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Stability of a fractional Diphtheria model under the natural immunity rate

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Abstract: In this paper, we are interested in a fractional diphtheria model where the exposed individuals have the natural immunity rate. This paper is the continuation of study in [14], by considering a fractional derivative in time for $0 < \alpha \leq 1$. The boundedness, non-negative, existence, and uniqueness of the proposed model are studied. Moreover, the existence of all equilibrium points, local, and global stability are also studied, where the local stability is proven by the Magniton's theorem and global stability is proven by applying the Lyapunov function. Finally, the predictor-corrector scheme is applied to establish the numerical simulations.

Keywords: fractional-order differential equation; stability analysis; Magniton's theorem; Lyapunov function; predictor-corrector scheme; good health and well-being.
AMS (2020) Subject Classification: 35A01, 35B40.

1 Introduction

The proposed model we study in this paper is adapted from the following former model studied in [14] under the case of integer-order derivative ($\alpha = 1$)

$$\begin{aligned}
\frac{ds}{dt} &= (1-p)\mu N - \eta\frac{si}{N} - \delta s + \phi e,\\
\frac{de}{dt} &= \eta\frac{si}{N} - (\beta + \phi + \delta)e,\\
\frac{di}{dt} &= \beta e - (\gamma + \delta + \theta)i,\\
\frac{dq}{dt} &= \gamma i - (\varepsilon + \delta)q,\\
\frac{dr}{dt} &= p\mu N + \varepsilon q - \delta r,
\end{aligned} \tag{1.1}$$

where the variables N, μ, δ, η, β, γ, ε, and ϕ represent the total population, the natural increase rate, the natural mortality rate, the probability of interaction between susceptible and infected, infection rate, treatment rate (rate of quarantined individuals for each time unit), the recovery rate, and the exposed individuals with natural immunity respectively.

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Moreover, the non-dimensional model of the system (1.1) is obtained by substituting the following non-dimensional variables

$$s = \frac{S}{N}, e = \frac{E}{N}, i = \frac{I}{N}, q = \frac{Q}{N}, r = \frac{R}{N},$$

where $s, e, i, q,$ and r denote the proportion of number of individuals for each subpopulation. Then (1.1) becomes

$$\begin{aligned}
\frac{dS}{dt} &= (1-p)\mu - \eta SI - \delta S + \phi E,\\
\frac{dE}{dt} &= \eta SI - (\beta + \phi + \delta)E,\\
\frac{dI}{dt} &= \beta E - (\gamma + \delta + \theta)I,\\
\frac{dQ}{dt} &= \gamma I - (\varepsilon + \delta)Q,\\
\frac{dR}{dt} &= p\mu + \varepsilon Q - \delta R,
\end{aligned} \tag{1.2}$$

Noting that the growth rates in the system (1.2) only capture on the current state. In fact, the growth rates of population also capture the memory effects and long-rate interactions which exist in the diphtheria model but are neglected in the case of integer-order derivative. The fractional derivatives are then applied to get fractional differentiation or to include such memory effects and long-rate interactions in most biological system [21]. We further notice that the fractional differential equations (FDEs) are a generalized differential equations by employing the application of fractional operator having the noninteger order [2,9]. In this paper, we consider the Caputo fractional derivative which is given by the following definition.

Definition 1. *(See [23]). Let $\alpha > 0$. Then the Caputo fractional derivative of order α of function $f \in C^n$ is given by*

$$D_*^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}}\, ds,$$

where $n = \lceil \alpha \rceil$. In particular, if $0 < \alpha \leq 1$, one has

$$D_*^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{'}(s)}{(t-s)^{\alpha}}\, ds.$$

Since the Caputo fractional derivative of order α is the nonlocal operator (it includes the initial and current state), then the Caputo fractional derivative of order α is employed in the dynamical system model to present the influence of the memory effects and long-rate interactions (see [10,11,18,19]). Moreover, the growth rates in the system (1.2) give the current condition and also the initial states to get the memory effects and long-rate interactions. The first derivatives of system (1.2) are then replaced by the Caputo fractional derivative of order α as shown as

follows

$$\begin{aligned} D_*^\alpha S(t) &= (1-p)\mu - \eta SI - \delta S + \phi E, \\ D_*^\alpha E(t) &= \eta SI - (\beta + \phi + \delta)E, \\ D_*^\alpha I(t) &= \beta E - (\gamma + \delta + \theta)I, \\ D_*^\alpha Q(t) &= \gamma I - (\varepsilon + \delta)Q, \\ D_*^\alpha R(t) &= p\mu + \varepsilon Q - \delta R. \end{aligned} \tag{1.3}$$

where $S(0) = S_0, E(0) = E_0, I(0) = I_0, Q(0) = Q_0, R(0) = R_0$, and $0 < \alpha < 1$. Moreover, we consider $0 \leq p \leq 1$ for the proportion of vaccinated individuals in a population. We notice that the nonlinear fractional system (1.3) can be very complicated to establish the analytical solution. In the case of such nonlinear dynamical systems, the Lie algebra can be employed to approach the exact solutions as studied in [25–27] and the fractional dynamical systems can be found in the study [3, 8]. The Lie algebra constructs the symmetric properties of the system to find the solution. Unfortunately, this Lie algebra can not widely be used because the coupled differential equations (1.3) are complicated and less symmetry. In this paper, we are not concerned with the analytical solution of fractional systems (1.3). The stability of systems (1.3) have been studied in [13,14] for the case of integer-order derivative in time ($\alpha = 1$) with or without optimal control.

We further organize this paper as follows. In Section 2, we give some basic defintions and the neccessary results which are useful to prove our main results (boundedness, non-negativity, existence, and uniqueness, local, and global stability). In Section 3, the proof of boundedness, non-negativity, and existence and uniqueness can be found. Moreover, the local stability is obtained through the theory of Routh-Hurwitz fractional-order criteria (for $0 < \alpha < 1$) for both disease-free and endemic equilibrium points. The local and global stability are proved based on basic definitions and necessary results in the previous section. Meanwhile, the equilibrium points for both disease-free and endemic condition are established based on Section 2 and the next generation matrix is applied to determine basic reproduction number. At the end of Section 3, we give the numerical scheme and results of system (1.3) through the predictor-corrector scheme.

2 Preliminaries

In this section, we present some lemmas, theorems, and corollary which are useful to prove the main results of our paper. We first give the comparison theorem which is useful to establish the boundedness.

Theorem 1. *(See [16]) Let $v(t) \in C([0, +\infty))$, $\alpha \in (0, 1]$, Φ, $\psi \in \mathbb{R}$, and $\Phi \neq 0$. If $v(t)$ satisfies*

$$D_*^\alpha v(t) \leq -\Phi v(t) + \psi, \; v(0) = v_0 \in \mathbb{R},$$

then

$$v(t) \leq \left(v_0 - \frac{\psi}{\Phi}\right) \mathcal{M}_\alpha[-\Phi t^\alpha] + \frac{\psi}{\Phi},$$

where $\mathcal{M}_\alpha(k)$ is said to be the Mittag-Leffler function of one parameter and is defined by

$$\mathcal{M}_\alpha(k) = \sum_{j=1}^{\infty} \frac{k^j}{\Gamma(\alpha j + 1)}.$$

In particular, if $\alpha = 1$ then $\mathcal{M}_\alpha(k)$ becomes the exponential function which has an important role in the classical calculus

$$e^k = \mathcal{M}_1(k) = \sum_{j=1}^{\infty} \frac{k^j}{\Gamma(j+1)}.$$

In [24], Saleh and Kilicman studied the correlations between the Mittag-Leffler and Wright functions. The further lemma and corollary are presented to show the non-negativity of the solution in the system (1.3).

Lemma 1. *(See [20]). Let $(v, D_*^\alpha)(t) \in C[0,\zeta]$, and $0 < \alpha \leq 1$. Then, one has*

$$v(t) = v(0) + \frac{1}{\Gamma(\alpha)} D_*^\alpha v(\xi) t^\alpha, \tag{2.1}$$

where $\xi \in [0, y]$, for all $y \in [0, \zeta)$.

Corollary 1. *(See [20]) $(v, D_*^\alpha)(t) \in C[0,\zeta]$, and $0 < \alpha \leq 1$. The function $v(t)$ is increased for all $t \in (0,\zeta)$ if $D_*^\alpha v(t) \geq 0$ and is decreased for all $t \in (0,\zeta)$ if $D_*^\alpha v(t) \leq 0$.*

Meanwhile, to prove the system (1.3) exists and unique then the following lemma is suitable.

Lemma 2. *(See [15]). Given a function $f : \mathcal{A} \to \mathbb{R}^n$, where $\mathcal{A} \in ((0,\infty) \times \Omega), \Omega \subseteq \mathbb{R}^n$ and a fractional system of order α*

$$D_*^\alpha v(t) = f(t, v(t)), \tag{2.2}$$

for $t > 0, v(0) \geq 0, \alpha \in (0, 1]$. Then the uniqueness in the system (1.3) can be established when the function $f(t, v(t))$ with respect to v satisfies the Lipschitz condition.

Theorem 2. *(See [17, 22]). Given a fractional system of order α*

$$D_*^\alpha \vec{v} = \vec{f}(\vec{v}), \vec{v}(0) = \vec{v}_0, 0 < \alpha \leq 1.$$

If the condition $\vec{f}(\vec{v}^*) = 0$ can be satisfied then a point $\vec{v}^*$ is known as equilibrium point. If the condition $|arg(\lambda_j)| > \alpha\pi/2$ can be satisfied by all eigenvalues λ_j obtained from the Jacobian matrix $J = \partial f/\partial v$ at the point $\vec{v}^*$ then this equilibrium point is asymptotic stable.

Lemma 3. *(See [28]). We consider $v(t) \in C(\mathbb{R}_+)$ and for any $0 < \alpha \leq 1$ the fractional system of order α exist. Then, one has*

$$D_*^\alpha \left[v(t) - v^* - v^* \ln \frac{v(t)}{v^*} \right] \leq \left(1 - \frac{v^*}{v(t)} \right) D_*^\alpha v(t), v^* \in \mathbb{R}_+, \text{ for any } t > 0.$$

Lemma 4. *(See [12]). Let Ω be a bounded and closed set and every solution of*

$$D_*^\alpha v(t) = f(v(t)),$$

start from and remain in Ω for all time. If there exists a continuous function $V(v) : \Omega \to \mathbb{R}$ satisfying

$$D_*^\alpha V(t)|_{D_*^\alpha v(t) = f(v(t))} \leq 0,$$

and $\mathcal{K}$ be the largest invariant set of $E := \{v | D_*^\alpha V(t)|_{D_*^\alpha v(t) = f(v(t))} = 0\}$. Then, for each solution $v(t)$ starting from Ω approaches to $\mathcal{K}$ as $t \to \infty$.

Noting that the Lyapunov function in Lemma 4 can be stated as the continuous function $V(v)$. It is necessary to construct a suitable Lyapunov function first in order to satisfy Lemma 4. Moreover, the negativity of the Lyapunov function is usually shown through Lemma 3.

3 Main results

In this section, we present the boundedness, non-negativity, existence, uniqueness, local stability, global stabilty, and equilibrium points, and basic reproduction number. The boundedness, non-negativity, and existence and uniqueness can be proved respectively by Theorem 1 (Comparison Theorem), Lemma 1, and Lemma 2. Moreover, the local stability is obtained through the theory of Routh-Hurwitz in fractional case (for $0 < \alpha \leq 1$) for both disease-free and endemic equilibrium points. The global stability is then proved by Lemma 4, Theorem 1 for disease-free equilibrium point, and Lemma 3 for endemic equilibrium point. Meanwhile, the equilibrium points for both disease-free and endemic condition are established by Theorem 2 and the next generation matrix is applied to determine basic reproduction number.

3.1 Boundedness

Let $w(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$. Then, one has

$$\begin{aligned} D_*^\alpha w(t) &= D_*^\alpha S(t) + D_*^\alpha E(t) + D_*^\alpha I(t) + D_*^\alpha Q(t) + D_*^\alpha R(t) \\ &= \mu - (S + E + I + Q + R)\delta - \theta I \\ &= \mu - w(t)\delta - \theta I(t) \\ &\leq \mu - w(t)\delta, \end{aligned}$$

which implies $D_*^\alpha w(t) + w(t)\delta \leq \mu$.

By the standard comparison theorem for fractional derivative of order α in Theorem 1, one has

$$w(t) \leq \left(w(0) - \frac{\mu}{\delta}\right) \mathcal{M}_\alpha[-\delta t^\alpha] + \frac{\mu}{\delta},$$

where $\mathcal{M}_\alpha$ is the Mittag-Leffler function. Since

$$\mathcal{M}_\alpha(-\delta t^\alpha) \to 0 \text{ as } t \to \infty,$$

then by Lemma 5 and Corollary 6 in [4], one has

$$w(t) \leq \frac{\mu}{\delta} \text{ as } t \to \infty$$

which is convergent to $\frac{\mu}{\delta}$ as $t \to \infty$. Then the system is bounded in the following region

$$\Phi = \left\{(S, E, I, Q, R) \in \mathbb{R}_+^5 : V(t) \leq \frac{\mu}{\delta} + \varepsilon, \varepsilon > 0\right\}.$$

3.2 Non-Negativity

By referring to (1.3) and the proof of boundedness, for $\varepsilon > 0$ small enough, we can prove the non-negative of S, E, I, Q, R.

1. Non-negativity of $(1.3)_1$.

$$\begin{aligned} D_*^\alpha S(t) &= (1-p)\mu + \phi E - (\eta I + \delta)S \\ &\geq -(\eta I + \delta)S \geq -\left(\frac{\eta\mu}{\delta} + \delta\right) S. \end{aligned}$$

Then, one gets

$$D_*^{\alpha} S(t) + \left(\frac{\eta\mu}{\delta} + \delta\right) S \geq 0.$$

By the standard comparison theorem for fractional derivative of order α in Theorem 1, one has

$$S(t) \geq [S(0) - 0]\, \mathcal{M}_{\alpha}\left[-\left(\frac{\eta\mu}{\delta} + \delta\right) t^{\alpha}\right] + 0 = [S(0) - 0]\, \mathcal{M}_{\alpha}\left[-\left(\frac{\eta\mu}{\delta} + \delta\right) t^{\alpha}\right].$$

Since $\mathcal{M}_{\alpha}\left[-\left(\frac{\eta\mu}{\delta} + \delta\right) t^{\alpha}\right] \approx 0$ as $(t \to \infty)$, then we can conclude that $S(t) \geq 0$.

2. Non-negativity of $(1.3)_2$

$$\begin{aligned} D_*^{\alpha} E(t) &= \eta SI - (\beta + \phi + \delta)E \\ &\geq -(\beta + \phi + \delta)E. \end{aligned}$$

Then, one gets

$$D_*^{\alpha} E(t) + (\beta + \phi + \delta)E \geq 0.$$

By the similar way, using the standard comparison theorem for fractional derivative of order α, one has $E(t) \geq 0$.

3. Non-negativity of $(1.3)_3$, $(1.3)_4$, and $(1.3)_5$ is similar to the previous ones, the one gets $I(t) \geq 0, Q(t) \geq 0, R(t) \geq 0$.

3.3 Existence and Uniqueness

Let we define new function $F_1(Y, \bar{Y}), F_2(Y, \bar{Y}), F_3(Y, \bar{Y}), F_4(Y, \bar{Y})$, and $F_5(Y, \bar{Y})$ as follows

$$\begin{aligned}
F_1(Y) &:= D_*^{\alpha} S(t) = (1-p)\mu - \eta SI - \delta S + \phi E, \\
F_1(\bar{Y}) &:= D_*^{\alpha} \bar{S}(t) = (1-p)\mu - \eta \bar{S}\bar{I} - \delta \bar{S} + \phi \bar{E}, \\
F_2(Y) &:= D_*^{\alpha} E(t) = \eta SI - (\beta + \phi + \delta)E, \\
F_2(\bar{Y}) &:= D_*^{\alpha} \bar{E}(t) = \eta \bar{S}\bar{I} - (\beta + \phi + \delta)\bar{E}, \\
F_3(Y) &:= D_*^{\alpha} I(t) = \beta E - (\gamma + \delta + \theta)I, \\
F_3(\bar{Y}) &:= D_*^{\alpha} \bar{I}(t) = \beta \bar{E} - (\gamma + \delta + \theta)\bar{I}, \\
F_4(Y) &:= D_*^{\alpha} Q(t) = \gamma I - (\varepsilon + \delta)Q, \\
F_4(\bar{Y}) &:= D_*^{\alpha} \bar{Q}(t) = \gamma \bar{I} - (\varepsilon + \delta)\bar{Q}, \\
F_5(Y) &:= D_*^{\alpha} R(t) = p\mu + \varepsilon Q - \delta R, \\
F_5(\bar{Y}) &:= D_*^{\alpha} \bar{R}(t) = p\mu + \varepsilon \bar{Q} - \delta \bar{R}, \\
&\text{where} \\
Y &= (S, E, I, Q, R), \quad \bar{Y} = (\bar{S}, \bar{E}, \bar{I}, \bar{Q}, \bar{R}).
\end{aligned}$$

Then, we can find the norm between $F(Y)$ and $F(\bar{Y})$,

$$\begin{aligned}
\left\|F(Y)-F(\bar{Y})\right\| &= |F_1(Y)-F_1(\bar{Y})|+|F_2(Y)-F_2(\bar{Y})|+|F_3(Y)-F_3(\bar{Y})|+|F_4(Y)-F_4(\bar{Y})| \\
&\quad +|F_5(Y)-F_5(\bar{Y})| \\
&= \left|-\eta SI-\delta S+\phi E+\eta\bar{S}\bar{I}+\delta\bar{S}-\phi\bar{E}\right| \\
&\quad +\left|\beta E-(\gamma+\delta+\theta)I-\beta\bar{E}+(\gamma+\delta+\theta)\bar{I}\right|+\left|\gamma I-(\varepsilon+\delta)Q-\gamma\bar{I}+(\varepsilon+\delta)\bar{Q}\right| \\
&\quad +\left|\varepsilon Q-\delta R-\varepsilon\bar{Q}+\delta\bar{R}\right|++\left|\eta SI-(\beta+\phi+\delta)E-\eta\bar{S}\bar{I}+(\beta+\phi+\delta)\bar{E}\right| \\
&= \left|-\eta(SI-\bar{S}\bar{I})-\delta(S-\bar{S})+\phi(E-\bar{E})\right|+\left|\eta(SI-\bar{S}\bar{I})-(\beta+\phi+\delta)(E-\bar{E})\right| \\
&\quad +\left|\beta(E-\bar{E})-(\gamma+\delta+\theta)(I-\bar{I})\right|+\left|\gamma(I-\bar{I})-(\varepsilon+\delta)(Q-\bar{Q})\right| \\
&\quad +\left|\varepsilon(Q-\bar{Q})-\delta(R-\bar{R})\right| \\
&\leq \left(\eta|SI-\bar{S}\bar{I}|+\delta|S-\bar{S}|+\phi|E-\bar{E}|\right)+\left(\eta|SI-\bar{S}\bar{I}|+(\beta+\phi+\delta)|E-\bar{E}|\right) \\
&\quad +\left(\beta|E-\bar{E}|+(\gamma+\delta+\theta)|I-\bar{I}|\right)+\left(\gamma|I-\bar{I}|+(\varepsilon+\delta)|Q-\bar{Q}|\right) \\
&\quad +\left(\varepsilon|Q-\bar{Q}|+\delta|R-\bar{R}|\right).
\end{aligned}$$

Noting that $max\left\{|S,\bar{S}|,|E,\bar{E}|,|I,\bar{I}|,|Q,\bar{Q}|,|R,\bar{R}|\right\}\leq\mathcal{M}$, where $\mathcal{M}$ is sufficiently large, then one has

$$\begin{aligned}
\left\|F(Y)-F(\bar{Y})\right\| &\leq (2\eta\mathcal{M}+\delta)|S-\bar{S}|+(2\beta+2\phi+\delta)|E-\bar{E}|+(2\eta\mathcal{M}+2\gamma+\delta+\theta)|I-\bar{I}| \\
&\quad +(2\varepsilon+\delta)|Q-\bar{Q}|+\delta|R-\bar{R}| \\
&= \mathcal{L}_1|S-\bar{S}|+\mathcal{L}_2|E-\bar{E}|+\mathcal{L}_3|I-\bar{I}|+\mathcal{L}_4|Q-\bar{Q}|+\mathcal{L}_5|R-\bar{R}| \\
&\leq \mathcal{L}\left\|Y-\bar{Y}\right\|,
\end{aligned}$$

where $\mathcal{L}_1=2\eta\mathcal{M}+\delta$, $\mathcal{L}_2=2(\beta+\phi)+\delta$, $\mathcal{L}_3=2(\eta\mathcal{M}+\gamma)+\delta+\theta$, $\mathcal{L}_4=2\varepsilon+\delta$, $\mathcal{L}_5=\delta$ and $max\left\{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4,\mathcal{L}_5\right\}\leq\mathcal{L}$. Therefore, the Lipschitz condition is achieved. By applying the Lemma 2, the fractional derivatives of order α in the system (1.3) with the initial data $Y_0=(S_0,E_0,I_0,Q_0,R_0)\geq 0$ has an unique solution $Y(t)=(S(t),E(t),I(t),Q(t),R(t))\in\Phi$. Thus, the existence and uniqueness are proved.

3.4 Basic reproduction number and Equilibrium points

The transmission of a disease is indicated by the value of $\mathcal{R}_0$ which is said to be basic reproduction number. The basic reproduction number $\mathcal{R}_0$ is obtained from the derivative of effected classes $(1.3)_2$ and $(1.3)_3$. By linearizing around the disease-free equilibrium $(\mathcal{E}_0)$, one has

$$\begin{aligned}
\begin{pmatrix}E\\I\end{pmatrix}_t &= \begin{pmatrix}\eta SI-(\beta+\phi+\delta)E\\ \beta E-(\gamma+\delta+\theta)I\end{pmatrix} \\
&= \begin{pmatrix}\eta SI\\0\end{pmatrix}-\begin{pmatrix}(\beta+\phi+\delta)E\\ -\beta E+(\gamma+\delta+\theta)I\end{pmatrix} \\
&= \begin{pmatrix}0&\eta S\\0&0\end{pmatrix}\begin{pmatrix}E\\I\end{pmatrix}-\begin{pmatrix}\beta+\phi+\delta&0\\-\beta&\gamma+\delta+\theta\end{pmatrix}\begin{pmatrix}E\\I\end{pmatrix}=(\mathcal{F}-\mathcal{V})\begin{pmatrix}E\\I\end{pmatrix},
\end{aligned}$$

where $\mathcal{F}$ is the transmission matrix of new infected individuals, and $\mathcal{V}$ is the transition matrix of individual displacements between groups of individuals. Moreover, the disease-free equilibrium point is obtained based on the Theorem 2 that $\mathcal{E}_0=(S^0,E^0,I^0,Q^0,R^0)=((1-$

$p)\mu/\delta, 0, 0, 0, p\mu/\delta)$. Therefore, the basic reproduction number $\mathcal{R}_0$ can be established from the following dominant eigen values of $\mathcal{F}\mathcal{V}^{-1}$

$$|\lambda I - \mathcal{F}\mathcal{V}^{-1}| = \begin{vmatrix} \lambda - \frac{\eta\beta(1-p)\mu}{\delta(\beta+\phi+\delta)(\gamma+\delta+\theta)} & -\frac{\eta(1-p)\mu}{\delta(\gamma+\delta+\theta)} \\ 0 & \lambda \end{vmatrix}$$

which gives $\lambda_1 = 0$ and $\lambda_2 = \frac{\eta\beta(1-p)\mu}{\delta(\beta+\phi+\delta)(\gamma+\delta+\theta)}$. Hence, the basic reproduction number of system (1.3) is $\mathcal{R}_0 = \frac{\eta\beta(1-p)\mu}{\delta(\beta+\phi+\delta)(\gamma+\delta+\theta)}$. By the similar ways, using Theorem 2, we can get the equilibrium point in endemic state $\mathcal{E}_1 = (S^*, E^*, I^*, Q^*, R^*)$ where

$$\begin{aligned}
S^* &= \frac{(\gamma+\delta+\theta)(\beta+\phi+\delta)}{\eta\beta}, \\
E^* &= \frac{(\gamma+\delta+\theta)(\beta+\phi+\delta)}{\eta\beta(\beta+\delta)}(\mathcal{R}_0 - 1), \\
I^* &= \frac{\delta(\beta+\phi+\delta)}{\eta(\beta+\delta)}(\mathcal{R}_0 - 1), \\
Q^* &= \frac{\gamma(\beta+\phi+\delta)}{\eta(\varepsilon+\delta)(\beta+\delta)}(\mathcal{R}_0 - 1), \\
R^* &= \frac{p\mu}{\delta} + \frac{\varepsilon\gamma(\beta+\phi+\delta)}{\eta\delta(\varepsilon+\delta)(\beta+\delta)}(\mathcal{R}_0 - 1).
\end{aligned}$$

Then the endemic equilibrium point $\mathcal{E}_1$ exists if $\mathcal{R}_0 > 1$.

3.5 Local stability

The theory of Routh-Hurwitz in fractional case is employed to establish the stability condition for the equilibrium point.

Theorem 3. *(See [1]). Based on the characteristic equation as follows*

$$\mathcal{L}(\lambda) = \mathcal{C}_0\lambda^n + \mathcal{C}_1\lambda^{n-1} + \mathcal{C}_2\lambda^{n-2} + ... + \mathcal{C}_n = 0.$$

Then equilibrium points of fractional system (1.3) *are local stable if one of the conditions given below are satisfied:*

1. $|arg(\lambda_n)| > \alpha\pi/2$ *for* $n = 1$ *and* $\mathcal{C}_1 > 0$.
2. $|arg(\lambda_n)| > \alpha\pi/2$ *for* $n = 2$:
   a). $\mathcal{C}_1 > 0$, *and* $\mathcal{C}_2 > 0$ *if* $\Delta(\mathcal{L}) > 0$ *or,*
   b). $\mathcal{C}_1 < 0, 4\mathcal{C}_2 > \mathcal{C}_1^2$, *and* $\left|tan^{-1}\left(\frac{\sqrt{4\mathcal{C}_2 - \mathcal{C}_1^2}}{\mathcal{C}_1}\right)\right| > \alpha\pi/2$ *if* $\Delta(\mathcal{L}) < 0$.
3. $|arg(\lambda_{1,2,3})| > \alpha\pi/2$ *for* $n = 3$:
   a). $\mathcal{C}_1 > 0, \mathcal{C}_3 > 0$, *and* $\mathcal{C}_1\mathcal{C}_2 > \mathcal{C}_3$ *if* $\Delta(\mathcal{L}) > 0$ *or,*
   b). $\mathcal{C}_1 \geq 0, \mathcal{C}_2 \geq 0, \mathcal{C}_3 > 0$, *and* $\alpha < 2/3$ *if* $\Delta(\mathcal{L}) < 0$ *or,*
   c). $\mathcal{C}_1 > 0, \mathcal{C}_2 > 0$, *and* $\mathcal{C}_1\mathcal{C}_2 = \mathcal{C}_3$ *for each* $0 < \alpha < 1$ *if* $\Delta(\mathcal{L}) < 0$,

with

$$\Delta(\mathcal{L}) = -\begin{vmatrix} 1 & \mathcal{C}_1 & \mathcal{C}_2 & \mathcal{C}_3 & 0 \\ 0 & 1 & \mathcal{C}_1 & \mathcal{C}_2 & \mathcal{C}_3 \\ 3 & 2\mathcal{C}_1 & \mathcal{C}_2 & 0 & 0 \\ 0 & 3 & 2\mathcal{C}_1 & \mathcal{C}_2 & 0 \\ 0 & 0 & 3 & 2\mathcal{C}_1 & \mathcal{C}_2 \end{vmatrix} = 18\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 + (\mathcal{C}_1\mathcal{C}_2)^2 - 4\mathcal{C}_3\mathcal{C}_1^3 - 4\mathcal{C}_2^3 - 27\mathcal{C}_3^3.$$

1. Local stability of $\mathcal{E}_0$
   For the equilibrium point in disease-free condition, the Jacobian matrix is given below

$$J(\mathcal{E}_0) = \begin{pmatrix} -\delta & \phi & -\frac{\eta(1-p)\mu}{\delta} & 0 & 0 \\ 0 & -(\beta+\phi+\delta) & \frac{\eta(1-p)\mu}{\delta} & 0 & 0 \\ 0 & \beta & -(\gamma+\delta+\theta) & 0 & 0 \\ 0 & 0 & \gamma & -(\varepsilon+\delta) & 0 \\ 0 & 0 & 0 & \varepsilon & -\delta \end{pmatrix}.$$

   The eigen values of $J(\mathcal{E}_0)$ are obtained by the solution of characteristic equation $|J(\mathcal{E}_0) - \lambda I| = 0$. We further find the minor values of matrix $J(\mathcal{E}_0)$ to get the cofactor expansion, then eigen values satisfy the following equation

$$(-\delta-\lambda)(-(\varepsilon+\delta)-\lambda)(-\delta-\lambda)det(A) = 0,$$

   where

$$A = \begin{pmatrix} -(\beta+\phi+\delta) & \frac{\eta(1-p)\mu}{\delta} \\ \beta & -(\gamma+\delta+\theta) \end{pmatrix}.$$

   Clearly, $J(\mathcal{E}_0)$ has three eigen values, they are $\lambda_{1,2} = -\delta < 0, \lambda_3 = -(\varepsilon+\delta) < 0$ and consequently, one has $|arg(\lambda_{1,2,3})| = \pi > \alpha\pi/2$. Therefore, the eigen values $\lambda_{4,5}$ determine the stability of equilibrium point $\mathcal{E}_0$ which can be found from the following characteristic equation

$$\lambda^2 + a_1\lambda + a_2 = 0,$$

   where

$$a_1 = \beta + \phi + \theta + \gamma + 2\delta,$$
$$a_2 = (\beta+\phi+\delta)(\gamma+\delta+\theta) - \frac{\eta\beta(1-p)\mu}{\delta}.$$

   Then, the roots can be stated as follows

$$\lambda_{4,5} = \frac{-a_1 \pm \sqrt{\Delta}}{2},$$

   where

$$\Delta = a_1^2 - 4a_2 > 0$$

Hence, the local stability at $\mathcal{E}_0$ is satisfied if $a_1 > 0$ and $a_2 > 0$. Therefore, the parameter $a_2 > 0$ can be stated as follows

$$(\beta + \phi + \delta)(\gamma + \delta + \theta) - \frac{\eta\beta(1-p)\mu}{\delta} > 0,$$

or

$$\frac{\eta\beta(1-p)\mu}{\delta(\beta + \phi + \delta)(\gamma + \delta + \theta)} < 1.$$

Then, the local stability at $\mathcal{E}_0$ is achieved only if $\mathcal{R}_0 < 1$.

2. Local stability of $\mathcal{E}_1$
Similarly, for the equilibrium point in endemic condition, the Jacobian matrix is given below

$$J(\mathcal{E}_1) = \begin{pmatrix} -(\eta I^* + \delta) & \phi & -\eta S^* & 0 & 0 \\ \eta I^* & -(\beta + \phi + \delta) & \eta S^* & 0 & 0 \\ 0 & \beta & -(\gamma + \delta + \theta) & 0 & 0 \\ 0 & 0 & \gamma & -(\varepsilon + \delta) & 0 \\ 0 & 0 & 0 & \varepsilon & -\delta \end{pmatrix}.$$

The eigen values of $J(\mathcal{E}_1)$ are obtained by the solution of characteristic equation $|J(\mathcal{E}_1) - \lambda I| = 0$. By the similar ways, we determine the minor values of matrix $J(\mathcal{E}_1)$ to get the cofactor expansion, then eigen values satisfy the following equation

$$(-\delta - \lambda)(-(\varepsilon + \delta) - \lambda)det(B) = 0,$$

where

$$B = \begin{pmatrix} -(\eta I^* + \delta) & \phi & -\eta S^* \\ \eta I^* & -(\beta + \phi + \delta) & \eta S^* \\ 0 & \beta & -(\gamma + \delta + \theta). \end{pmatrix}.$$

Obviously, $J(\mathcal{E}_1)$ has two eigen values, they are $\lambda_1 = -\delta < 0, \lambda_2 = -(\varepsilon + \delta) < 0$ and consequently, one has $|arg(\lambda_{1,2})| = \pi > \alpha\pi/2$. Therefore, the eigen values $\lambda_{3,4,5}$ determine the stability of equilibrium point $\mathcal{E}_1$ which can be found from the following characteristic equation

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0,$$

where

$$\begin{aligned} b_1 &= \eta I^* + 2\delta + \beta + \phi + \gamma + \theta, \\ b_2 &= (\eta I^* + \delta)(\beta + \delta) + (\eta I^* + 2\delta + \beta + \phi)(\gamma + \delta + \theta) - \eta\beta S^*, \\ b_3 &= (\eta I^* + \delta)(\beta + \delta)(\gamma + \delta + \theta) - \eta\beta S^*\delta. \end{aligned}$$

By referring to the Theorem 3, the eigen values of the characteristic equation satisfy $|arg(\lambda_{1,2,3})| > \alpha\pi/2$:
a). $b_1 > 0, b_3 > 0$, and $b_1 b_2 > b_3$ if $\Delta(\mathcal{L}) > 0$ or,

b). $b_1 \geq 0, b_2 \geq 0, b_3 > 0$, and $\alpha < 2/3$ if $\Delta(\mathcal{L}) < 0$ or,
c). $b_1 > 0, b_2 > 0$, and $b_1 b_2 = b_3$ for each $0 < \alpha < 1$ if $\Delta(\mathcal{L}) < 0$,
with
$\Delta(\mathcal{L}) = 18 b_1 b_2 b_3 + (b_1 b_2)^2 - 4 b_3 b_1^2 - 4 b_2^2 - 27 b_3^2$.

It is obvious that the equilibrium $\mathcal{E}_1$ is local stable if $b_1 > 0$, $b_2 > 0$, and $b_3 > 0$. Therefore, the condition $b_3 > 0$ is equivalent to

$$(\eta I^* + \delta)(\beta + \delta)(\gamma + \delta + \theta) - \eta\beta S^* \delta > 0,$$
$$\eta I^*(\beta + \delta)(\gamma + \delta + \theta) + \delta(\beta + \delta)(\gamma + \delta + \theta) - \eta\beta S^* \delta > 0.$$

By substituting the values of S^* and I^*, one can get

$$\delta(\beta + \phi + \delta)(\gamma + \delta + \theta)(\mathcal{R}_0 - 1) > 0.$$

Hence, $b_3 > 0$ is satisfied only if $\mathcal{R}_0 > 1$.

3.6 Global stability

In this section, we prove the global stability for disease-free and endemic equilibrium points. The proof of global stability is based on the Lemma 4 (Generalized Lasalle Invariance Principle) for both disease-free and endemic equilibrium points, Theorem 1 for disease-free equilibrium point and Lemma 3 for endemic equilibrium point.

1. Global stability of disease-free equilibrium point $\mathcal{E}_0$
   Consider the Lyapunov function

$$V(t) = mE + I,$$

where

$$m = \frac{\beta}{\beta + \phi + \delta}.$$

Then, the fractional derivative can be obtained as follows

$$\begin{aligned} D_*^\alpha V(t) &= m D_*^\alpha E(t) + D_*^\alpha I(t) \\ &= m(\eta S I - (\beta + \phi + \delta)E) + \beta E - (\gamma + \delta + \theta) I \\ &= (m\eta S - (\gamma + \delta + \theta)) I + \beta E - m(\beta + \phi + \delta) E \\ &= \left(\frac{\eta\beta S}{\beta + \phi + \delta} - (\gamma + \delta + \theta) \right) I \end{aligned} \tag{3.1}$$

It follows from $(1.3)_1$, one has

$$D_*^\alpha S(t) = (1 - p)\mu - \eta S I - \delta S + \phi E.$$

In the steady-state of (1.3), we can get $\phi E = \eta S I - (\beta + \delta) E$, which gives

$$\begin{aligned} D_*^\alpha S(t) &= (1 - p)\mu - (\beta + \delta) E - \delta S \\ &\leq (1 - p)\mu - \delta S. \end{aligned}$$

Based on the Theorem 1, then

$$S(t) \leq \left(S(0) - \frac{(1-p)\mu}{\delta}\right) \mathcal{M}_\alpha[-\delta t^\alpha] + \frac{(1-p)\mu}{\delta},$$

where $\mathcal{M}_\alpha$ is the Mittag-Leffler function. Since

$$\mathcal{M}_\alpha[-\delta t^\alpha] \to 0 \text{ as } t \to \infty,$$

then by Lemma 5 and Corollary 6 in [4], one gets

$$S(t) \leq \frac{(1-p)\mu}{\delta} \text{ as } t \to \infty, \tag{3.2}$$

Substituting (3.2) to (3.1), we have

$$\begin{aligned} D_*^\alpha V(t) &\leq \left(\frac{\eta\beta(1-p)\mu}{\delta(\beta+\phi+\delta)} - (\gamma+\delta+\theta)\right) I \\ &= \left(\frac{\eta\beta(1-p)\mu}{\delta(\beta+\phi+\delta)(\gamma+\delta+\theta)} - 1\right)(\gamma+\delta+\theta)I \\ &= (\mathcal{R}_0 - 1)(\gamma+\delta+\theta)I. \end{aligned}$$

Then one has $D_*^\alpha V(t) < 0$ if $\mathcal{R}_0 < 1$ for all positive parameters. Moreover, one has $D_*^\alpha V(t) = 0$ if $\mathcal{R}_0 = 1$. Therefore, if $\mathcal{R}_0 < 1$ then the global asymptotic $\mathcal{E}_0$ is satisfied.

2. Global stability of endemic equilibrium point $\mathcal{E}_1$
Given a Lyapunov function as follows

$$V(t) = V_1(S(t)) + V_2(E(t)) + V_3(I(t)) + V_4(Q(t)) + V_5(R(t)),$$

where

$$\begin{aligned} &V_1(S(t)) = S - S^* - S^* \ln\left(\frac{S}{S^*}\right), \quad V_2(E(t)) = E - E^* - E^* \ln\left(\frac{E}{E^*}\right), \\ &V_3(I(t)) = I - I^* - I^* \ln\left(\frac{I}{I^*}\right), \quad V_4(Q(t)) = Q - Q^* - Q^* \ln\left(\frac{Q}{Q^*}\right), \\ &V_5(R(t)) = R - R^* - R^* \ln\left(\frac{R}{R^*}\right), \end{aligned}$$

and $V(t)$ is continuous, and $S(t) > 0, E(t) > 0, I(t) > 0, Q(t) > 0, R(t) > 0$. By applying the Lemma 3, we get

$$\begin{aligned} D_*^\alpha V &\leq \left(1 - \frac{S^*}{S}\right) D_*^\alpha S + \left(1 - \frac{E^*}{E}\right) D_*^\alpha E + \left(1 - \frac{I^*}{I}\right) D_*^\alpha I + \left(1 - \frac{Q^*}{Q}\right) D_*^\alpha Q \\ &\quad + \left(1 - \frac{R^*}{R}\right) D_*^\alpha R \\ &= A_1 + A_2 + A_3 + A_4 + A_5, \end{aligned} \tag{3.3}$$

where

$$A_1 = \left(1 - \frac{S^*}{S}\right)((1-p)\mu - \eta SI - \delta S + \phi E), \quad A_2 = \left(1 - \frac{E^*}{E}\right)(\eta SI - \varphi_3 E),$$
$$A_3 = \left(1 - \frac{I^*}{I}\right)(\beta E - \varphi_2 I), \quad A_4 = \left(1 - \frac{Q^*}{Q}\right)(\gamma I - \varphi_1 Q), \tag{3.4}$$
$$A_5 = \left(1 - \frac{R^*}{R}\right)(p\mu + \varepsilon Q - \delta R).$$

We further estimate all the terms in (3.4) that

$$\begin{aligned} A_1 &= \left(1 - \frac{S^*}{S}\right)((1-p)\mu - \eta SI - \delta S + \phi E), \\ &= \left(1 - \frac{S^*}{S}\right)(\eta S^* I^* + \delta S^* - \phi E^* - \eta SI - \delta S + \phi E) \\ &= (\eta S^* I^* + \delta S^* - \phi E^* - \eta SI - \delta S + \phi E) - \\ &\quad \left(\frac{S^*}{S}\eta S^* I^* + \frac{S^*}{S}\delta S^* - \frac{S^*}{S}\phi E^* - \eta S^* I - \delta S^* + \frac{S^*}{S}\phi E\right) \\ &= \left(1 - \frac{S^*}{S}\right)\eta S^* I^* + \delta S^*\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \left(1 - \frac{S^*}{S}\right)\phi E + \\ &\quad \left(\frac{S^*}{S} - 1\right)\phi E^* \\ &\leq (\eta S^* I^* + \delta S^*)\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \phi E^*\left(1 - \frac{S^*}{S}\frac{E}{E^*}\right). \end{aligned} \tag{3.5}$$

$$A_2 = \left(1 - \frac{E^*}{E}\right)(\eta SI - \eta S^* I^*) \leq \eta S^* I^*\left(1 - \frac{E^*}{E}\frac{SI}{S^* I^*}\right). \tag{3.6}$$

$$A_3 = \left(1 - \frac{I^*}{I}\right)(\beta E - \beta E^*) \leq \beta E^*\left(1 - \frac{I^*}{I}\frac{E}{E^*}\right). \tag{3.7}$$

$$A_4 = \left(1 - \frac{Q^*}{Q}\right)(\gamma I - \gamma I^*) \leq \gamma I^*\left(1 - \frac{Q^*}{Q}\frac{I}{I^*}\right). \tag{3.8}$$

$$\begin{aligned} A_5 &= \left(1 - \frac{R^*}{R}\right)(p\mu + \varepsilon Q - \delta R) \\ &= \left(1 - \frac{R^*}{R}\right)(\delta R^* - \varepsilon Q^* + \varepsilon Q - \delta R) \\ &= (\delta R^* - \varepsilon Q^* + \varepsilon Q - \delta R) - \left(\frac{R^*}{R}\delta R^* - \frac{R^*}{R}\varepsilon Q^* + \frac{R^*}{R}\varepsilon Q - \delta R^*\right) \\ &\leq \delta R^*\left(2 - \frac{R}{R^*} - \frac{R^*}{R}\right) + \left(1 - \frac{R^*}{R}\frac{Q}{Q^*}\right)\varepsilon Q^*. \end{aligned} \tag{3.9}$$

All the terms in (3.5)-(3.9) are then substituted into (3.3) to get

$$
\begin{aligned}
D_*^{\alpha}V \leq\ & (\eta S^* I^* + \delta S^*)\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \phi E^*\left(1 - \frac{S^*}{S}\frac{E}{E^*}\right) + \\
& \eta S^* I^*\left(1 - \frac{E^*}{E}\frac{SI}{S^*I^*}\right) + \beta E^*\left(1 - \frac{I^*}{I}\frac{E}{E^*}\right) + \gamma I^*\left(1 - \frac{Q^*}{Q}\frac{I}{I^*}\right) + \\
& \delta R^*\left(2 - \frac{R}{R^*} - \frac{R^*}{R}\right) + \left(1 - \frac{R^*}{R}\frac{Q}{Q^*}\right)\varepsilon Q^*.
\end{aligned}
$$

Then one has $D_*^{\alpha}V(t) < 0$ if all the terms $\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) \leq 0$, $\left(1 - \frac{S^*}{S}\frac{E}{E^*}\right) \leq 0$, $\left(1 - \frac{E^*}{E}\frac{SI}{S^*I^*}\right) \leq 0$, $\left(1 - \frac{I^*}{I}\frac{E}{E^*}\right) \leq 0$, $\left(1 - \frac{Q^*}{Q}\frac{I}{I^*}\right) \leq 0$, $\left(2 - \frac{R}{R^*} - \frac{R^*}{R}\right) \leq 0$, $\left(1 - \frac{R^*}{R}\frac{Q}{Q^*}\right) \leq 0$ for all the positive parameters. Indeed, we have $D_*^{\alpha}V(t) = 0$ if $S = S^*, E = E^*, I = I^*, Q = Q^*$, and $R = R^*$. Therefore, the global asymptotic $\mathcal{E}_1$ is achieved.

3.7 Numerical scheme and results

In this section, we study the numerical scheme by employing the predictor-corrector studied in [5–7]. Then, the numerical schemes of system (1.3) are shown in the following equation.

$$
\begin{aligned}
S_h(\tau_{n+1}) &= S(0) + \frac{h^{\alpha}}{\Gamma(\alpha+2)}F_1\left(\tau_{n+1}, S_h^P(\tau_{n+1}), E_h^P(\tau_{n+1}), I_h^P(\tau_{n+1}), Q_h^P(\tau_{n+1}), R_h^P(\tau_{n+1})\right) \\
&\quad + \frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1}F_1\left(\tau_j, S_h^P(\tau_j), E_h^P(\tau_j), I_h^P(\tau_j), Q_h^P(\tau_j), R_h^P(\tau_j)\right), \\
E_h(\tau_{n+1}) &= E(0) + \frac{h^{\alpha}}{\Gamma(\alpha+2)}F_2\left(\tau_{n+1}, S_h^P(\tau_{n+1}), E_h^P(\tau_{n+1}), I_h^P(\tau_{n+1}), Q_h^P(\tau_{n+1}), R_h^P(\tau_{n+1})\right) \\
&\quad + \frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1}F_2\left(\tau_j, S_h^P(\tau_j), E_h^P(\tau_j), I_h^P(\tau_j), Q_h^P(\tau_j), R_h^P(\tau_j)\right), \\
I_h(\tau_{n+1}) &= I(0) + \frac{h^{\alpha}}{\Gamma(\alpha+2)}F_3\left(\tau_{n+1}, S_h^P(\tau_{n+1}), E_h^P(\tau_{n+1}), I_h^P(\tau_{n+1}), Q_h^P(\tau_{n+1}), R_h^P(\tau_{n+1})\right) \\
&\quad + \frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1}F_3\left(\tau_j, S_h^P(\tau_j), E_h^P(\tau_j), I_h^P(\tau_j), Q_h^P(\tau_j), R_h^P(\tau_j)\right), \\
Q_h(\tau_{n+1}) &= Q(0) + \frac{h^{\alpha}}{\Gamma(\alpha+2)}F_4\left(\tau_{n+1}, S_h^P(\tau_{n+1}), E_h^P(\tau_{n+1}), I_h^P(\tau_{n+1}), Q_h^P(\tau_{n+1}), R_h^P(\tau_{n+1})\right) \\
&\quad + \frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1}F_4\left(\tau_j, S_h^P(\tau_j), E_h^P(\tau_j), I_h^P(\tau_j), Q_h^P(\tau_j), R_h^P(\tau_j)\right), \\
R_h(\tau_{n+1}) &= R(0) + \frac{h^{\alpha}}{\Gamma(\alpha+2)}F_5\left(\tau_{n+1}, S_h^P(\tau_{n+1}), E_h^P(\tau_{n+1}), I_h^P(\tau_{n+1}), Q_h^P(\tau_{n+1}), R_h^P(\tau_{n+1})\right) \\
&\quad + \frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1}F_5\left(\tau_j, S_h^P(\tau_j), E_h^P(\tau_j), I_h^P(\tau_j), Q_h^P(\tau_j), R_h^P(\tau_j)\right),
\end{aligned}
$$

$$S_h^P(\tau_{n+1}) = S(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} F_1\left(\tau_j, S_h(\tau_j), E_h(\tau_j), I_h(\tau_j), Q_h(\tau_j), R_h(\tau_j)\right),$$

$$E_h^P(\tau_{n+1}) = E(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} F_2\left(\tau_j, S_h(\tau_j), E_h(\tau_j), I_h(\tau_j), Q_h(\tau_j), R_h(\tau_j)\right),$$

$$I_h^P(\tau_{n+1}) = I(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} F_3\left(\tau_j, S_h(\tau_j), E_h(\tau_j), I_h(\tau_j), Q_h(\tau_j), R_h(\tau_j)\right),$$

$$Q_h^P(\tau_{n+1}) = Q(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} F_4\left(\tau_j, S_h(\tau_j), E_h(\tau_j), I_h(\tau_j), Q_h(\tau_j), R_h(\tau_j)\right),$$

$$R_h^P(\tau_{n+1}) = R(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} F_5\left(\tau_j, S_h(\tau_j), E_h(\tau_j), I_h(\tau_j), Q_h(\tau_j), R_h(\tau_j)\right),$$

where S_h, E_h, I_h, Q_h, and R_h represent the corrector schemes for susceptible, exposed, infected, quarantine, and recovered sub population respectively. Meanwhile, $S_h^P, E_h^P, I_h^P, Q_h^P$, and R_h^P are respectively the predictor schemes for susceptible, exposed, infected, quarantine, and recovered sub population. The parameter h is a step size, and

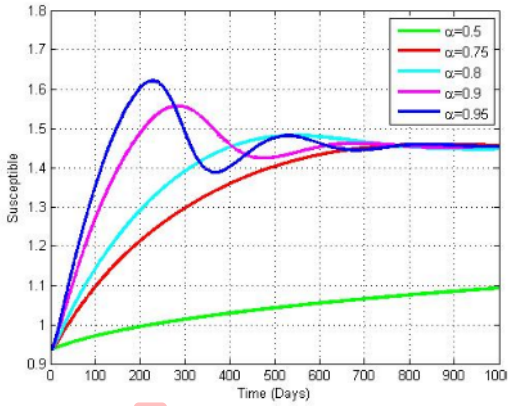
$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & \text{if } 1 \le j \le n, \\ 1, & \text{if } j = n+1, \end{cases}$$

$$b_{j,n+1} = \frac{h^{\alpha}}{\alpha}\left((n+1-j)^{\alpha} - (n-j)^{\alpha}\right).$$

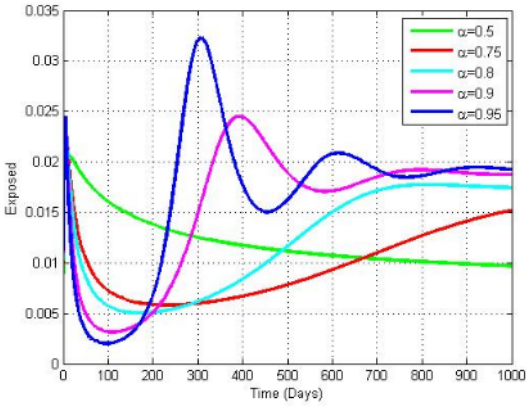
Here, the numerical simulations are based on the study in [14] and references there in. The initial values of S, E, I, Q, R are given $S(0) = 0.95, E(0) = 0, I(0) = 0.05, Q(0) = 0, R(0) = 0$, where the total population of susceptible and infected are 95% and 5% respectively. The parameter values are also adapted from the former study in [14] and references there in, i.e. $\eta = 0.57, \beta = 0.23, \varepsilon = 0.5, \mu = 0.019, \delta = 0.006, \theta = 0.05$, and some data are assumed in Table 1.

The Table 1 represents all possible scenarios for both disease-free and endemic condition (i.e. $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$). The scenario A_1 shows that there are no vaccination (p), treatment (γ), and natural immunity for exposed individuals (ϕ), where in such case there is no quarantine for infected individuals and basic reproduction number $\mathcal{R}_0 = 31.4127 > 1$. Meanwhile, the scenario A_2, the natural immunity for exposed individuals is considered, and the basic reproduction number $\mathcal{R}_0 = 3.4589 > 1$ is lower than scenario A_1. The scenarios B_1 and B_2 give same values of treatment and natural immunity but different values of vaccination, i.e. the vaccination in scenario B_2 higher than in scenario B_1. Then, it causes the basic reproduction number $\mathcal{R}_0$ in scenario B_2 lower than in scenario B_1. Moreover, the scenarios C_1 and C_2 only give same values of treatment, but the vaccination and natural immunity in scenario C_2 stronger than in scenario C_1. In this case, it causes the basic reproduction number $\mathcal{R}_0 = 0$ in scenario C_2 lower than in scenario C_1.

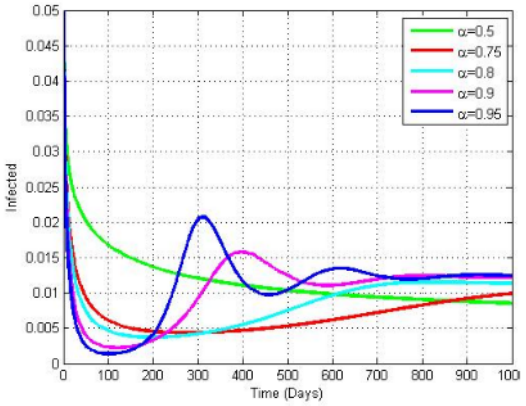
Figure 1 and 2 show the stability of endemic and disease-free equilibrium for $\alpha = 0.5$, $\alpha = 0.75$, $\alpha = 0.8$, $\alpha = 0.9$, $\alpha = 0.95$. The results are obtained based on the basic reproduction



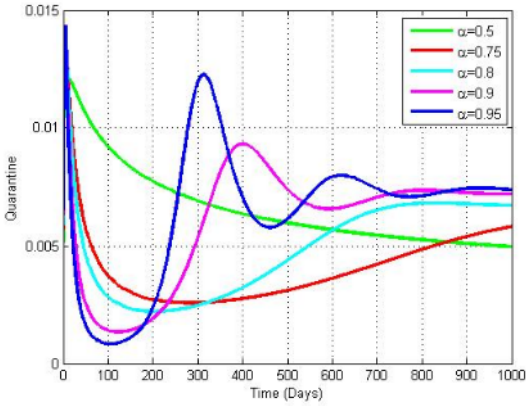
(a) Susceptible $S(t)$ profiles



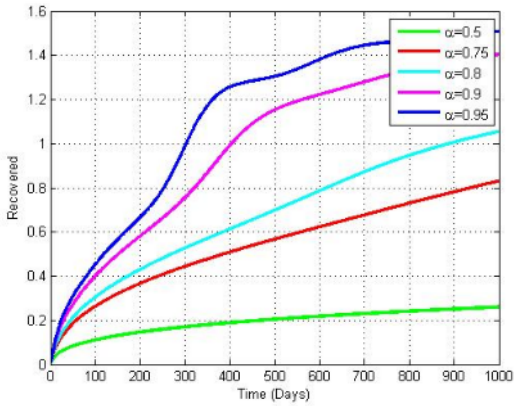
(b) Exposed $E(t)$ profiles



(c) Infected $I(t)$ profiles

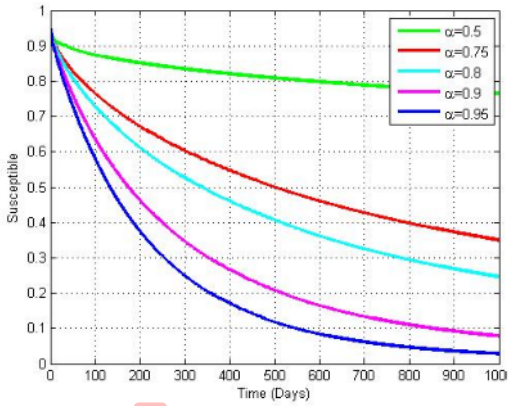


(d) Quarantine $Q(t)$ profiles

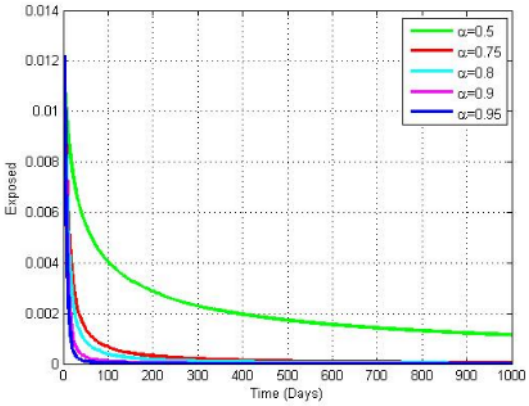


(e) Recovery $R(t)$ profiles

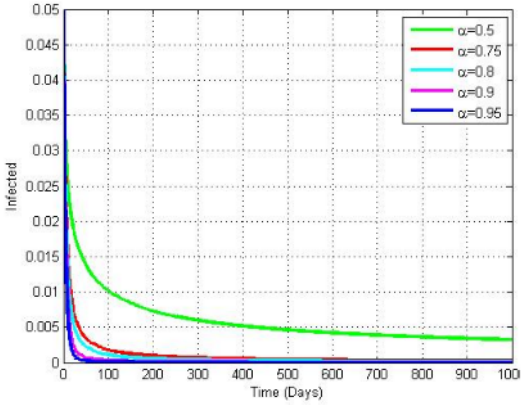
Figure 1: Profiles of the endemic equilibrium $\mathcal{E}_1$ for $\alpha = 0.5, 0.75, 0.8, 0.9, 0.95$



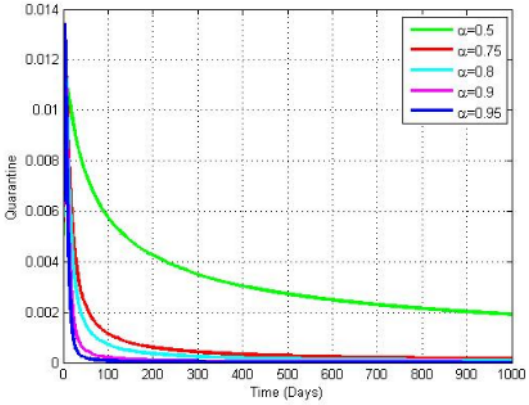
(a) Susceptible $S(t)$ profiles



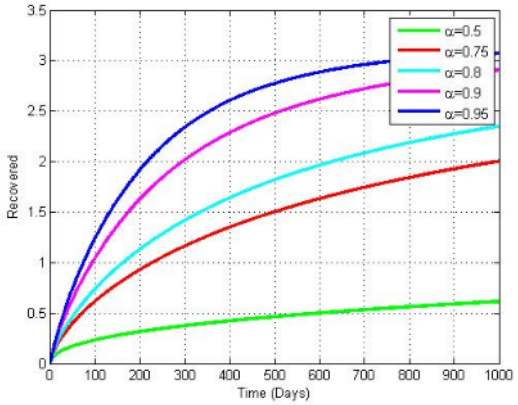
(b) Exposed $E(t)$ profiles



(c) Infected $I(t)$ profiles



(d) Quarantine $Q(t)$ profiles



(e) Recovery $R(t)$ profiles

Figure 2: Profiles of the disease-free equilibrium $\mathcal{E}_0$ for $\alpha = 0.5, 0.75, 0.8, 0.9, 0.95$

Table 1: Numerical simulation results in the Caputo sense for $\alpha = 0.95$ and the final values of S_f, E_f, I_f, Q_f, R_f

Parameter	A1	A2	B1	B2	C1	C2
γ	0	0.3	0.3	0.3	0.3	0.3
ϕ	0	0	0.3	0.3	0.9	1
p	0	0.3	0.3	0.9	0.3	1
$\mathcal{R}_0$	31.4127	3.4589	1.5230	0.2176	0.7186	0
S_f	0.1009	0.6409	1.4533	0.3346	2.1758	2.1758
E_f	0.0782	0.0401	0.0192	0	0	0
I_f	0.3208	0.0259	0.0124	0	0	0
Q_f	0	0.0154	0.0074	0	0	0
R_f	0	2.1677	1.5088	2.7653	0.9240	0.9240

number, i.e. if $\mathcal{R}_0 > 1$ then the existing infection can affect more new infection. Meanwhile, if $\mathcal{R}_0 < 1$ then the existing infection can affect less new infection. In Figure 1, we use $p = \phi = \gamma = 0.3$ with $\mathcal{R}_0 = 1.5230 > 1$ and $p = \phi = 1, \gamma = 0.3$ with $\mathcal{R}_0 = 0 < 1$ in Figure 2. Moreover, our results show that the higher the order of α, the less time it needs to converge.

4 Conclusions

We conclude our study based on the main results that the higher the natural immunity for exposed individual and vaccination, the smaller the basic reproduction number $\mathcal{R}_0$. The system has the disease-free if $\mathcal{R}_0 < 1$ and endemic if $\mathcal{R}_0 > 1$. The results of some values natural immunity for exposed individual (ϕ) and vaccination (p) can affect our dynamical systems. Hence, the bifurcation is very challenging to study where this birfurcation is not our concern in this paper.

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