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Local well-posedness of Boussinesq equations for MHD convection with fractional thermal diffusion in sobolev space $H^s(\mathbb{R}^n) \times H^{s+1-\epsilon}(\mathbb{R}^n) \times H^{s+\alpha-\epsilon}(\mathbb{R}^n)$

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A B S T R A C T

In this paper, we study the local well-posedness of the Boussinesq equation for MHD convection with fractional thermal diffusion in $H^s(\mathbb{R}^n) \times H^{s+1-\epsilon}(\mathbb{R}^n) \times H^{s+\alpha-\epsilon}(\mathbb{R}^n)$ with $s > \frac{n}{2} - 1$ and any small enough $\epsilon > 0$ such that $s+1-\epsilon > \frac{n}{2}$ and $s+\alpha-\epsilon \geq s+2-(\epsilon+\alpha) > \frac{n}{2}$. We present here the fractional operator $(-\Delta)^\alpha\theta$ for $\alpha > 1$ which is estimated by using Littlewood–Paley projection.

1. Introduction

The Boussinesq equation for magnetohydrodynamic convection (Boussinesq-MHD) is a dynamo model of turbulent plasma flows as stated in [1–3] and the references therein. Moreover, we consider the Boussinesq equation for magnetohydrodynamic convection with the fractional thermal diffusion:

$$\begin{cases} u_t + u \cdot \nabla u - b \cdot \nabla b + \nabla p - \nu \Delta u = \theta e_n, \\ b_t + u \cdot \nabla b - b \cdot \nabla u - \mu \Delta b = 0, \\ \theta_t + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{1.1}$$

and the initial conditions:

$$\begin{aligned} &u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \quad \theta(x,0) = \theta_0(x), \\ &\nabla \cdot u_0 = \nabla \cdot b_0 = 0. \end{aligned} \tag{1.2}$$

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for $x \in \mathbb{R}^n$ and $t \geq 0$. Here u, b, p, and θ respectively represent the fluid velocity, magnetic field, pressure, temperature in the content of thermal convection. Moreover $\nu > 0, \kappa > 0, \mu > 0$ are viscosity, thermal diffusivity, and electrical resistivity respectively. Meanwhile, $e_n = (0, 0, \dots, 0, 1)$ is the unit vector in the x_n direction.

The system of (1.1) is related to incompressible Boussinesq equations and electromagnetism of Maxwell's equations, where the Hall-MHD is neglected. The system of (1.1) reduces to the MHD equations when there are no temperature terms ($\theta \equiv 0$). Moreover, the system of (1.1) becomes the classical Boussinesq system, when there is no Lorentz force ($b \equiv 0$).

The system of (1.1) is also called as Bènard problem when $b \equiv 0$ and called as magnetic Bènard when the influence of the magnetic field for the behavior of the thermal instability is considered. Bènard problem has been under extensive investigation, more precisely concerning the stability (see [4] in Chapter 3). Moreover, magnetic Bènard problem has also been of much attention as in [5–7].

Meanwhile, the global well-posedness for weak or strong solutions was obtained by the stability and instability in a fully nonlinear 2D Boussinesq-MHD system as stated in Refs. [8–11]. In addition, Larios-Pei [12] established the local well-posedness in H^3, which was not optimal. Recently, the global well-posedness of axisymmetric Boussinesq-MHD system and the nonlinear damping for the Boussinesq-MHD equations have been studied in [13,14]. Hall-MHD system with or without fractional magnetic diffusion was studied by Dai in [15,16], as shown in the two following equations

$$\begin{cases} u_t + u \cdot \nabla u - b \cdot \nabla b + \nabla p = \nu \Delta u, \\ b_t + u \cdot \nabla b - b \cdot \nabla u + \nabla \eta \times ((\nabla \times b) \times b) = -\mu \Delta b, \end{cases} \tag{1.3}$$

and

$$\begin{cases} u_t + u \cdot \nabla u - b \cdot \nabla b + \nabla p = \nu \Delta u, \\ b_t + u \cdot \nabla b - b \cdot \nabla u + \nabla \eta \times ((\nabla \times b) \times b) = -\mu (-\Delta)^{\alpha} b, \end{cases} \tag{1.4}$$

where the functions (u, b) of (1.3) (without fractional magnetic) and (1.4) (with fractional magnetic) were in $H^s(\mathbb{R}^n) \times H^{s+1-\epsilon}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) = (H^s(\mathbb{R}^n))^2$ respectively. Moreover, Dai used $\frac{1}{2} < \alpha < 1$, $s > \frac{n}{2}$ in (1.4), and $s > \frac{n}{2} + 1$, $\alpha = 1$ in (1.3). Especially for the fractional case, this research studied by Dai in (1.4) was improvement of the research studied by Chae et al. [17]. The system of equations studied by Chae was similar to (1.4) and the space of $(H^s(\mathbb{R}^n))^2$ was also similar, the difference was in the condition of $s > \frac{n}{2} + 1$, where this condition was smaller than the condition $s > \frac{n}{2}$ and also the fractional magnetic was denoted by $\Lambda = (-\Delta)^{\frac{1}{2}}$ when $\alpha = \frac{1}{2}$.

Based on the previous works, our interest of this paper is to study locally well-posed Sobolev spaces $H^s(\mathbb{R}^n) \times H^{s+1-\epsilon}(\mathbb{R}^n) \times H^{s+\alpha-\epsilon}(\mathbb{R}^n)$ with $s > \frac{n}{2} - 1$ and any small enough $\epsilon > 0$ such that $s + 1 - \epsilon > \frac{n}{2}$ and $s + \alpha - \epsilon \geq s + 2 - (\epsilon + \alpha) > \frac{n}{2}$ of Boussinesq for MHD convection with fractional thermal diffusion. Moreover, we consider velocity (u), magnetic (b), and temperature (θ) in this research, where the fractional term is for the temperature (θ). The difference with the previous studies is in the Sobolev space. For the fractional cases, the previous works used the same space of u and b respectively in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) = (H^s(\mathbb{R}^n))^2$, but this research, we use the different spaces for u, b, and θ respectively in $H^s(\mathbb{R}^n) \times H^{s+1-\epsilon}(\mathbb{R}^n) \times H^{s+\alpha-\epsilon}(\mathbb{R}^n)$ with the condition $\alpha > 1$ and the condition of s mentioned before. This issue generates new analytical barriers that are difficult to handle. We provide a first attempt to break down this barrier through this paper.

This paper is organized as follows. In Section 2, we establish a priori estimates by using Littlewood–Paley decomposition, Bony's paraproduct calculus, commutator estimates which refer to [18–20]. Finally, by using Grönwall inequality, and identifying the Sobolev space H^s by Besov space $B^s_{2,2}$ we prove a priori estimates. In Section 3, we prove the unique solutions (u, b, θ) of (1.1) and also the continuity by the standard ways.

Remark 1.1. In general, the fractional diffusion cases admit the same Sobolev space for all functions (u, b), e.g. $(u, b) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) = (H^s(\mathbb{R}^n))^2$ (see [16,17]). Meanwhile, the current paper presents the different Sobolev spaces for all functions, e.g. $(u,b,\theta) \in H^s(\mathbb{R}^n) \times H^{s+1-\epsilon}(\mathbb{R}^n) \times H^{s+\alpha-\epsilon}(\mathbb{R}^n)$. Moreover, the difficulty of this paper is to find the appropriate conditions for estimate of functions (u,b,θ) in different Sobolev spaces. Especially, the optimal condition is determined for function θ involving the fractional Laplace operator with $\alpha > 1$.

2. Preliminaries

Throughout this paper, we always use $\|\cdot\|_p$ to denote $\|\cdot\|_{L^p}$. Then, we refer to [21] and [20] for briefly the Littlewood–Paley decomposition theory.

Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ be Fourier and inverse Fourier transforms. A non-negative radial function $\Phi \in C_0^\infty(\mathbb{R}^n)$ is given as

$$\Phi(\xi) = \begin{cases} 1, & \text{for } |\xi| \leq \frac{3}{4} \\ 0, & \text{for } |\xi| \geq 1. \end{cases}$$

Moreover, we present

$$\varphi(\xi) = \Phi\left(\frac{\xi}{2}\right) - \Phi(\xi)$$

and

$$\varphi_m(\xi) = \begin{cases} \varphi(\lambda_m^{-1}\xi), & \text{for } m \geq 0 \\ \Phi(\xi), & \text{for } m = -1. \end{cases}$$

where $\lambda_m = 2^m$ for integers m.

Then, the Little–Paley projection is defined as follows

$$u_m := \Delta_m u = \mathcal{F}^{-1}(\varphi(\lambda_m^{-1}\xi)\mathcal{F}u),$$
$$u_{-1} := \mathcal{F}^{-1}(\Phi(\xi)\mathcal{F}u).$$

Other useful notations in this paper are

$$u_{\leq q} = \sum_{m=-1}^{q} \Delta_m u, \quad u_m = \Delta_m u, \quad u_{(k,n]} = \sum_{k=q+1}^{n} u_k, \quad \widetilde{u}_m = \sum_{|m-k|\leq 1} u_k.$$

3. A priori estimates

In this section, we establish *a priori* estimates for smooth solutions of (1.1). We first present the notation $\mathcal{X}_{s,r,l} = H^s(\mathbb{R}^n) \times H^r(\mathbb{R}^n) \times H^l(\mathbb{R}^n)$ for simplification. Then we give Theorem 3.1 which is our main results of this paper.

Theorem 3.1. *Consider $\alpha > 1, n \geq 3$ to (1.1) and the initial data $(u_0,b_0,\theta_0) \in \mathcal{X}_{s,r,l}$ with $s > \frac{n}{2} - 1, \frac{n}{2} < r \leq s+1-\epsilon, \frac{n}{2} < l \leq s+\alpha-\epsilon$, for small enough $\epsilon > 0$. Then there exist $T = T(\nu,\mu,\kappa,\|(u_0,b_0,\theta_0)\|_{\mathcal{X}_{s,r,l}}) > 0$ and unique solution (u,b,θ) of (1.1) on $[0,T]$, such that*

$$\|(u(t),b(t),\theta(t))\|_{\mathcal{X}_{s,r,l}} \leq C(\alpha,\nu,\mu,\kappa,\|(u_0,b_0,\theta_0)\|_{\mathcal{X}_{s,r,l}})$$

Moreover, based on Theorem 3.1, we have Remark 3.1 for the special case when $\alpha = 1$.

Remark 3.1. Notice that $l \leq s+\alpha-\epsilon = s+1-\epsilon$ for $\alpha = 1$ and small enough ϵ. Moreover, $s+2-(\epsilon+\alpha) \leq s+\alpha-\epsilon$ for $\alpha > 1$. Thus, for $\alpha = 1$, we get the local well-posedness of (1.1) in $H^s \times H^{s+1-\epsilon} \times H^{s+1-\epsilon}$ with $s > \frac{n}{2} - 1$.

Then we want to prove Theorem 3.1 by identifying H^s with $B^s_{2,2}$. Let $m \in \mathbb{Z}$ be an integer and Δ_m be the localized operator of homogeneous frequency (see [21]). We first apply the operator Δ_m to Eqs. (1.1), then we multiply the result by $2^{2ms}\Delta_m u$, $2^{2mr}\Delta_m b$, and $2^{2ml}\Delta_m \theta$ respectively for the first, second, and third equations of (1.1). We further combine all the results, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{m\geq -1}\left(2^{2ms}\|\Delta_m u\|_2^2 + 2^{2mr}\|\Delta_m b\|_2^2 + 2^{2ml}\|\Delta_m \theta\|_2^2\right)\\
&\leq -C_0\sum_{m\geq -1}\left(\nu 2^{2ms+2m}\|\Delta_m u\|_2^2 + \mu 2^{2mr+2m}\|\Delta_m b\|_2^2 + \kappa 2^{2ml+2m\alpha}\|\Delta_m \theta\|_2^2\right)\\
&- \sum_{m\geq -1} 2^{2ms}\int_{\mathbb{R}^3}\Delta_m(u\cdot\nabla u)\cdot\Delta_m u\,dx + \sum_{m\geq -1} 2^{2ms}\int_{\mathbb{R}^3}\Delta_m(b\cdot\nabla b)\cdot\Delta_m u\,dx\\
&+ \sum_{m\geq -1} 2^{2ms}\int_{\mathbb{R}^3}\Delta_m \theta e_n\cdot\Delta_m u\,dx - \sum_{m\geq -1} 2^{2mr}\int_{\mathbb{R}^3}\Delta_m(u\cdot\nabla b)\cdot\Delta_m b\,dx\\
&+ \sum_{m\geq -1} 2^{2mr}\int_{\mathbb{R}^3}\Delta_m(b\cdot\nabla u)\cdot\Delta_m b\,dx - \sum_{m\geq -1} 2^{2ml}\int_{\mathbb{R}^3}\Delta_m(u\cdot\nabla \theta)\cdot\Delta_m \theta\,dx
\end{aligned}$$

Moreover, we write the final results of above inequalities as follows

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{m\geq -1}\left(2^{2ms}\|\Delta_m u\|_2^2 + 2^{2mr}\|\Delta_m b\|_2^2 + 2^{2ml}\|\Delta_m \theta\|_2^2\right)\\
&\leq -C_0\sum_{m\geq -1}\left(\nu 2^{2ms+2m}\|\Delta_m u\|_2^2 + \mu 2^{2mr+2m}\|\Delta_m b\|_2^2 + \kappa 2^{2ml+2m\alpha}\|\Delta_m \theta\|_2^2\right)\\
&- L_1 + L_2 + L_3 - L_4 + L_5 - L_6
\end{aligned} \tag{3.1}$$

Notice that the fractional thermal diffusion term follows from the following annulus for each $m \in \mathbb{Z}$ that,

$$\begin{aligned}
A_m &= \left\{\xi \in \mathbb{R}^d : 2^{m-1} \leq |\xi| \leq 2^{m+1}\right\}\\
&= \left\{\xi \in \mathbb{R}^d : \frac{2^m}{2} \leq |\xi| \leq 2(2^m)\right\} = \left\{\xi \in \mathbb{R}^d : C_1 2^m \leq |\xi| \leq C_2 2^m\right\}
\end{aligned}$$

and the fractional Laplacian operator of $(-\Delta)^\alpha$,

$$\mathcal{F}((-\Delta)^\alpha f(\xi)) = |\xi|^{2\alpha}\mathcal{F}(f(\xi))$$

we further can find the lower bounds for the fractional term of thermal diffusion

$$(-\Delta)^\alpha \Delta_m \theta \geq \mathcal{F}^{-1}\left(\varphi(\lambda_m^{-1}\xi)C_1^{2\alpha}2^{2m\alpha}\mathcal{F}\theta\right) \geq C_0 2^{2m\alpha}\theta_m.$$

where C_0 is a constant.

The estimates of L_1, L_2, L_4 and L_5 can be established with the similar way in [15]. We first present the estimate of L_1. We apply the following Bony's paraproduct

$$\begin{aligned}
\Delta_m(u\cdot\nabla v) := &\sum_{|m-k|\leq 2}\Delta_m(u_{\leq k-2}\cdot\nabla v_k) + \sum_{|m-k|\leq 2}\Delta_m(u_k\cdot\nabla v_{\leq k-2})\\
&+ \sum_{k\geq m-2}\Delta_m(u_k\cdot\nabla \widetilde{v}_k).
\end{aligned} \tag{3.2}$$

Then, one has

$$\begin{aligned} L_1 &= \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ms}\int_{\mathbb{R}^3} \Delta_m(u\cdot\nabla u)\Delta_m u\,dx \\ &= \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ms}\int_{\mathbb{R}^3} \Delta_m(u_{\leq k-2}\cdot\nabla u_k)\Delta_m u\,dx \\ &\quad + \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ms}\int_{\mathbb{R}^3} \Delta_m(u_k\cdot\nabla u_{\leq k-2})\Delta_m u\,dx \\ &\quad + \sum_{m\geq -1}\sum_{k\geq m-2} 2^{2ms}\int_{\mathbb{R}^3} \Delta_m(u_k\cdot\nabla \tilde{u}_k)\Delta_m u\,dx = L_{11}+L_{12}+L_{13} \end{aligned}$$

Due to the commutator

$$[\Delta_m, u_{\leq k-2}\cdot\nabla]v_k := \Delta_m(u_{\leq k-2}\cdot\nabla v_k) - u_{\leq k-2}\cdot\nabla\Delta_m v_k, \tag{3.3}$$

we get

$$\begin{aligned} L_{11} &= \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ms}\int_{\mathbb{R}^3} [\Delta_m, u_{\leq k-2}\cdot\nabla]u_k\Delta_m u\,dx \\ &\quad + \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ms}\int_{\mathbb{R}^3} u_{\leq k-2}\cdot\nabla\Delta_m u_k\Delta_m u\,dx = L_{111}+L_{112}. \end{aligned}$$

We further employ the following two lemmas for commutator and Bernstein's inequality.

Lemma 3.1. *The commutator satisfies the following estimate, for any $1<r<\infty$*

$$\|[\Delta_m, u_{\leq k-2}\cdot\nabla]v_k\|_r \lesssim \|\nabla u_{\leq k-2}\|_\infty \|v_k\|_r \tag{3.4}$$

Lemma 3.2. *Let n be the dimensional space and $r\geq s\geq 1$. Then for all distributions u, we have*

$$\|\Delta_m u\|_r \lesssim 2^{mn\left(\frac{1}{s}-\frac{1}{r}\right)}\|\Delta_m u\|_s. \tag{3.5}$$

Then, one has

$$\begin{aligned} |L_{111}| &\leq \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ms}\|\nabla u_{\leq k-2}\|_\infty\|u_k\|_2\|\Delta_m u\|_2 \\ &\lesssim \sum_{m\geq -1} 2^{2ms}\|\Delta_m u\|_2^2\sum_{k\leq m} 2^{\frac{n}{2}k+k}\|u_k\|_2 \\ &\lesssim \sum_{m\geq -1} 2^{(s+1)m\chi}\|\Delta_m u\|_2^{\chi}2^{ms(2-\chi)}\|\Delta_m u\|_2^{2-\chi} \\ &\quad \sum_{k\leq m} 2^{(s+1)k\psi}\|u_k\|_2^{\psi}2^{ks(1-\psi)}\|u_k\|_2^{1-\psi}\left(2^{-m\chi}2^{\frac{n}{2}k+k-ks-k\psi}\right) \\ &\lesssim \sum_{m\geq -1} 2^{(s+1)m\chi}\|\Delta_m u\|_2^{\chi}2^{ms(2-\alpha\chi)}\|\Delta_m u\|_2^{2-\chi} \\ &\quad \sum_{k\leq m} 2^{(s+1)k\psi}\|u_k\|_2^{\psi}2^{ks(1-\psi)}\|u_k\|_2^{1-\psi}2^{(k-m)\chi} \end{aligned}$$

where we have used

$$s\geq \frac{n}{2}+1-\chi-\psi,$$

for $0<\chi<2, 0<\psi<1$.

By Young's inequality,

$$|L_{111}| \le \frac{\nu}{32}\sum_{m\ge -1} 2^{2ms+2m}\|\Delta_m u\|_2^2 + C_\nu\left(\sum_{m\ge -1} 2^{2ms}\|\Delta_m u\|_2^2\right)^{\gamma_1} + C_\nu\left(\sum_{m\ge -1} 2^{2ms}\|\Delta_m u\|_2^2\right)^{\gamma_2},$$

for $\gamma_1 = \frac{1-\psi}{2} > 0$ and $\gamma_2 = \frac{2-\chi}{2} > 0$.

Based on the lemmas for the commutator, Bernstein's inequality, and Young's inequality, we have

$$\begin{aligned}
|L_{112}| &\le \sum_{m\ge -1}\sum_{|m-k|\le 2} 2^{2ms}\|\nabla u_k\|_\infty \|u_{\le k-2}\|_2\|\Delta_m u\|_2 \\
&\lesssim \sum_{m\ge -1} 2^{2ms}\|\Delta_m u\|_2 \sum_{k\le m} 2^{\frac{n}{2}k+k}\|u_{\le k-2}\|_2 \\
&\lesssim \sum_{m\ge -1} 2^{2ms}\|\Delta_m u\|_2 \sum_{k\le m}\sum_{m=-1}^{k-2} 2^{\frac{n}{2}k+k}\|\Delta_m u\|_2 \\
&\lesssim \sum_{m\ge -1} 2^{2ms+\frac{n}{2}m+m}\|\Delta_m u\|_2^2 \\
&\lesssim \sum_{m\ge -1} 2^{(ms+m)\chi}\|\Delta_m u\|_2^{\chi} 2^{(2-\chi)ms}\|\Delta_m u\|_2^{2-\chi} \\
&\le \frac{\nu}{32}\sum_{m\ge -1} 2^{2ms+2m}\|\Delta_m u\|_2^2 + C_\nu\left(\sum_{m\ge -1} 2^{2ms}\|\Delta_m u|_2^2\right)^{\frac{2-\chi}{2}}
\end{aligned}$$

The estimates of L_{12} and L_{13} are similar way with L_{111} and L_{112}. Especially for L_{13}, we use the notations

$$\widetilde{u}_m = \sum_{|m-k|\le 1} u_k.$$

Then, we finally have the following estimate of L_1

$$|L_1| \le \frac{\nu}{8}\sum_{m\ge -1} 2^{2ms+2m}\|\Delta_m u\|_2^2 + C_\nu\left(\sum_{m\ge -1} 2^{2ms}\|\Delta_m u\|_2^2\right)^{\gamma_1} + C_\nu\left(\sum_{m\ge -1} 2^{2ms}\|\Delta_m u\|_2^2\right)^{\gamma_2}, \tag{3.6}$$

where $\gamma_1 = \frac{1-\psi}{2} > 0, \gamma_2 = \frac{2-\chi}{2} > 0$ and $0 < \psi < 1, 0 < \chi < 2$.

Next, we establish the estimate of L_2. We first decompose L_2 as follows

$$\begin{aligned}
L_2 &= \sum_{m\ge -1}\sum_{|m-k|\le 2} 2^{2ms}\int_{\mathbb{R}^3} \Delta_m(b\cdot\nabla b)\Delta_m u\, dx \\
&= \sum_{m\ge -1}\sum_{|m-k|\le 2} 2^{2ms}\int_{\mathbb{R}^3} \Delta_m(b_{\le k-2}\cdot\nabla b_k)\Delta_m u\, dx \\
&\quad + \sum_{m\ge -1}\sum_{|m-k|\le 2} 2^{2ms}\int_{\mathbb{R}^3} \Delta_m(b_k\cdot\nabla b_{\le k-2})\Delta_m u\, dx \\
&\quad + \sum_{m\ge -1}\sum_{k\ge m-2} 2^{2ms}\int_{\mathbb{R}^3} \Delta_m(b_k\cdot\nabla \widetilde{b}_k)\Delta_m u\, dx = L_{21}+L_{22}+L_{23}
\end{aligned}$$

where we have used Bony's paraproduct and commutator notation.

By applying lemmas for commutator and Bernstein's inequality, then followed by Young's inequality, one has

$$\begin{aligned}
|L_{21}| &\leq \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ms}\|\nabla b_{\leq k-2}\|_\infty \|b_k\|_2\|\Delta_m u\|_2 \\
&\lesssim \sum_{m\geq -1} 2^{2ms}\|\Delta_m u\|_2^2 \sum_{k\leq m} 2^{\frac{n}{2}k}\|b_k\|_2 \\
&\lesssim \sum_{m\geq -1} 2^{2ms+m}\|\Delta_m u\|_2^2 \sum_{k\leq m} 2^{ks}\|b_k\|_2 2^{\frac{n}{2}k-ks} \\
&\leq \frac{\nu}{24}\sum_{m\geq -1} 2^{2ms+2m}\|\Delta_m u\|_2^2 + C_\nu\left(\sum_{m\geq -1} 2^{2ms}\|\Delta_m b\|_2^2\right)^{\frac{1}{2}}
\end{aligned}$$

where we have used $\frac{n}{2}-s\leq 0$.

Since the estimate of L_{22} is similar with the L_{21}, then we claim

$$\begin{aligned}
|L_{22}| &\leq \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ms}\|\nabla b_k\|_\infty \|b_{\leq k-2}\|_2\|\Delta_m u\|_2 \\
&\lesssim |L_{21}|.
\end{aligned}$$

Due to the lemmas for commutator and Bernstein's inequality, we get

$$\begin{aligned}
|L_{23}| &\leq \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ms}\|\nabla b_k\|_\infty \|\widetilde{b}_k\|_2\|\Delta_m u\|_2 \\
&\lesssim \sum_{m\geq -1} 2^{2ms}\|\Delta_m u\|_2^2 \sum_{k\leq m} 2^{\frac{n}{2}k}\|\widetilde{b}_k\|_2.
\end{aligned}$$

We use the following notation

$$\widetilde{b}_m = \sum_{|m-k|\leq 1} b_k,$$

to get

$$\begin{aligned}
|L_{23}| &\lesssim \sum_{m\geq -1} 2^{2ms}\|\Delta_m u\|_2^2 \sum_{k\leq m} 2^{\frac{n}{2}k}\|b_k\|_2 \\
&\leq \frac{\nu}{24}\sum_{m\geq -1} 2^{2ms+2m}\|\Delta_m u\|_2^2 + C_\nu\left(\sum_{m\geq -1} 2^{2ms}\|\Delta_m b\|_2^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Finally, the estimate of L_2 is given as follows

$$|L_2| \leq \frac{\nu}{8}\sum_{m\geq -1} 2^{2ms+2m}\|\Delta_m u\|_2^2 + C_\nu\left(\sum_{m\geq -1} 2^{2ms}\|\Delta_m u\|_2^2\right)^{\frac{1}{2}}. \tag{3.7}$$

As for L_1, by Bony's paraproduct, we decompose L_4 as follows

$$\begin{aligned}
L_4 &= \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2mr}\int_{\mathbb{R}^3} \Delta_m(u\cdot\nabla b)\Delta_m b\, dx \\
&= \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2mr}\int_{\mathbb{R}^3} \Delta_m(u_{\leq k-2}\cdot\nabla b_k)\Delta_m b\, dx \\
&\quad + \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2mr}\int_{\mathbb{R}^3} \Delta_m(u_k\cdot\nabla b_{\leq k-2})\Delta_m b\, dx \\
&\quad + \sum_{m\geq -1}\sum_{k\geq m-2} 2^{2mr}\int_{\mathbb{R}^3} \Delta_m(u_k\cdot\nabla \widetilde{b}_k)\Delta_m b\, dx = L_{41}+L_{42}+L_{43}
\end{aligned}$$

Based on the commutator, one has

$$
\begin{aligned}
L_{41} &= \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2mr}\int_{\mathbb{R}^3}[\Delta_m, u_{\leq k-2}\cdot\nabla]b_k\Delta_m b\,dx \\
&\quad + \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2mr}\int_{\mathbb{R}^3} u_{\leq k-2}\cdot\nabla\Delta_m b_k\Delta_m b\,dx = L_{411}+L_{412}.
\end{aligned}
$$

Apply the lemmas for commutator and Bernstein's inequality to get

$$
\begin{aligned}
|L_{411}| &\leq \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2mr}\|\nabla u_{\leq k-2}\|_\infty\|b_k\|_2\|\Delta_m b\|_2 \\
&\lesssim \sum_{m\geq -1} 2^{2mr}\|\Delta_m b\|_2^2\sum_{k\leq m} 2^{\frac{n}{2}k+k}\|u_k\|_2 \\
&\lesssim \sum_{m\geq -1} 2^{(r+1)m\chi}\|\Delta_m b\|_2^{\chi}2^{mr(2-\chi)}\|\Delta_m b\|_2^{2-\chi} \\
&\quad \sum_{k\leq m} 2^{(s+1)k\psi}\|u_k\|_2^{\psi}2^{ks(1-\psi)}\|u_k\|_2^{1-\psi}\left(2^{-m\chi}2^{\frac{n}{2}k+k-ks-k\psi}\right) \\
&\lesssim \sum_{m\geq -1} 2^{(r+1)m\chi}\|\Delta_m b\|_2^{\chi}2^{mr(2-\chi)}\|\Delta_m b\|_2^{2-\chi} \\
&\quad \sum_{k\leq m} 2^{(s+1)k\psi}\|u_k\|_2^{\psi}2^{ks(1-\psi)}\|u_k\|_2^{1-\psi}2^{(k-m)\chi}
\end{aligned}
$$

where we have used $s\geq \frac{n}{2}+1-\chi-\psi+\epsilon$ for small enough $\epsilon>0$ and $0<\chi<2, 0<\psi<1$.

By Young's inequality, one has

$$
\begin{aligned}
|L_{411}| &\leq \frac{\nu}{32}\sum_{m\geq -1}2^{2ms+2m}\|\Delta_m u\|_2^2+\frac{\mu}{32}\sum_{m\geq -1}2^{2mr+2m}\|\Delta_m b\|_2^2 \\
&\quad + C_{\nu,\mu}\left(\sum_{m\geq -1}2^{2ms}\|\Delta_m u\|_2^2\right)^{\frac{1-\psi}{2}} + C_{\nu,\mu}\left(\sum_{m\geq -1}2^{2mr}\|\Delta_m b\|_2^2\right)^{\frac{2-\chi}{2}}.
\end{aligned}
$$

We use the similar technique in L_{411} to estimate L_{412}.

Now, we decompose L_{42} as follows

$$
\begin{aligned}
|L_{42}| &\leq \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2mr}\|\nabla b_{\leq k-2}\|_\infty\|u_k\|_2\|\Delta_m b\|_2 \\
&\lesssim \sum_{m\geq -1} 2^{2mr}\|\Delta_m b\|_2\|\Delta_m u\|_2\sum_{k\leq m}2^{\frac{n}{2}k+k}\|b_k\|_2 \\
&\lesssim \sum_{m\geq -1} 2^{(s+1)m}\|\Delta_m u\|_2 2^{m\chi(r+1)}\|\Delta_m b\|_2^{\chi}2^{(2-\chi)mr}\|\Delta_m b\|_2^{1-\chi}\sum_{k\leq m}2^{\frac{n}{2}k+k}\|b_k\|_2.
\end{aligned}
$$

By Young's inequality, one has

$$
\begin{aligned}
|L_{42}| &\leq \frac{\nu}{32}\sum_{m\geq -1}2^{2ms+2m}\|\Delta_m u\|_2^2+\frac{\mu}{32}\sum_{m\geq -1}2^{2mr+2m}\|\Delta_m b\|_2^2 \\
&\quad + C_{\nu,\mu}\left(\sum_{m\geq -1}2^{2mr}\|\Delta_m b\|_2^2\right)^{\frac{1-\psi}{2}},
\end{aligned}
$$

since the estimate L_{43} is similar with L_{23}. Then, we claim for the estimates of L_{43} as follows

$$|L_{42}| \leq \frac{\nu}{32} \sum_{m\geq -1} 2^{2ms+2m} \|\Delta_m u\|_2^2 + \frac{\mu}{32} \sum_{m\geq -1} 2^{2mr+2m} \|\Delta_m b\|_2^2 + C_{\nu,\mu} \left(\sum_{m\geq -1} 2^{2mr} \|\Delta_m b\|_2^2 \right)^{\frac{1}{2}}.$$

Based on the results of L_{41}, L_{42}, and L_{43}, we present the following estimate of L_4

$$\begin{aligned} |L_4| \leq{}& \frac{\nu}{8} \sum_{m\geq -1} 2^{2ms+2m} \|\Delta_m u\|_2^2 + \frac{\mu}{8} \sum_{m\geq -1} 2^{2mr+2m} \|\Delta_m b\|_2^2 \\ &+ C_{\nu,\mu} \left(\sum_{m\geq -1} 2^{2ms} \|\Delta_m u\|_2^2 \right)^{\frac{1-\psi}{2}} + C_{\nu,\mu} \left(\sum_{m\geq -1} 2^{2mr} \|\Delta_m b\|_2^2 \right)^{\frac{2-\chi}{2}}. \end{aligned} \tag{3.8}$$

By using Bony's paraproduct as usual, we decompose L_5 as follows

$$\begin{aligned} L_5 &= \sum_{m\geq -1} \sum_{|m-k|\leq 2} 2^{2mr} \int_{\mathbb{R}^3} \Delta_m (b \cdot \nabla u) \Delta_m b \, dx \\ &= \sum_{m\geq -1} \sum_{|m-k|\leq 2} 2^{2mr} \int_{\mathbb{R}^3} \Delta_m (b_{\leq k-2} \cdot \nabla u_k) \Delta_m b \, dx \\ &\quad + \sum_{m\geq -1} \sum_{|m-k|\leq 2} 2^{2mr} \int_{\mathbb{R}^3} \Delta_m (b_k \cdot \nabla u_{\leq k-2}) \Delta_m b \, dx \\ &\quad + \sum_{m\geq -1} \sum_{k\geq m-2} 2^{2mr} \int_{\mathbb{R}^3} \Delta_m (b_k \cdot \nabla \widetilde{u}_k) \Delta_m b \, dx = L_{51} + L_{52} + L_{53} \end{aligned}$$

Notice that L_{52} and L_{53} can be estimated as L_{411} and L_{43} respectively. Then, we only need to estimate L_{51}. By applying the lemmas for commutator and Bernstein's inequality, one has

$$\begin{aligned} |L_{51}| \leq{}& \sum_{m\geq -1} \sum_{|m-k|\leq 2} 2^{2mr} \|\nabla b_{\leq k-2}\|_\infty \|b_k\|_2 \|\Delta_m b\|_2 \\ \lesssim{}& \sum_{m\geq -1} 2^{2mr} \|\Delta_m b\|_2^2 2^{2ms} \|\Delta_m u\|_2^2 \sum_{k\leq m} 2^{\frac{n}{2}k+k} \|u_k\|_2 \\ \lesssim{}& \sum_{m\geq -1} 2^{(r+1)m\chi} \|\Delta_m b\|_2^\chi 2^{mr(2-\chi)} \|\Delta_m b\|_2^{2-\chi} \sum_{m\geq -1} 2^{(s+1)m\zeta} \|\Delta_m u\|_2^\zeta 2^{ms(2-\zeta)} \|\Delta_m u\|_2^{2-\zeta} \\ & \sum_{k\leq m} 2^{(s+1)k\psi} \|u_k\|_2^\psi 2^{ks(1-\psi)} \|u_k\|_2^{1-\psi} \left(2^{-m\chi - m\zeta} 2^{\frac{n}{2}k+k-ks-k\psi-k\zeta} \right) \\ \lesssim{}& \sum_{m\geq -1} 2^{(r+1)m\chi} \|\Delta_m b\|_2^\chi 2^{mr(2-\alpha\chi)} \|\Delta_m b\|_2^{2-\chi} \\ & \sum_{k\leq m} 2^{(s+1)k\psi} \|u_k\|_2^\psi 2^{ks(1-\psi)} \|u_k\|_2^{1-\psi} 2^{(k-m)\chi} 2^{(k-m)\zeta} \end{aligned}$$

where we have used $\frac{n}{2}+1-s-\psi-\zeta \leq 0$ and $0<\chi<2, 0<\zeta<2, 0<\psi<1$. By Young's inequality and the results of L_{52}, L_{53}, we have estimate of L_5

$$\begin{aligned}|L_5| \leq &\frac{\nu}{8}\sum_{m\geq -1} 2^{2ms+2m}\|\Delta_m u\|_2^2 + \frac{\mu}{8}\sum_{m\geq -1} 2^{2mr+2m}\|\Delta_m b\|_2^2 \\ &+ C_{\nu,\mu}\left(\sum_{m\geq -1} 2^{2ms}\|\Delta_m u\|_2^2\right)^{\delta_1} + C_{\nu,\mu}\left(\sum_{m\geq -1} 2^{2ms}\|\Delta_m u\|_2^2\right)^{\delta_2} \\ &+ C_{\nu,\mu}\left(\sum_{m\geq -1} 2^{2mr}\|\Delta_m b\|_2^2\right)^{\delta_3}.\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}&\delta_1 = \frac{1-\psi}{2} > 0,\ \ \delta_2 = \frac{2-\zeta}{2} > 0,\ \ \delta_3 = \frac{2-\chi}{2} > 0 \\ &0<\chi<2,\ \ 0<\zeta<2,\ \ 0<\psi<1\end{aligned}$$

These two estimates of L_3 and L_6 contain the fractional thermal diffusion. It needs a little bit different ways to set the conditions so the results become more optimal. Let $l \leq s+\alpha-\epsilon$ for small enough $\epsilon>0$. We apply Hölder's inequality and Young's inequality, then the estimate of L_3 is obtained,

$$\begin{aligned}|L_3| &= \Big|\sum_{m\geq -1} 2^{2ms}\int_{\mathbb{R}^3} \Delta_m\theta e_n \Delta_m u\, dx\Big| \\ &\leq \sum_{m\geq -1} 2^{2ms+3m}\int_{\mathbb{R}^3} |\Delta_m\theta\Delta_m u| \quad dx \\ &\leq \sum_{m\geq -1} 2^{(-l-\alpha+s+2)m}\left(2^{(l+\alpha)m}\|\Delta_m\theta\|_2\right)\left(2^{(s+1)m}\|\Delta_m u\|_2\right) \\ &\leq \sum_{m\geq -1} 2^{(-l-\alpha+s+2)m}\left(2^{(2l+2\alpha)m}\|\Delta_m\theta\|_2^2 + 2^{(2s+2)m}\|\Delta_m u\|_2^2\right) \\ &\leq \frac{\nu}{8}\sum_{m\geq -1} 2^{(2s+2)m}\|\Delta_m u\|_2^2 + \frac{\kappa}{8}\sum_{m\geq -1} 2^{(2l+2\alpha)m}\|\Delta_m\theta\|_2^2\end{aligned} \tag{3.10}$$

where we have used $s+2-(\epsilon+\alpha) \leq l$ and $s+2-\alpha \leq l$.

We further establish the estimate of L_6 by using the assumptions $s>\frac{n}{2}-1, s+1-\epsilon>\frac{n}{2}$, and $l \leq s+\alpha-\epsilon$ for small enough $\epsilon>0$. Firstly, by applying Bony's paraproduct, we have

$$\begin{aligned}L_6 &= \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ml}\int_{\mathbb{R}^3} \Delta_m(u\cdot\nabla\theta)\Delta_m\theta\ dx \\ &= \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ml}\int_{\mathbb{R}^3} \Delta_m(u_{\leq k-2}\cdot\nabla\theta_k)\Delta_m\theta\ dx \\ &\quad + \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ml}\int_{\mathbb{R}^3} \Delta_m(u_k\cdot\nabla\theta_{\leq k-2})\Delta_m\theta\ dx \\ &\quad + \sum_{m\geq -1}\sum_{k\geq m-2} 2^{2ml}\int_{\mathbb{R}^3} \Delta_m(u_k\cdot\nabla\widetilde{\theta}_k)\Delta_m\theta\ dx = L_{61}+L_{62}+L_{63}\end{aligned}$$

with

$$\begin{aligned}L_{61} = &\sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ml}\int_{\mathbb{R}^3} [\Delta_m, u_{\leq k-2}\cdot\nabla]\theta_k\Delta_m\theta\ dx \\ &+ \sum_{m\geq -1}\sum_{|m-k|\leq 2} 2^{2ml}\int_{\mathbb{R}^3} u_{\leq k-2}\cdot\nabla\Delta_m\theta_k\Delta_m\theta\ dx = L_{611}+L_{612}.\end{aligned}$$

Set $\chi = 2 - \frac{2}{\alpha}$ and $\psi = 1 - \frac{1}{\alpha}$. For $\alpha > 1$, $\chi \in (0,2)$, $\psi \in (0,1)$, and $s \geq \frac{n}{2} + 1 - \chi - \psi$. By the commutator estimate, we obtain

$$\begin{aligned}
|L_{611}| &\leq \sum_{m\geq -1} \sum_{|m-k|\leq 2} 2^{2ml} \|\nabla u_{\leq k-2}\|_\infty \|\theta_k\|_2 \|\Delta_m \theta\|_2 \\
&\lesssim \sum_{m\geq -1} 2^{2ml} \|\Delta_m \theta\|_2^2 \sum_{k\leq m} 2^{\frac{n}{2}k+k} \|u_k\|_2 \\
&\lesssim \sum_{m\geq -1} 2^{(l+\alpha)m\chi} \|\Delta_m \theta\|_2^\chi 2^{ml(2-\alpha\chi)} \|\Delta_m \theta\|_2^{2-\chi} \\
&\quad \sum_{k\leq m} 2^{(s+1)k\psi} \|u_k\|_2^\psi 2^{ks(1-\psi)} \|u_k\|_2^{1-\psi} \left(2^{-m\chi} 2^{\frac{n}{2}k+k-ks-k\psi}\right) \\
&\lesssim \sum_{m\geq -1} 2^{(l+\alpha)m\chi} \|\Delta_m \theta\|_2^\chi 2^{ml(2-\alpha\chi)} \|\Delta_m \theta\|_2^{2-\chi} \\
&\quad \sum_{k\leq m} 2^{(s+1)k\psi} \|u_k\|_2^\psi 2^{ks(1-\psi)} \|u_k\|_2^{1-\psi} 2^{(k-m)\chi}
\end{aligned}$$

By Young's inequality, we get

$$\begin{aligned}
|L_{611}| &\leq \frac{\nu}{64} \sum_{m\geq -1} 2^{2ms+2m} \|\Delta_m u\|_2^2 + \frac{\kappa}{64} \sum_{m\geq -1} 2^{2ml+2m\alpha} \|\Delta_m \theta\|_2^2 \\
&\quad + C_{\nu,\kappa} \left(\sum_{m\geq -1} 2^{2ms} \|\Delta_m u\|_2^2\right)^{\frac{1-\psi}{2}} + C_{\nu,\kappa} \left(\sum_{m\geq -1} 2^{2ml} \|\Delta_m \theta\|_2^2\right)^{\frac{2-\chi}{2}}
\end{aligned}$$

Next, from the commutator, Bernstein's inequality, and Young's inequality, one has the following estimate of L_{612}

$$\begin{aligned}
|L_{612}| &\leq \sum_{m\geq -1} \sum_{|m-k|\leq 2} 2^{2ml} \|\nabla u_k\|_\infty \|u_{\leq k-2}\|_2 \|\Delta_m \theta\|_2 \\
&\lesssim \sum_{m\geq -1} 2^{2ml} \|\Delta_m \theta\|_2^2 \sum_{k\leq m} 2^{\frac{n}{2}k+k} \|u_{\leq k-2}\|_2 \\
&\lesssim \sum_{m\geq -1} 2^{(l+\alpha)m\chi} \|\Delta_m \theta\|_2^\chi 2^{ml(2-\alpha\chi)} \|\Delta_m \theta\|_2^{2-\chi} \\
&\quad \sum_{k\leq m} \sum_{m=-1}^{k-2} 2^{(s+1)k\psi} \|\Delta_m u\|_2^\psi 2^{ks(1-\psi)} \|\Delta_m u\|_2^{1-\psi} \left(2^{-m\chi} 2^{\frac{n}{2}k+k-ks-k\psi}\right)
\end{aligned}$$

By using the Young's inequality,

$$\begin{aligned}
|L_{612}| &\lesssim \sum_{m\geq -1} 2^{(l+\alpha)m\chi} \|\Delta_m \theta\|_2^\chi 2^{ml(2-\alpha\chi)} \|\Delta_m \theta\|_2^{2-\chi} \\
&\quad \sum_{k\leq m} \sum_{m=-1}^{k-2} 2^{(s+1)k\psi} \|\Delta_m u\|_2^\psi 2^{ks(1-\psi)} \|\Delta_m u\|_2^{1-\psi} 2^{(k-m)\chi} \\
&\leq \frac{\nu}{64} \sum_{m\geq -1} 2^{2ms+2m} \|\Delta_m u\|_2^2 + \frac{\kappa}{64} \sum_{m\geq -1} 2^{2ml+2m\alpha} \|\Delta_m \theta\|_2^2 \\
&\quad + C_{\nu,\kappa} \left(\sum_{m\geq -1} 2^{2ms} \|\Delta_m u\|_2^2\right)^{\frac{1-\psi}{2}} + C_{\nu,\kappa} \left(\sum_{m\geq -1} 2^{2ml} \|\Delta_m \theta\|_2^2\right)^{\frac{2-\chi}{2}}
\end{aligned}$$

Then, we combine the estimates of $|L_{611}|$ and $|L_{612}|$, one has the estimate of L_{61}

$$
\begin{aligned}
|L_{61}| \leq & \frac{\nu}{32} \sum_{m \geq -1} 2^{2ms+2m} \|\Delta_m u\|_2^2 + \frac{\kappa}{32} \sum_{m \geq -1} 2^{2ml+2m\alpha} \|\Delta_m \theta\|_2^2 \\
& + C_{\nu,\mu} \left(\sum_{m \geq -1} 2^{2ms} \|\Delta_m u\|_2^2 \right)^{\frac{1-\psi}{2}} + C_{\nu,\kappa} \left(\sum_{m \geq -1} 2^{2ml} \|\Delta_m \theta\|_2^2 \right)^{\frac{1-\chi}{2}}.
\end{aligned}
$$

The estimates of L_{62} and L_{63} are similar way with L_{611} and L_{612}. Especially for L_{63}, we use the notations

$$
\widetilde{\theta}_m = \sum_{|m-k| \leq 1} \theta_k.
$$

Finally, combining all the results, we can get the estimate of L_6 as follows

$$
\begin{aligned}
|L_6| \leq & \frac{\nu}{8} \sum_{m \geq -1} 2^{2ms+2m} \|\Delta_m u\|_2^2 + \frac{\kappa}{8} \sum_{m \geq -1} 2^{2ml+2m\alpha} \|\Delta_m \theta\|_2^2 \\
& + C_{\nu,\mu} \left(\sum_{m \geq -1} 2^{2ms} \|\Delta_m u\|_2^2 \right)^{\frac{1-\psi}{2}} + C_{\nu,\kappa} \left(\sum_{m \geq -1} 2^{2ml} \|\Delta_m \theta\|_2^2 \right)^{\frac{1-\chi}{2}}.
\end{aligned}
\tag{3.11}
$$

Proof of Theorem 3.1. Combining all the results of estimate and using the assumptions $s > \frac{n}{2} - 1$, $\frac{n}{2} < r \leq s + 1 - \epsilon$, $l \geq s + 2 - (\epsilon + \alpha) > \frac{n}{2}$, $l \leq s + \alpha - \epsilon$, Grönwall inequality, and identifying the Sobolev space H^s by Besov space $B^s_{2,2}$ that is

$$
\|u\|_{H^s}^2 \sim \sum_{m \geq -1} 2^{2ms} \|\Delta_m u\|_2^2
$$

for each $s \in \mathbb{R}$ and $u \in H^s$, then we have the following uniform estimates

$$
\|(u(t), b(t), \theta(t))\|_{\mathcal{X}_{s,r,l}} \leq C(\alpha, \nu, \mu, \kappa, \|(u_0, b_0, \theta_0)\|_{\mathcal{X}_{s,r,l}})
$$

for all $t \in [0, T]$ where $T = T(\nu, \mu, \kappa, \|(u_0, b_0, \theta_0)\|_{\mathcal{X}_{s,r,l}}) > 0$ and $\mathcal{X}_{s,r,l} = H^s \times H^{s+1-\epsilon} \times H^{s+\alpha-\epsilon}$ which proves Theorem 3.1. □

4. Uniqueness and continuity

We establish the uniqueness and continuity of solutions stated in Theorem 3.1 through a standard procedure.

Proposition 4.1. *Let $\epsilon > 0$ be small enough and $\alpha = 1$. Assume $(u_1, b_1, \theta_1, p_1)$ and $(u_2, b_2, \theta_2, p_2)$ are solutions of (1.1)–(1.2) in $C([0, T); \mathcal{X}_{s,s+1-\epsilon,s+1-\epsilon})$ satisfying the estimates in Theorem 3.1. Then $(u_1, b_1, \theta_1) = (u_2, b_2, \theta_2)$ for all $t \in [0, T)$.*

Proof. The difference $(U, B, \theta, \pi) = (u_1 - u_2, b_1 - b_2, \theta_1 - \theta_2, p_1 - p_2)$ satisfies

$$
\begin{aligned}
U_t + u_2 \cdot \nabla U - b_2 \cdot \nabla B + U \cdot \nabla u_1 - B \cdot \nabla b_1 + \nabla \pi &= \nu \Delta U + \theta e_n, \\
B_t + u_2 \cdot \nabla B - b_2 \cdot \nabla U + U \cdot \nabla b_1 - B \cdot \nabla u_1 &= \mu \Delta B, \\
\theta_t + u_2 \cdot \nabla \theta + U \cdot \nabla \theta_1 &= \kappa \Delta \theta.
\end{aligned}
\tag{4.1}
$$

Multiplying (4.1) by (U, B, θ) to get the inner product, then we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|U\|_2^2+\|B\|_2^2+\|\theta\|_2^2\right)+\nu\|\nabla U\|_2^2+\mu\|\nabla B\|_2^2+\kappa\|\nabla\theta\|_2^2\\
&=\int_{\mathbb{R}^n}(b_2\cdot\nabla)B\cdot U dx+\int_{\mathbb{R}^n}(B\cdot\nabla)b_1\cdot U dx-\int_{\mathbb{R}^n}(u_2\cdot\nabla)U\cdot U dx\\
&\quad-\int_{\mathbb{R}^n}(U\cdot\nabla)u_1\cdot U dx+\int_{\mathbb{R}^n}(b_2\cdot\nabla)U\cdot B dx+\int_{\mathbb{R}^n}(B\cdot\nabla)u_1\cdot B dx\\
&\quad-\int_{\mathbb{R}^n}(u_2\cdot\nabla)B\cdot B dx-\int_{\mathbb{R}^n}(U\cdot\nabla)b_1\cdot B dx-\int_{\mathbb{R}^n}(u_2\cdot\nabla)\theta\cdot\theta\, dx\\
&\quad-\int_{\mathbb{R}^n}(U\cdot\nabla)\theta_1\cdot\theta\, dx+\int_{\mathbb{R}^n}\theta e_n\cdot U dx=:\sum_{i=1}^{11}K_i.
\end{aligned}\tag{4.2}$$

Using integration by parts, the terms K_3, K_7, K_9 and $K_1 + K_5$ vanish. For the other terms, we have

$$|K_2|=\left|\int_{\mathbb{R}^n}(B\cdot\nabla)U\cdot b_1 dx\right|\le\nu\|\nabla U\|_2^2+C_\nu\|B\|_2^2\|b_1\|_{H^{s+1-\epsilon}}^2,$$

where we used the Sobolev embedding $H^{s+1-\epsilon}\subset L^\infty$ for $s+1-\epsilon>\frac{n}{2}$ which means that $\epsilon<s+1-\frac{n}{2}$ and $s>\frac{n}{2}-1$. Analogous computation shows that

$$\begin{aligned}
|K_{11}|\le&\|\theta\|_2^2+\|U\|_2^2,\\
|K_4|+|K_6|\le&\nu\|\nabla U\|_2^2+\mu\|\nabla B\|_2^2+C_{\mu,\nu}\|U,B\|_2^2\|u_1\|_{H^{s+1}}^2,\\
|K_8|+|K_{10}|\le&\mu\|\nabla B\|_2^2+\kappa\|\nabla\theta\|_2^2+C_{\mu,\kappa}\|U\|_2^2\|b_1,\theta_1\|_{H^{s+1-\epsilon}}^2.
\end{aligned}$$

Substituting all estimates of $K_1, K_2, \ldots, K_{11}$ into (4.2), and using Grönwall inequality for the result, then obtained

$$\|U\|_2^2+\|B\|_2^2+\|\theta\|_2^2\le C_{\nu,\mu,\kappa}(\|U(0)\|_2^2+\|B(0)\|_2^2+\|\theta(0)\|_2^2)$$

which indicates that $\|U(t)\|_2^2+\|B(t)\|_2^2+\|\theta(t)\|_2^2=0$ for all $t\in[0,T]$. Here we have used $U(0)=B(0)=\theta(0)=0$, $U=u_1-u_2\in L^2(0,T;H^{s+1})$ and $(B=b_1-b_2, \theta=\theta_1-\theta_2)\in\left(L^2(0,T;H^{s+1-\epsilon})\right)^2$. $\square$

We further establish the continuity by the following uniform estimates for (u, b, θ),

$$\|u\|_{H^s}^2+\|b\|_{H^r}^2+\|\theta\|_{H^l}^2\le C$$

where by identifying Sobolev space with Besov space, obtained $\|u\|_{H^s}^2\sim\sum_{m\ge-1}2^{2ms}\|\Delta_m u\|_2^2$, $\|b\|_{H^r}^2\sim\sum_{m\ge-1}2^{2mr}\|\Delta_m b\|_2^2$, and $\|\theta\|_{H^l}^2\sim\sum_{m\ge-1}2^{2ml}\|\Delta_q\theta\|_2^2$, for $m\in\mathbb{Z}$. Then we have $(u,b,\theta)\in C([0,T]; H^s(\mathbb{R}^n)\times H^r(\mathbb{R}^n)\times H^l(\mathbb{R}^n))$. Indeed, for any $M>0, R>0, S>0$, take N big enough such that

$$\sum_{m>N}2^{2ms}\|\Delta_m u\|_2^2\le\frac{M}{4},\quad\sum_{m>N}2^{2mr}\|\Delta_m b\|_2^2\le\frac{R}{4},\quad\sum_{m>N}2^{2ml}\|\Delta_m\theta\|_2^2\le\frac{S}{4}$$

For any $t\in(0,T^*)$ and ε such that $t+\varepsilon\in[0,T^*]$, we have

$$\begin{aligned}
\|u(t+\varepsilon)-u(t)\|_{H^s}^2&\le\sum_{m=-1}^{N}2^{2ms}\|\Delta_m u(t+\varepsilon)-\Delta_m u(t)\|_2^2+\frac{M}{2}\\
&\le\sum_{m=-1}^{N}2^{2ms}|\varepsilon|\,\|\partial_t u(t)\|_2^2+\frac{M}{2}\\
&\le 2N2^{2Ns}|\varepsilon|\,\|\partial_t u(t)\|_2^2+\frac{M}{2}\le M
\end{aligned}$$

for $|\varepsilon|$ small enough. Hence, $u(t)$ is continuous in $H^s(\mathbb{R}^n)$ at the time t. Similarly, the continuity of $b(t)$ and $\theta(t)$ at the time t is obtained

$$\|b(t+\varepsilon)-b(t)\|_{H^s}^2\le R,\quad\|\theta(t+\varepsilon)-\theta(t)\|_{H^s}^2\le S.\quad\square$$

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