Sketching : A framework for compressive learning

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M2 Advanced Mathematics

March 2022

Motivation and principles of compressive learning

What is the setup?

We are confronted with a dataset which comes in form of n d-dimensional vectors $\{x_i\}_{i=1}^n$. We would like to perform some kind of learning on it but we are scared of the complexity when n is huge.

What is compressive learning?

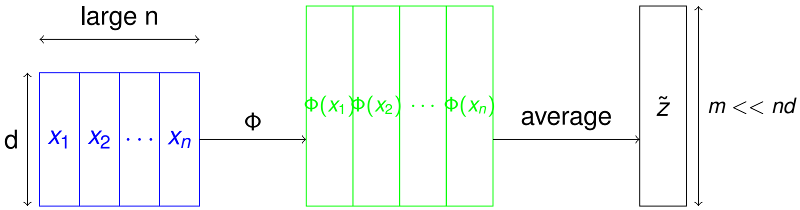
The principle of compressive learning consists of compressing (sketching) the dataset before applying any learning techniques. The sketch consists of a single vector $\tilde{z}$ which is constructed by transforming each vector of the dataset and averaging the results :

$$\tilde{z} = \frac{1}{n} \sum_{i=1}^{n} \Phi(x_i)$$

Motivation and principles of compressive learning

x_1	x_2	$\dots$	x_n
x_{11}	x_{12}	$\dots$	x_{1n}
x_{21}	x_{22}	$\dots$	x_{2n}
$\vdots$	$\vdots$	$\dots$	$\vdots$
x_{d1}	x_{d2}	$\dots$	x_{dn}

Table: Initial dataset

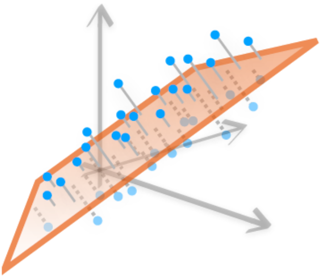


Principal Component Analysis (PCA)

PCA

The goal is to find the linear subspace P_k that best fits the d-dimensional data $\{x_i\}_{i=1}^N$ in the LS sense, i.e., find an orthogonal family of k vectors $\{u_l\}_{l=1}^k$ that maximizes

$$\sum_{l=1}^{k}\sum_{i=1}^{N}|u_l^T x_i|^2.$$



A solution is the k-principal eigenvectors of the empirical autocorrelation matrix

$$\hat{R} = \frac{1}{N}\sum_{i=1}^{N} x_i x_i^T =: \frac{1}{N}\sum_{i=1}^{N}\Phi(x_i).$$

$\hat{R}$ is a *sketch* of our data (of dim d^2).

The sketch

$$\hat{R} = \frac{1}{N}\sum_{i=1}^{N} x_i x_i^T =: \frac{1}{N}\sum_{i=1}^{N} \Phi(x_i) \in \mathbb{R}^{d^2}$$

is a very compressed version of the data $\{x_i\}_{i=1}^N$, *but*, it still contains the geometry of the data.

CS inspired idea

Take m random measurements[a] of each sample and use the sketch defined by $\Phi(x) = \mathcal{M}(xx^T)$. Provided $m > kd$, the principal eigenvectors can be recovered.

[a] $\mathcal{M} : \mathbb{R}^{d \times d} \to \mathbb{R}^m$ satisfying RIP on matrices of rank at most $2k$.

k-means centroids

The problem

The goal is to recover k centroids $\{c_l\}_{l=1}^{k}$ from some data $\{x_i\}_{i=1}^{N}$ that minimize

$$\sum_{i=1}^{N} \min_{l} ||x_i - c_l||^2.$$

For $N >> 1$ traditional algorithms are not very efficient because they take the whole dataset at once...
But, $N >> 1$ allows to use the laws of large numbers and concentration. It is reasonable to consider that the data will accumulate on small portions of the space.

The binning map

Assume the centroids are spaced by at least ε and have a norm smaller than r, then cover $[-r, r]^d$ by, $B = (\frac{2r}{\varepsilon})^d$, d-dimensional cubes (*bins*). For each bin, count the average number of points that belong to it. This defines the binning map $\hat{p} \in \mathbb{R}_+^B$.

This gives us a sketch of the data, but in a large dimensional space.
But, if the model that generated the data is "structured", i.e., the data concentrates in a few centroids, then the problem is a *sparse* problem.

CS inspired idea

Use a Gaussian random matrix in $\mathbb{R}^{m \times N}$ and define the sketch as $\tilde{z} = A\hat{p}$ and solve

$$\tilde{p} = argmin_{p \in \Sigma_k^+} ||\tilde{z} - Ap||^2$$

Sketching estimates the underlying data-distribution

We can do *k*-means clustering with *m* measures using:

- $m \geq k \log B$ for Gaussian sampling
- $m \geq k \log(k)^3 \log B$ for DFT of $\hat{p}$

What if we consider the *continuous* Fourier Tranform (FT) ?
We only know the empirical distribution $\bar{p}_{\mathcal{X}} = \frac{1}{N}\sum_{i=1}^{N} \delta(x_i - x)$, so

$$FT(\bar{p})(\omega) = \int_{\mathbb{R}^d} \bar{p}_{\mathcal{X}}(x) e^{-i2\pi\langle w,x\rangle} dx = \frac{1}{N}\sum_{i=1}^{N} e^{-i2\pi\langle w,x_i\rangle} =: \bar{\Psi}_{\bar{p}_{\mathcal{X}}}(\omega)$$

$$\xrightarrow{\mathbb{E}} \bar{\Psi}_{p^*}(\omega) \quad \text{The "true" characteristic function at } \omega.$$

CS inspired idea

If the true distribution p^* is "simple", then interpolating the characteristic function, using simple models, on "few" of its samples should give the true distribution.

Parallel with signal processing

- Recall that in signal processing our goal is to reconstruct a vector $x \in \mathbb{R}^d$ from $y = Ax + \epsilon$
- At first glance compressive learning setup is rather different as we deal with a large collection of vectors, rather than just one
- The analogy becomes clearer if we assume (as it is often the case in ML) that our vectors $\{x_i\}_{i=1}^n$ are modeled as i.i.d. random vectors having a probability measure $\mathbb{P}$
- In this case we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \Phi\left(x_i\right) \overset{a.s.}{=} \mathbb{E}_{\mathbb{P}}[\Phi(X)] = \mathcal{A}(\mathbb{P}),$$

where $\mathcal{A}$ is a linear operator matching a probability measure to the expectation over this measure of the feature map Φ

Parallel with signal processing

- In this manner we can write

$$\tilde{z} = \frac{1}{n}\sum_{i=1}^{n}\Phi(x_i) \approx \mathcal{A}(\mathbb{P}) = \mathcal{A}(\mathbb{P}) + \epsilon$$

- Thus instead of considering a linear projection of a vector measured with noise ($x \rightarrow Ax + \epsilon$), in compressive learning we consider a linear projection of the underlying probability measured with noise: $\mathbb{P} \rightarrow \mathcal{A}(\mathbb{P}) + \epsilon$
- In signal processing the linear measurement matrix A can be chosen at random to ensure good reconstruction properties with high probability. By analogy, in compressive sensing Φ is also often randomised

Task driven distances

Task driven distance

Given a loss function L and probability distributions p_X and $p_X^{'}$, we consider the distance

$$\rho(p_X, p_X^{'}) = \sup_{\theta} |R^*(\theta|p_X) - R^*(\theta|p_X^{'})|$$

where $R^*(\theta|p_X) = E_{X\sim p_X}(L(\theta)|X)$ is the expected risk under p_X.

- The loss function L is task specific
- Excess risk bounds $0 \leq R^*(\theta^{*'}|p_X) - R^*(\theta^*|p_X) \leq 2\rho(p_X, p_X^{'})$ where $\theta^* = argmin_\theta R^*(\theta|p_X)$ and $\theta^{*'} = argmin_\theta R^*(\theta|p_X^{'})$
- When $\hat{p}_X$ is the empirical distribution on the data, and p_X is the true distribution, under certain conditions $\rho(\hat{p}_X, p_X) = O(\frac{1}{\sqrt{n}})$.

In the compressive learning framework we are interested in an upper bound of the excess risk which is controlled by the task driven distance.

The Lower Restricted Isometry Property (LRIP)

The operator $\mathcal{A}$ is said to have the LRIP with constant C_0 and with respect to a parametric subfamily $\Sigma_\theta = \{p_\theta | \theta \in \Theta\}$ if

$$\rho(p_\theta, p_{\theta'}) \leq C_0 ||\mathcal{A}(p_\theta) - \mathcal{A}(p_{\theta'})||$$

for all $p_\theta, p_{\theta'} \in \Sigma_\theta$

Excess risk bound under LRIP: for all $\theta^{'} \in \Sigma_\theta$

$$R^*(\tilde{\theta}|p_X) - R^*(\theta^{'}|p_X) \leq 4C_0||\mathcal{A}(p_X) - \tilde{z}|| \\ + 4C_0||\mathcal{A}(p_{\theta'}) - \mathcal{A}(p_X)|| + 2\rho(p_{\theta'}, p_X)$$

with $\tilde{\theta} = argmin_\theta C(\theta|\tilde{z})$

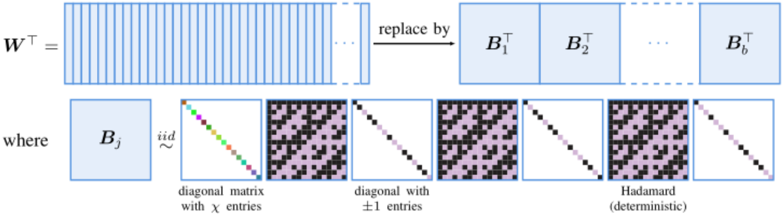
- Choosing $\theta^{'} = \theta^*$, this result is interpretable in terms of modeling and sampling error

Expected and mean kernel, MMD

- Two sources of randomness: the data and the random features used for the sketch
- For the random feature map we have $< \frac{1}{m}\Phi(x), \frac{1}{m}\Phi(x^{'}) >= \frac{1}{m}\sum_{j=1}^{m} e^{-j2\pi w_j(x-x^{'})}$
- Averaging over the random features gives the *expected kernel* $k(x, x^{'}) = E_w(\exp(-j2\pi < w, x - x^{'} >))$
- We define the *mean kernel* as $k(p, q) = E_{x\sim p, x^{'}\sim q} k(x, x^{'})$
- A quantity of interest is the *maximum mean discrepancy* $MMD(p, q) = \sqrt{k(p, p) - 2k(p, q) + k(q, q)}$
- It can be shown using concentration of measure that $\frac{1}{\sqrt{2}} MMD(p_\theta, p_{\theta'}) \leq \frac{1}{\sqrt{m}} ||\mathcal{A}(p_\theta) - \mathcal{A}(p_{\theta'})||$ when Σ_θ is a finite set
- When Σ_θ is infinite additional assumptions are required to ensure that the LRIP property holds

Compressed clustering, fast sketching

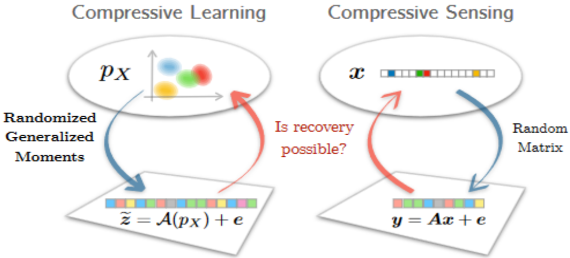
- In practice computing the sketch using random Fourier samples might be problematic (accurate implementations of the complex valued function $x \to \exp(-j2\pi x)$ is slow)
- Using the Fast Walsh-Hadamard transform it is possible to speed up the sketching process ($\Theta(d \log d)$ instead of $\Theta(d^2)$ for matrix products)
- Empirical studies on simulated and real data show that with proper setup, learning from sketch is comparable to classical approaches (k-means clustering)



Conclusion

Sketching is a framework that allows:

- to get leverage of assumptions on the simplicity of the underlying data generating process
- to apply ideas similar to Compressed Sensing
- learn the underlying data distribution through a compressed dataset
- not to reduce the dimension of the features, but the dimension of the dataset, without losing its characteristic (geometric) properties


Compressive Learning
Compressive Sensing
p_X
x
Randomized Generalized Moments
Is recovery possible?
Random Matrix
$\tilde{z} = \mathcal{A}(p_X) + e$
$y = Ax + e$