STAT641 - Homework 3

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Problem 1

a. The posterior distribution for λ is given by:

$$P(\lambda|y) = Gamma(\sum_i y_i + a, n + b) = \kappa\lambda^{\sum y_i + a - 1}\ e^{-(n+b)\lambda}$$

b. Given our specific prior and the moose count data we have:

$$a^* = 4(0) + 5(1) + 6(2) + 4(3) + (1)5 + 2 = 36$$

$$b^* = 20 + 3 = 23$$

$$P(\lambda|y) = Gamma(36, 23) = \frac{23^{36}}{\Gamma(36)}\lambda^{35}e^{-23\lambda}$$

The 95% credible interval for λ is given by:

```
quantile(rgamma(10000, 36, rate=23), probs=c(0.025, 0.975))
```

```
##     2.5%    97.5%
## 1.089387 2.112957
```

So there's a 95% change that the true value for λ falls between those values given the data.

The posterior mean is

$$a^*/b^* = 36/23 \approx 1.565$$

and the posterior variance is

$$a^*/b^{*2} = 36/23^2 \approx 0.068$$

Problem 2

We now have:

$$P(\lambda|y) = \frac{1}{m(y)}\left[\prod_i \frac{e^{-\lambda}\lambda^{y_i}}{y_i!}\right]\frac{1}{\Gamma(a)b^a}\lambda^{a-1}e^{-\lambda/b}$$

$$= \frac{1}{m(y)\Gamma(a)b^a\prod y_i!}e^{-n\lambda}\lambda^{\sum y_i}\lambda^{a-1}e^{-\lambda/b}$$

$$= \kappa e^{-\lambda(n+1/b)}\lambda^{\sum y_i + a - 1} = Gamma(\sum y_i + a - 1, (n + 1/b)^{-1})$$

Given $a = 2$ and $b = 1/3$ and our data we have:

$$a^* = 34 + 2 = 36$$

$$b^* = (20+3)^{-1} = 1/23$$

$$P(\lambda|y) = Gamma(36, 1/23) = \frac{1}{\Gamma(36)(1/23)^{36}}\lambda^{35}e^{-\lambda/(1/23)}$$

$$= \frac{23^{36}}{\Gamma(36)}\lambda^{35}e^{-23\lambda}$$

Which is precisely what we had in problem 1!

The 95% credible interval for λ is now given by:

```
quantile(rgamma(10000, 36, scale=1/23), probs=c(0.025, 0.975))
```

```
##     2.5%    97.5%
## 1.091437 2.107738
```

So there's a 95% change that the true value for λ falls between those values given the data.

The posterior mean is

$$a^* b^* = 36/23 \approx 1.565$$

and the posterior variance is

$$a^* (b^*)^2 = 36/23^2 \approx 0.068$$

.

Problem 3

The pdf for the Beta distribution is given by:

$$\pi(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}, 0 < \theta < 1, (a > 1, b > 1)$$

We want to solve:

$$\frac{d\pi}{d\theta} = 0$$

when $a > 1$ and $b > 1$.

$$\frac{d\pi}{d\theta} = C\left[(a-1)\theta^{a-2}(1-\theta)^{b-1} - \theta^{a-1}(b-1)(1-\theta)^{b-2}\right]$$

$$= C\theta^{a-2}(1-\theta)^{b-2}\left[(a-1)(1-\theta) - \theta(b-1)\right]$$

$$= C\theta^{a-2}(1-\theta)^{b-2}\left[a-1+(2-a-b)\theta\right]$$

where:

$$C = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

Now given $0 < \theta < 1$, the only way for this expression to equal 0 is if:

$$a - 1 + (2-a-b)\theta = 0$$

Therefore we have:

$$\theta = \frac{a-1}{a+b-2}$$

Which as the maximum of our pdf is the mode of our distribution.

Problem 4

Here our pdf is given by:

$$\pi(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}, \theta > 0, (a > 0, b > 0)$$

In this case we have:

$$\frac{d\pi}{d\theta} = C\left[(a-1)\theta^{a-2}e^{-b\theta} - be^{-b\theta}\theta^{a-1}\right]$$
$$= C\theta^{a-2}e^{-b}\,[(a-1) - b\theta]$$

where:

$$C = \frac{b^a}{\Gamma(a)}$$

In this case the expression can only be 0 if:

$$(a-1) - b\theta = 0$$

And therefore:

$$\theta = \frac{a-1}{b}$$

which gives us the mode of our distribution.

Problem 5

a. In general for a binomial experiment $y \sim Binomial(n, \theta)$ and a prior $\theta \sim Beta(a, b)$ our posterior distribution will be:

$$P(\theta|y) = Beta(a + y, b + n - y)$$

In our case we have:

$$a = 5, b = 3, y = 1, n = 15$$

So that our posterior distribution is:

$$P(\theta|y) = Beta(6, 17)$$

b. The prior mode is

$$\theta_{priormode} = \frac{5-1}{5+3-2} = 4/6 \approx 0.667$$

The posterior mode is

$$\theta_{postmode} = \frac{6-1}{6+17-2} = 5/21 \approx 0.238$$

For the maximum likelihood we have:

$$\theta_{MLE} = \frac{y}{n} = 1/15 \approx 0.067$$

c. We'll use R here:

```
quantile(rbeta(10000, 6, 17), c(0.025, 0.975))
```

```
##      2.5%     97.5%
## 0.1024829 0.4535012
```

Therefore there's a 95% chance our true value for θ lies between these two values.

d. The prior mean and variance:

$$\mu_{prior} = \frac{5}{5+3} = 5/8 \approx 0.625$$

$$var_{prior} = \frac{5(3)}{(5+3)^2(5+3+1)} = 15/576 \approx 0.026$$

The posterior mean and variance:

$$\mu_{post} = \frac{6}{6+17} = 6/23 \approx 0.261$$

$$var_{post} = \frac{6(17)}{(6+17)^2(6+17+1)} = 102/12696 \approx 0.008$$

Clearly the mean was brought much lower by the data (which makes sense given how few pieces of banana were caught) and the variance also shrunk quite a lot indicating that the boundaries we are putting on the true value have shrunk given the data.

Problem 6

a. Given an observation model that is normal with known variance:

$$y \sim N(\theta, \sigma_0^2)$$

and a prior:

$$\pi(\theta) \sim N(\mu_0, \tau_0^2)$$

we know that the posterior distribution will also be normal:

$$P(\theta | y_1, \ldots, y_n) = N(\mu_n, \tau_n^2)$$

where:

$$\mu_n = \frac{\mu_0/\tau_0^2 + n\bar{y}/\sigma_0^2}{1/\tau_0^2 + n/\sigma_0^2}$$

$$\tau_n^2 = (1/\tau_0^2 + n/\sigma_0^2)^{-1}$$

In our case:

$$n = 10, \bar{y} = 5.12, \sigma_0^2 = 0.9, \mu_0 = 5.3, \tau_0^2 = 100$$

So we have:

$$\mu_n = \frac{5.3/100 + 10(5.12)/0.9}{1/100 + 10/0.9} \approx 5.12$$

(which makes sense given τ_0^2 is very large and μ_0 is not)

$$\tau_n^2 = (1/100 + 10/0.9)^{-1} \approx 0.09$$

So our posterior distribution is:

$$P(\theta|y_1, \ldots, y_n) = N(5.12, 0.09)$$

b. The 95% credible interval is given by:

$$[5.12 - 1.96(0.3), 5.12 + 1.96(0.3)] \approx [4.53, 5.71]$$

and therefore we are asserting that there's a 95% probability that the true value of θ lies in that range.

c. The prior mode is $\mu_0 = 5.3$, the posterior mode is $\mu_n = 5.12$ and the maximum likelihood estimate is $\bar{y} = 5.12$.

d. Yes the mean of the data does fall in the credible interval. It doesn't always have to be this way given the fact that for small n we can always choose a prior that washes out our data. Specifically we can wash out the data by simple decreasing τ_0^2 to some very very small value as compared to σ_0^2.

For example let:

$\bar{y} = 0, n = 2, \sigma_0^2 = 1000, \tau_0^2 = 0.01, \mu_0 = 10$

Then we'll have:

$$\mu_n \approx 10$$

$$\tau_n^2 \approx 0.01$$

leaving us with a 95% credible interval of:

$$[9.804, 10.196]$$

which does not contain $\bar{y} = 0$.

Obviously choosing such a prior seems a tad insane but you could do it and therefore normal-normal problems don't guarantee that your mean falls in the credible interval.