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# Persistence, Extinction, and boundedness in pth moment of hybrid stochastic logistic systems by delay feedback control based on discrete-time observation

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**Abstract**

This paper focuses on the study of long-term behaviour of the stochastic logistic systems under Markov chain using delay feedback control based on discrete-time observations. Necessary conditions for extinction, persistence and boundedness in $p$th moment of the species in the time average is examined. Particularly, the upper bound of $(\tau + \tau_0)$ of the time lag between two successive observations $\tau$ and the delay time $\tau_0$, is obtained. The stochastic comparison theorem and asymptotic analysis are the main techniques applied. It has been observed that, the delay feedback control has an impact on the persistence of the species. Synthetic examples and virtual realities are given to support the findings.


*Keywords:* Persistence; Extinction; Hybrid systems; Logistic systems; Discrete-time feedback control

## 1. Introduction

Describing a single species of population growth precisely, the logistic model does a basic and crucial role, which also is the basis for population models involving multi-species such as Lotka–Volterra models, see, eg., [1,2] and references therein. The classical deterministic logistic model indicated by

$$dx(t) = x(t)(a - cx(t))dt, \tag{1.1}$$

where $t \geq 0$ and $x(0) = x_0 > 0$ has been researched broadly aiming to study its theoretical and practical significance, see [3,4] and other references included.

However, various population systems regularly suffer from external perturbations which lead to the changes of the behaviour of the population of species. Excessive harvesting, hunting or other forms of exploitation causes the extinction of species [5]. Due to the significance of sustainable source management, the feedback control of

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the Logistic system has been enticing a lot of attention in recent years. Authors in [1,6–8] and references therein, studied the deterministic logistic system with continuous time feedback control described by

$$\begin{cases} \dot{x}(t) = x(t)(a - cx(t) - eu(t)), \\ \dot{u}(t) = -f(t)u(t) + g(t)x(t), \end{cases} \tag{1.2}$$

where $x(t)$ represents the population density (size) at time $t$, $a$ represents the growth rate, $c$ denotes the coefficient for intra specific competition such that $a/c$ is the carrying capacity, $u(t)$ is the continuous time regulator or control, $a, c, e$ are all positive constants while $f(t)$ and $g(t)$ are bounded continuous positive functions called control parameters.

It is the fact tat, population systems are naturally unavoidably affected by environmental noises. Mao in [9] revealed that, different structures of the environmental noises such as white noise, colour noise also known as telegraph noise, may have different influences on the population systems. Telegraph noises involve swapping process of the species from one regime to another regime of environments, which are classified based on different circumstances such as nutrition or rainfall, see eg., [2,10–14]. The growth rates of some species may differ depending on the weather of a particular season or place. For instance, the grass in sufficiently rainy environment will grow more rapidly than in the dry. The swapping among different environments (regimes) of the species, is randomly and memoryless in which the waiting period for the next regime has an exponential distribution. The switching from one regime to another is often modelled by a Markov chain on a probability space picking values in a finite-state space on continuous-time.

Various researchers have considered the population systems affected by white noise, for examples, [2,11–15], where the intrinsic growth rate $a$ in (1.2) have been estimated by the sum of an error term and an average value. By using central limit theorem, it have been shown that, the error term follows a normal distribution, and the standard deviation of errors which are identified as the noise intensities, may depend on the population size. We hence replace $a$ by

$$a = a + \alpha x(t)\dot{B}(t),$$

where by $\dot{B}(t)$ is the white noise, $B(t)$ is the standard Brownian motion and $\alpha$ represents the white noise intensity in the specific regime.

H. Hu and L. Zhu [16,17] among others, extended the system (1.2) by including continuous time feedback control to study the survival and perish of species which are affected by the environmental noises. Their main concern was to investigate the effect of the continuous time feedback control to the survival (persistence) or perish (extinction) of species. Particularly, [16] studied single specie stochastic system with continuous feedback control under Markov chain of the form:

$$\begin{cases} dx(t) = x(t)\Big(a(r(t)) - c(r(t))x(t) - e(r(t))u(t)\Big)dt + \alpha(r(t))x(t)dB(t), \\ du(t) = \Big(-f(t)u(t) + g(t)x(t)\Big)dt, \end{cases} \tag{1.3}$$

where $r(t)$ is the Markov chain on the finite state space $\mathbb{S}$. For each $i \in \mathbb{S}$, $a(i)$ represent the intrinsic growth rate, $a(i)/c(i)$ is the measure of limit of the carrying capacity in regime $i$, $c(i)$, $e(i)$, $\alpha(i)$ are positive constants, and $f(t)$ and $g(t)$ are bounded, continuous positive functions on $[0, +\infty)$.

However, due to the fact that the population $x(t)$ require a continuous observation, very high frequent observation is needed whose cost is somehow expensive. For the sake of reducing cost in application, [18–20] pointed out that, the observation of the state should be done in discrete-time such as $0, \tau, 2\tau, \ldots$ where $\tau$ is the time interval between successive observations. Therefore, it is advisable to design a feedback control in the form of $u(x([t/\tau]\tau))$ such that the state is observed in discrete-time where $[t/\tau]$ is the integer part of $t/\tau$. Also, it is more realistic if we consider time delay $\tau_0(> 0)$ between the time when observation of the state is made and the time when the feedback control reaches the system [21]. We included the Markov chain in the system for the purpose of studying the behaviour of the specie in different regimes. So far no result yet has been developed concerning with the study of persistence and extinction of the hybrid logistic system (1.3) with delay feedback control which depend on discrete time observation of the state.

Being encouraged by the above discussion, our intension is to incorporate time delay in the designed feedback control which should be depending on discrete-time observations and explore sufficient conditions such that the system (1.4) should either persist or extinct. Hence, this work, studies hybrid stochastic logistic system (HSLS) also known as stochastic logistic system under regime switching by delay feedback control depending on discrete-time observation described by the system:

$$\begin{cases} dx(t) = x(t)\Big(a(r(t)) - c(r(t))x(t) - e(r(t))u(\delta_t - \tau_0)\Big)dt + \alpha(r(t))x(t)dB(t), \\ du(t) = \Big(-\beta(r(t))u(t) + \sigma(r(t))x(t)\Big)dt, \end{cases} \tag{1.4}$$

on $t \geq 0$ with an initial value

$$x(\theta) = x_0, \quad u(\theta) = u_0 \quad r(\theta) = r_0 \in \mathbb{S} \quad \text{for } -(\tau + \tau_0) \leq \theta \leq 0, \tag{1.5}$$

where $\delta_t = [t/\tau]\tau$, $\tau > 0$ represents the duration between the successive observations and $\tau_0$ is the delay time. $B(t)$ is a Brownian motion, Markov chain being represented by $r(t)$ where the detailed information about it, are explained in the next section and takes its values in the state space $\mathbb{S}$. For each $i \in \mathbb{S}$, we shall still use the definitions of $a(i)$, $c(i)$, $\alpha(i)$ and $e(i)$ as defined in the system (1.3), $\beta(i)$ and $\sigma(i)$ are all positive constants known as control parameters.

Before winding up this section, we would like to give distinguished key features between our work and that of the Refs. [16,17]. The distinguished features we made are highlighted as follows:

- The authors in [16,17] found that, the continuous time feedback control $u(t)$ have no effect to the permanence of the species. While in our case, the delayed feedback control based on discrete-time observation $u(\delta_t - \tau_0)$ has an impact on the persistence, and we obtained the bound of $(\tau + \tau_0)$ for the persistence of the system (1.4).
- The control parameter functions $f(t)$ and $g(t)$ of the second equation of the system 2 in [16] depend on time $(t)$ and are not flexible with the change of the state of the specie. But in our case, we have allowed flexibility to the control parameters $\beta(r(t))$ and $\sigma(r(t))$ at equation two of the system (1.4) to vary with the change of state.

Other parts of this work are organized in following aspects. Section 2 points the preliminary information including some notations and the proof of the existence of global positive solution; Section 3 tells about the sufficient condition of the extinction and the nonpersistent of the system (1.4) in the time average; Section 4 introduces favourable conditions for permanence of the system (1.4) in different ways such as persistence in the time average and weakly persistence in almost surely, and the upper bound of the system (1.4) in $pth$ moment; Section 5 gives examples, simulations and discussions about the results. Finally Section 6 makes a conclusion of this work.

## 2. Initial information

In this work, we shall use different notations as follows. For any given sequence of numbers $\{\nu(i)\}_{1\leq i\leq n}(n \in \mathbb{N})$, define $\check{\nu} = \max_{i\in\mathbb{S}}\{\nu(i)\}$ and $\hat{\nu} = \min_{i\in\mathbb{S}}\{\nu(i)\}$. Given any $p,\ q \in \mathbb{R}$, $p \vee q := \max\{p, q\}$, and $p \wedge q := \min\{p, q\}$. Define $\mathbb{R}_+ = (0, +\infty)$ and $\mathbb{R}^2_+ = (0, +\infty) \times (0, +\infty)$.

Define $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ as a complete probability space which satisfy the traditional conditions (that is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets) and $B(t)$ represents a Brownian motion defined on the probability space. Define $r(t)$, $t \geq 0$ as a right-continuous Markov chain on the probability space taking values in a finite-state space $\mathbb{S} = \{1, 2, \ldots, n\}(n < \infty)$ with the generator $\Gamma = (\gamma_{ij})_{n\times n}$ given by

$$\mathbb{P}\{r(t + \Delta) = j|r(t) = i\} = \begin{cases} \gamma_{ij} + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta \downarrow 0$, $o(\Delta)$ means $\lim_{\Delta\to 0} o(\Delta)/\Delta = 0$. Assume that $\Gamma$ is *conservative* (means that $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$, $\forall i \in \mathbb{S}$) and *irreducible* (i.e. the linear equations $\pi\Gamma = 0$ and $\sum_{i=1}^n \pi_i = 1$ have a unique solution $\pi = (\pi_1, \ldots, \pi_n) \in \mathbb{R}^{1\times n}$ (a row vector) satisfying $\pi_i > 0$ for each $i \in \mathbb{S}$. This solution is named as stationary distribution). We further assume that the Markov chain $r(t)$ does not depend on the Brownian motion $B(t)$. Here for simplicity, we also define $b(r(t)) = a(r(t)) - 0.5\alpha^2(r(t))$.

Since $x(t)$ represent the species' group size, and $u(t)$ stand as the watchdog, then $x(t)$ and $u(t)$ should not be negative. We now prove the system (1.4) has positive global solution (which means no explosion may occur when extended to finite time).

**Theorem 2.1.** *For any given initial value $x_0 > 0$, $u_0 \geq 0$ and $r_0 \in \mathbb{S}$, there is a unique continuous positive solution $(x(t), u(t)) \in \mathbb{R}^2_+$ to the system (1.4) on $t > 0$ a.s.*

**Proof.** Theorem 2.1 can be proved in similar way as the proof of Theorem 1 in [16], here we just give the highlights of the proof. For given initial data $(x_0, u_0) \in \mathbb{R}_+ \times [0, +\infty)$, there is a unique maximal local solution $(x(t), u(t)) \in (\mathbb{R}_+ \times \mathbb{R}_+)$ to the system (1.4) which may explode to infinity in finite time because its coefficients are locally Lipschitz continuous [22,23]. Referring to [13,16,17], we see that the nature of the solution of the first equation of the system (1.4) is given by

$$x(t) = \frac{\exp\left\{\int_0^t [b(r(s)) - e(r(s))u(\delta_s - \tau_0)]ds + \int_0^t \alpha(r(s))dB(s)\right\}}{1/x_0 + \int_0^t \left(c(r(s)) \exp\left\{\int_0^s [b(r(z)) - e(r(z))u(\delta_z - \tau_0)]dz + \int_0^s \alpha(r(z))dB(z)\right\}\right)ds} > 0, \tag{2.1}$$

$\forall\ t \in [0, \sigma_e)$, where $\sigma_e$ is the explosion time. Since $x(t) > 0$, we have

$$du(t) = \Big(-\beta(r(t))u(t) + \sigma(r(t))x(t)\Big)dt > -\beta(r(t))u(t)dt. \tag{2.2}$$

Now consider the comparison system having the initial value $m(0) = u(0)$,

$$dm(t) = -\beta(r(t))m(t)dt.$$

Solving this equation, we obtain

$$m(t) = \begin{cases} m(0)e^{-\int_0^t \beta(r(s))ds} > 0, & \text{if } m(0) > 0, \\ 0, & \text{if } m(0) = 0. \end{cases}$$

Using comparison theorem, we get

$$u(t) > m(t) \geq 0. \tag{2.3}$$

To get the existence of positive global solution, we have to show that $\sigma_e = \infty$ a.s. By (2.3), it is obvious that

$$dx(t) \leq x(t)\Big(a(r(t)) - c(r(t))x(t)\Big)dt + \alpha(r(t))x(t)dB(t). \tag{2.4}$$

Consider the comparison SDE with starting conditions $y(0) = x_0,\ r(0) = r_0$,

$$dy(t) = y(t)\Big(a(r(t)) - c(r(t))y(t)\Big)dt + \alpha(r(t))y(t)dB(t). \tag{2.5}$$

Using Theorem 2.1 in [13], we can reach to a conclusion that SDE (2.5) possess a unique positive global solution $y(t)$ for all $t \in [0, \infty)$. Applying comparison theorem of SDE (see e.g., [24]), we reach at $x(t) \leq y(t)$ a.s. for all $t \in [0, \infty)$ which implies that $\sigma_e = \infty$ a.s. Therefore, the first equation of system (1.4) has a unique positive global solution $x(t)$ a.s. On the other hand, solving equation two of the system (1.4), we obtain

$$u(t) = \int_0^t \left(\sigma(r(s))x(s)e^{-\int_s^t \beta(r(v))dv}\right)ds + u_0 e^{-\int_0^t \beta(r(s))ds}. \tag{2.6}$$

Since $\beta(i)$ and $\sigma(i)$ ($\forall\ i \in \mathbb{S}$) are all positive values, and $x(t)$ is unique positive global solution for the initial equation of system(1.4) a.s., we see that $u(t)$ is the unique positive global solution of the second equation of the system (1.4) for all $t > 0$ a.s. Thus conclusion of the Theorem holds. □

## 3. Extinction

This section, introduces the required conditions for the extinction of the system (1.4). By Theorem 2.1, we know $x(t)$ is a global positive solution for all $t \geq 0$. We therefore establish the extinction almost surely firstly.

**Theorem 3.1.**

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \leq \sum_{i=1}^{n} \pi_i b(i) \quad a.s. \tag{3.1}$$

*Moreover, if $\sum_{i=1}^{n}\pi_i b(i) < 0$, the system (1.4) will undergo extinction almost surely, that is*

$$\lim_{t\to\infty} x(t) = 0 \ a.s. \ \ and \ \ \lim_{t\to\infty} u(t) = 0 \ a.s.$$

*under initial data $x_0 > 0$, $u_0 \geq 0$ and $r_0 \in \mathbb{S}$.*

**Proof.** With respect to the first equation of the system (1.4), applying the Itô formula to $\ln x(t)$, we get

$$d(\ln x(t)) = \Big(b(r(t)) - c(r(t))x(t) - e(r(t))u(\delta_t - \tau_0)\Big)dt + \alpha(r(t))dB(t). \tag{3.2}$$

Using Integration techniques from 0 to $t$ on both sides yields

$$\ln x(t) = \ln x_0 + \int_0^t b(r(s))ds - \int_0^t c(r(s))x(s)ds - \int_0^t e(r(s))u(\delta_s - \tau_0)ds + \int_0^t \alpha(r(s))dB(s). \tag{3.3}$$

By Theorem2.1, we have

$$\ln x(t) \leq \ln x_0 + \int_0^t b(r(s))ds + M(t), \tag{3.4}$$

where $M(t) = \int_0^t \alpha(r(s))dB(s)$ is a local martingale. By the strong law of large numbers for local martingale and the strong ergodic theorem (see [22,23,25]), we obtain

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \ \ a.s. \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t b(r(s))ds = \sum_{i=1}^{n} \pi_i b(i) \ \ a.s. \tag{3.5}$$

let

$$\Omega_1 = \{\omega : \lim_{t\to\infty} \frac{M(t)}{t} = 0\}, \qquad \Omega_2 = \{\omega : \lim_{t\to\infty} \frac{1}{t}\int_0^t b(r(s))ds = \sum_{i=1}^{n} \pi_i b(i)\},$$

then we get

$$\mathbb{P}\{\omega : \omega \in \Omega_1 \cap \Omega_2\} = 1.$$

Dividing ((3.4)) by $t$ and applying limit to both sides yields

$$\begin{aligned}\limsup_{t\to\infty} \frac{\ln x(t)}{t} &\leq \limsup_{t\to\infty} \frac{\ln x_0}{t} + \limsup_{t\to\infty} \frac{1}{t}\int_0^t b(r(s))ds + \limsup_{t\to\infty} \frac{M(t)}{t}, \\ &= \sum_{i=1}^{n} \pi_i b(i), \qquad \forall \omega \in \Omega_1 \cap \Omega_2.\end{aligned} \tag{3.6}$$

Thus (3.1) holds. Now, let $\sum_{i=1}^{n} \pi_i b(i) < 0$, we arrive at

$$\lim_{t\to\infty} x(t) = 0 \ \ a.s. \tag{3.7}$$

This implies $x(t)$ will undergo extinction almost surely. We now turn to the second equation of the system (1.4) where from (2.6) we see that,

$$\begin{aligned}u(t) &= u_0 e^{-\int_0^t \beta(r(s))ds} + \int_0^t \sigma(r(s))x(s)e^{-\int_s^t \beta(r(v))dv}ds, \\ &\leq u_0 e^{-\hat{\beta}t} + \check{\sigma}\int_0^t x(s)e^{-\hat{\beta}(t-s)}ds.\end{aligned} \tag{3.8}$$

Taking the limit on both sides and using (3.7), we have:

$$\begin{aligned}\lim_{t\to\infty} u(t) &\leq \lim_{t\to\infty} u_0 e^{-\hat{\beta}t} + \lim_{t\to\infty} \check{\sigma}\int_0^t x(s)e^{-\hat{\beta}(t-s)}ds, \\ &= \lim_{t\to\infty} I_1(t),\end{aligned} \tag{3.9}$$

where

$$I_1(t) = \check{\sigma}\int_0^t x(s)e^{-\hat{\beta}(t-s)}ds.$$

We now turn to the analysis of $I_1(t)$. Since $\sum_{i=1}^n \pi_i b(i) < 0$, by (3.6), for $\forall\ \omega \in \Omega_1 \cap \Omega_2$, choose $\varepsilon = -\frac{1}{2}\sum_{i=1}^n \pi_i b(i) > 0$, $\exists\ T := T(\omega, \varepsilon) > 0$, $s.t.\ \forall\ t \geq\ T$,

$$x(t) \leq e^{\frac{1}{2}\sum_{i=1}^n \pi_i b(i)t}. \tag{3.10}$$

Then for $\forall\ \omega \in \Omega_1 \cap \Omega_2$ and $\forall\ t \geq\ T$, we have

$$\begin{aligned}
I_1(t) &= \check{\sigma}\int_0^t x(s)e^{-\hat{\beta}(t-s)}ds,\\
&= \check{\sigma}\int_0^T x(s)e^{-\hat{\beta}(t-s)}ds + \check{\sigma}\int_T^t x(s)e^{-\hat{\beta}(t-s)}ds,\\
&\leq \check{\sigma}(\max_{0\leq t\leq T} x(t))\int_0^T e^{-\hat{\beta}(t-s)}ds + \check{\sigma}\int_T^t e^{\frac{1}{2}\sum_{i=1}^n \pi_i b(i)s}e^{-\hat{\beta}(t-s)}ds,\\
&\leq \check{\sigma}(\max_{0\leq t\leq T} x(t))\int_0^T e^{-\hat{\beta}(t-s)}ds + \check{\sigma}\int_T^t e^{\frac{1}{2}\sum_{i=1}^n \pi_i b(i)s}ds,\\
&\leq \frac{\check{\sigma}(\max_{0\leq t\leq T} x(t))}{\hat{\beta}}e^{-\hat{\beta}(t-T)} + \frac{2\check{\sigma}}{\sum_{i=1}^n \pi_i b(i)}e^{\frac{1}{2}\sum_{i=1}^n \pi_i b(i)t}.
\end{aligned} \tag{3.11}$$

Noting that $\sum_{i=1}^n \pi_i b(i) < 0$, then we have

$$\begin{aligned}
&\lim_{t\to\infty}\frac{\check{\sigma}(\max_{0\leq t\leq T} x(t))}{\hat{\beta}}e^{-\hat{\beta}(t-T)} = 0,\\
&\lim_{t\to\infty}\frac{2\check{\sigma}}{\sum_{i=1}^n \pi_i b(i)}e^{\frac{1}{2}\sum_{i=1}^n \pi_i b(i)t} = 0.
\end{aligned} \tag{3.12}$$

Combining with (3.11) and (3.12), we have

$$\lim_{t\to\infty} I_1(t) = 0\ \ a.s.$$

It follows from (3.9) that

$$\lim_{t\to\infty} u(t) = 0\ \ a.s. \tag{3.13}$$

This is to say that $u(t)$ will go to extinction almost surely. □

Under different conditions, we can get the result of nonpersistent in the time average almost surely. Whence we state our new result;

**Theorem 3.2.** *If $\sum_{i=1}^n \pi_i b(i) = 0$, then the solutions of the system (1.4) will be nonpersistent in the time average almost surely, i.e.*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds = 0\ a.s. \quad \text{and} \quad \lim_{t\to\infty}\frac{1}{t}\int_0^t u(s)ds = 0\ a.s.$$

**Proof.** Noticing that $\sum_{i=1}^n \pi_i b(i) = 0$, then by the strong law of large numbers for local martingale and the strong ergodic theorem (see [22,23]), we have

$$\lim_{t\to\infty}\frac{M(t)}{t} = 0\ \ a.s. \qquad \lim_{t\to\infty}\frac{1}{t}\int_0^t b(r(s))ds = 0\ \ a.s. \tag{3.14}$$

For any $\varepsilon > 0$ and $\forall\ \omega \in \Omega_1$, there exist positive constants $T_1 = T_1(\varepsilon, \omega) > 0$ such that for $\forall\ t > T_1$

$$M(t) \leq \frac{\varepsilon}{2}t. \tag{3.15}$$

For $\forall\ \omega \in \Omega_2$, there exist positive constants $T_2 = T_2(\varepsilon, \omega) > 0$ such that for $\forall\ t > T_2$

$$\int_0^t b(r(s))ds \leq \frac{\varepsilon}{2}t. \tag{3.16}$$

Then $\forall\ \omega \in \Omega_1 \cap \Omega_2$ and $\forall\ t \geq t_1 := T_1 \vee T_2$, it follows from (3.3) that

$$\ln x(t) \leq \ln x_0 + \varepsilon t - \hat{c}\int_0^t x(s)ds. \tag{3.17}$$

Let $\phi(t) = \int_0^t x(s)ds$, we obtain

$$x(t) \leq x_0 e^{\varepsilon t} e^{-\hat{c}\phi(t)}. \tag{3.18}$$

A direct calculation arrives at

$$e^{\hat{c}\phi(t)} d\phi(t) \leq x_0 e^{\varepsilon t} dt.$$

Integrating both sides from $t_1$ to $t$ yields

$$e^{\hat{c}\phi(t)} \leq e^{\hat{c}\phi(t_1)} + \frac{\hat{c}x_0}{\varepsilon}(e^{\varepsilon t} - e^{\varepsilon t_1}). \tag{3.19}$$

Applying logarithm on both sides

$$\frac{1}{t}\phi(t) \leq \frac{1}{t\hat{c}}\ln\Big(e^{\hat{c}\phi(t_1)} + \frac{\hat{c}x_0}{\varepsilon}e^{\varepsilon t}\Big).$$

Taking the limit and applying L′Hôpital′s rule, for $\forall\ \omega \in \Omega_1 \cap \Omega_2$ we have

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds &\leq \limsup_{t\to\infty}\frac{1}{t\hat{c}}\ln\Big(e^{\hat{c}\phi(t_1)} + \frac{\hat{c}x_0}{\varepsilon}e^{\varepsilon t}\Big). \\ &\leq \frac{\varepsilon}{\hat{c}}.\end{aligned} \tag{3.20}$$

Since $\varepsilon$ is arbitrary and by Theorem 2.1, we get

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds = 0 \quad a.s. \tag{3.21}$$

Now, we turn to the second part of Theorem 3.2. It follows from the second equation of the system (1.4) that

$$u(t) \leq u_0 - \hat{\beta}\int_0^t u(s)ds + \check{\sigma}\int_0^t x(s)ds.$$

Multiplying by $\frac{1}{t}$ on both sides, we get

$$\frac{u(t)}{t} \leq \frac{u_0}{t} - \frac{\hat{\beta}}{t}\int_0^t u(s)ds + \frac{\check{\sigma}}{t}\int_0^t x(s)ds. \tag{3.22}$$

Rearranging (3.22), and noticing the fact that $u(t) > 0$ on $t > 0$, we have

$$\begin{aligned}\frac{1}{t}\int_0^t u(s)ds &\leq \frac{u_0}{\hat{\beta}t} - \frac{u(t)}{\hat{\beta}t} + \frac{\check{\sigma}}{\hat{\beta}t}\int_0^t x(s)ds, \\ &\leq \frac{u_0}{\hat{\beta}t} + \frac{\check{\sigma}}{\hat{\beta}t}\int_0^t x(s)ds.\end{aligned}$$

Applying the limit on both sides,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\int_0^t u(s)ds &\leq \frac{\check{\sigma}}{\hat{\beta}}\limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds + \frac{1}{\hat{\beta}}\limsup_{t\to\infty}\frac{u_0}{t}, \\ &= \frac{\check{\sigma}}{\hat{\beta}}\limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds.\end{aligned} \tag{3.23}$$

Using (3.21), we finally arrive at

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u(s)ds = 0 \quad a.s. \tag{3.24}$$

Hence, we conclude that the system ((1.4)) is nonpersistent in the time average almost surely. □

## 4. Persistence

Due to natural or artificial perturbations which affect the population, usually the population is either present (persistent) or absent (goes to extinction). In Section 3, we have discussed the extinction of the system (1.4) in different ways. Under this section, we shall introduce sufficient conditions for permanence of the system (1.4) in the time average and weakly persistence in almost surely. We shall further introduce the upper bound of the system (1.4) in $pth$ moment for any $p > 0$.

### 4.1. Persistence in the time average

Before stating our result on persistence, we first introduce the lower and upper bounds of the system (1.4) in the time average.

**Lemma 4.1.** *If $\sum_{i=1}^{n} \pi_i b(i) > 0$, choose $\tau > 0$ and $\tau_0 > 0$ sufficiently small such that*

$$\tau + \tau_0 < \frac{\hat{c}}{\check{\sigma}\check{e}}, \tag{4.1}$$

*then the solutions of the system (1.4) are ultimately upper bounded in the time average, that is*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \ \leq\ \check{\lambda}_1 \ a.s. \quad \text{and} \quad \limsup_{t\to\infty} \frac{1}{t}\int_0^t u(s)ds \ \leq\ \check{\lambda}_2 \ a.s. \tag{4.2}$$

*where $\check{\lambda}_1 = \frac{\sum_{i=1}^{n}\pi_i b(i)}{\hat{c}-(\tau+\tau_0)\check{e}\check{\sigma}}$ and $\check{\lambda}_2 = \frac{\check{\sigma}\sum_{i=1}^{n}\pi_i b(i)}{\hat{\beta}(\hat{c}-(\tau+\tau_0)\check{e}\check{\sigma})} = \check{\lambda}_2$.*

**Proof.** By using the strong law of large numbers for local martingale and the strong ergodic theorem (see [22,23,25]), we see that

$$\lim_{t\to\infty}\frac{M(t)}{t} = 0 \ \ a.s. \quad \text{and} \quad \lim_{t\to\infty}\int_0^t b(r(s))ds = \sum_{i=1}^{n}\pi_i b(i)t \ \ a.s. \tag{4.3}$$

Since $\sum_{i=1}^{n} \pi_i b(i) > 0$, let $\varepsilon = \frac{1}{3}\sum_{i=1}^{n}\pi_i b(i) > 0$, for $\forall\ \omega \in \Omega_1$, there exist positive constants $T_1 = T_1(\omega, \varepsilon) > 0$ such that $\forall\ t > T_1$,

$$M(t) \leq \frac{1}{3}\sum_{i=1}^{n}\pi_i b(i)t. \tag{4.4}$$

For $\forall\ \omega \in \Omega_2$, there exist positive constants $T_2 = T_2(\omega, \varepsilon) > 0$ such that $\forall\ t > T_2$,

$$\int_0^t b(r(s))ds \leq \frac{2}{3}\sum_{i=1}^{n}\pi_i b(i)t. \tag{4.5}$$

Then for $\forall\ \omega \in \Omega_1 \cap \Omega_2$ and $\forall\ t \geq t_1 := T_1 \vee T_2$, and note that

$$u(\delta_t - \tau_0) = u(t) - (u(t) - u(\delta_t - \tau_0)). \tag{4.6}$$

It follows from (3.3),

$$\begin{aligned}\ln x(t) \leq \ln x_0 &+ \sum_{i=1}^{n}\pi_i b(i)t - \int_0^t c(r(s))x(s)ds - \int_0^t e(r(s))u(s)ds \\ &+ \int_0^t e(r(s))u(t) - u(\delta_t - \tau_0)ds.\end{aligned} \tag{4.7}$$

From the second equation of (1.4), it is easy to realize that

$$u(t) - u(\delta_t - \tau_0) = -\int_{\delta_t-\tau_0}^{t}\beta(r(s))u(s)ds + \int_{\delta_t-\tau_0}^{t}\sigma(r(s))x(s)ds. \tag{4.8}$$

Substituting (4.8) into (4.6)

$$\begin{aligned}\ln x(t) &\leq \ln x_0 + \sum_{i=1}^{n} \pi_i b(i)t - \int_0^t c(r(s))x(s)ds - \int_0^t e(r(s))u(s)ds \\ &- \int_0^t e(r(s))(\int_{\delta_s-\tau_0}^{s} \beta(r(z))u(z)dz)ds + \int_0^t e(r(s))(\int_{\delta_s-\tau_0}^{s} \sigma(r(z))x(z)dz)ds, \\ &\leq \ln x_0 + \sum_{i=1}^{n} \pi_i b(i)t - \int_0^t c(r(s))x(s)ds + \int_0^t e(r(s))(\int_{\delta_s-\tau_0}^{s} \sigma(r(z))x(z)dz)ds. \end{aligned} \tag{4.9}$$

We then evaluate

$$\begin{aligned}\int_0^t e(r(s))(\int_{\delta_s-\tau_0}^{s} \sigma(r(z))x(z)dz)ds &\leq \int_0^t e(r(s))(\int_s -(\tau+\tau_0)^s \sigma(r(z))x(z)dz)ds, \\ &\leq \check{e}\check{\sigma} \int_{-(\tau+\tau_0)}^{t} x(z)\Big(\int_z^{z+(\tau+\tau_0)} ds\Big)dz, \\ &= \check{e}\check{\sigma}(\tau+\tau_0)^2 x_0 + (\tau+\tau_0)\check{e}\check{\sigma} \int_0^t x(z)dz. \end{aligned} \tag{4.10}$$

Substituting (4.10) into (4.9), we get

$$\ln x(t) \leq \ln x_0 + \check{e}\check{\sigma}(\tau+\tau_0)^2 x_0 + \sum_{i=1}^{n} \pi_i b(i)t - (\hat{c} - (\tau+\tau_0)\check{e}\check{\sigma}) \int_0^t x(s)ds. \tag{4.11}$$

By (4.1) we are sure that $(\hat{c} - (\tau+\tau_0)\check{e}\check{\sigma}) > 0$, let $\phi(t) = \int_0^t x(s)ds$, we directly obtain

$$x(t) \leq x_0 e^{\check{e}\check{\sigma}(\tau+\tau_0)^2 x_0} e^{\sum_{i=1}^{n} \pi_i b(i)t} e^{-(\hat{c}-(\tau+\tau_0)\check{e}\check{\sigma})\phi(t)}. \tag{4.12}$$

In similar way as we obtained (3.20) from (3.18), we integrate (4.12) from $t_1$ to $t$, taking the limits and applying L′Hôpital′s rule, we get

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t x(s)ds \leq \frac{\sum_{i=1}^{n} \pi_i b(i)}{\hat{c} - (\tau+\tau_0)\check{e}\check{\sigma}} = \check{\lambda}_1, \tag{4.13}$$

where $\check{\lambda}_1 > 0$. Hence, the first equation of the system ((1.4)) is upper bounded in the time average.

We further show that the second equation of the system ((1.4)) is also upper bounded in the time average. Substituting (4.13) into (3.23), we directly obtain

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t u(s)ds \leq \frac{\check{\sigma} \sum_{i=1}^{n} \pi_i b(i)}{\hat{\beta}(\hat{c} - (\tau+\tau_0)\check{e}\check{\sigma})} = \check{\lambda}_2 \quad a.s. \tag{4.14}$$

where $\check{\lambda}_2 > 0$. Hence, the system (1.4) is ultimately upper bounded in the time average. □

**Lemma 4.2.** *Let $\sum_{i=1}^{n} \pi_i b(i) > 0$ and $\hat{c}\hat{\beta} > \check{\sigma}\check{e}$. Choose $\tau > 0$ and $\tau_0 > 0$ sufficiently small such that*

$$\tau + \tau_0 < \frac{\hat{c}\hat{\beta} - \check{\sigma}\check{e}}{\check{\sigma}\check{e}\check{\beta}}. \tag{4.15}$$

*Then the solutions of the system (1.4) are ultimately lower bounded in the time average, i.e.*

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t x(s)ds \ \geq \ \hat{\lambda}_1 \ a.s. \quad \text{and} \quad \liminf_{t\to\infty} \frac{1}{t} \int_0^t u(s)ds \ \geq \ \hat{\lambda}_2 \ a.s. \tag{4.16}$$

*where* $\hat{\lambda}_1 := \left(\frac{\hat{\beta}\hat{c} - \check{\sigma}\check{e}(1+(\tau+\tau_0)\check{\beta})}{\hat{\beta}\hat{c}\check{c}}\right) \sum_{i=1}^{n} \pi_i b(i)$, $\hat{\lambda}_2 := \frac{\hat{\sigma}}{\check{\beta}} \hat{\lambda}_1$.

**Proof.** Applying Itô formula to $\ln x(t)$ with respect to the first equation of the system ((1.4)), we obtain

$$\begin{aligned}\ln x(t) &= \ln x_0 + \int_0^t b(r(s))ds - \int_0^t c(r(s))x(s)ds - \int_0^t e(r(s))u(\delta_s - \tau_0)ds + \int_0^t \alpha(r(s))dB(s).\\ &= \ln x_0 + \int_0^t b(r(s))ds - \int_0^t c(r(s))x(s)ds - \int_0^t e(r(s))u(s)ds\\ &\quad + \int_0^t e(r(s))(u(s) - u(\delta_s - \tau_0))ds + M(t).\end{aligned} \tag{4.17}$$

By Theorem 2.1, it follows from (4.8) that

$$u(t) - u(\delta_t - \tau_0) \geq -\int_{t-(\tau+\tau_0)}^t \beta(r(s))u(s)ds. \tag{4.18}$$

Then we have

$$\begin{aligned}\int_0^t e(r(s))(u(s) - u(\delta_s - \tau_0))ds &\geq -\int_0^t e(r(s))\Big(\int_{s-(\tau+\tau_0)}^s \beta(r(z))u(z)dz\Big)ds,\\ &\geq -\check{e}\check{\beta}\int_{-(\tau+\tau_0)}^t u(z)\Big(\int_z^{z+(\tau+\tau_0)} ds\Big)dz,\\ &= -(\tau+\tau_0)\check{e}\check{\beta}\int_{-(\tau+\tau_0)}^0 u(z)dz - (\tau+\tau_0)\check{e}\check{\beta}\int_0^t u(z)dz,\\ &= -\check{e}\check{\beta}(\tau+\tau_0)^2 u_0 - (\tau+\tau_0)\check{e}\check{\beta}\int_0^t u(z)dz.\end{aligned} \tag{4.19}$$

Substituting (4.19) into (4.17) yields

$$\begin{aligned}\ln x(t) &\geq \ln x_0 - \check{e}\check{\beta}(\tau+\tau_0)^2 u_0 + \int_0^t b(r(s))ds - \check{c}\int_0^t x(s)ds \ - \check{e}(1 + (\tau+\tau_0)\check{\beta})\int_0^t u(s)ds\\ &\quad + M(t).\end{aligned} \tag{4.20}$$

Similar to the analysis of (4.4), we see that for $\forall\, \varepsilon > 0$ and $\forall\, \omega \in \Omega_1 \cap \Omega_2$, there exists positive constant $t_2 = t_2(\omega, \varepsilon)$, such that for $\forall\, t \geq t_2$,

$$\begin{aligned}\int_0^t b(r(s))ds \ &\geq \ \Big(\sum_{i=1}^n \pi_i b(i) - \frac{\varepsilon}{3}\Big)t.\\ \int_0^t u(s)ds &\leq (\check{\lambda}_2 + \frac{\varepsilon}{3})t.\\ M(t) &\geq -\frac{\varepsilon}{3}t.\end{aligned} \tag{4.21}$$

Using (4.21), it follows from (4.20) that

$$\begin{aligned}\ln x(t) &\geq \ln x_0 - \check{e}\check{\beta}(\tau+\tau_0)^2 u_0 + \Big(\sum_{i=1}^n \pi_i b(i) - \frac{\varepsilon}{3}\Big)t - \check{e}(1 + (\tau+\tau_0)\check{\beta})(\check{\lambda}_2 + \frac{\varepsilon}{3})t\\ &\quad - \frac{\varepsilon}{3}t - \check{c}\int_0^t x(s)ds,\\ &= \ln x_0 - \check{e}\check{\beta}(\tau+\tau_0)^2 u_0 + \Big(1 - \frac{\check{\sigma}\check{e}(1 + (\tau+\tau_0)\check{\beta})}{\hat{\beta}\hat{c}}\Big)\sum_{i=1}^n \pi_i b(i)t\\ &\quad - (\frac{2}{3} + \frac{1}{3}\check{e}(1 + (\tau+\tau_0)\check{\beta}))\varepsilon t - \check{c}\int_0^t x(s)ds.\end{aligned}$$

Since $\varepsilon$ is arbitrary, choose $\varepsilon$ sufficiently large, then we have

$$\ln x(t) \geq \ln x_0 - \check{e}\check{\beta}(\tau+\tau_0)^2 u_0 + \Theta_2 t - \check{c}\int_0^t x(s)ds, \quad \forall\, \omega \in \Omega_1 \cap \Omega_2, \tag{4.22}$$

where $\Theta_2 = \left(1 - \frac{\check{\sigma}\check{e}(1+(\tau+\tau_0)\check{\beta})}{\hat{\beta}\hat{c}}\right)\sum_{i=1}^{n}\pi_i b(i)$. By (4.15), it is obvious that $\Theta_2 > 0$. Let $\phi(t) = \int_0^t x(s)ds$, it follows from ((4.22)) that for $\forall\ \omega \in \Omega_1 \cap \Omega_2$

$$x(t) \geq x_0 e^{-\check{e}\check{\beta}(\tau+\tau_0)^2 u_0 + \Theta_2 t} e^{-\check{c}\phi(t)}, \tag{4.23}$$

Rearranging (4.23), we get

$$e^{\check{c}\phi(t)} d\phi(t) \geq \mu e^{\Theta_2 t} dt.$$

where $\mu = x_0 e^{-\check{e}\check{\beta}(\tau+\tau_0)^2 u_0}$. In Similar way as (3.20) was obtained, apply L′Hôpital′s rule to get

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \geq \frac{1}{\check{c}} \liminf_{t\to\infty} \frac{1}{t} \ln\Big(e^{\check{c}\phi(t_2)} + \frac{\mu\check{c}}{\Theta_2}(e^{\Theta_2 t} - e^{\Theta_2 t_2})\Big) = \hat{\lambda}_1 \quad a.s. \tag{4.24}$$

On the other hand, considering the second equation of the system((1.4)),

$$u(t) = u_0 - \int_0^t \beta(r(s))u(s)ds + \int_0^t \sigma(r(s))x(s)ds. \tag{4.25}$$

By ((4.24)), we see that for $\forall\ \varepsilon > 0$ and $\forall\ \omega \in \Omega_1 \cap \Omega_2$, there exists a positive constant $t_3 = t_3(\omega, \varepsilon)$ such that for $\forall\ t \geq t_3$

$$\int_0^t x(s)ds \geq\ (\hat{\lambda}_1 - \varepsilon)t.$$

Substituting this into ((4.25)),

$$u(t) \geq u_0 + \hat{\sigma}(\hat{\lambda}_1 - \varepsilon)t - \check{\beta}\int_0^t u(s)ds.$$

Let $k(t) = \int_0^t u(s)ds$ and note $u_0 \geq 0$, it follows that

$$\frac{dk(t)}{dt} \geq \hat{\sigma}(\hat{\lambda}_1 - \varepsilon)t - \check{\beta}k(t). \tag{4.26}$$

By the comparison theorem, we have

$$k(t) \geq \hat{\sigma}(\hat{\lambda}_1 - \varepsilon)\int_{t_0}^t s e^{-(t-s)\check{\beta}} ds + e^{-\check{\beta}t}k(t_0). \tag{4.27}$$

We further compute

$$\int_{t_0}^t s e^{-(t-s)\check{\beta}} ds = \frac{t}{\check{\beta}} - \frac{t_0 e^{-\check{\beta}(t-t_0)}}{\check{\beta}} - \Big(\frac{1}{(\check{\beta})^2} - \frac{e^{-\check{\beta}(t-t_0)}}{(\check{\beta})^2}\Big).$$

Combining this with (4.27), to get

$$k(t) \geq \frac{\hat{\sigma}}{\check{\beta}}(\hat{\lambda}_1 - \varepsilon)\Big((t - \frac{1}{\check{\beta}}) - (t_0 - \frac{1}{\check{\beta}})e^{-\check{\beta}(t-t_0)}\Big) + e^{-\check{\beta}t}k(t_0). \tag{4.28}$$

For $\forall\ \omega \in \Omega_1 \cap \Omega_2$, dividing $t$ and taking the limits on both sides yield,

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t}\int_0^t u(s,\omega)ds &\geq \liminf_{t\to\infty} \frac{1}{t}\Big\{\frac{\hat{\sigma}}{\check{\beta}}(\hat{\lambda}_1 - \varepsilon)\Big((t - \frac{1}{\check{\beta}}) - (t_0 - \frac{1}{\check{\beta}})e^{-\check{\beta}(t-t_o)}\Big) + e^{-\check{\beta}t}k(t_0)\Big\},\\ &= \frac{\hat{\sigma}}{\check{\beta}}\Big(\hat{\lambda}_1 - \varepsilon\Big).\end{aligned}$$

Since $\varepsilon$ is arbitrary, we directly obtain

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t u(s,\omega)ds \geq \frac{\hat{\sigma}}{\check{\beta}}\hat{\lambda}_1 = \hat{\lambda}_2 \quad a.s. \tag{4.29}$$

Hence the system (1.4) is ultimately lower bounded in the time average almost surely. □

**Remark.** We notice that Li et al. [12,13] pointed out that, a stochastic equation is stochastically permanent if its solution is both stochastically ultimately upper bounded and lower bounded. Thus combining Lemmas 4.1 and 4.2, the system (1.4) will persist in the time average almost surely as far as it is bounded below and above in the time average. Thus, we state the following theorem.

**Theorem 4.1.** *If $\sum_{i=1}^{n} \pi_i b(i) > 0$ and $\hat{c}\hat{\beta} > \check{\sigma}\check{e}$, choose $\tau > 0$ and $\tau_0 > 0$ sufficiently small such that*

$$\tau + \tau_0 < \Big(\frac{\hat{c}}{\check{\sigma}\check{e}} \wedge \frac{\hat{c}\hat{\beta} - \check{\sigma}\check{e}}{\check{\sigma}\check{e}\check{\beta}}\Big), \tag{4.30}$$

*the system (1.4) will persist in the time average almost surely.*

Under Theorem 4.1, we can easily get the following corollary

**Corollary 4.1.** *Given the conditions of Theorem 4.1, the system (1.4) will be weakly persistence almost surely that is*

$$\limsup_{t\to\infty} x(t) > 0 \ a.s. \ \text{ and } \ \limsup_{t\to\infty} u(t) > 0 \ a.s. \tag{4.31}$$

*4.2. Boundedness in pth moment*

In this subsection we study sufficient conditions for the upper bound in $pth$ moment of the system (1.4) for any $p > 0$. We first cite the following useful lemma.

**Lemma 4.3** (*see [13]*). *If $c(i) > 0$ for any $i \in \mathbb{S}$, then for any given positive constant $p$ ($p > 0$), the solutions of the system (4.36) with any given positive initial value, $z(t)$ has the property that*

$$\limsup_{t\to\infty} \mathbb{E}(z^p(t)) \leq K(p), \tag{4.32}$$

*where*

$$K(p) := \begin{cases} (\frac{\check{a}}{\hat{c}})^p, & \text{for } 0 < p < 1; \\ (\frac{\check{a}+\frac{1}{2}(p-1)\check{\alpha}^2}{\hat{c}})^p, & \text{for } p \geq 1. \end{cases} \tag{4.33}$$

**Lemma 4.4.** *Given any arbitrary $p > 0$, the positive solutions of the system (1.4) satisfy*

$$\limsup_{t\to\infty} \mathbb{E}(x^p(t)) \leq \check{\Upsilon}_1, \quad \text{and} \quad \limsup_{t\to\infty} \mathbb{E}(u^p(t)) \ \leq \ \check{\Upsilon}_2, \tag{4.34}$$

*where $\check{\Upsilon}_1$ and $\check{\Upsilon}_2$ are all positive constants.*

**Proof.** By Theorem 2.1, it follows from the first equation of the system (1.4) that

$$dx(t) \leq x(t)\Big(a(r(t)) - c(r(t))x(t)\Big)dt + \alpha(r(t))x(t)dB(t). \tag{4.35}$$

Consider the auxiliary SDE with initial condition $z(0) = x_0, \quad r(0) = r_0$,

$$dz(t) = z(t)\Big(a(r(t)) - c(r(t))z(t)\Big)dt + \alpha(r(t))z(t)dB(t). \tag{4.36}$$

Applying the comparison theorem of SDE (see. e.g., [24]), we obtain

$$x(t) \leq z(t) \ a.s. \ \ \forall \ t \in [0, \infty).$$

This implies that for any given $p > 0$

$$x^p(t) \leq z^p(t) \ a.s. \ \ \forall \ t \in [0, \infty). \tag{4.37}$$

Then applying expectations and limits on both sides yields

$$\limsup_{t\to\infty} \mathbb{E}(x^p(t)) \leq \limsup_{t\to\infty} \mathbb{E}(z^p(t)), \tag{4.38}$$

By Lemma 4.3, we obtain

$$\limsup_{t\to\infty} \mathbb{E}(x^p(t)) \le K(p). \tag{4.39}$$

Recalling the remark 3.1 in Ref. [13], we see that for any given positive constant $p$ and continuity of $x(t)$, It follows from(4.39) that, there exist $T_1 := T_1(p) > 0$ and $\tilde{K}(p, x_0) > 0$, such that

$$\mathbb{E}(x^p(t)) \le 2K(p), \quad \forall t \ge T_1 \quad \text{and} \quad \mathbb{E}(x^p(t)) \le \tilde{K}(p, x_0), \qquad \text{for } t \in [0, T_1].$$

Thus

$$\mathbb{E}(x^p(t)) \le L_1(p, x_0) := \check{\Upsilon}_1, \qquad \forall t \in [0, \infty). \tag{4.40}$$

where $L_1(p, x_0) := \max\{2K(p), \tilde{K}(p, x_0)\}$. This verifies that the $pth$ moment of any positive solution of the first equation of the system (1.4) is bounded.

We now turn our discussion to the upper bound in $pth$ moment of the second equation of the system (1.4). For any $q > 1$ we evaluate

$$\begin{aligned} du^q(t) &= qu^{q-1}(t)du(t), \\ &= -q\beta(r(t))u^q(t)dt + q\sigma(r(t))u^{q-1}(t)x(t)dt. \end{aligned}$$

Integrating from 0 to $t$ and applying expectation on both sides

$$\mathbb{E}(u^q(t)) - u_0 = -q\int_0^t \beta(r(s))\mathbb{E}(u^q(s))ds + q\int_0^t \sigma(r(s))\mathbb{E}(u^{q-1}(s)x(s))ds. \tag{4.41}$$

Consider the term $\mathbb{E}(u^{q-1}(s)x(s))$ and by using (4.40), it can be observed that

$$\mathbb{E}(u^{q-1}(s)x(s)) \le (\mathbb{E}(u^q(s)))^{\frac{q-1}{q}}(\mathbb{E}(x^p(s)))^{\frac{1}{p}} \le (\mathbb{E}(u^q(s)))^{\frac{q-1}{q}}(\check{\Upsilon}_1)^{\frac{1}{p}}. \tag{4.42}$$

Let $v(t) = \mathbb{E}(u^q(t))$ and substituting (4.42) into (4.41) we get

$$v(t) - u_0 \le -\hat{\beta}q\int_0^t v(s)ds + \check{\sigma}q(\check{\Upsilon}_1)^{\frac{1}{p}}\int_0^t v^{\frac{q-1}{q}}(s)ds. \tag{4.43}$$

Differentiating (4.43), we arrive at

$$v'(t) \le -\hat{\beta}qv(t) + \check{\sigma}q(\check{\Upsilon}_1)^{\frac{1}{p}}v^{\frac{q-1}{q}}(t).$$

Dividing by $v^{\frac{q-1}{q}}(t)$ on both sides

$$v^{\frac{1}{q}-1}(t)v'(t) \le -\hat{\beta}qv^{\frac{1}{q}}(t) + \check{\sigma}q(\check{\Upsilon}_1)^{\frac{1}{p}}. \tag{4.44}$$

Let $h(t) = v^{\frac{1}{q}}(t)$ which implies that

$$\frac{dh(t)}{dt} \le \frac{1}{q}v^{\frac{1}{q}-1}(t)v'(t).$$

It follows from (4.44) that

$$\frac{dh(t)}{dt} \le -\hat{\beta}h(t) + \check{\sigma}(\check{\Upsilon}_1)^{\frac{1}{p}}, \tag{4.45}$$

with $h(0) = h_0 > 0$. By comparison method we solve (4.45) to get

$$h(t) \le h_0e^{-\hat{\beta}t} + \frac{\check{\sigma}(\check{\Upsilon}_1)^{\frac{1}{p}}}{\hat{\beta}}(1 - e^{-\hat{\beta}t}).$$

Recalling that $h(t) = v^{\frac{1}{q}}(t)$ and $v(t) = \mathbb{E}(u^q(t))$ and applying Holder's inequality see, e.g [23], we directly obtain

$$\mathbb{E}(u(t)) \le u_0e^{-\hat{\beta}t} + \frac{\check{\sigma}(\check{\Upsilon}_1)^{\frac{1}{p}}}{\hat{\beta}}(1 - e^{-\hat{\beta}t}).$$

Introducing limits and power $p$ on both sides, we directly arrive at

$$\limsup_{t\to\infty} \mathbb{E}(u^p(t)) \leq H(p), \tag{4.46}$$

where $H(p) = (\frac{\check{\sigma}}{\beta})^p \check{\Upsilon}_1$.

With the same procedures as we obtained (4.40), we also see that, for any given $p > 0$, there exist $T_2 := T_2(p) > 0$ and $\tilde{H}(p, x_0, u_0) > 0$, such that

$$\mathbb{E}(u^p(t)) \leq H(p), \quad \forall t \geq T_2 \quad \text{and} \quad \mathbb{E}(u^p(t)) \leq \tilde{H}(p, u_0), \qquad \text{for } t \in [0, T_2].$$

Thus

$$\mathbb{E}(u^p(t)) \leq L_2(p, u_0) := \check{\Upsilon}_2, \qquad \forall t \in [0, \infty), \tag{4.47}$$

where $L_2(p, u_0) := \max\{2H(p), \tilde{H}(p, u_0)\}$. Hence, by (4.40) and (4.47), we conclude that, any positive solutions of the system (1.4) is upper bounded in $pth$ moment $\square$.

## 5. Examples

In view of the findings in this work, it is a time to provide examples and verify the findings through simulations and illustrate the effect of Markov chain on the persistence and extinction of species.

**Example 5.1.** Consider the system ((1.4)) with Markov switching $r(t)$ taking values in a state space $S = \{1, 2\}$ with a generator $\Gamma = \begin{pmatrix} -5 & 5 \\ 1 & -1 \end{pmatrix}$ such that

$$\left.\begin{aligned} dx(t) &= x(t)\Big(a(1) - c(1)x(t) - e(1)u(\delta_t - \tau_0)\Big)dt + \alpha(1)x(t)dB(t), \\ du(t) &= \Big(-\beta(1)u(t) + \sigma(1)x(t)\Big)dt, \end{aligned}\right\} \quad \text{for } \ i = 1, \tag{5.1}$$

and

$$\left.\begin{aligned} dx(t) &= x(t)\Big(a(2) - c(2)x(t) - e(2)u(\delta_t - \tau_0)\Big)dt + \alpha(2)x(t)dB(t), \\ du(t) &= \Big(-\beta(2)u(t) + \sigma(2)x(t)\Big)dt, \end{aligned}\right\} \quad \text{for } \ i = 2. \tag{5.2}$$

Where by

$$\begin{aligned} &a(1) = 1.8,\ c(1) = 2.8,\ e(1) = 2.28,\ \alpha(1) = 1.7,\ \beta(1) = 4.8,\ \sigma(1) = 3, \\ &a(2) = 0.68,\ c(2) = 2.2,\ e(2) = 2.28,\ \alpha(2) = 1.8,\ \beta(2) = 3.2,\ \sigma(2) = 1.5. \end{aligned} \tag{5.3}$$

Initial values $x_0 = 1,\ u_0 = 1$.

By the definition of *irreducible Markov chain*, we compute

$$\pi = (\pi_1, \pi_2) = (\frac{1}{6}, \frac{5}{6}).$$

It can be seen that

$$b(1) = a(1) - \frac{1}{2}\alpha^2(1) = 0.355 \quad \text{and} \quad b(2) = a(2) - \frac{1}{2}\alpha^2(2) = -0.94.$$

We then compute

$$\sum_{i=1}^{n} \pi_i b(i) = \pi_1 b(1) + \pi_2 b(2) = \frac{1}{6} \times 0.355 + \frac{5}{6} \times -0.94 = -0.7242 < 0. \tag{5.4}$$

By the virtue of Theorem 3.1, the system ((1.4)) will go to extinction, see Fig. 5.1.

**Example 5.2.** Using the same coefficients as in Example 5.1 but using the generator $\Gamma = \begin{pmatrix} -1 & 1 \\ 5 & -5 \end{pmatrix}$, we obtain

$$\pi = (\pi_1, \pi_2) = (\frac{5}{6}, \frac{1}{6}).$$

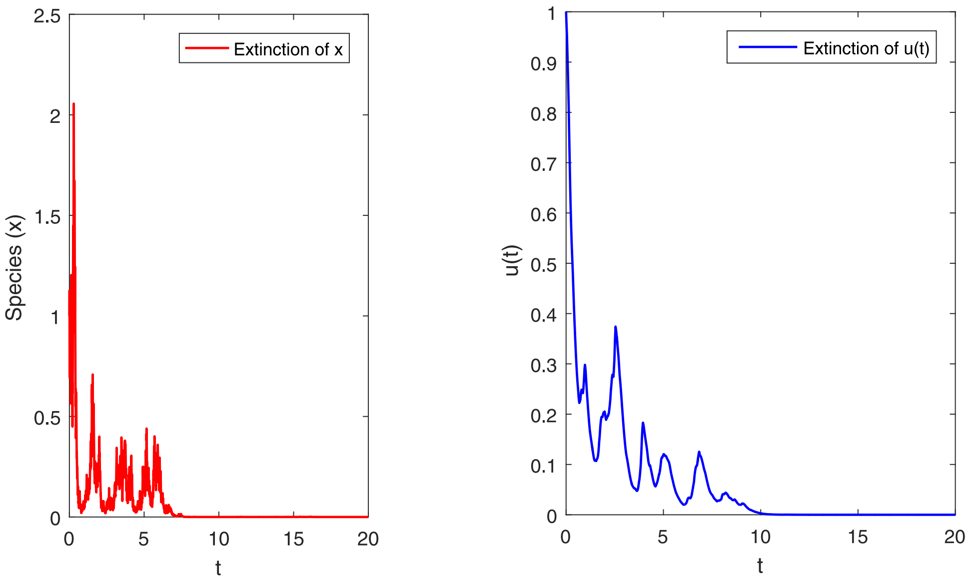


**Fig. 5.1.** Extinction of the species $x$ and the paths u(t) with $x_0 = 1$ and $u_0 = 1$.

In similar way as Example 5.1, we have $b(1) = 0.355$ and $b(2) = -0.94$. Then, we calculate

$$\sum_{i=1}^{n} \pi_i b(i) = \pi_1 b(1) + \pi_2 b(2) = \frac{5}{6} \times 0.355 + \frac{1}{6} \times -0.94 = 0.1392 > 0. \tag{5.5}$$

By Lemmas 4.1 and 4.2, compute $\frac{\hat{c}}{\breve{\sigma}\breve{e}} = 0.3216$, $\hat{c}\hat{\beta} = 7.04$, $\breve{\sigma}\breve{e} = 6.84$ and $\frac{\hat{c}\hat{\beta}-\breve{\sigma}\breve{e}}{\breve{\sigma}\breve{e}\breve{\beta}} = 0.0061$. Recalling (4.30), choosing $\tau = 0.0012$ and $\tau_0 = 0.0004$, we have

$$\check{\lambda}_1 = 0.0636 \quad \text{and} \quad \check{\lambda}_2 = 0.0596. \tag{5.6}$$

We further obtain

$$\Theta_2 = 0.00292, \ \hat{\lambda}_1 = 0.001041 \quad \text{and} \quad \hat{\lambda}_2 = 0.000325. \tag{5.7}$$

It can be observed that all conditions of Lemma 4.1 and Lemma 4.2 have been satisfied. Hence Theorem 4.1 has been verified. This is to say that, the system (1.4) persist in the time average almost surely. Furthermore, Corollary 4.1 is also satisfied, meaning that, the system (1.4) persist almost surely see Fig. 5.2.

## 6. Conclusion

This paper mainly studies the persistence and extinction of the HSLS (1.4) by delay feedback control which depend on discrete-time observation. We have observed that, the extinction or persistence of the HSLS (1.4) depend on $\sum_{i=1}^{n} \pi_i b(i)$. This indicates that, the stationary distribution $(\pi_1, \ldots, \pi_n)$ of the Markov chain $r(t)$ has an impact on the extinction or persistence of the HSLS (1.4). If $r(t)$ stays much longer in the bad state that is when $\sum_{i=1}^{n} \pi_i b(i) < 0$, then the HSLS (1.4) will extinct. If $r(t)$ spends much time in the good state let us say when $\sum_{i=1}^{n} \pi_i b(i) > 0$, then the system will survive.

On the other hand, we realized that persistence of the system (1.4) also depends on the choice of the time difference $\tau$ between the successive observations and the time delay $\tau_0$. Furthermore, we manage to obtain the bound of $(\tau + \tau_0)$. Contrast to the results in [17] where the continuous-time feedback control is harmless to the persistence or extinction of the species, our results show that the delay feedback control based on discrete-time observations has an influence on the persistence of the species.

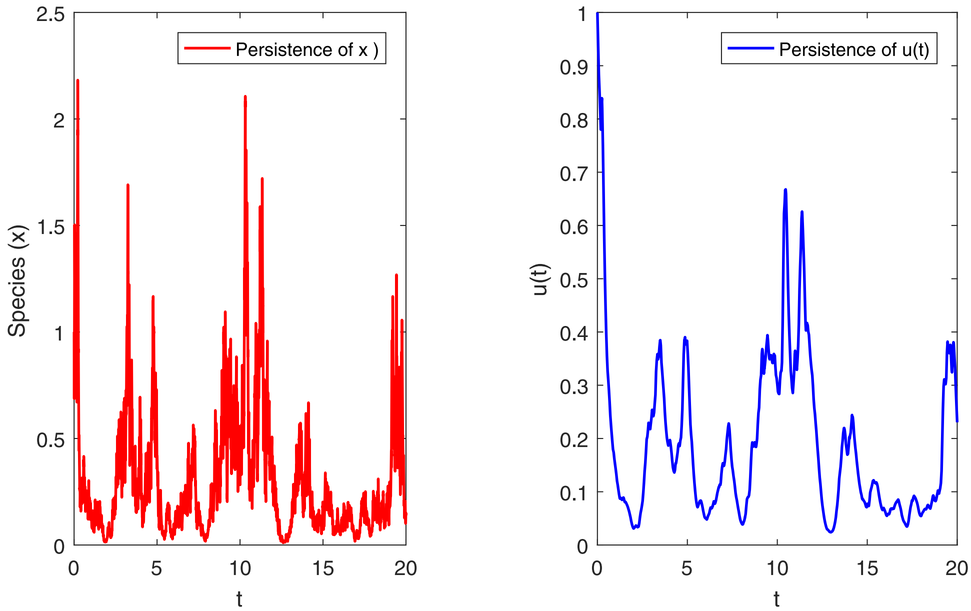


**Fig. 5.2.** Persistence of the species $x$ and the paths u(t) with $x_0 = 1$ and $u_0 = 1$.

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