Differential Equations and Dynamical Systems (October 2024) 32(4):965–977
https://doi.org/10.1007/s12591-022-00602-1

ORIGINAL RESEARCH

Asymptotic Stability of Singular Traveling Waves to Degenerate Advection-Diffusion Equations Under Small Perturbation

Mohammad Ghani[1]

Accepted: 16 April 2022 / Published online: 9 May 2022

Abstract
The main equation of this paper is the special case of equation studied by Il'in and Oleinik for single model equation with convex nonlinearity [4], by considering $f(u) = u^m$ and nonlinear diffusion, for $m > 0$. We are interested in the stability of the degenerate advection-diffusion equation by dealing with the singular term when $u_+ = 0$. We first transform the original equation into the traveling waves by using the ansatz transformation. The weighted energy estimates of the transformed equation are then established, where the aim of this weighted function is to avoid the singular term when $u_+ = 0$. At the final stage, the stability of traveling waves is shown based on the weighted energy estimates and appropriate perturbations.

Keywords Stability · Degenerate advection-diffusion · Small perturbation · Large wave amplitude

Mathematics Subject Classification 35A01 · 35B40

Introduction

Our goal is to establish the stability of traveling waves to the following advection-diffusion equation with porous medium diffusion,

$$u_t + (u^m)_x = (u^m)_{xx}, \tag{1}$$

where $m > 0$, $u = u(x, t)$ and the initial condition

$$u(x, 0) = u_0(x) \to u_\pm \text{ as } x \to \pm\infty. \tag{2}$$

The Eq. (1) is the special case of following equation with convex nonlinearity,

✉ Mohammad Ghani
mohammad.ghani2013@gmail.com

1 Faculty of Advanced Technology and Multidiscipline, Universitas Airlangga, Surabaya 60115, Indonesia

$$u_t + f(u)_x = \mu u_{xx}, \tag{3}$$

where a constant $\mu > 0$ and a smooth function $f''(u) > 0$. Il'in and Oleinik studied the stability of shock profiles to (3) based on the maximum principle [4] and spectral analysis studied by Sattinger in [19].

Moreover, the Eq. (1) reduces to Burger's equation, when $m = 2$ (advection term) and $m = 1$ (diffusion term). Mickens and Oyedeji [17] studied traveling wave solutions to Burgers containing square root and non-diffusion Fisher,

$$\begin{aligned} u_t + a_1\sqrt{u}u_x &= D_1 u_{xx}, \\ u_t + a_2\sqrt{u}u_x &= \lambda_1\sqrt{u} - \lambda_2 u, \end{aligned} \tag{4}$$

where $a_1 > 0, a_2 > 0, D_1 > 0, \lambda_1 > 0, \lambda_2 > 0$.

Moreover, other researches containing the square root term were studied by Buckmire et. al [1], Jordan [7], and Mickens [15, 16]. This current paper, we are concerned with the stability of traveling waves of (1) with porous medium diffusion. Since there is assumption of singularity, we use the weighted energy estimates to deal with the stability of traveling waves of (1) under small perturbation and arbitrary wave amplitude as in [6, 8]. The energy method was also used to establish the stability of traveling waves to coupled Burgers equation as in [5],

$$\begin{aligned} &u_t + \left(\frac{1}{2}u^2 + \frac{1}{2}b^2\right)_x = \mu u_{xx}, \\ &b_t + (ub)_x = \nu b_{xx}, \end{aligned} \tag{5}$$

where the small coefficient was not required.

A similar system to (5) was studied by Li and Wang [12], as shown the following equation

$$\begin{aligned} u_t - (uv)_x &= D u_{xx}, \\ v_t + (\varepsilon v^2 - u)_x &= \varepsilon v_{xx}, \end{aligned} \tag{6}$$

where this Eq. (6) was derived from Keller-Segel model, so-called chemotaxis model and the coefficients ε was assumed small enough. This research was contrast with the research in [5] where the smallness of coefficients was not required. We refer to [9, 10, 13] for other references employing the elementary energy method to study the stability problem.

The main issue of this paper are nonlinear diffusion and singular term when $u_+ = 0$. We employ the weighted function to handle the singular term as studied in [6]. Moreover, the nonlinear diffusion has been studied in [3] for the traveling wave problem to chemotaxis model with $u_+ > 0$.

We organize this paper as follows. We first present the existence of traveling waves to the transformed nonlinear advection-diffusion Eq. (1) in the section "Transformation of the Original Equation "by the ansatz traveling waves, derive the appropriate perturbations, and the stability of traveling waves of (1). Moreover, we establish the a priori estimate by using the weighted energy estimates in the section" Weighted Energy Estimate".

Notation 1 Throughout this paper, we present the usual notation of Sobolev space $H^r(\mathbb{R})$ where the norms are defined as $\|f\|_r := \sum_{k=0}^{r} \|\partial_x^k f\|$ and $\|f\| := \|f\|_{L^2(\mathbb{R})}$. Moreover, we present the notation $H^r_w(\mathbb{R})$ to represent the weighted Sobolev space where the norms are given as $\|f\|_{r,w} := \sum_{k=0}^{r} \|\sqrt{w(x)}\partial_x^k f\|$ and $\|f\|_w := \|f\|_{L^2_w(\mathbb{R})}$.

Transformation of the Original Equation

The traveling wave solution of (1) can be stated in the form

$$u(x,t) = U(z), \quad z = x - st, \tag{7}$$

satisfying

$$-sU_z + (U^m)_z = (U^m)_{zz}, \tag{8}$$

with the boundary conditions

$$U(z) \to u_\pm \ \text{ as } \ z \to \pm\infty. \tag{9}$$

Now, integrating (8) with respect to z,

$$(U^m)_z = -sU + U^m + G, \tag{10}$$

where $G = su_\pm - u_\pm^m$. By using the fact that $U_z \to 0$ as $z \to \pm\infty$, one has Rankine-Hugoniot

$$s(u_+ - u_-) = u_+^m - u_-^m. \tag{11}$$

Then by (11) with $u_+ = 0$, yields

$$s = u_-^{m-1} > 0. \tag{12}$$

We further, present the following proposition to deal with the existence of traveling waves

Proposition 1 *Let $u_\pm$ satisfy (11) and $u_+ = 0$. Then there exists a monotone traveling waves $U(x - st)$ to (8), which is unique up to a translation and satisfies $U_z < 0$, where s is given by (12). Moreover, U monotonicity behavior at $z = \pm\infty$ with rates*

$$U - u_\pm \sim e^{\lambda_\pm z},$$

where $\lambda_- = \frac{m-1}{m}$ and $\lambda_+ = 1$.

Proof It follows from (10), one has

$$U_z = \frac{U^{1-m}}{m} \cdot (-sU + U^m + G) := M(U).$$

We differentiate $M(U)$ with respect to U, substitute $G = su_\pm - u_\pm^m$ and the points $u_\pm$ into the results, then one yields

$$\lambda := \frac{dM(U)}{dU} = \frac{m - su_\pm^{1-m}}{m}.$$

By substituting the wave speed s in (12) into λ, and using the fact $u_+ = 0$, then at the points $u_\pm$, one can derive

$$\lambda_- = \frac{m-1}{m}, \quad \lambda_+ = 1,$$

which completes the proof. □

Remark 1 Since we consider $u_+ = 0$. Then we construct the following weighted function to handle the singularity in energy estimates

$$w(z) := 1 + e^{\eta z}, \quad \text{where} \quad \eta = \frac{m-1}{m} > 0 \quad \text{for} \quad z \in \mathbb{R}$$

which gives

$$C_1 w(z) \le \frac{1}{U(z)} \le C_2 w(z),$$

where $C_2 > C_1 > 0$.

By referring to [11, 14], let us denote the following perturbation of zero mass of ϕ as

$$\phi_0(z) = \int_{-\infty}^{z} (u_0 - U)(y)dy.$$

Then to achieve our goal of this paper for the stability of traveling waves of (1), we give the following theorems.

Theorem 1 *Let $U(x - st)$ be the traveling waves obtained in Proposition 1. Then there exists a constant $\varepsilon_0 > 0$ such that if $\|u_0 - U\|_{1,w} + \|\phi_0\|_w \le \varepsilon_0$, then the Cauchy problem (1), (2) has an unique global solution $u(x, t)$ satisfying*

$$u - U \in C\big([0, \infty); H^1_w\big) \cap L^2([0, \infty); H^1_w),$$

and the asymptotic stability

$$\sup_{x\in\mathbb{R}} |u(x, t) - U(x - st)| \to 0 \quad \text{as} \quad t \to +\infty.$$

By changing the variables $(x, t) \to (z = x - st, t)$, then the equation (1) becomes

$$u_t - su_z + (u^m)_z = (u^m)_{zz} \tag{13}$$

By decomposing the solution u of (13) as

$$u(z, t) = U(z) + \phi_z(z, t), \tag{14}$$

then, one has

$$\phi(z, t) = \int_{-\infty}^{z} (u(y, t) - U(y))dy. \tag{15}$$

Now, we substitute (14) into (13) and integrating the resuls in z, one has

$$\phi_t = s\phi_z - (\phi_z + U)^m + m\big(U^{m-1}\phi_z\big)_z + F \tag{16}$$

where $F = \big((\phi_z + U)^m - U^m - mU^{m-1}\phi_z\big)_z$ and initial data of ϕ

$$\phi(z,0)=\phi_0(z)=\int_{-\infty}^{z}(u_0-U)dy \tag{17}$$

with $\phi_0(\pm\infty)=0$. We further find the solution of transformed problem (16), (17) in the space

$$X(0,T):=\left\{\phi(z,t)\in C([0,T),H_w^2)\,:\,\phi_z\in L^2((0,T);H_w^2))\right\}$$

with $0<T\le+\infty$. Let

$$N(t):=\sup_{0\le\tau\le t}\left\{\|\phi(.,\tau)\|_{2,w}\right\}.$$

From the Sobolev inequality $\|f\|_{L^\infty}\le\sqrt{2}\|f\|_{L_w^2}^{\frac{1}{2}}\|f_x\|_{L_w^2}^{\frac{1}{2}}$, it follows that

$$\sup_{\tau\in[0,t]}\{\|\phi(\cdot,\tau)\|_{L^\infty},\|\phi_z(\cdot,\tau)\|_{L^\infty}\}\le N(t).$$

For (16), (17), we have the following global well-posedness.

Theorem 2 *There exists a constant* $\delta_1>0$, *such that if* $N(0)\le\delta_1$, *then Cauchy problem* (16), (17) *has an unique global solution* $\phi\in X(0,+\infty)$ *such that*

$$\|\phi(.,t)\|_{2,w}^2+\int_0^t\|\phi_z(.,\tau)\|_{2,w}^2\,d\tau\le C\|\phi_0\|_{2,w}^2\le CN^2(0), \tag{18}$$

for any $t\in[0,\infty)$ *and the asymptotic stability*

$$\sup_{z\in R}|\phi_z(z,t)|\to 0 \text{ as } t\to+\infty. \tag{19}$$

The global existence of ϕ stated in Theorem 2 follows from the local existence theorem and the a priori estimates which are given below.

Proposition 2 (Local existence) *For any* $\varepsilon_1>0$, *there exists a positive constant T depending on* ε_1 *such that if* $\phi\in H_w^2$ *with* $N(0)\le\varepsilon_1/2$, *then problem* (16), (17) *has an unique solution* $\phi\in X(0,T)$ *satisfying* $N(t)\le 2N(0)$ *for any* $0\le t\le T$.

By referring to [18], we can establish the local existence under a standard way.

Proof We differentiate (16) in z twice that

$$\begin{aligned}\phi_{zzt}&=s\phi_{zzz}-m(\phi_z+U)^{m-1}(\phi_{zzz}+U_{zz})\\&\quad-m(m-1)(\phi_z+U)^{m-2}(\phi_{zz}+U_z)^2+m(U^{m-1}\phi_z)_{zzz}+F_{zz}.\end{aligned}$$

Then multiplying by ϕ_{zz}/U, integrating the results in t, and applying $N(t)\ll 1$, one yields

$$\int\phi_{zz}^2w+\int_0^t\int\phi_{zzz}^2w\le C\int\phi_{0zz}^2w+CN(t)\int_0^t\int\phi_{zz}^2w,$$

for some $C>0$. By Sobolev embedding theorem, all the terms on the right hand side can be bounded by

$$CN(t)\int_0^t\int \phi_{zzz}^2 w,$$

then one gets

$$N^2(t)+\int_0^t\int \phi_{zzz}^2 w \le CN^2(0)+CN(t)\int_0^t\int \phi_{zzz}^2 w.$$

By choosing $C=4$ such that $N(t)\le 1/4$, one has

$$N(t)\le 2N(0),$$

which completes the proof of Proposition 2. □

We further need to establish the a priori estimate.

Proposition 3 (A priori estimates) *Let $\phi\in X(0,T)$ be a solution obtained in Proposition 2 for some time $T>0$. Then there exists a constant $\varepsilon_2>0$, independent of T such that if $N(t)<\varepsilon_2$, then ϕ satisfies (18) for any $0\le t\le T$.*

The proof of Proposition 3 is based on the following energy estimates.

Lemma 1 *Let $\phi\in H_w^2(\mathbb{R})$ and ϕ be solution of (16), (17), then there exists a constant $C>0$ such that*

$$\|\phi(.,t)\|_{2,w}^2+\int_0^t \|\phi_z(.,\tau)\|_{2,w}^2 d\tau \le C\|\phi_0\|_{2,w}^2+CN(t)\int_0^t\int \phi_{zz}^2 w \tag{20}$$

Then, we can prove the Proposition 3 through the Lemma 1.

Proof To establish the result, it only suffices to show that the a priori estimate (18) holds. By applying the Sobolev embedding theorem, we noting that all the nonlinear terms on the right-hand side of (20) can be bounded by

$$CN(t)\int_0^t \|\phi_z(.,\tau)\|_{2,w}^2 d\tau,$$

for some $C>0$. Then it follows from Lemma 1, one has

$$N^2(t)+\int_0^t \|\phi_z(.,\tau)\|_{2,w}^2 d\tau \le CN^2(0)+CN(t)\int_0^t \|\phi_z(.,\tau)\|_{2,w}^2 d\tau,$$

for $0\le t\le T$ and some constant $C>0$. For any $t\in[0,T]$, we choose $N(t)\le 1/2C$ to get

$$N^2(t)+\int_0^t \|\phi_z(.,\tau)\|_{2,w}^2 d\tau \le CN^2(0),$$

which gives the result (18). □

We are now ready to prove Theorem 2 which is the main theorem in this paper.

Proof By having the transformation (14), Theorem 1 is a consequence of Theorem 2. The a priori estimate (18) guarantees that $N(t)$ is small if $N(0)$ is small enough. Thus, applying the standard extension procedure, we get the global well-posedness of (16), (17) in $X(0, +\infty)$.

Next, we prove the convergence (19). Clearly, if $\phi \in H^2_w$, then $\phi \in H^2$, since $w \geq 1$. Owing to the global estimate (18), we get

$$\int_0^t \int_{-\infty}^{\infty} \phi_z^2(z,\tau)dzd\tau \leq C\|\phi_0\|^2_{2,w} \leq CN^2(0). \tag{21}$$

In view of the first equation of (16), one has

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty} \phi_z^2(z,t)dz &= -2\int_{-\infty}^{\infty} \phi_t\phi_{zz}dz \\
&= -2\int_{-\infty}^{\infty} \phi_{zz}\big(s\phi_z - (\phi_z + U)^m + m(U^{m-1}\phi_z)_z + F\big).
\end{aligned}$$

Then, by employing the binomial theorem and Young's inequality, one yields

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty} \phi_z^2(z,t)dz &\leq 2\int_{-\infty}^{\infty} \phi_{zz}\left(s\phi_z + u_-^m(m!)^2\phi_z^2\sum_{l=0}^{m}\frac{1}{l!}\left(\frac{1}{u_-}\right)^l + \phi_{zz}\right) \\
&\quad + 2\int_{-\infty}^{\infty} \phi_{zz}\big(m(U^{m-1}\phi_z)_z + 2C\phi_z\phi_{zz} + C\phi_{zz}\big) \\
&= 2\int_{-\infty}^{\infty} \phi_{zz}\big(s\phi_z + u_-^m(m!)^2\phi_z^2 e^{1/u_-} + \phi_{zz}\big) \\
&\quad + 2\int_{-\infty}^{\infty} \phi_{zz}\big(m(U^{m-1}\phi_z)_z + 2C\phi_z\phi_{zz} + C\phi_{zz}\big) \\
&\leq C\int_{-\infty}^{\infty} (\phi_{zz}^2 + \phi_z^2).
\end{aligned}$$

It then follows from the global estimate (18) that

$$\int_0^{\infty} \left|\frac{d}{dt}\int_{-\infty}^{\infty} \phi_z^2(z,t)dz\right| \leq C\int_0^{\infty}\int_{-\infty}^{\infty} (\phi_{zz}^2 + \phi_z^2) \leq C\|\phi_0\|^2_{2,w} \leq CN^2(0). \tag{22}$$

From (21) and (22), we get

$$\int_{-\infty}^{\infty} \phi_z^2(z,t)dz \to 0 \text{ as } t \to +\infty.$$

By Cauchy-Schwarz inequality, we further have

$$
\begin{aligned}
\phi_z^2(z,t) &= 2\int_{-\infty}^{z} \phi_z\phi_{zz}(y,t)dy \\
&\le 2\Big(\int_{-\infty}^{+\infty} \phi_z^2(y,t)dy\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{+\infty} \phi_{zz}^2(y,t)dy\Big)^{\frac{1}{2}} \\
&\le C\Big(\int_{-\infty}^{+\infty} \phi_z^2(y,t)dy\Big)^{\frac{1}{2}} \to 0 \text{ as } t\to+\infty
\end{aligned}
$$

Hence (19) is proved. □

Weighted Energy Estimate

Now, we are concerned with the a priori estimates for solution ϕ of (16), (17), and hence prove Proposition 3. To avoid the singular term, we apply the weighted function $1/U(z) \le Cw$ for all $z \in \mathbb{R}$ as studied in [6, 8].

L^2-estimate of ϕ

Lemma 2 *Under the assumptions of Lemma 1, if $N(t) \ll 1$, then*

$$
\|\phi(.,t)\|_w^2 + \int_0^t \|\phi_z(.,\tau)\|_w^2 d\tau \le C\|\phi_0\|_w^2 + CN(t)\int_0^t\int \phi_{zz}^2 w \tag{23}
$$

Proof We multiply the Eq. (16) by ϕ/U and integrate the result in z to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\frac{\phi^2}{U} + m\int U^{m-2}\phi_z^2 \\
&= \int\frac{s\phi\phi_z}{U} - \int\frac{(\phi_z+U)^m\phi}{U} - \int mU^{m-1}\phi\phi_z\Big(\frac{1}{U}\Big)_z + \int\frac{F\phi}{U}.
\end{aligned} \tag{24}
$$

We estimate the term $(\phi_z+U)^m$ in (24) that

$$
(\phi_z+U)^m \le (\phi_z+u_-)^m = u_-^m\Big(\frac{\phi_z}{u_-}+1\Big)^m = \sum_{l=0}^{m} u_-^m\frac{P_l^m}{l!}\Big(\frac{\phi_z}{u_-}\Big)^l, \tag{25}
$$

where $P_l^m = \frac{m!}{(m-l)!}$.

Since $N(t) \ll 1$, which implies $\|\phi_z(\cdot,t)\|_{L^\infty} \le 1$, then (25) becomes

$$
(\phi_z+U)^m \le u_-^m(m!)^2\phi_z^2\sum_{l=0}^{m}\frac{1}{l!}\Big(\frac{1}{u_-}\Big)^l = u_-^m(m!)^2\phi_z^2 e^{1/u_-} \le C\phi_z^2. \tag{26}
$$

Substituting (26) into (24), one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \frac{\phi^2}{U} + m\int U^{m-2}\phi_z^2 \\
&\leq C\int \frac{\phi\phi_z^2}{U} - \int \frac{\phi^2}{2}\left[\left(\frac{s}{U}\right)_z - mU^{m-1}\left(\frac{1}{U}\right)_{zz}\right].
\end{aligned}
\tag{27}
$$

It follows from (10), and the fact $u_+ = 0$, noting that

$$
\begin{aligned}
\left(\frac{s}{U}\right)_z - mU^{m-1}\left(\frac{1}{U}\right)_{zz} &= \left[\left(\frac{s}{U}\right) + mU^{m-1}\frac{U_z}{U^2}\right]_z \\
&= \left[\frac{s}{U} + \frac{-sU + U^m + su_+ - u_+^m}{U^2}\right]_z \\
&= \left[\frac{U^m + su_+ - u_+^m}{U^2}\right]_z \\
&= (m-2)U^{m-2}\frac{U_z}{U}.
\end{aligned}
\tag{28}
$$

By Young's inequality, $\|\phi(\cdot,t)\|_{L^\infty} \leq N(t) \ll 1$, and (26), it holds

$$
\left|\int \frac{F\phi}{U}\right| \leq CN(t)\int \left(\frac{|\phi_z|^2 + |\phi_{zz}|^2}{U}\right).
\tag{29}
$$

Substituting (28), (29) into (27), then applying $1/U(z) \leq Cw$ for all $z \in \mathbb{R}$, one has

$$
\frac{1}{2}\frac{d}{dt}\int \phi^2 w + (1 - CN(t))\int \phi_z^2 w \leq CN(t)\int \phi_{zz}^2 w.
\tag{30}
$$

We further integrate (30) in t and using the fact $N(t) \ll 1$, then the proof of Lemma 2 is finished. □

H^1-estimate of ϕ

Lemma 3 *Under the assumptions of Lemma 1, if $N(t) \ll 1$, then*

$$
\|\phi(.,t)\|_{1,w}^2 + \int_0^t \|\phi_z(.,\tau)\|_{1,w}^2 d\tau \leq C\|\phi_0\|_{1,w}^2.
\tag{31}
$$

Proof We differentiate (16) in z to get

$$
\phi_{zt} = s\phi_{zz} - m(\phi_z + U)^{m-1}(\phi_{zz} + U_z) + m(U^{m-1}\phi_z)_{zz} + F_z.
\tag{32}
$$

Now, multiplying (31) by ϕ_z/U, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \frac{\phi_z^2}{U} + m\int U^{m-2}\phi_{zz}^2 \\
&= \int \frac{s\phi_z\phi_{zz}}{U} - \int mU^{m-1}\phi_z\phi_{zz}\Big(\frac{1}{U}\Big)_z - \int \frac{m(\phi_z+U)^{m-1}(\phi_{zz}+U_z)\phi_z}{U} \\
&\quad + \int m(m-1)U^{m-1}\Big(\frac{1}{U}\Big)_z\phi_z\phi_{zz} + \int m(m-1)U^m\bigg(\phi_z\Big(\frac{1}{U}\Big)_z\bigg)^2 \\
&\quad + \int |F|\bigg(|\phi_z| + \frac{|\phi_{zz}|}{U}\bigg).
\end{aligned}
\tag{33}
$$

By the similar way in L^2 estimate, we can also estimate the term $(\phi_z + U)^{m-1}$ in (33) which consists of two conditions: $(\phi_z + U)^{m-1} \le (\phi_z + u_-)^{m-1}$ for $m \ge 1$ and $(\phi_z + U)^{m-1} \le (\phi_z + Cw(z))^{m-1}$ for $0 < m < 1$, where $w(z)$ is defined in Remark 1. Then, it follows from (16), $0 < U(z) \le u_-$ and (26), we can derive

$$
|F| \le C\big(|\phi_z||\phi_{zz}| + |\phi_z|^2\big). \tag{34}
$$

By Young's inequality and $\|\phi(\cdot,t)\|_{L^\infty} \le N(t)$, one has

$$
\int |F|\bigg(|\phi_z| + \frac{|\phi_{zz}|}{U}\bigg) \le CN(t)\int \bigg(\frac{|\phi_z|^2 + |\phi_{zz}|^2}{U}\bigg). \tag{35}
$$

Substituting (34), (35) into (33) to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \frac{\phi_z^2}{U} + (1 - CN(t))\int \frac{\phi_{zz}^2}{U} \\
&\le C\int \frac{\phi_z^4 + U_z\phi_z^3}{U} - \int \frac{\phi_z^2}{2}\bigg[\Big(\frac{s}{U}\Big) - mU^{m-1}\Big(\frac{1}{U}\Big)_z\bigg]_z \\
&\quad + \int \frac{CN(t)}{U}(|\phi_z|^2 + |\phi_{zz}|^2).
\end{aligned}
\tag{36}
$$

Substituting (28) into (36), applying Young's inequality, $\|\phi_z\|_{L^\infty} \ll N(t)$, $1/U(z) \le Cw$ for all $z \in \mathbb{R}$, and the results with (23), one has

$$
\frac{1}{2}\frac{d}{dt}\int \phi_z^2 w + (1 - CN(t))\int \phi_{zz}^2 w \le CN(t)\int \phi_z^2 w. \tag{37}
$$

Integrating (37) with respect to t and combining the results with Lemma 2, one has

$$
\int \phi_z^2 w + (1 - CN(t))\int_0^t \int \phi_{zz}^2 w \le C\int \phi_{0z}^2 w. \tag{38}
$$

Using the fact $N(t) \ll 1$, then the proof of Lemma 3 is finished. □

H^2-estimate of ϕ

Lemma 4 *Under the assumptions of Lemma 1, if $N(t) \ll 1$, then*

$$\|\phi(.,t)\|_{2,w}^2 + \int_0^t \|\phi_z(.,\tau)\|_{2,w}^2 d\tau \le C\|\phi_0\|_{2,w}^2 \tag{39}$$

Proof We differentiate (32) in z to get

$$\begin{aligned} \phi_{zzt} &= s\phi_{zzz} - m(\phi_z + U)^{m-1}(\phi_{zzz} + U_{zz}) \\ &\quad - m(m-1)(\phi_z + U)^{m-2}(\phi_{zz} + U_z)^2 + m(U^{m-1}\phi_z)_{zzz} + F_{zz}. \end{aligned} \tag{40}$$

Now, multiplying (31) by ϕ_{zz}/U, one has

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \frac{\phi_{zz}^2}{U} + m\int U^{m-2}\phi_{zzz}^2 \\ &= \int \frac{s\phi_{zz}\phi_{zzz}}{U} - \int mU^{m-1}\phi_{zz}\phi_{zzz}\Big(\frac{1}{U}\Big)_z - \int \frac{m(\phi_z + U)^{m-1}(\phi_{zzz} + U_{zz})\phi_{zz}}{U} \\ &\quad + \int m(m-1)U^{m-1}\Big(\frac{1}{U}\Big)_z\phi_{zz}\phi_{zzz} + \int m(m-1)U^m\Big(\phi_{zz}\Big(\frac{1}{U}\Big)_z\Big)^2 \\ &\quad + \int m^2(m-1)U^m\Big(\frac{1}{U}\Big)_z^2\frac{\phi_z\phi_{zz}\phi_{zzz}}{U} + \int m^2(m-1)U^{m-1}U_z\Big(\frac{1}{U}\Big)_z^2\frac{\phi_z\phi_{zz}^2}{U} \\ &\quad + \int m(m-1)U^m\Big(\frac{1}{U}\Big)_{zz}\frac{\phi_z\phi_{zz}\phi_{zzz}}{U^2} + \int m(m-1)U^{m-1}\Big(\frac{1}{U}\Big)_{zz}\Big(\frac{1}{U}\Big)_z\frac{\phi_z\phi_{zz}^2}{U} \\ &\quad + \int \frac{F_{zz}\phi_{zz}}{U} - \int \frac{m(m-1)(\phi_z + U)^{m-2}(\phi_{zz} + U_z)^2\phi_{zz}}{U}. \end{aligned} \tag{41}$$

We employ the similar technique as in L^2 and H^1 estimates to establish the estimate of the term $(\phi_z + U)^{m-2}$. Noting that

$$\int \frac{F_{zz}\phi_{zz}}{U} = -\int \frac{F_z\phi_{zzz}}{U} + \int \frac{F_z\phi_{zz}U_z}{U^2}, \tag{42}$$

and

$$\begin{aligned} F_z &= m((U + \phi_z)^{m-1} - U^{m-1})\phi_{zzz} + m(m-1)(U + \phi_z)^{m-2}\phi_{zz}^2 \\ &\quad + m(m-1)U_z^2\big((U + \phi_z)^{m-2} - U^{m-2} - (m-2)U^{m-3}\phi_z\big) \\ &\quad + mU_{zz}((U + \phi_z)^{m-1} - U^{m-1} - (m-1)U^{m-2}\phi_z) \\ &\quad + 2m(m-1)U_z((U + \phi_z)^{m-2} - U^{m-2})\phi_{zz}. \end{aligned} \tag{43}$$

By Young's inequality, $\|\phi_z(\cdot,t)\|_{L^\infty} \le N(t)$, (43), and (34), then (42) becomes

$$\Big|\int \frac{F_{zz}\phi_{zz}}{U}\Big| \le CN(t)\int \Big(\frac{|\phi_{zz}|^2 + |\phi_{zzz}|^2}{U}\Big). \tag{44}$$

Substituting (42)–(44) into (41), one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\frac{\phi_{zz}^2}{U}+(1-CN(t))\int\frac{\phi_{zzz}^2}{U}\\
&\leq-\int\frac{\phi_{zz}^2}{2}\left[\left(\frac{s}{U}\right)-mU^{m-1}\left(\frac{1}{U}\right)_z\right]_z+\int CN(t)\frac{|\phi_{zz}|^2}{U}.
\end{aligned}
\tag{45}
$$

Substituting (28) into (45), applying Young's inequality, $\|\phi_z\|_{L^\infty} \ll N(t)$, $1/U(z) \leq Cw$ for all $z \in \mathbb{R}$, the results in (23) and (31), and integrating with respect to t, then one has

$$
\int\phi_{zz}^2 w+(1-CN(t))\int_0^t\int\phi_{zzz}^2 w\leq C\int\phi_{0zz}^2 w. \tag{46}
$$

Using the fact $N(t) \ll 1$, then the proof of Lemma 4 is finished. □

Acknowledgements Author would like to thank to the reviewer for the valuable comments and suggestions which helped to improve the paper.

References

1. Buckmire, R., McMurtry, K., Mickens, R.E.: Numerical studies of a nonlinear heat equation with square root reaction term. Numerical Methods for Partial Differential Equations **25**, 598–609 (2009)
2. Debnath, L.: Nonlinear Partial Differential Equations for Scientists and Engineers. Birkhauser, Boston (1997)
3. Ghani, M., Li, J., Zhang, K.: Asymptotic stability of traveling fronts to a chemotaxis model with nonlinear diffusion. Discrete and Continuous Dynamical Systems - B **26**, 6253–6265 (2021)
4. Il'in, A.M., Oleinik, O.A.: Asymptotic behavior of solutions of the Cauchy problem for certain quasilinear equationsfor large time (in Russian). Mat. Sb. **51**, 191–216 (1960)
5. Hu, Y.: Asymptotic nonlinear stability of traveling waves to a system of coupled Burgers equations. Journal of Mathematical Analysis and Applications. **397**, 322–333 (2013)
6. Jin, H.Y., Li, J., Wang, Z.A.: Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity. J. Differential Equations. **255**, 193–219 (2013)
7. Jordan, P.M.: A Note on the Lambert W-function: Applications in the mathematical and physical sciences. Contemporary Mathematics. **618**, 247–263 (2014)
8. Li, J., Li, T., Wang, Z.A.: Stability of traveling waves of the Keller-Segel system with logarithmic sensitivity. Math. Models Methods Appl. Sci. **24**, 2819–2849 (2014)
9. Li, J., Wang, Z.A.: Convergence to traveling waves of a singular PDE-ODE hybrid chemotaxis system in the half space. J. Differential Equations. **268**, 6940–6970 (2020)
10. Li, T., Wang, Z.A.: Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis. SIAM J. Appl. Math. **70**, 1522–1541 (2010)
11. Kawashima, S., Matsumura, A.: Stability of shock profiles in viscoelasticity with non-convex constitutive relations. Comm. Pure Appl. Math. **47**, 1547–1569 (1994)
12. Li, T., Wang, Z.A.: Asymptotic nonlinear stability of traveling waves to conservation laws arising from chemotaxis. J. Differential Equations **250**, 1310–1333 (2011)
13. Martinez, V.R., Wang, Z.-A., Zhao, K.: Asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology. Indiana Univ. Math. J. **67**, 1383–1424 (2018)
14. Matsumura, A., Nishihara, K.: On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas, Japan. J. Appl. Math. **2**, 17–25 (1985)
15. Mickens, R.E.: Exact finite difference scheme for an advection equation having square-root dynamics. Journal of Difference Equations and Applications. **14**, 1149–1157 (2008)
16. Mickens, R.E.: Wave front behavior of traveling waves solutions for a PDE having square-root dynamics. Mathematics and Computers in Simulation. **82**, 1271–1277 (2012)
17. Mickens, R.E., Oyedeji, K.: Traveling wave solutions to modified Burgers and diffusionless Fisher PDE's. Evolution Equations and Control Theory. **8**, 139–147 (2019)

18. Nishida, T.: Nonlinear Hyperbolic Equations and Related Topics in Fluid Dynamics, Publications Math'ematiques d'Orsay 78–02. D'epartement de Math'ematique. Universit'e de ParisSud. Orsay, France (1978)
19. Sattinger, D.H.: On the stability of waves of nonlinear parabolic systems. Adv. Math. **22**, 312–355 (1976)
20. Whitham, G.B.: Linear and Nonlinear Waves. Wiley-Interscience, New York (1974)