

# $\mathcal{U}$ -Bootstrap percolation : critical probability, exponential decay and applications, by Ivailo HARTARSKY

Leo Davy    Martin Gjorgjevski    Aleksandr Pak

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# Update rules

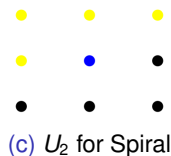
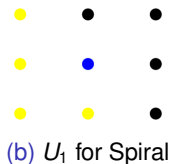
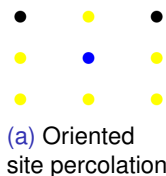
- An update rule is a finite set  $X \subseteq \mathbb{Z}^2 - \{0\}$
- An update family is a finite collection of update rules  
 $\mathcal{U} = \{X \subseteq \mathbb{Z}^2 - \{0\}\}$

$\mathcal{U}$ -Bootstrap percolation initialized at  $A$  refers to the following process:

- $A_0 = A$
- $A_{t+1} = A_t \cup \{x \in \mathbb{Z}^2 : x + X \subseteq A_t \text{ for some } X \in \mathcal{U}\}$

- The set  $A$  is known as the set of initially infected sites
- The closure of  $A$  is defined as  $[A] = \cup_{t \geq 0} A_t$
- The initialization is random i.e. each site (vertex) in  $\mathbb{Z}^2$  is infected with probability  $p$  independently from the other vertices
- The process is monotone i.e. if a site gets infected, it stays infected forever
- After the initialization, the process is deterministic in the sense that a site will get infected if and only if there is some rule  $X$  in  $\mathcal{U}$  such that  $x + X$  is infected

# Examples



- r-Neighbour models for  $r=1,2,3,4$
- Oriented site  $\mathcal{U} = \{(-1, 1), (1, 1)\}$
- Spiral  $\mathcal{U} = \{U_1, U_2, U_3, U_4\}$ , where
  - $U_1 = \{(1, -1), (1, 0), (1, 1), (0, 1)\}$
  - $U_2 = \{(1, -1), (1, 0), (-1, -1), (0, -1)\}$
  - $U_3 = -U_1, U_4 = -U_2$
- Directed triangular bootstrap percolation

# Stable directions, basic properties

For a vector  $u \in \mathbb{S}^1$ , we define  $\mathbb{H}_u = \{x \in \mathbb{Z}^2 \mid \langle x, u \rangle < 0\}$ .

## Definition

Given an update family  $\mathcal{U}$ , a direction  $u \in \mathbb{S}^1$  is

- stable if  $[\mathbb{H}_u] = \mathbb{H}_u$ . The set of stable directions is denoted by  $\mathcal{S} = \mathcal{S}(\mathcal{U})$
  - strongly stable if  $u \in \text{int}\mathcal{S}$
  - unstable if it is not stable
- 
- Dichotomy  $[\mathbb{H}_u] \in \{\mathbb{H}_u, \mathbb{Z}^2\}$
  - $\mathcal{S} \subseteq \mathbb{S}^1$  is a set of stable directions for some update family  $\mathcal{U}$  if and only if it can be expressed as a union of closed intervals with rational endpoints<sup>1</sup> in  $\mathbb{S}^1$

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<sup>1</sup>A direction  $u \in \mathbb{S}^1$  is said to be rational if there is a point in the grid  $\mathbb{Z}^2 \cap \{\lambda u \mid \lambda \in \mathbb{R}\}$  1

# Classification of $\mathcal{U}$ -Bootstrap percolation

$\mathcal{U}$ -bootstrap percolation update families exhibit different properties based on their stable sets. Let  $\mathcal{U}$  be an update family with a set of stable directions  $\mathcal{S}$

- If there is a open semicircle  $C$  such that  $\mathcal{S} \cap C = \emptyset$  then  $\mathcal{U}$  is said to be *supercritical*
- If every open semicircle  $C$  intersects  $\mathcal{S}$ , but there is an open semicircle  $C_0$  that doesn't intersect  $\text{int}\mathcal{S}$  then  $\mathcal{U}$  is said to be *critical*
- If every open semicircle  $C$  intersects  $\text{int}\mathcal{S}$  then  $\mathcal{U}$  is said to be *subcritical*

# Supercritical and critical families

## Infection time of the origin

The infection time of 0 is defined as  $\tau_p = \inf\{t \in \mathbb{N} : 0 \in A_t\}$ , given that  $A_0 = A$  is sampled according to a Bernoulli  $p$  distribution

- For supercritical families,  $\tau_p = p^{-\Theta(1)}$  as  $p \rightarrow 0$  with high probability
- For critical families,  $\tau_p = \exp(p^{-\Theta(1)})$  as  $p \rightarrow 0$  with high probability

Corollary<sup>2</sup>: For supercritical and critical families,  $p_c = \inf\{p > 0 \mid P_p([A] = \mathbb{Z}^2) = 1\} = 0$  i.e. for any  $p > 0$  we have percolation.

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<sup>2</sup>BOLLOBÁS, SMITH, UZZEL, Monotone Cellular automata in a random environment, *Combinatorics, Probability and Computing*, 2015

# Subcritical Families

- For subcritical families  $p_c > 0$ <sup>3</sup>
- Percolation at  $p = p_c$  is open
- Behaviour of  $\tau_p$  as  $p \rightarrow p_c$  from above is open
- Exponential decay answered
- Infinite component without percolation answered

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<sup>3</sup>BALISTER, BOLLOBÁS, PRZYKUCKI, SMITH, Subcritical  $\mathcal{U}$ -Bootstrap percolation models have non-trivial phase transitions, *arXiv*, 2019



# $d_u^\theta$ measures directions that are difficult to infect

## Critical densities with conic boundary conditions

For  $u \in \mathbb{S}^1$  and  $\theta \in [-\pi, \pi]$

$$d_u^\theta := \inf \left\{ q \in [0, 1], \sum_n n \mathbb{P}_q(0 \notin [(A \cup V_{u, u+\theta}) \cap B_n]) < \infty \right\}$$

Morally, the critical probability with infection of

$$V_{u, u+\theta} = \mathbb{H}_u \cap \mathbb{H}_{u+\theta}.$$

- The summand decays slowly in  $n$  when it is hard to infect the origin using only infections at distance less than  $n$ . So, when it is hard to infect 0,  $d_u^\theta$  is large<sup>4</sup>.
- When  $\theta \sim \pm\pi$ , few sites are infected, so it is easy for the origin not to be infected, the summand can be large. Hence,  $d_u^\theta$  decreases when  $\theta \rightarrow 0$ .

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<sup>4</sup>non zero...

## Theorem

For any  $\mathcal{U}$ -bootstrap percolation model, its critical probability

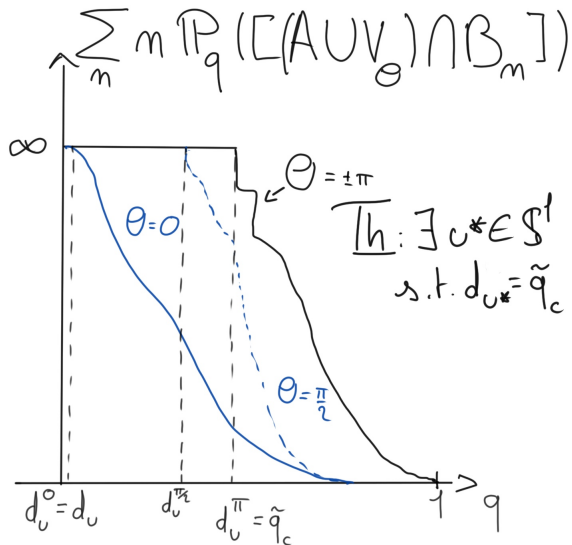
$$\tilde{q}_c = \inf\{q \in [0, 1], \sum_n n \mathbb{P}_q(0 \notin [A \cap B_n]) < \infty\}$$

is equal to the maximal value of its critical density function

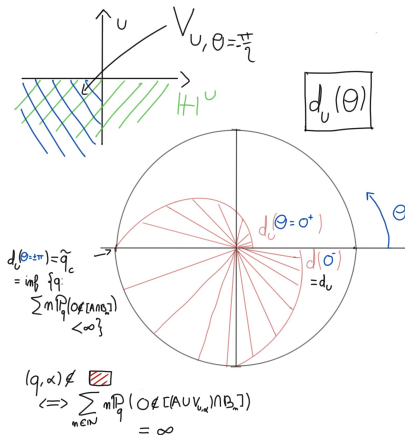
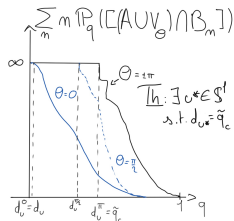
$$d_u = \max_{0^\pm} \inf\{q \in [0, 1], \sum_n n \mathbb{P}_q(0 \notin [(A \cup V_{u, u+0^\pm}) \cap B_n] < \infty\}$$

for  $u$  in any semicircle  $C$ , i.e.,

$$\tilde{q}_c = \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u.$$

Phase diagram in a fixed direction  $u$ 

# Phase diagram in a fixed direction $u$



# Monotonicity

Let's denote  $E_{u,\theta} = \{0 \notin [(A \cup V_{u,u+\theta}) \cap B_n]\}$ . Then,

$$E_{u,\pm\pi} = \{0 \notin [A \cap B_n]\} \supset E_{u,\theta}$$

which gives that the following holds for any  $u$

$$\tilde{q}_c \geq \sup_{\theta} \sup_u d_u^{\theta} \geq \limsup_{\theta \rightarrow 0} \sup_u d_u^{\theta} = \sup_u d_u.$$

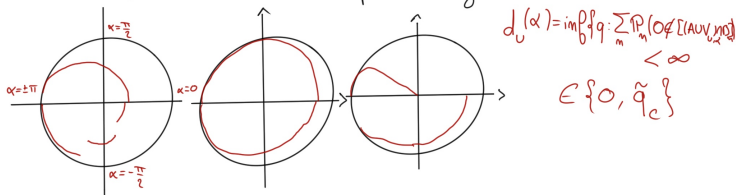
The theorem states that all those quantities are equal.

## Meaning of the theorem

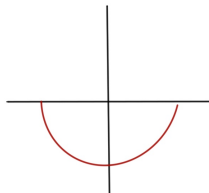
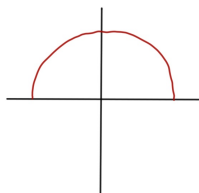
The difficulty of the model is as hard as its most difficult direction. In this direction, infecting a half plane doesn't affect the infection of the origin.

# Consequence of monotonicity

For  $u_1, u_2, \dots \in \mathcal{S}^1$ , we have phase diagrams



Th:  $\exists u^* \in \mathcal{S}^1$  with one of the following phase diagrams on  $[0, \pi]$  or  $[-\pi, \pi]$



Th: (Bollobás... & Hartarsky)

For any update family  $\mathcal{U}$ ,  
the family is:

- subcritical if and only if

its phase diagram  $(u, \theta) \in \mathbb{S}^1 \times \mathbb{S}^1 \mapsto d_u^\theta$   
is contained in a torus and contains a half disk  
(of radius  $1 - \tilde{q}_c$  and  $\tilde{q}_c$ ) (of radius  $\tilde{q}_c$ )

- supercritical or critical if and only if

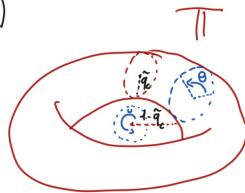
its phase diagram  $(u, \theta) \in \mathbb{S}^1 \times \mathbb{S}^1 \mapsto d_u^\theta$   
is contained in a circle ( $\cong \mathbb{S}^1$ )

and it is supercritical if its phase diagram



contains a half circle

$$u \in \mathbb{S}^1 \mapsto \mathbb{P}_{q=0}(0 \notin [V_{u,0}]) = \begin{cases} 1, & \text{if } u \text{ unstable} \\ 0, & \text{otherwise} \end{cases}$$



# Proving $\sup d_u \geq \tilde{q}_c$

The goal is to show, that for any  $q' > \sup d_u$  it holds that

$$\sum_n n \mathbb{P}_{q'}(0 \notin [A \cap B_n]) < \infty.$$

The idea is to show that, at  $q'$ , the origin is infected most of the time.

**2-step percolation :  $q' = \sup d_u + \varepsilon$**

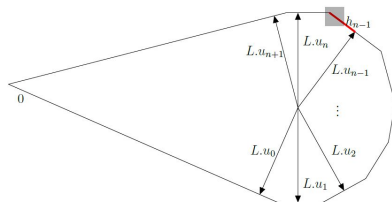
- 1 Infect sites with probability  $\varepsilon$  to find some structures
- 2 Infecting new sites with probability  $q$  allows structures to grow



# Some details on the proof

- The structures that grow are droplets, with sides  $(u_i)_{i=1}^n$  depending on  $\sup d_u$ .
- In the second percolation, droplets of size  $L$  grow into droplets of size  $\geq (1 + \delta)L$ , for some  $\delta > 0$ .
- The proof can be done in any semi-circle, so we can get  $\tilde{q}_c = \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u$
- The proof contains that  $\forall q > \sup d_u$ , there exists a constant  $c(q) > 0$  such that

$$\theta_n(q) \leq e^{-c(q)n}$$



# Applying the theorem

## Theorem

For any update rules  $\mathcal{U}$ ,

$$q_c \leq \tilde{q}_c = \sup_{u \in \mathbb{S}^1} d_u = \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u.$$

In particular, if  $\mathcal{U}$  is not subcritical, then  $\tilde{q}_c = q_c = 0$

So, having knowledge on  $u \mapsto d_u$  allows to upper bound  $q_c$ ...

## Proposition : (It's harder for submodels to infect)

For any sub-collection of rules  $\mathcal{U}' \subset \mathcal{U}$

$$q_c(\mathcal{U}) \leq \tilde{q}_c(\mathcal{U}) \leq \inf_C \sup_{u \in C} d_u(\mathcal{U}')$$

... and it is not even necessary to know the critical density for the whole set of rules to get such bounds.

# First level bound

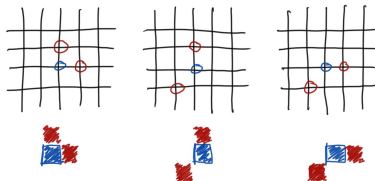
## *DTBP* : Directed Triangular Bootstrap Percolation

Let  $\mathcal{U}' = \{(-1, -1), (0, 1)\}$ , one of the rules of *DTBP*, then

$$q_c(DTBP) \leq \tilde{q}_c(DTBP) \leq \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u(\mathcal{U}').$$

Applying a general formula for one rule families (using OP) gives  $q_c(DTBP) \leq 0.245\dots^a$

<sup>a</sup>Previous known bound was 0.312



# Second level bound

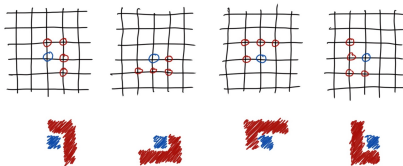
However, knowing subfamilies with a single rule is not enough.

## Spiral

For spiral, it is possible to compute  $d_u$  for all pairs of rules, such that the difficulty on pairs is the same as the difficulty of some Bidirectional OP :

$$q_c(\text{Spiral}) \leq \tilde{q}_c(\text{Spiral}) \leq 1 - p_c^{OP}.$$

And the result is tight.



# Oriented percolation as an example of bootstrap percolation

- Oriented percolation (OP) is one of the simplest subcritical BP model.
- It is a BP model with one-family rule  
 $\mathcal{U} = \{U\} = \{\{-1, 1\}, \{1, 1\}\}$
- Some of the well-known results in the field are reviewed in Durrett's article <sup>5</sup>
- We will follow this article to introduce some non-trivial results about OP

## Remark

In the article of Durrett bond percolation is considered rather than site. However, it can be shown that the results obtained also apply to site OP.

<sup>5</sup>R. DURRETT, Oriented Percolation in Two Dimensions, *The Annals of Probability*, 1984

# Edge speed

- We remind that the parameter of OP -  $p = 1 - q$  stands for the intensity of **healthy** sites
- We let  $p_c^{OP}$  be the critical probability for OP
- We say that  $x \rightarrow y$  if there exist  $x_0 = x, x_1, \dots, x_n = y$  such that  $x_i - x_{i-1} \in \mathcal{U}$  and  $x_i$  open for  $0 < i \leq n$ , i.e. there exists an OP path between  $x$  and  $y$ .
- We are naturally interested in the event  $\{0 \rightarrow \infty\} = \{\cap_{i=1}^{\infty} L_i \neq \emptyset\}$ , where  $L_n = \{x : (0, 0) \rightarrow (x, n)\}$  - the set of sites on the height  $n$  connected to 0.
- Notice that saying that  $0 \rightarrow \infty$  is equivalent to saying that 0 is not infected in the BP model (duality BP - OP).

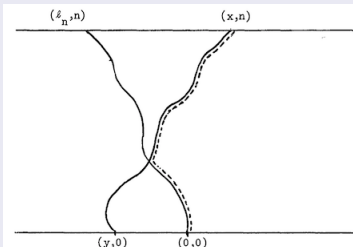
# Edge Speed

- We denote  $r_n = \sup\{x \in \mathbb{Z}, \exists y \leq 0, (y, 0) \rightarrow (x, n)\}$  - the rightmost edge with the convention  $\sup\{\emptyset\} = -\infty$
- This definition may seem a bit artificial at first glance (why would we look at  $(y, 0) \rightarrow (x, n)$  for some  $y \leq 0$  rather than  $(0, 0) \rightarrow (x, n)$ )?

## Property

Provided that  $L_n \neq \emptyset$ , we have  $r_n = \sup L_n = \sup\{x : (0, 0) \rightarrow (x, n)\}$

## Proof by a picture



# Edge speed

- We now want to quantify the asymptotic behaviour of  $\frac{r_n}{n}$

## Theorem - definition

There exists a function  $\alpha : [0, 1] \rightarrow [-\infty, 1]$  called edge speed with the following property:

$$\frac{r_n}{n} \rightarrow \alpha(p) = \inf_n \mathbb{E}_p[r_n/n]$$

Moreover, we have that  $\alpha$  is continuous and strictly increasing on  $[p_c^{OP}, 1]$  with  $\alpha(p_c^{OP}) = 0$ ,  $\alpha(1) = 1$  and  $\alpha(p) = -\infty$  for  $p < p_c^{OP}$ .

- Intuitively edge speed tells us how far the rightmost path (starting from height 0 and to the left of the origin) is expected to go.
- Edge speed has the following "criticality" properties which we state without proof



# Properties of edge speed

## Property 1

For  $p > p_c^{OP}$  (above criticality for OP) and  $\alpha_0 < \alpha(p)$  with positive probability there exists an infinite OP path  $((a_i, i))_{i \in \mathcal{N}}$  with  $a_0 = 0$  and  $\inf_n \frac{a_n}{n} \geq \alpha_0$ . Intuitively this means that we have chances to get an infinite path that goes sufficiently far (but below edge speed) to the right.

## Property 2

For  $\alpha_0 > \alpha(p)$ , for some  $\gamma > 0$  we have

$$\mathbb{P}_p(r_n \geq \alpha_0 n) \leq e^{-\gamma n}$$

This exponential decay shows that it is unlikely to find a path that goes too far (further than edge speed) to the right.

# Critical densities of OP

- We let  $\psi(u) = 1 - \alpha^{-1}(|\tan(u)|)$ , where  $\alpha^{-1}$  is the inverse of the edge speed function  $\alpha$ .
- The following theorem expresses the critical density of OP depending on the direction  $u$  which is parametrized by the angle it makes with the origin:  $u \in [-\pi, \pi]$ .

## Theorem

The critical densities of the BP percolation with  $\mathcal{U} = \{(1, 1), (-1, 1)\}$  (dual to OP) are given by

$$d_u(\mathcal{U}) = \begin{cases} 0, & u \in [-3\pi/4, -\pi/4] \\ 1 - p_c^{\text{OP}} = q_c, & u \in [0, \pi] \\ \psi(u), & \text{otherwise} \end{cases} \quad (1)$$

# Outline of the proof

- If  $u \in (-3\pi/4, -\pi/4)$ , we have  $[\mathbb{H}_u] = \mathbb{Z}^2$  (the directions are unstable), so in this case  $d_u = 0$
- By symmetry it suffices to treat  $u \in [-\pi/4, \pi/2]$
- The critical density  $d_u$  can be thought as the value  $\tilde{q}$  above which a.s there is no oriented infinite path from the origin which does not pass by  $\mathbb{H}_u$
- It suffices then to show that below  $\tilde{q}$  there is such an infinite path with positive probability and above it does not exist a.s.
- For  $q < \tilde{q}$  one can prove that it is in order not to pass by  $\mathbb{H}_u$  it suffices to go to the right with the speed below edge speed which is possible (with positive probability) by the property 2
- Conversely, for  $q > \tilde{q}$ , we would have to walk to the right above edge speed to get around  $\mathbb{H}_u$  which is not possible (exponential decay)

# Open questions and conjectures about OP

- We remind that  $q_c = \inf \{q \in [0, 1], \mathbb{P}_q([A] = \mathbb{Z}^2) = 1\}$ ,  
whereas  $\tilde{q}_c = \inf \{q \in [0, 1], \sum_n n\theta_n(q) < \infty\}$

## Conjecture

For all BP models (all update families) we have:  $q_c = \tilde{q}_c$ .

- It would be practical to know if the complication of taking right and left limits to define the critical density  $d_u = \max(d_u^+, d_u^-)$  is necessary.

## Question

What are the continuity properties of the function  $(u, \theta) \rightarrow d_u^\theta$ ?