THE EXTRAPOLATION FOR BOUNDARY FINITE ELEMENTS

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THE EXTRAPOLATION FOR BOUNDARY FINITE ELEMENTS

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I. Introduction

It has been shown in [1]-[3] that the Richardson extrapolation can be applied to the elliptic Ritz projection with linear finite elements and increase the second-order accuracy of linear finite elements

$$U_h(z) = u(z) + 0(h^2 |\ln {}^1/h|)$$

in mesh point z at least to order three or four

$$U_h(z) \equiv \frac{1}{3} (4\; U_{h/2}(z) - U_h(z)) = u(x) + 0(h^3 \ln 1/h)(\text{or } 0(h^4 \ln 1/h))$$

Where Th is uniform triangulation and $T_{h/2}$ is generated from Th by dividing each triangle as usual into four congruent subtriangles.

In the present paper above basic results have been extended to boundary finite element and the coupling of finite elements and boundary finite elements.

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* Research undertaken during visit to the Institute for Mathematics and its Applications

2. The Extrapolation for Boundary Finite Elements

Let us consider the following boundary value problem

$$
\begin{aligned}
-\Delta u &= f \qquad \text{in } \Omega \\
u &= u_0 \qquad \text{on } \Gamma
\end{aligned}
\tag{2.1}
$$

where $\Omega \subset R^2$ is a bounded domain with Lipschitz boundary. The equivalent integral equation is that

$$\int_\Gamma q(x)E(x;y)ds_x = \int_\Gamma u_0(x) \frac{\partial E(x;y)}{\partial n} ds_x + \frac{1}{2} u_0(y) - \int_\Omega f(x)E(x;y)dx \tag{2.2}$$

$$\forall\, y \in \Gamma$$

$$u(y) = \int_\Omega f(x)\ E(x;y)dx + \int_\Gamma q(x)E(x;y)ds_x - \int_\Gamma u_0(x) \frac{\partial}{\partial n} E(x;y)ds_x \tag{2.3}$$

$$\forall\, y \in \Omega$$

Where $E(x;y)$ is a fundamental solution of Laplace equation, i.e.

$$\Delta_x E(x;y) + \delta(x-y) = 0$$

and $q = \frac{\partial u}{\partial n}\Big|_\Gamma$. For two dimensional problem it is known

$$E(x;y) = -\frac{1}{2\pi} \ell n\ |x-y| \tag{2.4}$$

The Galerkin variational formulation of (2.2) is that

$$\text{find } q \subset H^{-1/2}(\Gamma) \text{ such that} \tag{2.5}$$

$$b(q,\ q^*) = \ell((u_0,f),\ q^*) \qquad \forall q^* \in H^{-1/2}(\Gamma)$$

where

$$b(q,q^*) = \int_\Gamma \int_\Gamma q(x)q^*(y)\ E(x;y)\ ds_x\ ds_y$$

is a symmetry and coercive bilinear form on $H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$

$$\ell((u_0,f),\ q^*) = \int_\Gamma \int_\Gamma u_0(x)\ q^*(y)\ \frac{\partial}{\partial n_x}\ E(x;y)ds_x\ ds_y + \frac{1}{2}\int_\Gamma u_0(x)ds_x$$

$$- \int_\Gamma \int_\Omega f(x)q^*(y)\ E(x;y)dx\ ds_y$$

is a bounded linear functional on $H^{-1/2}(\Gamma)$.

The approximation problem corresponding to (2.5) can be expressed as the following

$$\text{find } q_h \in V_h \subset H^{-1/2}(\Gamma) \text{ such that} \tag{2.6}$$

$$b(q_h,\ q_h^*) = \ell((u_0,\ f),\ q_h^*)\ , \qquad \forall\ q_h^* \in V_h$$

Where V_h is a boundary finite element subspace on Γ , and h is mesh size parameter. Later on, we suppose Vh is a linear elements subspace on Γ.

Lemma 1 Assume $E(x;y)$ is a fundamental solution defined by (2.4). Then there exists $\tilde{E}(x;y) \in V_h$ such that

$$\|\tilde{E}(.;y)\|_{0,2,\Gamma} \leq c|\ln h|^{1/2}$$

$$\|\tilde{E}(.,y)\|_{1/2,\Gamma} \leq c|\ln h|^{1/2} \tag{2.7}$$

$$\|\tilde{E}(.;y) - E(.,y)\|_{0,1,\Gamma} \leq c\, h|\ln h|$$

where c is a positive constant independent of h and y.

Proof. Suppose

$$My = \{x| \ \|x-y\| \leq 1 \}, \quad \Gamma_1 = \Gamma \cap M_y, \quad \Gamma_2 = \Gamma \setminus \Gamma_1$$

Because of $E(x;y)$ satisfying

$$-\Delta E(x;y) = \delta\,(x-y) \quad \text{in} \quad M_y$$

$$E(x;y) = 0 \quad \text{on} \quad \partial M_y$$

we have $E(x;y) \in H_0^{1-\varepsilon}(M_y)$ and

$$(\nabla E(x;y)\ , \ \nabla v) = v(y) \qquad \forall\ v \in H_0^{1-\varepsilon}(M_y)$$

By [5], there exists $E_h(x;y) \in W_h \subset H_0^1(M_y)$ (where W_h is piecewise linear interpolation finite element subspace of $H_0^1(M_y)$), such that

$$(\nabla E_h(.;y)\ ,\ \nabla v\ ) = v(y) \qquad \forall\ v\ c\ W_h$$

and

$$\| E_h(.;y) \|_{1,2,M_y} \leqslant c\ |\ln h|^{1/2}$$

$$\| E_h(.;y) - E(x;y) \|_{1,1,M_y} \leqslant ch\ |\ell n h|$$

so

$$\| E_h(.;y) \|_{1,2,M_y \cap \Omega} \leqslant c\ |\ \ln h|^{1/2}$$

$$\| E(.;y) - E_h(.;y) \|_{1,1,M_y\ \ \Omega} \leqslant ch\ |\ \ln h\ |$$

By virtue of trace theory, we can get

$$\| E_h(.;y) \|_{0,2,\ \partial(M_y \cap \Omega)} \leqslant c\ |\ \ln h\ |^{1/2}$$

$$\| E_h(.;y) - E(.;y) \|_{0,1,\ \partial(M_y \cap \Omega)} \leqslant ch\ |\ \ln h\ |$$

In view of $\Gamma_1 \subset (M_y \cap \Omega)$, we have

$$\| E_h(.;y) \|_{0,2,\Gamma_1} \leqslant c\ |\ \ln h\ |^{1/2}$$

$$\| E_h(.;y) - E(.;y) \|_{0,1,\Gamma_1} \leqslant ch\ |\ \ln h\ |$$

Because of $\|x-y\| > 1, \quad \forall\ x\ \varepsilon\ \Gamma_2$

so $\qquad \| E(.;y) \|_{2,\infty,\Gamma_2} \leqslant 1$

Suppose $E^1(.;y)$ is a picewise linear interpolation of $E(x;y)$, then

$$\| E^I(.;y) - E(.;y)\|_{0,1,\Gamma_2} \leqslant ch^2 \|E(.;y)\|_{2,1,\Gamma_2} \leqslant \tilde{c}h^2$$

$$\| E^I(.;y)\|_{0,\Gamma_2} \leqslant c$$

$$\| E^I(.;y)\|_{1,\Omega \setminus (My \quad \Omega)} \leqslant c$$

Setting

$$\tilde{E} |x;y) = \begin{cases} E_h(x;y) & x \in \Gamma_1 \\ \\ E^I \ (x;y) & x \in \Gamma_2 \end{cases}$$

then

$$\|\tilde{E}(.;y)\|_{0,\Gamma} \leqslant \|E_h(.;y)\|_{0,\Gamma_1} + \|E^I(.;y)\|_{0,\Gamma_2} \leqslant c|\ell n \ h|^{1/2}$$

$$\|\tilde{E} \ (.;y)\|_{1,\Omega} \leqslant c \ |\ell n \ h|^{1/2}$$

$$\|\tilde{E}(.;y)\|_{1/2,\Gamma} = \inf\{\|v\|_{1,\Omega} \ , \ \tilde{\gamma_0}E(t,y) = \gamma_0 v\}$$

$$\leqslant \|E(.;y)\|_{1,\Omega} \leqslant c|\ell n \ h|^{1/2}$$

where γ_0 is the trace operator of 0-order on the Γ. In addition,

$$\|\tilde{E}(.;y) - E(.;y)\|_{0,1,\Gamma} \leqslant \|\tilde{E}(.;y) - E(.;y)\|_{0,1,\Gamma_1}$$

$$+ \| \tilde{E} \ (.;y) - E(.;y)\|_{0,1,\Gamma_2} \leqslant ch\|\ell n h\|$$

The proof ends.

Remark 1 In view of (7) it is clear that

$$\| \tilde{E}\,(.;y)\|_{0,1,\Gamma} \leqslant c|\ell n h|^{1/2}$$

Lemma 2 Suppose G_z is the Green function of boundary integral equation (2.2) ,

$$b(q,Gz) = q(z) \qquad \forall\, q \in H^{-1/2}\,(\Gamma)$$

$G_{zh} \in Vh$ is the approximation of G_z

$$b(q,\ G_{zh}) = q(z) \qquad \forall\, q \in Vh \subset H^{-1/2}(\Gamma)$$

Then we have

$$\|G_{zh}\|_{-3/2,\Gamma} \leqslant c\ |\ \ell n\ h|^{1/2} \qquad (2.8)$$

$$\|G_{zh} - G_z\|_{-3/2,\Gamma} \leqslant c\ h$$

where c is a positive constant independent of h and z.

Proof. (1). $\forall\ g \in H^{3/2}\,(\Gamma)$, there exists a $q_g \in Vh \subset H^{1/2}(\Gamma)$

such that

$$b(q_g,\ q') = <g,\ q'> \qquad \forall\ q' \in V_h$$

and

$$\|q_g\|_{1/2,\Gamma} \leqslant c\|g\|_{3/2,\Gamma}$$

By [6] ,

$$<g,G_{zh}> = b(q_g,\ G_{zh}) = g_g(z) \leqslant \|q_g\|_{0,\infty,\Gamma} \overset{([6])}{\leqslant}$$

$$c\ |\ \ell n\ h|^{1/2}\ \|g_g\|_{1/2,\Gamma} \leq c|\ell n\ h|^{1/2}\ \|g\|_{3/2,\Gamma}$$

So

$$\|G_{zh}\|_{-3/2,\Gamma} = \sup_{g\varepsilon H^{3/2}(\Gamma)} \frac{< g, G_{zh} >}{\|g\|_{3/2,\Gamma}} \leq c\ |\ell nh|^{1/2}$$

(2) In similar manner, $\forall\ g \in H^{3/2}(\Gamma)$, there exist $q_g \in H^{1/2}(\Gamma)$, $q_h\ \varepsilon\ V_h \subset H^{1/2}(\Gamma)$ such that

$$b(q_g,\ q') = < g,\ q' > \quad \forall\ q'\ \varepsilon\ H^{-1/2}\ (\Gamma)$$

$$b(q_h,\ q') = < g,\ q' > \quad \forall\ q'\ \varepsilon\ V_h$$

By [11],

$$\|q_g - q_h\|_{0,\Gamma} \leq ch^{3/2}\ \|q_g\|_{1/2,\Gamma} \leq c\ h^{3/2}\ \|g\|_{3/2,\Gamma}$$

So

$$< g,\ G_Z - G_{ZR} > = b(q_g - q_h,\ G_Z - G_{Zh}) = b(q_g - q_h,\ G_Z)$$

$$= q_g(Z) - q_h(Z) \leq \|q_g - q_h\|_{0,\infty,\Gamma} \leq ch^{-1/2}\ \|q_g - q_h\|_{0,\Gamma}$$

$$\leq c\ h\|g\|_{3/2,\Gamma}$$

$$\|G_Z - G_{Zh}\|_{-3/2,\Gamma} = \sup_{g\varepsilon H^{3/2}(\Gamma)} \frac{< g,\ G_Z - G_{Zh} >}{\|g\|_{3/2,\Gamma}} \leq ch \qquad \#$$

Assume $F(\Gamma_h)$ is a finite element regular family on Γ which satisfies that the distance from any point on the arc S_iS_{i+1} to line $\overline{S_i\ S_{i+1}}$ is shorter than $\theta(h^2)$. Then we have

Lemma 3 Suppose $q \in c^3(\Gamma)$, $q^I \in Vh$ is the interpolation of q, then

$$b(q^I - q, G_{zh}) = \frac{h^2}{12} b(q'', G_{zh}) + 0(h^3 \, |\ln h|^{3/2}) \, \|q\|_{3,\infty,\Gamma} \tag{2.9}$$

Proof. In fact, on the $\overline{S_i \, S_{i+1}}$

$$q^I(S) = \frac{S_{i+1} - S}{h} q \, (S_i) + \frac{S - S_i}{h} q \, (S_{i+1})$$

$$= \frac{S_{i+1} - S}{h} (q(s) + (S_i - S) \, q'(s) + \frac{1}{2} (S_i - s)^2 \, q''(s) + \frac{1}{6} (S_i - s)^3 \, q'''(\xi))$$

$$+ \frac{S - S_i}{h} (q(s) + (S_{i+1} - S) q'(s) + \frac{1}{2} (S_{i+1} - s)^2 q''(s) + \frac{1}{6} (S_{i+1} - S)^3 q'''(\eta))$$

$$= q(S) + \frac{1}{2} (S_{i+1} - S)(S - S_i) q''(s) + 0(h^3) \, \|q\|_{3,\infty,\Gamma}$$

Therefore we have

$$I(y) = \int_\Gamma (q^I(s) - q(s)) E(x;y) ds_x = \int_\Gamma (q^I - q) \tilde{E}(\chi;y) ds_x$$

$$+ \int_\Gamma (q^I - q)(E(\chi;y) - \tilde{E}(\chi;y)) ds_x = I_1 + I_2 \tag{2.10}$$

By lemma 1 and Remark 1 we have

$$I_2 \leq c \, h^2 \, \|q\|_{2,\infty,\Gamma} \int_\Gamma |\tilde{E}(x;y) - E(x;y)| dS_x = 0(h^3 |\ln h|) \|q\|_{2,\infty,\Gamma}$$

$$I_1 = \sum_i \int_{\widehat{S_iS_{i+1}}} (q^I - q)\ \tilde{E}(\chi;y)ds_x$$

$$= \sum_i \int_{\overline{S_iS_{i+1}}} (q^I - q)\ \tilde{E}(x;y)ds_x + \sum_i (\int_{\widehat{S_iS_{i+1}}} - \int_{\overline{S_iS_{i+1}}})(q^I-q)\tilde{E}(x;y)ds_x = L_1 + L_2$$

$$L_2 \leq c\ h^2 \int_\Gamma |\frac{\partial}{\partial n}((q^I - q)\tilde{E}(\chi;y))|dS_x$$

$$= ch^2 \{\|q - q\|_{1,\infty,\Gamma} \|\tilde{E}\|_{1,1,\Gamma} + \|q^I-q\|_{1/2,\Gamma} \|\frac{\partial\tilde{E}}{\partial n}\|_{1/2,\Gamma}\}$$

$$\leq ch^3|\ln h|^{1/2}\|q\|_{2,\infty,\Gamma} + ch^{3+1/2}\ \|q\|_{2,\Gamma}\ \|\tilde{E}\|_{1/2,\Gamma}\} \leq ch^3|\ln h|^{1/2}\ \|q\|_{2,\infty,\Gamma} \qquad (2.11)$$

on the $\overline{S_iS_{i+1}}$, by virtue of $\tilde{E}(x;y) \varepsilon Vh$,

$$\tilde{E}(x;y) = \frac{S_{i+1} - S}{h}\ \tilde{E}\ (x_i;y) + \frac{S - S_i}{h}\tilde{E}\ (X_{i+1};\ y) = N_i(s)\tilde{E}(x_i;y) + N_{i+1}(S)\tilde{E}(x_{i+1};y)$$

$$\int_{\overline{S_iS_{i+1}}} N_i(s)ds = \frac{1}{h} \int_{S_i}^{S_{i+1}} (S_{i+1} - s)ds = h/2$$

$$\int_{\overline{S_iS_{i+1}}} N_{i+1}\ (s)ds = h/2$$

Hence

$$L_1 = \sum_i \int_{\overline{S_iS_{i+1}}} (q^I-q)\tilde{E}(x;y)ds_x = \sum_i \int_{\overline{S_iS_{i+1}}} (\frac{1}{2}\ (S_{i+1} - S)(S-S_i)q''(s).$$

$$(N_i(s)\ \tilde{E}\ (X_i;y) + N_{i+1}(s)\ \tilde{E}\ (X_{i+1};y)))ds_x + O(h^3)\|q\|_{3,\infty,\Gamma} \int_\Gamma |\tilde{E}(x;y)|ds_x$$

$$= \sum_i q''(\xi)\left(\frac{h^3}{24}\tilde{E}(X_i;y) + \frac{h^3}{24}\tilde{E}(x_{i+1},y)\right) + 0\,(h^3|\ln h|^{1/2})\ \|q\|_{3,\infty,\Gamma}$$

$$= \sum_i \frac{h^2}{12}\{q''(\xi)(\int_{\overline{S_iS_{i+1}}}(N_i(s)\tilde{E}(x_i;y)+N_{i+1}(s)\tilde{E}(x_{i+1};y))ds_x)\}+0(h^3(\ln h|^{1/2})\|q\|_{3,\infty,\Gamma}$$

$$L_1 = \frac{h^2}{12}\int_\Gamma q''(s)\tilde{E}(x;y)ds_x + \frac{h^2}{12}\sum_i \int_{\overline{S_iS_{i+1}}} (q''(\xi) - q''(s))\tilde{E}(x;y)ds_x$$

$$+ 0(h^3|\ln h|^{1/2})\|q\|_{3,\infty,\Gamma}$$

but

$$\int_{\overline{S_iS_{i+1}}} (q''(\xi) - q''(s))\tilde{E}(x;y)ds_x = 0(h|\ln h|^{1/2})$$

So

$$L_1 = \frac{h^2}{12}\int_\Gamma q''(s)\tilde{E}(x;y)ds_x + 0(h^3|\ln h|^{1/2})\ \|q\|_{3,\infty,\Gamma}$$

$$= \frac{h^2}{12}\int_\Gamma q''(s)E(x;y)ds_x + \frac{h^2}{12}\int_\Gamma q''(s)(\tilde{E}(x,y) - E(x;y))ds_x$$

$$+ 0(h^3|\ln h|^{1/2})\ \|q\|_{3,\infty,\Gamma} \qquad \text{(by lemma 1)}$$

$$= \frac{h^2}{12}\int_\Gamma q''(s)E(x;y)ds_x + 0(h^3|\ln h|^{1/2})\ \|q\|_{3,\infty,\Gamma} \tag{2.12}$$

Consequently,

$$I(y) = I_1 + I_2 = L_1 + L_2 + I_2 = \frac{h^2}{12} \int_\Gamma q''(s)E(x;y)ds_x + O(h^3|\ln h|)\|q\|_{3,\infty,\Gamma} \quad (2.13)$$

Therefore,

$$b(q^I - q_1, G_{zh}) = \int_\Gamma I(y)G_{zh}(y)ds_y = \frac{h^2}{12} \int_\Gamma \int_\Gamma q''(x)G_{zh}(y)E(x;y)ds_x\, ds_y$$

$$+ O(h^3 \, |\ln n|)\|q\|_{3,\infty,\Gamma} \|G_{zh}\|_{-3/2,\Gamma}$$

$$= \frac{h^2}{12} \int_\Gamma \int_\Gamma q''(x)G_Z(y)\, E(x;y)ds_x\, ds_y + \frac{h^2}{12} \int_\Gamma \int_\Gamma g''(x)(G_{zh}(y) - G_z(y))E(x;y)\, ds_x ds_y$$

$$+ O(h^3 \, |\ln h|^{3/2}) \, \|q\|_{3,\infty,\Gamma}$$

$$= \frac{h^2}{12} b(q'', G_Z) + O(h^2|\ln h|^{1/2})\|q\|_{3,\infty,\Gamma} \|G_{zh} - G_z\|_{0,1,\Gamma} + O(h^3|\ln h|^{3/2})$$

Owing to that

$$\|G_{zh} - G_z\|_{0,1,\Gamma} \le |\Omega| \, \|G_{zh} - G_h\|_{-3/2} \le c\, h$$

Finally, we obtain

$$b(q^I - q, G_{zh}) = \frac{h^2}{12} b(q'', G_z) + O(h^3|\ln h|^{3/2}) \, \|q\|_{3,\infty,\Gamma}$$

Theorem 1 Suppose q is the solution of (2.5), $q \in C^3(\Gamma)$, $q_h \in Vh \subset H^{-1/2}(\Gamma)$ is a solution of (2.6), then for any mesh point Z we have

$$\frac{1}{3} (4\, q_{h/2}(Z) - q_h(Z)) = q(Z) + O(h^3|\ln h|^{3/2}) \, \|q\|_{3,\infty,\Gamma} \quad (2.14)$$

Proof since

$$b(q - q_h, G_{2h}) = 0$$

$$b(q,G) = q(z), \qquad b(q_h, G_{zh}) = q_h(Z)$$

$\forall$ mesh point z we have

$$q(Z) - q_h(Z) = q^I(Z) - q_h(Z) = b(q^I - q_h, G_{Zh})$$

$$= b(q^I - q, G_{Zh}) = \frac{h^2}{12} b(q'', G_{Zh}) + O(h^3|\ln h|^{3/2})\|q\|_{3,\infty,\Gamma}$$

Hence (2.14) is obtained.

3. Extrapolation for the Collocation Method

The integral equation (2.2) can be expressed as the following

find $q \varepsilon H^{-1/2}(\Gamma)$ such that

$$\int_\Gamma q(x)\, E(x;y)dS_x = F(y) \tag{3.1}$$

Where

$$F(y) = \int_\Gamma u_0(x) \frac{\partial}{\partial n} E(x;y)ds_x + \frac{1}{2} u_0(y) - \int_\Omega f(x)\, E(x;y)dx$$

The collocation method for (3.1) is that

find $q_h \varepsilon V_h \subset H^{-1/2}(\Gamma)$ such that

$$\int_\Gamma q_h(x)\, E(x;y;)ds_x = F(y_j) \qquad j = 1,2,\ldots n \tag{3.2}$$

where y_j are mesh points of Γ_h .

From (3.1) and (3.2) we have

$$\int_\Gamma (q(x) - q_h(x)\,)\, E(x;y_j)ds_x = 0\ , \qquad j = 1,2,\ldots$$

Hence

$$\int_\Gamma (q^I(x) - q_h(x))\, E(x;y_j)ds_x = \int_\Gamma (q^I(x) - q(x))\, E(x;y_j)ds_x,\ j=1,2,..n \tag{3.3}$$

i.e.

$$\int_\Gamma (q^I(x) - q_h(x))E(x;y_j)ds_x = I(y_j) \qquad j = 1,2,3,\ldots n \tag{3.4}$$

where $I(y)$ is defined by (2.10). By virtue of (2.13) we have

$$\int_\Gamma [q^I(x) - q_h(x) - \frac{h^2}{12} q''(s)] E(x;y_j)ds_x = 0(h^3 \ln|h|) \|q\|_{3,\infty,\Gamma} \tag{3.5}$$

$$j = 1,2,\ldots n.$$

Let N_i, N_{i+1} are shape function of lineare element in one dimensional space.

$$N_i(s) = \frac{S_{i+1} - S}{h} \quad , \quad N_{i+1}(s) = \frac{s - s_i}{h}$$

Then

$$(q^I(x) - q_h(x) - \frac{h^2}{12} q''(s))|_{\overline{s_i s_i}} = \alpha_i N_i(s) - \alpha_{i+1} N_{i+1}(s)$$

The algebraic equations (3.5) can be expressed as

$$AX = F$$

where $X^T = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$, $F^T = \{f_1, f_2, \ldots f_n\}$

$$A = (a_{ij}),$$

$$a_{ij} = \int_{\overline{s_{i-1}s_i}} N_{i+1}(s)\ E(x;y_j)ds_x + \int_{\overline{s_i s_{i+1}}} N_i(s)\ E(x;y_j)ds_x$$

$$f_i = 0(h^3 \ln|h|) \|q\|_{3,\infty,\Gamma}$$

It is clear that

$$\left| \int_{\overline{s_i s_{i+1}}} N_i(s)\ E(x;y_j)ds_x \right| = \left| \int_{\overline{s_i s_{i+1}}} N_i(s)(E(x,y_j) - \tilde{E}(x_i y_j))ds_y \right.$$

$$+ \int_{\overline{s_i s_{i+1}}} N_i(s)\ \tilde{E}\ (x;y_j) ds_x| \leq c\ h| \ln h| + c\ |\ln h|^{1/2}$$

Hence

$$a_{ij} = 0\ (\ln|h|^{1/2})$$

$$\alpha_i = 0\ (\ \frac{(\ln|h|^{1/2}\ )^{n-1}}{(\ln|h^{1/2})^n}\ \ h^3\ |\ln h|)\ \| q \|_{3,\infty,\Gamma}$$

$$= 0\ (h^3|\ln|^{1/2})$$

Finally, we obtain: $\forall$ mesh point z

$$q^I(z) - q_h(z) = \frac{h^2}{12} q''(x) + 0(h^3\ |\ln\ h|^{1/2}\ ) \| q \|_{3,\infty,\Gamma}$$

By virtue of $q^I(z) = q(z)$ we have

Lemma 4 Suppose $q \in c^\infty(\Omega)$ is a solution of (2.5) and q_h is a collocation solution of (3.2), then

$$q(z) - q_h(z) = \frac{h^2}{12} q''(x) + 0(h^3|\ln h|^{1/2}\ )\ \| q \|_{3,\infty,\Gamma} \qquad (3.6)$$

for any mesh ponts z

From lemma 4 we deduce the extrapolation formula for collocation method

$$\frac{1}{3}\ (4\ q_{h/2}(z) - q_h(z)) - q(z) = 0(h^3|\ln\ h|^{1/2}\ ) \| q \|_{3,\infty,\Gamma} \qquad (3.7)$$

4. Numerical Test

Let us consider the following problems

$$-\Delta u(x) = -2\pi\delta(x) \qquad \forall\ x \subset \Omega = \{x|\ |x| \leq 0.5\}.$$

$$U|_{\Gamma} = \ln(0.5) \qquad \text{on} \quad \Gamma = \partial\Omega$$

Exact and Approximation solutions

$\tilde{q}_h(x) = \frac{1}{3}(4q_{h/2}(x)+q_h(x))$	$q_h(x)$	$q_{h/2}(x)$	$q(x)$
1.9999950	1.9986870	1.9996680	2.0000000
1.9999770	1.9986910	1.9996560	2.0000000
1.9999810	1.9986830	1.9996570	2.0000000
1.9999740	1.9986800	1.9996520	2.0000000
1.9999790	1.9986900	1.9996570	2.0000000
1.9999790	1.9986830	1.9996550	2.0000000
1.9999700	1.9986910	1.9996500	2.0000000
1.9999800	1.9986860	1.9996570	2.0000000
1.9999640	1.9986880	1.9996450	2.0000000
1.9999880	1.9986890	1.9996630	2.0000000
1.9999650	1.9986860	1.9996450	2.0000000
1.9999800	1.9986890	1.9996580	2.0000000
1.9999720	1.9986830	1.9996500	2.0000000
1.9999800	1.9986920	1.9996580	2.0000000
1.9999550	1.9986830	1.9996370	2.0000000
1.9999860	1.9986880	1.9996620	2.0000000
1.9999720	1.9986880	1.9996510	2.0000000

$q_h(x) = \frac{1}{3}(4q_{h/2}(x)+q_h(x))$	$q_h(x)$	$q_{h/2}(x)$	$q(x)$
1.9999510	1.9986830	1.9996340	2.0000000
1.9999760	1.9986910	1.9996550	2.0000000
1.9999840	1.9986860	1.9996590	2.0000000
1.9999930	1.9986870	1.9996660	2.0000000
2.0000000	1.9986920	1.9996730	2.0000000
1.9999860	1.9986880	1.9996610	2.0000000
1.9999890	1.9986800	1.9996610	2.0000000
1.9999580	1.9986900	1.9996410	2.0000000
1.9999960	1.9986870	1.9996680	2.0000000
1.9999810	1.9986880	1.9996580	2.0000000
1.9999690	1.9986880	1.9996490	2.0000000
1.9999740	1.9986860	1.9996520	2.0000000
1.9999830	1.9986850	1.9996580	2.0000000
1.9999720	1.9986910	1.9996520	2.0000000
1.9999820	1.9986860	1.9996580	2.0000000

Error

$q(x)-\tilde{q}_h(x)$	$q(x)-q_h(x)$	$q(x)-q_{h/2}(x)$
.0000049	.0013126	.0003319
.0000225	.0013094	.0003443
.0000190	.0013167	.0003433
.0000263	.0013115	.0003477
.0000210	.0013096	.0003432

.0000215	.0013167	.0003452
.0000302	.0013092	.0003500
.0000196	.0013143	.0003432
.0000362	.0013115	.0003550
.0000119	.0013114	.0003368
.0000352	.0013142	.0003549
.0000197	.0013107	.0003425
.0000283	.0013169	.0003505
.0000197	.0013076	.0003417
.0000452	.0013165	.0003630
.0000137	.0013118	.0003382
.0000285	.0013118	.0003493
.0000491	.0013167	.0003660
.0000240	.0013086	.0003451
.0000161	.0013142	.0003406
.0000070	.0013133	.0003335
.0000001	.0013083	.0003272
.0000144	.0013117	.0003388
.0000114	.0013201	.0003387
.0000423	.0013101	.0003593
.0000044	.0013126	.0003315
.0000190	.0013118	.0003421
.0000305	.0013120	.0003510
.0000257	.0013144	.0003480
.0000170	.0013150	.0003415
.0000283	.0013087	.0003483
.0000185	.0013142	.0003424

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