We want to show that if (joint density)

$$X, Y \sim BN(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$$

then (marginal densities)

$$X \sim N(\mu_x, \sigma_x^2) \qquad Y \sim N(\mu_y, \sigma_y^2)$$

We show this by computing

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

Mathematical Interlude

Factoid 1: Completing the Square

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Take $\left(\frac{1}{2}\frac{b}{a}\right)^2$ add it in and subtract it out to obtain the completed square in x

$$x^2 + \frac{b}{a}x + \frac{c}{a} + \frac{1}{4}\frac{b^2}{a^2} - \frac{1}{4}\frac{b^2}{a^2} = 0$$

$$x^2 + \frac{b}{a}x + \frac{1}{4}\frac{b^2}{a^2} + \frac{c}{a} - \frac{1}{4}\frac{b^2}{a^2} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{1}{4}\frac{b^2}{a^2} - \frac{c}{a} \quad \Rightarrow \quad x = -\frac{b}{2a} \pm \left(\frac{b^2 - 4ac}{2a}\right)^{\frac{1}{2}}$$

$\boxed{\text{Quadratic Formula}}$

Factoid 2:

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\frac{x^2}{\sigma^2}\right\} dx = (2\pi\sigma)^{\frac{1}{2}}$$

the normalizing constant for a Gaussian density $X \sim N(0, \sigma^2)$

Factoid 3:

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\} dx = \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\frac{x'^2}{\sigma^2}\right\} dx$$

$x' = x - \mu \qquad x \in (-\infty, \infty)$

$dx' = dx \qquad x' \in (-\infty, \infty)$

Shifting x by μ does not change the value of the integral

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= [2\pi\sigma_x\sigma_y(1-\rho^2)^{\frac{1}{2}}]^{-1} \times \int_{-\infty}^{\infty} \exp\left\{-\tfrac{1}{2}(1-\rho^2)\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\} dy$$

Let $u = \frac{x-\mu_x}{\sigma_x}$ $\quad v = \frac{y-\mu_y}{\sigma_y}$ $\quad x \in (-\infty,\infty) \Rightarrow u \in (-\infty,\infty)$

and $\quad y \in (-\infty,\infty) \Rightarrow v \in (-\infty,\infty)$

$dv = \frac{dy}{\sigma_y}$

Substituting yields

$$f_X(x) = [2\pi\sigma_x(1-\rho^2)^{\frac{1}{2}}]^{-1} \times \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{1}{1-\rho^2}\right)[u^2+v^2-2\rho uv]\right\} dv$$

Now complete the square in v (because we are integrating w.r.t. v)

$v^2+u^2-2\rho uv$

$= v^2+u^2-2\rho uv + \left(\frac{1}{2}\cdot 2\rho u\right)^2 - \frac{1}{2}(2\rho u)^2 = v^2 - 2\rho u + u^2 + \rho^2u^2 - \rho^2u^2$

$= v^2 - 2\rho uv + \rho^2u^2 + u^2 - \rho^2u^2 = (v-\rho u)^2 + u^2(1-\rho^2)$

Substituting back gives

$$f_X(x) = \frac{1}{2\pi\sigma_x(1-\rho^2)^{-\frac{1}{2}}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[(v-\rho u)^2 + u^2(1-\rho^2)\right]\right\} dv$$

$$= \left(\frac{1}{2\pi\sigma_x^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\frac{u^2(1-\rho^2)}{(1-\rho^2)}\right\} \cdot \left(\frac{1}{2\pi(1-\rho^2)}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left\{-\tfrac{1}{2}(1-\rho^2)[v-\rho u]^2\right\} dv$$

Apply Factoid 3

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\frac{[v-\rho u]^2}{(1-\rho^2)}\right\} dv = \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\frac{v'^2}{(1-\rho^2)}\right\} dv'$$

Apply Factoid 2

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\frac{v'^2}{(1-\rho^2)}\right\} dv' = (2\pi(1-\rho^2))^{\frac{1}{2}}$$

Hence

$$f_X(x) = \left(\frac{1}{2\pi\sigma_X^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2}u^2\right\} \cdot \frac{1}{[2\pi(1-\rho^2)]^{\frac{1}{2}}} \cdot [2\pi(1-\rho^2)]^{\frac{1}{2}}$$

$$= \left(\frac{1}{2\pi\sigma_X^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X^2}\right)^2\right\}$$

Because $u = \left(\frac{x-\mu_X}{\sigma_X}\right)$

Q. E. D.
C.F.Q.D.