# Mathematics for Data Science

Lecture 1: Probability theory and Random Variables

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# Introduction to probability theory

# Randomness (Uncertainty)

Fundamental example: consider the game of a die throw.

- Fundamental example $\varepsilon$ : "throw a balanced die" $\longleftarrow$ `Action`.
- Sample space: the set of all possible results of this random experiment

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

- Events: In this random experiment, one can be interested in more complex events than just a simple result of the experiment.
- The The Power set $\Omega$, called $\mathcal{P}(\Omega)$, is the set of all subsets of $\Omega$.
- A family of subsets $\mathcal{A}$ of $\Omega$. These subsets are called events. We say that the event $A$ has occured if and only if the result $\omega$ of $\Omega$ that has occurred belongs to $A$.
- $\sigma$-**Algebra**: We call $\sigma$-Algebra any family $\mathcal{A}$ of subsets of $\Omega$ satisfying:
  1. $\Omega \in \mathcal{A}$.
  2. if $A \in \mathcal{A}$, then $\bar{A} \in \mathcal{A}$.
  3. if $(A_n)_{n\in\mathbb{N}}$ is a sequence of elements in $\mathcal{A}$, then $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{A}$.
- $(\Omega, \mathcal{A})$ is a measurable space (or a Borel space).

# Notions on Events

- Let $(\Omega, \mathcal{A})$ be a measurable space:
  - The set $\mathcal{A}$ is called $\sigma$-Algebra of events. The elements of $\mathcal{A}$ are called the events.
  - The event $\Omega$ is called certain event. The event $\emptyset$ is called impossible event.
- Operations on events. Let $A$ and $B$ be two events:
  - $\bar{A}$ is the complement event of $A$ (we also note $A^c$). $\bar{A} = \Omega \setminus A$.
    $barA$ occurs if and only if $A$ does not occur.
  - $A \cap B$ is the event «$A$ and $B$».
    $A \cap B$ occurs when both events occur.
  - $A \cup B$ is the event «$A$ or $B$».
    $A \cup B$ occurs when at least one of the two events occurs.
- Mutually Exclusive Events: $A$ and $B$ are mutually exclusive if their simultaneous realization is impossible: $A \cap B = \emptyset$.
- Implication: $A$ implies $B$ means that if $A$ occurs, then $B$ also occurs: $A \subset B$.

# Probability Space

- Let $(\Omega, \mathcal{A})$ a measurable space. A probability function on $(\Omega, \mathcal{A})$, is any map

$$P : \mathcal{A} \to \mathbb{R}$$

  such that:

  1. $\forall A \in \mathcal{A}, P(A) \geqslant 0$.
  2. $P(\Omega) = 1$.
  3. $\forall (A_n)_{n \in \mathbb{N}^*} \in \mathcal{A}^{\mathbb{N}^*}$, a familty of pairwise disjoint (mutually exclusive) events, we have:

$$P(\bigcup_{n \in \mathbb{N}^*} A_n) = \sum_{n=1}^{+\infty} P(A_n)$$

- The triplet $(\Omega, \mathcal{A}, P)$ is called a **probability space**.

# Probability: Properties

1. $P(\emptyset) = 0$.
2. $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$.
3. If $A_1$ and $A_2$ are mutually exclusive, $A_1 \cap A_2 = \emptyset$, $P(A_1 \cup A_2) = P(A_1) + P(A_2)$.
4. $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$.
5. $P(\bar{A}) = 1 - P(A)$.
6. $P(B \setminus A) = P(B) - P(B \cap A)$.
7. $A \subset B \Rightarrow P(A) \leqslant P(B)$.

## Uniform probability on finite $\Omega$

- Let $\Omega$ be a finite sample space. We say that $P$ is the **uniform probability** on the measurable space $(\Omega, P(\Omega))$ if:

$$\forall \omega, \omega' \in \Omega, \qquad P(\omega'\}) = P(\omega'\})$$

One also says that there is **equiprobability** of elementary events.

- Let $(\Omega, \mathcal{P}(\Omega), P)$ be a finite probability space. If $P$ is the uniform probability, then

$$\forall A \in \mathcal{A}, \qquad P(A) = \frac{\mathrm{Card}(A)}{\mathrm{Card}(\Omega)}$$

# Conditional probability

- Let $(\Omega, \mathcal{A}, P)$ be a probability space and $B \in \mathcal{A}$ such that $P(B) > 0$. The map function $P_B$ defined on $\mathcal{A}$ by:
$$P_B(A) = P(A|B) = \frac{P(A \cap B)}{P(B)}, \qquad \forall A \in \mathcal{A}$$
is a probability function on $(\Omega, \mathcal{A})$; it is called the conditional probability given $B$. It is the probability of event $A$ occurring given that event $B$ has occurred.
- Remark: $(A|B)$ is not an event! We use the notation $P(A|B)$ for simplicity, but the notation $P_B(A)$ is the correct one.
- Chain rule:
$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$
- Law of total probability:
  - $\forall A \in \mathcal{A}, \quad P(A) = P(A \cap B) + P(A \cap \bar{B})$
  - We call partition of $\Omega$, a set of events that are pairwise disjoint and whose union is the sample space $\Omega$. The partition is said to be "countable" if its cardinality is at most equal to that of $\mathbb{N}$.
  - Let $(B_n)_{n \geqslant 0}$ a partition of $\Omega$. We have:
$$\forall A \in \mathcal{A}, \qquad P(A) = \sum_{n \geqslant 0} P(A \cap B_n)$$
- Independence: Events $A$ and $B$ are independent iff $P(A \cap B) = P(A)P(B)$.

## First Bayes' theorem

Let $(\Omega, \mathcal{A}, P)$ a probability space. For all events $A$ and $B$ such that $P(A) \neq 0$ and $P(B) \neq 0$, we have:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

## Second Bayes' theorem

Let $(\Omega, \mathcal{A}, P)$ a probability space and $(B_n)_{n\geqslant 0}$ a partition of $\Omega$ s.t. for all $n \geqslant 0$ $P(B_n) \neq 0$. We have for all $A \in \mathcal{A}$ s.t. $P(A) \neq 0$

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{n\geqslant 0} P(A|B_n)P(B_n)} \qquad \forall i \geqslant 0$$

# Real Random Variable

## Definition

*Let* $\varepsilon$ *an experiment and* $(\Omega, \mathcal{A}, P)$ *the associated probability space. . In many situations, one associates to each result* $\omega \in \Omega$ *a real number denoted* $X(\omega)$*; In this way, one builds a map* $X : \Omega \to \mathbb{R}$*. Historically,* $\varepsilon$ *was a game and* $X$ *représented the earning of a player.*

## Example: a die throw

A player throws a fair six faces dice and we observe the obtained number:

- If the result is 1,3 or 5, the player earns 1 euro.
- If the result is 2 or 4, the player earns 5 euros.
- If the result is 6, the player loses 10 euros.

### Analysis

- $\varepsilon$: "throw a fair die".
- $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- $\mathcal{A} = \mathcal{P}(\Omega)$.
- $\mathrm{P}$ is the equiprobability on $(\Omega, \mathcal{A})$.

Let $X$ the map function from $\Omega$ to $\mathbb{R}$ that associates to each result the corresponding earning.

So we have

- $X(1) = X(3) = X(5) = 1$
- $X(2) = X(4) = 5$
- $X(6) = -10$

We say that $X$ is a **random variable** on $\Omega$.

One can ask what is the probability for the player to win 1 euro:

$\Rightarrow$ $X(\omega) = 1$.

- this is the case if and only if $\omega \in \{1, 3, 5\}$.
- The sought-for probability is therefore $P(\{1, 3, 5\}) = 1/2$.
- Which can also be written as $P(X = 1) = 1/2$.

Thus, we will consider the event:

$$\{X = 1\} = \{\omega \in \Omega / X(\omega) = 1\} = \{\omega \in \Omega / X(\omega) \in \{1\}\} = X^{-1}(\{1\}) = \{1, 3, 5\}.$$

Similarly, we have:

- $P(X = 5) = 1/3$.
- $P(X = -10) = 1/6$.

One can present the three previous probabilities in a table:

| $x_i$ | -10 | 1 | 5 |
|---|---|---|---|
| $p_i = P(X = x_i)$ | 1/6 | 1/2 | 1/3 |

This is tantamount of considering a new sample space:

$$\Omega_X = X(\Omega) = \{-10, 1, 5\}$$

equipped with the probability $P_X$ defined in the table above. This new probability is called the **probability distribution** of $X$.

Notice that

$$P(\bigcup_{x_i \in \Omega_X} \{X = x_i\}) = \sum_{x_i \in \Omega_X} P(X = x_i) = 1$$

In this chapter:

- We treat the case where $X(\Omega)$ is countable.
- The random variable in this case is ***discrete***.
- We define its probability law by its probability distribution.
- We will define the two main numerical characteristics of a discrete random variable:
  - Expected value: characteristic of centrality (the *mean*).
  - Variance: characteristic of dispersion.
- We will also define the couples of random variables.

# Discrete Random Variables

**Definition**

*We say that a real random variable $X$ is* ***discrete*** *if the set of all possible values that $X$ can take is finite or countable.*

*If we suppose that the set $X(\Omega)$ of all possible values of $X$ admits a smallest element $x_1$. Then the discrete real random variable $X$ is completely defined by:*

- *The set $X(\Omega)$ of all possible values of $X$, sorted in ascending order: $X(\Omega) = \{x_1, x_2, \dots, x_i, \dots\}$ with $x_1 \leqslant x_2 \leqslant \dots \leqslant x_i \leqslant \dots$.*
- *The probability distribution defined on $X(\Omega)$ by*

$$p_i = p(x_i) = P(X = x_i) \quad \forall\, i = 1, 2, \dots$$

**Remarks**:

- $B \subset \mathbb{R}$, $P(X \in B) = \sum_{i/x_i \in B} p(x_i)$.
- $P(a < X \leqslant b) = \sum_{i/a < x_i \leqslant b} p(x_i)$.
- $p(x_i) \geqslant 0$ and $\sum_{i=1}^{\infty} p(x_i) = 1$.
- If the number of possible values of $X$ is small enough, the probability distribution of $X$ is often presented as a table.

## Definition

*Given a discrete random variable $X$, we call cumulative distribution function of $X$ (or simply distribution function), denoted $F_X$, the function defined by: for any real a,*

$$F(a) = P(X \leqslant a) = \sum_{i/x_i \leqslant a} P(X = x_i)$$

The value $F_X(a)$ represents the probability that $X$ takes a value smaller or equal to $a$.

## Properties

1. It is a staircase function.
2. $F(a) \leqslant 1$ since it is a probability.
3. $F(a)$ is continuous from the right.
4. $\lim_{a \to -\infty} F(a) = 0$ and $\lim_{a \to \infty} F(a) = 1$

The distribution function characterizes the distribution of $X$. In other words, if $X$ and $Y$ are two random variables, we have $F_X = F_Y$ if and only if their probability distributions are the same.

## Distribution function and probabilities over $X$

All the computations of probabilities about $X$ can be carried out using the distribution function. For example,

$$P(a < X \leqslant b) = F(b) - F(a) \qquad \text{pour tout } a < b$$

This is easier to understand if one writes the event $\{X \leqslant b\}$ as a union of two incompatible events $\{X \leqslant a\}$ and $\{a < X \leqslant b\}$, Let

$$\{X \leqslant b\} = \{X \leqslant a\} \cup \{a < X \leqslant b\}$$

In this way,

$$P(X \leqslant b) = P(X \leqslant a) + P(a < X \leqslant b)$$

which proves the equality above.

**Remark**

One can compute the individual probabilities by:

$$p_i = P(X = x_i) = F(x_i) - F(x_{i-1}) \qquad \text{pour } 1 \leqslant i \leqslant n$$

## Distribution function and probabilities over $X$

### Example

We play three times to "heads or tails" $\Rightarrow$

- $\Omega = \{P, F\}^3$.
- $\mathrm{card}(\Omega) = |\Omega| = 2^3 = 8$.

Let $X$ the random variable "number of tails obtained" $\Rightarrow X(\Omega) = \{0, 1, 2, 3\}$.

- Let's calculate $P(X = 1)$.
- $X^{-1}(1) = \{(P, F, F), (F, P, F), (F, F, P)\}$.

$\Rightarrow P(X = 1) = \frac{3}{8}$

Using the same method we obtain the probability distribution of $X$:

| $k$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X = k)$ | $1/8$ | $3/8$ | $3/8$ | $1/8$ |

The distribution function $X$ is therefore given by:

$$F(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1/8 & \text{si } 0 \leqslant x < 1 \\ 1/2 & \text{si } 1 \leqslant x < 2 \\ 7/8 & \text{si } 2 \leqslant x < 3 \\ 1 & \text{si } x \geqslant 3 \end{cases}$$

One can represent both the probability distribution and the distribution function of $X$ in the same table:

| $k$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X = k)$ | $1/8$ | $3/8$ | $3/8$ | $1/8$ |
| $F_X(x)$ | $1/8$ | $1/2$ | $7/8$ | $1$ |

The graph of the distribution function is represented below:

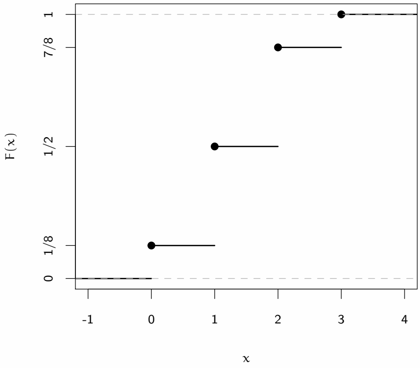



*Figure 1: Distribution function*

Here is another slightly different representation of the distribution function:

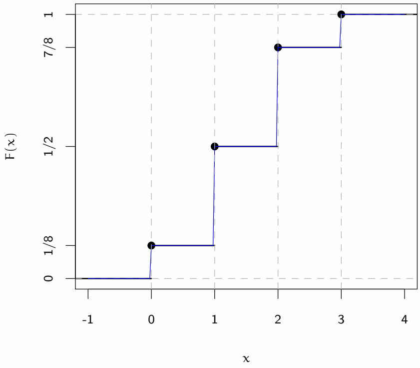



*Figure 2: Distribution function*

# The indicator random variable

**Definition**

*Let $A$ an event. We call indicator random variable of the event $A$, the random variable $X = \mathbb{1}_A$ defined by:*

$$X(\omega) = \begin{cases} 1 & \text{si } \omega \in A \\ 0 & \text{si } \omega \in \bar{A} \end{cases}$$

Therefore:

- $P(X = 1) = P(A) = p$
- $P(X = 0) = P(\bar{A}) = 1 - p$

The distribution function of the indicator random variable is therefore:

$$F(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1 - p & \text{si } 0 \leqslant x < 1 \\ 1 & \text{si } x \geqslant 1 \end{cases}$$

### Example

- Let $U$ an urn containing two white ball and three red balls.
- We randomly take one ball out of the box.
- Let $A$ : "take one white ball out".
- Let $X$ be the indicator random variable of $A$.

Find the probability distribution and the distribution function of $X$.

The probability distribution of $X$ is

| $k$ | $0$ | $1$ |
|---|---|---|
| $P(X=k)$ | $\frac{3}{5}$ | $\frac{2}{5}$ |

and its distribution function is:

$$F(x) = \begin{cases} 0 & \text{si } x < 0 \\ 3/5 & \text{si } 0 \leqslant x < 1 \\ 1 & \text{si } x \geqslant 1 \end{cases}$$

# Moments of a discrete random variable

### Definition

*For a discrete random variable $X$ with probability distribution $p(.)$, we define the expected value of $X$, called $E(X)$, by*

$$E(X) = \sum_{i \in \mathbb{N}} x_i p(x_i)$$

*In concrete terms, the expected value of $X$ is the weighted mean of the values of $X$, the weights being the probabilities associated to the values of $X$.*

### Examples

1. In the previous example where we play three times to "heads or tails", the expected value of $X$ is:

$$E(X) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = 1.5$$

2. For the indicator random variable of $A$:

$$E(X) = 0 \times P(X = 0) + 1 \times P(X = 1) = P(A) = p$$

which means that the expected value of the indicator of an event $A$ corresponds to the probability that $A$ occurs.

**Theorem**

*Let $X$ be a discrete random variable whose possible values are $x_i$, $i \geqslant 1$, and denote by $p(x_i)$ the probability that $X = x_i$ occurs. Then, for any real function $g$, we have*

$$E(g(X)) = \sum_i g(x_i)p(x_i)$$

**Example**

Let $X$ be a random variable that can take three values $\{-1, 0, 1\}$ with the following probabilities:

$$P(X = -1) = 0.2 \qquad P(X = 0) = 0.5 \qquad P(X = 1) = 0.3$$

Calculate $E(X^2)$.

**Solution**

***First method:*** Let $Y = X^2$. The probability distribution of $Y$ is given by

$$P(Y = 1) = P(X = -1) + P(X = 1) = 0.5$$
$$P(Y = 0) = P(X = 0) = 0.5$$

So

$$E(X^2) = E(Y) = 1(0.5) + 0(0.5) = 0.5$$

***Second method:*** Using the theorem

$$\begin{aligned} E(X^2) &= (-1)^2(0.2) + 0^2(0.5) + 1^2(0.3) \\ &= 1(0.2 + 0.3) + 0(0.5) = 0.5 \end{aligned}$$

**Remark**

$$0.5 = E(X^2) \neq (E(X))^2 = 0.01$$

### Properties

1. $E(X+a) = E(X) + a, \quad a \in \mathbb{R}$
   results which follows from:

$$\sum_i p_i(x_i + a) = \sum_i p_i x_i + \sum_i a p_i = \sum_i p_i x_i + a \sum_i p_i = \sum_i p_i x_i + a$$

2. $E(aX) = aE(X), \quad a \in \mathbb{R}$
   to prove it, just write:

$$\sum_i p_i a x_i = a \sum_i p_i x_i$$

3. $E(X+Y) = E(X) + E(Y)$, $X$ and $Y$ being two random variables.

All these three properties are summarised in the claim that the expected value is linear:

$$E(\lambda X + \mu Y) = \lambda E(X) + \mu E(Y), \quad \forall \lambda \in \mathbb{R}, \forall \mu \in \mathbb{R}.$$

**Definition**

*Let $X$ be a discrete random variable. We call variance of $X$, denoted $V(X)$, the quantity defined by, when it exists,*

$$V(X) = E\left[(X - E(X))^2\right]$$

*Thus, the variance is the expected value of the square of the centered random variable $X - E(X)$. The variance can be interpreted as a measure of the dispersion of the possible values of $X$ around its expected value.*

**Remark**

Equivalently, the variance might be defined by the following formula:

$$V(X) = E(X^2) - E^2(X)$$

Indeed:

$$\begin{aligned} V(X) &= E\left[X^2 - 2XE(X) + E^2(X)\right] \\ &= E(X^2) - E[2XE(X)] + E[E^2(X)] \\ &= E(X^2) - 2E^2(X) + E^2(X) \end{aligned}$$

# Variance

## Example

OLet us compute $V(X)$ in the case where $X$ is the number obtained when throwing a fair die.

Previously, we saw that $E(X) = \frac{7}{2}$. Moreover,

$$
\begin{aligned}
E(X^2) &= \sum_i x_i^2 p(x_i) \\
&= 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{6}\right) + 3^2\left(\frac{1}{6}\right) + 4^2\left(\frac{1}{6}\right) + 5^2\left(\frac{1}{6}\right) + 6^2\left(\frac{1}{6}\right) \\
&= \left(\frac{1}{6}\right)(91) = \frac{91}{6}
\end{aligned}
$$

And therefore

$$
\begin{aligned}
V(X) &= E(X^2) - E^2(X) \\
&= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}
\end{aligned}
$$

## Properties

1. $V(X) \geqslant 0$

2. $\forall\, a \in \mathbb{R}, \quad V(X+a) = V(X)$
   en effet:

$$\begin{aligned} V(X+a) &= E\left[[X+a-E(X+a)]^2\right] \\ &= E\left[[X+a-E(X)-a]^2\right] \\ &= E\left[[X-E(X)]^2\right] = V(X) \end{aligned}$$

3. $\forall\, a \in \mathbb{R}, \quad V(aX) = a^2 V(X)$
   en effet:

$$\begin{aligned} V(aX) &= E\left[[aX-E(aX)]^2\right] \\ &= E\left[[aX-aE(X)]^2\right] \\ &= E\left[a^2\,[X-E(X)]^2\right] \\ &= a^2\left[E\,[X-E(X)]^2\right] = a^2 V(X) \end{aligned}$$

### Definition

*Let $X$ be a discrete random variable. The square root of the variance is called the **standard deviation** of $X$ and is denoted*

$$\sigma_X = \sqrt{V(X)}$$

*$\sigma_X$ has the same physical units as the random variable $X$.*

- The standard deviation allows to measure the dispersion of a set of data.
- The smaller sigma is, the closer to each other the values of the data are.
- Example: the dispersion of the grades in an exam. The smaller sigma is, the more homogeneous the class is.
- \- Expected value and standard deviation are linked through *Bienaymé-Tchebychev inequality*.

**Theorem**

*Let $X$ a random variable of expected value $\mu$ and variance $\sigma^2$. For all $\varepsilon > 0$, We have:*

$$P\left(|X - E(X)| \geqslant \varepsilon\right) \leqslant \frac{\sigma^2}{\varepsilon^2}$$

**Remark**

This inequality can be written in a slightly different fashion. Let $k = \varepsilon/\sigma$.

$$P\left(|X - E(X)| \geqslant k\sigma\right) \leqslant \frac{1}{k^2}$$

**Importance**

This inequality relates the probability for $X$ to deviate from its expected value $E(X)$ to its variance, which is precisely an indicator of the dispersion around the expected value. The inequality makes quantitatively precise the statement "the smaller the variance is, the less likely it is to find values far away from the expected value".

# Centred and non-centred moments

**Definition**

*We call non centered moment of order $r \in \mathbb{N}^*$ of $X$ the quantity, when it exists:*

$$m_r(X) = \sum_{i \in \mathbb{N}} x_i^r p(x_i) = E(X^r).$$

**Definition**

*The centered moment of order $r \in \mathbb{N}^*$ the quantity, when it exists:*

$$\mu_r(X) = \sum_{i \in \mathbb{N}} p_i \left[x_i - E(X)\right]^r = E\left[X - E(X)\right]^r.$$

**Remark**

The first moments are:

- $m_1(X) = E(X), \quad \mu_1(X) = 0.$
- $\mu_2(X) = V(X) = m_2(X) - m_1^2(X).$

# Two Random Variables

# Two Discrete Random Variables

So far, we have dealt with one random variable. However, it is often necessary to consider events related to two variables simultaneously, or even to more than two variables.

## Definition

*Let $X$ and $Y$ two discrete random variables, defined on probability space $(\Omega, \mathcal{A}, P)$ and that $X(\Omega) = \{x_1, x_2, \dots, x_l\}$ and $Y(\Omega) = \{y_1, y_2, \dots, y_k\}$, $l$ and $k \in \mathbb{N}$.*

*The **probability law of** $(X, Y)$ is defined by joint probabilities:*

$$p_{ij} = P(X = x_i; Y = y_j) = P(\{X = x_i\} \cap \{Y = y_j\})$$

*We have*

$$p_{ij} \geqslant 0 \quad \textit{et} \quad \sum_{i=1}^{l} \sum_{j=1}^{k} p_{ij} = 1$$

*The pair $(X, Y)$ is called two dimensional random vector and can have $l \times k$ valeurs.*

# Joint probability distribution table

The probabilities $p_{ij}$ can be presented in a two dimensional table than we call joint probability distribution table:

*Table 1: Joint probability distribution table*

| $X \backslash Y$ | $y_1$ | $y_2$ | $\dots$ | $y_j$ | $\dots$ | $y_k$ |
|---|---|---|---|---|---|---|
| $x_1$ | $p_{11}$ | $p_{12}$ | | $p_{1j}$ | | $p_{1k}$ |
| $x_2$ | $p_{21}$ | $p_{22}$ | | $p_{2j}$ | | $p_{2k}$ |
| $\vdots$ | | | | | | |
| $x_i$ | $p_{i1}$ | $p_{i2}$ | | $p_{ij}$ | | $p_{ik}$ |
| $\vdots$ | | | | | | |
| $x_l$ | $p_{l1}$ | $p_{l2}$ | | $p_{lj}$ | | $p_{lk}$ |

In the header we have the possible values of $Y$ and in the first column the possible values of $X$. The probability $p_{ij} = P(X = x_i; Y = y_j)$ is at the intersection of $i^{th}$ line and $j^{th}$ column.

# Example: pair of random variables

### Example

Three balls are drawn at random from an urn containing 3 red, 4 white and 5 black balls. $X$ and $Y$ are respectively the number of red and white balls drawn. Determine the joint probability distribution of the pair $(X, Y)$.

### Solution

- $\varepsilon$: "draw 3 balls from an urn containing 12 balls".
- $|\Omega| = C_{12}^{3} = 220$.
- $X(\Omega) = \{0, 1, 2, 3\}$ and $Y(\Omega) = \{0, 1, 2, 3\}$.
- $p(X = 0, Y = 0) = p(0, 0) = C_{5}^{3}/C_{12}^{3} = \frac{10}{220}$.
- $p(0, 1) = C_{4}^{1}C_{5}^{2}/C_{12}^{3} = \frac{40}{220}$.
- $p(1, 0) = C_{3}^{1}C_{5}^{2}/C_{12}^{3} = \frac{30}{220}$.

## Example

Three balls are drawn at random from an urn containing 3 red, 4 white and 5 black balls. $X$ and $Y$ are respectively the number of red and white balls drawn. Determine the joint probability distribution of the pair $(X, Y)$.

## Solution

*Table 2: Joint probability distribution table*

| $X\backslash Y$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $\frac{10}{220}$ | $\frac{40}{220}$ | $\frac{30}{220}$ | $\frac{4}{220}$ |
| 1 | $\frac{30}{220}$ | $\frac{60}{220}$ | $\frac{18}{220}$ | 0 |
| 2 | $\frac{15}{220}$ | $\frac{12}{220}$ | 0 | 0 |
| 3 | $\frac{1}{220}$ | 0 | 0 | 0 |

When we know the joint distribution of the random variables $X$ and $Y$, we can also look at the probability distribution of $X$ alone and $Y$ alone. These are the marginal probability distributions.

- Marginal distribution of $X$:

$$p_{i.} = P(X = x_i) = P[\{X = x_i\} \cap \Omega] = \sum_{j=1}^{k} p_{ij} \qquad \forall\, i = 1, 2, \ldots, l$$

- Marginal distribution of $Y$:

$$p_{.j} = P(Y = y_j) = P[\Omega \cap \{Y = y_j\}] = \sum_{i=1}^{l} p_{ij} \qquad \forall\, j = 1, 2, \ldots, k$$

We can calculate the marginal distributions directly from the table of the joint distribution.

# Marginal Probability Distributions

*Table 3: Joint distribution table with marginal distributions*

| $X \backslash Y$ | $y_1$ | $y_2$ | $\dots$ | $y_j$ | $\dots$ | $y_k$ | Marginal of X |
|---|---|---|---|---|---|---|---|
| $x_1$ | $p_{11}$ | $p_{12}$ | | $p_{1j}$ | | $p_{1k}$ | $p_{1.}$ |
| $x_2$ | $p_{21}$ | $p_{22}$ | | $p_{2j}$ | | $p_{2k}$ | $p_{2.}$ |
| $\vdots$ | | | | | | | |
| $x_i$ | $p_{i1}$ | $p_{i2}$ | | $p_{ij}$ | | $p_{ik}$ | $p_{i.}$ |
| $\vdots$ | | | | | | | |
| $x_l$ | $p_{l1}$ | $p_{l2}$ | | $p_{lj}$ | | $p_{lk}$ | $p_{l.}$ |
| Marginal of Y | $p_{.1}$ | $p_{.2}$ | | $p_{.l}$ | | $p_{.k}$ | 1 |

### Example

Three balls are drawn at random from an urn containing 3 red, 4 white and 5 black balls. $X$ and $Y$ are respectively the number of red and white balls drawn. Determine the joint probability distribution of the pair $(X, Y)$.

### Solution

*Table 4: Joint distribution table*

| $X \backslash Y$ | 0 | 1 | 2 | 3 | $p_{i.} = P(X = x_i)$ |
|---|---|---|---|---|---|
| 0 | $\frac{10}{220}$ | $\frac{40}{220}$ | $\frac{30}{220}$ | $\frac{4}{220}$ | $\frac{84}{220}$ |
| 1 | $\frac{30}{220}$ | $\frac{60}{220}$ | $\frac{18}{220}$ | 0 | $\frac{108}{220}$ |
| 2 | $\frac{15}{220}$ | $\frac{12}{220}$ | 0 | 0 | $\frac{27}{220}$ |
| 3 | $\frac{1}{220}$ | 0 | 0 | 0 | $\frac{1}{220}$ |
| $p_{.j} = P(Y = y_j)$ | $\frac{56}{220}$ | $\frac{112}{220}$ | $\frac{48}{220}$ | $\frac{4}{220}$ | 1 |

**Definition**

*For each value* $y_j$ *of* $Y$ *such that* $p_{.j} = P(Y = y_j) \neq 0$ *we can define the conditional distribution of* $X$ *given* $Y = y_j$ *by*

$$p_{i/j} = P(X = x_i / Y = y_j) = \frac{P(X = x_i; Y = y_j)}{P(Y = y_j)} = \frac{p_{ij}}{p_{.j}} \qquad \forall i = 1, 2, \ldots, l$$

*Same for* $Y$ *given* $X = x_i$*:*

$$p_{j/i} = P(Y = y_j / X = x_i) = \frac{P(X = x_i; Y = y_j)}{P(X = x_i)} = \frac{p_{ij}}{p_{i.}} \qquad \forall j = 1, 2, \ldots, k$$

# Independence of random variables

**Definition**

*We say that two random variables are independent iff*

$$P(X = x_i; Y = y_j) = P(X = x_i)P(Y = y_j) \qquad \forall\, i = 1, 2, \ldots, l \textit{ and } j = 1, 2, \ldots, k$$

*One demonstrates that*

$$P(\{X \in A\} \cap \{Y \in B\}) = P(\{X \in A\})P(\{Y \in B\}) \qquad \forall\ A \textit{ and } B \in \mathcal{A}$$

**Properties**

Let two random variables $X$ and $Y$,

1. $E(X + Y) = E(X) + E(Y)$
2. If $X$ and $Y$ are independent so $E(XY) = E(X)E(Y)$. But the reciprocal is not always true.

## Definition

*Let two random variables $X$ and $Y$. The **covariance** of $X$ and $Y$, when it exists, is*

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))] = \sum_i \sum_j (x_i - E(X))(y_j - E(Y))p_{ij}$$

*that we can calculate using the formula*

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

## Properties

- $Cov(X, Y) = Cov(Y, X)$
- $Cov(aX_1 + bX_2, Y) = aCov(X_1, Y) + bCov(X_2, Y)$
- $V(X + Y) = V(X) + V(Y) + 2Cov(X, Y)$
- If $X$ and $Y$ are independant so
  - $Cov(X, Y) = 0$ (the reciprocal is not always true)
  - $V(X + Y) = V(X) + V(Y)$ (the reciprocal is not always true)

# Linear Correlation (Correlation of Spearman)

## Definition

*The correlation between $X$ and $Y$ is defined by*

$$\rho = \rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{V(X)V(Y)}} = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

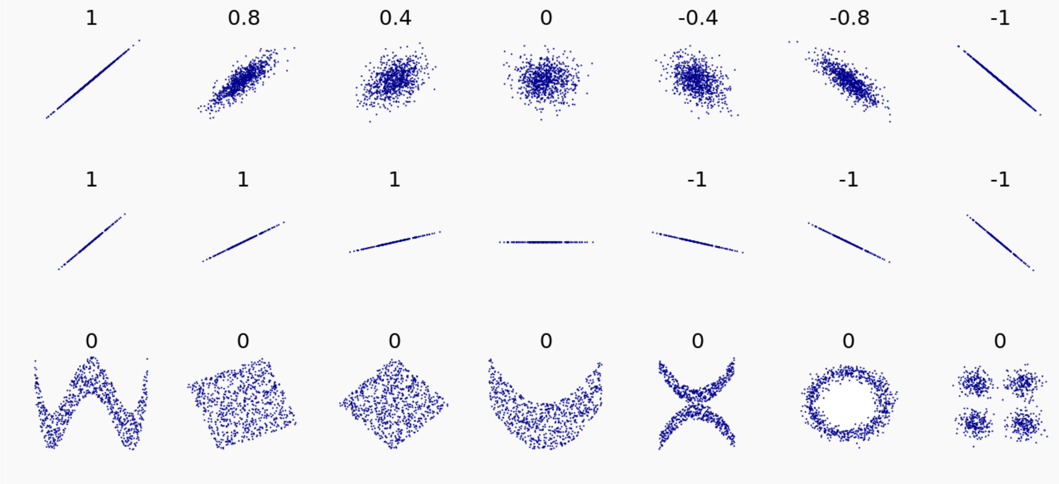
*We can demonstrate that*

$$-1 \leqslant \rho(X, Y) \leqslant 1$$

## Interpretation of $\rho$

- The correlation coefficient is a measure of the degree of linearity between $X$ and $Y$.
- Values of $rho$ close to 1 or $-1$ indicate an almost rigorous linearity between $X$ and $Y$.
- Values of $rho$ close to 0 indicate the absence of any linear relationship.
- When $\rho(X, Y)$ is positive, $Y$ tends to increase if $X$ does the same.
- When $\rho(X, Y) < 0$, $Y$ tends to decrease if $X$ increases.

# Linear correlation



## Loi Uniforme Discrète $\mathcal{U}(n)$

### Definition

*Une distribution de probabilité suit une loi uniforme lorsque toutes les valeurs prises par la variable aléatoire sont équiprobables. Si $n$ est le nombre de valeurs différentes prises par la variable aléatoire alors on a:*

$$P(X = x_i) = \frac{1}{n} \qquad \forall i \in \{1, \dots, n\}$$

*On dit $X \sim \mathcal{U}(n)$.*

### Example

La distribution des chiffres obtenus au lancer de dé (si ce dernier est non pipé) suit une loi uniforme dont la loi de probabilité est la suivante :

| $x_i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $P(X = x_i)$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

# Moments de loi uniforme discrète

## Cas particulier

Dans le cas particulier d'une loi uniforme discrète où chaque valeur de la variable aléatoire $X$ correspond à son rang, i.e. $x_i = i \ \forall i \in \{1, \ldots, n\}$, on a:

$$E(X) = \frac{n+1}{2} \quad \text{et} \quad V(X) = \frac{n^2-1}{12}$$

## Démonstration

La démonstration de ces résultats est établie en utilisant les égalités:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{et} \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Vous avez la démonstration de ces égalités dans l'Annexe.

## Example

L'exemple du lancer du dé: on peut calculer directement les moments de $X$:

$$E(X) = \frac{6+1}{2} = 3.5 \quad \text{et} \quad V(X) = \frac{6^2-1}{12} = \frac{35}{12} \simeq 2.92.$$

# Loi de Bernoulli $\mathcal{B}(p)$

# Loi de Bernoulli $\mathcal{B}(p)$

## Variable Indicatrice

Soit $A$ un événement quelconque; on appelle v.a. indicatrice de l'événement $A$, la v.a. définie par $X = \mathbb{1}_A$, c'est à dire:

$$X(\omega) = \mathbb{1}_A(\omega) = \begin{cases} 0 & \text{si } \omega \in \bar{A} \\ 1 & \text{si } \omega \in A \end{cases}$$

Ainsi $X(\Omega) = \{0, 1\}$ avec:

$$\begin{aligned} &P(X = 1) = P\{\omega \in \Omega / X(\omega) = 1\} = P(A) = p \\ &P(X = 0) = P\{\omega \in \Omega / X(\omega) = 0\} = P(\bar{A}) = 1 - P(A) = q \\ &\text{avec } p + q = 1 \end{aligned}$$

## Definition

*On dit que $X$ suit une loi de Bernoulli de paramètre $p = P(A)$, ce qu'on écrit symboliquement $X \sim \mathcal{B}(p)$. Une distribution de Bernoulli est associée à la notion "épreuve de Bernoulli", qui est une épreuve aléatoire à deux issues: succès $(X = 1)$ et échec $(X = 0)$.*

## Fonction de répartition de loi de Bernoulli

$$F(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1 - p & \text{si } 0 \leqslant x < 1 \\ 1 & \text{si } x \geqslant 1. \end{cases}$$

## Espérance de loi de Bernoulli

$$E(X) = 1 \times P(A) + 0 \times P(\bar{A}) = P(A) = p$$

## Variance de loi de Bernoulli

$$V(X) = E(X^2) - E^2(X) = p - p^2 = p(1-p) = pq$$

car

$$E(X^2) = 1^2 \times P(A) + 0^2 \times P(\bar{A}) = P(A) = p$$

## Loi Binomiale $\mathcal{B}(n, p)$

- Décrite pour la première fois par *Isaac Newton* en 1676 et démontrée pour la première fois par le mathématicien suisse *Jacob Bernoulli* en 1713.
- La loi binomiale est l'une des distributions de probabilité les plus fréquemment rencontrées en statistique appliquée.
- On exécute $n$ épreuves **indépendantes** de Bernoulli.
- Chaque épreuve a $p$ pour probabilité de succès et $1 - p$ pour probabilité d'échec.

$$A \quad A \quad \bar{A} \quad A \quad \bar{A} \quad \ldots \quad \bar{A} \quad A \quad A$$

$$S \quad S \quad E \quad S \quad E \quad \ldots \quad E \quad S \quad S$$

- $X =$ **le nombre de succès** sur l'ensemble des $n$ épreuves.
- $X$ dépend de deux paramètres $n$ et $p$.

$$S \quad S \quad E \quad S \quad E \quad \dots \quad E \quad S \quad S$$

- $X =$**le nombre de succès** sur l'ensemble des $n$ épreuves.
- $X(\Omega) = \{0, 1, \dots, n\}$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad 0 \leqslant k \leqslant n$$

- $\binom{n}{k}$ est le nombre d'échantillons de taille $n$ comportant exactement $k$ succès, de probabilité $p^k$, indépendamment de l'ordre, et donc $n-k$ échecs, de probabilité $(1-p)^{n-k}$.
- On écrit $X \sim \mathcal{B}(n, p)$.

**Remark**

Une variable de Bernoulli n'est donc qu'une variable binomiale de paramètres $(1, p)$.

$$X \sim \mathcal{B}(p) \iff X \sim \mathcal{B}(1, p)$$

# Triangle de Pascal & Binôme de Newton

## Triangle de Pascal

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad \forall\, n \geqslant 1 \text{ et } 1 \leqslant k \leqslant n-1$$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $n = 0$: | | | | | 1 | | | | |
| $n = 1$: | | | | 1 | | 1 | | | |
| $n = 2$: | | | 1 | | 2 | | 1 | | |
| $n = 3$: | | 1 | | 3 | | 3 | | 1 | |
| $n = 4$: | 1 | | 4 | | 6 | | 4 | | 1 |

## Binôme de Newton

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Cette formule permet de vérifier que la loi Binomiale est une loi de probabilité:

$$\sum^{n} P(X=k) = \sum^{n} \binom{n}{k} p^k (1-p)^{n-k} = [p + (1-p)]^n = 1$$

**Example**

On jette cinq pièces équilibrées. Les résultats sont supposés indépendants. Donner la loi de probabilité de la variable $X$ qui compte le nombre de piles obtenus.

**Solution**

- $X =$ nombre de piles (*succès*).
- $n = 5$.
- $p = 1/2$.
- $X \sim \mathcal{B}(5, \frac{1}{2})$.
- $X(\Omega) = \{0, 1, \ldots, 5\}$
- $P(X = 0) = \binom{5}{0}\left(\frac{1}{2}\right)^0\left(1 - \frac{1}{2}\right)^{5-0} = \frac{1}{32}$
- $P(X = 1) = \binom{5}{1}\left(\frac{1}{2}\right)^1\left(1 - \frac{1}{2}\right)^{4} = \frac{5}{32}$
- $P(X = 2) = \binom{5}{2}\left(\frac{1}{2}\right)^2\left(1 - \frac{1}{2}\right)^{3} = \frac{10}{32}$
- $P(X = 3) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(1 - \frac{1}{2}\right)^{2} = \frac{10}{32}$
- $P(X = 4) = \binom{5}{4}\left(\frac{1}{2}\right)^4\left(1 - \frac{1}{2}\right)^{1} = \frac{5}{32}$
- $P(X = 5) = \binom{5}{5}\left(\frac{1}{2}\right)^5\left(1 - \frac{1}{2}\right)^{0} = \frac{1}{32}$

Si $X \sim \mathcal{B}(n,p)$ alors $E(X) = np$ et $V(X) = np(1-p)$

**Démonstration**

***Première approche:*** On associe à chaque épreuve $i$, $1 \leqslant i \leqslant n$, une v.a. de Bernoulli.

$$\mathbb{1}_A = X_i = \begin{cases} 1 & \text{si } A \text{ est réalisé} \\ 0 & \text{si } \bar{A} \text{ est réalisé} \end{cases}$$

On peut écrire alors: $X = \sum_{i=1}^{n} X_i = X_1 + X_2 + \ldots + X_n$
Donc

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = np$$

et

$$V(X) = V\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} V(X_i) = np(1-p)$$

car les v.a. $X_i$ sont indépendantes.

***Deuxième approche:*** Calcul direct.

- $E(X) = \sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k} = \ldots = np$
- $V(X) = E(X^2) - E^2(X)$
- Pour obtenir $E(X^2)$ par un procédé de calcul identique, on passe par l'intermédiaire du moment factoriel $E[X(X-1)]$.
- $V(X) = E(X^2) - E^2(X) = E[X(X-1)] + E(X) - E(X^2)$
- $E[X(X-1)] = \sum_{k=0}^{n} k(k-1)\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k} = \ldots = n(n-1)p^2$
- $V(X) = n(n-1)p^2 + np - (np)^2 = np(1-p)$

**Example**

Le nombre de résultats pile apparus au cours de $n$ jets d'une pièce de monnaie suit une loi binomiale $\mathcal{B}(n, 1/2)$:

$$P(X=k) = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} = \frac{\binom{n}{k}}{2^n}, \quad 0 \leqslant k \leqslant n$$

avec $E(X) = n/2$ et $V(X) = n/4$.

**Example**

Le nombre $N$ de boules rouges apparues au cours de $n$ tirages avec remise dans une urne contenant deux rouges, trois vertes et une noire suit une loi binomiale $\mathcal{B}(n, 1/3)$:

$$P(N=k) = \binom{n}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{n-k} = \binom{n}{k}\frac{2^{n-k}}{3^n}, \quad 0 \leqslant k \leqslant n$$

avec $E(X) = n/3$ et $V(X) = 2n/9$.

**Remark**

Si $X_1 \sim \mathcal{B}(n_1, p)$ et $X_2 \sim \mathcal{B}(n_2, p)$, les v.a. $X_1$ et $X_2$ étant **indépendantes**, alors $X_1 + X_2 \sim \mathcal{B}(n_1 + n_2, p)$. Ceci résulte de la définition d'une loi binomiale puisqu'on totalise ici le résultat de $n_1 + n_2$ épreuves indépendantes.

## Loi de Poisson $\mathcal{P}(\lambda)$

## Definition

*Une v.a. $X$ suit une loi de Poisson de paramètre $\lambda > 0$ si c'est une variable à valeurs entières, $X(\Omega) = \mathbb{N}$, donc avec une infinité de valeurs possibles, de probabilité:*

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k \in \mathbb{N}$$

*Cette loi ne dépend qu'un seul paramètre réel positif $\lambda$, avec l'écriture symbolique $X \sim \mathcal{P}(\lambda)$.*

## Remark

$$e^x = \sum_{i=0}^{+\infty} \frac{x^i}{i!}$$

Donc

$$\sum_{k=0}^{\infty} P(X = k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1$$

# Moments de loi de Poisson

Si $X \sim \mathcal{P}(\lambda)$ alors $E(X) = \lambda$ et $V(X) = \lambda$

## Espérance de loi de Poisson

$$\begin{aligned} E(X) &= \sum_{k=0}^{\infty} k P(X = k) \\ &= \dots \\ &= \lambda. \end{aligned}$$

## Variance de loi de Poisson

- On calcule d'abord $E(X^2) = \sum_{k=0}^{\infty} k^2 P(X = k) = \dots = \lambda(\lambda + 1)$.

Ensuite

$$V(X) = \lambda(\lambda + 1) - \lambda^2 = \lambda$$

# Loi de Poisson $\mathcal{P}(\lambda)$

## Example

- $X =$ nombre de micro-ordinateurs vendus chaque jour dans un magasin.
- On suppose $X \sim \mathcal{P}(5)$.
- La probabilité associée à la vente de 5 micro-ordinateurs est

$$P(X=5) = e^{-5}\frac{5^5}{5!} = e^{-5} \simeq 0.1755$$

- La probabilité de vendre au moins 2 micro-ordinateurs est

$$P(X \geqslant 2) = 1 - \left(e^{-5}\frac{5^0}{0!} + e^{-5}\frac{5^1}{1!}\right) \simeq 0.9596$$

- Le nombre moyen de micro-ordinateurs vendus chaque jour dans le magasin est égal à 5 puisque $E(X) = \lambda = 5$.

## Properties

Si $X$ et $Y$ sont deux variables **indépendantes** suivant des lois de Poisson, $X \sim \mathcal{P}(\lambda)$ et $Y \sim \mathcal{P}(\mu)$, alors leur somme suit aussi une loi de Poisson: $X + Y \sim \mathcal{P}(\lambda + \mu)$.

$$\text{Si } n \to \infty \text{ et } p \to 0 \text{ alors } X : \mathcal{B}(n, p) \sim \mathcal{P}(\lambda).$$

**Remark**

Une bonne approximation est obtenue si $n \geqslant 50$ et $np \leqslant 5$.

Dans ce contexte, la loi de Poisson est souvent utilisée pour modéliser le nombre de succès lorsqu'on répète un très grand nombre de fois une expérience ayant une chance très faible de réussir par une loi de Poisson.

**Applications de la loi de Poisson**

- Le nombre d'individus dépassant l'âge de 100 ans dans une communauté.
- Le nombre de faux numéros téléphoniques composés en un jour.
- Le nombre de clients pénétrant dans un bureau de poste donné en l'espace d'un jour.
- Le nombre de particules $\alpha$ émises par un matériau radioactif pendant un certain laps de temps.

La v.a. dans ces exemples est répartie de manière approximativement poissonienne car: on approxime par là une variable binomiale.

# Loi Géométrique ou de Pascal $\mathcal{G}(p)$

- $\varepsilon$ : "On répéte l'épreuve de Bernoulli jusqu'à avoir le premier succès".
- Exemple:

$$\begin{array}{ccccccccc} \bar{A} & \bar{A} & \bar{A} & \bar{A} & \bar{A} & \ldots & \bar{A} & \bar{A} & A \\ E & E & E & E & E & \ldots & E & E & S \end{array}$$

- Chaque épreuve a $p$ pour probabilité de succès et $1-p$ pour probabilité d'échec.
- $X =$"le nombre d'épreuves effectuées".

$$\underbrace{E \quad E \quad E \quad E \quad E \quad \ldots \quad E \quad E}_{k-1} \quad S$$

- $X(\Omega) = \mathbb{N}^* = \{1, 2, 3, \ldots\}$. On dit $X \sim \mathcal{G}(p)$.
- $\forall\, k \in \mathbb{N}^* \quad P(X = k) = (1-p)^{k-1}p$
- Attention: Parfois $X =$ "nombre d'épreuves effectuées avant obtenir le premier succès". Dans ce cas $X(\Omega) = \mathbb{N}$. On dit $X \sim \mathcal{G}(p)$ sur $\mathbb{N}$.
- Cette loi peut servir à modéliser des temps de vie, ou des temps d'attente, lorsque le temps est mesuré de manière discrète (nombre de jours par exemple).
- Série entière : $\sum_{k=0}^{\infty} x^k = 1/(1-x)$ pour $|x| < 1$
- $\sum_{k=1}^{\infty} P(X = k) = \sum_{k=1}^{\infty} (1-p)^{k-1}p = p \sum_{j=0}^{\infty} (1-p)^j \sum_{k=1}^{\infty} (1-p)^{k-1}p = p \sum_{j=0}^{\infty} (1-p)^j = p\,\frac{1}{1-(1-p)} = 1$

## Espérance de loi Géométrique

- $E(X) = \sum_{k=1}^{\infty} kP(X = k) = \sum_{k=1}^{\infty} kp(1-p)^{k-1} = p\sum_{k=1}^{\infty} k(1-p)^{k-1}$
- Série entière: $\sum_{k=0}^{\infty} x^k = 1/(1-x)$ pour $|x| < 1$
- Dérivée première de la série entière: $\sum_{k=1}^{\infty} kx^{k-1} = 1/(1-x)^2$
- Donc $E(X) = \frac{p}{[1-(1-p)]^2} = \frac{1}{p}$

En d'autres termes, si des épreuves indépendantes ayant une probabilité $p$ d'obtenir un succès sont réalisés jusqu'à ce que le premier succès se produise, le nombre espéré d'essais nécessaires est égal à $1/p$. Par exemple, le nombre espéré de jets d'un dé équilibré qu'il faut pour obtenir la valeur 1 est 6.

## Variance de loi Géométrique

- $V(X) = E(X^2) - E^2(X) = E[X(X-1)] + E(X) - E^2(X)$. Or,

$$E[X(X-1)] = \sum_{k=2}^{\infty} k(k-1)p(1-p)^{k-1}$$
$$= p(1-p)\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2}$$

- Dérivée première de la série entière: $\sum_{k=1}^{\infty} kx^{k-1} = 1/(1-x)^2$
- Dérivée seconde de la série entière: $\sum_{k=2}^{\infty} k(k-1)x^{k-2} = 2/(1-x)^3$
- Donc $E[X(X-1)] = \frac{2p(1-p)}{[1-(1-p)]^3} = \frac{2(1-p)}{p^2}$
- Et alors $V(X) = E[X(X-1)] + E(X) - E^2(X) = \frac{1-p}{p^2}$.

# Loi Binomiale Négative $\mathcal{BN}(r, p)$

# Loi Binomiale Négative $\mathcal{BN}(r, p)$

- $\varepsilon$ : "On répéte l'épreuve de Bernoulli jusqu'à obtenir un total de $r$ succès".
- Exemple avec $r = 3$:

$$\begin{matrix} \bar{A} & A & \bar{A} & \bar{A} & \bar{A} & A & \bar{A} & \bar{A} & A \\ E & S & E & E & E & S & E & E & S \end{matrix}$$

- Mais on peut obtenir $r$ succès d'autres façons:

$$S \quad E \quad E \quad E \quad E \quad E \quad S \quad E \quad S$$

$$E \quad E \quad E \quad E \quad S \quad E \quad S \quad E \quad S$$

- Chaque épreuve a $p$ pour probabilité de succès et $1 - p$ pour probabilité d'échec.
- Désignons $X =$"le nombre d'épreuves nécessaires pour attendre ce résultat".

$$\underbrace{\overbrace{E \quad S \quad E \quad E \quad E \quad S \quad E \quad E}^{r-1 \text{ succès et } k-r \text{ échecs}} \quad S}_{X=k}$$

- $X(\Omega) = \{r, r+1, r+2, \ldots\}$. On dit $X \sim \mathcal{BN}(r, p)$.
- $\forall\, k \in X(\Omega)$,

$$P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

- $\varepsilon$ : “On répéte l’épreuve de Bernoulli jusqu’à obtenir un total de $r$ succès”.
- Soit,

$$E \quad \ldots \quad E \quad S \quad E \quad \ldots \quad E \quad S \ldots \quad E \ldots \quad E \quad S$$

- Soit, $Y_1$ le nombre d’épreuves nécessaires jusqu’au premier succès, $Y_2$ le nombre d’épreuves supplémentaires nécessaires pour obtenir un deuxième succès, $Y_3$ celui menant au 3ème et ainsi de suite.
- Càd,

$$\underbrace{E \quad \ldots \quad E \quad S}_{Y_1} \; \underbrace{E \quad \ldots \quad E \quad S}_{Y_2} \; \underbrace{\ldots}_{\ldots} \; \underbrace{E \quad \ldots \quad E \quad S}_{Y_r}$$

- Les tirages étants indépendantes et ayant toujours la même probabilité de succès, chacune des variables $Y_1, Y_2, \ldots, Y_r$ est géométrique $\mathcal{G}(p)$.
- $X =$“le nombre d’épreuves nécessaires à l’obtention de $r$ succès”$= Y_1 + Y_2 + \ldots + Y_r$.
- Donc,

$$E(X) = E(Y_1) + E(Y_2) + \ldots + E(Y_r) = \sum_{i=1}^{r} \frac{1}{p} = \frac{r}{p}$$

et

$$V(X) = \sum_{i=1}^{r} V(Y_i) = \frac{r(1-p)}{p^2}$$

car les $Y_i$ sont indépendantes.