DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **26**, Number **12**, December **2021**

doi:10.3934/dcdsb.2021017

pp. **6253–6265**

ASYMPTOTIC STABILITY OF TRAVELING FRONTS TO A CHEMOTAXIS MODEL WITH NONLINEAR DIFFUSION

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(Communicated by Zhian Wang)

Abstract. We are interested in the existence and stability of traveling waves of arbitrary amplitudes to a chemotaxis model with porous medium diffusion. We first make a complete classification of traveling waves under specific relations among the biological parameters. Then we show all these traveling waves are asymptotically stable under appropriate perturbations. The proof is based on a Cole-Hopf transformation and the energy method.

1. Introduction.

In this paper, we consider the following PDE-ODE hybrid chemotaxis model

$$\begin{cases} u_t = D(u^m)_{xx} - \chi\left(u(\ln c)_x\right)_x, \\ c_t = -uc + \beta c \end{cases} \tag{1}$$

with $m > 0$ and initial data

$$(u, c)(x, 0) = (u_0, c_0)(x) \to (u_\pm, c_\pm) \text{ as } x \to \pm\infty. \tag{2}$$

System (1) could model the reinforced movement of cells (or bacterial) in porous media, where u is the population density of cells, and c is the concentration of chemical signals (e.g. nutrient) with growth rate $\beta > 0$. $D > 0$ is the diffusion rate of cells and χ is the chemotactic coefficient. The chemotaxis is said to be attractive if $\chi > 0$ and repulsive if $\chi < 0$. The logarithmic sensitivity $\ln c$ comes from the pervasiveness of Weber-Fechner law [9], and has been verified by experimental data [7].

When $m = 1$, system (1) is exactly the chemotaxis model proposed in [19] to describe the reinforced random walks. There are lots of interesting analytical works in this case. Othmer and Stevens [19] derived the model from random walk, and carried out the numerical simulations of the formation of spikes and blowup. Subsequently, Levine and Sleeman [10] presented analytical results supporting some numerical results in [19]. Yang etc. [25, 26] investigated the global existence and blowup of classical solutions on a bounded domain with no-flux boundary conditions. [13] and [27] further studied the global existence of smooth solutions and weak solutions to system (1) with Robin boundary condition, respectively. Global dynamics including well-posedness and large time behaviors of solutions in the whole

2020 *Mathematics Subject Classification.* Primary: 35A01, 35B40, 35Q92; Secondary: 92C17.
Key words and phrases. Chemotaxis, nonlinear diffusion, asymptotic stability, traveling fronts.
The third author is supported by National Science Foundation of China (No. 11771071).
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space, were considered in [3, 11, 15, 24]. Besides the spike solution and blowup solution, traveling wave is another biological pattern observed in chemotaxis [9]. Wang and Hillen [23] first established the existence of traveling fronts to (1) when $m = 1$. The stability of such traveling front in the case of $u_+ > 0$ was obtained in [14]. If $u_+ = 0$, the appearance of the vacuum end-state results in extreme difficulty, which was overcome in [6] by a tedious weighted energy method. Recently, [12] considered the half-space problem of (1) (with $m = 1$) under non-zero flux boundary condition. By introducing a wave selection mechanism, the authors [12] showed that the system still admits traveling wave profiles on the half-space. For more related works on traveling waves of chemotaxis models, we refer the interested readers to [5, 22].

When $m \neq 1$, system (1) is the chemotaxis model with diffusion in porous media. Chemotaxis phenomena in porous media are both important in experiments and mathematical modelings. For example, [18, 21] took experiments to quantify the bacterial chemotaxis in porous media, and [1, 4] introduced chemotaxis models with nonlinear diffusion to prevent overcrowding. Tao and Winkler [20] established the global existence and boundedness of solutions to a chemotaxis model of self-aggregation with arbitrary porous medium diffusion. However, few results are available to the chemotaxis model (1) except for the existence of compactly supported traveling waves in [2]. The main issue is that the nonlinear diffusion of u and the absence of diffusion of c generate new analytical barriers that are difficult to handle. The purpose of this paper is to provide a first attempt to break down these barriers, and prove the existence and stability of traveling waves to system (1) with $m > 0$ and $u_+ > 0$.

The existence of traveling waves to system (1) is stated as follows.

Theorem 1.1 (Existence)**.** *Let $D > 0$, $m > 0$, $\chi > 0$, $\beta > 0$ and $u_+ > 0$. Assume that $(u_\pm, c_\pm)$ satisfies one of the following conditions:*

(i) $u_- > u_+ = \beta$ *and* $0 = c_- < c_+$*;*
(ii) $u_- > \beta > u_+$ *and* $c_\pm = 0$*;*
(iii) $\beta = u_- > u_+$ *and* $c_- > c_+ = 0$.

Then system (1)-(2) has a unique (up to a translation) traveling wave solution $(U, \mathcal{C})(x - st)$ with $U' < 0$. The chemical concentration $\mathcal{C}$ satisfies: $\mathcal{C}' > 0$ in case (i); $\mathcal{C}' > 0$ on $(-\infty, z_0)$, $\mathcal{C}' < 0$ on $(z_0, +\infty)$ for some number z_0 in case (ii); $\mathcal{C}' < 0$ in case (iii). Moreover, the wave speed s is given by $s = \sqrt{\chi(u_- + u_+ - \beta)}$.

Remark 1. The traveling waves in cases (i) and (ii) are both invasion patterns, since at any finite spatial position, U goes to $u_- > 0$, while $\mathcal{C}$ shrinks to 0 as $t \to \infty$. However, the wave patterns of $\mathcal{C}$ are different: $\mathcal{C}$ is a front in case (i), while it is a pulse in case (ii). The traveling wave in case (iii) is a coexistence pattern, since at any finite spatial position, U goes to $u_- > 0$ and $\mathcal{C}$ goes to $c_- > 0$ as $t \to \infty$. This result indicates that the growth rate of signals has significant impact on the pattern formations of the chemotaxis model.

Remark 2. The chemotaxis model (1) does not have a traveling wave solution if $\chi < 0$. Otherwise, if $(U, \mathcal{C})$ is a traveling wave of (1) with $\chi < 0$, then equalities (13) and (54) still hold. If the wave speed $s > 0$, it then follows from (13) that $U' > 0$ and hence $u_- < u_+$. But owing to (54), to ensure $\mathcal{C}$ is bounded, we need $u_+ \leq \beta$ and $u_- \geq \beta$, which leads to a contradiction that $\beta \leq u_- < u_+ \leq \beta$. Similarly, if $s < 0$, (13) implies $U' < 0$ and $u_- > u_+$. But (54) implies $u_- \leq \beta$ and $u_+ \geq \beta$, which also leads to a contradiction $\beta \geq u_- > u_+ \geq \beta$.

Denote by $H^m(\mathbb{R})$ the usual Sobolev space with norm $\|f\|_m := \sum_{k=0}^{m} \|\partial_x^k f\|$ and $\|f\| := \|f\|_{L^2(\mathbb{R})}$. Our main result on the stability of traveling wave is stated as follows.

Theorem 1.2 (Stability)**.** *Assume that $D > 0$, $m > 0$, $\chi > 0$, $\beta > 0$ and $u_+ > 0$. Let $(U, \mathcal{C})(x - st)$ be a traveling wave obtained in Theorem 1.1. Then there exists a constant $\varepsilon_0 > 0$ such that if $\|u_0 - U\|_2 + \|(\ln c_0)_x - (\ln \mathcal{C})_x\|_2 + \|(\phi_0, \psi_0)\|_3 \le \varepsilon_0$, where*

$$\phi_0(x) = \int_{-\infty}^{x} (u_0 - U)(y)dy, \;\; \psi_0(x) = \ln \mathcal{C}(x) - \ln c_0(x),$$

then the Cauchy problem (1)-(2) has a unique global solution $(u, c)(x, t)$ satisfying

$$(u - U, (\ln c)_x - (\ln \mathcal{C})_x) \in C([0, \infty); H^2) \,\cap\, L^2([0, \infty); H^2),$$

and

$$\sup_{x\in\mathbb{R}} |(u, c)(x, t) - (U, \mathcal{C})(x - st)| \to 0 \text{ as } t \to +\infty.$$

Remark 3. We need $u_+ > 0$ essentially to derive the existence and stability of traveling waves to system (1). It is interesting to investigate the dynamics of system (1) when $\beta = 0$ (and hence $u_+ = 0$). However, this problem is quite challenging since it has singularities if $m < 1$, while it has degeneracies if $m > 1$. We leave this problem for the future work.

To show Theorems 1.1 and 1.2, we take the Cole-Hopf transformation as in [6, 14],

$$v = -(\ln c)_x, \tag{3}$$

to get the parabolic-hyperbolic system

$$\begin{cases} u_t - \chi(uv)_x = D(u^m)_{xx}, \\ v_t - u_x = 0, \end{cases} \tag{4}$$

with initial data

$$(u, v)(x, 0) = (u_0, v_0)(x) \to (u_\pm, v_\pm) \text{ as } x \to \pm\infty. \tag{5}$$

We first prove the existence of traveling fronts to system (4) by the phase-plane analysis; and then we derive the stability of such traveling fronts via energy estimates; finally we transfer the results of system (4) back to the original chemotaxis model (1). During this process, one can see that under the influence of porous medium diffusion, we have to establish the estimate for the third order derivative to close the a priori estimates. Moreover, notice the transformed system (4) does not involve β, when we transfer the results from system (4) to system (1), certain conditions on $u_\pm$, $c_\pm$ and β are needed to obtain biological meaningful results.

The rest of paper is organized as follows. In Section 2, we present the existence and some basic properties of traveling waves to the transformed parabolic-hyperbolic system (4), and derive the perturbation equations. Section 3 is devoted to establishing the a priori estimate. In Section 4, we prove the stability of traveling waves to the parabolic-hyperbolic system (4), and transfer the results to the original chemotaxis model (1).

2. **Transformation of the problem.** We first seek the traveling wave $(U,V)(x-st)$ of the parabolic-hyperbolic system (4). Substituting the following traveling wave ansatz

$$(u,v)(x,t)=(U,V)(z),\ z=x-st \tag{6}$$

into (4), where s denotes the traveling wave speed and z is the moving coordinate, we have

$$\begin{cases} -sU'-\chi(UV)'=D\left(U^m\right)'', \\ -sV'=U', \end{cases} \tag{7}$$

where $' := \frac{d}{dz}$, and the boundary conditions are imposed as

$$(U,V)(z)\to(u_\pm,v_\pm) \text{ as } z\to\pm\infty. \tag{8}$$

Now, integrating (7) with respect to z and using the fact that $U'(z)\to 0$ as $z\to\pm\infty$, we get

$$\begin{cases} D\left(U^m\right)'=-sU-\chi UV+su_\pm+\chi u_\pm v_\pm, \\ -sV=U-sv_\pm-u_\pm, \end{cases} \tag{9}$$

and the Rankine-Hugoniot condition

$$\begin{aligned} s(u_+-u_-)&=\chi(u_-v_--u_+v_+), \\ s(v_+-v_-)&=u_--u_+, \end{aligned} \tag{10}$$

which gives

$$s^2+s\chi v_+-\chi u_-=0. \tag{11}$$

Without loss of generality, we only consider the case $s>0$ and

$$s=-\frac{\chi v_+}{2}+\frac{\sqrt{\chi^2v_+^2+4\chi u_-}}{2}. \tag{12}$$

From (9) and (11), we get

$$U'=\frac{\chi U^{1-m}}{Dms}\cdot(U-u_-)(U-u_+). \tag{13}$$

Then employing the classical phase-plane analysis, one can easily prove the existence and uniqueness of traveling wave solutions to system (4).

Lemma 2.1. *Assume that $D>0$, $m>0$, $\chi>0$, $\beta>0$ and $u_+>0$. Suppose that $u_\pm$ and $v_\pm$ satisfy (10). Then there exists a monotone traveling wave solution $(U,V)(x-st)$ to system (7), which is unique up to a translation and satisfies $U'<0$, $V'>0$, where the wave speed s is given by (12). Moreover, (U,V) decays exponentially fast with rates*

$$U-u_\pm\sim e^{\lambda_\pm z}, V-v_\pm\sim e^{\lambda_\pm z} \text{ as } z\to\pm\infty,$$

where $\lambda_\pm=\frac{\chi u_\pm^{1-m}}{Dms}\cdot(u_\pm-u_\mp)$.

For the parabolic-hyperbolic system (4), we define

$$(\phi_0,\psi_0)(z):=\int_{-\infty}^{z}(u_0-U,v_0-V)(y)dy,$$

which is the so-called zero mass perturbation (see [8, 16]). Then we have the following stability result.

Theorem 2.2. *Assume that $D>0$, $m>0$, $\chi>0$, $\beta>0$ and $u_+>0$. Let $(U,V)(x-st)$ be the traveling wave obtained in Lemma 2.1. Then there exists a constant $\varepsilon_0>0$ such that if $\|u_0-U\|_2+\|v_0-V\|_2+\|(\phi_0,\psi_0)\|_3\le\varepsilon_0$, then the Cauchy problem (4)-(5) has a unique global solution $(u,v)(x,t)$ satisfying*

$$(u-U,v-V)\in C([0,\infty);H^2)\cap L^2([0,\infty);H^2),$$

and

$$\sup_{x\in\mathbb{R}}|(u,v)(x,t)-(U,V)(x-st)|\to 0 \quad \text{as} \quad t\to+\infty.$$

By changing the variables $(x,t)\to(z=x-st,t)$, system (4) becomes

$$\begin{cases} u_t-su_z-\chi(uv)_z=D\left(u^m\right)_{zz}, \\ v_t-sv_z-u_z=0. \end{cases} \tag{14}$$

We decompose the solution (u,v) of (14) as

$$(u,v)(z,t)=(U,V)(z)+(\phi_z,\psi_z)(z,t). \tag{15}$$

Then

$$\phi(z,t)=\int_{-\infty}^{z}(u(y,t)-U(y))dy,\ \ \psi(z,t)=\int_{-\infty}^{z}(v(y,t)-V(y))dy. \tag{16}$$

Substituting (15) into (14) and integrating the resultant equation with respect to z, one has

$$\begin{cases} \phi_t-(s+\chi V)\phi_z-\chi U\psi_z=Dm\left(U^{m-1}\phi_z\right)_z+G+\chi\phi_z\psi_z, \\ \psi_t-s\psi_z-\phi_z=0, \end{cases} \tag{17}$$

where $G=D\left((U+\phi_z)^m-U^m-mU^{m-1}\phi_z\right)_z$. The initial datum of (ϕ,ψ) is given by

$$(\phi,\psi)(z,0)=(\phi_0,\psi_0)(z)=\int_{-\infty}^{z}(u_0-U,v_0-V)dy \tag{18}$$

with $(\phi_0,\psi_0)(\pm\infty)=0$. We seek the solution of reformulated problem (17)-(18) in the space

$$X(0,T):=\left\{(\phi,\psi)\in C([0,T),H^3):\phi_z\in L^2((0,T);H^3)),\psi_z\in L^2((0,T);H^2))\right\}$$

with $0<T\le+\infty$. Let

$$N(t):=\sup_{0\le\tau\le t}\left\{\|\phi(.,\tau)\|_3+\|\psi(.,\tau)\|_3\right\}.$$

From the Sobolev inequality $\|f\|_{L^\infty}\le\sqrt{2}\|f\|_{L^2}^{\frac12}\|f_x\|_{L^2}^{\frac12}$, it follows that

$$\sup_{\tau\in[0,t]}\left\{\|\phi(\cdot,\tau)\|_{W^{2,\infty}},\|\psi(\cdot,\tau)\|_{W^{2,\infty}}\right\}\le N(t).$$

For system (17)-(18), we have the following global well-posedness.

Theorem 2.3. *There exists a constant $\delta_1>0$ such that if $N(0)\le\delta_1$, then the Cauchy problem (17)-(18) has a unique global solution $(\phi,\psi)\in X(0,+\infty)$ such that*

$$\|(\phi,\psi)(.,t)\|_3^2+\int_0^t\left(\|\phi_z(.,\tau)\|_3^2+\|\psi_z(.,\tau)\|_2^2\right)d\tau\le C(\|\phi_0\|_3^2+\|\psi_0\|_3^2) \tag{19}$$

for any $t>0$. Moreover, it holds that

$$\sup_{z\in R}|(\phi_z,\psi_z)(z,t)|\to 0 \ \text{as}\ t\to+\infty. \tag{20}$$

According to the classical works (see [14]), the global smooth solution can be constructed by the local wellposedness, the a priori estimate and an extension procedure. Since the local wellposedness can be proved in a standard way (e.g. see [17]), we need to establish the following a priori estimate.

Proposition 1. *Assume that $(\phi,\psi)\in X(0,T)$ is a solution of (17)-(18) for some time $T>0$. Then there is a constant $\varepsilon_1>0$, independent of T, such that if $N(T)<\varepsilon_1$, then (ϕ,ψ) satisfies (19) for any $0\le t\le T$.*

3. **Energy estimates.** In this section, we establish the a priori estimates for solution (ϕ,ψ) of (17)-(18), and hence prove Proposition 1. We first derive the basic L^2 estimate.

Lemma 3.1. *Under the same assumptions of Proposition 1, if $N(t)\ll 1$, then*

$$\|(\phi,\psi)(.,t)\|^2+\int_0^t\|\phi_z(.,\tau)\|^2d\tau\le C(\|\phi_0\|^2+\|\psi_0\|^2)+CN(t)\int_0^t\int(\psi_z^2+\phi_{zz}^2). \tag{21}$$

Proof. Multiplying $(17)_1$ by $\frac{\phi}{U}$ and $(17)_2$ by $\chi\psi$, adding them, and integrating the resulting equations, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi^2}{U}+\chi\psi^2\right)+Dm\int U^{m-2}\phi_z^2\\&=-\int\frac{\phi^2}{2}\left(\left(\frac{s+\chi V}{U}\right)_z-Dm\left(U^{m-1}\left(\frac{1}{U}\right)_z\right)_z\right)+\int\left(\frac{G\phi}{U}+\chi\frac{\phi_z\psi_z\phi}{U}\right).\end{aligned} \tag{22}$$

By (12), $s+\chi v_+>0$. It then follows from the first equation of (9) and $U_z<0$ that

$$\begin{aligned}\left(\frac{s+\chi V}{U}\right)_z-Dm\left(U^{m-1}\left(\frac{1}{U}\right)_z\right)_z&=\left(\frac{s+\chi V}{U}-DmU^{m-1}\left(\frac{1}{U}\right)_z\right)_z\\&=\left(\frac{su_++\chi u_+v_+}{U^2}\right)_z\\&=-\frac{2u_+(s+\chi v_+)U_z}{U^3}>0.\end{aligned} \tag{23}$$

Noting that $U\ge u_+>0$ and $\|\phi_z(\cdot,t)\|_{L^\infty}\le N(t)\ll 1$, it holds

$$|G|\le C(|\phi_{zz}||\phi_z|+|\phi_z|^2). \tag{24}$$

And then by Young's inequality, we have

$$\left|\int\frac{G\phi}{U}\right|\le CN(t)\int(|\phi_z^2|+|\phi_{zz}|^2), \tag{25}$$

where we have used $\|\phi(\cdot,t)\|_{L^\infty}\le N(t)$. Similarly,

$$\left|\int\frac{\phi_z\psi_z\phi}{U}\right|\le CN(t)\int(\phi_z^2+\psi_z^2). \tag{26}$$

Substituting (23), (25)-(26) into (22) and using $U\ge u_+>0$, we get (21). ◻

The next lemma gives the estimate of the first order derivative of (ϕ,ψ).

Lemma 3.2. *Under the same assumptions of Proposition 1, if $N(t)\ll 1$, it holds that*

$$\|(\phi,\psi)(.,t)\|_1^2+\int_0^t(\|\phi_z(.,\tau)\|_1^2+\|\psi_z(.,\tau)\|^2)d\tau\le C(\|\phi_0\|_1^2+\|\psi_0\|_1^2). \tag{27}$$

Proof. Differentiating (17) in z gives

$$\begin{cases} \phi_{zt} - \chi U \psi_{zz} - Dm\left(U^{m-1}\phi_{zz}\right)_z \\ \quad = Dm((U^{m-1})_z\phi_z)_z + \chi U_z\psi_z + ((s+\chi V)\phi_z)_z + (G + \chi\phi_z\psi_z)_z, \\ \psi_{zt} - s\psi_{zz} - \phi_{zz} = 0. \end{cases} \tag{28}$$

Multiplying $(28)_1$ by $\frac{\phi_z}{U}$ and $(28)_2$ by $\chi\psi_z$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi_z^2}{U} + \chi\psi_z^2\right) + Dm\int U^{m-2}\phi_{zz}^2 \\ &= \int \frac{\phi_z^2}{2}\left(Dm\left((U^{m-1})_{zz}\frac{1}{U} + U^{m-1}\left(\frac{1}{U}\right)_{zz}\right) - \left(\frac{s+\chi V}{U}\right)_z\right) \\ &\quad + \chi\int\frac{V_z\phi_z^2}{U} + \chi\int\frac{U_z\psi_z\phi_z}{U} - \int(G+\chi\phi_z\psi_z)\left(\frac{\phi_z}{U}\right)_z. \end{aligned} \tag{29}$$

By Young's inequality,

$$\chi\int\frac{U_z\psi_z\phi_z}{U} \le \frac{\delta\chi}{2}\int U\psi_z^2 + \frac{\chi}{2\delta}\int\frac{U_z^2\phi_z^2}{U^3},$$

where δ is a small constant to be determined later. Substituting this inequality into (29) leads to

$$\begin{aligned} &\int\left(\frac{\phi_z^2}{U} + \chi\psi_z^2\right) + 2Dm\int_0^t\int U^{m-2}\phi_{zz}^2 \\ &\le C\|(\phi_{0z},\psi_{0z})\|^2 + C(1+\frac{1}{\delta})\int_0^t\int\phi_z^2 + \delta\chi\int_0^t\int U\psi_z^2 \\ &\quad + C\int_0^t\int|(G+\chi\phi_z\psi_z)|\left(|\phi_z| + \frac{|\phi_{zz}|}{U}\right). \end{aligned} \tag{30}$$

It remains to estimate the term $\int_0^t\int U\psi_z^2$. Multiplying the first equation of (17) by ψ_z yields

$$\chi U\psi_z^2 = \phi_t\psi_z - (s+\chi V)\phi_z\psi_z - Dm\left(U^{m-1}\phi_z\right)_z\psi_z - (G+\chi\phi_z\psi_z)\psi_z. \tag{31}$$

By the second equation of (28), we have

$$\begin{aligned} \phi_t\psi_z &= (\phi\psi_z)_t - \phi\psi_{zt} = (\phi\psi_z)_t - \phi(s\psi_{zz} + \phi_{zz}) \\ &= (\phi\psi_z)_t - s(\phi\psi_z)_z + s\phi_z\psi_z - (\phi\phi_z)_z + \phi_z^2. \end{aligned} \tag{32}$$

Combining (31) with (32) and integrating the results, we get

$$\begin{aligned} \chi\int_0^t\int U\psi_z^2 = \int\phi\psi_z - \int\phi_0\psi_{0z} + \int_0^t\int\phi_z^2 - Dm\int_0^t\int\left(U^{m-1}\phi_z\right)_z\psi_z \\ - \chi\int_0^t\int V\phi_z\psi_z - \int_0^t\int(G+\chi\phi_z\psi_z)\psi_z. \end{aligned}$$

By Young's inequality, noting $u_- \ge U \ge u_+ > 0$, we have

$$\begin{aligned} -Dm\int\left(U^{m-1}\phi_z\right)_z\psi_z &= -Dm\int U^{m-1}\phi_{zz}\psi_z - Dm\int(U^{m-1})_z\phi_z\psi_z \\ &\le \frac{\chi}{4}\int U\psi_z^2 + \frac{D^2m^2A_m}{\chi}\int U^{m-2}\phi_{zz}^2 + C\int\phi_z^2, \end{aligned}$$

where $A_m = u_-^{m-1}$ if $m \geq 1$, $A_m = u_+^{m-1}$ if $0 < m < 1$. By Young's inequality again,

$$\chi \int |V\phi_z\psi_z| \leq \frac{\chi}{4}\int U\psi_z^2 + \chi \int V^2\phi_z^2.$$

Thus,

$$\begin{aligned}\chi \int_0^t \int U\psi_z^2 \leq & \int \psi_z^2 + \int \phi^2 + 2\int |\phi_0\psi_{0z}| + \frac{2D^2m^2A_m}{\chi}\int_0^t \int U^{m-2}\phi_{zz}^2 \\ & + C\int_0^t \int \phi_z^2 + C\int_0^t \int |(G+\chi\phi_z\psi_z)\psi_z|.\end{aligned} \tag{33}$$

Substituting (33) into (30), and choosing $\delta = \min\{\frac{\chi}{2DmA_m}, \frac{\chi}{2}\}$, by Lemma 3.1, when $N(t) \ll 1$, we have

$$\begin{aligned}&\int (\phi_z^2 + \chi\psi_z^2) + \int_0^t \int U^{m-2}\phi_{zz}^2 \\ &\leq C(\|\phi_0\|_1^2 + \|\psi_0\|_1^2) + CN(t)\int_0^t \int \psi_z^2 \\ &\quad + C\int |(G+\chi\phi_z\psi_z)| \left(|\phi_z| + \frac{|\phi_{zz}|}{U} + |\psi_z|\right).\end{aligned} \tag{34}$$

Substituting (34) into (33) gives

$$\int_0^t \int \psi_z^2 \leq C(\|\phi_0\|_1^2 + \|\psi_0\|_1^2) + C\int |(G+\chi\phi_z\psi_z)| \left(|\phi_z| + \frac{|\phi_{zz}|}{U} + |\psi_z|\right), \tag{35}$$

which in combination with (34) further leads to

$$\begin{aligned}&\int (\phi_z^2 + \psi_z^2) + \int_0^t \int \phi_{zz}^2 + \int_0^t \int \psi_z^2 \\ &\leq C(\|\phi_0\|_1^2 + \|\psi_0\|_1^2) + C\int_0^t \int |(G+\chi\phi_z\psi_z)| \left(|\phi_z| + \frac{|\phi_{zz}|}{U} + |\psi_z|\right).\end{aligned} \tag{36}$$

In view of (24), by Young's inequality, the fact that $\|\phi(\cdot,t)\|_{L^\infty} \leq N(t)$ and Lemma 3.1, we get

$$\int |(G+\chi\phi_z\psi_z)| \left(|\phi_z| + \frac{|\phi_{zz}|}{U} + |\psi_z|\right) \leq CN(t)\int_0^t \int (\phi_{zz}^2 + \psi_z^2).$$

Finally, substituting this inequality into (36), when $N(t) \ll 1$, we obtain the desired (27). $\square$

Next, we estimate the second order derivative of (ϕ, ψ).

Lemma 3.3. *If $N(t) \ll 1$, then it follows that*

$$\|(\phi,\psi)(.,t)\|_2^2 + \int_0^t (\|\phi_z(.,\tau)\|_2^2 + \|\psi_z(.,\tau)\|_1^2) \leq C(\|\phi_0\|_2^2 + \|\psi_0\|_2^2). \tag{37}$$

Proof. Differentiating (28) with respect to z gives

$$\begin{cases}\phi_{zzt} - \chi U\psi_{zzz} - Dm\left(U^{m-1}\phi_{zzz}\right)_z \\ \quad = Dm(2(U^{m-1})_z\phi_{zz} + (U^{m-1})_{zz}\phi_z)_z + \chi(2U_z\psi_{zz} + U_{zz}\psi_z) \\ \quad\quad + ((s+\chi V)\phi_z)_{zz} + (G+\chi\phi_z\psi_z)_{zz}, \\ \psi_{zzt} - s\psi_{zzz} - \phi_{zzz} = 0.\end{cases} \tag{38}$$

Multiplying $(38)_1$ by $\frac{\phi_{zz}}{U}$ and $(38)_2$ by $\chi\psi_{zz}$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi_{zz}^2}{U}+\chi\psi_{zz}^2\right)+Dm\int U^{m-2}\phi_{zzz}^2-\chi\int(2U_z\psi_{zz}+U_{zz}\psi_z)\frac{\phi_{zz}}{U}\\&=\int\left[Dm\left(\frac{3(U^{m-1})_{zz}}{U}-\left(\frac{(U^{m-1})_z}{U}\right)_z\right)-\left(\frac{s+\chi V\phi_z}{2U}\right)_z+\frac{2\chi V_z}{U}\right]\phi_{zz}^2\\&\quad+\int\left(\frac{\chi V\phi_{zz}}{U}+\frac{Dm(U^{m-1})_{zzz}}{U}\right)\phi_z\phi_{zz}+\int(G+\chi\phi_z\psi_z)_z\left(\frac{\phi_{zz}}{U}\right)_z.\end{aligned}\tag{39}$$

By Young's inequality,

$$\chi\left|(2U_z\psi_{zz}+U_{zz}\psi_z)\frac{\phi_{zz}}{U}\right|\le\frac{\delta\chi}{2}U\psi_{zz}^2+\chi\left(\frac{2}{\delta}\cdot\frac{U_z^2}{U^3}+\frac{U_{zz}^2}{U^2}\right)\phi_{zz}^2+\chi\psi_z^2,\tag{40}$$

where δ is a small constant. Noting

$$\begin{aligned}G_z=&Dm((U+\phi_z)^{m-1}-U^{m-1})\phi_{zzz}+Dm(m-1)(U+\phi_z)^{m-2}\phi_{zz}^2\\&+Dm(m-1)U_z^2((U+\phi_z)^{m-2}-U^{m-2}-(m-2)U^{m-3}\phi_z)\\&+DmU_{zz}((U+\phi_z)^{m-1}-U^{m-1}-(m-1)U^{m-2}\phi_z)\\&+2Dm(m-1)U_z((U+\phi_z)^{m-2}-U^{m-2})\phi_{zz},\end{aligned}\tag{41}$$

we have

$$\int(G+\chi\phi_z\psi_z)_z\left(\frac{\phi_{zz}}{U}\right)_z\le CN(t)\int(\phi_z^2+\phi_{zz}^2+\phi_{zzz}^2+\psi_{zz}^2),\tag{42}$$

where we have used $\|\phi_z(\cdot,t)\|_{L^\infty},\|\psi_z(\cdot,t)\|_{L^\infty},\|\phi_{zz}(\cdot,t)\|_{L^\infty}\le N(t)$. Substituting (40) and (42) into (39), by (27), we get

$$\begin{aligned}&\int\left(\frac{\phi_{zz}^2}{U}+\chi\psi_{zz}^2\right)+2Dm\int_0^t\int U^{m-2}\phi_{zzz}^2\\&\le C(\|\phi_0\|_2^2+\|\psi_0\|_2^2)+\delta\chi\int_0^t\int U\psi_{zz}^2+CN(t)\int_0^t\int(\psi_{zz}^2+\phi_{zzz}^2).\end{aligned}\tag{43}$$

Next we estimate $\int_0^t\int U\psi_{zz}^2$. Multiplying $(28)_1$ by ψ_{zz}, we get

$$\begin{aligned}\chi U\psi_{zz}^2=&\phi_{zt}\psi_{zz}-Dm\left(U^{m-1}\phi_{zz}\right)_z\psi_{zz}-Dm((U^{m-1})_z\phi_z)_z\psi_{zz}\\&-(\chi U_z\psi_z+((s+\chi V)\phi_z)_z+(G+\chi\phi_z\psi_z)_z)\psi_{zz}.\end{aligned}\tag{44}$$

By the second equation of (38),

$$\begin{aligned}\phi_{zt}\psi_{zz}&=(\phi_z\psi_{zz})_t-\phi_z\psi_{zzt}\\&=(\phi_z\psi_{zz})_t-s\phi_z\psi_{zzz}-\phi_z\phi_{zzz}\\&=(\phi_z\psi_{zz})_t-s(\phi_z\psi_{zz})_z+s\phi_{zz}\psi_{zz}-(\phi_z\phi_{zz})_z+\phi_{zz}^2.\end{aligned}$$

By Young's inequality,

$$\begin{aligned}Dm\left(U^{m-1}\phi_{zz}\right)_z\psi_{zz}&=DmU^{m-1}\phi_{zzz}\psi_{zz}+Dm(U^{m-1})_z\phi_{zz}\psi_{zz}\\&\le\frac{\chi U\psi_{zz}^2}{4}+\frac{2D^2m^2U^{2m-3}\phi_{zzz}^2}{\chi}+\frac{2D^2m^2|(U^{m-1})_z|^2\phi_{zz}^2}{\chi}.\end{aligned}$$

Similarly,

$$|Dm((U^{m-1})_z\phi_z)_z\psi_{zz}+(\chi U_z\psi_z+((s+\chi V)\phi_z)_z|\le\frac{\chi U\psi_{zz}^2}{4}+C(\psi_z^2+\phi_{zz}^2+\phi_z^2).$$

In view of (41), since $\|\phi_z(\cdot,t)\|_{L^\infty}, \|\psi_z(\cdot,t)\|_{L^\infty}, \|\phi_{zz}(\cdot,t)\|_{L^\infty} \leq N(t)$, we get

$$|(G+\chi\phi_z\psi_z)_z)\psi_{zz}| \leq CN(t)(\psi_{zz}^2+\phi_{zz}^2+\phi_{zzz}^2+\phi_z^2).$$

Thus, integrating (44) gives

$$\begin{aligned}\chi\int_0^t\int U\psi_{zz}^2 \leq &\int(\frac{1}{\delta}\phi_z^2+\delta\psi_{zz}^2+\phi_{0z}^2+\psi_{0zz}^2)+\frac{2D^2m^2}{\chi}\int_0^t\int U^{2m-3}\phi_{zzz}^2\\ &+C\int_0^t\int(\phi_z^2+\psi_z^2+\phi_{zz}^2)+CN(t)\int_0^t\int(\psi_{zz}^2+\phi_{zzz}^2).\end{aligned} \tag{45}$$

Substituting (45) into (43), choosing $\delta \ll 1$ and $N(t) \ll 1$, since $U \geq u_+ > 0$, by Lemmas 3.1 and 3.2, we have

$$\int(\phi_{zz}^2+\chi\psi_{zz}^2)+\int_0^t\int\phi_{zzz}^2 \leq C\|(\phi_0,\psi_0)(\cdot,t)\|_2, \tag{46}$$

which in combination with (45) further gives

$$\int_0^t\int\psi_{zz}^2 \leq C\|(\phi_0,\psi_0)(\cdot,t)\|_2. \tag{47}$$

The desired estimate (37) finally follows from (46) and (47). □

Under the influence of nonlinear diffusion, we need to estimate the third order derivative of (ϕ,ψ) so as to close the energy estimates.

Lemma 3.4. *If $N(t) \ll 1$, we have*

$$\|(\phi,\psi)(.,t)\|_3^2+\int_0^t(\|\phi_z(.,\tau)\|_3^2+\|\psi_z(.,\tau)\|_2^2) \leq C(\|\phi_0\|_3^2+\|\psi_0\|_3^2). \tag{48}$$

Proof. Differentiating (38) with respect to z gives

$$\begin{cases}\phi_{zzzt}-\chi U\psi_{zzzz}-Dm\left(U^{m-1}\phi_{zzzz}\right)_z\\ =Dm(3(U^{m-1})_{zz}\phi_{zz}+3(U^{m-1})_z\phi_{zzz}+(U^{m-1})_{zzz}\phi_z)_z\\ \quad+\chi(3U_{zz}\psi_{zz}+3U_z\psi_{zzz}+U_{zzz}\psi_z)+((s+\chi V)\phi_z)_{zzz}+(G+\chi\phi_z\psi_z)_{zzz},\\ \psi_{zzzt}-s\psi_{zzzz}-\phi_{zzzz}=0.\end{cases} \tag{49}$$

Multiplying $(49)_1$ by $\frac{\phi_{zzz}}{U}$ and $(49)_2$ by $\chi\psi_{zzz}$, applying the same argument as that of Lemma 3.3, one can get the third order estimate (48). We omit the details here. □

Proposition 1 follows from Lemma 3.1 to Lemma 3.4.

4. **Proof of main results.** We now prove the main theorems in the current paper. Owing to the transformation (15), Theorem 2.2 is a consequence of Theorem 2.3.

Proof of Theorem 2.3. The a priori estimate (19) guarantees that $N(t)$ is small if $N(0)$ is small enough. Thus, applying the standard extension procedure, we get the global well-posedness of (17)-(18) in $X(0,+\infty)$.

Next, we prove the convergence (20). Owing to the global estimate (19), we get

$$\int_0^t\int_{-\infty}^{\infty}\phi_z^2(z,\tau)dzd\tau \leq C(\|\phi_0\|_3^2+\|\psi_0\|_3^2) < \infty,\ \forall\, t>0. \tag{50}$$

In view of the first equation of (17), by Young's inequality,

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{\infty}\phi_z^2(z,t)dz\\
&=-2\int_{-\infty}^{\infty}\phi_t\phi_{zz}dz\\
&=-2\int_{-\infty}^{\infty}\phi_{zz}(Dm\left(U^{m-1}\phi_z\right)_z+(s+\chi V)\phi_z+\chi U\psi_z+G+\chi\phi_z\psi_z)\\
&\le C\int_{-\infty}^{\infty}(\phi_{zz}^2+\phi_z^2+\psi_z^2).
\end{aligned}$$

It then follows from the global estimate (19) that

$$\int_0^{\infty}\left|\frac{d}{dt}\int_{-\infty}^{\infty}\phi_z^2(z,t)dz\right|\le C\int_0^{\infty}\int_{-\infty}^{\infty}(\phi_{zz}^2+\phi_z^2+\psi_z^2)\le C(\|\phi_0\|_3^2+\|\psi_0\|_3^2)<\infty. \tag{51}$$

By (50) and (51), we get

$$\int_{-\infty}^{\infty}\phi_z^2(z,t)dz\to 0 \text{ as } t\to+\infty.$$

By Cauchy-Schwarz inequality, we further have

$$\begin{aligned}
\phi_z^2(z,t)&=2\int_{-\infty}^{z}\phi_z\phi_{zz}(y,t)dy\\
&\le 2\left(\int_{-\infty}^{\infty}\phi_z^2(y,t)dy\right)^{\frac12}\left(\int_{-\infty}^{\infty}\phi_{zz}^2(y,t)dy\right)^{\frac12}\\
&\le C\left(\int_{-\infty}^{\infty}\phi_z^2(y,t)dy\right)^{\frac12}\\
&\to 0 \text{ as } t\to+\infty.
\end{aligned}$$

Applying the same argument to ψ_z yields

$$\sup_{z\in\mathbb{R}}|\psi_z(z,t)|\to 0 \text{ as } t\to+\infty. \tag{52}$$

Hence (20) is proved. □

Proof of Theorem 1.1. The existence of U follows from Lemma 2.1. We next show the existence of $\mathcal{C}$. The second equation of (1) implies

$$s\mathcal{C}'=\mathcal{C}(U-\beta), \tag{53}$$

from which one can easily calculate that

$$\mathcal{C}(z)=\mathcal{C}(0)e^{\frac{1}{s}\int_0^z(U(y)-\beta)dy}. \tag{54}$$

Since U converges to $u_\pm$ exponentially fast as $z\to\pm\infty$, and $s>0$, to ensure $\mathcal{C}$ is bounded, we need

$$u_+\le\beta,\ u_-\ge\beta.$$

(i) If $u_+=\beta$, since $U'<0$, we have $u_->U>\beta$. It then follows from (53) that $\mathcal{C}'>0$. Noting that (53) also implies

$$c_\pm(u_\pm-\beta)=0, \tag{55}$$

we further get $c_-=0$. Hence $u_->u_+=\beta$ and $0=c_-<c_+$.

(ii) If $u_- > \beta > u_+ > 0$, then we get from (55) that $c_\pm = 0$. Since U is monotone, there is only one point z_0 such that $U(z_0) = \beta$. Hence by (53), $\mathcal{C}'(z) > 0$ on $(-\infty, z_0)$ and $\mathcal{C}'(z) < 0$ on $(z_0, +\infty)$.

(iii) If $u_- = \beta$, then $\beta = u_- > U > u_+ > 0$. We get from (53) that $\mathcal{C}' < 0$. (55) also implies $c_+ = 0$. Hence, $\beta = u_- > u_+ > 0$ and $c_- > c_+ = 0$.

We next compute the wave speed s. Owing to (53) and the Cole-Hopf transformation (3), by the second equation of (9), we get $\beta = sv_\pm + u_\pm$, which together with (11) gives

$$s^2 + \chi(\beta - u_+ - u_-) = 0.$$

Hence, $s = \sqrt{\chi(u_+ + u_- - \beta)}$. $\square$

Proof of Theorem 1.2. The stability of u has been proved in Theorem 2.2. It remains to pass the results from v to c. In view of the transformations (3) and (15), we have

$$\frac{c(x,t)}{\mathcal{C}(x-st)} = e^{\int_{-\infty}^{x}(V(y-st)-v(y,t))dy} = e^{\psi(x,t)}.$$

By Cauchy-Schwarz inequality, the global estimate (19) and (52), we get

$$\begin{aligned}\sup_{x\in\mathbb{R}} \psi^2(x,t) &= 2\sup_{x\in\mathbb{R}} \int_{-\infty}^{x} \psi\psi_y(y,t)dy \\ &\le 2\Big(\int_{\mathbb{R}} \psi^2(y,t)dy\Big)^{1/2}\Big(\int_{\mathbb{R}} \psi_y^2(y,t)dy\Big)^{1/2} \\ &\le C\|\psi_x(\cdot,t)\| \\ &\to 0 \text{ as } t\to\infty.\end{aligned}$$

Then for all $x \in \mathbb{R}$,

$$\begin{aligned}|c(x,t) - \mathcal{C}(x-st)| &= |\mathcal{C}(x-st)e^{\psi(x,t)} - \mathcal{C}(x-st)| \\ &= \mathcal{C}(x-st)|1-e^{\psi(x,t)}| \\ &\le C|1-e^{\psi(x,t)}| \\ &\to 0 \text{ as } t\to\infty.\end{aligned}$$

The proof is completed. $\square$

Acknowledgements. The authors are grateful to the referee's insightful comments and suggestions, which lead to improvements of this manuscript.

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Received August 2020; revised November 2020.

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