9.07 Introduction to Probability and Statistics for Brain and Cognitive Sciences

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Addendum to Lecture 3: Continuous Probability Models

A. Gamma Function Factoids

The gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

Factoid 1 (Generalized Factorial)

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$$

This result is easy to establish using integration by parts. In particular if $\alpha = n$, an integer we have

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1)\\ &= (n-1)(n-2)\Gamma(n-2)\\ &= (n-1)(n-2)\cdots 2\ 1 = (n-1)!\end{aligned}$$

since $\Gamma(1) = \int_0^\infty e^{-x} dx = 1.$

Factoid 2

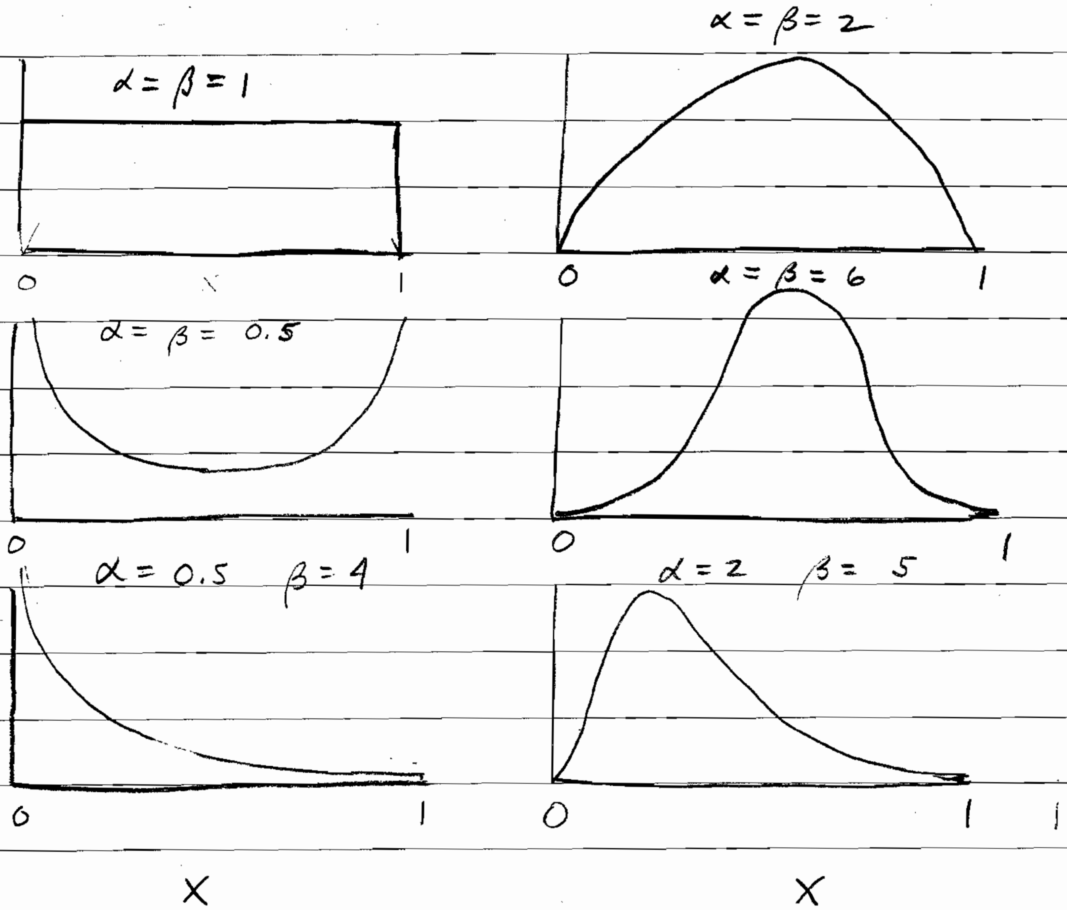
$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Factoid 3 If n is odd

$$\Gamma\left(\frac{n}{2}\right) = \frac{\sqrt{\pi}\,(n-1)!}{2^{n-1}\left(\frac{n-1}{2}\right)!}$$

B. Beta Distribution

1. Shapes of the Beta Distribution



$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad x \in (0, 1)$$

$$\alpha > 0 \quad \beta > 0$$

2. Evaluating the normalizing constant for the beta distribution.

We want to show that

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

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Consider the "trick" of writing

$$\Gamma(\alpha)\,\Gamma(\beta) = \int_0^\infty u^{\alpha-1} e^{-u}\,du \int_0^\infty v^{\alpha-1} e^{-v}\,dv.$$

$$= \int_0^\infty \int_0^\infty u^{\alpha-1} v^{\beta-1} e^{-(u+v)}\,du\,dv$$

Make the change of variables

$x = u\,(u+v)^{-1}$

$y = u+v$

then

$$xy = u\,(u+v)^{-1}(u+v) = u \qquad\qquad y = u+v$$

$$= xy + v$$

$u \to \infty \Rightarrow x = 1$

$x = \dfrac{u}{u+v}$ $\quad$ u fixed

$v \to \infty \Rightarrow x \approx 0$ $\qquad y - xy = v$

$u = 0$

v fixed $\Rightarrow x = 0$ $\qquad y(1-x) = v$

Recall from Calculus that the Jacobian of the transformation is

$$J(x,y) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

is the determinant of J. Now,

where $|J|_A$ = under the transformation (change-of-variables) the integral is

$$\iint_R f(u,v)\,du\,dv = \iint_{R'} f(u(x,y), v(x,y))\,|J(x,y)|\,dx\,dy$$

We have $\dfrac{du}{dx} = y \qquad \dfrac{du}{dy} = x \qquad \dfrac{dv}{dx} = -y \qquad \dfrac{dv}{dy} = (1-x)$

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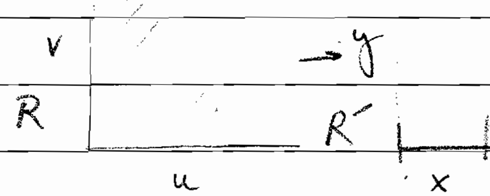
$ad - bc$

$$J(x,y) = \begin{vmatrix} y & x \\ -y & (1-x) \end{vmatrix} = y - xy + xy = y \qquad \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

The regim R is $u \in (0, \infty)$ $v \in (0, \infty)$ and the region R' is $x \in (0, 1)$ $y \in (0, \infty)$. This is because $u = v = 0$ or u fixed and $v \to \infty$ $\Rightarrow x = 0$ and $u \to \infty$ for v fixed $\Rightarrow x = 1$. Because $y = u + v$ the range of y is the union of the ranges of u and v which is $(0, \infty)$.



Hence,

$$\Gamma(\alpha)\,\Gamma(\beta) = \int_0^\infty \int_0^\infty u^{\alpha-1} v^{\beta-1} e^{-(u+v)}\, du\, dv$$

$$= \int_0^\infty \int_0^1 (xy)^{\alpha-1} [(1-x)y]^{\beta-1} e^{-y} \cdot y\, dx\, dy$$

$$= \int_0^\infty y^{\alpha+\beta-1} e^{-y} dy \cdot \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx$$

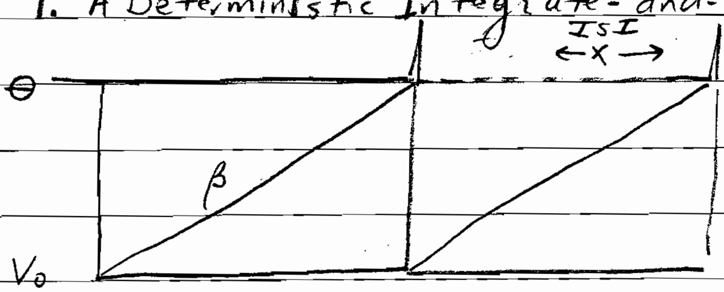
$$= \Gamma(\alpha+\beta) \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx$$

Or

$$\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

C. Inverse Gaussian Distribution

1. A Deterministic Integrate-and-Fire Neuron



$V(t) = V_0 + \beta t$

If $\Theta = V_0 + \beta t$ then

$t = \beta^{-1}(\Theta - V_0)$ is when a spike is fired

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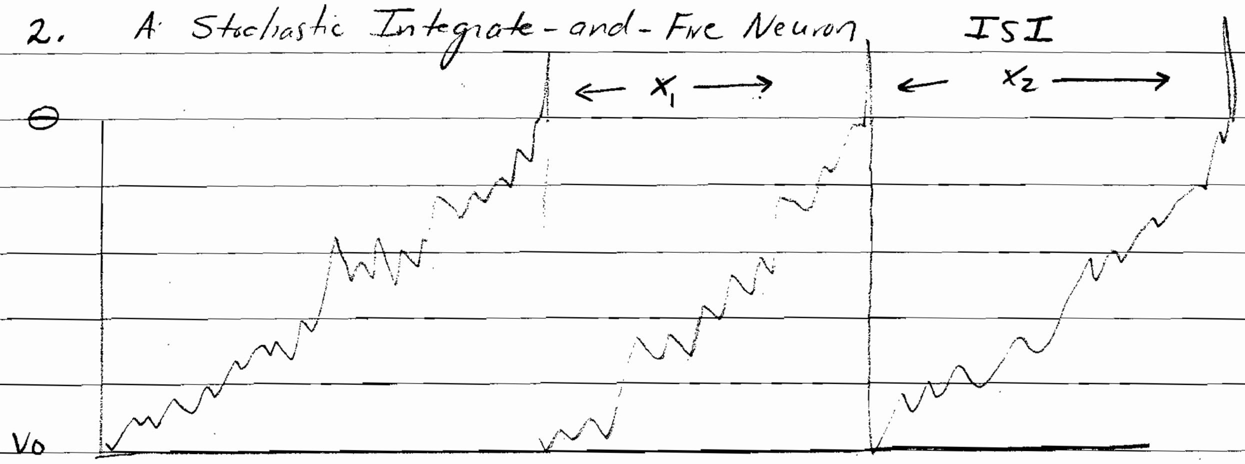
2. A Stochastic Integrate-and-Fire Neuron



$$V(t) = V_0 + \beta t + W(t)$$

where $W(t)$ a Wiener process (time-dependent) Gaussian r.v. with $E(W(t)) = 0$ and $Var(W(t) - W(s)) = \sigma^2(t-s)$ for $s < t$. The ISI's are now random variables and their probability density is the inverse Gaussian defined as

$$f(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\frac{\lambda(x-\mu)^2}{\mu^2 x}\right\} \qquad \mu > 0 \quad \lambda > 0$$

$$E(X) = \mu$$

$$Var(X) = \mu^3 \lambda^{-1}$$

It turns out that

$$\mu = \frac{(\Theta - V_0)}{\beta} \quad \text{and} \quad \lambda = \frac{(\Theta - V_0)^2}{\sigma^2}$$