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Lecture 4

Functions of a Random Variable

Examples of Important Probability Densities Derived as Functions of Random Variables

Transformations of Random Variables

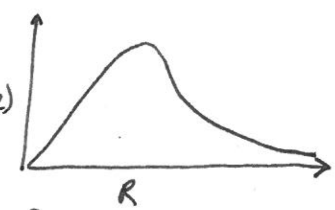
Motivation

$V = I \cdot R$ (Ohm's Law)

$I = \frac{V}{R}$ Assume

$R \sim f_R(R)$

$\Rightarrow I$ is a R.V.



What is $f_I(I)$?

$X \sim f_X(x)$

We consider

$Y = aX + b$ (Linear) in general

$Y = |X|$ (Absolute Value) $X \sim N(\mu, \sigma^2)$

$Y = X^2$ (Quadratic) $X \sim N(\mu, \sigma^2)$

$Y = g(X)$ (Monotonic) monotonic general

We want to find $f_Y(y)$ and $F_Y(y)$

i) Find $F_Y(y)$ in terms of $F_X(x)$

ii) Apply the chain rule to find $f_Y(y) = F_Y'(y)$

Ex. 4.1 $X \sim f_X(x)$ $Y = aX + b$

Assume $x \in (-\infty, \infty)$

$$F_Y(y) = Pr(Y \le y)$$
$$= Pr(aX + b \le y)$$
$$= Pr\left(X \le \frac{y-b}{a}\right)$$
$$= F_X\left(\frac{y-b}{a}\right) = \int_{-\infty}^{\frac{y-b}{a}} f_X(x)\,dx$$

To find $f_Y(y)$ consider two cases

i) $a > 0$

Let $z = F_Y(y)$ $g(\ ) = F_X(\ )$

$h(y) = \frac{y-b}{a}$

$$f_Y(y) = \frac{dz}{dy} = F_X'\left(\frac{y-b}{a}\right) \cdot \frac{d\left(\frac{y-b}{a}\right)}{dy}$$

Recall Chain Rule for Differentiation

$z = g(h(y))$

$$\frac{dz}{dy} = g'(h(y)) \cdot \frac{d(h(y))}{dy}$$
$$= g'(h(y)) \cdot h'(y)$$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

ii) $a < 0$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

$a<0 \Rightarrow f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} < 0$ because $f_X(x) \geq 0$ p.d.f.

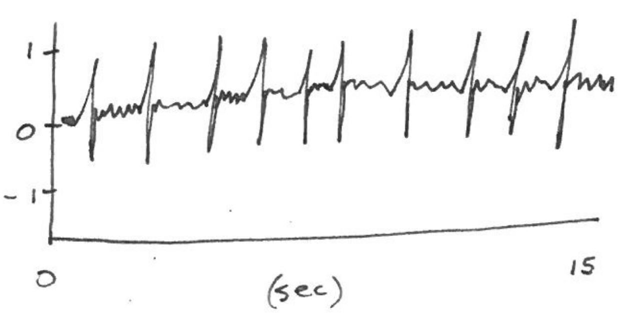
$\Rightarrow f_Y(y)$ is not a p.d.f. Therefore, multiply by $-\frac{1}{a}$ if $a<0$

$$f_Y(y) = \begin{cases} \frac{1}{a} f_X\left(\frac{y-a}{b}\right) & a > 0 \\ -\frac{1}{a} f_Y\left(\frac{y-b}{a}\right) & a < 0 \end{cases} = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$\boxed{\text{Ex 4A} \quad X \sim \text{Gamma } \alpha; \beta = 1 \quad \text{Find} \quad Y = \beta X \Rightarrow \beta \text{ scale parameter}}$

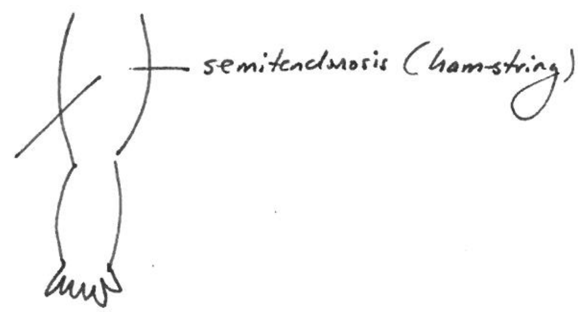
Question: Suppose $a = 0$ Find $f_Y(y)$.

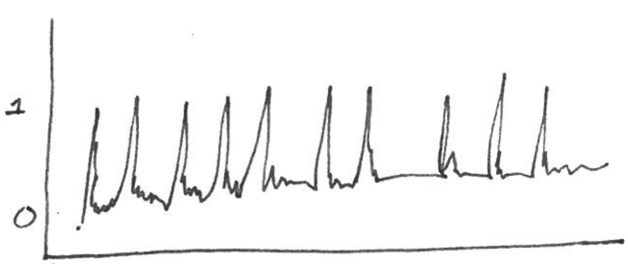
Ex. 4.1 Special Case of Transformation of Variables in Prop. 4.1



Raw

X





Rectified

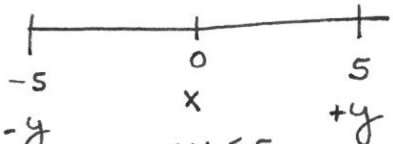
$Y = |X|$

Fig 4 A

Ex 4.2 Assume $X \sim N(\mu, \sigma^2)$ Take $Y = |X|$

Find $F_Y(y)$ and $f_Y(y)$

$|X| \le 5$



$-5 \le X \le 5$

$|X| \le y$

$-y \le x \le y$

$F_Y(y) = Pr(|X| \le y)$ equivalent events

$= Pr(-y \le X \le y)$

$= Pr(-y-\mu \le X-\mu \le y-\mu)$

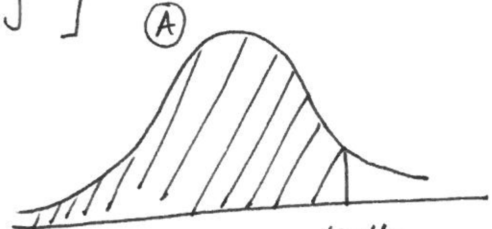
$= Pr\left(\frac{-y-\mu}{\sigma} \le \overset{z}{\frac{X-\mu}{\sigma}} \le \frac{y-\mu}{\sigma}\right) = \boxed{\cancel{F_X\left(\frac{y-\mu}{\sigma}\right) - F_X\left(\frac{-y-\mu}{\sigma}\right)}}$

standard Gaussian c.d.f.

$= \Phi\left(\frac{y-\mu}{\sigma}\right) - \Phi\left(\frac{-y-\mu}{\sigma}\right)$ ← this is the c.d.f of Y.

$= \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \left[\underbrace{\int_{-\infty}^{\frac{y-\mu}{\sigma}} \exp\left\{-\frac{1}{2}x^2\right\} dx}_{A} - \underbrace{\int_{-\infty}^{\frac{-y-\mu}{\sigma}} \exp\left\{-\frac{1}{2}x^2\right\} dx}_{B} \right]$

A



B

Find $f_Y(y)$

$z = g(h(y)) = f_Y(y)$

$g(\ ) = \Phi(\ )$ $g' = \Phi' = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2}x^2\right\}$

$h(y) = \frac{y-\mu}{\sigma}$ $\frac{dh}{dy} = \frac{1}{\sigma}$

$h(y) = \frac{-y-\mu}{\sigma}$ $\frac{dh}{dy} = -\frac{1}{\sigma}$

$f_Y(y) = \phi\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} - \left(-\frac{1}{\sigma}\right)\phi\left(\frac{-y-\mu}{\sigma}\right)$

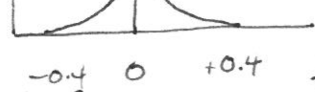
$= \frac{1}{\sigma}\left[\phi\left(\frac{y-\mu}{\sigma}\right) + \phi\left(\frac{-y-\mu}{\sigma}\right)\right]$

half-normal (Gaussi

If $\mu = 0$ $f_Y(y) = \frac{2}{\sigma}\phi\left(\frac{y}{\sigma}\right)$ $y > 0$

N. B. Not the p.d.f of the sum of two Gaussian random variables

$f_X(x)$



Ex. 4.3

Is $f_X(x)$ Gaussian?

$f_Y(y)$

Ex. 4.3 $X \sim N(0,1)$ $X \in (-\infty, \infty)$ $Y = X^2$ Find $F_Y(y)$ and $f_Y(y)$

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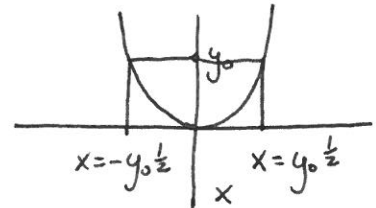


$Y \le y_0$

$X^2 \le y_0$

$-y_0^{\frac{1}{2}} \le X \le y_0^{\frac{1}{2}}$

$F_Y(y) = Pr(Y \le y)$

$= Pr(X^2 \le y)$

$= Pr(-y^{\frac{1}{2}} \le X \le y^{\frac{1}{2}})$

$= F_X(y^{\frac{1}{2}}) - F_X(-y^{\frac{1}{2}})$

$= \Phi(y^{\frac{1}{2}}) - \Phi(-y^{\frac{1}{2}})$

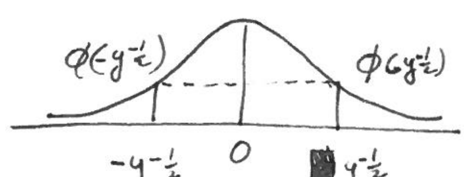
On differentiating

$z = g(h(y)) = F_Y(y)$ $\quad f_Y(y) = \frac{dz}{dy} = g'(h(y)) \cdot \frac{dh}{dy}$

$g(\ ) = \Phi(\ )$ $\quad g'(\ ) = \phi$

$h(y) = y^{\frac{1}{2}}$ $\quad \frac{dh}{dy} = \frac{1}{2} \cdot y^{-\frac{1}{2}}$

$h(y) = -y^{\frac{1}{2}}$ $\quad \frac{dh}{dy} = -\frac{1}{2} y^{-\frac{1}{2}}$



$f(y) = \frac{1}{2} y^{-\frac{1}{2}} \left[\phi(y^{\frac{1}{2}}) + \phi(-y^{\frac{1}{2}})\right]$

$= y^{-\frac{1}{2}} \phi(y^{\frac{1}{2}}) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} y^{-\frac{1}{2}} \exp\{-\frac{1}{2} y\}$

$\Rightarrow \Gamma(\alpha = \frac{1}{2}, \beta = \frac{1}{2})$ $\quad \chi^2_{(1)}$

Lecture 3

Lecture 3 Addendum

Factoid 2

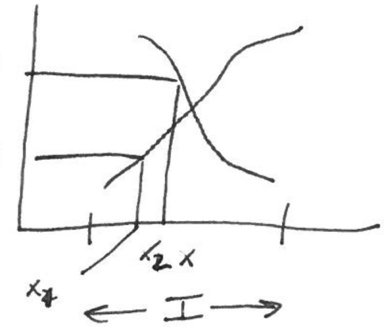
$(2\pi)^{\frac{1}{2}} = \Gamma(\frac{1}{2})$

$y^{-\frac{1}{2}} = y^{\frac{1}{2} - 1}$ $\quad \alpha = \frac{1}{2}$

$\exp\{-\frac{1}{2} y\}$ $\quad \beta = 1/2$

Proposition 4.1 X continuous R.V. w/p.d.f $f_X(x)$ Let $Y = g(X)$ g is differentiable and strictly monotonic on some interval I. Suppose $f(x) = 0$, if $x \notin I$. Then

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right|$$



Proof: $F_Y(y) = Pr(Y \le y)$

$= Pr(g(X) \le y)$

$= Pr(X \le g^{-1}(y))$ 1-to-1 ($\Rightarrow$ inverse exists)

$= F_X(g^{-1}(y))$

$x_1 < x_2$

$g(x_1) < g(x_2)$

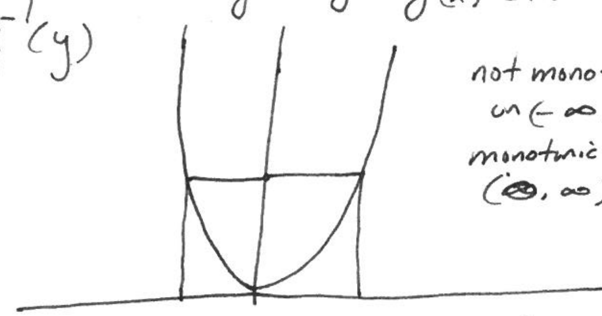
$g(x_1) > g(x_2)$

$\Rightarrow$ $F_Y(y) = z = g(h(y))$ $g = F_X(\ )$

$f_Y(y) = F_X'(g^{-1}(y)) \cdot \frac{d[g^{-1}(y)]}{dy}$ h[redacted]$(y) = g^{-1}(y)$

$= g'(h(y)) \cdot \frac{d(g^{-1}(y))}{dy}$

$y = x^2 = g(x)$ $g'(x)$ exists $\forall x \in I$



not monotonic on $(-\infty, \infty)$

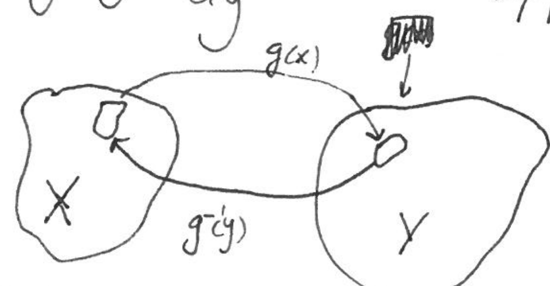
monotonic on $(0, \infty)$

If $\frac{d(g^{-1}(y))}{dy} < 0$ then multiply by [redacted] $-\frac{d(g^{-1}(y))}{dy}$

$x = g^{-1}(y) = y^{\frac{1}{2}}$

This mapping is 1-1

Intuitive Derivation



$f_Y(y)\,dy = f_X(x)\,dx$

$f_Y(y) = f_X(x)\frac{dx}{dy}$

$x = g^{-1}(y)$

$\Rightarrow f_Y(y) = f_X(g^{-1}(y)) \frac{d\,g^{-1}(y)}{dy}$

Ex. 4A $X \sim \beta(\alpha, \beta)$

[redacted] $Y = -X^{\frac{1}{3}}$ $y \in (-1, 0)$

$x = -y^3$ $\frac{dx}{dy} = -3y^2$

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

$$f_Y(y) = 3\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(y^3)^{\alpha-1}(1-y^3)^{\beta-1} \cdot y^2$$