN(t)\int_0^t\int \pi_{zz}^2 \tag{27}$$

Proof Multiplying Eq. (22) by π/U, integrating the results with respect to z, one can provide

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \frac{\pi^2}{U} + Dm\int U^{m-2}\pi_z^2 &= \int \frac{s\pi\pi_z}{U} + \int \frac{f'(U)\pi_z\pi}{U} \\ &\quad - \int DmU^{m-1}\pi\pi_z\left(\frac{1}{U}\right)_z + \int \frac{F_1\pi}{U} + \int \frac{F_2\pi}{U},\end{aligned} \tag{28}$$

By simplifying the calculations, one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \frac{\pi^2}{U} + Dm\int U^{m-2}\pi_z^2 &\le C\int \frac{\pi\pi_z^2}{U} - \int \frac{\pi^2}{2}\left[\left(\frac{s+f'(U)}{U}\right)_z \right. \\ &\quad \left. -DmU^{m-1}\left(\frac{1}{U}\right)_{zz}\right] \\ &\quad + \int \frac{F_1\pi}{U} + \int \frac{F_2\pi}{U}.\end{aligned} \tag{29}$$

It follows from (13), noting that

$$\begin{aligned}&\left(\frac{s+f'(U)}{U}\right)_z - mU^{m-1}\left(\frac{1}{U}\right)_{zz} \\ &= \left[\left(\frac{s+f'(U)}{U}\right) + DmU^{m-1}\frac{U_z}{U^2}\right]_z \\ &= \left[\frac{s+f'(U)}{U} + \frac{-sU+f(U)+su_+ - u_+^m}{U^2}\right]_z \\ &= \left[\frac{f(U)+Uf'(U)+su_+ - u_+^m}{U^2}\right]_z \\ &= -(f(U)+Uf'(U)+su_\pm - f(u_\pm))\frac{U_z}{U^3} > 0.\end{aligned} \tag{30}$$

Moreover, we estimate the following term

$$\left|\int \frac{F_1\pi}{U}\right| = \left|\int (f(U+\pi_z) - f'(U)\pi_z - f(U))\frac{\pi}{U}\right|,$$

by applying the following Taylor expansion and $\|\pi(\cdot,t)\|_{L^\infty} \le N(t)$ into the above equation,

$$\begin{aligned}F_1 &:= f(U+\pi_z) - f'(U)\pi_z - f(U) \\ &= \int_0^1 \left((1-s)f''(U+s\pi_z)ds\right)\pi_z^2 = \mathcal{O}(1)\pi_z^2,\end{aligned}$$

then one can derive

$$\left|\int \frac{F_1\pi}{U}\right| \le CN(t)\int \left(|\pi_z|^2 + |\pi_{zz}|^2\right). \tag{31}$$

The estimate of $|F_2|$ can be derived through the following binomial for the term $(\pi_z + U)^m$ in (28),

$$\begin{aligned}(\pi_z+U)^m &\le (\pi_z+u_-)^m = u_-^m\left(\frac{\pi_z}{u_-}+1\right)^m \\ &= \sum_{l=0}^m u_-^m \frac{P_l^m}{l!}\left(\frac{\pi_z}{u_-}\right)^l,\end{aligned} \tag{32}$$

where $P_l^m = \frac{m!}{(m-l)!}$. Because of $N(t) \ll 1$ and $\|\pi_z(\cdot,t)\|_{L^\infty} \le 1$, the inequality (32) can be rewritten as follows

$$\begin{aligned}(\pi_z+U)^m &\le u_-^m(m!)^2\pi_z^2\sum_{l=0}^m \frac{1}{l!}\left(\frac{1}{u_-}\right)^l \\ &= u_-^m(m!)^2\pi_z^2 e^{1/u_-} \le C\pi_z^2,\end{aligned} \tag{33}$$

which gives

$$\left|\int \frac{F_2\pi}{U}\right| \le CN(t)\int \left(|\pi_z|^2 + |\pi_{zz}|^2\right). \tag{34}$$

Substituting (30)-(34) into (29), one has

$$\frac{1}{2}\frac{d}{dt}\int \pi^2 + \int \left(DmU^{m-2} - \frac{CN(t)}{U} \right)\pi_z^2 \leq CN(t)\int \pi_{zz}^2. \tag{35}$$

The term U^{m-2} consists of two conditions as presented as follows

$$U^{m-2} = \left(\frac{1}{U}\right)^{2-m} \leq Ku_+ \leq Cu_+,\ \text{if } 0 < m < 2,$$
$$U^{m-2} \leq Lu_- \leq Cu_-,\ \text{if } m \geq 2,$$

where $C = max\{K, L\} = max\left\{ \frac{b}{m-b}, (m+b)^m \right\}$ for $b > 0$ and $m > b$. Then, (35) becomes

$$\frac{1}{2}\frac{d}{dt}\int \pi^2 + \int C\big(Dm(u_+ + u_-) - N(t)\big)\pi_z^2 \leq CN(t)\int \pi_{zz}^2. \tag{36}$$

By employing $N(t) \leq Dm(u_+ + u_-)$, further calculation of integration of (36) with respect to t, the proof of estimate π in L^2 is completed. □

H^1-estimate of π

Lemma 3 *Under the assumptions of Lemma 1, if $N(t) \leq Dm(u_+ + u_-)$, then*

$$\|\pi(.,t)\|_1^2 + \int_0^t \|\pi_z(.,\tau)\|_1^2 d\tau \leq C\|\pi_0\|_1^2. \tag{37}$$

Proof Firstly, differentiating Eq. (22) with respect to z, then one provides

$$\pi_{zt} = s\pi_{zz} + f'(U)\pi_{zz} + f''(U)\pi_z U_z + m(U^{m-1}\pi_z)_{zz} + (F_1)_z + (F_2)_z. \tag{38}$$

Multiplying Eq. (38) by π_z/U, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \frac{\pi_z^2}{U} + Dm\int U^{m-2}\pi_{zz}^2 = &\int \frac{s\pi_z\pi_{zz}}{U} - \int DmU^{m-1}\pi_z\pi_{zz}\left(\frac{1}{U}\right)_z \\ &- \int \frac{(f'(U)\pi_{zz} + f''(U)\pi_z U_z)\pi_z}{U} \\ &+ \int m(m-1)U^{m-1}\left(\frac{1}{U}\right)_z \pi_z\pi_{zz} \\ &+ \int m(m-1)U^m\left(\pi_z\left(\frac{1}{U}\right)_z\right)^2 \\ &+ \int |F_1 + F_2|\left(|\pi_z| + \frac{|\pi_{zz}|}{U}\right).\end{aligned} \tag{39}$$

The similar strategy of the estimate in L^2, provide the estimate of $(\pi_z + U)^{m-1}$ in (39) having two results when $m \geq 1$ and $0 < m < 1$, respectively: $(\pi_z + U)^{m-1} \leq (\pi_z + u_-)^{m-1}$ and $(\pi_z + U)^{m-1} \leq (\pi_z + u_+)^{m-1}$. Following from (22), $u_+ \leq U(z) \leq u_-$, Taylor expansion, and (33), one can provide

$$|F_1| \leq C(|\pi_{zz}|^2 + |\pi_z|^2),\quad |F_2| \leq C(|\pi_z||\pi_{zz}| + |\pi_z|^2). \tag{40}$$

The Young's inequality and the estimate of $\|\pi(\cdot,t)\|_{L^\infty} \leq N(t)$ are employed to provide

$$\int |F_1 + F_2|\left(|\pi_z| + \frac{|\pi_{zz}|}{U}\right) \leq CN(t)\int \left(|\pi_z|^2 + |\pi_{zz}|^2\right). \tag{41}$$

Moreover, by inserting (40)-(41) into (39), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \frac{\pi_z^2}{U} + \int \left(DmU^{m-2} - \frac{CN(t)}{U}\right)\pi_{zz}^2 \leq &\int \frac{CN(t)}{U}(|\pi_z|^2 + |\pi_{zz}|^2) \\ &- \int \frac{\pi_z^2}{2}\left[\left(\frac{s + f'(U)}{U}\right) - DmU^{m-1}\left(\frac{1}{U}\right)_z\right]_z.\end{aligned} \tag{42}$$

Conducting (30) into (42), the Young's inequality, $\|\pi_z\|_{L^\infty} \ll N(t)$, $f'(U) > 0$, $f''(U) > 0$, and joining the results with (27), one has

$$\frac{1}{2}\frac{d}{dt}\int \pi_z^2 + \int C(Dm(u_+ + u_-) - N(t))\pi_{zz}^2 \leq CN(t)\int \pi_z^2. \tag{43}$$

Integrating (43) in t, by joining the results with the estimate of π in L^2, one provides

$$\int \pi_z^2 + \int_0^t \int C(Dm(u_+ + u_-) - N(t))\pi_{zz}^2 \leq C\int \pi_{0z}^2. \tag{44}$$

Employing the fact $N(t) \leq Dm(u_+ + u_-)$, the estimate of π in H^1 is provided. □

H^2-estimate of π

Lemma 4 *Under the assumptions of Lemma 1, if $N(t) \leq Dm(u_+ + u_-)$, then*

$$\|\pi(.,t)\|_2^2 + \int_0^t \|\pi_z(.,\tau)\|_2^2 d\tau \le C\|\pi_0\|_2^2 \tag{45}$$

Proof Differentiating Eq. (38) with respect to z, then one has

$$\begin{aligned}\pi_{zzt} &= s\pi_{zzz} + f'''(U)\pi_z U_z^2 + f''(U)(2\pi_{zz}U_z \\ &\quad + \pi_z U_{zz}) + f'(U)\pi_{zzz} \\ &\quad + m(U^{m-1}\pi_z)_{zzz} + (F_1)_{zz} + (F_2)_{zz}.\end{aligned} \tag{46}$$

Now, multiplying (37) by π_{zz}/U, one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \frac{\pi_{zz}^2}{U} + Dm\int U^{m-2}\pi_{zzz}^2 \\ &= \int \frac{s\pi_{zz}\pi_{zzz}}{U} - \int mU^{m-1}\pi_{zz}\pi_{zzz}\Big(\frac{1}{U}\Big)_z \\ &\quad + \int \frac{(F_1)_{zz}\pi_{zz}}{U} + \int \frac{(F_2)_{zz}\pi_{zz}}{U} \\ &\quad + \int \frac{(f'''(U)\pi_z U_z^2 + f''(U)(2\pi_{zz}U_z + \pi_z U_{zz}) + f'(U)\pi_{zzz})\pi_{zz}}{U} \\ &\quad + \int m(m-1)U^{m-1}\Big(\frac{1}{U}\Big)_z \pi_{zz}\pi_{zzz} \\ &\quad + \int Dm(m-1)U^m\Big(\pi_{zz}\Big(\frac{1}{U}\Big)_z\Big)^2 \\ &\quad + \int m^2(m-1)U^m\Big(\frac{1}{U}\Big)_z^2 \frac{\pi_z\pi_{zz}\pi_{zzz}}{U} \\ &\quad + \int m^2(m-1)U^{m-1}U_z\Big(\frac{1}{U}\Big)_z^2 \frac{\pi_z\pi_{zz}^2}{U} \\ &\quad + \int m(m-1)U^m\Big(\frac{1}{U}\Big)_{zz}\frac{\pi_z\pi_{zz}\pi_{zzz}}{U^2} \\ &\quad + \int m(m-1)U^{m-1}\Big(\frac{1}{U}\Big)_{zz}\Big(\frac{1}{U}\Big)_z\frac{\pi_z\pi_{zz}^2}{U}.\end{aligned} \tag{47}$$

The similar strategy in L^2 and H^1 is employed to provide the estimate of $(\pi_z + U)^{m-2}$. Then, one can derive

$$\begin{aligned}\int \frac{(F_1)_{zz}\pi_{zz}}{U} &= -\int \frac{(F_1)_z\pi_{zzz}}{U} + \int \frac{(F_1)_z\pi_{zz}U_z}{U^2}, \\ \int \frac{(F_2)_{zz}\pi_{zz}}{U} &= -\int \frac{(F_2)_z\pi_{zzz}}{U} + \int \frac{(F_2)_z\pi_{zz}U_z}{U^2},\end{aligned} \tag{48}$$

and

$$\begin{aligned}(F_1)_z &= f'(U+\pi_z)(U_z + \pi_{zz}) \\ &\quad - f''(U)\pi_z U_z - f'(U)(\pi_{zz} + U_z) \\ (F_2)_z &= m((U+\pi_z)^{m-1} - U^{m-1})\pi_{zzz} \\ &\quad + m(m-1)(U+\pi_z)^{m-2}\pi_{zz}^2 \\ &\quad + m(m-1)U_z^2((U+\pi_z)^{m-2} \\ &\quad - U^{m-2} - (m-2)U^{m-3}\pi_z) \\ &\quad + mU_{zz}((U+\pi_z)^{m-1} \\ &\quad - U^{m-1} - (m-1)U^{m-2}\pi_z) \\ &\quad + 2m(m-1)U_z((U+\pi_z)^{m-2} \\ &\quad - U^{m-2})\pi_{zz}.\end{aligned} \tag{49}$$

Employing Young's inequality and Taylor expansion as in the L^2-estimate of π

$$\begin{aligned}|(F_1)_z| &\le \left|f'(U+\pi_z) - f''(U)\pi_z - f'(U)\right|\left|U_z + \pi_{zz}\right| \\ &= \left|\int_0^1 ((1-s)f'''(U+s\pi_z)ds)\pi_z^2\right|\left|U_z + \pi_{zz}\right| \\ &= \mathcal{O}(1)\pi_z^2\left|U_z + \pi_{zz}\right|,\end{aligned}$$

and applying $\|\pi_z(\cdot,t)\|_{L^\infty} \le N(t)$, (49), $U_z \le C$, and (40), one can rewrite (48) as follows

$$\begin{aligned}\left|\int \frac{(F_1)_{zz}\pi_{zz}}{U}\right| &\le CN(t)\int\left(\frac{|\pi_{zz}|^2 + |\pi_{zz}||\pi_{zzz}|}{U}\right) \\ \left|\int \frac{(F_2)_{zz}\pi_{zz}}{U}\right| &\le CN(t)\int\left(\frac{|\pi_{zz}|^2 + |\pi_{zzz}|^2}{U}\right).\end{aligned} \tag{50}$$

Substituting (48)-(50) into (47), one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \frac{\pi_{zz}^2}{U} + \int C(Dm(u_+ + u_-) - N(t))\pi_{zzz}^2 \\ &\le -\int \frac{\pi_{zz}^2}{2}\left[\left(\frac{s + f'(U)}{U}\right) - mU^{m-1}\Big(\frac{1}{U}\Big)_z\right]_z \\ &\quad + \int CN(t)\frac{|\pi_{zz}|^2}{U}.\end{aligned} \tag{51}$$

Conducting (30) into (51), the Young's inequality, the estimate of π in L^2 and H^1 respectively in (27)-(37), $f'(U) > 0$, $f''(U) > 0$, and by employing the integration in t, one can provide

$$\int \pi_{zz}^2 + (Dm(u_+ + u_-) - N(t))\int_0^t \int \pi_{zzz}^2 \le C\int \pi_{0zz}^2. \tag{52}$$

Using the fact $N(t) \le Dm(u_+ + u_-)$, the proof of estimate of π in H^2 is completed. □

Proof The proof of main Theorem 2 is now a concern to be studied. Theorem 1 has the consequence to provide the proof of Theorem 2 based on the transformation (20). Then, the global well-posedness of (22)-(23) in $X(0,+\infty)$ is provided in the standard strategy. For $\pi \in H^2$, then the global estimate (24) is given as follows

$$\int_0^t \int_{-\infty}^{\infty} \pi_z^2(z,\tau)dzd\tau \le C\|\pi_0\|_2^2 \le CN^2(0). \tag{53}$$

In view of the first equation of (22), one has

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty}\pi_z^2(z,t)dz &= -2\int_{-\infty}^{\infty}\pi_t\pi_{zz}dz \\ &= -2\int_{-\infty}^{\infty}\pi_{zz}\big(s\pi_z + f'(U)\pi_z + m(U^{m-1}\pi_z)_z + F_1 + F_2\big).\end{aligned}$$

Moreover, $f'(U) > 0$, $f''(U) > 0$, $f(U) > 0$, the following Taylor expansion and binomial theorem, respectively

$$
\begin{aligned}
&f(U+\pi_z) - f'(U)\pi_z - f(U) \\
&\quad = \int_0^1 \big((1-s)f''(U+s\pi_z)ds\big)\pi_z^2 = \mathcal{O}(1)\pi_z^2, \\
&(\pi_z + U)^m \le u_-^m (m!)^2 \pi_z^2 \sum_{l=0}^{m} \frac{1}{l!}\left(\frac{1}{u_-}\right)^l \\
&\quad = u_-^m (m!)^2 \pi_z^2 e^{1/u_-} \le C\pi_z^2,
\end{aligned}
$$

and Young's inequality are employed to provide

$$
\frac{d}{dt}\int_{-\infty}^{\infty} \pi_z^2(z,t)dz \le C\int_{-\infty}^{\infty} (\pi_{zz}^2 + \pi_z^2).
$$

Applying the global estimate in (24), we have

$$
\begin{aligned}
&\int_0^\infty \left|\frac{d}{dt}\int_{-\infty}^{\infty} \pi_z^2(z,t)dz\right| \le C\int_0^\infty \int_{-\infty}^{\infty} (\pi_{zz}^2 + \pi_z^2) \\
&\quad \le C\|\pi_0\|_{2,w}^2 \le CN^2(0).
\end{aligned} \tag{54}
$$

From (53) and (54), one gives

$$
\int_{-\infty}^{\infty} \pi_z^2(z,t)dz \to 0 \text{ as } t \to +\infty.
$$

The Cauchy–Schwarz inequality is conducted, then one can provide

$$
\begin{aligned}
\pi_z^2(z,t) &= 2\int_{-\infty}^{z} \pi_z \pi_{zz}(y,t)dy \\
&\le 2\left(\int_{-\infty}^{+\infty} \pi_z^2(y,t)dy\right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \pi_{zz}^2(y,t)dy\right)^{\frac{1}{2}} \\
&\le C\left(\int_{-\infty}^{+\infty} \pi_z^2(y,t)dy\right)^{\frac{1}{2}} \to 0 \text{ as } t \to +\infty,
\end{aligned}
$$

which complete the proof of (25). □

Examples

Given the findings in this paper, it is time to provide examples, validate the findings using simulations, and demonstrate the effect of nonlinear degenerate viscosity. For simplification, we assume that $f(u) = \frac{u^2}{2}$ in (1), then one has

$$
u_t + uu_x = Dm\big((m-1)^{m-2}(u_x)^2 + u^{m-1}u_{xx}\big), \tag{55}
$$

where $m > 0$, $u = u(x,t)$, and the initial conditions

$$
\begin{aligned}
&u(x,0) = u_0(x) = 0.5 + \frac{1}{1+e^{(x-80)}}, \\
&u_0(-\infty) = u_- = 1.5,\; u_0(+\infty) = u_+ = 0.5.
\end{aligned} \tag{56}
$$

Moreover, the initial perturbation is given below

$$
(u_0 - U)(x) = \frac{0.5\sin(x)}{((x-80)/10)^2 + 1}. \tag{57}
$$

By employing the finite difference in Eq. (55), one can derive

$$
\begin{aligned}
u(j,n+1) = u(j,n) &+ \frac{Dm\Delta t}{4\Delta x^2}\big((m-1)u(j,n)^{m-2} \\
&(u(j+1,n) - u(j-1,n))^2\big) \\
&+ \frac{Dm\Delta t}{\Delta x^2}\big(u(j,n)^{m-1}(u(j+1,n) \\
&-u(j,n) - u(j-1,n))\big) \\
&- \frac{\Delta t}{2\Delta x}(u(j,n)(u(j+1,n) - u(j-1,n))).
\end{aligned}
$$

Figure 1a shows the traveling fronts with the initial traveling fronts in Eq. (56) and initial perturbation in Eq. (57). Moreover, Figure 1b provides the profile of traveling fronts with the varied values of nonlinear degenerate viscosity $m = 1,2,3,4,5,6,7$, where the profiles of traveling fronts move from right to left when the varied values of m go higher. Figures 1c–1f give the illustration of traveling fronts for various of wave speed s with the current values of m where the wave speed s is stated as $z = x - st$. We can see that the profiles of traveling fronts have the movement from left to right.

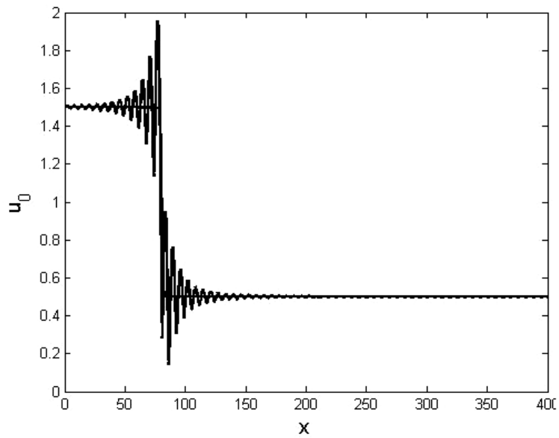
Moreover, we modify Eq. (55) by assuming $f(u) = \frac{u^m}{m}$, then one has

$$
u_t + u^{m-1}u_x = Dm\big((m-1)^{m-2}(u_x)^2 + u^{m-1}u_{xx}\big), \tag{58}
$$

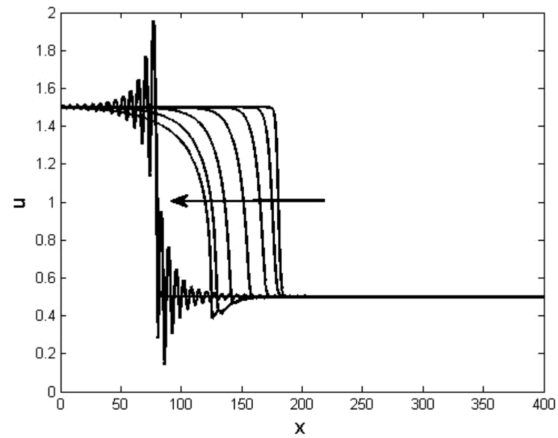
which provides the numerical scheme

$$
\begin{aligned}
u(j,n+1) = u(j,n) &+ \frac{Dm\Delta t}{4\Delta x^2}\big((m-1)u(j,n)^{m-2} \\
&(u(j+1,n) - u(j-1,n))^2\big) \\
&+ \frac{Dm\Delta t}{\Delta x^2}\big(u(j,n)^{m-1}(u(j+1,n) \\
&-u(j,n) - u(j-1,n))\big) \\
&- \frac{\Delta t}{2\Delta x}\big(u(j,n)^{m-1}(u(j+1,n) \\
&-u(j-1,n))\big).
\end{aligned}
$$

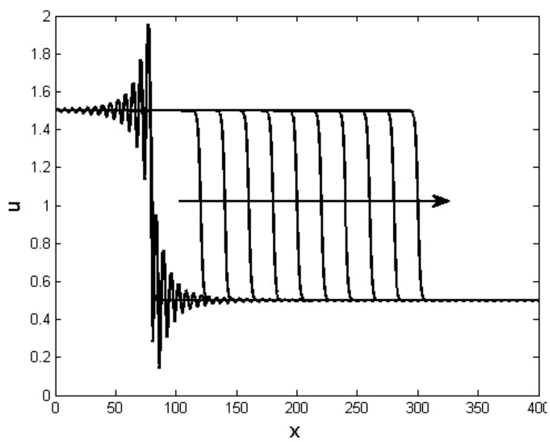
According to the initial conditions in Eq. (56) and initial perturbations in Eq. (57), we establish the numerical simulations of Eq. (58) as shown in Fig. 2 with the varied values of wave speed s and nonlinear degenerate viscosity $m = 1,2,3,4,5,6$.



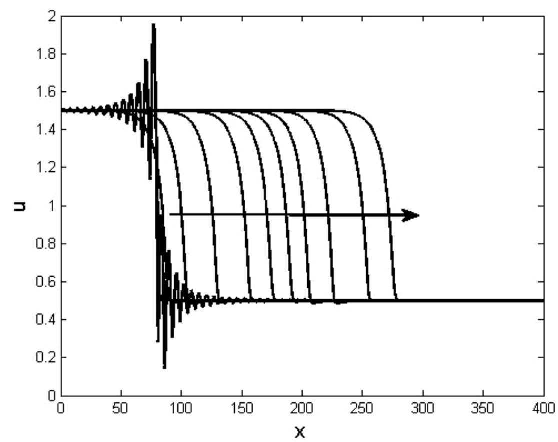
(a) $u_0(x)$



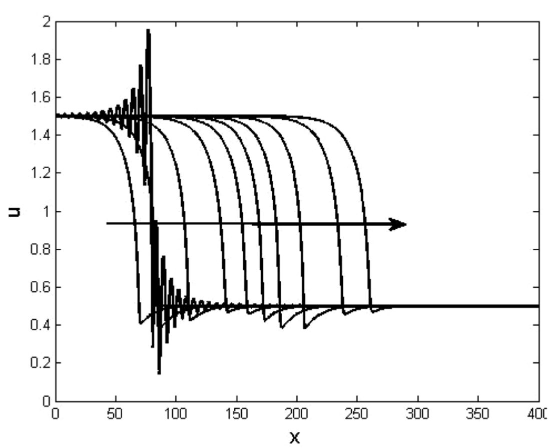
(b) $u(x,t)$ for varied values of m



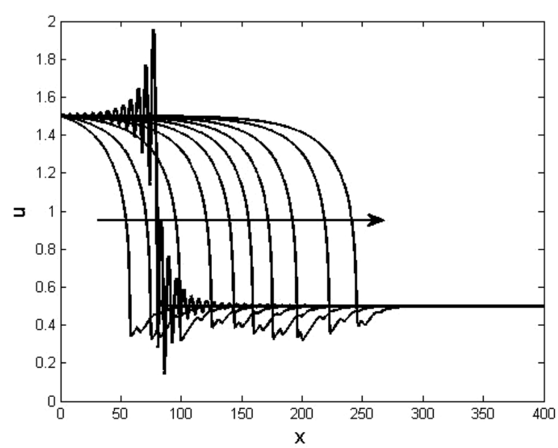
(c) $u(x,t)$ for varied values of s with $m=1$



(d) $u(x,t)$ for varied values of s with $m=4$

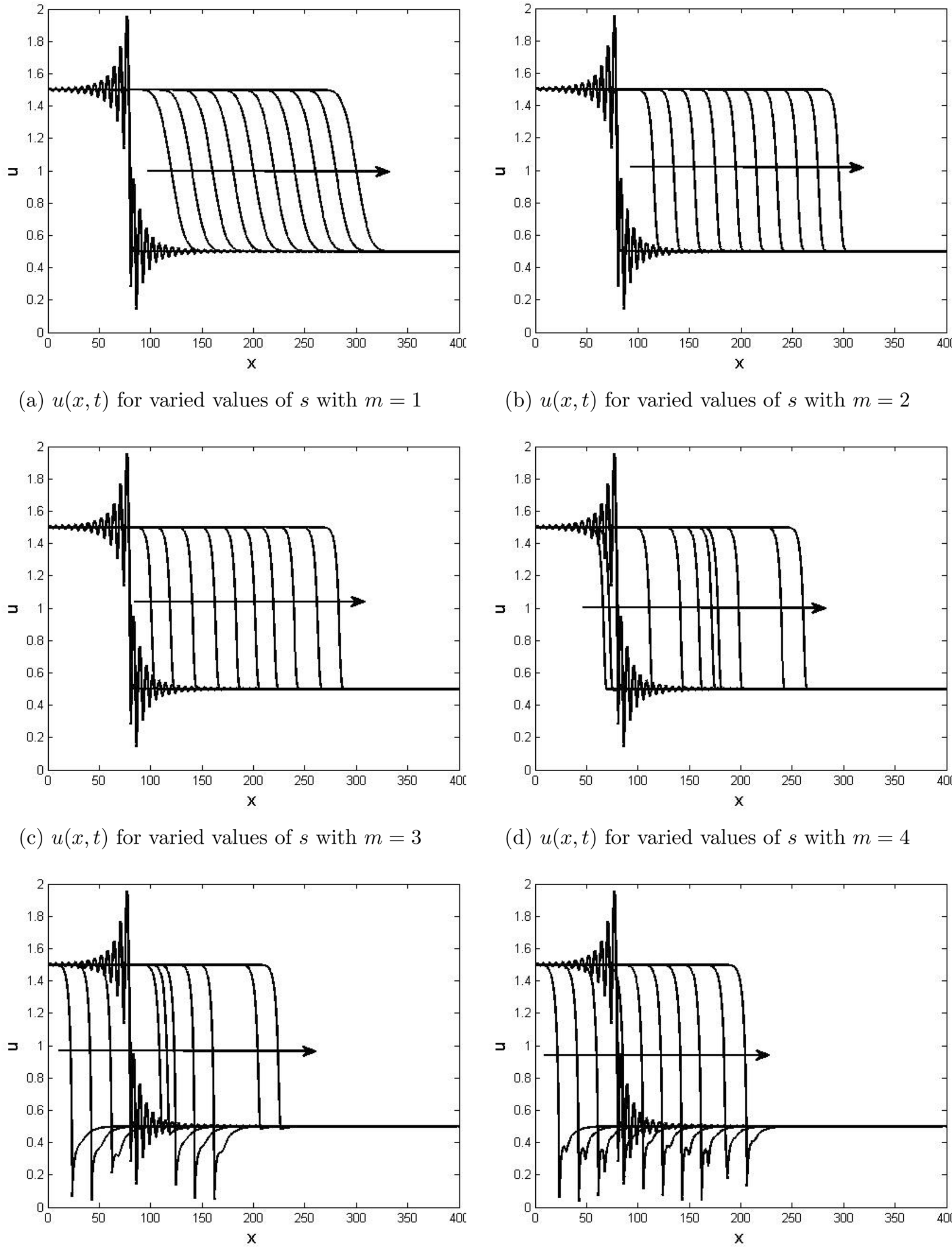


(e) $u(x,t)$ for varied values of s with $m=5$



(f) $u(x,t)$ for varied values of s with $m=7$

Fig. 1 1a–1f Initial and time evolution of traveling fronts $u_0(x)$ and $u(x, t)$, respectively, with the varied values of m and s



(a) $u(x,t)$ for varied values of s with $m=1$

(b) $u(x,t)$ for varied values of s with $m=2$

(c) $u(x,t)$ for varied values of s with $m=3$

(d) $u(x,t)$ for varied values of s with $m=4$

(e) $u(x,t)$ for varied values of s with $m=5$

(f) $u(x,t)$ for varied values of s with $m=6$

Fig. 2 2a–2f Time evolution of traveling fronts $u(x, t)$ with the varied values of m and s

Conclusions

In this paper, we study the stability of traveling fronts to viscous Burgers' equations with the nonlinear degenerate viscosity. The estimate of terms $f(U)_x$ and $D(u^m)_{xx}$ are introduced in Remark 1 to overcome the result of energy estimate. The stability of this paper is based on the energy estimate in L^2, H^1, and H^2 under the condition $N(t) \leq Dm(u_+ + u_-)$. Moreover, the numerical simulation is provided in this paper to validate our works and demonstrate the effect of nonlinear degenerate viscosity. The future study of this paper is to apply the large perturbation in the traveling fronts of Eq. (1).

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