Solution to Problem 1:

The updated hyperparameters are:

$$\alpha_1 = \alpha_0 + \frac{n}{2} \qquad \beta_1 = \left(\frac{1}{\beta} + \frac{1}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2 + \frac{nk}{2(n+k)}(\bar{x} - \mu)^2\right)^{-1}$$

$$k_1 = k_0 + n \qquad \mu_1 = \frac{k\mu + n\bar{x}}{k + n}$$

$$Degrees\ of\ freedome = 2\alpha$$

	α_1	β_1	μ_1	k_1
Surface	$-0.5 + \frac{10}{2} = 4.5$	$\left(0 + \frac{3.59}{2} + 0\right)^{-1} = 0.557$	$\frac{0 + 10 \times 4.804}{0 + 10} = 4.804$	$0 + 10 = 10$
Bottom	$-0.5 + \frac{10}{2} = 4.5$	$\left(0 + \frac{9.25}{2} + 0\right)^{-1} = 0.216$	$\frac{0 + 10 \times 5.839}{0 + 10} = 5.839$	$0 + 10 = 10$

The spread for the predictive non-standard t distribution is:

$$\left(\sqrt{\frac{kn}{k+n}\alpha\beta}\right)^{-1}$$

	Spread Factor (Surface)	Spread Factor (Bottom)
n=10	0.2824	0.4535
n=40	0.2232	0.3585

The resulting theoretical distributions for the case of n=10 and n=40 are plotted in figures 1.

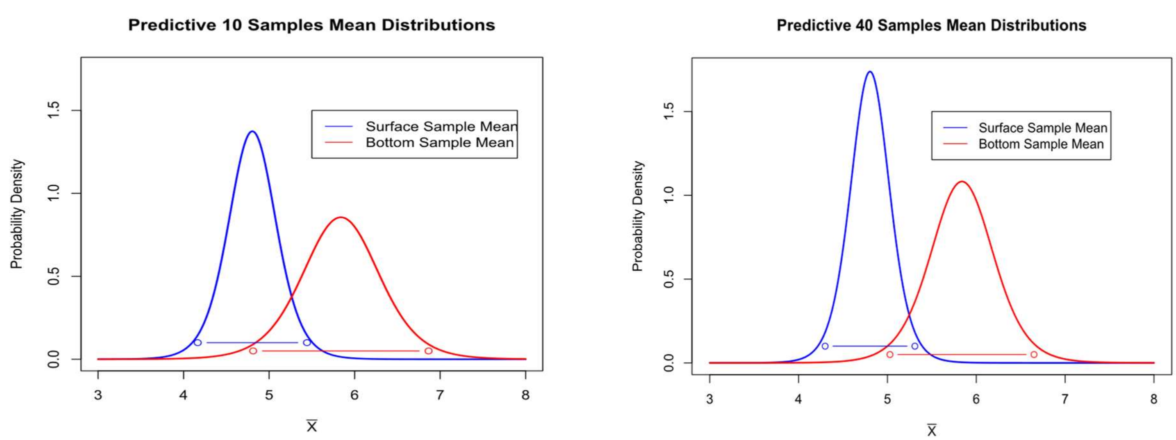


Figure 1

The 95% credible interval for the predicted sample means for four cases are reported in the following table.

	95% Interval (Surface)	95% Interval (Bottom)
n=10	[4.17,5.44]	[4.81,6.86]
n=40	[4.30,5.31]	[5.03,6.65]

As it was expected, the predictive distributions un case of 40 observations are taller and narrower than their counterparts with 10 observations. It is because we are making predictions about the mean of future samples, and as samples become larger in number, their means get closer to the real ample of population. Hence, the predictive distributions will have less uncertainty.

Solution to Problem 2:

The direct Monte Carlo sampling was performed, and the resulting four distributions, for surface and bottom samples in case of 10 and 40 samples are depicted in the figure 2, along with the theoretical distributions from problem 1. The 95% credible interval of the four distributions are presented in the following table, and they match the results from theoretical inference in the last problem.

	95% Interval (Surface)	95% Interval (Bottom)
n=10	[4.17,5.45]	[4.80,6.86]
n=40	[4.30,5.31]	[5.04,6.64]

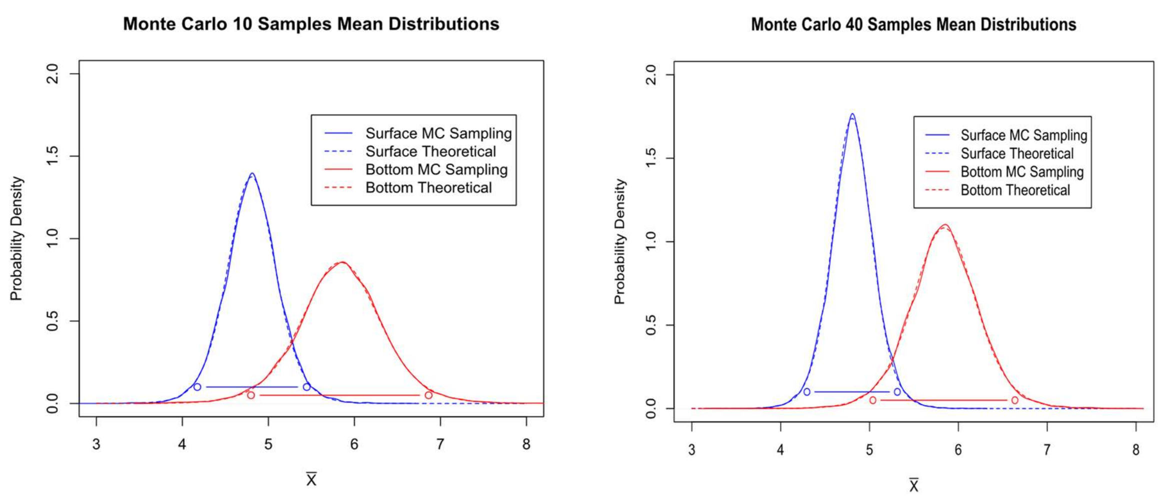
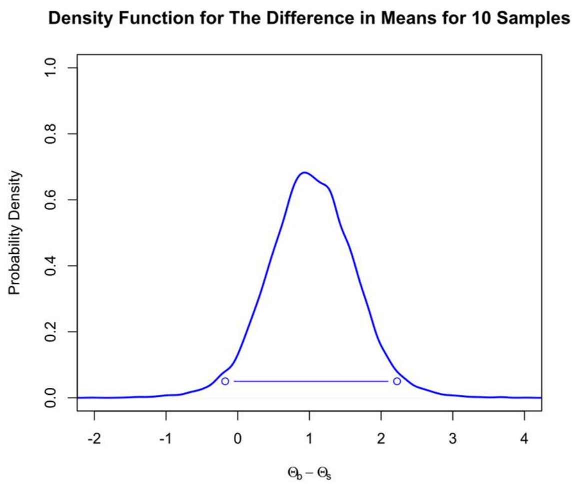


Figure 2

The two difference kernel densities are presented in the figure 3. And its 95% credible interval is in the following table.



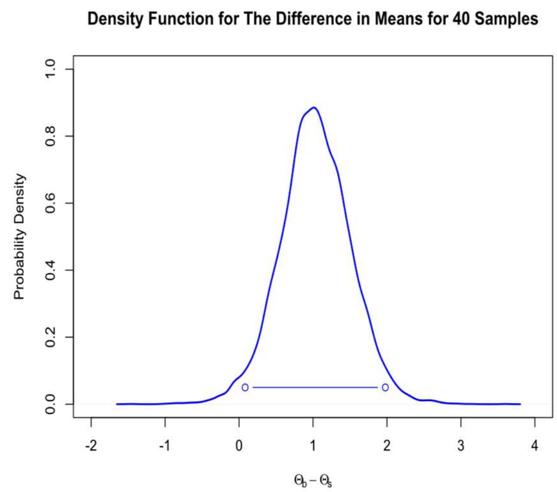


Figure 3

Again, as the number of samples increases, the certainty in the density distribution of the difference increases. It is interesting that if we rely on the 10 future sample prediction, we may decide that it is probable that the concentration on the surface would be greater than the concentration at the bottom (the 95% credible interval has some negative values). But, as the number of future samples is increased to 40, we can almost be sure that the concentration at the bottom is always greater than that on the surface.

	2.5% Quantile	97.5% Quantile
n=10	-0.175	2.222
n=40	0.082	1.978

Solution to Problem 3:

Problem 1 was repeated, but with known standard deviation equal to the standard deviation of the observations, and a uniform normal prior (mean=0 and SD=Inf). Four resulting theoretical distributions are shown in figure 4, and the 95% credible intervals are in the following table.

Posterior parameters:

$$\mu_1 = \frac{\frac{\mu}{\tau^2} + \frac{\sum x_i}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}} \rightarrow \begin{cases} \mu_{1_s} = 4.804 \\ \mu_{1_b} = 5.839 \end{cases} \qquad \tau_1 = \left(\frac{1}{\tau^2} + \frac{n}{\sigma^2}\right)^{-\frac{1}{2}} \rightarrow \begin{cases} \tau_{1_s} = 0.20 \\ \tau_{1_b} = 0.32 \end{cases}$$

Predictive normal distribution:

$$\begin{cases} mean_s = 4.804 \\ mean_b = 5.839 \end{cases}$$

	Standard Deviation (surface)	Standard Deviation (Bottom)
n=10	0.2824	0.4535
n=40	0.2232	0.3585

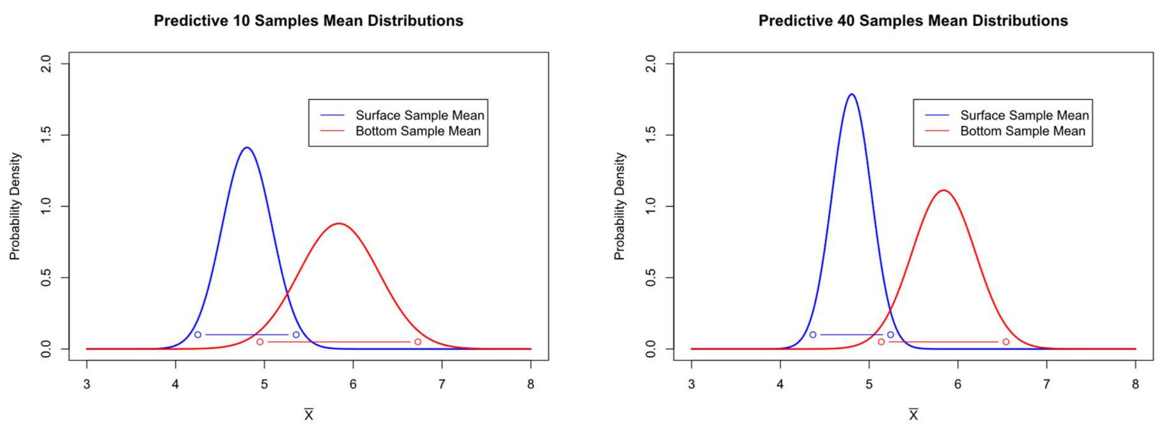


Figure 4

	95% Interval (Surface)	95% Interval (Bottom)
n=10	[4.25,5.36]	[4.95,6.73]
n=40	[4.37,5.24]	[5.14,6.54]

The standard deviations for predictive normal distributions in this problem, are equal to spread factors for the non-standard t distributions from problem 1.

In all four cases, the 95% credible interval is narrower of the ones from problem 1. Results of this problem are showing more certainty, and it is because we have eliminated some uncertainty in the model by assuming that we know the standard deviation of the population. However, it seems that in this case this difference is negligible, and simplifying the model by replacing the population standard deviation by the sample standard deviation won't add much error to the results.