On radial RSD

we start from the redshift space distortions (RSD) map:

$$\vec{s} = \vec{x} - f\,\hat{x}\;\vec{u}\cdot\hat{x}$$

where $\hat{x}$ denotes the radial coordinate direction

and do the continuous case first:

$$\delta_D(\vec{k}) + \delta_s(\vec{k}) = \int e^{-i\vec{k}\cdot\vec{x}} \frac{d^3x}{(2\pi)^3}\, e^{i f(\vec{k}\cdot\hat{x})(\vec{u}\cdot\hat{x})} [1+\delta(\vec{x})]$$

$$= \int e^{-i\vec{k}\cdot\vec{x}} \frac{d^3x}{(2\pi)^3}\, e^{i f k_i u_j \hat{x}_i \hat{x}_j} [1+\delta(\vec{x})]$$

using

$(1+\delta_s(\vec{s}))\, d^3s = (1+\delta(\vec{x}))\, d^3x$

$\times \int e^{-i\vec{k}\cdot\vec{s}}$ both sides

in linear perturbation theory, we have then

$$\delta_s(\vec{k}) = \delta(\vec{k}) + \int \frac{d^3x}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}}\, i f k_i u_j \hat{x}_i \hat{x}_j$$

with $\vec{u}(\vec{x}) = \int \frac{-i\vec{k}}{k^2}\theta(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} d^3k$

then

$$\delta_s(\vec{k}) = \delta(\vec{k}) + f\int d^3q\; \theta(\vec{q})\, \frac{k_i q_j}{q^2} \underbrace{\int \frac{d^3x}{(2\pi)^3} e^{-i(\vec{k}-\vec{q})\cdot\vec{x}}\, \hat{x}_i \hat{x}_j}_{\equiv I_{ij}(\vec{k}-\vec{q})}$$

The object I_{ij} can be calculated to give,

$$I_{ij}(\vec{p}) = \frac{1}{3}\delta_{ij}\,\delta_D(\vec{p}) + \frac{H(p)}{\left(\frac{4\pi}{3}p^3\right)}\left(\frac{1}{3}\delta_{ij} - \hat{p}_i \hat{p}_j\right)$$

$H(p)$: Heaviside

So we have,

$$\boxed{\delta_s(\vec{k}) = \delta(\vec{k}) + f\int d^3q\; \theta(\vec{q})\, \frac{k_i q_j}{q^2}\, I_{ij}(\vec{k}-\vec{q})}$$

Note: The fact that the 2nd term is not proportional to $\theta(\vec{k})$ is purely due to radial character, in plane-parallel (PP) limit we have

$I_{ij}(\vec{k}-\vec{q}) = \hat{z}_i \hat{z}_j\, \delta_D(\vec{k}-\vec{q})$

$\Rightarrow \delta_s(\vec{k}) = \delta(\vec{k}) + f\mu_k^2\,\theta(\vec{k})$

We can also write things in configuration space,

$$\delta_s(\vec{x}) = \delta(\vec{x}) + f\int d^3k\; e^{i\vec{k}\cdot\vec{x}} \int \frac{d^3x'}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}'}\, i k_i u_j' \hat{x}_i' \hat{x}_j'$$

$$= \delta(\vec{x}) + f\int \frac{d^3x'}{(2\pi)^3}\, \hat{x}'\cdot\vec{u}(\vec{x}')\;\; \hat{x}'\cdot\vec{\nabla} \underbrace{\int d^3k\; e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}_{(2\pi)^3\,\delta_D(\vec{x}-\vec{x}')}$$

$$\Rightarrow \quad \boxed{\delta_s(\bar{x}) = \delta(\bar{x}) + f\, \bar{\nabla}\cdot\left[\hat{x}\,(\hat{x}\cdot\bar{u})\right]}$$ ②

This is the simplest way to write down the linear RSD formula, equally valid in radial and PP case (taking $\hat{x}\to\hat{z}$).

We can massage this a bit more, using that

$$\nabla_i \hat{x}_i = \frac{2}{|\bar{x}|} \qquad \text{(this vanishes in PP case)}$$

$$\nabla_i \hat{x}_j = \nabla_i \frac{x_j}{|\bar{x}|} = \frac{\delta_{ij} - \hat{x}_i\hat{x}_j}{|\bar{x}|}$$

$$\Rightarrow \bar{\nabla}_i\left[\hat{x}_i\hat{x}_j u_j\right] = \frac{2}{|\bar{x}|}\hat{x}_j u_j + \underbrace{\hat{x}_i \frac{\delta_{ij} - \hat{x}_i\hat{x}_j}{|\bar{x}|}}_{=0\ !} + \hat{x}_i\hat{x}_j\nabla_i u_j$$

$$\underset{\substack{\uparrow \\ \hat{x}_i\hat{x}_j\nabla_{ij} = \partial^2/\partial r^2}}{=} \frac{2}{r}u_r + \frac{\partial u_r}{\partial r} \qquad (u_r \equiv \hat{x}\cdot\bar{u})$$

So written in radial coordinates we have,

$$\boxed{\delta_s(\bar{x}) = \delta(\bar{x}) + f\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right]\nabla^{-2}\theta(\bar{x})}$$

This agrees (upon using $\theta=\delta$ in linear theory) with the well-known Hamilton & Culhane (1996) formula for the radial RSD operator:

$$1 + f\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right]\nabla^{-2}$$

All this of course can be written in Kaiser's paper notation as

$$\delta_s(\bar{x}) = \delta(\bar{x}) + f\left(\frac{2}{r}u_r + \frac{\partial u_r}{\partial r}\right)$$

Now, let's step back and rederive the Fourier-space result in a more convenient form, i.e. discrete rather than continuos Fourier transform.

The discrete Fourier transform (DFT) is just

$$\delta_{\vec{k}} = \frac{1}{N_g^3} \sum_{\vec{x}} e^{-i\vec{k}\cdot\vec{x}} \, \delta(\vec{x})$$

where we denote the DFT of $\delta(\vec{x})$ by $\delta_{\vec{k}}$ and the continuous FT by $\delta(\vec{k})$, note that they have different dimensions.

In the continuous limit we have

$$\frac{1}{N_g^3} \sum_{\vec{x}} \rightarrow \underbrace{\left(\frac{2\pi}{L}\right)^3}_{\equiv k_F^3} \int \frac{d^3x}{(2\pi)^3}$$

so $\delta_{\vec{k}} \rightarrow k_F^3 \, \delta(\vec{k})$

$\left(\frac{L}{\delta x}\right)^3 = N_g^3$

[For a box of side L we have $\vec{k}_i = \frac{2\pi}{L}(i_1, i_2, i_3)$
$i_j = 0, \dots, \pm N_g/2$
$\vec{x}_i = \frac{L}{N_g}(j_1, j_2, j_3)$ $j_i = 1, \dots, N_g$]

Now for continuous case we have,

$$\langle \delta(\vec{k}) \, \delta(\vec{k}') \rangle = \underbrace{\delta_D(\vec{k}+\vec{k}')}_{\text{by transl. invariance}} \, P(k)$$

↑ power spectrum (depends only on $|\vec{k}|$ due to isotropy)

while in discrete case,

$$\langle \delta_{\vec{k}} \, \delta_{\vec{k}'} \rangle = \frac{1}{N_g^6} \sum_{\vec{x},\vec{x}'} e^{-i\vec{k}\cdot\vec{x}} \, e^{-i\vec{k}'\cdot\vec{x}'} \underbrace{\langle \delta(\vec{x}) \, \delta(\vec{x}') \rangle}_{\xi(|\vec{x}-\vec{x}'|)}$$

↑ again by transl. inv.: $\vec{x}-\vec{x}'$
$|\vec{x}-\vec{x}'|$: isotropy

let $\vec{r} \equiv \vec{x}-\vec{x}'$

$$\Rightarrow \langle \delta_{\vec{k}} \, \delta_{\vec{k}'} \rangle = \underbrace{\frac{1}{N_g^3} \sum_{\vec{x}'} e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}'}}_{\delta^K_{\vec{k},-\vec{k}'}} \; \underbrace{\frac{1}{N_g^3} \sum_{\vec{r}} e^{-i\vec{k}\cdot\vec{r}} \, \xi(r)}_{k_F^3 P(k) \equiv \hat{P}(k)}$$

(dimensionless power)

where δ^K is the Kronecker delta.

$$\Rightarrow \boxed{\langle \delta_{\vec{k}} \, \delta_{\vec{k}'} \rangle = \delta^K_{\vec{k},-\vec{k}'} \, k_F^3 P(k) = \delta^K_{\vec{k},-\vec{k}'} \, \hat{P}(k)}$$

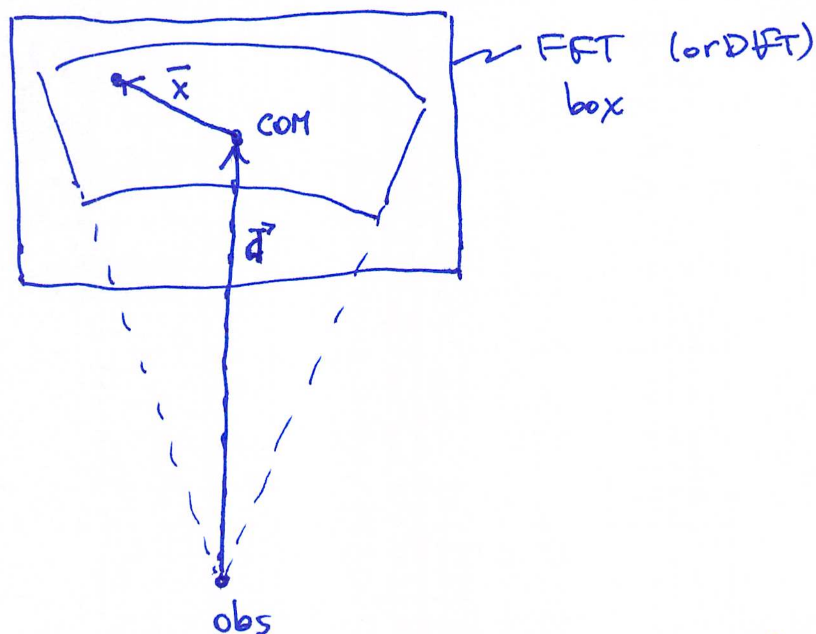
in continuous limit $\delta^K_{\vec{k},\vec{k}'} \, k_F^3 \rightarrow \delta_D(\vec{k}+\vec{k}')$

OK, let's now start from configuration space RSD fluctuations

and do the DFT of it

$$\delta^s_{\vec{k}} = \delta_{\vec{k}} + f \frac{1}{N_g^3} \sum_{\vec{x}} \vec{\nabla} \cdot [\hat{x} (\hat{x} \cdot \vec{u})] \, e^{-i \vec{k} \cdot \vec{x}}$$

Now, here we must be careful, since the box in which the survey or data is may not include the observer, e.g.



For the fields $\delta(\vec{x})$ and $u(\vec{x})$ we can use any coordinates we want, when doing an FFT they will be typically going from 0 to L in each dimension (for a cubic box) or at most $-\frac{L}{2}$ to $\frac{L}{2}$ -

But of course the radial direction will not be the unit vector of such coordinates. So here for simplicity we take coordinates in the box which are centered at the "COM" (center of mass) of the survey [so for a box coordinates go from $-\frac{L}{2}$ to $\frac{L}{2}$ in each direction]. Let $\vec{d}$ be the location of COM with respect to the observer. Then we have:

$$\hat{x} \longrightarrow \frac{\vec{x} + \vec{d}}{|\vec{x} + \vec{d}|}$$

So to be more precise, what we have is

$$\delta^s_k = \delta_k + f \frac{1}{N_g^3} \sum_{\bar{x}} \nabla \cdot \left[\frac{\bar{x}+\bar{d}}{|\bar{x}+\bar{d}|} \left(\frac{\bar{x}+\bar{d}}{|\bar{x}+\bar{d}|} \cdot \bar{u}_{(x)} \right) \right] e^{-ik\cdot\bar{x}}$$

in the limit $d \to \infty$ we recover the PP approximation

$$\hat{r} \equiv \frac{(\bar{x}+\bar{d})}{|\bar{x}+\bar{d}|} \to \hat{d} \qquad \bar{r} \equiv \bar{x}+\bar{d}$$

also it is recovered if range of $\bar{x}$ is $\ll d$ (narrow survey)

So using the $\hat{r}$ notation we have

$$\delta^s_k = \delta_k + f \frac{1}{N_g^3} \sum_{\bar{x}} \bar{\nabla} \cdot [\hat{r} (\hat{r} \cdot \bar{u})] e^{-ik\cdot\bar{x}}$$

Now, we use (again) that

$$\nabla_i \hat{r}_j = \frac{\delta_{ij} - \hat{r}_i \hat{r}_j}{|\bar{x}+\bar{d}|}$$

$$\Rightarrow \bar{\nabla} \cdot [\hat{r} (\hat{r} \cdot \bar{u})] = \frac{2 \hat{r} \cdot \bar{u}}{|\bar{x}+\bar{d}|} + \hat{r}_i \hat{r}_j \cdot \nabla_i u_j \qquad (*)$$

Or we can do instead

$$\bar{\nabla} \cdot [\hat{r} (\hat{r} \cdot \bar{u})] e^{-ik\cdot\bar{x}} = \bar{\nabla} \cdot \{ \hat{r} (\hat{r} \cdot \bar{u}) e^{-ik\cdot\bar{x}} \} + i \bar{k} \cdot \hat{r} (\hat{r} \cdot \bar{u}) e^{-ik\cdot\bar{x}}$$

$$\Rightarrow \delta^s_{\vec{k}} = \delta_{\vec{k}} + i f \frac{1}{N_g^3} \sum_{\bar{x}} k \cdot \hat{r} (\hat{r} \cdot \bar{u}) e^{-ik\cdot\bar{x}} + \underbrace{f \frac{1}{N_g^3} \sum_{\bar{x}} \bar{\nabla} \cdot [\hat{r} (\hat{r} \cdot \bar{u}) e^{-ik\cdot\bar{x}}]}$$

in continuous case this is just a boundary term that vanishes, not sure we can drop this now.

$$\Rightarrow \delta^s_k = \delta_{\vec{k}} + f \frac{1}{N_g^3} \sum_{\bar{x}} \sum_{\bar{q}} \frac{\bar{q} \cdot \hat{r}}{q^2} \, k \cdot \hat{r} \, \theta_{\bar{q}} \, e^{-i(\bar{k}-\bar{q})\cdot\bar{x}}$$

$+ B.T.$

"boundary term"

$$\Rightarrow \delta^s_{\vec{k}} = \delta_{\vec{k}} + f \sum_{\vec{q}} \theta_{\vec{q}} \frac{q_j k_i}{q^2} \underbrace{\frac{1}{N_g^3} \sum_{\vec{x}} \hat{r}_i \hat{r}_j e^{-i(\vec{k}-\vec{q})\cdot\vec{x}}}_{\equiv J_{ij}(\vec{k}-\vec{q})} + \text{B.T.}$$

$$\Rightarrow \boxed{\delta^s_{\vec{k}} = \delta_{\vec{k}} + f \sum_{\vec{q}} \theta_{\vec{q}} \frac{k_i q_j}{q^2} J_{ij}(\vec{k}-\vec{q}) + \text{B.T.}}$$

We can go back to Eq. (*) in previous page and derive an alternative result without boundary terms

$$\delta^s_{\vec{k}} = \delta_{\vec{k}} + f \frac{1}{N_g^3} \sum_{\vec{x}} \hat{r}_i \hat{r}_j \nabla_i u_j e^{-i\vec{k}\cdot\vec{x}} + 2f \frac{1}{N_g^3} \sum_{\vec{x}} \frac{\hat{r}\cdot\vec{u}}{r} e^{-i\vec{k}\cdot\vec{x}}$$

$$\Rightarrow \delta^s_{\vec{k}} = \delta_{\vec{k}} + f \sum_{\vec{q}} \theta_{\vec{q}} \frac{q_i q_j}{q^2} J_{ij}(\vec{k}-\vec{q}) - 2if \sum_{\vec{q}} \theta_{\vec{q}} \frac{\sum_{\vec{x}} \frac{\hat{r}\cdot\vec{q}}{q^2 r} e^{-i(\vec{k}-\vec{q})\cdot\vec{x}}}{N_g^3}$$

$$\Rightarrow \delta^s_{\vec{k}} = \delta_{\vec{k}} + f \sum_{\vec{q}} \theta_{\vec{q}} \frac{q_i q_j}{q^2} J_{ij}(\vec{k}-\vec{q}) - 2if \sum_{\vec{q}} \theta_{\vec{q}} \frac{q_i}{q^2} \left(\frac{1}{N_g^3} \sum_{\vec{x}} \frac{\hat{r}_i}{r} e^{-i(\vec{k}-\vec{q})\cdot\vec{x}} \right)$$

from this expression we can identify what the boundary term is. We are going to use the formula above with B.T. $=0$ in what follows (since B.T. $=0$ in continuous limit).

So, we take $\delta^s_{\vec{k}} = \delta_{\vec{k}} + f \sum_{\vec{q}} \theta_{\vec{q}} \frac{k_i q_j}{q^2} J_{ij}(\vec{k}-\vec{q})$

Now, consider the (artificial) case when there is only one mode in the fluctuation spectrum, say $\vec{p}$. We have $(\vec{k}\neq\vec{p})$

$$\delta^s_{\vec{k}} = f\, \theta_{\vec{p}} \frac{k_i p_j}{p^2} J_{ij}(\vec{k}-\vec{p})$$

So, even though the real-space spectrum has only $\vec{k}=\vec{p}$, the RS spectrum has $\vec{k}\neq\vec{p}$ waves, due to the fact that RSD act on radial direction so a wave in some direction $\vec{p}$ generates others ($\vec{k}$) through a geometric coupling

$$\frac{k_i p_j}{p^2} J_{ij}(\vec{k}-\vec{p}) = \frac{1}{N_g^3} \sum_{\vec{x}} \left(\frac{k}{p}\right) (\hat{k}\cdot\hat{r})(\hat{p}\cdot\hat{r})\, e^{-i(\vec{k}-\vec{p})\cdot\vec{x}}$$

This coupling has an interesting dependence on $\vec{k}$ & $\vec{p}$, which we should explore. But this "creation" of waves from a single one is the reason the RS power is not diagonal any more.

In order for that coupling to be strong the sum should act constructively, so want:

$$\begin{cases} \dfrac{k}{p} \Rightarrow & k>p \\ (\hat{k}\cdot\hat{r})(\hat{p}\cdot\hat{r}) \rightarrow & \hat{k} \parallel \hat{p} \quad \text{so we get something positive} \\ (\vec{k}-\vec{p})\cdot\vec{x} \simeq 0 \Rightarrow & \vec{k}\simeq\vec{p} \end{cases}$$

not all these things are possible simultaneously, of course.

OK, let's calculate the power spectrum now.

$$\langle \delta^s_{\vec{k}_1} \delta^s_{\vec{k}_2} \rangle = \langle \delta_{\vec{k}_1} \delta_{\vec{k}_2} \rangle + f \sum_{\vec{q}} \left\langle \theta_{\vec{q}} \frac{k_{1i}\, q_j}{q^2} J_{ij}(\vec{k}_1-\vec{q})\, \delta_{\vec{k}_2} \right\rangle + \vec{k}_1 \leftrightarrow \vec{k}_2$$

$$+ f^2 \sum_{\vec{q}_1} \sum_{\vec{q}_2} \frac{k_{1i}\, q_{1j}}{q_1^2} \frac{k_{2k}\, q_{2l}}{q_2^2} J_{ij}(\vec{k}_1-\vec{q}_1) J_{kl}(\vec{k}_2-\vec{q}_2) \langle \theta_{\vec{q}_1} \theta_{\vec{q}_2} \rangle$$

$$= \hat{P}(k_1)\, \delta^K_{\vec{k}_1,-\vec{k}_2} + f \sum_{\vec{q}} \frac{k_{1i}\, q_j}{q^2} J_{ij}(\vec{k}_1-\vec{q})\, \delta^K_{\vec{q}_1,-\vec{k}_2}\, \hat{P}_{\delta\theta}(k_2) + \vec{k}_1 \leftrightarrow \vec{k}_2$$

$$+ f^2 \sum_{\vec{q}_1} \sum_{\vec{q}_2} \frac{k_{1i}\, q_{1j}}{q_1^2} \frac{k_{2k}\, q_{2l}}{q_2^2} J_{ij}(\vec{k}_1-\vec{q}_1)\, J_{kl}(\vec{k}_2-\vec{q}_2)\, \delta^K_{\vec{q}_1,-\vec{q}_2}\, \hat{P}_{\theta\theta}(q_1)$$

$$= \hat{P}(k_1)\, \delta^K_{\vec{k}_1,-\vec{k}_2} - f \frac{k_{1i}\, k_{2j}}{k_2^2} J_{ij}(\vec{k}_{12})\, \hat{P}_{\delta\theta}(k_2) + \vec{k}_1 \leftrightarrow \vec{k}_2$$

$\vec{k}_{12} \equiv \vec{k}_1 + \vec{k}_2$

$\vec{q}_1 \to \vec{q}$

$$- f^2 \sum_{\vec{q}} \frac{k_{1i}\, k_{2k}\, q_j\, q_l}{q^4} J_{ij}(\vec{k}_1-\vec{q})\, J_{kl}(\vec{k}_2+\vec{q})\, \hat{P}_{\theta\theta}(q)$$

Now, let $\hat{J}(\vec{k}_1,\vec{k}_2) \equiv -\frac{k_{1i}\, k_{2j}}{k_1\, k_2} J_{ij}(\vec{k}_{12}) = \frac{-1}{N_g^3} \sum_{\vec{x}} (\hat{k}_1\cdot\hat{r})(\hat{k}_2\cdot\hat{r})\, e^{-i\vec{k}_{12}\cdot\vec{x}}$

which in PP limit goes to

$$\hat{J}(\vec{k}_1,\vec{k}_2) \xrightarrow[PP]{} (\hat{k}_1\cdot\hat{d})(\hat{k}_2\cdot\hat{d}) \underbrace{\frac{-1}{N_g^3} \sum_{\vec{x}} e^{-i\vec{k}_{12}\cdot\vec{x}}}_{-\delta^K_{\vec{k}_1,-\vec{k}_2}} = \delta^K_{\vec{k}_1,-\vec{k}_2} (\hat{k}_1\cdot\hat{d})^2$$

Thus we have,

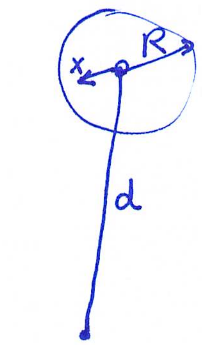
$$\boxed{P^s(\vec{k}_1,\vec{k}_2) = \hat{P}(k_1)\, \delta^K_{\vec{k}_1,-\vec{k}_2} + f \left[\frac{k_1}{k_2} \hat{P}_{\delta\theta}(k_2) + \frac{k_2}{k_1} \hat{P}_{\delta\theta}(k_1) \right] \hat{J}(\vec{k}_1,\vec{k}_2) + f^2 \sum_{\vec{q}} \frac{k_1 k_2}{q^2} \hat{P}_{\theta\theta}(q)\, \hat{J}(\vec{k}_1,-\vec{q})\, \hat{J}(\vec{k}_2,\vec{q})}$$

let's look first @ diagonal: let $\vec{k}_1 = \vec{k}$ $\vec{k}_2 = -\vec{k}$

$$\Rightarrow \boxed{P^s(\vec{k},-\vec{k}) = \hat{P}(k) + 2f\, \hat{P}_{\delta\theta}(k)\, \hat{J}(\vec{k},-\vec{k}) + f^2 \sum_{\vec{q}} \frac{k^2}{q^2} \hat{P}_{\theta\theta}(q)\, |\hat{J}(\vec{k},-\vec{q})|^2}$$

where: $\hat{J}(\vec{k},-\vec{k}) = \frac{1}{N_g^3} \sum_{\vec{x}} (\hat{k}\cdot\hat{r})^2$

Now, let's look at $\hat{J}(\hat{k},-\hat{k})$ in the limit where the survey region is spherical, and let us take the continuous limit so



$$\frac{1}{N_g^3}\sum_{\vec{x}} (\hat{k}\cdot\hat{r})^2 \longrightarrow \int_V \frac{d^3x}{V}\,\frac{[\hat{k}\cdot(\vec{x}+\vec{d})]^2}{(\vec{x}+\vec{d})^2}$$

after angular integration over $d\Omega_x$

$$\Rightarrow \hat{J}(\hat{k},-\hat{k}) = \int \frac{d^3x}{8V}\left\{\left(1+\frac{x^2}{d^2}\right) + \left(5-3\frac{x^2}{d^2}\right)(\hat{d}\cdot\hat{k})^2 - \left(1-\frac{x^2}{d^2}\right)\frac{1-3(\hat{d}\cdot\hat{k})^2}{4}\,\frac{d}{x}\ln\left(\frac{d+x}{d-x}\right)^2\right\}$$

$$\underset{x\ll d}{=} \int \frac{d^3x}{\cancel{8}V}\left\{(\hat{k}\cdot\hat{d})^2 + \underbrace{\left[1-3(\hat{k}\cdot\hat{d})^2\right]}_{-2\mathcal{P}_2(\hat{k}\cdot\hat{d})}\left[\frac{x^2}{3d^2} - \frac{x^4}{15d^4} - \frac{x^6}{105d^6} + \cdots\right]\right\}$$

For a spherical region: $\displaystyle\int \frac{d^3x}{V}\,\frac{x^{2n}}{d^{2n}} = \frac{3}{R^3}\int_0^R r^2dr\,\frac{r^{2n}}{d^{2n}} = \frac{3}{d^{2n}}\,\frac{R^{2n}}{2n+3}$

$$\Rightarrow \hat{J}(\hat{k},-\hat{k}) = (\hat{k}\cdot\hat{d})^2 - 2\mathcal{P}_2(\hat{k}\cdot\hat{d})\left[\frac{1}{5}\frac{R^2}{d^2} - \frac{3}{7\times 15}\frac{R^4}{d^4} - \frac{\cancel{3}}{\cancel{9}_{3}\times 105}\times\frac{R^6}{d^6} + \cdots\right]\Big\}$$

$$\Rightarrow \boxed{\hat{J}(\hat{k},-\hat{k}) = (\hat{k}\cdot\hat{d})^2 + \mathcal{P}_2(\hat{k}\cdot\hat{d})\left[-\frac{2}{5}\frac{R^2}{d^2} + \frac{6}{105}\frac{R^4}{d^4} + \frac{2}{315}\frac{R^6}{d^6}\cdots\right]}$$

this makes a lot of sense: to lowest order in $\frac{R}{d}$ it gives back the PP result: $\mu^2 \equiv (\hat{k}\cdot\hat{d})^2$. All higher orders are proportional to $\mathcal{P}_2(\hat{k}\cdot\hat{d})$, i.e. orthogonal to a monopole, so when we compute the monopole they don't contribute and we recover Kaiser's coefficient $\frac{2f}{3}\hat{P}_{\delta\theta}(k)$.

The higher order terms describing deviations from PP, look like a fairly well-behaved series, the coefficients go down pretty fast.

So even for $R \sim d$ seem well converging!

This will have to be cross-checked for a particular survey geometry, right now not clear if sphere is a reasonable approximation (say for R giving same V as survey) to the real case.

Dealing with $|\hat{J}(\vec{k},-\vec{q})|^2$ in the f^2 term is tougher, but one should be able to make some progress (please take a look at this).

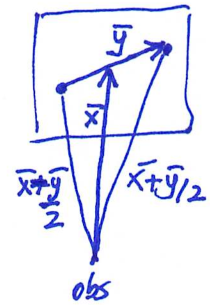
Now let's look at the off-diagonal power. Naively, I would expect that off-diagonal terms are suppressed by $\frac{R^2}{d^2}$, so they vanish in PP limit. So I would expect that there is something we can do to $\mathcal{O}(f)$ term to recover the amplitude we lost in the diagonal term, i.e. to recover a $+\frac{2}{5}\frac{R^2}{d^2}\mathcal{P}_2(\hat{k}\cdot\hat{d})$ type-contribution

(but maybe this is wishful thinking)

One idea is to multiply by $\vec{J}(-\vec{k}_1,\vec{k}_2)$ to make $\mathcal{O}(f)$ term real, i.e. $|\vec{J}(\vec{k}_1,\vec{k}_2)|^2$ and then integrate over $\hat{k}_1$ & $\hat{k}_2$ for fixed k_2 & k_1 ?

In general, we want to multiply $\hat{P}(\vec{k}_1,\vec{k}_2)$ by something that is orthogonal to the 1st term $[\hat{P}(k)]$ and get a quantity that is fully sensitive to f different from monopole.
(orthogonal)

We can define a space-dependent power spectrum estimator,



$$\hat{S}(\bar{k},\bar{x}) \equiv \frac{1}{N_g^3}\sum_{\bar{y}} \delta(\bar{x}+\bar{y}/2)\,\delta(\bar{x}-\bar{y}/2)\, e^{-i\bar{k}\cdot\bar{y}}$$

which we can write in terms of Fourier coefficients,

$$\hat{S}(\bar{k},\bar{x}) = \frac{1}{N_g^3}\sum_{y} e^{-i\bar{k}\cdot\bar{y}} \frac{1}{N_g^3}\sum_{\bar{k}_1}\frac{1}{N_g^3}\sum_{\bar{k}_2} e^{i\bar{k}_1\cdot(\bar{x}+\bar{y}/2)}\, e^{i\bar{k}_2\cdot(\bar{x}-\bar{y}/2)}\, \delta_{\bar{k}_1}\delta_{\bar{k}_2}$$

now, $\displaystyle \frac{1}{N_g^3}\sum_{\bar{y}} e^{i(-\bar{k}+\frac{\bar{k}_1}{2}-\frac{\bar{k}_2}{2})\cdot\bar{y}} = \delta^{K}_{\bar{k}_1-2\bar{k},\bar{k}_2}$

$$\Rightarrow \boxed{\hat{S}(\bar{k},\bar{x}) = \frac{1}{N_g^3}\sum_{\bar{k}_1} e^{i2(\bar{k}_1-\bar{k})\cdot\bar{x}}\,\delta_{\bar{k}_1}\,\delta_{\bar{k}_1-2\bar{k}} = \frac{1}{N_g^3}\sum_{\bar{q}} e^{i\bar{q}\cdot\bar{x}}\,\delta_{\bar{k}+\frac{\bar{q}}{2}}\,\delta^*_{\bar{k}-\frac{\bar{q}}{2}}}$$

Take expectation value,

$$\langle \hat{S}(\bar{k},\bar{x})\rangle = \frac{1}{N_g^3}\sum_{\bar{q}} e^{i\bar{q}\cdot\bar{x}}\,\langle \delta_{\bar{k}+\frac{\bar{q}}{2}}\,\delta^*_{\bar{k}-\frac{\bar{q}}{2}}\rangle$$

suppose there is translation invariance, then

$$\langle \hat{S}(\bar{k},\bar{x})\rangle = \frac{1}{N_g^3}\sum_{\bar{q}} e^{i\bar{q}\cdot\bar{x}}\,\hat{P}(\bar{k}+\tfrac{\bar{q}}{2})\,\delta^K_{\bar{q},\vec{0}} = \hat{P}(\bar{k}) \quad \checkmark$$

in the absence of translation invariance we have,

$$\boxed{\langle \hat{S}(\bar{k},\bar{x})\rangle = \frac{1}{N_g^3}\sum_{\bar{q}} e^{i\bar{q}\cdot\bar{x}}\,\hat{P}(\bar{k}+\tfrac{\bar{q}}{2},\,-\bar{k}+\tfrac{\bar{q}}{2})}$$

We can build multipoles from $\hat{S}$ using $\hat{x}$ as the radial direction, leading to

$$\hat{S}_\ell(k) = \int\frac{d\Omega_k}{4\pi}\frac{1}{N_g^3}\sum_{\bar{x}} \hat{S}(\bar{k},\bar{x})\,\mathcal{P}_\ell(\hat{k}\cdot\hat{x})$$

$$\Rightarrow \langle \hat{S}_\ell(k) \rangle = \int \frac{d^2\hat{k}}{4\pi} \frac{1}{N_g^3} \sum_{\vec{x}} \mathcal{P}_\ell(\hat{k}\cdot\hat{x}) \frac{1}{N_g^3} \sum_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} \hat{P}(\vec{k}+\tfrac{\vec{q}}{2}; -\vec{k}+\tfrac{\vec{q}}{2})$$

$$= \int \frac{d^2\hat{k}}{4\pi} \frac{1}{N_g^3} \sum_{\vec{q}} \hat{P}(\vec{k}+\tfrac{\vec{q}}{2}, -\vec{k}+\tfrac{\vec{q}}{2}) \left(\frac{1}{N_g^3} \sum_{\vec{x}} \mathcal{P}_\ell(\hat{k}\cdot\hat{x})\, e^{i\vec{q}\cdot\vec{x}} \right)$$

From here we can see that the Y06 estimator is a particular way of combining the ~~ex~~diagonal & extra-diagonal power spectrum we defined!

Some questions,

1) Since $\hat{P}$ will presumably decay fast for large q, we don't need to sum over all q's, and we can do ( ) for a given geometry since it only involves geometrical factors

$\Rightarrow$ this looks like it will give us a fast way of implementing Y06 !

2) does this recover the full info on $\hat{P}$?
Or is it better to study off diag + diag as expressed before?