Solution to Problem 1:

The posteriors for precision (P) are gamma with hyperparameters:

$$\alpha_1 = \alpha_0 + \frac{n}{2}$$

$$\beta_1 = \left(\frac{1}{\beta} + \frac{1}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2 + \frac{nk}{2(n+k)}(\bar{x} - \mu)^2\right)^{-1}$$

	Shape	Scale
Surface	$\alpha_1 = -0.5 + \frac{10}{2} = 4.5$	$\beta_1 = \left(0 + \frac{3.59}{2} + 0\right)^{-1} = 0.557$
Bottom	$\alpha_1 = -0.5 + \frac{10}{2} = 4.5$	$\beta_1 = \left(0 + \frac{9.25}{2} + 0\right)^{-1} = 0.216$

The posterior precision distributions are presented in the figure 1.

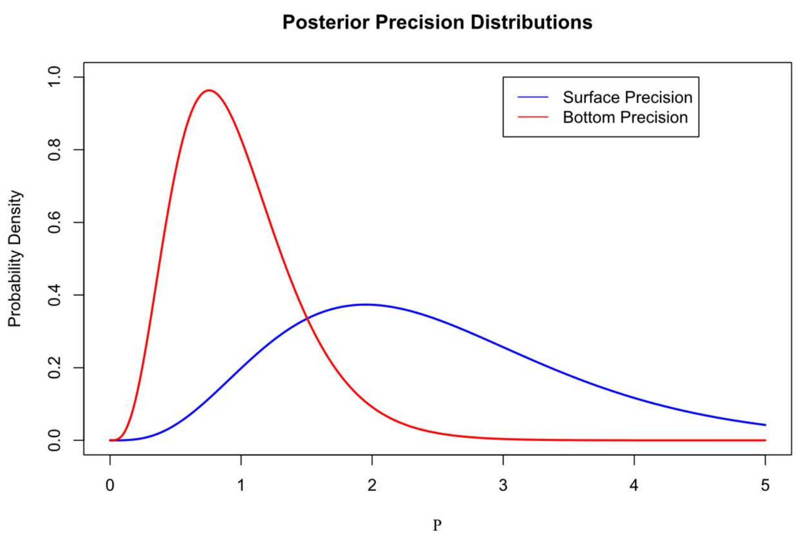

Figure 1

90% posterior credible intervals for precisions are:

	5% quantile	95% quantile
Surface	0.927	4.715
Bottom	0.359	1.828

It seems the precision is more spread on the surface than it is at the bottom, but its expected value is smaller at the bottom than it is on the surface, hence the normal distribution of the concentration is more spread at the bottom.

The posterior marginal distribution for mean (Θ) is a non-standard t distribution with hyperparameters:

$$k_1 = k_0 + n$$

$$\mu_1 = \frac{k\mu + n\bar{x}}{k + n}$$

$$Spread = \left(\sqrt{k\alpha\beta}\right)^{-1}$$

$$Degrees\ of\ freedome = 2\alpha$$

	Center	Precision multiplier	Spread	DOF
Surface	$\mu_1 = \frac{0 + 10 \times 4.804}{0 + 10} = 4.804$	$k_1 = 0 + 10 = 10$	$\left(\sqrt{10 \times 4.5 \times 0.557}\right)^{-1} = 0.2$	$2 \times 4.5 = 9$
Bottom	$\mu_1 = \frac{0 + 10 \times 5.839}{0 + 10} = 5.839$	$k_1 = 0 + 10 = 10$	$\left(\sqrt{10 \times 4.5 \times 0.216}\right)^{-1} = 0.321$	$2 \times 4.5 = 9$

The posterior mean marginal distributions are presented in the figure 2.

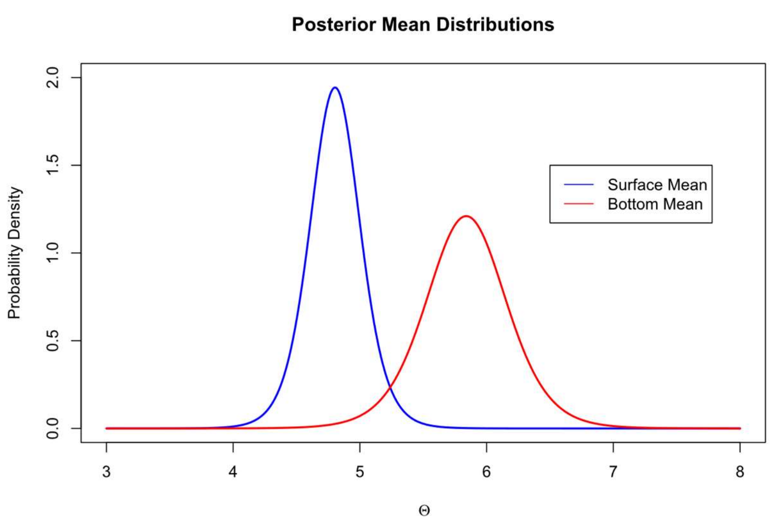

Figure 2

90% posterior credible intervals for mean are:

	5% quantile	95% quantile
Surface	4.44	5.17
Bottom	5.25	6.43

The mode of the mean posterior is smaller on the surface, so it is expected that the concentration on the surface has a normal distribution with a mean and standard deviation smaller than the distribution at the bottom.

Solution to Problem 2:

The resulting density distributions for precision are presented in the figure 3 along with the theoretical distribution from last problem. The 90% credible intervals are,

	5% quantile	95% quantile
Surface	0.359	1.818
Bottom	0.927	4.869

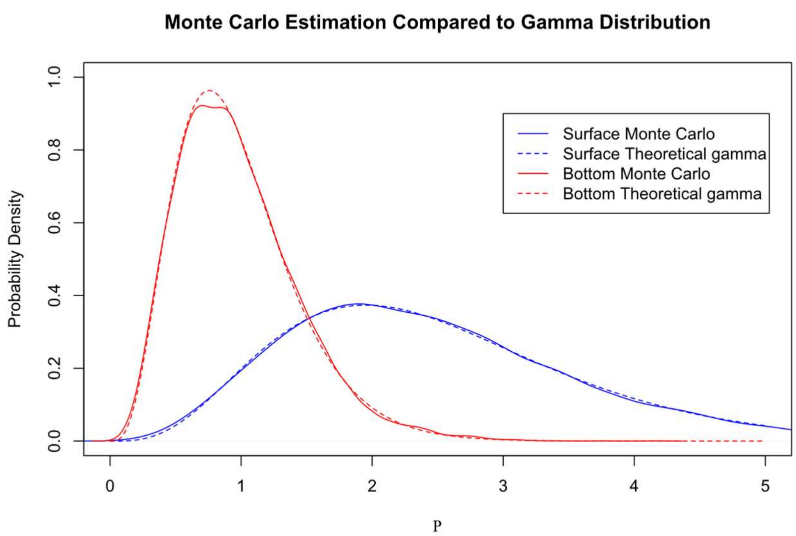

Figure 3

The resulting density distributions for mean are presented in the figure 4 along with the theoretical distribution from last problem. The 90% credible intervals are,

	5% quantile	95% quantile
Surface	4.44	5.17
Bottom	5.25	6.41

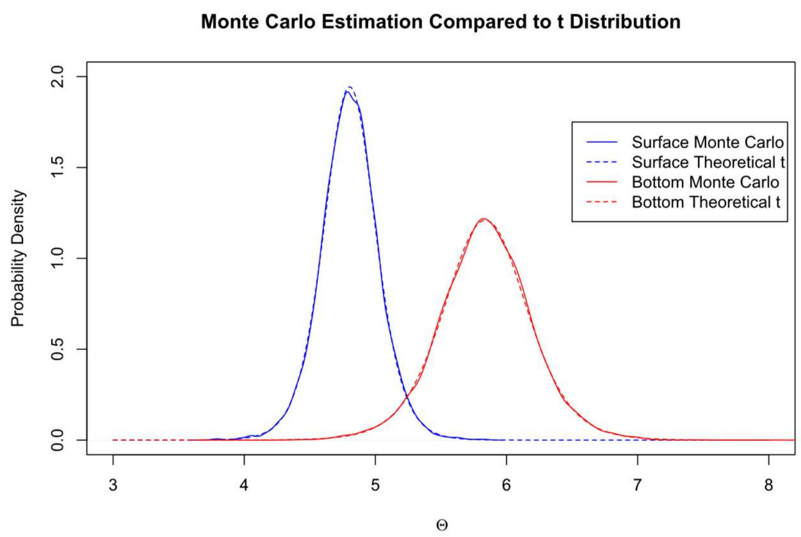

Figure 4

Results are resembling each other fairly enough. The only areas with a notable difference are around the distribution peaks.

Solution to Problem 3:

After finding the cases that values at the bottom are greater than the ones on the surface, among the 10000 samples, there is a 91.17% chance that standard deviation at the bottom is greater than the surface, and there is a 99.1% chance that the concentration at the bottom has a greater mean.

Solution to Problem 4:

The distribution at the bottom should have a greater mean, it is like the contamination builds up at the bottom. At the same time, distribution at the bottom is more uncertain, maybe it is because of the noise in measurements that are there because of the greater depth of water. On the contrary, the distribution on the surface should have a greater mean and a smaller standard deviation. The smaller variance may be a result of easier and more precise sampling process, and the lower concentration may be because of evaporation or the greater density of the chemical.

Q-Q plots for surface and bottom data are presented in figures 5 and 6 respectively. Assumption of normality seems to be reasonable for both datasets, except for the first two data points in surface dataset, and the first and the last datapoints in the bottom dataset.

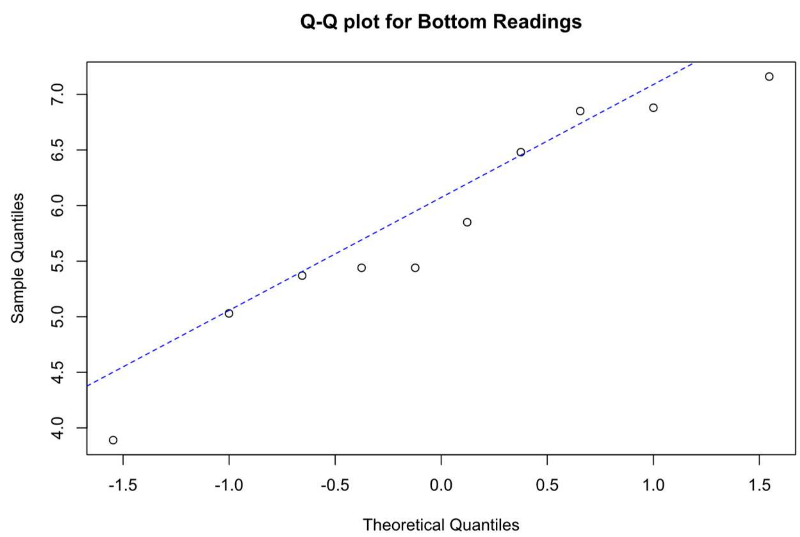

Figure 5

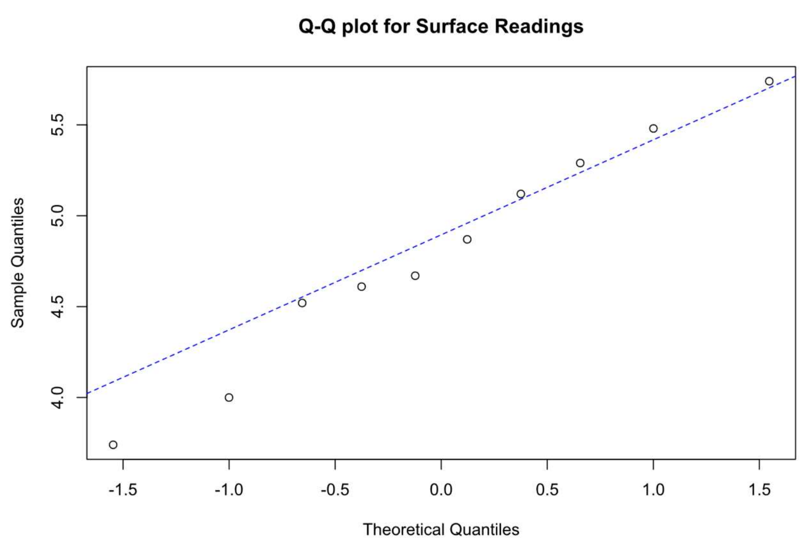

Figure 6