1. Consider the function:

$$f(x) = \sin(\pi x) + \cos(\pi x), \quad x \in \left[\frac{1}{2}, \frac{3}{2}\right]. \tag{1}$$

a) Compute the coefficients of the polynomial interpolator of degree 2 of f, $P(x) = ax^2 + bx + c$, on the points $x_0 = \frac{1}{2}$, $x_1 = \frac{3}{4}$ and $x_2 = \frac{3}{2}$. Solve the associated linear system $\underline{\underline{A}}\ \underline{y} = \underline{q}$ by using $\underline{\underline{A}}^{-1}$.

b) Compute the 1-norm condition number of $\underline{\underline{A}}$.

c) Compute the solution of $f(x) = 0$, correct to 2 decimal places, by using the Newton method. Choose $x = 1$ as starting point.

2. Consider the Cauchy problem:

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T], \\ y(t_0) = y_0. \end{cases} \tag{2}$$

a) Write a pseudocode that uses the forward Euler method with stepsize Δt to solve (**??**).

b) Consider the following Cauchy problem:

$$\begin{cases} y'(t) = \lambda y(t), & t \in [t_0, T], \\ y(0) = y_0, \end{cases} \tag{3}$$

for $\lambda \in \mathcal{R}$. Discuss briefly the stability of the forward Euler method.

c) Set $\lambda = -1$, $t_0 = 0$, $T = 0.3$, $y_0 = 1$, and $\Delta t = 0.1$. Let Y_n, $n = 0, \ldots, 3$, the resulting Forward Euler solution starting from $Y_0 = y_0$. Compute the final time error $|y(T) - Y_3|$.

d) The forward Euler approximation of (**??**) obeys the error bound $|y(t_n) - Y_n| \leq \frac{M_2 \Delta t}{2L}(e^{L(t_n - t_0)} - 1)$, with $M_2 = \max_{t \in [t_0, T]} |y''(t)|$, L the Lipschitz constant of f with respect to y, and $t_n = t_0 + n\Delta t$, for $n = 0, 1, \ldots, \lfloor (T - t_0)/\Delta t \rfloor$. Explain why this bound is of limited value in providing an estimation of the true numerical error using problem (**??**) with $\lambda < 0$ as an example.

3. Consider the unsteady incompressible Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \dfrac{1}{Re} \Delta \mathbf{u} & \text{in } \Omega \times [t_0, T], \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \times [t_0, T], \\ \mathbf{u}(\mathbf{x}, t) = \mathbf{0} & \text{on } \partial\Omega \times [t_0, T], \\ \mathbf{u}(t_0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) & \text{on } \partial\Omega \times \{t_0\}. \end{cases} \tag{4}$$

a) Choose a discretization method for the space approximation and derive a semi-discrete formulation of the problem (**??**).

b) Suppose that $Re >> 1$. Provides a brief overview of the main turbulence modeling approaches by highlighting the advantages and disadvantages.