Solutions Manual to

Introduction to Mathematical Analysis

Original Text by

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Contents

Preface

This document is a solutions manual to accompany

Douglass, Steven A. *Introduction to Mathematical Analysis.* Addison-Wesley, 1996. ("IMA")

IMA was the textbook of the introductory analysis courses—MAS241 Analysis I and MAS242 Analysis II—offered by the Department of Mathematical Sciences in KAIST. This paper is a collection of my solutions to exercises studying introductory analysis with this book.

All solutions are based on the author's definitions, assuming that all theorems, including lemmas and corollaries, and examples covered in the text can be cited without proof. If one quotes a proposition introduced in another exercise, see the solution to that problem for proof. In particular, there are many equivalent propositions about notions of the elementary topology. Please keep that in mind when you solve problems from other sources.

This paper is unofficial and might still contain some errors, although I have worked on minimizing them. You can contribute to it via my email. All reports and suggestions are welcome.

"The first principle is that you must not fool yourself
—and you are the easiest person to fool."
Richard P. Feynman

본 문서는 상기 도서(이하 "IMA")의 연습 문제 해설집입니다. IMA는 KAIST 수리과학과 기초 해석학 과목인 MAS241 해석학 I 및 MAS242 해석학 II 수업에서 교재로 사용했었습니다. 이 해설집은 제가 IMA로 해석학을 공부하면서 풀었던 일부 연습 문제의 풀이를 모은 것입니다.

모든 풀이는 저자가 내린 정의를 바탕으로 보조 정리와 따름 정리를 포함한 모든 정리 및 본문에서 다룬 예시를 증명 없이 인용할 수 있을 때를 가정하고 작성했습니다. 특히 2장에서 배우는 기초 위상수학은 개념마다 수많은 동치 명제가 있으므로 출처가 다른 문제를 푸실 때 유의하시기를 바랍니다. 다른 연습 문제에서 소개한 명제를 인용하면 증명은 해당 문제의 풀이를 참고하세요.

이 해설집은 공인되지 않았습니다. 여러 번 퇴고하고 검수했지만, 여전히 논리에 오류가 있거나 계산을 실수했을 수 있습니다. 제 메일로 이 해설집에 기여하실 수 있습니다. 모든 제보와 제안을 환영합니다.

"제1 원칙은 자기 자신을 속이지 않는 것이다
—그리고 당신은 속이기 가장 쉬운 사람이다."
리처드 P. 파인만

Chapter 1

The Structure of the Real Numbers: Sequences

1.1 Completeness of the Real Numbers

1.4 Suppose S is a nonempty set of real numbers that is bounded above. Let $\mu = \sup S$. Prove that μ is unique. (Assume there exists a second number having the properties of $\mu = \sup S$ and show that it must equal μ.)

Solution. Suppose that ν is also a supremum of S. μ is an upper bound for S, so $\nu \leq \mu$. ν is an upper bound for S, so $\mu \leq \nu$. Therefore, $\mu = \nu$. ■

1.5 Use Axiom 1.1.1 and Theorem 1.1.1 to derive Archimedes' principle. Hint: Given $\epsilon > 0$ and any positive constant M, form the set $S = \{k\epsilon : k \text{ in } \mathbb{N}\}$. Show that M cannot possibly be an upper bound for S.

Solution. Suppose, to the contrary, that M is an upper bound for S. Since S is a nonempty set of real numbers that is bounded above, by Axiom 1.1.1, $\mu = \sup S$ exists in $\mathbb{R}$. By Theorem 1.1.1, there exists a $k \in \mathbb{N}$ such that $\mu - \epsilon < k\epsilon \leq \mu$. Then we have a contradiction that $\mu < (k+1)\epsilon \in S$. Therefore, M cannot possibly be an upper bound for S, so Archimedes' principle holds. ■

1.6 Show that Archimedes' principle is equivalent to the following statement: For any $c > 0$, there exists a k in $\mathbb{N}$ such that $k - 1 \leq c < k$.

Solution.

($\Longrightarrow$) Let $c > 0$ be given, and let $S = \{k \in \mathbb{N} \mid c < k\}$. By Archimedes' principle, there exists a $k \in \mathbb{N}$ such that $k \in S$, so $S \neq \varnothing$. Then the inequalities hold for $k = \min S$.

($\Longleftarrow$) Let ϵ and M be any two positive real numbers. For $c = M/\epsilon$, there exists a $k \in \mathbb{N}$ such that $k - 1 \leq M/\epsilon < k$ which implies $M < k\epsilon$. ■

1.7 Use Theorem 1.1.2 to treat the cases when $c \leq 0 < d$ and $c < d \leq 0$ and thus show that, if $c < d$, there exists a rational number x such that $c < x < d$.

Solution. The theorem is proved in the case where $0 < c < d$. If $c \leq 0 < d$, there exists a rational number x such that $c \leq 0 < d/2 < x < d$. If $c < d \leq 0$, there exists a rational number x such that $0 < -\dfrac{c+d}{2} < x < -c$. Therefore, $c < -x < \dfrac{c+d}{2} < d$. This completes the proof of Theorem 1.1.2. ■

1.8 Suppose that c and d are real numbers with $c < d$. Show that there exists an irrational number x such that $c < x < d$.

Solution. By Theorem 1.1.2, there exists a rational number x such that $c < x < d$, and there exists a rational number y such that $x < y < d$. We know that $\sqrt{2}$ is irrational. Since

$$\begin{aligned} 1 < \sqrt{2} < 2 &\iff y - x < \sqrt{2}(y - x) < 2(y - x) \\ &\iff c < x < 2x - y + \sqrt{2}(y - x) < y < d, \end{aligned}$$

we claim that $z = 2x - y + \sqrt{2}(y - x)$ is irrational. Suppose, to the contrary, that z is rational. Then we have a contradiction that $\sqrt{2} = \dfrac{z - 2x + y}{y - x}$ is rational. This completes the proof. ■

1.12 If S is a nonempty, bounded set in $\mathbb{R}$ and if T is a nonempty subset of S, prove that

$$\inf S \le \inf T \le \sup T \le \sup S.$$

Solution. S is bounded, and so is T. It is evident that $\inf T \le \sup T$. Since T is nonempty, let $t \in T$. Then $t \in S$ implies $\inf S \le t \le \sup S$ for all $t \in T$. Hence, $\inf S$ is a lower bound for T, and $\sup S$ is an upper bound for T. Therefore, $\inf T \ge \inf S$, and $\sup T \le \sup S$. ■

1.14 Suppose that S_1 and S_2 are nonempty, bounded sets in $\mathbb{R}$. Show that $\sup(S_1 \cup S_2) = \max\{\sup S_1, \sup S_2\}$ and that $\inf(S_1 \cup S_2) = \min\{\inf S_1, \inf S_2\}$. Is an analogous theorem for intersections possible? Explain.

Solution. Without loss of generality, assume that $\sup S_1 \le \sup S_2$. For all $x_1 \in S_1$, $x_1 \le \sup S_1 \le \sup S_2$. For all $x_2 \in S_2$, $x_2 \le \sup S_2$. Hence, $\sup S_2$ is an upper bound for $S_1 \cup S_2$, so $\sup(S_1 \cup S_2)$ exists, and $\sup(S_1 \cup S_2) \le \sup S_2$. By Exercise 1.12, $\sup(S_1 \cup S_2) \ge \sup S_2$. Therefore, $\sup(S_1 \cup S_2) = \sup S_2 = \max\{\sup S_1, \sup S_2\}$.

Likewise, without loss of generality, assume that $\inf S_1 \ge \inf S_2$. For all $x_1 \in S_1$, $x_1 \ge \inf S_1 \ge \inf S_2$. For all $x_2 \in S_2$, $x_2 \ge \inf S_2$. Hence, $\inf S_2$ is a lower bound for $S_1 \cup S_2$, so $\inf(S_1 \cup S_2)$ exists, and $\inf(S_1 \cup S_2) \ge \inf S_2$. By Exercise 1.12, $\inf(S_1 \cup S_2) \le \inf S_2$. Therefore, $\inf(S_1 \cup S_2) = \inf S_2 = \min\{\inf S_1, \inf S_2\}$.

Define $\sup \varnothing := -\infty$ and $\inf \varnothing := \infty$. For intersections, $\sup(S_1 \cap S_2) \le \min\{\sup S_1, \sup S_2\}$ and $\inf(S_1 \cap S_2) \ge \max\{\inf S_1, \inf S_2\}$. The inequalities can be strict. Consider $S_1 = \{-1, 0, 1\}$ and $S_2 = \{-2, 0, 2\}$. Then $S_1 \cap S_2 = \{0\}$. We have $\sup(S_1 \cap S_2) = \inf(S_1 \cap S_2) = 0$, but $\sup S_k = k$ and $\inf S_k = -k$ for each $k \in \{1, 2\}$. ■

1.15 Let S_1 and S_2 be two nonempty, bounded sets in $\mathbb{R}$. Define $S_1 + S_2 = \{x_1 + x_2 : x_i \text{ in } S_i, i = 1, 2\}$. Prove that $S_1 + S_2$ is bounded and that $\sup(S_1 + S_2) = \sup S_1 + \sup S_2$ and $\inf(S_1 + S_2) = \inf S_1 + \inf S_2$.

Solution. The completeness axiom for $\mathbb{R}$ guarantees the existence of $\sup S_i \in \mathbb{R}$ and $\inf S_i \in \mathbb{R}$ for each $i \in \{1, 2\}$, so we have $\inf S_i \le x_i \le \sup S_i$ for all $x_i \in S_i$. Adding two inequalities gives

$$\inf S_1 + \inf S_2 \le x_1 + x_2 \le \sup S_1 + \sup S_2$$

for all $x_1 + x_2 \in S_1 + S_2$. Therefore, $S_1 + S_2$ is bounded. Also, $\varnothing \ne S_1 \subset \mathbb{R}$ and $\varnothing \ne S_2 \subset \mathbb{R}$ implies $\varnothing \ne S_1 + S_2 \subset \mathbb{R}$. Hence, both $\sup(S_1 + S_2) \in \mathbb{R}$ and $\inf(S_1 + S_2) \in \mathbb{R}$ exist as a result of the completeness axiom for $\mathbb{R}$.

(Proof by definition) $\sup S_1 + \sup S_2$ is an upper bound for $S_1 + S_2$, so $\sup(S_1 + S_2) \leq \sup S_1 + \sup S_2$. For all $x = x_1 + x_2$, fix $x_2 \in S_2$. Then we have $x_1 \leq \sup(S_1 + S_2) - x_2$. $\sup(S_1 + S_2) - x_2$ is an upper bound for S_1, so $\sup S_1 \leq \sup(S_1 + S_2) - x_2$.

We also have $x_2 \leq \sup(S_1 + S_2) - \sup S_1$. $\sup(S_1 + S_2) - \sup S_1$ is an upper bound for S_2, so $\sup S_2 \leq \sup(S_1 + S_2) - \sup S_1$. Therefore, $\sup(S_1 + S_2) = \sup S_1 + \sup S_2$.

Likewise, $\inf S_1 + \inf S_2$ is a lower bound for $S_1 + S_2$, so $\inf(S_1 + S_2) \geq \inf S_1 + \inf S_2$. For all $x = x_1 + x_2$, fix $x_2 \in S_2$. Then we have $x_1 \geq \inf(S_1 + S_2) - x_2$. $\inf(S_1 + S_2) - x_2$ is a lower bound for S_1, so $\inf S_1 \geq \inf(S_1 + S_2) - x_2$.

We also have $x_2 \geq \inf(S_1 + S_2) - \inf S_1$. $\inf(S_1 + S_2) - \inf S_1$ is a lower bound for S_2, so $\inf S_2 \leq \inf(S_1 + S_2) - \inf S_1$. Therefore, $\inf(S_1 + S_2) = \inf S_1 + \inf S_2$.

(Proof by Theorem 1.1.1) Let $\epsilon = \sup S_1 + \sup S_2 - \sup(S_1 + S_2)$. It is obvious that $\epsilon \geq 0$ as $\sup S_1 + \sup S_2$ is an upper bound for $S_1 + S_2$. Suppose $\epsilon > 0$. Then, by Theorem 1.1.1, there exists $x_i \in S_i$ for each $i \in \{1, 2\}$ such that $\sup S_i - \epsilon/2 < x_i \leq \sup S_i$. We have

$$\sup(S_1 + S_2) = \sup S_1 + \sup S_2 - \epsilon < x_1 + x_2 \leq \sup S_1 + \sup S_2$$

that is impossible. Therefore, $\epsilon = 0$, that is, $\sup(S_1 + S_2) = \sup S_1 + \sup S_2$.

Likewise, for the case of infima, let $\epsilon = \inf(S_1 + S_2) - (\inf S_1 + \inf S_2)$. It is obvious that $\epsilon \geq 0$ as $\inf S_1 + \inf S_2$ is a lower bound for $S_1 + S_2$. Suppose $\epsilon > 0$. Then, by Theorem 1.1.1, there exists $x_i \in S_i$ for each $i \in \{1, 2\}$ such that $\inf S_i \leq x_i < \inf S_i + \epsilon/2$. We have

$$\inf S_1 + \inf S_2 \leq x_1 + x_2 < \inf S_1 + \inf S_2 + \epsilon = \inf(S_1 + S_2)$$

that is impossible. Therefore, $\epsilon = 0$, that is, $\inf(S_1 + S_2) = \inf S_1 + \inf S_2$. ■

1.16 **a) Bernoulli's Inequality** Prove by induction that $(1 + x)^n \geq 1 + nx$ for all $x > -1$ and all n in $\mathbb{N}$.

b) Use part (a) to show that, as in Example 5, if $1 < \mu k$ and $\mu > 0$, then

$$\left(\mu - \frac{1}{k}\right)^n \geq \mu^n - \frac{n\mu^{n-1}}{k}$$

for all n in $\mathbb{N}$.

Solution.

a) (i) It holds for $n = 1$.

(ii) Suppose that $(1 + x)^k \geq 1 + kx$ for all $x > -1$ and some $k \in \mathbb{N}$. Then

$$(1 + x)^{k+1} = (1 + x)^k(1 + x) \geq (1 + kx)(1 + x) = 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x.$$

By induction, $(1 + x)^n \geq 1 + nx$ for all $x > -1$ and all n in $\mathbb{N}$.

b) $1 < \mu k$ implies $-1/(\mu k) > -1$. By Bernoulli's inequality,

$$\left(\mu - \frac{1}{k}\right)^n = \mu^n \left[1 + \left(-\frac{1}{\mu k}\right)\right]^n \geq \mu^n \left[1 + n\left(-\frac{1}{\mu k}\right)\right] = \mu^n - \frac{n\mu^{n-1}}{k}.$$

■

1.19 Prove by induction that, for every j in $\mathbb{N}$, $2^{j-1} \leq j!$.

Solution.

(i) The inequality holds for $j = 1$.

(ii) Suppose that $2^{k-1} \leq k!$ for some $k \in \mathbb{N}$. Then

$$2^{(k+1)-1} = 2 \cdot 2^{k-1} \leq 2 \cdot k! \leq (k+1)k! = (k+1)!.$$

By induction, $2^{j-1} \leq j!$ for every j in $\mathbb{N}$. ■

1.20 **a)** Use the Binomial Theorem to show that $(1 + 1/k)^k$ expands to become

$$1 + 1 + \sum_{j=2}^{k} \frac{1}{j!} \left(1 - \frac{1}{k}\right) \left(1 - \frac{2}{k}\right) \cdots \left(1 - \frac{j-1}{k}\right).$$

b) Prove that, for each $j = 2, 3, \cdots, k$,

$$\frac{1}{j!} \left(1 - \frac{1}{k}\right) \left(1 - \frac{2}{k}\right) \cdots \left(1 - \frac{j-1}{k}\right) \leq \frac{1}{j!}.$$

c) Show that $(1 + 1/k)^k \leq \sum_{j=0}^{k} 1/j!$ for all k in $\mathbb{N}$.

Solution.

a) By the Binomial Theorem,

$$\left(1 + \frac{1}{k}\right)^k = \sum_{j=0}^{k} \binom{k}{j} \frac{1}{k^j} = 1 + 1 + \sum_{j=2}^{k} \frac{k!}{j!(k-j)!} \frac{1}{k^j} = 1 + 1 + \sum_{j=2}^{k} \frac{1}{j!} \prod_{i=1}^{j-1} \left(1 - \frac{i}{k}\right).$$

b) For each $j \in \{2, 3, \cdots, k\}$, $\prod_{i=1}^{j-1} \left(1 - \frac{i}{k}\right) \leq 1$. Therefore, $\frac{1}{j!} \prod_{i=1}^{j-1} \left(1 - \frac{i}{k}\right) \leq \frac{1}{j!}$.

c) It follows from the above that

$$\begin{aligned} \left(1 + \frac{1}{k}\right)^k &= 1 + 1 + \sum_{j=2}^{k} \frac{1}{j!} \prod_{i=1}^{j-1} \left(1 - \frac{i}{k}\right) \\ &\leq \frac{1}{0!} + \frac{1}{1!} + \sum_{j=2}^{k} \frac{1}{j!} \\ &= \sum_{j=0}^{k} \frac{1}{j!}. \end{aligned}$$

■

1.2 Neighborhood and Limit Points

1.23 Suppose that S is a bounded, infinite subset of $\mathbb{R}$. Let $\mu = \sup S$. Is μ necessarily a limit point of S? If so, show that, for every k in $\mathbb{N}$, there exists a point x in S such that $\mu - 1/k < x < \mu$. If not, exhibit a bounded, infinite set S in $\mathbb{R}$ whose supremum is not a limit point of S.

Solution. Let $S = \{1/n : n \in \mathbb{N}\}$. S is bounded as $0 < 1/n \leq 1$ for all $n \in \mathbb{N}$ and infinite. However, $\sup S = 1$ is not a limit point of S because $N'(1; 1/3) \cap S = \varnothing$. ■

1.26 Prove that a nonempty finite set has no limit points.

Solution. Let $S = \{x_1, \cdots, x_n\}$ be a nonempty finite set. Suppose, to the contrary, that there exists a limit point x_0 of S. Let

$$\epsilon = \begin{cases} 1 & \text{if } S = \{x_0\}, \\ \min\{|x_0 - x_k| : x_k \in S \setminus \{x_0\}\} & \text{otherwise.} \end{cases}$$

Then we have a contradiction, $N'(x_0; \epsilon) \cap S = \varnothing$. Therefore, a nonempty finite set has no limit points. ■

1.27 Find all limit points of the following sets. In each case prove that your answer is correct.

- **a)** The set $S = \{x : x \text{ is irrational and } a < x < b\}$ where $a < b$
- **b)** The set $\mathbb{Q}$ of all rational numbers
- **c)** The set of all irrational numbers
- **d)** The set $S = \{p/2^k : p \text{ in } \mathbb{Z}, k \text{ in } \mathbb{N}\}$
- **e)** The set $S = \{1/m + 1/n : m, n \text{ in } \mathbb{N}\}$

Solution. Let $\epsilon > 0$ be given.

- **a)** By Exercise 1.8, for each $x \in [a, b]$, there exists an irrational number in $N'(x; \epsilon) \cap S = (\max\{x - \epsilon, a\}, \min\{x + \epsilon, b\})$. For any $x \in \mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$, let $\epsilon = \min\{|x - a|, |x - b|\}$. Then $N'(x; \epsilon) \cap S = \varnothing$. Therefore, $[a, b]$ is the set of all its limit points.

- **b)** By Theorem 1.1.2, for each $x \in \mathbb{R}$, there exists a rational number in $N'(x; \epsilon)$. Therefore, $N'(x; \epsilon) \cap \mathbb{Q} \neq \varnothing$, so $\mathbb{R}$ is the set of all its limit points.

- **c)** By Exercise 1.8, for each $x \in \mathbb{R}$, there exists an irrational number in $N'(x; \epsilon)$. Therefore, $N'(x; \epsilon) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \varnothing$, so $\mathbb{R}$ is the set of all its limit points.

- **d)** By Archimedes' principle, choose $k \in \mathbb{N}$ such that $1/2^k < \epsilon$. By Exercise 1.6, for each $x \in \mathbb{R}$, there exists a $p \in \mathbb{Z}$ such that $p - 1 \leq 2^k x < p$. Then we have $p/2^k - 1/2^k \leq x < p/2^k \iff x < p/2^k \leq x + 1/2^k < x + \epsilon$. Therefore, $N'(x; \epsilon) \cap S \neq \varnothing$, so $\mathbb{R}$ is the set of all its limit points.

- **e)** By Archimedes' principle, choose $k \in \mathbb{N}$ such that $2/k < \epsilon$. $1/k + 1/k \in N'(0; \epsilon)$, so 0 is a limit point of S. By Archimedes' principle, choose $n \in \mathbb{N}$ such that $1/n < \epsilon$. Then, for each $m \in \mathbb{N}$, $1/m < 1/m + 1/n < 1/m + \epsilon$ implies $N'(1/m; \epsilon) \cap S \neq \varnothing$. Therefore, $S_1 = \{0\} \cup \{1/m : m \in \mathbb{N}\}$ is contained in the set of all its limit points.

  We claim that S_1 is, in fact, the set of all limit points of S.

(i) If $x < 0$ or $x > 2$, let $\epsilon = \min\{|x|, |x-2|\}$. $S \subset (0, 2]$, so $N'(x;\epsilon) \cap S = \varnothing$.

(ii) If $x = 2$, since $(3/2, 2) \cap S = \varnothing$, $N'(x; 1/2) \cap S = \varnothing$.

(iii) If $1 < x < 2$, by Exercise 1.6, there exists an $n \in \mathbb{N}$ such that $n \leq 1/(x-1) < n+1$. Then

$$1 + \frac{1}{n+1} < x \leq 1 + \frac{1}{n}.$$

Letting $\epsilon = \min\left\{x - \left(1 + \frac{1}{n+1}\right), 1 + \frac{1}{n} - x\right\}$ gives $N'(x;\epsilon) \cap S = \varnothing$.

(iv) If $x \in (0,1) \setminus S_1$, by Exercise 1.6, there exists an $m \in \mathbb{N}$ such that $m < 1/x < m+1 \iff 1/(m+1) < x < 1/m$. There exists an $n \in \mathbb{N}$ such that

$$n \leq \frac{1}{x - \frac{1}{m+1}} < n+1.$$

Then we have

$$\frac{1}{m+1} + \frac{1}{n+1} < x \leq \frac{1}{m+1} + \frac{1}{n}.$$

Letting $\epsilon = \min\left\{x - \left(\frac{1}{m+1} + \frac{1}{n+1}\right), \frac{1}{m+1} + \frac{1}{n} - x\right\}$ gives $N'(x;\epsilon) \cap S = \varnothing$.

Therefore, $\{0\} \cup \{1/m : m \in \mathbb{N}\}$ is the set of all its limit points.

■

1.29 Suppose that S is an unbounded, infinite set in $\mathbb{R}$. Does S necessarily have a limit point? If so, prove it. If not, provide an example of an unbounded, infinite set having no limit points.

Solution. Consider $\mathbb{N}$, an unbounded, infinite set in $\mathbb{R}$. Suppose, to the contrary, that there exists a limit point x_0 of $\mathbb{N}$. Let

$$\epsilon = \begin{cases} \frac{1}{2}\min\{x-n, n+1-x\} & \text{if } n < x_0 < n+1 \text{ for some } n \in \mathbb{N}, \\ \frac{1}{2} & \text{if } x_0 \in \mathbb{N}. \end{cases}$$

Then we have a contradiction, $N'(x_0;\epsilon) \cap \mathbb{N} = \varnothing$. Therefore, $\mathbb{N}$ has no limit points. ■

1.30 Suppose that S is a bounded, infinite set in $\mathbb{R}$ having exactly one limit point x_0. Prove that, for any $\epsilon > 0$, the neighborhood $N'(x_0;\epsilon)$ contains all but finitely many of the points of S.

Solution. Suppose, to the contrary, that $S \setminus N'(x_0;\epsilon_0)$ contains infinitely many of the points of S for some $\epsilon_0 > 0$. Then, by the Bolzano–Weierstrass theorem, $S \setminus N'(x_0;\epsilon_0)$ has a limit point x_1 in $\mathbb{R}$. $x_1 \neq x_0$ because $N'(x_0;\epsilon_0) \cap (S \setminus N'(x_0;\epsilon_0)) = \varnothing$. Since $S \setminus N'(x_0;\epsilon_0) \subset S$, x_1 is also a limit point of S. It contradicts the assumption that S has exactly one limit point. Therefore, for any $\epsilon > 0$, the neighborhood $N'(x_0;\epsilon)$ contains all but finitely many of the points of S. ■

1.3 The Limit of a Sequence

1.35 Complete the following steps proving that $\lim_{k\to\infty} k^{1/k} = 1$:

- **a)** For each $k \geq 2$, show that there exists a positive y_k such that $k^{1/k} = 1 + y_k$.
- **b)** Use the binomial theorem to show that, for $k \geq 2$,

$$k > 1 + \frac{1}{2}k(k-1)y_k^2.$$

- **c)** Show that $y_k < \sqrt{2/k}$ for all $k \geq 2$.
- **d)** Use Archimedes' principle to show that $\lim_{k\to\infty} \sqrt{2/k} = 0$.
- **e)** Use the Squeeze Play to show that $\lim_{k\to\infty} k^{1/k} = 1$.

Solution.

- **a)** For each $k \geq 2$, $k^{1/k} > k^0 = 1$ implies $y_k = k^{1/k} - 1 > 0$.
- **b)** By the binomial theorem,

$$k = (1 + y_k)^k = \sum_{j=0}^{k} \binom{k}{j} y_k^j > 1 + \frac{1}{2}k(k-1)y_k^2.$$

- **c)** From part (b), $(k-1)(ky_k^2 - 2) < 0$. For all $k \geq 2$, $ky_k^2 - 2 < 0$. Therefore, $y_k < \sqrt{2/k}$.
- **d)** Let $\epsilon > 0$ be given. By Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that $2/k_0 < \epsilon^2$. For all $k \geq k_0$, $\sqrt{2/k} < \sqrt{2/k_0} < \epsilon$. Therefore, $\lim_{k\to\infty} \sqrt{2/k} = 0$.
- **e)** From part (a) and (c), $1 < k^{1/k} = 1 + y_k < 1 + \sqrt{2/k}$. By the Squeeze Play, $\lim_{k\to\infty} k^{1/k} = 1$.

■

1.39 Prove that a sequence $\{x_k\}$ in $\mathbb{R}$ converges to 0 if and only if $\{|x_k|\}$ converges to 0.

Solution. Note that $x_k \in N(0;\epsilon)$ if and only if $|x_k| \in N(0;\epsilon)$ for every $\epsilon > 0$. Therefore, the statement holds trivially. ■

1.41 Let x_0 be a limit point of a nonempty subset S in $\mathbb{R}$. Prove that there exists a sequence of distinct points of S that converges to x_0.

Solution. Recall that x_0 is a limit point of a nonempty subset S in $\mathbb{R}$ if, for each $\epsilon > 0$, $N'(x_0;\epsilon) \cap S \neq \varnothing$. (Definition 1.2.3) Start with $\epsilon_1 = 1$. Choose a point $x_1 \in N'(x_0;\epsilon_1) \cap S$. For each $k \geq 2$, let $\epsilon_k = \min\{1/k, |x_0 - x_{k-1}|\}$, and choose a point $x_k \in N'(x_0;\epsilon_k) \cap S$. Note that $\epsilon_k \leq |x_0 - x_{k-1}|$ implies $x_k \notin N'(x_0;\epsilon_j)$ for all $j > k$. Therefore, $\{x_k\}$ is a sequence of distinct points of S.

It remains to prove that $\lim_{k\to\infty} x_k = x_0$. Let $\epsilon > 0$ be given. By Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that $1/k_0 < \epsilon$. For all $k \geq k_0$, $\epsilon_k \leq 1/k \leq 1/k_0 < \epsilon$ implies $x_k \in N'(x_0;\epsilon_k) \subset N(x_0;\epsilon)$. Therefore, a sequence $\{x_k\}$ of distinct points of S converges to x_0. ■

1.47 Suppose that $\{x_k\}$ is a sequence of positive numbers for which $\lim_{k\to\infty} x_{k+1}/x_k = L$ exists.

a) Prove that if $L < 1$, then $\{x_k\}$ converges. Find $\lim_{k\to\infty} x_k$.

b) Prove that if $L > 1$, then $\{x_k\}$ diverges to ∞.

c) Exhibit two sequences $\{y_k\}$ and $\{z_k\}$ such that

$$\lim_{k\to\infty} \frac{y_{k+1}}{y_k} = \lim_{k\to\infty} \frac{z_{k+1}}{z_k} = 1$$

and such that $\{y_k\}$ converges while $\{z_k\}$ diverges. Thus, confirm that $L = 1$ gives no information.

d) Apply this test to determine the convergence of $\{k/2^k\}$.

Solution.

a) Since $\{x_k\}$ is a sequence of positive numbers, $x_{k+1}/x_k > 0$. By Theorem 1.3.9,

$$L = \lim_{k\to\infty} \frac{x_{k+1}}{x_k} \geq 0.$$

Let $\epsilon = (1-L)/2$, then there exists an index $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} k \geq k_0 \implies \left|\frac{x_{k+1}}{x_k} - L\right| < \frac{1-L}{2} &\iff L - \frac{1-L}{2} < \frac{x_{k+1}}{x_k} < L + \frac{1-L}{2} \\ &\implies 0 < x_{k+1} < \frac{L+1}{2} x_k. \end{aligned}$$

So, applying the inequality recursively gives

$$0 < x_k < \frac{L+1}{2} x_{k-1} < \left(\frac{L+1}{2}\right)^2 x_{k-2} < \cdots < \left(\frac{L+1}{2}\right)^{k-k_0} x_{k_0}$$

for $k > k_0$. According to Example 18 in Chapter 1, $\lim_{k\to\infty} r^k = 0$ if $0 < r < 1$. Therefore,

$$\lim_{k\to\infty} \left(\frac{L+1}{2}\right)^{k-k_0} x_{k_0} = \left(\frac{L+1}{2}\right)^{-k_0} x_{k_0} \lim_{k\to\infty} \left(\frac{L+1}{2}\right)^k = 0.$$

By the Squeeze Play (Theorem 1.3.8), we conclude that $\{x_k\}$ converges and

$$\lim_{k\to\infty} x_k = 0.$$

b) Let $\epsilon = (L-1)/2$, then there exists an index $k_0 \in \mathbb{N}$ such that

$$k \geq k_0 \implies \left|\frac{x_{k+1}}{x_k} - L\right| < \frac{L-1}{2} \implies x_{k+1} > \frac{L+1}{2} x_k.$$

So, $\{x_k\}$ is eventually strictly increasing. Applying the inequality recursively gives

$$x_k > \frac{L+1}{2} x_{k-1} > \left(\frac{L+1}{2}\right)^2 x_{k-2} > \cdots > \left(\frac{L+1}{2}\right)^{k-k_0} x_{k_0}$$

for $k > k_0$. By the Bernoulli's inequality, since $k - k_0 \in \mathbb{N}$ for $k > k_0$ and $(L-1)/2 > 0$,

$$x_k > \left(\frac{L+1}{2}\right)^{k-k_0} x_{k_0} = \left(1 + \frac{L-1}{2}\right)^{k-k_0} x_{k_0} \geq \left[1 + (k-k_0)\frac{L-1}{2}\right] x_{k_0}.$$

Suppose that $\{x_k\}$ is bounded above, that is, there exists an upper bound $M \in \mathbb{R}$ such that $x_k \leq M$ for all $k \in \mathbb{N}$. M must be positive because $\{x_k\}$ is a sequence of positive numbers. However, by Archimedes' principle, there is $k - k_0 \in \mathbb{N}$ such that

$$\frac{M}{x_{k_0}} < (k-k_0)\frac{L-1}{2}$$

which leads to a contradiction

$$\frac{M}{x_{k_0}} < 1 + (k-k_0)\frac{L-1}{2} \implies M < \left[1 + (k-k_0)\frac{L-1}{2}\right] x_{k_0} < x_k.$$

So, $\{x_k\}$ is not bounded above. Therefore, the unbounded sequence $\{x_k\}$ that is eventually strictly increasing must diverge to ∞.

c) Let $y_k = 1/k$ and $z_k = k$. It follows from Example 17 that

$$\lim_{k\to\infty} \frac{y_{k+1}}{y_k} = \lim_{k\to\infty} \frac{1/(k+1)}{1/k} = \lim_{k\to\infty} \frac{k}{k+1} = 1,$$

and that

$$\lim_{k\to\infty} \frac{z_{k+1}}{z_k} = \lim_{k\to\infty} \frac{k+1}{k} = 1.$$

$\{y_k\}$ converges from Example 12, and $\{z_k\}$ diverges from Example 16 as a consequence of Theorem 1.3.4. Therefore, $L = 1$ gives no information.

d) Let $\{x_k\} = k/2^k$. We have

$$\lim_{k\to\infty} \frac{x_{k+1}}{x_k} = \lim_{k\to\infty} \frac{k+1}{2^{k+1}}\frac{2^k}{k} = \frac{1}{2}\lim_{k\to\infty} \frac{k+1}{k} = \frac{1}{2} < 1,$$

so $\{k/2^k\}$ converges to 0.

■

1.50 **Cantor's Nested Interval Theorem.** For each k in $\mathbb{N}$, let $I_k = [a_k, b_k]$ be a closed interval such that for each k, $I_k \supseteq I_{k+1}$. Apply Theorem 1.3.7 to the sequences $\{a_k\}$ and $\{b_k\}$ to prove that $\bigcup_{k=1}^{\infty} I_k \neq \varnothing$. Furthermore, prove that if $\{a_k\}$ and $\{b_k\}$ both converge to x_0, then $\bigcup_{k=1}^{\infty} I_k = \{x_0\}$.

Solution. Refer to the proofs of Theorem 2.3.1 and Corollary 2.3.2. ■

1.59 Let $\{x_k\}$ be any sequence for which $\nu = \liminf x_k$ is finite. Fix any $\epsilon > 0$.

a) Prove that there exists a k_0 in $\mathbb{N}$ such that, for all $k \geq k_0$, $x_k > \nu - \epsilon$.

b) Prove that, for any k in $\mathbb{N}$, there exists a $k_1 > k$ such that $x_{k_1} < \nu + \epsilon$.

Solution. $\nu = \liminf x_k$ is finite, so ν is in the set C of cluster points of $\{x_k\}$ by Theorem 1.3.16.

a) $\{x_k\}$ is bounded below. Suppose that, for any $k \in \mathbb{N}$, there exists a $k_1 > k$ such that $x_{k_1} \leq \nu - \epsilon$. Then the subsequence $\{x_{k_j} : x_{k_j} \leq \nu - \epsilon\}$ is bounded, so it has a cluster point less than $\inf C$ by Theorem 1.3.11. That is a contradiction. Therefore, we deduce that there must exist a k_0 in $\mathbb{N}$ such that, for $k \geq k_0$, we have $x_k > \nu - \epsilon$.

b) Since $\nu = \liminf x_k$ is a cluster point of $\{x_k\}$, given any $\epsilon > 0$ and any $k \in \mathbb{N}$, there must exist a $k_1 > k$ such that $x_{k_1} < \nu + \epsilon$.

■

1.65 Let $\{y_k\}$ be any bounded sequence in $\mathbb{R}$. For each k in $\mathbb{N}$, let $x_k = \inf\{y_j : j \geq k\}$ and $z_k = \sup\{y_j : j \geq k\}$.

a) Prove that $x_k \leq y_k \leq z_k$ for all k in $\mathbb{N}$.

b) Prove that $\{x_k\}$ is monotone increasing and bounded above and that $\{z_k\}$ is monotone decreasing and bounded below.

c) Prove that $\{y_k\}$ converges if and only if $\lim_{k\to\infty} x_k = \lim_{k\to\infty} z_k$ in which case $\lim_{k\to\infty} y_k$ is equal to this common limit.

Solution. Since $\{y_k\}$ is bounded, there exist a lower bound m and an upper bound M for $\{y_k\}$ such that, for each $k \in \mathbb{N}$, $m \leq y_k \leq M$. By Exercise 1.14, for all $k \in \mathbb{N}$, $x_k = \inf\{y_j : j \geq k\} = \min\{y_k, x_{k+1}\}$ and $z_k = \sup\{y_j : j \geq k\} = \max\{y_k, z_{k+1}\}$.

a) It follows from the above that $x_k \leq y_k \leq z_k$.

b) It follows from the above that $x_k \leq x_{k+1} \leq z_{k+1} \leq z_k$. $\{x_k\}$ is bounded above by M, and $\{z_k\}$ is bounded below by m.

c)($\Longrightarrow$) Suppose that $\{y_k\}$ converges to y_0. By Theorem 1.3.7, $\{x_k\}$ and $\{z_k\}$ converge, and

$$\lim_{k\to\infty} x_k = \sup\{x_k : k \text{ in } \mathbb{N}\} = \sup\{\inf\{y_j : j \geq k\} : k \text{ in } \mathbb{N}\} = x_0,$$
$$\lim_{k\to\infty} z_k = \inf\{z_k : k \text{ in } \mathbb{N}\} = \inf\{\sup\{y_j : j \geq k\} : k \text{ in } \mathbb{N}\} = z_0.$$

By Exercise 1.66, there exist a subsequence of $\{y_k\}$ that converges to x_0 and another subsequence of $\{y_k\}$ that converges to z_0. By Theorem 1.3.13, the two subsequences must converge to y_0. Therefore, $\lim\limits_{k\to\infty} x_k = \lim\limits_{k\to\infty} y_k = \lim\limits_{k\to\infty} z_k$.

($\Longleftarrow$) By the Squeeze Play, $\lim\limits_{k\to\infty} x_k = \lim\limits_{k\to\infty} y_k = \lim\limits_{k\to\infty} z_k$.

■

1.66 Let $\{x_k\}$ be a bounded sequence in $\mathbb{R}$. For each k in $\mathbb{N}$, $y_k = \sup\{x_j : j \geq k\}$. Let $c = \inf\{y_k : k \text{ in } \mathbb{N}\}$. That is, $c = \inf\{\sup\{x_j : j \geq k\} : k \text{ in } \mathbb{N}\}$. Show that there exists a subsequence of $\{x_k\}$ that converges to c.

Solution. By Exercise 1.65, $\{y_k\}$ is monotone decreasing and bounded below, so $\{y_k\}$ converges to c. Let $\epsilon > 0$ be given. There exists an index j_0 such that, whenever $j \geq j_0$, $y_j \in N(c; \epsilon/2)$. By Theorem 1.1.1, for each $j \in \mathbb{N}$, there exists an $x_{k_j} \in \{x_i : i \geq j\}$ such that $y_j - \epsilon/2 < x_{k_j} \leq y_j$. For $j \geq j_0$, $|x_{k_j} - c| \leq |x_{k_j} - y_j| + |y_j - c| < \epsilon/2 + \epsilon/2 = \epsilon$. Therefore, $\{x_{k_j}\}$ converges to c. ■

1.67 Suppose that $\{x_k\}$ is a sequence such that the two subsequences $\{x_{2j}\}$ and $\{x_{2j-1}\}$ converge to x_0. Prove that $\{x_k\}$ converges to x_0 also.

Solution. This is a special case of Exercise 1.68. ∎

1.68 **a)** Suppose that $\{x_k\}$ is a sequence which has two subsequences $\{x_{k_j}\}$ and $\left\{x_{k'_j}\right\}$ with the following properties:

i) $\{k_j : j \text{ in } \mathbb{N}\} \cup \{k'_j : j \text{ in } \mathbb{N}\} = \mathbb{N}$.

ii) $\lim_{j\to\infty} x_{k_j} = \lim_{j\to\infty} x_{k'_j} = x_0$.

Prove that $\{x_k\}$ converges to x_0 also.

b) Generalize the result in part (a) to the case where there are a finite number of subsequences all converging to x_0.

Solution.

a) Let $\epsilon > 0$ be given. By (ii), there exists an index $j_1 \in \mathbb{N}$ such that $x_{k_j} \in N(x_0;\epsilon)$ for all $j \ge j_1$, and there exists an index $j_2 \in \mathbb{N}$ such that $x_{k'_j} \in N(x_0;\epsilon)$ for all $j \ge j_2$. Let $k_0 = \max\left\{k_{j_1}, k'_{j_2}\right\}$. Then $x_k \in N(x_0;\epsilon)$ whenever $k \ge k_0$. Therefore, $\lim_{k\to\infty} x_k = x_0$.

b) Suppose that $\{x_k\}$ is a sequence which has a finite number of subsequences $\{x_{k_{1j}}\}, \cdots, \{x_{k_{mj}}\}$ with the following properties:

i) $\bigcup_{i=1}^{m}\{k_{ij} : j \text{ in } \mathbb{N}\} = \mathbb{N}$.

ii) $\lim_{j\to\infty} x_{k_{ij}} = x_0$ for each $1 \le i \le m$.

Then $\{x_k\}$ converges to x_0 also.

Proof. Let $\epsilon > 0$ be given. By (ii), for each $1 \le i \le m$, there exists an index $j_i \in \mathbb{N}$ such that $x_{k_{ij}} \in N(x_0;\epsilon)$ for all $j \ge j_i$. Let $k_0 = \max\{k_{ij_i} : 1 \le i \le m\}$. Then $x_k \in N(x_0;\epsilon)$ whenever $k \ge k_0$. Therefore, $\lim_{k\to\infty} x_k = x_0$.

∎

1.69 Let $\{x_k\}$ be a bounded sequence of positive numbers. For each k in $\mathbb{N}$, define $y_k = x_{k+1}/x_k$ and $z_k = (x_k)^{1/k}$.

a) Prove that

$$\liminf y_k \le \liminf z_k \le \limsup z_k \le \limsup y_k.$$

b) Give examples to show that the inequalities in part (a) can be strict.

c) Prove that, if $\{y_k\}$ converges, then $\{z_k\}$ also converges.

d) Find a sequence $\{x_k\}$ such that $\{z_k\}$ converges but $\{y_k\}$ diverges.

Solution.

a) Let $\nu = \liminf y_k$. If $\nu = 0$, there is nothing to prove. Assume that $\nu > 0$. Fix any $0 < \epsilon < \nu$. By Theorem 1.3.17, there exists a $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, $y_k > \nu - \epsilon$. It follows from

$$y_k y_{k-1} \cdots y_{k_0} = \frac{x_{k+1}}{x_k} \frac{x_k}{x_{k-1}} \cdots \frac{x_{k_0+1}}{x_{k_0}} = \frac{x_{k+1}}{x_{k_0}} > (\nu - \epsilon)^{k-k_0+1}$$

that, for all $k \geq k_0$,

$$x_k \geq \frac{x_{k_0}}{(\nu-\epsilon)^{k_0}} (\nu - \epsilon)^k \iff z_k = (x_k)^{1/k} \geq \left[\frac{x_{k_0}}{(\nu-\epsilon)^{k_0}} \right]^{1/k} (\nu - \epsilon).$$

We have $\liminf z_k \geq \nu - \epsilon$ for any $0 < \epsilon < \nu$. Therefore, $\liminf z_k \geq \nu = \liminf y_k$.

The second inequality is evident from the definitions.

Let $\mu = \limsup y_k$. If $\mu = \infty$, there is nothing to prove. Assume that μ is finite. Fix any $\epsilon > 0$. By Theorem 1.3.17, there exists a $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, $y_k < \mu + \epsilon$. It follows from

$$y_k y_{k-1} \cdots y_{k_0} = \frac{x_{k+1}}{x_k} \frac{x_k}{x_{k-1}} \cdots \frac{x_{k_0+1}}{x_{k_0}} = \frac{x_{k+1}}{x_{k_0}} < (\mu + \epsilon)^{k-k_0+1}$$

that, for all $k \geq k_0$,

$$x_k \leq \frac{x_{k_0}}{(\mu+\epsilon)^{k_0}} (\mu + \epsilon)^k \iff z_k = (x_k)^{1/k} \leq \left[\frac{x_{k_0}}{(\mu+\epsilon)^{k_0}} \right]^{1/k} (\mu + \epsilon).$$

We have $\limsup z_k \leq \mu + \epsilon$ for any $\epsilon > 0$. Therefore, $\limsup z_k \leq \mu = \limsup y_k$.

b) Consider $\{x_k\}$ where

$$x_k = \begin{cases} 2^{-\frac{k+1}{2}} & \text{if } k \text{ is odd,} \\ 3^{-\frac{k}{2}} & \text{if } k \text{ is even,} \end{cases}$$

for which

$$\begin{aligned} \liminf y_k &= \lim_{k\to\infty} \left(\frac{2}{3}\right)^k = 0, \\ \liminf z_k &= \lim_{k\to\infty} 3^{-\frac{k}{2k}} = \frac{1}{\sqrt{3}}, \\ \limsup z_k &= \lim_{k\to\infty} 2^{-\frac{k+1}{2k}} = \frac{1}{\sqrt{2}}, \\ \limsup y_k &= \lim_{k\to\infty} \left(\frac{3}{2}\right)^k = \infty. \end{aligned}$$

■

c) By Theorem 1.3.18, if $\{y_k\}$ converges, then $\liminf y_k = \limsup y_k$. By the inequalities in part (a), $\liminf z_k = \limsup z_k$. Therefore, $\{z_k\}$ also converges.

d) Consider $\{x_k\}$ where

$$x_k = \begin{cases} 2^{-k} & \text{if } k \text{ is odd,} \\ 2^{-(k-2)} & \text{if } k \text{ is even,} \end{cases}$$

for which

$$\liminf y_k = \frac{1}{8},$$

$$\limsup y_k = 2,$$
$$\liminf z_k = \limsup z_k = \frac{1}{2}.$$

By Theorem 1.3.18, $\{z_k\}$ converges but $\{y_k\}$ diverges.

1.70 Let $\{x_k\}$ be any bounded sequence. For each k in $\mathbb{N}$, define $\sigma_k = (1/k)\sum_{j=1}^{k} x_j$, the *arithmetic average* of the first k terms of the sequence. If $\{\sigma_k\}$ converges to x_0, then $\{x_k\}$ is said to converge $(C,1)$ to x_0.

a) Prove that
$$\liminf x_k \le \liminf \sigma_k \le \limsup \sigma_k \le \limsup x_k.$$

b) Give examples to show that the inequalities in part (a) can be strict.

c) Prove that, if $\{x_k\}$ converges, then $\{x_k\}$ converges $(C,1)$.

d) Find a divergent sequence that converges $(C,1)$.

Solution.

a) Let $\nu = \liminf x_k$. Fix any $\epsilon > 0$. By Theorem 1.3.17, there exists a $k_0 \in \mathbb{N}$ such that, for all $k \ge k_0$, $x_k > \nu - \epsilon$. It follows from
$$\sigma_k = \frac{1}{k}\sum_{j=1}^{k} x_j > \frac{1}{k}\left[\sum_{j=1}^{k_0-1} x_k + \sum_{j=k_0}^{k} (\nu - \epsilon)\right] = \nu - \epsilon + \frac{1}{k}\sum_{j=1}^{k_0-1}(x_k - \nu + \epsilon).$$
that $\liminf \sigma_k \ge \nu - \epsilon$ for any $\epsilon > 0$. Therefore, $\liminf \sigma_k \ge \nu = \liminf x_k$.

The second inequality is evident from the definitions.

Let $\mu = \limsup x_k$. Fix any $\epsilon > 0$. By Theorem 1.3.17, there exists a $k_0 \in \mathbb{N}$ such that, for all $k \ge k_0$, $x_k < \mu + \epsilon$. It follows from
$$\sigma_k = \frac{1}{k}\sum_{j=1}^{k} x_j < \frac{1}{k}\left[\sum_{j=1}^{k_0-1} x_k + \sum_{j=k_0}^{k} (\mu + \epsilon)\right] = \mu + \epsilon + \frac{1}{k}\sum_{j=1}^{k_0-1}(x_k - \mu - \epsilon).$$
that $\limsup \sigma_k \le \mu + \epsilon$ for any $\epsilon > 0$. Therefore, $\limsup \sigma_k \le \mu = \limsup x_k$.

b) Consider $\{x_k\}$: $1, -1, -1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, \cdots$ (one 1, two -1's, four 1's, eight -1's, $\cdots$). Then we have
$$\frac{\sum_{j=0}^{k}(-2)^j}{\sum_{j=0}^{k} 2^j} = \frac{(-2)^{k+1} - 1}{3(2^{k+1} - 1)},$$
so
$$\liminf x_k = -1 < \liminf \sigma_k = -\frac{1}{3} < \limsup \sigma_k = \frac{1}{3} < \limsup x_k = 1.$$

c) By Theorem 1.3.18, if $\{x_k\}$ converges, then $\liminf x_k = \limsup x_k$. By the inequalities in part (a), $\liminf \sigma_k = \limsup \sigma_k$. Therefore, $\{\sigma_k\}$ also converges.

d) Consider $\{x_k\}$ where

$$x_k = \begin{cases} 1, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even} \end{cases}$$

for which

$$\begin{aligned} \liminf x_k &= 0, \\ \limsup x_k &= 1, \\ \liminf \sigma_k = \limsup \sigma_k &= \frac{1}{2}. \end{aligned}$$

By Theorem 1.3.18, $\{\sigma_k\}$ converges but $\{x_k\}$ diverges.

■

Remark. Refer to Cesàro summation to learn more about (C, α) summation.

1.71 Let $\{x_k\}$ be a bounded sequence of positive numbers. For each k in $\mathbb{N}$, define $\tau_k = (x_1 x_2 x_3 \cdots x_k)^{1/k}$, the *geometric average* of the first k terms. Show that, if $\{x_k\}$ converges, then so also does $\{\tau_k\}$. Find a divergent sequence $\{x_k\}$ for which $\{\tau_k\}$ converges.

Solution. Since $\{x_k\}$ is a bounded sequence of positive numbers, we have a bounded sequence $\{\ln \tau_k\}$ with

$$\ln \tau_k = \frac{1}{k} \sum_{j=1}^{k} \ln x_j.$$

By Exercise 1.70, $\{\ln x_k\}$ converges, then $\{\ln \tau_k\}$ converges. Therefore, if $\{x_k\}$ converges, then so also does $\{\tau_k\}$.

Consider $\{x_k\}$ where

$$x_k = \begin{cases} e, & \text{if } k \text{ is odd} \\ 1, & \text{if } k \text{ is even} \end{cases}$$

for which

$$\begin{aligned} \liminf x_k &= 1, \\ \limsup x_k &= e, \\ \liminf \tau_k = \limsup \tau_k &= \sqrt{e}. \end{aligned}$$

By Theorem 1.3.18, $\{\tau_k\}$ converges but $\{x_k\}$ diverges. ■

1.4 Cauchy Sequences

1.73 Let $\{x_k\}$ be a sequence of real numbers. For each k in $\mathbb{N}$, define $y_k = \sum_{j=1}^{k} x_j$ and $z_k = \sum_{j=1}^{k} |x_j|$. Prove that, if $\{z_k\}$ is a Cauchy sequence, then $\{y_k\}$ is also. Is the converse true? If so, prove it; if not, provide a counterexample.

Solution. Since $\{z_k\}$ is Cauchy, for any $\epsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that $k \geq m \geq k_0$ implies

$$|y_k - y_m| = \left| \sum_{j=m+1}^{k} x_j \right| \leq \sum_{j=m+1}^{k} |x_j| = z_k - z_m = |z_k - z_m| < \epsilon.$$

Therefore, $\{y_k\}$ is also Cauchy.

However, the converse is false. Consider $x_k = (-1)^k/k$. We claim that $\{y_k\}$ is Cauchy. Let $\epsilon > 0$ be given. By Archimedes' principle, there is a $k_0 \in \mathbb{N}$ such that $1/k_0 < \epsilon$. For $m \geq k \geq k_0$,

$$|y_m - y_k| = \left| \sum_{j=k+1}^{m} \frac{(-1)^j}{j} \right| \leq \frac{1}{k+1} < \frac{1}{k} \leq \frac{1}{k_0} < \epsilon.$$

By Example 22, $\{z_k\}$ diverges. Hence, it is not Cauchy by Theorem 1.4.4. ■

Remark. Refer to Definition 11.3.1 and Theorem 11.3.1. See Example 13 and 14 in Section 11.4. The following presents another proof by the comparison test (Theorem 11.2.1). It follows from $-|x_j| \leq x_j \leq |x_j| \iff 0 \leq x_j + |x_j| \leq 2|x_j|$ that

$$0 \leq \sum_{j=1}^{k}(x_j + |x_j|) \leq 2\sum_{j=1}^{k}|x_j|$$
$$0 \leq y_k + z_k \leq 2z_k.$$

If $\{z_k\}$ is Cauchy, by Theorem 1.4.4, $\{z_k\}$ converges to, say, z_0. $\{y_k + z_k\}$ is monotone increasing and is bounded above by $2z_0$. By Theorem 1.3.7, $\{y_k + z_k\}$ converges, and so does $\{y_k\}$ by Theorem 1.5.1. Therefore, by Theorem 1.4.4, $\{y_k\}$ is Cauchy.

1.75 Suppose that $\{x_k\}$ is a sequence with the property that $\lim_{k\to\infty}(x_{k+1} - x_k) = 0$. Does it follow that $\{x_k\}$ is Cauchy? If so, prove it; if not, provide a counterexample.

Solution. Consider $x_k = \sum_{j=1}^{k} 1/j$. It follows from Example 12 in Section 1.3 that $\lim_{k\to\infty}(x_{k+1} - x_k) = \lim_{k\to\infty} 1/(k+1) = 0$. By Example 22 in that section, $\lim_{k\to\infty} x_k = \infty$. By Theorem 1.4.2, $\{x_k\}$ is not Cauchy. ■

1.78 Fix any number w in $(0, 1)$. Fix any two number x_1 and x_2. For $k \geq 2$, define $x_{k+1} = wx_{k-1} + (1-w)x_k$. (The numbers w and $1-w$ are called *weights* and x_{k+1} is said to be a *weighted average* of x_{k-1} and x_k.) Compute several terms of this sequence and identify the general form of x_k in terms of x_1, x_2, and w. Prove that $\{x_k\}$ is Cauchy. Find $\lim_{k\to\infty} x_k$.

Solution. $\{x_k\}$ is contractive as

$$\begin{aligned} x_{k+1} = wx_{k-1} + (1-w)x_k &\iff x_{k+1} - x_k = -w(x_k - x_{k-1}) \\ &\implies |x_{k+1} - x_k| = w|x_k - x_{k-1}|. \end{aligned}$$

Hence, it is Cauchy by Theorem 1.4.5. Since, for each $k \in \mathbb{N}$,

$$x_{k+1} - x_k = -w(x_k - x_{k-1}) = (-w)^2(x_{k-1} - x_{k-2}) = \cdots = (-w)^{k-1}(x_2 - x_1),$$

we have the general form of x_k

$$\begin{aligned} x_k &= x_1 + \sum_{j=1}^{k-1}(x_{k+1} - x_k) \\ &= x_1 + \sum_{j=1}^{k-1}(-w)^{j-1}(x_2 - x_1) \\ &= x_1 + (x_2 - x_1)\frac{1-(-w)^{k-1}}{1+w} \\ &= \frac{\left[w + (-w)^{k-1}\right]x_1 + \left[1-(-w)^{k-1}\right]x_2}{w+1}. \end{aligned}$$

Therefore, $\lim\limits_{k\to\infty} x_k = \dfrac{wx_1 + x_2}{w+1}$. ■

1.79 Fix any $c > 0$. Define a sequence $\{x_k\}$ recursively as follows: Let x_1 be any positive number and, for k in $\mathbb{N}$, let $x_{k+1} = 1/(c + x_k)$. Prove that $\{x_k\}$ is a contractive sequence.

Solution. By simple induction, for each $k \geq 2$, $0 < x_k < 1/c$. We have

$$\begin{aligned} |x_{k+1} - x_k| &= \left|\frac{1}{c+x_k} - \frac{1}{c+x_{k-1}}\right| = \frac{|x_k - x_{k-1}|}{(c+x_k)(c+x_{k-1})} \\ &= \frac{1}{1+c/x_k}|x_k - x_{k-1}| \leq \frac{1}{1+c^2}|x_k - x_{k-1}|. \end{aligned}$$

Note that $0 < \dfrac{1}{1+c^2} < 1$. Therefore, $\{x_k\}$ is a contractive sequence. ■

1.82 Suppose that $\{x_k\}$ is a sequence in $\mathbb{R}$. Suppose also that there exists a strictly monotone increasing sequence $\{C_k\}$ such that $\lim_{k\to\infty} C_k = 1$ and with the property that, for all $k \geq 2$,

$$|x_{k+1} - x_k| \leq C_k|x_k - x_{k-1}|.$$

Determine whether $\{x_k\}$ is necessarily Cauchy. If so, prove it; if not, provide a counterexample.

Solution. Consider $\{x_k\}$ with $x_k = \sum_{j=1}^{k} 1/j$ and let $C_k = k/(k+1)$ for each $k \in \mathbb{N}$. Then $\{C_k\}$ is a strictly increasing sequence such that $\lim_{k\to\infty} C_k = 1$ by Exercise 17 in Section 1.3. For all $k \geq 2$,

$$|x_{k+1} - x_k| = \frac{1}{k+1} \leq C_k|x_k - x_{k-1}| = \frac{k}{k+1}\cdot\frac{1}{k}.$$

However, by Exercise 22 in that section, $\lim_{k\to\infty} x_k = \infty$. Therefore, by Theorem 1.4.4, $\{x_k\}$ is not necessarily Cauchy. ■

1.83 Suppose that $\{x_k\}$ is a sequence in $\mathbb{R}$. Suppose also that there exists a strictly monotone decreasing sequence $\{C_k\}$ that converges to some number in $[0, 1)$ and with the property that, for all $k \geq 2$,

$$|x_{k+1} - x_k| \leq C_k|x_k - x_{k-1}|.$$

Determine whether $\{x_k\}$ is necessarily Cauchy. If so, prove it; if not, provide a counterexample.

Solution. Assume that $\lim_{k\to\infty} C_k = C_0 \in [0,1)$. There exists a $k_0 \in \mathbb{N}$ such that $k \geq k_0$ implies

$$C_k - C_0 \leq |C_k - C_0| < \frac{1-C_0}{2} \implies C_k < \frac{C_0+1}{2} < 1.$$

Hence, $k \geq k_0$,

$$|x_{k+1} - x_k| \leq C_k|x_k - x_{k-1}| < \frac{C_0+1}{2}|x_k - x_{k-1}|.$$

$\{x_k\}$ is eventually contractive, so it is Cauchy by Theorem 1.4.5. ■

Remark. The monotonicity is unnecessary. $\{x_k\}$ is contractive if $\limsup_{k\to\infty} C_k < 1$. ■

1.84 Suppose that S is any nonempty set in $\mathbb{R}$ and that x_0 is any limit point of S. Prove that there exists a Cauchy sequence of *distinct* points in S that converges to x_0.

Solution. Every convergent sequence is Cauchy by Theorem 1.4.4. See Exercise 1.41. ■

1.85 Consider the following possible theorem:

A sequence $\{x_k\}$ in $\mathbb{R}$ is Cauchy if and only
if each of its proper subsequences is Cauchy.

Determine whether this statement is, in fact, a theorem. If it is, prove it; if not, explain exactly how it fails.

Solution. The statement is indeed true, that is, a theorem.

($\Longrightarrow$) Suppose that a sequence $\{x_k\}$ in $\mathbb{R}$ is Cauchy. Let $\epsilon > 0$ be given. There exists a $k_0 \in \mathbb{N}$ such that $k \geq k_0$ and $m \geq k_0$ implies $|x_k - x_m| < \epsilon$. For each proper subsequence $\{x_{k_j}\}$, choose two indices $k_i \geq k_0$ and $k_j \geq k_0$. Then $|x_{k_i} - x_{k_j}| < \epsilon$. Therefore, each of its proper subsequences is Cauchy.

($\Longleftarrow$) Suppose that a sequence $\{x_k\}$ in $\mathbb{R}$ is not Cauchy. Let $k_0 \in \mathbb{N}$ be given. There exists an $\epsilon > 0$ such that, for some $k \geq k_0$ and $m \geq k_0$, $|x_k - x_m| \geq \epsilon$. Then we can construct a proper subsequence $\{x_{k_j}\}$ such that, for all $j \in \mathbb{N}$, $|x_{k_{2j}} - x_{k_{2j-1}}| \geq \epsilon$. Therefore, there exists a proper subsequence that is not Cauchy.

This completes the proof. ■

1.5 The Algebra of Convergent Sequences

1.91 Prove that, if $\{x_k\}$ converges to x_0, then $\{|x_k|\}$ converges to $|x_0|$. Is the converse true? If so, prove it; otherwise, provide a counterexample.

Solution. This is a special case of Corollary 1.5.5. By Theorem 1.5.3 and Theorem 1.5.4,

$$\lim_{k\to\infty}|x_k| = \lim_{k\to\infty}\sqrt{x_k^2} = \sqrt{\lim_{k\to\infty} x_k^2} = \sqrt{\left(\lim_{k\to\infty} x_k\right)^2} = \sqrt{x_0^2} = |x_0|.$$

More precisely, let $\epsilon > 0$ be given. Since $\{x_k\}$ converges to x_0, there exists an index k_0 such that, whenever $k \geq k_0$, $x_k \in N(x_0;\epsilon)$. We have $||x_k| - |x_0|| \leq |x_k - x_0| < \epsilon$ for all $k \geq k_0$. Therefore, $\{|x_k|\}$ converges to $|x_0|$.

The converse is false. Consider $x_k = (-1)^k$. $\{|x_k|\}$ converges to 1 as $|x_k| = 1$ for all $k \in \mathbb{N}$. However, $\{x_k\}$ has two cluster points, -1 and 1. Therefore, by Theorem 1.3.2, $\{x_k\}$ diverges. ■

1.93 Let $x_1 = 1$ and define $x_{k+1} = (2x_k + 3)/4$ for k in $\mathbb{N}$. Determine whether $\{x_k\}$ converges. If it does, find its limit; otherwise, prove that it diverges.

Solution 1. $\{x_k\}$ is contractive as $x_{k+1} - x_k = \frac{1}{2}(x_k - x_{k-1})$. By Theorem 1.4.5, $\{x_k\}$ is Cauchy, so it is convergent by Theorem 1.4.4. Let $\lim\limits_{k\to\infty} x_k = L$. Then we have

$$\lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} \frac{2x_k + 3}{4} \iff L = \frac{2L+3}{4} \iff L = \frac{3}{2}.$$

■

Solution 2. For each $k \in \mathbb{N}$,

$$x_{k+1} - \frac{3}{2} = \frac{1}{2}\left(x_k - \frac{3}{2}\right) = \frac{1}{2^2}\left(x_{k-1} - \frac{3}{2}\right) = \cdots = \frac{1}{2^k}\left(x_1 - \frac{3}{2}\right) = -\frac{1}{2^{k+1}}.$$

Therefore,

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} \left(\frac{3}{2} - \frac{1}{2^k}\right) = \frac{3}{2}.$$

■

1.94 Choose any $x_1 > 0$ and define $x_{k+1} = x_k + 1/x_k$ for k in $\mathbb{N}$. Determine whether $\{x_k\}$ converges. If it does, find its limit; otherwise, prove that it diverges.

Solution. It follows from

$$x_{k+1}^2 = \left(x_k + \frac{1}{x_k}\right)^2 = x_k^2 + 2 + \frac{1}{x_k^2}$$

that $x_{k+1}^2 - x_k^2 > 2$ for every $k \in \mathbb{N}$. Then we have

$$x_k^2 = x_1^2 + \sum_{j=1}^{k-1} \left(x_{k+1}^2 - x_k^2\right) \geq x_1^2 + 2(k-1).$$

$\{x_k^2\}$ is unbounded, so is $\{x_k\}$. Therefore, by Theorem 1.3.4, $\{x_k\}$ diverges. ■

1.95 Fix any $c > 0$. Let x_1 be any positive number and define $x_{k+1} = (x_k + c/x_k)/2$.

a) Prove that $\{x_k\}$ converges and find its limit.

b) Use this sequence to calculate $\sqrt{5}$, accurate to six decimal places.

Solution.

a) Since $x_1 > 0$, by simple induction, $x_k > 0$ for all $k \in \mathbb{N}$. By the AM-GM inequality,

$$x_k = \frac{1}{2}\left(x_k + \frac{c}{x_k}\right) \geq \frac{1}{2} \cdot 2\sqrt{x_k \cdot \frac{c}{x_k}} = \sqrt{c}$$

for all $k \geq 2$. Hence, we have

$$x_{k+1} - \sqrt{c} = \frac{1}{2}\left(x_k + \frac{c}{x_k}\right) - \sqrt{c} = \frac{x_k^2 - 2\sqrt{c}x_k + c^2}{2x_k}$$

$$
\begin{aligned}
&= \frac{x_k - \sqrt{c}}{2x_k}(x_k - \sqrt{c}) = \frac{1}{2}\left(1 - \frac{\sqrt{c}}{x_k}\right)(x_k - \sqrt{c}) \\
&\leq \frac{1}{2}(x_k - \sqrt{c})
\end{aligned}
$$

for all $k \geq 2$. Recursively, one can deduce that

$$0 \leq x_k - \sqrt{c} \leq \frac{1}{2}(x_{k-1} - \sqrt{c}) \leq \frac{1}{2^2}(x_{k-2} - \sqrt{c}) \leq \cdots \leq \frac{1}{2^{k-2}}(x_2 - \sqrt{c})$$

for all $k \geq 2$. By the Squeeze Play, taking limits yields

$$\lim_{k\to\infty}(x_k - \sqrt{c}) = 0 \iff \lim_{k\to\infty} x_k = \sqrt{c}.$$

b) Let $c = 5$, and let $x_1 = 1$.

$$
\begin{aligned}
x_2 &= \frac{1}{2}\left(x_1 + \frac{5}{x_1}\right) = 3 \\
x_3 &= \frac{1}{2}\left(x_2 + \frac{5}{x_2}\right) = \frac{7}{3} = 2.3333333... \\
x_4 &= \frac{1}{2}\left(x_3 + \frac{5}{x_3}\right) = \frac{47}{21} = 2.2380952... \\
x_5 &= \frac{1}{2}\left(x_4 + \frac{5}{x_4}\right) = \frac{2207}{987} = 2.2360688... \\
&\vdots \\
\sqrt{5} &= 2.2360679...
\end{aligned}
$$

■

1.96 Let x_1 be any positive number. For k in $\mathbb{N}$, define $x_{k+1} = 3 + 4/x_k$. Determine whether $\{x_k\}$ converges. If so, find its limit; otherwise, show that it diverges.

Solution. By simple induction, $x_k > 3$ for $k \geq 2$. Observe that

$$|x_{k+1} - x_k| = 4\left|\frac{1}{x_k} - \frac{1}{x_{k-1}}\right| = \frac{4}{x_k x_{k-1}}|x_k - x_{k-1}| < \frac{4}{9}|x_k - x_{k-1}|$$

for $k \geq 3$. Since $\{x_k\}$ is eventually contractive, it is Cauchy by Theorem 1.4.5, and hence convergent by Theorem 1.4.4. Let $\lim_{k\to\infty} x_k = L$. Then we have

$$\lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty}\left(3 + \frac{4}{x_k}\right) \iff L = 3 + \frac{4}{L} \iff L^2 - 3L - 4 = (L+1)(L-4) = 0.$$

By Theorem 1.3.9, $L \geq 3$. Therefore, $L = 4$. ■

1.99 Let a and b be constants such that $0 < a < b$. Prove that $\lim_{k\to\infty}\left(a^k + b^k\right)^{1/k} = b$.

Solution. We have $b^k < a^k + b^k < 2b^k \implies b < \left(a^k + b^k\right)^{1/k} < b \cdot 2^{1/k}$. Since

$$b \leq \lim_{k\to\infty}\left(a^k + b^k\right)^{1/k} \leq b \lim_{k\to\infty} 2^{1/k} = b,$$

by the Squeeze Play, $\lim_{k\to\infty} \left(a^k + b^k\right)^{1/k} = b$. ■

1.6 Cardinality

1.108 **a)** Suppose that $S = \{s_1, s_2, s_3, \cdots, s_n\}$ is any set with n elements. The power set of S, denoted here $P(S)$, is the collection of all subsets of S, including the empty set and S itself. Let $\mathbf{B}_n = \{(b_1, b_2, b_3, \cdots, b_n) : b_j = 0 \text{ or } 1\}$ denote the set of all binary strings of length n. Show that the cardinality of $P(S)$ is the same as that of $\mathbf{B}_n$.

b) Prove by induction on n that $P(S)$ has cardinality 2^n. (For this reason, $P(S)$ is often denoted 2^S.)

Solution.

a) For each $T \in P(S)$, define a function $f_k : P(S) \to \{0, 1\}$ by

$$f_k(T) = \begin{cases} 1, & \text{if } s_k \in T \\ 0, & \text{if } s_k \notin T. \end{cases}$$

We claim that the function $F : T \mapsto (f_1(T), f_2(T), \cdots, f_n(T))$ maps $P(S)$ one-to-one onto $\mathbf{B}_n$.

Suppose that T_1 and T_2 are two distinct subsets of S. Without loss of generality, assume that $s_k \in T_1$ but $s_k \notin T_2$. Such an element $s_k \in S$ must exist; otherwise, $T_1 = T_2$. $f_k(T_1) \neq f_k(T_2)$ implies $F(T_1) \neq F(T_2)$, so F is one-to-one.

For each $\mathbf{b} = (b_1, b_2, \cdots, b_n) \in \mathbf{B}_n$, let $T = \{s_k \in S \mid y_k = 1, 1 \le k \le n\}$. Note that $T \in P(S)$ and $f(T) = \mathbf{b}$. Hence, F is onto.

We conclude that $P(S)$ is in one-to-one correspondence with $\mathbf{B}_n$. Therefore, the cardinality of $P(S)$ is the same as that of $\mathbf{B}_n$.

b) For the sake of clarity, let $S_n = \{s_1, s_2, \cdots, s_n\}$.

(i) For $n = 1$, $P(S_1) = \{\varnothing, \{s_1\}\}$, so $P(S_1)$ has cardinality 2.

(ii) Suppose that $P(S_k)$ has cardinality 2^k for some $k \in \mathbb{N}$. We have

$$P(S_{k+1}) = \{T \mid T \in P(S_k)\} \cup \{T \cup \{s_k\} \mid T \in P(S_k)\}.$$

Therefore, $P(S_{k+1})$ has cardinality $2^k + 2^k = 2^{k+1}$.

By induction, $P(S_n)$ has cardinality 2^n for all $n \in \mathbb{N}$.

■

1.109 **a)** Suppose that $S = \{s_1, s_2, s_3, \cdots\}$ is any countably infinite set and let $P(S)$ denote the collection of all subsets of S. Let $\mathbf{B} = \{(b_1, b_2, b_3, \cdots) : b_j = 0 \text{ or } 1\}$ denote the set of all binary sequences. Show that the cardinality of $P(S)$ is the same as that of $\mathbf{B}$. (Again, $P(S)$ is sometimes denoted 2^S for this reason.)

b) Use Cantor's diagonalization technique to show that $\mathbf{B}$ is uncountable.

c) Hence show that a countably infinite set S has uncountably many subsets.

Solution.

a) For each $T \in P(S)$, define a function $f_k : P(S) \to \{0, 1\}$ by

$$f_k(T) = \begin{cases} 1, & \text{if } s_k \in T \\ 0, & \text{if } s_k \notin T. \end{cases}$$

We claim that the function $F : T \mapsto (f_1(T), f_2(T), \cdots)$ maps $P(S)$ one-to-one onto $\mathbf{B}$.

Suppose that T_1 and T_2 are two distinct subsets of S. Without loss of generality, assume that $s_k \in T_1$ but $s_k \notin T_2$. Such an element $s_k \in S$ must exist; otherwise, $T_1 = T_2$. $f_k(T_1) \neq f_k(T_2)$ implies $F(T_1) \neq F(T_2)$, so F is one-to-one.

For each $\mathbf{b} = (b_1, b_2, \cdots) \in \mathbf{B}$, let $T = \{s_k \in S \mid y_k = 1, k \in \mathbb{N}\}$. Note that $T \in P(S)$ and $f(T) = \mathbf{b}$. Hence, F is onto.

We conclude that $P(S)$ is in one-to-one correspondence with $\mathbf{B}$. Therefore, the cardinality of $P(S)$ is the same as that of $\mathbf{B}$.

b) $\mathbf{B}$ is clearly infinite. Suppose, to the contrary, that $\mathbf{B}$ is countably infinite. Let $\{\mathbf{b}_k = (b_{k1}, b_{k2}, \cdots) \in \mathbf{B} \mid k \in \mathbb{N}\}$ be any enumeration of elements in $\mathbf{B}$. Construct a binary sequence $\mathbf{a} = (a_1, a_2, \cdots)$ by choosing a_k for each $k \in \mathbb{N}$ according to the following rule:

$$a_k = \begin{cases} 1, & \text{if } b_{kk} = 0 \\ 0, & \text{if } b_{kk} = 1. \end{cases}$$

It is evident that $\mathbf{a} \in \mathbf{B}$. However, for each $k \in \mathbb{N}$, $\mathbf{a} \neq \mathbf{b}_k$ because $a_k \neq b_{kk}$. $\mathbf{a}$ does not belong to the enumeration. This contradiction implies that $\mathbf{B}$ must be uncountable.

c) By part (a), $P(S)$ is also uncountable. Therefore, S has uncountably many subsets.

■

1.110 Prove that an uncountable set of real numbers must have a limit point.

Solution. For an uncountable set S in $\mathbb{R}$, let $S_k = S \cap [k, k+1)$ for each $k \in \mathbb{Z}$. Then there must exist a $k \in \mathbb{Z}$ such that S_k is infinite. Otherwise, S is countable as we can enumerate elements in S like $S_0 \to S_{-1} \to S_1 \to S_{-2} \to S_2 \to \cdots$. S_k is bounded and infinite, so S_k has a limit point by the Bolzano–Weierstrass Theorem. Since $S_k \subset S$, it is also a limit point of S. ■

Chapter 2

Euclidean Spaces

2.1 Euclidean n-Space

2.2 For $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$, define $\|\mathbf{x}\|_1 = |x_1| + |x_2|$. Prove that $\|\cdot\|_1$ is a norm on $\mathbb{R}^2$.

Solution. Let $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. $\|\cdot\|_1$ has the following properties:

i) *Positive Definiteness*

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| \geq 0;$$
$$\|\mathbf{x}\|_1 = 0 \iff x_1 = x_2 = 0 \iff \mathbf{x} = \mathbf{0}.$$

ii) *Absolute Homogeneity*

$$\|c\mathbf{x}\|_1 = |cx_1| + |cx_2| = |c|(|x_1| + |x_2|) = |c| \cdot \|\mathbf{x}\|_1.$$

iii) *Subadditivity*

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_1 &= |x_1 + y_1| + |x_2 + y_2| \\ &\leq |x_1| + |y_1| + |x_2| + |y_2| \\ &= (|x_1| + |x_2|) + (|y_1| + |y_2|) \\ &= \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1.\end{aligned}$$

By Definition 2.1.3, $\|\cdot\|_1$ is indeed a norm on $\mathbb{R}^2$. ■

2.3 For $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$, define

$$d_1(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_1.$$

a) Show that d_1 is a metric on $\mathbb{R}^2$.

b) Using this metric, define an 1-neighborhood $N_1(\mathbf{x}; r)$ of $\mathbf{x}$ to be the set $N_1(\mathbf{x}; r) = \{\mathbf{y} \text{ in } \mathbb{R}^2 : d_1(\mathbf{x}, \mathbf{y}) < r\}$. Sketch the neighborhood $N_1(\mathbf{x}; r)$.

c) Let $N(\mathbf{x}; r)$ be any (Euclidean) neighborhood of $\mathbf{x}$. Show that there exist positive r_1 and r_2 such that

$$N_1(\mathbf{x}; r_1) \subset N(\mathbf{x}; r) \subset N_1(\mathbf{x}; r_2).$$

d) Let $N_1(\mathbf{x}; r)$ be any 1-neighborhood of $\mathbf{x}$. Show that there are Euclidean neighborhoods with radii r_1 and r_2 so that

$$N(\mathbf{x}; r_1) \subset N_1(\mathbf{x}; r) \subset N(\mathbf{x}; r_2).$$

Solution.

a) Let $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. d_1 has the following properties:

i) *Positive Definiteness*

$$\begin{aligned} &d_1(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_1 \geq 0; \\ &d_1(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x} - \mathbf{y} = \mathbf{0} \iff \mathbf{x} = \mathbf{y}. \end{aligned}$$

ii) *Symmetry*

$$d_1(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_1 = \|(-1)(\mathbf{y} - \mathbf{x})\|_1 = \|\mathbf{y} - \mathbf{x}\|_1 = d_1(\mathbf{y}, \mathbf{x}).$$

iii) *The Triangle Inequality*

$$\begin{aligned} d_1(\mathbf{x}, \mathbf{z}) &= \|\mathbf{x} - \mathbf{z}\|_1 = \|(\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z})\|_1 \\ &\leq \|\mathbf{x} - \mathbf{y}\|_1 + \|\mathbf{y} - \mathbf{z}\|_1 = d_1(\mathbf{x}, \mathbf{y}) + d_1(\mathbf{y}, \mathbf{z}). \end{aligned}$$

By Definition 2.1.4, d_1 is indeed a metric on $\mathbb{R}^2$.

b) See the remark below.

c) Let $r_1 = r$.

$$\begin{aligned} \mathbf{y} \in N_1(\mathbf{x}; r_1) &\iff d_1(\mathbf{x}, \mathbf{y}) = |y_1 - x_1| + |y_2 - x_2| < r_1 = r \\ &\implies d(\mathbf{x}, \mathbf{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \leq \sqrt{(|y_1 - x_1| + |y_2 - x_2|)^2} < r^2 \\ &\iff \mathbf{y} \in N(\mathbf{x}; r). \end{aligned}$$

Therefore, $N_1(\mathbf{x}; r_1) \subset N(\mathbf{x}; r)$. Let $r_2 = \sqrt{2}r$.

$$\begin{aligned} \mathbf{y} \in N(\mathbf{x}; r) &\iff d(\mathbf{x}, \mathbf{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < r \\ &\implies 2|y_1 - x_1||y_2 - x_2| \leq |y_1 - x_1|^2 + |y_2 - x_2|^2 < r^2 \end{aligned}$$

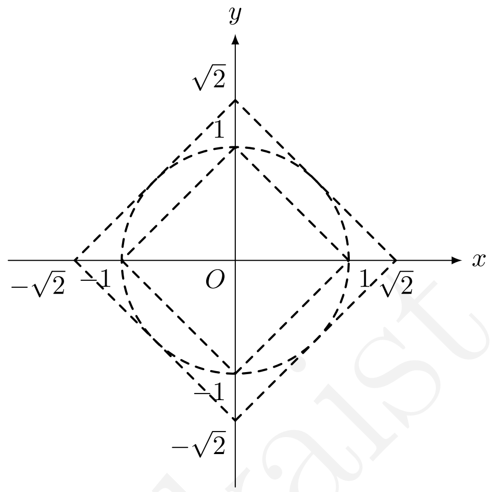
$$\begin{aligned}&\Longrightarrow d_1(\mathbf{x},\mathbf{y}) = \left[|y_1-x_1|^2+|y_2-x_2|^2+2|y_1-x_1||y_2-x_2|\right]^{1/2} < \sqrt{2}r = r_2\\ &\Longleftrightarrow \mathbf{y}\in N_1(\mathbf{x};r_2).\end{aligned}$$

Therefore, $N(\mathbf{x};r)\subset N_1(\mathbf{x};r_2)$.

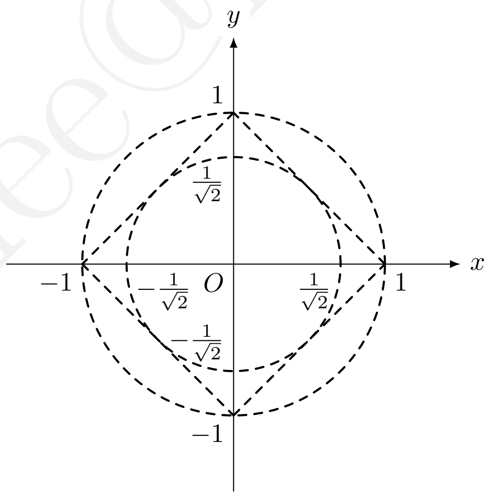
d) By part (c), let $r_1 = r/\sqrt{2}$ and let $r_2 = r$.

■

Remark. Let $\mathbf{x}=\mathbf{0}$ and $r=1$. The figures show the geometric interpretations of part (c)



and (d).



2.4 For $\mathbf{x}=(x_1,x_2)$ in $\mathbb{R}^2$, define

$$\|\mathbf{x}\|_\infty = \max\{|x_1|,|x_2|\}.$$

Prove that $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^2$.

Solution. Let $\mathbf{y}=(y_1,y_2)\in\mathbb{R}^2$. Recall that

$$\max\{a,b\} = \frac{a+b}{2}+\frac{|a-b|}{2}.$$

$\|\cdot\|_\infty$ has the following properties:

i) *Positive Definiteness*

$$\|\mathbf{x}\|_\infty = \frac{|x_1|+|x_2|}{2} + \frac{||x_1|-|x_2||}{2} \geq 0;$$
$$\|\mathbf{x}\|_\infty = 0 \iff x_1 = x_2 = 0 \iff \mathbf{x} = \mathbf{0}.$$

ii) *Absolute Homogeneity*

$$\|c\mathbf{x}\|_\infty = \max\{|cx_1|, |cx_2|\} = |c| \max\{|x_1|, |x_2|\} = |c| \cdot \|\mathbf{x}\|_\infty.$$

iii) *Subadditivity*

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_\infty &= \max\{|x_1+y_1|, |x_2+y_2|\} \\ &\leq \max\{|x_1|+|y_1|, |x_2|+|y_2|\} \\ &\leq \max\{|x_1|, |x_2|\} + \max\{|y_1|, |y_2|\} \\ &= \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty.\end{aligned}$$

By Definition 2.1.3, $\|\cdot\|_\infty$ is indeed a norm on $\mathbb{R}^2$. ■

2.5 For $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$, define

$$d_\infty(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|_\infty.$$

a) Show that d_∞ is a metric on $\mathbb{R}^2$.

b) Using this metric, define an ∞-neighborhood $N_\infty(\mathbf{x}; r)$ of $\mathbf{x}$ to be the set $N_\infty(\mathbf{x}; r) = \{\mathbf{y} \text{ in } \mathbb{R}^2 : d_\infty(\mathbf{x}, \mathbf{y}) < r\}$. Sketch the neighborhood $N_\infty(\mathbf{x}; r)$.

c) Let $N(\mathbf{x}; r)$ be any (Euclidean) neighborhood of $\mathbf{x}$. Show that there exist positive r_1 and r_2 such that

$$N_\infty(\mathbf{x}; r_1) \subset N(\mathbf{x}; r) \subset N_\infty(\mathbf{x}; r_2).$$

d) Let $N_\infty(\mathbf{x}; r)$ be any ∞-neighborhood of $\mathbf{x}$. Show that there are Euclidean neighborhoods with radii r_1 and r_2 so that

$$N(\mathbf{x}; r_1) \subset N_\infty(\mathbf{x}; r) \subset N(\mathbf{x}; r_2).$$

Solution.

a) Let $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. d_∞ has the following properties:

i) *Positive Definiteness*

$$d_\infty(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|_\infty \geq 0;$$
$$d_\infty(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x}-\mathbf{y} = \mathbf{0} \iff \mathbf{x} = \mathbf{y}.$$

ii) *Symmetry*

$$d_\infty(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|_\infty = \|(-1)(\mathbf{y}-\mathbf{x})\|_\infty = \|\mathbf{y}-\mathbf{x}\|_\infty = d_\infty(\mathbf{y},\mathbf{x}).$$

iii) *The Triangle Inequality*

$$\begin{aligned} d_\infty(\mathbf{x},\mathbf{z}) &= \|\mathbf{x}-\mathbf{z}\|_\infty = \|(\mathbf{x}-\mathbf{y})+(\mathbf{y}-\mathbf{z})\|_\infty \\ &\le \|\mathbf{x}-\mathbf{y}\|_\infty + \|\mathbf{y}-\mathbf{z}\|_\infty = d_\infty(\mathbf{x},\mathbf{y}) + d_\infty(\mathbf{y},\mathbf{z}). \end{aligned}$$

By Definition 2.1.4, d_∞ is indeed a metric on $\mathbb{R}^2$.

b) See the remark below.

c) Let $r_1 = r/\sqrt{2}$.

$$\begin{aligned} \mathbf{y} \in N_\infty(\mathbf{x}; r_1) &\iff d_\infty(\mathbf{x},\mathbf{y}) = \max\{|y_1 - x_1|, |y_2 - x_2|\} < r_1 = r/\sqrt{2} \\ &\implies d(\mathbf{x},\mathbf{y}) = \sqrt{(y_1-x_1)^2 + (y_2-x_2)^2} \le r^2/2 + r^2/2 = r^2 \\ &\iff \mathbf{y} \in N(\mathbf{x}; r). \end{aligned}$$

Therefore, $N_\infty(\mathbf{x}; r_1) \subset N(\mathbf{x}; r)$. Let $r_2 = r$.

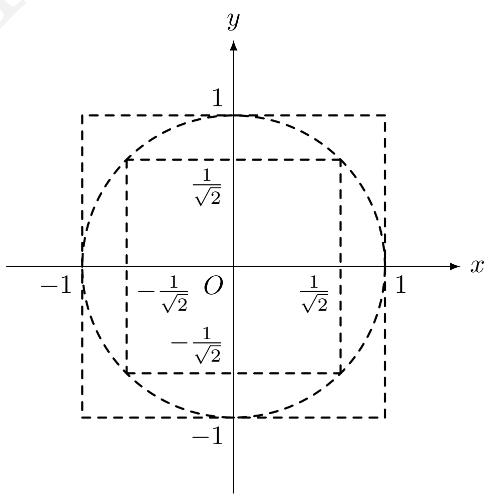
$$\begin{aligned} \mathbf{y} \in N(\mathbf{x}; r) &\iff d(\mathbf{x},\mathbf{y}) = \sqrt{(y_1-x_1)^2 + (y_2-x_2)^2} < r \\ &\implies d_\infty(\mathbf{x},\mathbf{y}) = \max\{|y_1 - x_1|, |y_2 - x_2|\} < r = r_2 \\ &\iff \mathbf{y} \in N_\infty(\mathbf{x}; r_2). \end{aligned}$$

Therefore, $N(\mathbf{x}; r) \subset N_\infty(\mathbf{x}; r_2)$.

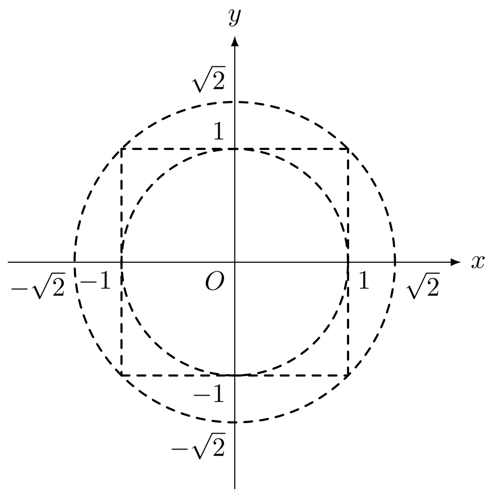
d) By part (c), let $r_1 = r$ and let $r_2 = \sqrt{2}r$.

■

Remark. Let $r = 1$. The figures show the geometric interpretations of part (c)



and (d).



2.7 Prove that, for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, $\|\mathbf{x}+\mathbf{y}\| \geq \|\mathbf{x}\| - \|\mathbf{y}\|$.

Solution. By the triangle inequality,

$$\|\mathbf{x}\| = \|(\mathbf{x}+\mathbf{y}) + (-\mathbf{y})\| \leq \|\mathbf{x}+\mathbf{y}\| + \|-\mathbf{y}\| = \|\mathbf{x}+\mathbf{y}\| + \|\mathbf{y}\|.$$

Therefore, $\|\mathbf{x}+\mathbf{y}\| \geq \|\mathbf{x}\| - \|\mathbf{y}\|$. ■

2.8 Let $\mathbf{x}$ be a limit point of a nonempty subset S of $\mathbb{R}^n$. Prove that any deleted neighborhood of $\mathbf{x}$ contains infinitely many points of S.

Solution. Suppose, to the contrary, that $N'(\mathbf{x}) \cap S$ were to contain only finitely many points. Let $N'(\mathbf{x}) \cap S = \{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ and let

$$\epsilon = \begin{cases} 1 & \text{if } S = \{\mathbf{x}\}, \\ \min\{\|\mathbf{x} - \mathbf{x}_k\| : \mathbf{x}_k \in S \setminus \{\mathbf{x}\}\} & \text{otherwise.} \end{cases}$$

Then we have a contradiction, $N'(\mathbf{x}; \epsilon) \cap S = \varnothing$. Therefore, $N'(\mathbf{x}) \cap S$ must contain infinitely many points. ■

2.12

a) Prove that the limit of a convergent sequence in $\mathbb{R}^n$ is unique.

b) Prove that a convergent sequence in $\mathbb{R}^n$ is bounded.

c) Prove that a convergent sequence in $\mathbb{R}^n$ is Cauchy.

Solution.

a) We have to prove that if $\{\mathbf{x}_k\}$ converges to $\mathbf{x}_0$ and also converges to $\mathbf{x}_0'$, then $\mathbf{x}_0 = \mathbf{x}_0'$. If, to the contrary, we assume that $\mathbf{x}_0 \neq \mathbf{x}_0'$, then, by choosing ϵ to be a positive number less than $\|\mathbf{x}_0 - \mathbf{x}_0'\|/2$, we will obtain a contradiction.

Because $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}_0$, we can choose a $k_1 \in \mathbb{N}$ such that, when $k \geq k_1$, then $\mathbf{x}_k \in N(\mathbf{x}_0; \epsilon)$. Similarly, we can choose a $k_2 \in \mathbb{N}$ such that, when $k \geq k_2$, then $\mathbf{x}_k \in N(\mathbf{x}_0'; \epsilon)$. Let $k_0 = \max\{k_1, k_2\}$. For any $k \geq k_0$, it follows that $\mathbf{x}_k \in N(\mathbf{x}_0; \epsilon) \cap N(\mathbf{x}_0'; \epsilon)$. Thus, for $k \geq k_0$, we obtain the contradictory inequality

$$\|\mathbf{x}_0 - \mathbf{x}_0'\| = \|\mathbf{x}_0 - \mathbf{x}_k + \mathbf{x}_k - \mathbf{x}_0'\| \leq \|\mathbf{x}_0 - \mathbf{x}_k\| + \|\mathbf{x}_k - \mathbf{x}_0'\|$$

$$< \epsilon + \epsilon = 2\epsilon < \|\mathbf{x}_0 - \mathbf{x}'_0\|.$$

The contradiction results from our having assumed, falsely, that $\mathbf{x}_0 \neq \mathbf{x}'_0$. This completes the proof.

b) Suppose that $\{\mathbf{x}_k\}$ converges to $\mathbf{x}_0$. We must show there exists some real number M such that $\|\mathbf{x}_k\| \leq M$, for $k \in \mathbb{N}$. Choose $\epsilon = 1$. Since $\{\mathbf{x}_k\}$ converges to $\mathbf{x}_0$, there exists an index k_0 such that, if $k \geq k_0$, then $\mathbf{x}_k$ belongs to $N(\mathbf{x}_0; 1)$. For $k \geq k_0$,

$$\|\mathbf{x}_k\| = \|\mathbf{x}_k - \mathbf{x}_0 + \mathbf{x}_0\| \leq \|\mathbf{x}_k - \mathbf{x}_0\| + \|\mathbf{x}_0\| < 1 + \|\mathbf{x}_0\|.$$

It follows that $\mathbf{x}_k \in N(\mathbf{0}; \|\mathbf{x}_0\| + 1)$. Let $M = \max\{\|\mathbf{x}_1\|, \|\mathbf{x}_2\|, \cdots, \|\mathbf{x}_{k_0-1}\|, \|\mathbf{x}_0\| + 1\}$. It is easy to see that $\{\mathbf{x}_k\}$ must be contained in $N(\mathbf{0}; M)$ and is therefore bounded.

c) Suppose that $\{\mathbf{x}_k\}$ is a convergent sequence with limit $\mathbf{x}_0$. Fix any positive ϵ. Suppose that k_0 is chosen so that, for $k \geq k_0$, $\mathbf{x}_k \in N(\mathbf{x}_0; \epsilon/2)$. Choose any two indices k and m each greater than k_0. It follows that $\mathbf{x}_k \in N(\mathbf{x}_0; \epsilon/2)$ and $\mathbf{x}_m \in N(\mathbf{x}_0; \epsilon/2)$. Consequently,

$$\begin{aligned}\|\mathbf{x}_k - \mathbf{x}_m\| &= \|\mathbf{x}_k - \mathbf{x}_0 + \mathbf{x}_0 - \mathbf{x}_m\| \leq \|\mathbf{x}_k - \mathbf{x}_0\| + \|\mathbf{x}_0 - \mathbf{x}_m\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

In other words, if k and m are each greater than k_0, then $\|\mathbf{x}_k - \mathbf{x}_m\| < \epsilon$. Therefore, $\{\mathbf{x}_k\}$ is Cauchy.

■

2.15 Prove that every Cauchy sequence in $\mathbb{R}^n$ is bounded.

Solution. Assume that $\{\mathbf{x}_k\}$ is a Cauchy sequence. Let $\epsilon = 1$. By the Cauchy condition, choose a k_0 such that, whenever $k, m \geq k_0$, we have $\|\mathbf{x}_k - \mathbf{x}_m\| < 1$. Let $m = k_0$ and observe that for any $k \geq k_0$.

$$\|\mathbf{x}_k\| = \|\mathbf{x}_k - \mathbf{x}_0 + \mathbf{x}_0\| \leq \|\mathbf{x}_k - \mathbf{x}_0\| + \|\mathbf{x}_0\| < 1 + \|\mathbf{x}_0\|.$$

Let $M = \max\{\|\mathbf{x}_1\|, \|\mathbf{x}_2\|, \cdots, \|\mathbf{x}_{k_0-1}\|, \|\mathbf{x}_0\| + 1\}$. Then it is immediate that $\|\mathbf{x}_k\| \leq M$ for all k in $\mathbb{N}$. Therefore, $\{\mathbf{x}_k\}$ is bounded. ■

2.17 Suppose that S is any nonempty set in $\mathbb{R}^n$ and that $\mathbf{x}_0$ is any limit point of S. Prove that there exists a Cauchy sequence of distinct points of S that converges to $\mathbf{x}_0$.

Solution. Recall that $\mathbf{x}_0$ is a limit point of a nonempty subset S in $\mathbb{R}^n$ if, for each $\epsilon > 0$, $N'(\mathbf{x}_0; \epsilon) \cap S \neq \varnothing$. (Definition 2.1.8) Start with $\epsilon_1 = 1$. Choose a point $\mathbf{x}_1 \in N'(\mathbf{x}_0; \epsilon_1) \cap S$. For each $k \geq 2$, let $\epsilon_k = \min\{1/k, \|\mathbf{x}_0 - \mathbf{x}_{k-1}\|\}$, and choose a point $\mathbf{x}_k \in N'(\mathbf{x}_0; \epsilon_k) \cap S$. Note that $\epsilon_k \leq \|\mathbf{x}_0 - \mathbf{x}_{k-1}\|$ implies $\mathbf{x}_k \notin N'(\mathbf{x}_0; \epsilon_j)$ for all $j > k$. Therefore, $\{\mathbf{x}_k\}$ is a sequence of distinct points of S.

By Theorem 2.1.8, as we proved in part (c) of Exercise 2.12, a convergent sequence is Cauchy, so it remains to prove that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}_0$. Let $\epsilon > 0$ be given. By Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that $1/k_0 < \epsilon$. For all $k \geq k_0$, $\epsilon_k \leq 1/k \leq 1/k_0 < \epsilon$ implies $\mathbf{x}_k \in N'(\mathbf{x}_0; \epsilon_k) \subset N(\mathbf{x}_0; \epsilon)$. Hence, a sequence $\{\mathbf{x}_k\}$ of distinct points of S converges to $\mathbf{x}_0$ and is therefore Cauchy. ■

2.2 Open and Closed Sets

2.19 Let U be any open set in $\mathbb{R}$. Show that U is the union of some open neighborhoods of its points. Likewise, show that any open set in $\mathbb{R}^n$ is the union of neighborhoods of its points.

Solution. Let U be any open set in $\mathbb{R}^n$. $\mathbb{R}$ is a special case of $\mathbb{R}^n$. For all $\mathbf{x} \in U$, there exists an open neighborhood $N(\mathbf{x}) \subseteq U$ of $\mathbf{x}$. We have $\bigcup_{x \in U} N(\mathbf{x}) \subseteq U$. Since $\mathbf{x} \in N(\mathbf{x}) \subseteq \bigcup_{x \in U} N(\mathbf{x})$ for every $\mathbf{x} \in U$, we have $U \subseteq \bigcup_{x \in U} N(\mathbf{x})$. Therefore, $U = \bigcup_{x \in U} N(\mathbf{x})$. ■

2.20 Prove that the union of the sets $C_k = [1/k, 3 - 1/k]$, for $k = 3, 4, 5, \cdots$ is the open interval $(0, 3)$.

Solution. For all $k \geq 3$, $C_k \subset (0, 3)$, so $\bigcup_{k=3}^{\infty} C_k \subseteq (0, 3)$. It remains to show $(0, 3) \subseteq \bigcup_{k=3}^{\infty} C_k$. For each $x \in (0, 3)$, by Archimedes' principle, choose $k_1 \in \mathbb{N}$ such that $1/k_1 < x$ and choose $k_2 \in \mathbb{N}$ such that $1/k_2 < 3 - x$. Let $k_0 = \max\{k_1, k_2, 3\}$. Then $x \in (1/k_1, 3 - 1/k_2) \subset C_{k_0} \subset \bigcup_{k=3}^{\infty} C_k$. Therefore, $(0, 3) \subseteq \bigcup_{k=3}^{\infty} C_k$, so $\bigcup_{k=3}^{\infty} C_k = (0, 3)$. ■

2.21 Is the set $S = \{(x_1, x_2, 0) : x_1^2 + x_2^2 < r^2\}$ open in $\mathbb{R}^3$? If so, prove it; otherwise, explain why not.

Solution. Let $\epsilon > 0$ be given. For each $\mathbf{x} = (x_1, x_2, 0) \in S$, we have $(x_1, x_2, \epsilon/2) \in N(\mathbf{x}; \epsilon) \cap S^c$. No neighborhood of $\mathbf{x} \in S$ is contained in S. Therefore, S is not open in $\mathbb{R}^3$. ■

2.24 Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two distinct points in $\mathbb{R}^n$. Prove that there exist disjoint open sets U_1 and U_2 such that $\mathbf{x}_1$ is in U_1 and $\mathbf{x}_2$ is in U_2.

Solution. Let $r = \dfrac{d(\mathbf{x}_1, \mathbf{x}_2)}{2}$. $r > 0$ since $\mathbf{x}_1 \neq \mathbf{x}_2$. Consider $N(\mathbf{x}_1; r)$ and $N(\mathbf{x}_2; r)$.

i) Every *open ball*—what our textbook defines as a *neighborhood*—is open.

ii) $\mathbf{x}_1 \in N(\mathbf{x}_1; r)$ and $\mathbf{x}_2 \in N(\mathbf{x}_2; r)$.

iii) Suppose that there is a point $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y} \in N(\mathbf{x}_1; r) \cap N(\mathbf{x}_2; r)$. Then we have

$$\begin{aligned} d(\mathbf{y}, \mathbf{x}_1) &< \frac{d(\mathbf{x}_1, \mathbf{x}_2)}{2} \\ d(\mathbf{y}, \mathbf{x}_2) &< \frac{d(\mathbf{x}_1, \mathbf{x}_2)}{2} \\ d(\mathbf{y}, \mathbf{x}_1) + d(\mathbf{y}, \mathbf{x}_2) &< d(\mathbf{x}_1, \mathbf{x}_2), \end{aligned}$$

which contradicts the triangle inequality. Therefore, such a point $\mathbf{y}$ does not exist, that is, $N(\mathbf{x}_1; r) \cap N(\mathbf{x}_2; r) = \varnothing$.

This completes the proof. ■

2.25 Prove that, if $\{\mathbf{x}_k\}$ is a convergent sequence in $\mathbb{R}^n$ with limit $\mathbf{x}_0$, then $S = \{\mathbf{x}_k : k \text{ in } \mathbb{N}\} \cup \{\mathbf{x}_0\}$ is closed in $\mathbb{R}^n$. (Evidently, $\mathbf{x}_0$ is a limit point of S. How do you know it is the only limit point, not just of $\{\mathbf{x}_k : k \text{ in } \mathbb{N}\}$, but of S? Exercise 1.67 and/or Exercise 2.17 are relevant.)

Solution 1. Construct a sequence $\{\mathbf{y}_k\}$ by

$$\mathbf{y}_k = \begin{cases} \mathbf{x}_0 & \text{if } k \text{ is odd,} \\ \mathbf{x}_{k/2} & \text{if } k \text{ is even.} \end{cases}$$

The two subsequences $\{\mathbf{y}_{2j}\} = \{\mathbf{x}_j\}$ and $\{\mathbf{y}_{2j-1}\} = \{\mathbf{x}_0\}$ converge to $\mathbf{x}_0$. By Exercise 1.67, $\{\mathbf{y}_k\}$ also converges to $\mathbf{x}_0$. $\{\mathbf{y}_k \mid k \in \mathbb{N}\} = S$, so $\mathbf{x}_0$ is the only limit point of S. By Theorem 2.2.4, S is closed in $\mathbb{R}^n$. ■

Solution 2. Suppose that S has another limit point $\mathbf{x}_1 \neq \mathbf{x}_0$. By Exercise 2.17, there exists a Cauchy sequence $\{\mathbf{x}_{k_j}\}$ of distinct points in S that converges to $\mathbf{x}_1$. $\{\mathbf{x}_{k_j}\} \setminus \{\mathbf{x}_0\}$ is a subsequence of $\{\mathbf{x}_k\}$ that converges to $\mathbf{x}_1$. By Theorem 2.1.10, $\mathbf{x}_1$ is a cluster point of $\{\mathbf{x}_k\}$. This contradicts that $\mathbf{x}_0$ is the only cluster point of $\{\mathbf{x}_k\}$. Therefore, $\mathbf{x}_0$ is the only limit point of S. By Theorem 2.2.4, S is closed in $\mathbb{R}^n$. ■

2.31 Prove that, for any open set U in $\mathbb{R}^n$, $U^0 = U$. Explain why, for any set S in $\mathbb{R}^n$, it follows that $\left(S^0\right)^0 = S^0$.

Solution. By Definition 2.2.1, $U \subseteq U^0$. By Theorem 2.2.5, $U^0 \subseteq U$. Therefore, $U^0 = U$. S^0 is open in $\mathbb{R}^n$ by Corollary 2.2.6, so $\left(S^0\right)^0 = S^0$. ■

2.32 Prove that, for any closed set C in $\mathbb{R}^n$, $\overline{C} = C$. Explain why, for any set S in $\mathbb{R}^n$, it follows that $\overline{\left(\overline{S}\right)} = \overline{S}$.

Solution. By Definition 2.2.2, $C \subseteq \overline{C}$. By Theorem 2.2.4, $C' \subseteq C$ implies $\overline{C} \subseteq C$. Therefore, $\overline{C} = C$. $\overline{S}$ is closed in $\mathbb{R}^n$ by Corollary 2.2.8, so $\overline{\left(\overline{S}\right)} = \overline{S}$. ■

2.33 Prove that, for any set S in $\mathbb{R}^n$, $S^0 \cap \operatorname{bd}(S) = \varnothing$.

Solution. For all $\mathbf{x} \in S^0$, there exists an $r > 0$ such that $N(\mathbf{x}; r) \subseteq S \implies N(\mathbf{x}; r) \cap S^c = \varnothing$. Therefore, $\mathbf{x} \notin \operatorname{bd}(S)$, so $S^0 \cap \operatorname{bd}(S) = \varnothing$. ■

2.34 Prove that, for any set S in $\mathbb{R}^n$, $S^0 \cup \operatorname{bd}(S) = \overline{S}$.

Solution. First, we have $S^0 \subseteq S \subseteq \overline{S}$. Let $\mathbf{x} \in \left(\overline{S}\right)^c$. Since $\overline{S}$ is closed, $\left(\overline{S}\right)^c$ is open by Theorem 2.2.2, so $\mathbf{x}$ is an interior point of $\left(\overline{S}\right)^c$. $S \subseteq \overline{S}$ implies $\left(\overline{S}\right)^c \subseteq S^c$, so $\mathbf{x}$ is also an interior point of S^c. By Exercise 2.33, $(S^c)^0 \cap \operatorname{bd}(S^c) = \varnothing$. By Exercise 2.37, $\operatorname{bd}(S) = \operatorname{bd}(S^c)$. Hence, $\mathbf{x} \in \operatorname{bd}(S)^c$. $\left(\overline{S}\right)^c \subseteq \operatorname{bd}(S)^c$ implies $\operatorname{bd}(S) \subseteq \overline{S}$. Therefore, $S^0 \cup \operatorname{bd}(S) \subseteq \overline{S}$.

Let $\mathbf{x} \in \overline{S}$. Then every neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ contains points in S. If there exists an $r > 0$ such that $N(\mathbf{x}; r) \cap S^c = \varnothing$ or, in other words, $N(\mathbf{x}; r) \subseteq S$, then $\mathbf{x} \in S^0$. Otherwise, $S \in \operatorname{bd}(S)$. Therefore, $\overline{S} \subseteq S^0 \cup \operatorname{bd}(S)$. We conclude that $S^0 \cup \operatorname{bd}(S) = \overline{S}$. ■

2.35 Prove that, for any set S in $\mathbb{R}^n$, $\overline{S} \cap \overline{(S^c)} = \operatorname{bd}(S)$.

Solution. By Exercise 2.37, $\operatorname{bd}(S) = \operatorname{bd}(S^c)$. By Exercise 2.34, we have

$$\begin{aligned}\overline{S} \cap \overline{(S^c)} &= \left[S^0 \cup \operatorname{bd}(S)\right] \cap \left[(S^c)^0 \cup \operatorname{bd}(S^c)\right] \\ &= \operatorname{bd}(S) \cup \left[S^0 \cap (S^c)^0\right] \\ &= \operatorname{bd}(S).\end{aligned}$$

■

2.36 Prove or disprove: For every S in $\mathbb{R}^n$, $\overline{(S^0)} = \overline{S}$.

Solution. False. Consider $S = \{\mathbf{0}\}$. Then $S^0 = \varnothing$. $\overline{(S^0)} = \varnothing$, but $\overline{S} = S = \{\mathbf{0}\}$. ■

2.37 Prove or disprove: For every S in $\mathbb{R}^n$, $\operatorname{bd}(S) = \operatorname{bd}(S^c)$.

Solution. True. It follows from the definition that

$$\begin{aligned}\operatorname{bd}(S) &= \{\mathbf{x} \in \mathbb{R}^n : N(\mathbf{x}) \cap S \neq \varnothing,\, N(\mathbf{x}) \cap S^c \neq \varnothing\} \\ &= \{\mathbf{x} \in \mathbb{R}^n : N(\mathbf{x}) \cap S^c \neq \varnothing,\, N(\mathbf{x}) \cap (S^c)^c \neq \varnothing\} \\ &= \operatorname{bd}(S^c).\end{aligned}$$

■

2.38 Prove or disprove: For every S in $\mathbb{R}^n$, $\operatorname{bd}\left(\overline{S}\right) = \operatorname{bd}(S^c)$.

Solution. False. Consider $S = (-1,0) \cup (0,1)$. Then $\overline{S} = [-1,1]$. $\operatorname{bd}\left(\overline{S}\right) = \{-1,1\}$, but $\operatorname{bd}(S) = \{-1,0,1\}$. ■

2.41 Let S be a bounded set in $\mathbb{R}^n$.

a) Prove that $\overline{S}$ is also bounded.

b) If M is a bound for S, is it true that M is necessarily a bound for $\overline{S}$ also? Prove your answer.

Solution. Let M be a bound for S. For all $\mathbf{x}_0 \in S'$, by Exercise 2.17, there exists a Cauchy sequence $\{\mathbf{x}_k\}$ of distinct points in S that converges to $\mathbf{x}_0$. Let $\epsilon > 0$ be given. There exists an index $k_0 \in \mathbb{N}$ such that, whenever $k \geq k_0$, $\mathbf{x}_k \in N(\mathbf{x}_0; \epsilon)$. For $k \geq k_0$, we have $\|\mathbf{x}_0\| \leq \|\mathbf{x}_0 - \mathbf{x}_k\| + \|\mathbf{x}_k\| < M + \epsilon$. Therefore, $\overline{S}$ is also bounded. Moreover, since ϵ is arbitrary, M is also a bound for $\overline{S}$. ■

2.42 Let S be any bounded set in $\mathbb{R}^n$. Prove that $d\left(\overline{S}\right) = d(S)$, where $d(A)$ denotes the diameter of the set A.

Solution. Since $S \subseteq \overline{S}$, it is clear that $d(S) \leq d\left(\overline{S}\right)$. Let $\epsilon > 0$ be given. Choose $\mathbf{x}$ and $\mathbf{y}$ in $\overline{S}$. Then there are points $\mathbf{x}' \in N(\mathbf{x}; \epsilon) \cap S$ and $\mathbf{y}' \in N(\mathbf{y}; \epsilon) \cap S$. It follows from

$$\begin{aligned}\|\mathbf{x} - \mathbf{y}\| &\leq \|\mathbf{x} - \mathbf{x}'\| + \|\mathbf{x}' - \mathbf{y}'\| + \|\mathbf{y}' - \mathbf{y}\| \\ &< \epsilon + \|\mathbf{x}' - \mathbf{y}'\| + \epsilon \leq d(S) + 2\epsilon\end{aligned}$$

that $d\left(\overline{S}\right) \leq d(S) + 2\epsilon$. Since ϵ is arbitrary, $d\left(\overline{S}\right) \leq d(S)$. Therefore, $d\left(\overline{S}\right) = d(S)$. ■

2.44 Suppose that S is a closed subset of $\mathbb{R}^n$ and that $\mathbf{x}$ is a point of S^c. Prove that there exists a $\mathbf{y}_0$ in S such that $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}_0\|$. (*Hint:* Consider separately the two cases when S is a finite set and when S is an infinite set. If S is an infinite set, assume that no such $\mathbf{y}_0$ exists. Let $\{\epsilon_k\}$ be a sequence of positive numbers that converges monotonically to 0. Explain how to choose distinct points $\mathbf{y}_k$ in S such that $d(\mathbf{x}, S) < \|\mathbf{x} - \mathbf{y}_k\| < d(\mathbf{x}, S) + \epsilon_k$ for all k in $\mathbb{N}$. Apply the Bolzano–Weierstrass theorem, then extract a convergent subsequence $\{\mathbf{y}_{k_j}\}$. What properties must the limit of the subsequence $\{\mathbf{y}_{k_j}\}$ have?)

Solution. If S is a finite set, then $A = \{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in S\}$ is also a finite set. Therefore, $\min A \in A$, so there exists a $\mathbf{y}_0$ in S such that $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}_0\|$.

Assume that S is an infinite set. Let $A = \{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in S\}$. A is nonempty and bounded below by 0, so $\inf A = d(\mathbf{x}, S)$ exists in $\mathbb{R}$ by Axiom 1.1.1. Suppose, to the contrary, that there is no $\mathbf{y}_0$ such that $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}_0\|$. Let $\{\epsilon_k\}$ be a sequence of positive numbers that converges monotonically to 0. By Theorem 1.1.1, choose $\mathbf{y}_1 \in S$ such that $d(\mathbf{x}, S) < \|\mathbf{x} - \mathbf{y}_1\| < d(\mathbf{x}, S) + \epsilon_1$. For each $k \geq 2$, choose $\mathbf{y}_k \in S$ such that $d(\mathbf{x}, S) < \|\mathbf{x} - \mathbf{y}_k\| < \min\{d(\mathbf{x}, S) + \epsilon_k, \|\mathbf{x} - \mathbf{y}_{k-1}\|\}$. The left inequalities are strict because we assumed that no such $\mathbf{y}_0$ exists.

Let $T = \{\mathbf{y}_k \mid k \in \mathbb{N}\}$. As $\{\|\mathbf{x} - \mathbf{y}_k\|\}$ is strictly decreasing, $\{\mathbf{y}_k\}$ are distinct points in S, so T is infinite. For all $k \in \mathbb{N}$, $\|\mathbf{y}_k\| \leq \|\mathbf{y}_k - \mathbf{x}\| + \|\mathbf{x}\| < d(\mathbf{x}, S) + \epsilon_1 + \|\mathbf{x}\|$, so T is bounded. By the Bolzano–Weierstrass theorem, T has a limit point. Let $\mathbf{y}' \in T'$. $T \subseteq S$ implies $\mathbf{y}' \in S'$. Since S is closed, by Theorem 2.2.4, $\mathbf{y}' \in S$. By Exercise 2.17, there exists a Cauchy sequence $\{\mathbf{y}_{k_j}\}$ of distinct points of $T \subseteq S$ that converges to $\mathbf{y}'$.

Let $\epsilon > 0$ be given. Since $\lim\limits_{k\to\infty} \epsilon_k = 0$, there exists an index $k_0 \in \mathbb{N}$ such that, whenever $k \geq k_0$, $\epsilon_k < \epsilon$. Then, for all $k \geq k_0$, $0 < \|\mathbf{x} - \mathbf{y}_k\| - d(\mathbf{x}, S) < \epsilon_k < \epsilon$. Hence, $\lim\limits_{k\to\infty} \|\mathbf{x} - \mathbf{y}_k\| = d(\mathbf{x}, S)$. We have $\lim\limits_{j\to\infty} \|\mathbf{x} - \mathbf{y}_{k_j}\| = \|\mathbf{x} - \mathbf{y}'\| = d(\mathbf{x}, S)$, so $\mathbf{y}_0 = \mathbf{y}' \in S$. Therefore, our original assumption was wrong. This completes the proof. ∎

2.45 Suppose that S is an open subset of $\mathbb{R}^n$ and that $\mathbf{x}$ is a point of S^c. Prove that there exists no $\mathbf{y}$ in S such that $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}\|$.

Solution. Suppose, to the contrary, that there exists a $\mathbf{y} \in S$ such that $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}\|$. $d(\mathbf{x}, S)$ must be positive because $\mathbf{x} \in S^c$ and $\mathbf{y} \in S$. Since S is open, $\mathbf{y} \in S^0$, so there exists an $r > 0$ such that $N(\mathbf{y}; r) \subseteq S$. Let

$$\mathbf{y}' = \mathbf{y} + \frac{r}{2} \cdot \frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|}.$$

It is clear that $\mathbf{y}' \in N(\mathbf{y}; r) \subseteq S$. Then we have a contradiction,

$$\|\mathbf{x} - \mathbf{y}'\| = \left(1 - \frac{r}{2\|\mathbf{x} - \mathbf{y}\|}\right) \|\mathbf{x} - \mathbf{y}\| < \|\mathbf{x} - \mathbf{y}\|.$$

Therefore, there exists no $\mathbf{y}$ in S such that $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}\|$. ∎

2.46 Prove that no point of Cantor's set C is an isolated point of C.

Remark. Refer to Example 8 in Section 2.2.

Solution. Let $\epsilon > 0$ be given. By Archimedes' principle, choose $k \in \mathbb{N}$ such that $1/3^k < \epsilon$. Let $x \in C$. Then $x \in C_k = [0, 1] \setminus U_k$. Let $y \neq x$ be an endpoint of a closed interval in C_k that contains x. Then

$|x-y| \le 1/3^k < \epsilon$. Note that C contains the endpoints of the closed intervals in C_k, so $y \in C$. Therefore, every point of C is its limit point. It follows that C has no isolated point. ■

2.47 Here is one way of proving that Cantor's set C is uncountable.

a) Show that

$$.t_1t_2t_3\cdots t_k\overline{2} = \sum_{j=1}^{k} \frac{t_j}{3^j} + \sum_{j=k+1}^{\infty} \frac{2}{3^j}$$

and

$$.t_1t_2t_3\cdots (t_k+1)\overline{0} = \sum_{j=1}^{k-1} \frac{t_j}{3^j} + \frac{t_k+1}{3^k}$$

where $t_k \neq 2$, represent the same x in $[0,1]$.

b) Show, in general, that x in $[0,1]$ is in one of the $I(k,j)$ and is excluded from Cantor's set if and only if the ternary expansion of x has the form

$$.t_1t_2\cdots t_{k-1}1t_{k+1}t_{k+2}\cdots,$$

where *none* of the $t_1, t_2, \cdots, t_{k-1}$ is 1. (For $j > k$, the t_j are any of 0, 1, 2, using as few 1's as possible in accord with our convention.) Thus show that Cantor's set consists of exactly those x for which the ternary expansion contains no 1's.

c) Show that Cantor's set C can be put in one-to-one correspondence with $\mathbf{B}$, the set of infinite binary sequences. (Refer to Exercise 1.109.)

d) Hence prove that C is uncountable.

Solution.

a) It is evident that

$$\begin{aligned} .t_1t_2t_3\cdots t_k\overline{2} &= \sum_{j=1}^{k} \frac{t_j}{3^j} + \sum_{j=k+1}^{\infty} \frac{2}{3^j} = \sum_{j=1}^{k-1} \frac{t_j}{3^j} + \frac{t_k}{3^k} + \sum_{j=k+1}^{\infty} \frac{2}{3^j} \\ &= \sum_{j=1}^{k-1} \frac{t_j}{3^j} + \frac{t_k}{3^k} + \frac{2/3^{k+1}}{1-1/3} = \sum_{j=1}^{k-1} \frac{t_j}{3^j} + \frac{t_k+1}{3^k} = .t_1t_2t_3\cdots(t_k+1)\overline{0}. \end{aligned}$$

b) None of the $t_1, t_2, \cdots, t_{k-1}$ is 1, so regard

$$\left(\frac{t_1}{2}\right)\left(\frac{t_2}{2}\right)\cdots\left(\frac{t_{k-1}}{2}\right) = \sum_{j=1}^{k-1}\left(\frac{t_{k-j}}{2}\right)2^{j-1}$$

as a binary number. Each binary expansion uniquely represents an integer between 0 and $2^{k-1}-1$. Therefore, x of the form $.t_1t_2\cdots t_{k-1}1t_{k+1}t_{k+2}\cdots$ is in $I(k, \sum_{j=1}^{k-1} t_{k-j}2^{j-2}+1)$. Moreover, $C = \{.t_1t_2t_3\cdots$ (ternary expansi
$t_k \in \{0,2\}$ for all $k \in \mathbb{N}\}$.

c) For each $.t_1t_2t_3\cdots \in C$, form a binary sequence $\{b_k\}$ with $b_k = t_k/2$. It is clear that this relation puts C in one-to-one correspondence with $\mathbf{B}$.

d) The cardinality of C is the same as that of $\mathbf{B}$. Therefore, C is uncountable. ∎

2.3 Completeness

2.48 Find a nested sequence of nonempty, bounded, open sets in $\mathbb{R}$ whose intersection is empty.

Solution. Consider $\{C_k\}$ with $C_k = (0, 1/k)$. For each $k \in \mathbb{N}$,

(i) $\{C_k\}$ is nested as $C_k = (0, \frac{1}{k}) \supseteq \left(0, \frac{1}{k+1}\right) = C_{k+1}$.

(ii) C_k is nonempty as $\frac{1}{2k} \in C_k$.

(iii) C_k is bounded as $(0, 1/k) \subseteq (0, 1)$.

(iv) C_k is open as it is an open interval.

We claim that $\bigcap_{k=1}^{\infty} C_k = \varnothing$. Suppose, to the contrary, that there exists some real number x such that $x \in \bigcap_{k=1}^{\infty} C_k$. It is clear that x must be positive. By Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that $1/k_0 < x$. Then, for all $k \geq k_0$, $x \notin C_k$. Therefore, $x \notin \bigcap_{k=1}^{\infty} C_k$. ∎

2.49 Find a nested sequence of nonempty, closed sets in $\mathbb{R}$ whose intersection is empty.

Solution. Consider $\{C_k\}$ with $C_k = [k, \infty)$. For each $k \in \mathbb{N}$,

(i) $\{C_k\}$ is nested as $C_k = [k, \infty) \supseteq [k+1, \infty) = C_{k+1}$.

(ii) C_k is nonempty as $k \in C_k$.

(iii) C_k is closed, by Theorem 2.2.2, as its complement $(-\infty, k)$ is open.

We claim that $\bigcap_{k=1}^{\infty} C_k = \varnothing$. Suppose, to the contrary, that there exists some real number x such that $x \in \bigcap_{k=1}^{\infty} C_k$. It is clear that x must be positive. By Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that $k_0 > x$. Then, for all $k \geq k_0$, $x \notin C_k$. Therefore, $x \notin \bigcap_{k=1}^{\infty} C_k$. ∎

2.50 Let $\mathbf{x}_0$ be an interior point of a set S in $\mathbb{R}^n$. Exhibit a nested sequence $\{C_k\}$ of nonempty, closed, bounded sets, each contained in S such that $\bigcap_{k=1}^{\infty} C_k = \{\mathbf{x}_0\}$.

Remark. For a trivial sequence, consider $\{C_k\}$ with $C_k = \{\mathbf{x}_0\}$. The solution exhibits a strictly nested sequence.

Solution. Since $\mathbf{x}_0 \in S^0$, there exists some $\epsilon_0 > 0$ such that $N(\mathbf{x}_0; 2\epsilon_0) \subseteq S$. Consider $\{C_k\}$ with $C_k = \overline{N(\mathbf{x}_0; \epsilon_0/k)}$. For each $k \in \mathbb{N}$,

(i) $\{C_k\}$ is nested as $C_k = \overline{N(\mathbf{x}_0; \frac{\epsilon_0}{k})} \supseteq \overline{N\left(\mathbf{x}_0; \frac{\epsilon_0}{k+1}\right)} = C_{k+1}$.

(ii) C_k is nonempty as $\mathbf{x}_0 \in C_k$.

(iii) C_k is closed and bounded as it is a closure of $N(\mathbf{x}_0; \epsilon_0/k)$.

(iv) C_k is contained in S as $C_k \subseteq C_1 \subset N(\mathbf{x}_0; 2\epsilon_0) \subseteq S$.

We claim that $\bigcap_{k=1}^{\infty} C_k = \{\mathbf{x}_0\}$. By (ii), $\{\mathbf{x}_0\} \subseteq \bigcap_{k=1}^{\infty} C_k$. Suppose, to the contrary, that there exists some $\mathbf{x}' \in \bigcap_{k=1}^{\infty} C_k$ such that $\epsilon = \|\mathbf{x}_0 - \mathbf{x}'\| > 0$. By Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that $\epsilon_0/k_0 < \epsilon$. Then, for all $k \geq k_0$, $\mathbf{x}' \notin C_k$. Hence, $\bigcap_{k=1}^{\infty} C_k \subseteq \{\mathbf{x}_0\}$. Therefore, $\bigcap_{k=1}^{\infty} C_k = \{\mathbf{x}_0\}$. ∎

2.51 Let $\mathbf{x}_0$ be a limit point of a set S in $\mathbb{R}^n$. Exhibit a nested sequence $\{C_k\}$ of nonempty, closed, bounded sets, each of which contains points of S, whose intersection is $\{\mathbf{x}_0\}$.

Solution. Since $\mathbf{x}_0 \in S'$, for each $\epsilon > 0$, $N'(x_0;\epsilon) \cap S \neq \varnothing$. Consider $\{C_k\}$ with $C_k = \overline{N(\mathbf{x}_0; 1/k)}$. For each $k \in \mathbb{N}$,

(i) $\{C_k\}$ is nested as $C_k = \overline{N(\mathbf{x}_0; \frac{1}{k})} \supseteq \overline{N\left(\mathbf{x}_0; \frac{1}{k+1}\right)} = C_{k+1}$.

(ii) C_k is nonempty as $\mathbf{x}_0 \in C_k$.

(iii) C_k is closed and bounded as it is a closure of $N(\mathbf{x}_0; 1/k)$.

(iv) C_k contains points of S as $C_k \cap S \neq \varnothing$.

We claim that $\bigcap_{k=1}^{\infty} C_k = \{\mathbf{x}_0\}$. By (ii), $\{\mathbf{x}_0\} \subseteq \bigcap_{k=1}^{\infty} C_k$. Suppose, to the contrary, that there exists some $\mathbf{x}' \in \bigcap_{k=1}^{\infty} C_k$ such that $\epsilon = \|\mathbf{x}_0 - \mathbf{x}'\| > 0$. By Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that $1/k_0 < \epsilon$. Then, for all $k \geq k_0$, $\mathbf{x}' \notin C_k$. Hence, $\bigcap_{k=1}^{\infty} C_k \subseteq \{\mathbf{x}_0\}$. Therefore, $\bigcap_{k=1}^{\infty} C_k = \{\mathbf{x}_0\}$. ∎

2.53 Assume Theorem 2.3.1 and its corollary as an axiom. Also assume Archimedes' principle. Let S be a nonempty set in $\mathbb{R}$ that is bounded above. We prove here that $\sup S$ exists in $\mathbb{R}$.

a) Let B denote the set of all upper bounds for S and let $A = B^c$. Show that $A \neq \varnothing$. Also show that if a is in A and if b is in B, then $a < b$ and $S \cap [a,b] \neq \varnothing$.

b) Use the bisection method to construct a nested sequence $\{[a_k, b_k]\}$ of closed, bounded intervals such that a_k is in A, b_k is in B, and $b_k - a_k = (b_1 - a_1)/2^{k-1}$ for all k in $\mathbb{N}$.

c) Show that $\bigcap_{k=1}^{\infty} [a_k, b_k]$ consists of a single point x_0.

d) Show that x_0 is an upper bound for S.

e) Show that $x_0 \leq b$, for all b in B. Thus $x_0 = \sup S$.

Solution.

a) $\varnothing \neq S \subset \mathbb{R}$, so there must be an element of S, say x. For every $\epsilon > 0$, clearly $x - \epsilon$ is not an upper bound for S. $x - \epsilon \notin B$; therefore, $x - \epsilon \in B^c = A$. Hence $A \neq \varnothing$.

Suppose $a \geq b$ for $a \in A$ and $b \in B$. For all $x \in S$, $x \leq b \leq a$, so a is an upper bound for S. Then we have a contradiction, $a \in B = A^c$. Therefore, $a < b$.

Since a is not an upper bound for S, there exists an $x \in S$ such that $x > a$. But b is an upper bound for S, $x \leq b$. $x \in S \cap [a,b]$; therefore, $S \cap [a,b] \neq \varnothing$.

b) Choose any $a_1 \in A$ and $b_1 \in B$. Bisect the interval $I_1 = [a_1, b_1]$ at the midpoint $c_1 = (a_1 + b_1)/2$. Either $S \cap [a_1, c_1] \neq \varnothing$ or $S \cap [c_1, b_1] \neq \varnothing$. Let $I_2 = [a_2, b_2]$ denote whichever of these two intervals contains at least one point of S. If both subintervals do so, choose the rightmost interval. Note that $I_2 \subset I_1$ and that the length of I_2 is $b_2 - a_2 = (b_1 - a_1)/2$.

Continue this procedure, repeatedly bisecting each interval $I_k = [a_k, b_k]$ and choosing $I_{k+1} = [a_{k+1}, b_{k+1}]$ to be the half of $[a_k, b_k]$ that contains at least one point of S, or, if both do so, choose the rightmost half. In this way we obtain a sequence $\{I_k\} = \{[a_k, b_k]\}$ of closed intervals with the following properties: For each k in $\mathbb{N}$,

i) $S \cap I_k \neq \varnothing$.

ii) $I_{k+1} \subset I_k$.

iii) The length of I_k is $b_k - a_k = (b_1 - a_1)/2^{k-1}$.

The first two properties are self-evident from our construction of the sequence of intervals $\{I_k\}$. The inductive proof of (iii) is straightforward.

c) For each k in $\mathbb{N}$, $\{[a_k, b_k]\}$ is a nested sequence in $\mathbb{R}$ with $a_k < b_k$, and we have in addition that

$$\lim_{k\to\infty}(b_k - a_k) = \lim_{k\to\infty}\frac{b_1 - a_1}{2^{k-1}} = 2(b_1 - a_1)\lim_{k\to\infty}\frac{1}{2^k} = 0.$$

To prove that $\lim_{k\to\infty} 1/2^k = 0$ rigorously, fix $\epsilon > 0$. By Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that $1 < \epsilon k_0$. It follows from simple induction that $k < 2^k$. Then, for $k \geq k_0$,

$$\left|\frac{1}{2^k} - 0\right| = \frac{1}{2^k} < \frac{1}{k} \leq \frac{1}{k_0} < \epsilon.$$

By Corollary 2.3.2, $\bigcap_{k=1}^{\infty}[a_k, b_k] = \{x_0\}$.

d) Suppose, to the contrary, $x_0 \notin B$. Then there exists $x \in S$ such that $x > x_0$. By Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that

$$\frac{b_1 - a_1}{2^{k_0 - 1}} = b_{k_0} - a_{k_0} < x - x_0$$

Since $b_{k_0} \in B$, $x \leq b_{k_0}$. However, we have a contradiction that

$$a_{k_0} \leq x_0 < x \leq b_{k_0} \implies b_{k_0} - a_{k_0} \geq x - x_0.$$

Therefore, such an element $x \in S$ does not exist, and x_0 is indeed an upper bound for S.

e) Suppose, to the contrary, there exists $b_0 \in B$ such that $b_0 < x_0$. By Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that

$$\frac{b_1 - a_1}{2^{k_0 - 1}} = b_{k_0} - a_{k_0} < x_0 - b_0$$

Since $b_0 \in B$, $a_{k_0} \leq b_0$. However, we have a contradiction that

$$a_{k_0} \leq b_0 < x_0 \leq b_{k_0} \implies b_{k_0} - a_{k_0} \geq x_0 - b_0.$$

Therefore, such an upper bound $b_0 \in B$ does not exist. Thus $x_0 = \sup S$.

■

2.4 Relative Topology and Connectedness

2.55 Let $\mathbf{x}_0$ be any point in $\mathbb{R}^n$. Let $\{\epsilon_k\}$ be any sequence of positive numbers that converge monotonically to 0. For each k in $\mathbb{N}$, let $C_k = \overline{N(\mathbf{x}_0; \epsilon_k)}$.

a) Prove that $\{C_k\}$ is a nested sequence of nonempty, bounded, closed subsets of $\mathbb{R}^n$.

b) Prove that $\bigcap_{k=1}^{\infty} C_k = \{\mathbf{x}_0\}$.

Solution.

a) For each $k \in \mathbb{N}$,

(i) $\{C_k\}$ is nested as $C_k = \overline{N(\mathbf{x}_0; \epsilon_k)} \supseteq \overline{N(\mathbf{x}_0; \epsilon_{k+1})} = C_{k+1}$.

(ii) C_k is nonempty as $\mathbf{x}_0 \in C_k$.

(iii) C_k is closed and bounded as it is a closure of $N(\mathbf{x}_0; \epsilon_k)$.

b) We claim that $\bigcap_{k=1}^{\infty} C_k = \{\mathbf{x}_0\}$. By (ii), $\{\mathbf{x}_0\} \subseteq \bigcap_{k=1}^{\infty} C_k$. Suppose, to the contrary, that there exists some $\mathbf{x}' \in \bigcap_{k=1}^{\infty} C_k$ such that $\epsilon = \|\mathbf{x}_0 - \mathbf{x}'\| > 0$. Since $\{\epsilon_k\}$ converges monotonically to 0, there exists a $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, $\epsilon_{k_0} < \epsilon$, so $\mathbf{x}' \notin C_k$. Hence, $\bigcap_{k=1}^{\infty} C_k \subseteq \{\mathbf{x}_0\}$. Therefore, $\bigcap_{k=1}^{\infty} C_k = \{\mathbf{x}_0\}$.

■

2.58 Let $X = [-1, 0) \cup (0, 1)$.

a) Show that both $[-1, 0)$ and $(0, 1)$ are relatively open in X. Show that both are also relatively closed in X.

b) Let $S = (-1/2, 1/2) \cap X$. Is S relatively open in X? Is S relatively closed in X? Prove your answers.

c) For each k in $\mathbb{N}$, let $x_k = (-1)^k[1 - 1/(k+1)]$. Let $S = \{x_k : k \text{ in } \mathbb{N}\}$. Show that S has exactly one relative limit point x_0 in X. What is it?

d) Thus the sequence $\{x_k\}$ defined in part (c) is a bounded sequence having exactly one cluster point x_0 in X. Does this sequence converge to x_0? If so, prove it; otherwise, explain why not.

e) Exhibit a Cauchy sequence in X that fails to converge in X.

f) Show that the sets $C_k = [-1/k, 1/k] \cap X$ form a nested sequence of bounded, relatively closed sets with an empty intersection.

Solution.

a) By Definition 2.4.1,

- $(-2, 0)$ is open in $\mathbb{R}$, and $[-1, 0) = (-2, 0) \cap X$, so $[-1, 0)$ is relatively open in X.
- $[-1, 0]$ is closed in $\mathbb{R}$, and $[-1, 0) = [-1, 0] \cap X$, so $[-1, 0)$ is relatively closed in X.
- $(0, 1)$ is open in $\mathbb{R}$, and $(0, 1) = (0, 1) \cap X$, so $(0, 1)$ is relatively open in X.

- $[0,1]$ is closed in $\mathbb{R}$, and $(0,1) = [0,1] \cap X$, so $(0,1)$ is relatively closed in X.

b) $S \cap X = S$. $(-1/2, 1/2)$ is open but not closed in $\mathbb{R}$. By Definition 2.4.1, S is relatively open but not relatively closed in X.

c) Fix any $\epsilon > 0$. By Archimedes' principle, choose $k \in \mathbb{N}$ such that $1/(2k+1) < \epsilon$. Then we have $1 - \epsilon < (-1)^{2k}[1 - 1/(2k+1)] < 1$, so 1 is a limit point of S. By that principle, choose $k \in \mathbb{N}$ such that $1/(2k) < \epsilon$. Then we have $-1 < (-1)^{2k-1}[1 - 1/(2k)] < -1 + \epsilon$. Hence, 1 and -1 are the two limit points of S in $\mathbb{R}$. Only -1 is in X. Therefore, $x_0 = -1$.

d) $\{x_k\}$ in X diverges in $\mathbb{R}$. Therefore, it does not converge to x_0.

e) Consider $\{x_k\}$ with $x_k = -1/k$. It is in X as $x_k \in [-1, 0) \subset X$ for all $k \in \mathbb{N}$, and it is Cauchy as $\lim_{k\to\infty} x_k = 0$. However, $0 \notin X$, so it fails to converge in X.

f) For each $k \in \mathbb{N}$,

i) $\{C_k\}$ is nested as $C_k = [-1/k, 1/k] \cap X \supseteq [-1/(k+1), 1/(k+1)] \cap X = C_{k+1}$.

ii) C_k is bounded and relatively closed because $C_k \cap X = C_k$, and $[-1/k, 1/k]$ is compact.

iii) By Exercise 2.55, $\bigcap_{k=1}^{\infty}[-1/k, 1/k] = \{0\}$. However, $0 \notin X$, so $\bigcap_{k=1}^{\infty} C_k = \varnothing$.

■

2.61 Prove that $\mathbb{R}$ itself is connected.

Solution. Suppose, to the contrary, that $\mathbb{R}$ is disconnected. Then, by Definition 2.4.4, there exist two nonempty, disjoint open sets U and V such that $\mathbb{R} \subseteq U \cup V$. It follows from $\varnothing \neq U \subseteq \mathbb{R}$ and $\varnothing \neq V \subseteq \mathbb{R}$ that $\mathbb{R} \cap U \neq \varnothing$, $\mathbb{R} \cap V \neq \varnothing$ and $U \cup V \subseteq \mathbb{R}$. We have $U \cup V = \mathbb{R}$ and $U \cap V = \varnothing$, so $U = V^c$. Since V is open, by Theorem 2.2.2, U is closed. Likewise, V is also clopen. Note that $\varnothing \neq U \subsetneq \mathbb{R}$ and $\varnothing \neq V \subsetneq \mathbb{R}$.

Let U be the one such that $a \in U$ and $b \in U^c = V$ for some real numbers a and b with $a < b$. Let $A = \{x \in [a,b] : [a,x] \subseteq X\}$. A is nonempty and bounded above, so $c = \sup A$ exists in $\mathbb{R}$ by Axiom 1.1.1. Then $c \in \operatorname{bd}(U) = \operatorname{bd}(U^c)$. Since U and $U^c = V$ are closed, we have both $c \in U$ and $c \in U^c = V$. That is a contradiction. Therefore, $\mathbb{R}$ is connected. ■

2.5 Compactness

2.68 If possible, give an example of each of the following. If no example exists, explain why.

a) A closed subset S of $\mathbb{R}^n$ that is not compact.

b) A compact subset S of $\mathbb{R}^n$ that is not closed.

c) An open subset S of $\mathbb{R}^n$ that is not compact.

d) A compact subset S of $\mathbb{R}^n$ that is not open.

e) Two compact subsets C_1 and C_2 of $\mathbb{R}^n$ such that $C_1 \cup C_2$ is not compact.

f) Two compact subsets C_1 and C_2 of $\mathbb{R}^n$ such that $C_1 \cap C_2$ is not compact.

Solution.

a) $\mathbb{R}^n$ is an unbounded, closed subset of itself. By Theorem 2.5.4, $\mathbb{R}^n$ is not compact.

b) By Theorem 2.5.4, if S is a compact subset of $\mathbb{R}^n$, then it must be closed.

c) $(0,1)$ is open but not closed. By Theorem 2.5.4, $(0,1)$ is not compact.

d) $[0,1]$ is closed and bounded. It is compact by Theorem 2.5.3 but is not open.

e) By Theorem 2.5.4, both C_1 and C_2 are closed and bounded. By Theorem 2.2.3, $C_1 \cup C_2$ is closed. By Example 2 in Section 2.1, $C_1 \cup C_2$ is bounded. Therefore, by Theorem 2.5.3, $C_1 \cup C_2$ is compact.

f) By Theorem 2.5.4, both C_1 and C_2 are closed and bounded. By Theorem 2.2.3, $C_1 \cap C_2$ is closed. $C_1 \cap C_2 \subseteq C_1$, so it is bounded. Therefore, by Theorem 2.5.3, $C_1 \cap C_2$ is compact.

■

2.69 Prove that every closed subset of a compact set in $\mathbb{R}^n$ is compact.

Solution. By Theorem 2.5.4, a compact set in $\mathbb{R}^n$ is bounded. Every closed subset of a compact set is bounded, so it is compact by Theorem 2.5.3. ■

2.72 Let C_1 and C_2 be two disjoint compact subsets of $\mathbb{R}^n$. Prove that there exist disjoint open sets U_1 and U_2 such that $C_1 \subseteq U_1$ and $C_2 \subseteq U_2$.

Solution. By Theorem 2.5.4, both C_1 and C_2 are closed and bounded. By Theorem 2.2.10, the *distance* between C_1 and C_2, $\epsilon = \inf\{\|\mathbf{x}-\mathbf{y}\| : \mathbf{x} \in C_1, \mathbf{y} \in C_2\}$, is positive. Let $U_1 = \bigcup_{\mathbf{x}\in C_1} N(\mathbf{x};\epsilon/2)$ and $U_2 = \bigcup_{\mathbf{x}\in C_2} N(\mathbf{x};\epsilon/2)$. By Theorem 2.2.1, both U_1 and U_2 are open. It remains to show that $U_1 \cap U_2 = \varnothing$. Suppose, to the contrary, that there exists some $\mathbf{z} \in U_1 \cap U_2$ such that $\mathbf{z} \in N(\mathbf{x};\epsilon/2)$ for some $\mathbf{x} \in C_1$ and $\mathbf{z} \in N(\mathbf{y};\epsilon/2)$ for some $\mathbf{y} \in C_2$. By the triangle inequality,

$$\epsilon \le \|\mathbf{x}-\mathbf{y}\| \le \|\mathbf{x}-\mathbf{z}\| + \|\mathbf{z}-\mathbf{y}\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

That is a contradiction. Therefore, $U_1 \cap U_2 = \varnothing$. ■

2.73 Complete the proof of Theorem 2.5.5 (that is, show that (i) and (iii) of that theorem are equivalent) by completing the following steps.

a) Let S be a nonempty, closed, bounded set in $\mathbb{R}^n$ and let T be any infinite subset of S. Use the Bolzano–Weierstrass theorem to show that T must have a limit point $\mathbf{x}_0$ and that $\mathbf{x}_0$ must be a point in S.

b) Let S be a nonempty subset of $\mathbb{R}^n$ with the property that every infinite subset T of S has a limit point of S.

i) Assume that S is unbounded and construct an infinite subset T of S that cannot have a limit point. Thus show that S must be bounded.

ii) Suppose that $\mathbf{x}_0$ is a limit point of S. Use Exercise 2.17 to find an infinite subset T of S having $\mathbf{x}_0$ as its only limit point. Thus show that $\mathbf{x}_0$ must already be in S and that S must be closed.

Solution.

a) S is bounded, so is its subset T. By the Bolzano–Weierstrass theorem, T has a limit point $\mathbf{x}_0$. Since $T \subseteq S$, $\mathbf{x}_0$ is also a limit point of S. By Theorem 2.2.4, $\mathbf{x}_0 \in S$.

b) **i)** Assume that S is unbounded. Choose $\mathbf{x}_1 \in S$. For each $k \geq 2$, there must exist some $\mathbf{x}_k \in S$ such that $\|\mathbf{x}_k\| > \|\mathbf{x}_{k-1}\| + 1$; otherwise, S is bounded by $\|\mathbf{x}_{k-1}\| + 1$. Let $T = \{\mathbf{x}_k : k \in \mathbb{N}\}$. For any deleted neighborhood $N'(\mathbf{x})$ of a point $\mathbf{x} \in \mathbb{R}^n$, $N'(\mathbf{x}) \cap T$ is finite. By Exercise 2.8, T cannot have a limit point. Therefore, S must be bounded.

ii) Suppose that $\mathbf{x}_0$ is a limit point of S. By Exercise 2.17, there exists a Cauchy sequence of distinct points of S that converges to $\mathbf{x}_0$. Let $T = \{\mathbf{x}_k : k \in \mathbb{N}\} \cup \{\mathbf{x}_0\}$. By Exercise 2.25, $\mathbf{x}_0$ is the only limit point of T. Since $T \subseteq S$, $\mathbf{x}_0 \in S$. By Theorem 2.2.4, S must be closed.

This completes the proof.

■

2.74 **Definition.** A set S is said to be *locally compact* if, for every $\mathbf{x}$ in S, there is an open set U containing $\mathbf{x}$ such that $\overline{U}$ is compact. Prove that $\mathbb{R}^n$ is locally compact.

Solution. For any neighborhood $N(\mathbf{x})$ of a point $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \in N(\mathbf{x})$ and $\overline{N(\mathbf{x})}$ is compact. Therefore, $\mathbb{R}^n$ is locally compact. ■

2.75 In 1908, Frigyes Riesz, one of the great mathematicians of the fist half of the twentieth century, proposed the following approach to compactness.

Definition. A collection $\mathcal{C} = \{S_\alpha : \alpha \text{ in } A\}$ has the finite intersection property if every finite subcollection of $\mathcal{C}$ has a nonempty intersection.

Theorem. F. Riesz. A set S in $\mathbb{R}^n$ is compact if and only if every collection $\mathcal{C} = \{S_\alpha : S_\alpha \text{ closed}, S_\alpha \subseteq S\}$ of closed subsets of S with the *finite intersection property* has a nonempty intersection: $\cap_\alpha S_\alpha \neq \varnothing$.

Prove Riesz's theorem.

Solution. Suppose that S is compact and that there is a collection $\mathcal{C}$ of closed subsets of S with the finite intersection property satisfying $\bigcap_{\alpha \in A} S_\alpha = \varnothing$. Let $\mathcal{U} = \{S_\alpha^c : S_\alpha \in \mathcal{C}\}$. Clearly $\mathcal{U}$ is a collection of open sets. By De Morgan's law,

$$\bigcup_{\alpha \in A} S_\alpha^c = \left(\bigcap_{\alpha \in A} S_\alpha \right)^c = \varnothing^c = \mathbb{R}^n,$$

so $\mathcal{U}$ is an open cover of S. Since S is compact, there exists a finite subcover $\{U_i \in \mathcal{U}\}_{i=1}^n$ of $\mathcal{U}$. It follows from the finite intersection property of $\mathcal{C}$ that

$$S \subseteq \bigcup_{i=1}^n U_i^c = \left(\bigcap_{i=1}^n S_i \right)^c \iff \varnothing \neq \bigcap_{i=1}^n S_i \subseteq S^c$$

which is a contradiction. Therefore, if S is compact, every collection of closed subsets S_α of S with the finite intersection property must have a nonempty intersection: $\cap_\alpha S_\alpha \neq \varnothing$. ■

Chapter 3

Continuity

3.1 Limit and Continuity

3.4 Prove parts (iv) and (v) of Theorem 3.1.3.

Solution. Part (v) of Theorem 3.1.3 follows from parts (iii) and (iv), so it suffices to prove part (iv). Suppose that f is a real-valued function defined on $S \subseteq \mathbb{R}^n$, that $\mathbf{c} \in S'$, and that $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L \neq 0$. Let $\epsilon > 0$ be given. Choose $\delta_1 > 0$ such that, for $\mathbf{x} \in S \cap N'(\mathbf{c}; \delta_1)$,

$$\begin{gathered}
f(\mathbf{x}) \in N(L; |L|/2) \\
||f(\mathbf{x})| - |L|| \leq |f(\mathbf{x}) - L| < \frac{|L|}{2} \\
-\frac{|L|}{2} < |f(\mathbf{x})| - |L| < \frac{|L|}{2} \\
\frac{|L|}{2} < |f(\mathbf{x})| < \frac{3|L|}{2} \\
|L| < 2|f(\mathbf{x})| < 3|L| \\
\frac{1}{|f(\mathbf{x})|} < \frac{2}{|L|} < \frac{3}{|f(\mathbf{x})|}.
\end{gathered}$$

Choose $\delta_2 > 0$ such that, for $\mathbf{x} \in S \cap N'(\mathbf{c}; \delta_2)$, $|f(\mathbf{x}) - L| < \dfrac{\epsilon|L|^2}{2}$. Take $\delta = \min\{\delta_1, \delta_2\}$. Then, for $\mathbf{x} \in S \cap N'(\mathbf{c}; \delta)$,

$$\left|\frac{1}{f(\mathbf{x})} - \frac{1}{L}\right| = \frac{|f(\mathbf{x}) - L|}{|L||f(\mathbf{x})|} < \frac{1}{|L|} \cdot \frac{2}{|L|} \cdot \frac{\epsilon|L|^2}{2} = \epsilon.$$

Therefore, $\lim_{\mathbf{x}\to\mathbf{c}} 1/f(\mathbf{x}) = 1/L$. ■

3.12 Find each limit, if it exists; if a limit fails to exist, prove it.

a) $\displaystyle\lim_{x\to c}\frac{\sin(x-c)}{x^2-c^2}$

b) $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$

c) $\displaystyle\lim_{x\to 1}\frac{(1+x)^{1/2}-(1-x)^{1/2}}{x}$

d) $\displaystyle\lim_{x\to 0}[x+\operatorname{sgn}(x)]$, where

$$\operatorname{sgn}(x)=\begin{cases}-1, & \text{if } x<0\\ 0, & \text{if } x=0\\ 1, & \text{if } x>0.\end{cases}$$

Solution. The limit laws listed in Theorem 3.1.3 are keys to the solution.

a) $\displaystyle\lim_{x\to c}\frac{\sin(x-c)}{x-c}=\lim_{h\to 0}\frac{\sin h}{h}=1$. If $c\neq 0$, $\displaystyle\lim_{x\to c}\frac{1}{x+c}=\frac{1}{2c}$.

$$\lim_{x\to c}\frac{\sin(x-c)}{x^2-c^2}=\lim_{x\to c}\left[\frac{\sin(x-c)}{x-c}\cdot\frac{1}{x+c}\right]=\lim_{x\to c}\frac{\sin(x-c)}{x-c}\lim_{x\to c}\frac{1}{x+c}=1\cdot\frac{1}{2c}=\frac{1}{2c}$$

If $c=0$, $\displaystyle\lim_{x\to c}\frac{\sin(x-c)}{x^2-c^2}=\lim_{x\to 0}\frac{\sin x}{x^2}$ does not exist. Suppose, to the contrary, that there exists some $L\in\mathbb{R}$ such that $\displaystyle\lim_{x\to 0}\frac{\sin x}{x^2}=L$. However, it leads to a contradiction,

$$1=\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\left(\frac{\sin x}{x^2}\cdot x\right)=\lim_{x\to 0}\frac{\sin x}{x^2}\lim_{x\to 0}x=L\cdot 0=0.$$

Therefore, such a limit L fails to exist.

b) It follows from Example 9 in Section 3.1 that

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x^2}&=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}\\&=\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}\right)\\&=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\lim_{x\to 0}\frac{1}{1+\cos x}\\&=1^2\cdot\frac{1}{1+1}=\frac{1}{2}.\end{aligned}$$

c) This is an arithmetic combination of limits of functions that are continuous at 1.

$$\lim_{x\to 1}\frac{(1+x)^{1/2}-(1-x)^{1/2}}{x}=\frac{\sqrt{1+1}-\sqrt{1-1}}{1}=\sqrt{2}$$

d) The origin 0 is clearly an interior point of $\mathbb{R}$, the domain of the sign function. Both the left-hand and the right-hand limit of $\operatorname{sgn}(x)$ at $x=0$ exist but are different as

$$-1=\lim_{x\to 0^-}\operatorname{sgn}(x)\neq\lim_{x\to 0^+}\operatorname{sgn}(x)=1.$$

$\lim_{x\to 0} \operatorname{sgn}(x)$ does not exist, and neither does $\lim_{x\to 0}[x+\operatorname{sgn}(x)]$. For the sake of contradiction, suppose that there exists some $L \in \mathbb{R}$ such that $\lim_{x\to 0}[x+\operatorname{sgn}(x)] = L$. Then we have

$$\lim_{x\to 0} \operatorname{sgn}(x) = \lim_{x\to 0}[x+\operatorname{sgn}(x)-x] = \lim_{x\to 0}[x+\operatorname{sgn}(x)] - \lim_{x\to 0} x = L - 0 = L.$$

Therefore, such a limit L fails to exist.

■

3.13 For x in $\mathbb{R}$, define $f(x) = 0$ if x is irrational and $f(x) = 1/q$ if $x = p/q$ in lowest terms. Prove that f is continuous at every irrational point and is discontinuous at every rational point.

Solution. Let an arbitrary rational $c = p/q$ with a unique pair of coprime integers $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then $f(c) = 1/q$. Pick $\epsilon = f(c)/2$. No matter how small $\delta > 0$ is taken, there is an irrational number $x \in N(c;\delta)$ which implies

$$|f(x) - f(c)| = \left|0 - \frac{1}{q}\right| = \frac{1}{q} > \frac{1}{2q} = \epsilon.$$

Therefore, f is discontinuous on $\mathbb{Q}$.

Lemma. f is periodic with period 1.

Proof. For all $x \in \mathbb{R}\setminus\mathbb{Q}$, $f(x+1) = f(x) = 0$, since $x+1 \in \mathbb{R}\setminus\mathbb{Q}$ implies $x \in \mathbb{R}\setminus\mathbb{Q}$. For all $x \in \mathbb{Q}$, there exists a unique pair of coprime integers $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $x = p/q$. Consider $x+1 = (p+q)/q$. If d divides p and q, it divides $p+q$ and p. Conversely, if d divides $p+q$ and q, it divides $(p+q)-q = q$ and p. $\gcd(p+q, q) = \gcd(p, q) = 1$, so $f(x+1) = 1/q = f(x)$. ■

Since f is periodic with period 1, and $0 \in \mathbb{Q}$, it suffices to check all irrational points in the interval $[0, 1]$. Assume an arbitrary irrational $c \in [0, 1]\setminus\mathbb{Q}$. Then $f(c) = 0$. Let $\epsilon > 0$ be given. By Archimedes' principle, choose $k \in \mathbb{N}$ such that $1/k < \epsilon$. Let

$$S = \left\{ r \in [0,\, 1] \cap \mathbb{Q} \,\middle|\, f(r) \geq \frac{1}{k} \right\}.$$

There exist only finitely many rational numbers (reduced to lowest terms) in I having denominator not greater than k, so the set S is finite. Take $\delta = \min\{|c-r| : r \in S\}$. Then,

$$|x - c| < \delta \implies |f(x) - f(c)| \leq \frac{1}{k} < \epsilon.$$

Therefore, f is continuous on $\mathbb{R}\setminus\mathbb{Q}$. ■

3.15 **a)** Show that, for each k in $\mathbb{N}$, $f_k(x) = 1/\left(1 + x^{2k}\right)$ is continuous on all of $\mathbb{R}$.

b) Define

$$f_0(x) = \begin{cases} 1, & \text{for } |x| < 1 \\ 1/2, & \text{for } |x| = 1 \\ 0, & \text{for } |x| > 1. \end{cases}$$

Sketch the graph of f_0. Identify where it is continuous and identify the type and location of discontinuities of f_0. Prove your assertions.

c) Superimpose on the graph of f_0, the graphs of f_k for $k = 1$, 2, 3, and 4.

d) Fix any x in $\mathbb{R}$ and consider the sequence $\{y_k\}$ where, for each k in $\mathbb{N}$, $y_k = f_k(x)$. Show that, whatever the choice of x, the sequence $\{y_k\}$ converges. (Consider the three case: $|x| > 1$, $|x| = 1$, and $|x| < 1$ separately.)

e) Show that for each x in $\mathbb{R}$, $\lim_{k\to\infty} f_k(x) = f_0(x)$.

Solution.

a) Let $\epsilon > 0$ be given. For each $x_0 \in [0, \infty)$,

$$\begin{aligned} |f_k(x) - f_k(x_0)| &= \left| \frac{1}{x^{2k}+1} - \frac{1}{x_0^{2k}+1} \right| = \left| \frac{x^{2k} - x_0^{2k}}{(x^{2k}+1)(x_0^{2k}+1)} \right| \\ &< \left| x^{2k} - x_0^{2k} \right| = |x - x_0| \left| \sum_{j=0}^{2k-1} x^{2k-j-1} x_0^j \right| \\ &< |x - x_0| \sum_{j=0}^{2k-1} |x|^{2k-j-1} |x_0|^j. \end{aligned}$$

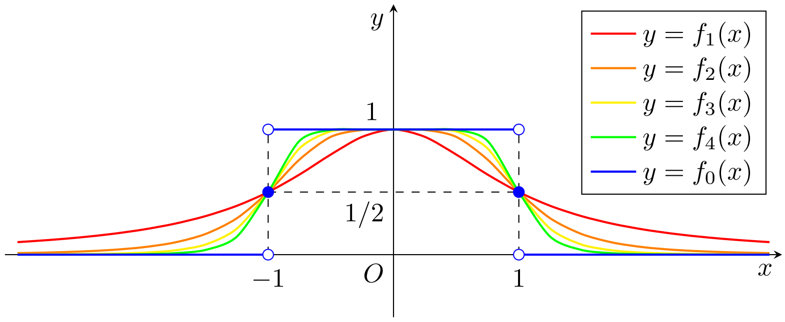
We have $|x - x_0| < 1 \implies |x| < 1 + |x_0|$. Let $M = \sum_{j=0}^{2k-1} (1+|x_0|)^{2k-j-1}|x_0|^j$. If we take $\delta = \min\{1, \epsilon/M\}$, then $|x - x_0| < \delta$ implies

$$|f_k(x) - f_k(x_0)| < |x - x_0| \sum_{j=0}^{2k-1} |x|^{2k-j-1}|x_0|^j < \delta M \le \epsilon.$$

Therefore, f_k is continuous on $[0, \infty)$. $f_k(x) = f_k(-x)$, so f_k is continuous on $(-\infty, \infty)$.

b) See part (c). f_0 has jump discontinuities at ± 1 as $|f(\pm 1^+) - f(\pm 1^-)| = 1$. f_0 is continuous elsewhere, that is, on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

c) See the figure on the next page.



d) **i)** $|x| > 1 \implies y_k > y_{k+1}$. $\{y_k\}$ is monotone decreasing and bonded below by 0. By Theorem 1.3.7, $\{y_k\}$ converges.

ii) $|x| = 1 \implies y_k = 1/2$ for all $k \in \mathbb{N}$.

iii) $|x| < 1 \implies y_k < y_{k+1}$. $\{y_k\}$ is monotone increasing and bonded above by 1. By Theorem 1.3.7, $\{y_k\}$ converges.

e) Let $\epsilon > 0$ be given. Since $0 < f_k(x) \le 1$ for all $k \in \mathbb{N}$, we may assume that $0 < \epsilon < 1$.

i) If $|x| = 1$, by part (d), $\lim_{k\to\infty} y_k = f_0(x) = 1/2$.

ii) If $|x| > 1$, by Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that $\left(x^2\right)^{k_0} > 1/\epsilon - 1$. Then, for $k \ge k_0$, $y_k < \epsilon$. Therefore, $\lim_{k\to\infty} y_k = f_0(x)$.

iii) If $|x| < 1$, by Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that $\left(x^2\right)^{k_0} < 1/(1-\epsilon) - 1$. Then, for $k \ge k_0$, $1 - y_k < \epsilon$. Therefore, $\lim_{k\to\infty} y_k = f_0(x)$.

■

3.16 For each $k = 0, 1, 2, \cdots$ and for x in $[-1, 1]$, define $f_k(x) = \cos\left(k\cos^{-1}x\right)$.

a) Show that $|f_k(x)| \le 1$ for all x in $[-1, 1]$, all $k \ge 0$.

b) Let $\theta = \cos^{-1}x$ and use trigonometric identities for $\cos(k+1)\theta$ and $\cos(k-1)\theta$ to show that, for $k \ge 1$,

$$f_{k+1}(x) = 2xf_k(x) - f_{k-1}(x).$$

c) Calculate $f_0(x)$, $f_1(x)$, $f_2(x)$ and $f_3(x)$ as polynomials in x.

d) By induction show that f_k is a polynomial of degree k, with leading coefficient 2^{k-1}. Hence, each f_k is continuous on $[-1, 1]$.

e) Prove that $f_k(1) = 1$ and $f_k(-1) = (-1)^k$ for all $k \ge 0$.

f) Prove that f_k has k roots in the interval $(-1, 1)$. Find them.

g) Sketch the graphs of f_k for $k \in \{0, 1, 2, \cdots, 6\}$. The polynomials f_k are variants of *Chebyshev's polynomials*.

Solution.

a) $x \in [-1, 1] \implies \cos^{-1}x \in [0, \pi] \implies \cos([0, k\pi]) \subseteq [-1, 1]$ for all $k \ge 0$.

b) By trigonometric identities,

$$\begin{aligned}
\cos(k+1)\theta &= \cos k\theta \cos\theta - \sin k\theta \sin\theta \\
\cos(k-1)\theta &= \cos k\theta \cos\theta + \sin k\theta \sin\theta \\
\cos(k+1)\theta &= 2\cos k\theta \cos\theta - \cos(k-1)\theta \\
\therefore f_{k+1}(x) &= 2x f_k(x) - f_{k-1}(x).
\end{aligned}$$

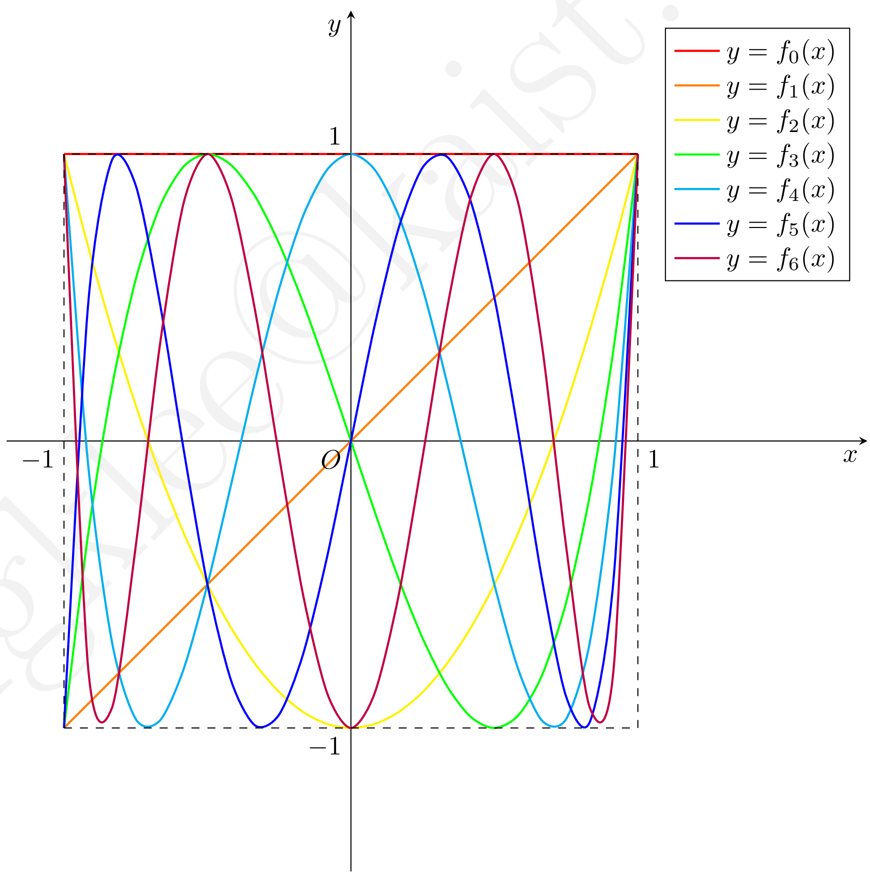
c) $f_0(x) = 1,\ f_1(x) = x,\ f_2(x) = 2x^2 - 1$, and $f_3(x) = 4x^3 - 3x$.

d) It follows from trivial induction.

e) For all $k \geq 0$, $f_k(1) = \cos(k \cdot 0) = 1$ and $f_k(-1) = \cos(k\pi) = (-1)^k$.

f) $f_k(x) = 0 \iff k\cos^{-1} x = m\pi + \pi/2$ where $0 \leq m < k$. Therefore, $\cos\dfrac{\pi}{2k}, \cos\dfrac{3\pi}{2k}, \cdots, \cos\dfrac{(2k-1)\pi}{2k}$ are the roots in $(-1, 1)$.

g) See the figure on the next page.



■

3.17 Fix any $m > 0$. Define f on $[0, 1]$ to be $f(x) = mx$ if x is rational and $f(x) = m(1 - x)$ otherwise. Prove that f is continuous only at $x = 1/2$.

Solution. Let $\epsilon > 0$ be given. Take $\delta = \epsilon/m$. Then $x \in N(1/2;\delta) \cap [0,1]$ implies

$$|f(x) - f(1/2)| = \left\{\begin{array}{ll} |mx - m/2| & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ |m - mx - m/2| & \text{otherwise} \end{array}\right\} = m|x - 1/2| < m\delta = \epsilon.$$

Therefore, f is continuous at $x = 1/2$.

Let $1/2 \neq c \in [0,1] \cap \mathbb{Q}$. Construct a Cauchy sequence $\{x_k\}$ in $[0,1] \setminus \mathbb{Q}$ that converges to c. Then

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} m(1 - x_k) = m(1-c) \neq mc = f(c).$$

Let $c \in [0,1] \setminus \mathbb{Q}$. Construct a Cauchy sequence $\{x_k\}$ in $[0,1] \cap \mathbb{Q}$ that converges to c. Then

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} mx_k = mc \neq m(1-c) = f(c).$$

By Theorem 3.1.6, f is discontinuous on $[0,1/2) \cup (1/2,1]$. ■

3.18 Let f be continuous on $[a,b]$. Define a function g as follows: $g(a) = f(a)$ and, for x in $(a,b]$,

$$g(x) = \sup\{f(y) : y \text{ in } [a,x]\}.$$

Prove that g is monotone increasing and continuous on $[a,b]$.

Solution. If $a \le x_1 \le x_2 \le b$, then $[a,x_1] \subseteq [a,x_2]$ implies $f([a,x_1]) \subseteq f([a,x_2])$. By Exercise 1.12, $g(x_1) \le g(x_2)$. Therefore, g is monotone increasing on $[a,b]$.

Let $c \in [a,b]$. Note that $f(c) \le g(c)$. Since f is continuous at c, for any $\epsilon > 0$, there exists some $\delta > 0$ such that

$$x \in N(c;\delta) \cap [a,b] \implies f(c) - \epsilon < f(x) < f(c) + \epsilon.$$

(i) Assume that $f(c) < g(c)$. Let $\epsilon = g(c) - f(c)$. Note that $f(c) + \epsilon = g(c)$.

If $x \in (c-\delta,c) \cap [a,b]$, then $g(x) \le g(c)$ by monotonicity of g. Since $f(c) < g(c)$, $g(x)$ is an upper bound for $f([a,c])$, so $g(x) \ge g(c)$.

If $x \in (c,c+\delta) \cap [a,b]$, then $g(c) \le g(x)$ by monotonicity of g. Since $f(x) < g(c)$, $g(c)$ is an upper bound for $f([a,x])$, so $g(c) \ge g(x)$.

Hence, $g(x) = g(c)$ on $N(c;\delta) \cap [a,b]$. g being locally constant is continuous at c.

(ii) Assume that $f(c) = g(c)$.

If $x \in (c-\delta,c) \cap [a,b]$, then

$$g(c) - \epsilon = f(c) - \epsilon < f(x) \le g(x) \le g(c).$$

If $x \in (c,c+\delta) \cap [a,b]$, then

$$g(c) \le g(x) = \sup f([c,x]) \le f(c) + \epsilon = g(c) + \epsilon.$$

The first equality is because $g(x) = \max\{g(c), \sup f([c,x])\}$ and $g(c) = f(c) \le \sup f([c,x])$.

Hence, $x \in N(c;\delta) \cap [a,b] \implies |g(x) - g(c)| < 2\epsilon$, so g is continuous at c.

Therefore, g is continuous on $[a, b]$. ■

3.20 Suppose that f is a real-valued function defined on all of $\mathbb{R}$ and satisfying the identity

$$f(x + y) = f(x)f(y),$$

for all x, y in $\mathbb{R}$.

a) Prove that $f(x) \geq 0$ for all x in $\mathbb{R}$.

b) Prove that, if f has value 0 at even one point, then f is identically 0 on $\mathbb{R}$.

c) Prove that, if f has a nonzero value at even one point, then $f(0) = 1$.

d) Prove that, if f is continuous at $x = 0$, then f is continuous on all of $\mathbb{R}$.

e) Prove that, if f is continuous and nonzero, then $f(x) = a^x$, where $a = f(1)$.

Solution.

a) $f(x) = f(x/2)^2 \geq 0$.

b) Assume that $f(a) = 0$ for some $a \in \mathbb{R}$. Then $f(x) = f(x - a)f(a) = 0$ for all $x \in \mathbb{R}$.

c) Assume that $f(a) > 0$ for some $a \in \mathbb{R}$. Then $f(a) = f(a)f(0) \iff f(a)[f(0) - 1] = 0$. Therefore, $f(0) = 1$.

d) $f(0) = f(0)^2 \iff f(0)[f(0) - 1] = 0$, so $f(0) = 0$ or $f(0) = 1$. If $f(0) = 0$, then f is continuous by part (b). Assume that $\lim_{x\to 0} f(x) = f(0) = 1$. Then

$$\lim_{x\to c} f(x) = \lim_{h\to 0} f(c + h) = f(c) \lim_{h\to 0} f(h) = f(c)f(0) = f(c)$$

for all $c \in \mathbb{R}$. Therefore, f is continuous on $\mathbb{R}$.

e) (i) f is nonzero, so $f(0) = 1 = a^0$ by part (c).

(ii) For $x \in \mathbb{Q} \cap (0, \infty)$, let $x = p/q$ with $p \in \mathbb{N}$ and $q \in \mathbb{N}$ in lowest terms. Note that

$$a = f(1) = f\Bigg(\underbrace{\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}}_{q \text{ times}}\Bigg) = f\left(\frac{1}{q}\right)^q.$$

Then we have

$$f(x) = f\left(\frac{p}{q}\right) = f\Bigg(\underbrace{\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}}_{p \text{ times}}\Bigg) = f\left(\frac{1}{q}\right)^p = \left[f\left(\frac{1}{q}\right)^q\right]^{\frac{p}{q}} = f(1)^x = a^x.$$

(iii) For $x \in \mathbb{Q} \cap (-\infty, 0)$, it follows from $f(0) = f(x)f(-x) = 1$ that

$$f(x) = \frac{1}{f(-x)} = \frac{1}{a^{-x}} = a^x.$$

Hence, $f(x) = a^x$ for all $x \in \mathbb{Q}$. For $x \in \mathbb{R} \setminus \mathbb{Q}$, construct a Cauchy sequence $\{x_k\}$ of rational numbers that converges to x. $f(x) = a^x$ is continuous on $\mathbb{R}$ by Exercise 3.19. Since f is continuous, by Theorem 3.1.6,

$$f(x) = \lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} a^{x_k} = a^x.$$

This completes the proof.

■

3.21 A function f on a set S in $\mathbb{R}^n$ is said to *satisfy a Lipschitz condition of order* k at a point $\mathbf{c}$ in S if there exists a constant K and neighborhood $N(\mathbf{c})$ such that, for all $\mathbf{x}$ in $N(\mathbf{c})$,

$$|f(\mathbf{x}) - f(\mathbf{c})| \le K\|\mathbf{x} - \mathbf{c}\|^k.$$

a) Prove that if f satisfies a Lipschitz condition of order $k > 0$ at some $\mathbf{c}$ in its domain, then f is continuous at $\mathbf{c}$.

b) Show that a function f on $\mathbb{R}$ satisfies a Lipschitz condition of order $k = 1$ at a point c if and only if, for x in a neighborhood of c, the graph of f lies between the lines $y = K(x-c)+f(c)$ and $y = -K(x-c) + f(c)$.

c) Find an example of a function f on $\mathbb{R}$ that satisfies a Lipschitz condition of order $k = 2$ at $c = 0$. Find an example of a function f on $\mathbb{R}$ that satisfies a Lipschitz condition of order $k = 1/2$ at $c = 1$.

Solution. Note that $K \ge 0$.

a) If $K = 0$, $f(\mathbf{x}) = f(\mathbf{c})$ for all $\mathbf{x} \in N(\mathbf{c})$. Assume $K > 0$ and fix any $\epsilon > 0$. Let $\delta = (\epsilon/K)^{1/k}$. Then $\|\mathbf{x} - \mathbf{c}\| < \delta$ implies $|f(\mathbf{x}) - f(\mathbf{c})| \le K\|\mathbf{x} - \mathbf{c}\|^k < K\delta^k = \epsilon$. Therefore, f is continuous at $\mathbf{c}$.

b) A function f on $\mathbb{R}$ satisfies a Lipschitz condition of order $k = 1$ at a point c.

$$\begin{aligned}
&\iff |f(x) - f(c)| \le K|x - c|. \\
&\iff -K|x - c| \le f(x) - f(c) \le K|x - c|. \\
&\iff \begin{cases} K(x-c) + f(c) \le f(x) \le -K(x-c) + f(c) & \text{if } x < c, \\ -K(x-c) + f(c) \le f(x) \le K(x-c) + f(c) & \text{if } x \ge c. \end{cases}
\end{aligned}$$

In either case, the graph of f, $y = f(x)$, lies between the lines $y = K(x-c) + f(c)$ and $y = -K(x-c) + f(c)$.

c) Consider $f(x) = x^2$. We have $|f(x) - f(0)| = x^2 = |x - 0|^2$. f satisfies a Lipschitz condition of order $k = 2$ at $c = 0$.

Consider $f(x) = \sqrt{|x-1|}$. We have $|f(x) - f(1)| = \sqrt{|x-1|} = |x-1|^{1/2}$. f satisfies a Lipschitz condition of order $k = 1/2$ at $c = 1$.

■

3.23 (See Example 8 of Section 2.2 and Exercise 2.47.) Let C denote Cantor's set. Any x in C has a ternary expansion of the form $x = .t_1t_2t_3\cdots$ with $t_k = 0$ or 2. Define $f(x) = \sum_{j=1}^{\infty} t_j/2^{j+1}$.

a) Show that, if x and y are in C with $x < y$, then $f(x) \leq f(y)$.

b) Show that f maps C onto $[0,1]$. (*Hint:* Every number in $[0,1]$ has a binary expansion $.b_1b_2b_3\cdots$ where $b_k = 0$ or 1.)

c) Show that, if $a(k,j)$ and $b(k,j)$ are the endpoints of any of the intervals $I(k,j)$ removed from $[0,1]$ in the construction of Cantor's set, then $f(a(k,j)) = f(b(k,j))$. (Thus, for example, $f(1/3) = f(2/3) = 1/2$, $f(1/9) = f(2/9) = 1/4$, $f(7/9) = f(8/9) = 1/2 + 1/4 = 3/4$, $\cdots$.)

d) Extend f to be defined on $C^c \cap [0,1]$ as follows: For x in $I(k,j)$, define $f(x) = f(a(k,j)) = f(b(k,j))$. Thus f is constant on each of the open intervals $I(k,j)$. With this definition, f maps all of $[0,1]$ onto $[0,1]$. Prove that f is continuous on $[0,1]$. Sketch the graph of f. The function f is called *Cantor's function* or *the devil's staircase*.

Solution.

a) If $x = .t_1t_2t_3\cdots$ and $y = .s_1s_2s_3\cdots$ are in C with $x < y$, then there exists a $k \in \mathbb{N}$ such that $k = \min\{j \in \mathbb{N} \mid 0 = t_j < s_j = 2\}$, so

$$\begin{aligned} f(y) - f(x) &= \sum_{j=1}^{\infty} \frac{s_j - t_j}{2^{j+1}} = \frac{2}{2^k} + \sum_{j=k+1}^{\infty} \frac{s_j - t_j}{2^{j+1}} \\ &\geq \frac{2}{2^{k+1}} - \sum_{j=k+1}^{\infty} \frac{2}{2^{j+1}} = \frac{2}{2^{k+1}} - \frac{2/2^{k+2}}{1 - 1/2} = 0. \end{aligned}$$

b) Let $c \in [0,1]$ have a binary expansion $c = .b_1b_2b_3\cdots = \sum_{j=1}^{\infty} b_j/2^j$ where $b_k = 0$ or 1. Let $x \in C$ have a ternary expansion $x = .t_1t_2t_3\cdots$ where $t_k = 2b_k$ for all $k \in \mathbb{N}$. Then $f(x) = c$. Therefore, f maps C onto $[0,1]$.

c) In part (a), the equality holds if $x = a(k,j)$ and $y = b(k,j)$, that is, $f(a(k,j)) = f(b(k,j))$.

d) Let $c \in C$ have a ternary expansion $c = .t_1t_2t_3\cdots$ where $t_k \in \{0,2\}$ for all $k \in \mathbb{N}$. Fix any $\epsilon > 0$. Choose $k_0 \in \mathbb{N}$ such that $1/2^{k_0-1} < \epsilon$ and take $\delta = 1/3^{k_0}$. Then, for $x \in N(c;\delta) \cap [0,1]$ with a ternary expansion $x = .s_1s_2s_3\cdots$, we have $k_0 \leq \min\{j \in \mathbb{N} \mid s_j \neq t_j\}$, so

$$\begin{aligned} |f(x) - f(c)| &= \left| \sum_{j=1}^{\infty} \frac{s_j - t_j}{2^{j+1}} \right| = \left| \sum_{j=1}^{k_0-1} \frac{s_j - t_j}{2^{j+1}} + \sum_{j=k_0}^{\infty} \frac{s_j - t_j}{2^{j+1}} \right| = \left| \sum_{j=k_0}^{\infty} \frac{s_j - t_j}{2^{j+1}} \right| \\ &\leq \sum_{j=k_0}^{\infty} \frac{|s_j - t_j|}{2^{j+1}} \leq \sum_{j=k_0}^{\infty} \frac{2}{2^{j+1}} = \frac{1/2^{k_0}}{1 - 1/2} = 1/2^{k_0-1} < \epsilon. \end{aligned}$$

Hence, f is continuous on C.

Since f is constant on each of the open intervals $I(k,j)$, f is continuous on $[0,1] \setminus C$. Therefore, f is continuous on $[0,1]$. We omit the graph.

■

3.26 For $\mathbf{x} = (x_1, x_2) \neq \mathbf{0}$ in $\mathbb{R}^2$, let

$$f(\mathbf{x}) = \frac{2x_1x_2}{x_1^2 + x_2^2}.$$

Let $f(\mathbf{0}) = 0$. Let $\mathbf{c} = \mathbf{0}$ and define $g_1(t) = f(t, 0)$ and $g_2(t) = f(0, t)$. Prove that, for $j = 1, 2$, $\lim_{t\to 0} g_j(t) = f(\mathbf{0})$. Does it follow that $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = 0$? (See Example 3.)

Solution. We have $g_1(t) = g_2(t) = 0$ for all $t \in \mathbb{R}$. Hence, $\lim\limits_{t\to 0} g_1(t) = \lim\limits_{t\to 0} g_2(t) = f(\mathbf{0})$. However, the limit $\lim\limits_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ does not exist according to Example 3 in Section 3.1. ■

3.28 Define f on the open first quadrant $(0, \infty) \times (0, \infty)$ by $f(\mathbf{x}) = x_1/x_2$, where $\mathbf{x} = (x_1, x_2)$. Examine the limiting behavior of f as $\mathbf{x}$ approaches $\mathbf{0}$ along various paths. Prove that $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ does not exist. Sketch the graph of f.

Solution. For $t > 0$ and $m > 0$, $f(t, mt) = 1/m$. In other words, f assumes every positive value on any deleted neighborhood of $\mathbf{0}$. Therefore, $\lim\limits_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ does not exist by similar argument as Example 3 in Section 3.1. We omit the graph. ■

3.31 Let f be a real-valued function defined on a rectangle $S = I_1 \times I_2$ in $\mathbb{R}^2$, where $I_j = [a_j, b_j]$. Assume that f factors into a product $f(x_1, x_2) = f_1(x_1)f_2(x_2)$ of functions f_1 on I_1 and f_2 on I_2. Assume also that f_1 is continuous at c_1 in I_1 and that f_2 is continuous at c_2 in I_2. Is f necessarily continuous at $\mathbf{c} = (c_1, c_2)$? If so, prove it; otherwise, provide a counterexample.

Solution. Let $\epsilon > 0$ be given. Since f_1 is continuous at $c_1 \in I_1$, by Theorem 3.1.1, there exists a $\delta_1 > 0$ and an $M > 0$ such that $x_1 \in N(c_1, \delta_0) \cap I_1 \implies |f_1(x_1)| \leq M$.

f_1 is continuous at $c_1 \in I_1$, so there exists a $\delta_1 > 0$ such that

$$x_1 \in N(c_1, \delta_1) \cap I_1 \implies |f_1(x_1) - f_1(c_1)| < \frac{\epsilon}{2(1 + |f_2(c_2)|)}.$$

f_2 is continuous at $c_2 \in I_2$, so there exists a $\delta_2 > 0$ such that

$$x_2 \in N(c_2, \delta_2) \cap I_2 \implies |f_2(x_2) - f_2(c_2)| < \frac{\epsilon}{2M}.$$

Then $\mathbf{x} \in N(\mathbf{c}; \min\{\delta_0, \delta_1, \delta_2\})$ implies

$$\begin{aligned}
|f(\mathbf{x}) - f(\mathbf{c})| &= |f_1(x_1)f_2(x_2) - f_1(c_1)f_2(c_2)| \\
&= |f_1(x_1)f_2(x_2) - f_1(x_1)f_2(c_2) + f_1(x_1)f_2(c_2) - f_1(c_1)f_2(c_2)| \\
&\leq |f_1(x_1)f_2(x_2) - f_1(x_1)f_2(c_2)| + |f_1(x_1)f_2(c_2) - f_1(c_1)f_2(c_2)| \\
&= |f_1(x_1)||f_2(x_2) - f_2(c_2)| + |f_1(x_1) - f_1(c_1)||f_2(c_2)| \\
&< M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2(1 + |f_2(c_2)|)} \cdot |f_2(c_2)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Therefore, f is continuous at $\mathbf{c}$. ■

3.32 Suppose that f is a real-valued function defined on a set S in $\mathbb{R}^n$ and that $\mathbf{c}$ is a limit point of S. Prove that $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$ if and only if, for every Cauchy sequence $\{\mathbf{x}_k\}$ in $S \setminus \{\mathbf{c}\}$ that converges to $\mathbf{c}$, $\lim_{k\to\infty} f(\mathbf{x}_k) = L$.

Remark. Theorem 3.1.6 needs modification. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Let $c = 0$ and let $x_k = 0$ for all $k \in \mathbb{N}$. It is clear that $\{x_k\}$ is Cauchy and that $\lim\limits_{x\to 0} f(x) = 0$. However, $\lim\limits_{k\to\infty} f(x_k) = f(0) = 1$. Therefore, the original statement may fail if a Cauchy sequence is not in $S \setminus \{\mathbf{c}\}$.

Solution.

($\Longrightarrow$) Assume that $\lim\limits_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$. Let $\epsilon > 0$ be given. There exists a $\delta > 0$ such that $S \cap N'(\mathbf{c};\delta) \subseteq f^{-1}(N(L;\epsilon))$. Let $\{\mathbf{x}_k\}$ be a Cauchy sequence in $S \setminus \{\mathbf{c}\}$ that converges to $\mathbf{c}$. Then there exists an index $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, $\mathbf{x}_k \in N'(\mathbf{c};\delta) \subseteq f^{-1}(N(L;\epsilon))$. Therefore, $\lim\limits_{k\to\infty} f(\mathbf{x}_k) = L$.

($\Longleftarrow$) Assume that, for every Cauchy sequence $\{\mathbf{x}_k\}$ in $S \setminus \{\mathbf{c}\}$ that converges to $\mathbf{c}$, $\lim\limits_{k\to\infty} f(\mathbf{x}_k) = L$. Suppose, to the contrary, that $\lim\limits_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) \neq L$. Then there exists an $\epsilon > 0$ such that, for any $\delta > 0$, $S \cap N'(\mathbf{c},\delta) \cap f^{-1}(N(L;\epsilon))^c \neq \varnothing$. Construct $\{\mathbf{x}_k\}$ by choosing a point $\mathbf{x}_k \in S \cap N'(\mathbf{c}, 1/k) \cap f^{-1}(N(L;\epsilon))^c$ for each $k \in \mathbb{N}$. $\{\mathbf{x}_k\}$ is a Cauchy sequence in $S \setminus \{\mathbf{c}\}$ that converges to $\mathbf{c}$; however, $\lim\limits_{k\to\infty} f(\mathbf{x}_k) \neq L$. Therefore, $\lim\limits_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$.

This completes the proof of Theorem 3.1.6. ■

3.33 Use Exercise 3.32 to prove that f is continuous at a point $\mathbf{c}$ in S if and only if, for every Cauchy sequence $\{\mathbf{x}_k\}$ in S which converges to $\mathbf{c}$, $\lim_{k\to\infty} f(\mathbf{x}_k) = f(\mathbf{c})$.

Solution.

($\Longrightarrow$) Let $\{\mathbf{x}_k\}$ be a Cauchy sequence in S which converges to $\mathbf{c}$. If finitely many terms of $\{\mathbf{x}_k\}$ are $\mathbf{c}$, then $\lim\limits_{k\to\infty} f(\mathbf{x}_k) = f(\mathbf{c})$ by Exercise 3.32. Otherwise, $\{\mathbf{x}_k\}$ must be eventually constant; there exists a $k_0 \in \mathbb{N}$ such that $k \geq k_0 \implies \mathbf{x}_k = \mathbf{c} \implies f(\mathbf{x}_k) = f(\mathbf{c})$, which is equivalent to $\lim\limits_{k\to\infty} f(\mathbf{x}_k) = f(\mathbf{c})$.

($\Longleftarrow$) If $\mathbf{c} \in S'$, then $\lim\limits_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$ is immediate from Exercise 3.32. Otherwise, if $\mathbf{c}$ is an isolated point, eventually constant sequences with value $\mathbf{c}$ are the only Cauchy sequences that satisfy the condition. It is vacuously true that f is continuous at isolated points.

■

3.2 The Topological Description of Continuity

3.37 Suppose that f is continuous on $[a, b]$ and that $m = \min\{f(x) : x \text{ in } [a, b]\}$ and $M = \max\{f(x) : x \text{ in } [a, b]\}$. Prove that $f([a, b]) = [m, M]$.

Solution. It is evident that $f([a, b]) \subseteq [m, M]$. By Theorem 2.4.3, $[a, b]$ is connected, so $f([a, b])$ is also connected by Theorem 3.2.2. Let $y \in [m, M]$. Assume that there is no $x \in [a, b]$ such that $f(x) = y$. Then $U = (-\infty, y)$ and $V = (y, \infty)$ are nonempty, disjoint open sets such that $f([a, b]) \subseteq U \cup V$, $f([a, b]) \cap U \neq \varnothing$, and $f([a,b]) \cap V \neq \varnothing$. However, these conditions imply that, contrary to our hypothesis, $f([a,b])$ is not connected. The contradiction implies that $[m, M] \subseteq f([a,b])$. Therefore, $f([a,b]) = [m, M]$. ■

3.38 Suppose that f and g are real-valued functions that are each continuous on a common domain S in $\mathbb{R}^n$. Determine whether the set $\{\mathbf{x} \text{ in } S : f(\mathbf{x}) > g(\mathbf{x})\}$ is relatively open in S. If so, prove it; otherwise, provide a counterexample.

Solution. $f \in C(S)$ and $g \in C(S)$, so $h = f - g \in C(S)$ by Theorem 3.3.1. By Theorem 3.2.1, the set $\{\mathbf{x} \in S \mid f(\mathbf{x}) > g(\mathbf{x})\} = \{\mathbf{x} \in S \mid h(\mathbf{x}) > 0\} = h^{-1}((0,\infty))$ is relatively open in S because $(0,\infty) \subset \mathbb{R}$ is open. ■

3.39 Suppose that f is a real-valued function that is continuous on a nonempty set S in $\mathbb{R}^n$. Suppose that U is a relatively open set in S. Is $f(U) = \{f(\mathbf{x}) : \mathbf{x} \text{ in } U\}$ necessarily relatively open in $f(S)$? If so, prove it; otherwise, provide a counterexample.

Solution. Consider $f(x) = ||x| - 1|$. By Theorem 3.2.7 and Theorem 3.3.1, f is continuous on $\mathbb{R}$. Let $U = (-1,1)$ be a relatively open set in $\mathbb{R}$. However, $f(U) = (0,1]$ is not relatively open in $f(\mathbb{R}) = [0,\infty)$. ■

3.40 Let f be a real-valued function that is continuous on a nonempty set S in $\mathbb{R}^n$. Assume that C is a relatively closed subset of $f(S)$. Prove that $f^{-1}(C)$ is relatively closed in S.

Solution. If C is a relatively closed subset of $f(S)$, then $C^c \cap f(S) = f(S) \setminus C$ is a relatively open subset of $f(S)$ by Theorem 2.2.2. Since f is continuous on S, by Theorem 3.2.1, $f^{-1}(f(S) \setminus C)$ is relatively open in S. Therefore, it follows from $f^{-1}(f(S) \setminus C) = S \setminus f^{-1}(C) = f^{-1}(C)^c \cap S$ that $f^{-1}(C)$ is relatively closed in S by Theorem 2.2.2. ■

3.41 Suppose that f is a real-valued function defined on a nonempty set S in $\mathbb{R}^n$ such that, for every relatively closed subset C of $f(S)$, $f^{-1}(C)$ is relatively closed in S. Does it necessarily follow that f is continuous on S? If so, prove it; otherwise, provide a counterexample.

Solution. If U is a relatively open subset of $f(S)$, then $U^c \cap f(S) = f(S) \setminus U$ is a relatively closed subset of $f(S)$ by Theorem 2.2.2. Since $f^{-1}(f(S) \setminus U) = S \setminus f^{-1}(U) = f^{-1}(U)^c \cap S$ is relatively closed in S, $f^{-1}(U)$ is relatively open in S by Theorem 2.2.2. By Theorem 3.2.1, f is continuous on S. ■

3.42 Suppose that f is a real-valued function that is continuous on a nonempty set S in $\mathbb{R}^n$ and suppose that $f(S)$ is compact in $\mathbb{R}$. Does it necessarily follow that S must be compact? If so, prove it; otherwise, provide a counterexample.

Solution. Consider $f(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^n$. f is constant, so continuous on $\mathbb{R}^n$. $f(\mathbb{R}^n) = \{0\}$ is closed and bounded, so it is compact in $\mathbb{R}$ by Heine–Borel Theorem. However, by Theorem 2.5.4, $\mathbb{R}^n$ is not compact because it is unbounded. ■

3.47 Let f be a continuous real-valued function on $[a,b]$ that is one-to-one. Prove that f must be strictly monotone.

Solution. Suppose that there exist x_1, x_2 and x_3 in $[a, b]$ such that $a \leq x_1 < x_2 < x_3 \leq b$ and $f(x_1) < f(x_2) > f(x_3)$ or $f(x_1) > f(x_2) < f(x_3)$. The inequalities must be strict because f is one-to-one. Let $k \in (\min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\}) \cap (\min\{f(x_2), f(x_3)\}, \max\{f(x_2), f(x_3)\})$. By the Intermediate Value Theorem, there exist a $c_1 \in (x_1, x_2)$ and a $c_2 \in (x_2, x_3)$ such that $f(c_1) = f(c_2) = k$. However, we assumed that f is one-to-one. The contradiction implies that, if $a \leq x_1 < x_2 < x_3 \leq b$, then $f(x_1) < f(x_2) < f(x_3)$ or $f(x_1) > f(x_2) > f(x_3)$. Therefore, f must be strictly monotone. ■

3.59 Suppose that f is any continuous function that maps $[a, b]$ into $[a, b]$. Prove that there must exist an x in the interval $[a, b]$ such that $f(x) = x$. (This is your first example of an important class of theorems called *fixed-point theorems*; there is a point x left fixed by f. Such theorems treat questions of *stability* and *equilibrium*.)

Solution. Let $g(x) = f(x) - x$. g is continuous on $[a, b]$ by Theorem 3.3.1. $g(a) \geq 0$ and $g(b) \leq 0$. By the Intermediate Value Theorem, there exists an $x \in [a, b]$ such that $g(x) = 0$, that is, $f(x) = x$. ■

3.60 Weaken the hypothesis in Exercise 3.59 as follows: Assume that f is a continuous function on $[a, b]$ and that $f(a) \geq a$ and $f(b) \leq b$. Prove that there must exist an x in (a, b) such that $f(x) = x$. Does the same result follow if $f(a) < a$ and $f(b) > b$? Prove your answer.

Solution. Let $g(x) = f(x) - x$. g is continuous on $[a, b]$ by Theorem 3.3.1.

If $f(a) \geq a$ and $f(b) \leq b$, then $g(a) \geq 0$ and $g(b) \leq 0$. By the Intermediate Value Theorem, there exists an $x \in [a, b]$ such that $g(x) = 0$, that is, $f(x) = x$.

If $f(a) < a$ and $f(b) > b$, then $g(a) < 0$ and $g(b) > 0$. By the Intermediate Value Theorem, there exists an $x \in (a, b)$ such that $g(x) = 0$, that is, $f(x) = x$. ■

Chapter 4

Differentiation

4.1 The Derivative

4.2 Define a function f on $[0,1]$ by setting $f(x)=0$ if x is irrational and $f(x)=1/q$ if $x=p/q$ is rational with p and q in lowest terms. Determine where f is differentiable.

Solution. f is nowhere differentiable. For rational numbers, this follows from discontinuity. (See Exercise 3.13.) Let $x \in [0,1] \setminus \mathbb{Q}$ and let $\delta > 0$ be given. By Archimedes' principle, choose $q \in \mathbb{N}$ such that $1/q < \delta$. Then choose $p \in \mathbb{Z}$ such that $p-1 \leq qx < p$. Then $x < r = p/q \leq x + 1/q$ implies $r \in N(x,\delta) \cap \mathbb{Q}$ and $f(r) \geq 1/q$. We have

$$\left|\frac{f(r)-f(x)}{r-x}\right| \geq 1,$$

but, for $y \in N(x,\delta) \setminus \mathbb{Q}$,

$$\left|\frac{f(y)-f(x)}{y-x}\right| = 0.$$

Therefore, the limit $\lim_{t\to x} \dfrac{f(t)-f(x)}{t-x}$ does not exist, so f is not differentiable on $[0,1]\setminus\mathbb{Q}$. ■

4.6 Suppose that f is a function defined in a neighborhood of a point c and that

$$\lim_{h\to 0} \frac{f(c+h)-f(c-h)}{2h} = L$$

exists. Does it necessarily follow that $f'(c)$ exists? If so, prove it; otherwise, provide a counterexample.

Solution. Consider $f(x)=|x|$ with $c=0$.

$$\lim_{h\to 0} \frac{f(c+h)-f(c-h)}{2h} = 0$$

exists. However, it is easy to see that

$$f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = -1,$$

and that

$$f'\left(0^+\right) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = 1.$$

Since $f'(0^-) \neq f'\left(0^+\right)$, we deduce that $f'(0)$ fails to exist. ■

4.7 Suppose that f is differentiable on a neighborhood $N(c)$ of a point c. Prove that

$$\lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h} = f'(c).$$

Solution. Since $f'(c)$ exists,

$$\begin{aligned}\lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h} &= \frac{1}{2}\lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h} - \frac{f(c)-f(c-h)}{h}\right]\\ &= \frac{1}{2}\left[\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} - \lim_{h\to 0}\frac{f(c)-f(c-h)}{h}\right]\\ &= \frac{f'(c)+f'(c)}{2}\\ &= f'(c).\end{aligned}$$

■

4.2 Composition of Functions: The Chain Rule

4.13 Let $f(x) = x\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Prove that f is differentiable at every x in $\mathbb{R}$ except 0.

Solution. By Theorem 4.1.4 and Theorem 4.2.1, f is differentiable at every x in $\mathbb{R}$ except 0. It remains to check that f is not differentiable at 0.

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\sin\frac{1}{h}$$

does not exist by Example 12 in Section 3.1. Therefore, $f'(0)$ fails to exist. ■

4.14 Let $f(x) = x^2\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Prove that f is differentiable at all x in $\mathbb{R}$, but that $\lim_{x\to 0} f'(x)$ does not exist.

Solution. By Theorem 4.1.4 and Theorem 4.2.1, f is differentiable at every x in $\mathbb{R}$ except 0. It remains to check that f is differentiable at 0. It follows from

$$0 \le \lim_{h\to 0}\left|\frac{f(h)-f(0)}{h}\right| = \lim_{h\to 0}\left|h\sin\frac{1}{h}\right| \le \lim_{h\to 0}|h| = 0$$

that $f'(0) = 0$ by the Squeeze Play. For $x \neq 0$,

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}.$$

Suppose, to the contrary, that $L = \lim\limits_{x\to 0} f'(x)$ exists. Then it leads to a contradiction that

$$\lim_{x\to 0}\cos\frac{1}{x} = \lim_{x\to 0}\left[2x\sin\frac{1}{x} - f'(x)\right] = 0 - L = -L$$

exists. Therefore, $\lim\limits_{x\to 0} f'(x)$ does not exist. ■

4.15 Let $f(x)=x^2\sin(1/x^2)$ for $x\neq 0$ and $f(0)=0$. Prove that f is differentiable at all x in $\mathbb{R}$, but that f' is not bounded on any neighborhood $N(0)$ of 0.

Solution. By Theorem 4.1.4 and Theorem 4.2.1, f is differentiable at every x in $\mathbb{R}$ except 0. It remains to check that f is differentiable at 0. It follows from

$$0\le \lim_{h\to 0}\left|\frac{f(h)-f(0)}{h}\right|=\lim_{h\to 0}\left|h\sin\frac{1}{h^2}\right|\le \lim_{h\to 0}|h|=0$$

that $f'(0)=0$ by the Squeeze Play. For $x\neq 0$,

$$f'(x)=2x\sin\frac{1}{x^2}-\frac{2}{x}\cos\frac{1}{x^2}.$$

Let $\delta>0$ be given. Suppose, to the contrary, that there exists some $M>0$ such that $|f'(x)|\le M$ for all $x\in N(0;\delta)$. By Archimedes' principle, choose $k_1\in\mathbb{N}$ such that

$$\frac{1}{2k_1\pi}<\delta^2 \iff \frac{1}{\sqrt{2k_1\pi}}<\delta,$$

and choose $k_2\in\mathbb{N}$ such that

$$8k_2\pi>M^2 \iff 2\sqrt{2k_2\pi}>M.$$

Let $k=\max\{k_1,k_2\}$. Then $(2k\pi)^{-1/2}\in N(0;\delta)$, but

$$\left|f'\left(\frac{1}{\sqrt{2k\pi}}\right)\right|=2\sqrt{2k\pi}>M.$$

Therefore, f' is not bounded on any neighborhood $N(0)$ of 0. ■

4.17 Suppose that a function g maps an interval I into an interval J and that f maps J into $\mathbb{R}$. Suppose also that g is differentiable on a neighborhood $N(c)$ of c and is twice differentiable at c and that f is differentiable on a neighborhood of $d=g(c)$ and is twice differentiable at d. Prove that $f\circ g$ is twice differentiable at c and find a formula for $(f\circ g)''(c)$.

Solution. By Theorem 4.2.1, $(f\circ g)'(x)=f'(g(x))g'(x)=(f'\circ g)(x)g'(x)$ for all x in some neighborhood $N(c)$ of c. Recall the conditions:

- g is differentiable on $N(c)$ and is twice differentiable at c.
- f' is differentiable at $d=g(c)$.

By Theorem 4.1.4 and Theorem 4.2.1, $f\circ g$ is twice differentiable at c and

$$\begin{aligned}(f\circ g)''(c)&=(f'\circ g)'(c)g'(c)+(f'\circ g)(c)g''(c)\\&=f''(g(c))g'(c)^2+f'(g(c))g''(c).\end{aligned}$$

■

4.3 The Mean Value Theorem

4.20 Suppose that f is continuous on $[0, b]$ and differentiable on $(0, b)$. Suppose also that $f(0) = 0$ and f' is increasing on $(0, b)$. Define $g(x) = f(x)/x$ for x in $(0, b)$. Prove that g is increasing on $(0, b)$.

Solution. For all x and y such that $0 < x < y < b$, by the Mean Value Theorem, there exist a $c_1 \in (0, x)$ such that

$$f'(c_1) = \frac{f(x) - f(0)}{x} = \frac{f(x)}{x},$$

and a $c_2 \in (x, y)$ such that

$$f'(c_2) = \frac{f(y) - f(x)}{y - x}.$$

Since f' is increasing on $(0, b)$, $0 < c_1 < x < c_2 < y < b$ implies

$$f'(c_1) = \frac{f(x)}{x} \leq \frac{f(y) - f(x)}{y - x} = f'(c_2).$$

Then we have

$$(y - x)f(x) \leq xf(y) - xf(x) \iff g(x) = \frac{f(x)}{x} \leq \frac{f(y)}{y} = g(y).$$

Therefore, g is increasing on $(0, b)$. ∎

4.24 **a)** Suppose that f is monotone increasing on an open interval I and differentiable at a point c in I. Prove that $f'(c) \geq 0$.

a) Suppose that, in part (a), f is assumed to be strictly monotone increasing. Does it follow that $f'(c) > 0$? If so, prove it; otherwise, provide a counterexample.

Solution.

a) Since f is monotone increasing,

$$f(c^-) = \lim_{h \to 0^-} \frac{f(c + h) - f(c)}{h} \geq 0,$$

and

$$f\left(c^+\right) = \lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h} \geq 0.$$

$f'(c)$ exists, so $f(c) = f\left(c^+\right) = f(c^-) \geq 0$.

b) Consider $f(x) = x^3$. f is strictly monotone increasing, but $f'(0) = 0$.

∎

4.25 Suppose that f is a function defined on an interval I such that f' exists and is bounded on I. Prove that f must be uniformly continuous on I.

Solution. Since f' exists and is bounded on I, there exists some $M \geq 0$ such that

$$M = \|f'\|_\infty = \sup\{|f'(x)| : x \in I\} < \infty.$$

For all x and y in I with $x < y$, by the Mean Value Theorem, there exists a $c \in (x, y)$ such that

$$\left|\frac{f(y)-f(x)}{y-x}\right| = |f'(c)| \le M.$$

If $M = 0$, then f must be a constant function that is uniformly continuous by Corollary 4.3.4. Assume $M > 0$. Let $\epsilon > 0$ be given and take $\delta = \epsilon/M$. For all x and y in I, $|x - y| < \delta$ implies

$$|f(x) - f(y)| \le M|x - y| < M\delta = \epsilon.$$

Therefore, f must be uniformly continuous on I. ■

4.26 Suppose that f is a function defined on an interval I such that f' exists and is bounded on I. Prove that f must satisfy a Lipschitz condition of order 1 on I.

Solution. Since f' exists and is bounded on I, there exists some $M \ge 0$ such that

$$M = \|f'\|_\infty = \sup\{|f'(x)| : x \in I\} < \infty.$$

For all x and y in I with $x < y$, by the Mean Value Theorem, there exists a $c \in (x, y)$ such that

$$\left|\frac{f(y)-f(x)}{y-x}\right| = |f'(c)| \le M.$$

For all x and y in I,

$$|f(x) - f(y)| \le M|x - y|.$$

Therefore, f must satisfy a Lipschitz condition of order 1 on I. ■

4.27 Suppose that f' is bounded on $(0, 1]$. Prove that $\lim_{k\to\infty} f(1/k)$ exists.

Solution. Suppose that there exists some $M \ge 0$ such that $|f'(x)| \le M$ for all $x \in (0, 1]$. $\{1/k\}$ is a Cauchy sequence that converges to 0. Fix any $\epsilon > 0$. There exists a $k_0 \in \mathbb{N}$ such that $k \ge k_0$ and $m \ge k_0$ implies $|1/k - 1/m| < \epsilon/M$. By the Mean Value Theorem, there exists a c strictly between $1/k$ and $1/m$ such that $|f(1/k) - f(1/m)| = |f'(c)||1/k - 1/m| \le M|1/k - 1/m| < \epsilon$. Therefore, $\{f(1/k)\}$ is also a Cauchy sequence. Therefore, by Theorem 1.4.4, $\lim_{k\to\infty} f(1/k)$ exists. ■

4.28 Suppose that f, g, and h are continuous on $[a, b]$ and differentiable on (a, b). For x in $[a, b]$, define

$$F(x) = \begin{vmatrix} f(x) & f(a) & f(b) \\ g(x) & g(a) & g(b) \\ h(x) & h(a) & h(b) \end{vmatrix}.$$

a) Prove that $F'(c) = 0$ for some c in (a, b).

b) Show that, for an appropriate choice of g and h, the result in part (a) implies the Mean Value Theorem.

c) Show that, for an appropriate choice of h, the result in part (a) implies the Cauchy's Generalized Mean Value Theorem.

Solution.

a) F is continuous on $[a,b]$ and differentiable on (a,b). $F(a)$ and $F(b)$ are determinants, each with two identical columns, so $F(a)=F(b)=0$. By Rolle's Theorem, there exists a $c\in(a,b)$ such that $F'(c)=0$.

b) Let $g(x)=x$ and let $h(x)=1$. We have

$$F(x)=\begin{vmatrix} f(x) & f(a) & f(b) \\ x & a & b \\ 1 & 1 & 1 \end{vmatrix} = (a-b)f(x)-[f(a)-f(b)]x+bf(a)-af(b).$$

g and h are continuous on $[a,b]$ and differentiable on (a,b). There exists a $c\in(a,b)$ such that $F'(c)=(a-b)f'(c)-[f(a)-f(b)]=0$, or $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

c) Let $h(x)=1$. We have

$$F(x)=\begin{vmatrix} f(x) & f(a) & f(b) \\ g(x) & g(a) & g(b) \\ 1 & 1 & 1 \end{vmatrix} = [g(a)-g(b)]f(x)-[f(a)-f(b)]g(x)+f(a)g(b)-f(b)g(a).$$

h is continuous on $[a,b]$ and differentiable on (a,b). There exists a $c\in(a,b)$ such that $F'(c)=[g(a)-g(b)]f'(c)-[f(a)-f(b)]g'(c)=0$, or $[g(b)-g(a)]f'(c)=[f(b)-f(a)]g'(c)$.

■

4.30 Suppose that a function f is monotone increasing, bounded, and differentiable on (a,∞). Does it necessarily follow that $\lim_{x\to\infty} f'(x)=0$? If so, prove it; otherwise, construct a counterexample.

Solution. The limit $\lim\limits_{x\to\infty} f'(x)$ may not exist. The following counterexample involves some knowledge covered in succeeding chapters. Let

$$g(x)=\begin{cases}\sin^2(\pi x) & \text{if } x\in(0,1),\\ 0 & \text{otherwise.}\end{cases}$$

Then let

$$f(x)=\int_0^x \sum_{k=1}^{\infty} g(k^2(t-k))\,dt.$$

f is monotone increasing because $f'(x)\geq 0$ on $(0,\infty)$. f is bounded above by

$$\begin{aligned}\sum_{k=1}^{\infty}\int_k^{k+1} g(k^2(t-k))\,dt &= \sum_{k=1}^{\infty}\int_0^1 g(k^2t)\,dt = \sum_{k=1}^{\infty}\int_0^{k^2} g(u)\,\frac{du}{k^2}\\ &= \sum_{k=1}^{\infty}\frac{1}{k^2}\left[\int_0^1 \sin^2(\pi u)\,du + \int_1^{k^2} 0\,du\right]\\ &= \frac{\pi^2}{6}\int_0^1 \frac{1-\cos(2\pi u)}{2}\,du = \frac{\pi^2}{12}\end{aligned}$$

and bounded below by 0. f is differentiable on $(0,\infty)$ because it is the definite integral of a continuous function. However, the limit $\lim\limits_{x\to\infty} f'(x)$ fails to exist since $f'(k)=g(0)=0$ and $f'\left(k+\dfrac{1}{2k^2}\right)=g\left(\dfrac{1}{2}\right)=1$ for each $k\in\mathbb{N}$. ■

4.32 Use the Mean Value Theorem to prove Bernoulli's inequality: For every $x > -1$ and every k in $\mathbb{N}$,

$$(1+x)^k \geq 1 + kx.$$

Solution. Let $f(x) = (1+x)^k$. For every $x > -1$ and every k in $\mathbb{N}$, f is differentiable. The inequality trivially holds for $x = 0$. Assume $x \neq 0$. By the Mean Value Theorem, there exists a c strictly between 0 and x such that

$$\frac{f(x) - f(0)}{x} = \frac{(1+x)^k - 1}{x} = k(1+c)^{k-1} \iff (1+x)^k = 1 + kx(1+c)^{k-1}.$$

(i) If $-1 < x < 0$, then $c \in (x, 0)$. $(1+c)^{k-1} \leq 1$ implies $(1+x)^k = 1 + kx(1+c)^{k-1} \geq 1 + kx$.

(ii) If $x > 0$, then $c \in (0, x)$. $(1+c)^{k-1} \geq 1$ implies $(1+x)^k = 1 + kx(1+c)^{k-1} \geq 1 + kx$.

This completes the proof. ■

4.33 Suppose that f is continuous on $[a, b]$, that f is differentiable on (a, b), and that $f(a) = f(b) = 0$. Prove that, for every k in $\mathbb{R}$, there exists a c in (a, b) such that $f'(c) = kf(c)$. (Consider the function $g(x) = e^{-kx} f(x)$ for x in $[a, b]$.)

Solution. Let $g(x) = e^{-kx} f(x)$ for $x \in [a, b]$. g is continuous on $[a, b]$ and is differentiable on (a, b). Since $g(a) = g(b) = 0$, by Rolle's theorem, there exists a $c \in (a, b)$ such that

$$g'(c) = -ke^{-kc} f(c) + e^{-kc} f'(c) = e^{-kc}[f'(c) - kf(c)] = 0.$$

$e^{-kc} > 0$, so $f'(c) = kf(c)$. ■

4.34 Prove *Darboux's Intermediate Value Theorem for derivatives*: If f is differentiable on $[a, b]$ and if d is some number between $f'(a)$ and $f'(b)$, then there exists a c in $[a, b]$ such that $f'(c) = d$. (*Hint*: For x in $[a, b]$, let $g(x) = d(x - a) - f(x)$. Show that g must have a critical point in (a, b).)

Solution. If d equals $f'(a)$ or $f'(b)$, then setting c equal to a or b, respectively, gives the desired result.

Now assume that d is strictly between $f'(a)$ and $f'(b)$, and in particular that $f'(a) < d < f'(b)$. Let $g(x) = d(x - a) - f(x)$. g is also differentiable and continuous on $[a, b]$. By Theorem 3.2.4, g has its maximum on $[a, b]$. Since $g'(a) = d - f'(a) > 0$ and $g'(b) = d - f'(b) < 0$, by Theorem 4.3.1, g cannot attain its maximum value at neither a nor b. Therefore, g must attain its maximum value at some point $c \in (a, b)$. Therefore, by Theorem 4.3.1, $g'(c) = 0$, that is, $f'(c) = d$.

If it is the case that $f'(a) > d > f'(b)$, we adjust our above proof, instead asserting that g has its minimum on (a, b). ■

4.46 Suppose that f is three times differentiable on $[a, b]$ and that $f(a) = f(b) = f'(a) = f'(b)$. Prove or disprove that there exists a c in (a, b) such that $f^{(3)}(c) = 0$.

Solution. The statement is false. Consider $f(x) = x^3 - x + 2$ on $[-1, 1]$. It is clear that f is three times differentiable on $[-1, 1]$. We have $f'(x) = 3x^2 - 1$, so $f(-1) = f(1) = f'(-1) = f'(1) = 2$. However, $f''(x) = 6x$, and $f^{(3)}(x) = 6$ for all $x \in [-1, 1]$. ■

4.47 For $0 < a < b$, let f be continuous on $[a, b]$ and differentiable on (a, b). Prove that there exists a c in (a, b) such that $f(b) - f(a) = cf'(c)\ln(b/a)$. (Use Cauchy's mean value theorem.) Hence prove that $\lim_{k\to\infty} k\left(a^{1/k} - 1\right) = \ln a$.

Solution. Let $g(x) = \ln x$. Since both f and g are continuous on $[a, b]$ and differentiable on (a, b), by Theorem 4.3.7, there exists a c in (a, b) such that

$$\frac{f(b) - f(a)}{c} = f'(c)(\ln b - \ln a) \iff f(b) - f(a) = cf'(c)\ln(b/a).$$

Let $f(x) = x^{1/k}$. If $a = 1$, then the equality is trivial. Assume that $a \neq 1$. Since f is continuous on $[1, a]$ (or $[a, 1]$) and differentiable on $(1, a)$ (or $(a, 1)$), it follows from the above that

$$a^{1/k} - 1 = c \cdot \frac{1}{k}c^{1/k-1}\ln a \iff k\left(a^{1/k} - 1\right) = c^{1/k}\ln a.$$

We have $\lim\limits_{k\to\infty} c^{1/k} = 1$. Therefore, $\lim\limits_{k\to\infty} k\left(a^{1/k} - 1\right) = \ln a$. ■

4.50 Suppose that f is differentiable on (a, b) and that c is a point of (a, b) such that $\lim_{x\to c} f'(x)$ exists. Prove that this limit must be $f'(c)$.

Solution. See Exercise 4.51. ■

4.51 Suppose that f is continuous on (a, b) and that f' is known to exist everywhere on (a, b) except possibly at the point c. Suppose that $\lim_{x\to c} f'(x) = L$ exists. Prove that f is differentiable at c and that $f'(c) = L$.

Solution. Let $\epsilon > 0$ be given. There exists some $\delta > 0$ such that $N(c; \delta) \subset (a, b)$ and

$$0 < |x - c| < \delta \implies |f'(x) - L| < \epsilon.$$

For $y \in N(c; \delta)$, by the Mean Value Theorem, there exists z strictly between c and y such that

$$\frac{f(y) - f(c)}{y - c} = f'(z).$$

Since $z \in N(c; \delta)$, we have

$$\left|\frac{f(y) - f(c)}{y - c} - L\right| = |f'(z) - L| < \epsilon.$$

Therefore, $f'(c) = \lim\limits_{x\to c} f'(x) = L$. ■

4.52 If f is twice differentiable on $[a, a + h]$, prove that there exists a c in $(a, a + h)$ such that

$$f(a + h) = f(a) + hf'(a) + \frac{1}{2}h^2 f''(c).$$

Solution. Assume $h > 0$. Let $F(x) = f(x) - [f(a) + f'(a)(x - a)]$ and let $G(x) = (x - a)^2$. By Cauchy's mean value theorem, there exists a $c_1 \in (a, a + h)$ such that

$$G'(c_1)[F(a + h) - F(a)] = F'(c_1)[G(a + h) - G(a)]$$

$$\begin{aligned} 2(c_1 - a)[f(a+h) - f(a) - hf'(a)] &= [f'(c_1) - f'(a)]h^2 \\ f(a+h) - f(a) - hf'(a) &= \frac{1}{2}h^2 \cdot \frac{f'(c_1) - f'(a)}{c_1 - a}. \end{aligned}$$

By the Mean Value Theorem, there exists a $c \in (a, c_1)$ such that

$$\frac{f'(c_1) - f'(a)}{c_1 - a} = f''(c).$$

Therefore, there exists a $c \in (a, c_1) \subset (a, a+h)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{1}{2}h^2 f''(c).$$

■

4.53 Suppose that f maps $[a, b]$ continuously into $[a, b]$ and that f is differentiable on (a, b) with

$$\|f'\|_\infty = \sup\{|f'(x)| : x \text{ in } (a, b)\} < 1.$$

By Exercise 3.59, we know that f has a fixed point x_0 in $[a, b]$, that is, a point such that $f(x_0) = x_0$. Here we show how to compute x_0. First choose any x_1 in $[a, b]$. Define $x_2 = f(x_1)$. In general, for k in $\mathbb{N}$, define $x_{k+1} = f(x_k)$.

a) Prove that the resulting sequence $\{x_k\}$ is contractive, therefore convergent.

b) Prove that $\lim_{k\to\infty} x_k = x_0$ is a fixed point of f.

Solution. Let $M = \|f'\|_\infty = \sup\{|f'(x)| : x \text{ in } (a, b)\}$. We have $0 < M < 1$.

a) By the Mean Value Theorem, there exists a c strictly between x_k and x_{k+1} such that

$$|x_{k+1} - x_k| = |f(x_k) - f(x_{k-1})| = |f'(c)||x_k - x_{k-1}| \le M|x_k - x_{k-1}|.$$

Therefore, $\{x_k\}$ is contractive. By Theorem 1.4.4 and Theorem 1.4.5, it is convergent.

b) Since $f : [a, b] \to [a, b]$ is continuous, $x_0 = \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} f(x_k) = f\left(\lim_{k\to\infty} x_k\right) = f(x_0)$. Therefore, x_0 is a fixed point of f.

■

4.4 L'Hôpital's rule

4.58 Find $\lim_{x\to 0^+} x^x$.

Solution. By L'Hôpital's rule (Theorem 4.4.2),

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{t\to\infty} \frac{-\ln t}{t} = \lim_{t\to\infty} \frac{-1/t}{1} = 0.$$

Therefore, by Theorem 3.2.7,

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} \exp(x\ln x) = \exp\left(\lim_{x\to 0^+} x\ln x\right) = e^0 = 1.$$

■

4.68 Suppose that f is differentiable in some neighborhood of c and that $f''(c)$ exists. Use l'Hôpital's rule and Exercise 4.7 to prove that

$$\lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2} = f''(c).$$

Solution. Let $F(h) = f(c+h)+f(c-h)-2f(c)$ and let $G(h) = h^2$. Check the conditions:

- Both F and G are differentiable on some neighborhood of 0.
- $F(0) = G(0) = 0$.
- $G'(h) = 2h$ does not vanish on some deleted neighborhood of 0.

By L'Hôpital's rule,

$$\lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2} = \lim_{h\to 0} \frac{f'(c+h)-f'(c-h)}{2h}.$$

Since $f''(c)$ exists, according to Exercise 4.7,

$$\lim_{h\to 0} \frac{f'(c+h)-f'(c-h)}{2h} = f''(c).$$

Therefore,

$$\lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2} = f''(c).$$

■

4.69 Suppose that f is twice differentiable on some neighborhood of a point c. Prove that

$$\lim_{h\to 0} \frac{f(c+h)-f(c)-f'(c)h}{h^2} = \frac{f''(c)}{2}.$$

Solution. Let $F(h) = f(c+h)-f(c)-f'(c)h$ and let $G(h) = h^2$. Check the conditions:

- Both F and G are differentiable on some neighborhood of 0.
- $F(0) = G(0) = 0$.
- $G'(h) = 2h$ does not vanish on some deleted neighborhood of 0.

Since $f''(c)$ exists, by L'Hôpital's rule,

$$\lim_{h\to 0} \frac{f(c+h)-f(c)-f'(c)h}{h^2} = \lim_{h\to 0} \frac{f'(c+h)-f'(c)}{2h} = \frac{f''(c)}{2}.$$

■

4.70 Suppose that f is three times differentiable on some neighborhood of a point c. Prove that

$$\lim_{h\to 0}\frac{f(c+h)-f(c)-f'(c)h-f''(c)h^2/2}{h^3}=\frac{f^{(3)}(c)}{6}.$$

Solution. Let $F(h)=f(c+h)-f(c)-f'(c)h-f''(c)h^2/2$ and let $G(h)=h^3$. Check the conditions:

- Both F and G are differentiable on some neighborhood of 0.
- $F(0)=G(0)=0$.
- $G'(h)=3h^2$ does not vanish on some deleted neighborhood of 0.

Since $f''(c)$ exists, by L'Hôpital's rule,

$$\lim_{h\to 0}\frac{f(c+h)-f(c)-f'(c)h-f''(c)h^2/2}{h^3}=\lim_{h\to 0}\frac{f'(c+h)-f'(c)-f''(c)h}{3h^2}.$$

f' is twice differentiable on some neighborhood of a point c. According to Exercise 4.69,

$$\lim_{h\to 0}\frac{f'(c+h)-f'(c)-f''(c)h}{h^2}=\frac{f^{(3)}(c)}{2}.$$

Therefore,

$$\lim_{h\to 0}\frac{f(c+h)-f(c)-f'(c)h-f''(c)h^2/2}{h^3}=\frac{f^{(3)}(c)}{6}.$$

■

Chapter 5

Functions of Bounded Variation

5.1 Partitions

5.3 Prove that $\preceq$ is a partial order on $\Pi[a,b]$.

Solution. $\preceq$ is just $\subseteq$. ■

5.4 Total Variation as a Function

5.10 Define $f(0) = 0$ and $f(x) = \sin(1/x)$ for $x \neq 0$. Confirm the assertion made in Example 7 to show that the function f is not in $BV(0,1)$.

Solution. Choose any $k \in \mathbb{N}$ and form the partition with points $x_0 = 0$, $x_j = \dfrac{1}{(k-j+1/2)\pi}$ for $j \in \{1, 2, \cdots, k\}$, and $x_{k+1} = 1$. Then

$$\begin{aligned}|f(x_j) - f(x_{j-1})| &= |\sin((k-j+1/2)\pi) - \sin((k-j+3/2)\pi)| \\ &= \left|(-1)^{k-j} - (-1)^{k-j+1}\right| \\ &= 2\end{aligned}$$

for $j \in \{2, \cdots, k\}$. We have

$$\sum_{j=1}^{k+1} |\Delta f_j| = 1 + \sum_{j=2}^{k} |\Delta f_j| + |f(1) - f(x_k)| \geq 1 + 2(k-1) = 2k - 1.$$

Therefore, $V(f, 0, 1)$ cannot be finite, so $f \notin BV(0,1)$. ■

5.11 Define $f(0) = 0$ and $f(x) = x\sin(1/x)$ for $x \neq 0$. Show that f is of unbounded variation on any interval containing $x_0 = 0$. (See Example 22 in Section 1.3.)

Solution. First, consider the interval $[0,b]$. By Archimedes' principle, choose $n \in \mathbb{N}$ such that $\dfrac{1}{(n+1/2)\pi} < b$. Choose any k in $\mathbb{N}$ and form the partition with partition points $x_0 = 0$, $x_j = \dfrac{1}{(n+k-j+1/2)\pi}$ for $j \in$

$\{1, 2, \cdots, k\}$. Then

$$\begin{aligned} |f(x_j) - f(x_{j-1})| &= \left| \frac{(-1)^{n+k-j}}{(n+k-j+1/2)\pi} - \frac{(-1)^{n+k-j-1}}{(n+k-j+3/2)\pi} \right| \\ &= \frac{1}{(n+k-j+3/2)\pi} + \frac{1}{(n+k-j+1/2)\pi} \end{aligned}$$

for $j \in \{2, \cdots, k\}$. We have

$$\begin{aligned} \sum_{j=1}^{k} |\Delta f_j| &= \frac{1}{(n+k-1/2)\pi} + \sum_{j=2}^{k} \left[\frac{1}{(n+k-j+3/2)\pi} + \frac{1}{(n+k-j+1/2)\pi} \right] \\ &> \sum_{j=1}^{k-1} \frac{2}{(n+k-j+1/2)\pi} = \sum_{j=1}^{k-1} \frac{2}{(n+j+1/2)\pi}. \end{aligned}$$

By Example 22 in Section 1.3, $V(f, 0, b)$ cannot be finite. Therefore, $f \notin BV(0, b)$.

Consider the interval $[a, 0]$. It follows from the above that $f \notin BV(0, -a)$. Since $f(x) = f(-x)$, $f \notin BV(a, 0)$. We conclude that f is of unbounded variation on any interval containing $x_0 = 0$. ■

5.12 Define $f(0) = 0$ and $f(x) = x^2 \sin(1/x)$ for $x \neq 0$. Show that f is of bounded variation on any interval $[a, b]$, including those that contain $x_0 = 0$.

Solution. According to Exercise 4.14, f is continuous on any interval $[a, b] \subset \mathbb{R}$ and differentiable on (a, b). From

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

we have, for any $x \in (a, b) \setminus \{0\}$,

$$\begin{aligned} |f'(x)| &= |2x \sin(1/x) - \cos(1/x)| \\ &\leq 2|x||\sin(1/x)| + |\cos(1/x)| \\ &\leq 2 \max\{|a|, |b|\} + 1. \end{aligned}$$

Therefore, $\|f'\|_\infty = \sup\{|f'(x)| : x \in (a, b)\} < \infty$. By Theorem 5.3.3, $f \in BV(a, b)$. ■

Chapter 6

The Riemann Integral

6.1 Suppose that f and g are bounded functions on $[a, b]$ such that $f + g$ is in $R[a, b]$. Does it follow that f and g are also in $R[a, b]$? If so, prove it; otherwise provide a counterexample.

Solution. Consider

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

and $g = 1 - f$. $f([a, b]) = g([a, b]) \subset N(0; 2)$. Clearly f and g are bounded. $f + g$ is continuous as $f + g = 1$ for all $x \in [a, b]$. By Theorem 6.2.7, $f + g \in R[a, b]$. However, f is the "Dirichlet's dancing function" that is proved to be **NOT** Riemann integrable in Example 1. Therefore, $f \notin R[a, b]$ and $g \notin R[a, b]$. ■

6.2 Suppose that f and g are bounded functions on $[a, b]$ such that f and fg are in $R[a, b]$. Does it follow that g is in $R[a, b]$? If so, prove it; otherwise provide a counterexample.

Solution. Consider $f = 0$ and

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

$f([a, b]) \subset g([a, b]) \subset N(0; 2)$. Clearly f and g are bounded. f and fg are continuous as $f = fg = 0$ for all $x \in [a, b]$. By Theorem 6.2.7, f and fg are in $R[a, b]$. However, g is the "Dirichlet's dancing function" that is proved to be **NOT** Riemann integrable in Example 1. Therefore, $g \notin R[a, b]$. ■

6.7 Suppose that g is in $R[a, b]$, that $g([a, b]) \subseteq [c, d]$, and that f is in $R[c, d]$. Does it follow that $f \circ g$ is in $R[a, b]$? *Hint:* Consider the following pair of functions on $[0, 1]$:

$$g(x) = \begin{cases} 1, & x = 0 \\ 1/q, & x = p/q > 0,\ p,\ q \text{ relatively prime} \\ 0, & x \text{ irrational} \end{cases}$$

and

$$f(y) = \begin{cases} 0, & y = 0 \\ 1, & 0 < y < 1. \end{cases}$$

Identify $f \circ g$ on $[0, 1]$ and determine its integrability.

Solution. It follows from Example 2 in Section 6.2 that $g \in R[0,1]$. We have that $g([0,1]) \subseteq [0,1]$. Since f is monotone increasing on $[0,1]$, by Theorem 5.3.2, $f \in BV[0,1]$. Then $f \in R[0,1]$ by Theorem 6.2.8. However, the composition

$$(f \circ g)(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is the "Dirichlet's dancing function" that is proved to be **NOT** Riemann integrable on $[0,1]$ in Example 1. ■

6.8 Prove or disprove that, if g is monotone on $[a,b]$ and if f is continuous on $g([a,b])$, then $f \circ g$ is integrable on $[a,b]$.

Solution. The statement is false. Consider

$$g(x) = \begin{cases} x & \text{if } x \in [-1,0), \\ x+1 & \text{if } x \in [0,1]. \end{cases}$$

g is obviously monotone on $[-1,1]$. Define $f(x) = 1/x$. Then f is continuous on $g([-1,1]) = [-1,0) \cup [1,2]$. However,

$$(f \circ g)(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \in [-1,0), \\ \dfrac{1}{x+1} & \text{if } x \in [0,1] \end{cases}$$

is unbounded, so $f \circ g \notin R[a,b]$. ■

Remark. If $g \in R[a,b]$ and $f \in C(S)$ for some compact set $S \supseteq g([a,b])$ (such that f is uniformly continuous on S by Theorem 3.4.2), then $f \circ g \in R[a,b]$.

6.9 Prove or disprove that, if g is continuous on $[a,b]$ and if f is integrable on $g([a,b])$, then $f \circ g$ is integrable on $[a,b]$.

Solution. The statement is false. Refer to Lu, J. (1999). Is the Composite Function Integrable? *The American Mathematical Monthly, 106*(8), 763-766. doi:10.2307/2589023. ■

6.12 Suppose that f and g are two continuous functions on $[a,b]$ such that $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$. Prove that there must exist a c in $[a,b]$ such that $f(c) = g(c)$.

Solution. Let $h(x) = f(x) - g(x)$. Since h is continuous on $[a,b]$, by Theorem 6.3.2, there exists a $c \in [a,b]$ such that

$$h(c) = \frac{1}{b-a} \int_a^b h(x)\,dx$$
$$f(c) - g(c) = \frac{1}{b-a} \int_a^b [f(x) - g(x)]\,dx = 0,$$

that is, $f(c) = g(c)$. ■

6.14 Suppose that f is continuous on $[a,b]$ and that $\int_a^b f(x)g(x)\,dx = 0$ for all g in $R[a,b]$. Prove that $f(x) = 0$ for all x in $[a,b]$.

Solution. Since f is continuous on $[a,b]$, by Theorem 6.2.7, $f \in R[a,b]$. Let $g = f$. Then we have

$$\int_a^b f(x)g(x)\,dx = \int_a^b f(x)^2\,dx = 0.$$

f^2 is continuous and nonnegative on $[a,b]$. By Theorem 6.2.9, $f(x)^2 = 0$ for all x in $[a,b]$. Therefore, $f(x) = 0$ for all x in $[a,b]$. ∎

6.21 Suppose that f is a bounded function on $[a,b]$ such that either $f^+ = \max\{f,0\}$ or $f^- = \max\{-f,0\}$ (but not necessarily both) is in $R[a,b]$. Does it necessarily follow that f is also in $R[a,b]$? If so, prove it; otherwise, provide a counterexample.

Solution. Consider

$$f(x) = \begin{cases} 1 & x \in [a,b] \cap \mathbb{Q}, \\ 0 & x \in [a,b] \setminus \mathbb{Q}. \end{cases}$$

$f([a,b]) \subset N(0;2)$. Clearly f is bounded. f^- is continuous as $f^- = 0$ for all $x \in [a,b]$. By Theorem 6.2.7, $f^- \in R[a,b]$. However, f is the "Dirichlet's dancing function" that is proved to be **NOT** Riemann integrable in Example 1. Therefore, $f \notin R[a,b]$. ∎

6.23 Prove that, for any two bounded functions f and g on $[a,b]$,

$$L(f) + L(g) \le L(f+g) \le U(f+g) \le U(f) + U(g).$$

Solution. Consider a partition $\pi = \{a = x_0, x_1, \cdots, x_k = b\}$.

$$\inf f([x_{j-1},x_j]) + \inf g([x_{j-1},x_j]) \le \inf\{f(x)+g(x) : x \in [x_{j-1},x_j]\}$$

implies $L(f,\pi) + L(g,\pi) \le L(f+g,\pi)$. For any partitions π_1 and π_2 in $\Pi[a,b]$, by Theorem 6.2.1,

$$\begin{aligned} L(f,\pi_1) + L(g,\pi_2) &\le L(f,\pi_1 \vee \pi_2) + L(g,\pi_1 \vee \pi_2) \\ &\le L(f+g,\pi_1 \vee \pi_2) \\ &\le L(f+g) \end{aligned}$$

We have $L(f,\pi_1) \le L(f+g) - L(g,\pi_2)$ for some fixed $\pi_2 \in \Pi[a,b]$. $L(f+g) - L(g,\pi_2)$ is an upper bound for $\{L(f,\pi) : \pi \in \Pi[a,b]\}$, so $L(f) \le L(f+g) - L(g,\pi_2)$.

Then we have $L(g,\pi_2) \le L(f+g) - L(f)$. $L(f+g) - L(f)$ is an upper bound for $\{L(g,\pi) : \pi \in \Pi[a,b]\}$, so $L(g) \le L(f+g) - L(f)$. Therefore, $L(f) + L(g) \le L(f+g)$.

By Theorem 6.2.3, $L(f+g) \le U(f+g)$.

Again, consider a partition $\pi = \{a = x_0, x_1, \cdots, x_k = b\}$.

$$\sup\{f(x)+g(x) : x \in [x_{j-1},x_j]\} \le \sup f([x_{j-1},x_j]) + \sup g([x_{j-1},x_j])$$

implies $U(f+g,\pi) \le U(f,\pi) + U(g,\pi)$. For any partitions π_1 and π_2 in $\Pi[a,b]$, by Theorem 6.2.1,

$$\begin{aligned} U(f+g) &\le U(f+g,\pi_1 \vee \pi_2) \\ &\le U(f,\pi_1 \vee \pi_2) + U(g,\pi_1 \vee \pi_2) \end{aligned}$$

$$\leq U(f, \pi_1) + U(g, \pi_2)$$

We have $U(f, \pi_1) \geq U(f+g) - U(g, \pi_2)$ for some fixed $\pi_2 \in \Pi[a,b]$. $U(f+g) - U(g, \pi_2)$ is a lower bound for $\{U(f, \pi) : \pi \in \Pi[a,b]\}$, so $U(f) \leq U(f+g) - U(g, \pi_2)$.

Then we have $U(g, \pi_2) \leq U(f+g) - U(f)$. $U(f+g) - U(f)$ is a lower bound for $\{U(g, \pi) : \pi \in \Pi[a,b]\}$, so $U(g) \leq U(f+g) - U(f)$. Therefore, $U(f) + U(g) \leq U(f+g)$. ■

6.29 **a)** Suppose that f is continuous on $[0,1]$. Prove that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} f\left(\frac{j}{k}\right) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} f\left(\frac{j}{k}\right) = \int_0^1 f(x)\,dx.$$

b) Suppose that f is continuous on $[a,b]$. For k in $\mathbb{N}$, let $h_k = (b-a)/k$. Prove that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} f(a + jh_k) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} f(a + jh_k) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Solution.

a) By Theorem 6.2.7, $f \in R[0,1]$. Since f is continuous on a compact interval $[0,1]$, by Theorem 3.4.2, f is uniformly continuous on $[0,1]$. Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that, for all x, y in $[0,1]$, $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.

By Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that $1/k_0 < \delta$. For $k \geq k_0$, form the partition with partition points $x_j = j/k$ for $j \in \{0, 1, \cdots, k\}$. By Theorem 6.3.2, there exists a c_j in each subinterval $[x_{j-1}, x_j]$ such that

$$\frac{f(c_j)}{k} = \int_{(j-1)/k}^{j/k} f(x)\,dx.$$

Then we have

$$\begin{aligned}
\left| \frac{1}{k} \sum_{j=0}^{k-1} f\left(\frac{j}{k}\right) - \int_0^1 f(x)\,dx \right| &= \left| \frac{1}{k} \sum_{j=1}^{k} f\left(\frac{j-1}{k}\right) - \sum_{j=1}^{k} \int_{(j-1)/k}^{j/k} f(x)\,dx \right| \\
&= \frac{1}{k} \sum_{j=1}^{k} \left| f\left(\frac{j-1}{k}\right) - f(c_j) \right| \\
&< \frac{1}{k} \sum_{j=1}^{k} \epsilon = \epsilon,
\end{aligned}$$

and

$$\begin{aligned}
\left| \frac{1}{k} \sum_{j=1}^{k} f\left(\frac{j}{k}\right) - \int_0^1 f(x)\,dx \right| &= \left| \frac{1}{k} \sum_{j=1}^{k} f\left(\frac{j}{k}\right) - \sum_{j=1}^{k} \int_{(j-1)/k}^{j/k} f(x)\,dx \right| \\
&= \frac{1}{k} \sum_{j=1}^{k} \left| f\left(\frac{j}{k}\right) - f(c_j) \right| \\
&< \frac{1}{k} \sum_{j=1}^{k} \epsilon = \epsilon.
\end{aligned}$$

b) Define $g(x) = f(a + (b-a)x)$ on $[0,1]$. Since g is continuous on $[0,1]$, by part (a),

$$\text{(LHS)} = \int_0^1 f(a+(b-a)x)\,dx = \int_a^b f(t)\,\frac{dt}{b-a}. \qquad \blacksquare$$

6.33 Suppose that f satisfies a Lipschitz condition of order 1 with Lipschitz constant K at every x in $[0,1]$. That is, $|f(y) - f(x)| \le K|y - x|$ for all x, y in $[0,1]$. Prove that f is in $R[0,1]$ and that, for every k in $\mathbb{N}$,

$$\left| \int_0^1 f(x)\,dx - \frac{1}{k}\sum_{j=1}^{k} f\left(\frac{j}{k}\right) \right| \le \frac{K}{2k}.$$

Solution. Let $\epsilon > 0$ be given. Take $\delta = \epsilon/K$. Then, for all x, y in $[0,1]$, $|x-y| < \delta$ implies

$$|f(x) - f(y)| \le K|x-y| < K\delta = \epsilon.$$

Therefore f is uniformly continuous on $[0,1]$. By Theorem 6.2.7, $f \in R[0,1]$.

Observe that

$$\begin{aligned}
\left| \int_{(j-1)/k}^{j/k} f(x)\,dx - \frac{1}{k} f\left(\frac{j}{k}\right) \right| &= \left| \int_{(j-1)/k}^{j/k} \left[f(x) - f\left(\frac{j}{k}\right) \right] dx \right| \\
&\le \int_{(j-1)/k}^{j/k} \left| f(x) - f\left(\frac{j}{k}\right) \right| dx \\
&\le \int_{(j-1)/k}^{j/k} K \left| x - \frac{j}{k} \right| dx \\
&= -K \int_{(j-1)/k}^{j/k} \left(x - \frac{j}{k} \right) dx \\
&= -K \left[\frac{1}{2}\left(x - \frac{j}{k} \right)^2 \right]_{(j-1)/k}^{j/k} \\
&= \frac{K}{2k^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left| \int_0^1 f(x)\,dx - \frac{1}{k}\sum_{j=1}^{k} f\left(\frac{j}{k}\right) \right| &= \left| \sum_{j=1}^{k} \left[\int_{(j-1)/k}^{j/k} f(x)\,dx - \frac{1}{k} f\left(\frac{j}{k}\right) \right] \right| \\
&\le \sum_{j=1}^{k} \left| \int_{(j-1)/k}^{j/k} f(x)\,dx - \frac{1}{k} f\left(\frac{j}{k}\right) \right| \\
&\le \sum_{j=1}^{k} \frac{K}{2k^2} \\
&= \frac{K}{2k}.
\end{aligned}$$

This completes the proof. $\blacksquare$

6.41 Let f be continuously differentiable on $[a,b]$. We know by Theorem 5.3.2 that f is in $BV(a,b)$. Prove that $V(f;a,b) = \int_a^b |f'(x)|\,dx$.

Solution. Let $\pi = \{a = x_0, x_1, x_2, \cdots, x_p = b\}$ be any partition of $[a,b]$. We may assume that $\{x_j\}_{j=0}^p$ is a strictly increasing sequence. Since f is differentiable, by the mean value theorem, for each subinterval $[x_{j-1}, x_j]$, there exists $t_j \in (x_{j-1}, x_j)$ such that

$$|f(x_j) - f(x_{j-1})| = |f'(t_j)|(x_j - x_{j-1}).$$

Then we have a Riemann sum for the function $|f'|$,

$$\sum_{j=1}^p |f(x_j) - f(x_{j-1})| = \sum_{j=1}^p |f'(t_j)|(x_j - x_{j-1}) = S(|f'|, \pi).$$

Therefore,

$$L(|f'|, \pi) \le \sum_{j=1}^p |\Delta f_j| \le U(|f'|, \pi).$$

Let $\epsilon > 0$ be given. By Theorem 1.1.1, we can choose partitions π_1, π_2 and π_3 in $\Pi[a,b]$ such that

$$\begin{aligned} L(|f'|, \pi_1) &> L(|f'|) - \epsilon, \\ U(|f'|, \pi_2) &< U(|f'|) + \epsilon, \\ \sum_{\pi_3} |\Delta f_j| &> V(f;a,b) - \epsilon. \end{aligned}$$

Let $\pi_0 = \pi_1 \vee \pi_2 \vee \pi_3$. For any refinement π of π_0,

$$\begin{aligned} L(|f'|) - \epsilon < L(|f'|, \pi_1) \le L(|f'|, \pi) \le \sum_{\pi} |\Delta f_j| &\le V(f;a,b), \\ V(f;a,b) - \epsilon < \sum_{\pi_3} |\Delta f_j| \le \sum_{\pi} |\Delta f_j| &\le U(|f'|, \pi) \le U(|f'|, \pi_2) < U(|f'|) + \epsilon, \\ \therefore L(|f'|) - \epsilon < &V(f;a,b) < U(|f'|) + 2\epsilon. \end{aligned}$$

f' is continuous, so is $|f'|$ by Theorem 3.2.7. By Theorem 6.2.7, $|f'| \in R[a,b]$, and

$$\int_a^b |f'(x)|\,dx = L(|f'|) = U(|f'|)$$

exists. Taking limits as $\epsilon \to 0^+$, by the Squeeze Play,

$$V(f;a,b) = \int_a^b |f'(x)|\,dx.$$

This completes the proof. ■

6.59 Let f be a continuous function on $[a, b]$. Suppose that there exists a positive constant K such that

$$|f(x)| \leq K \int_a^x |f(t)|\, dt,$$

for all x in $[a, b]$. Prove that $f(x) = 0$ for all x in $[a, b]$.

Solution. Let $F(x) = \displaystyle\int_a^x |f(t)|\, dt$. Note that F is nonnegative and $F(a) = 0$. f is continuous on $[a, b]$, so is $|f|$. By Theorem 6.3.3, F is differentiable on $[a, b]$ and $F'(x) = |f(x)|$. Then we have

$$\begin{aligned} F'(x) - KF(x) &\leq 0 \\ e^{-Kx}F'(x) + (-Ke^{-Kx})F(x) &\leq 0 \\ \frac{d}{dx}\left[e^{-Kx}F(x)\right] &\leq 0. \end{aligned}$$

$e^{-Kx}F(x)$ is nonincreasing on $[a, b]$, so $e^{-Kx}F(x) \leq e^{-Ka}F(a) = 0$. Hence, F is nonpositive. It must follow that $F(x) = 0$ for all x in $[a, b]$. Therefore, $F'(x) = |f(x)| = 0$. We conclude that $f(x) = 0$ for all x in $[a, b]$. ∎

6.63 Define $F(x) = x^2 \sin\left(1/x^2\right)$ for x in $(0, 1]$ and $F(0) = 0$. Prove that $F'(x)$ exists on $[0, 1]$ but F' is not integrable on $[0, 1]$. (*Not every derivative is integrable.*)

Solution. For $x \in (0, 1]$, $F'(x) = 2x \sin \dfrac{1}{x^2} - \dfrac{1}{x} \cos \dfrac{1}{x^2}$. $F'(0)$ exists as

$$0 \leq \lim_{h\to 0^+} \left|\frac{F(h) - F(0)}{h}\right| = \lim_{h\to 0^+} \left|h \sin \frac{1}{h^2}\right| \leq \lim_{h\to 0^+} |h| = 0,$$

and, by the Squeeze Play, $F'(0) = 0$. By Exercise 4.15, F' is not bounded on any neighborhood $N(0)$ of 0. Therefore, F' is not integrable on $[0, 1]$. ∎

6.71 For x in $[-1, 1]$ and k in $\mathbb{N}$, define

$$f_k(x) = \frac{x^{2k}}{1 + x^{2k}}.$$

a) Find a function f_0 on $[-1, 1]$ such that $\{f_k\}$ converges pointwise to f_0.

b) Determine whether $\{f_k\}$ converges uniformly to f_0.

c) Use the Squeeze Play to find $\lim_{k\to\infty} \int_{-1}^1 f_k(x)\, dx$.

d) Calculate $\int_{-1}^1 f_0(x)\, dx$ and determine whether

$$\lim_{k\to\infty} \int_{-1}^1 f_k(x)\, dx = \int_{-1}^1 f_0(x)\, dx.$$

Solution.

a) We proved in Exercise 3.15(e) that

$$\left\{1 - f_k(x) = \frac{1}{1 + x^{2k}}\right\}$$

converges pointwise to the function

$$x \longmapsto \begin{cases} 1 & \text{if } |x| < 1, \\ 1/2 & \text{if } |x| = 1. \end{cases}$$

Define

$$f_0(x) = \begin{cases} 0 & \text{for } |x| < 1, \\ 1/2 & \text{for } |x| = 1. \end{cases}$$

Then $\{f_k\}$ must converge pointwise to f_0. For $x = 0$ or $|x| = 1$, $f_0(x) = f_k(x)$ for each $k \in \mathbb{N}$. For $0 < |x| < 1$,

$$1 - \frac{1}{1 + x^{2k}} < 1,$$

so we may assume that $0 < \epsilon < 1$. Let $c = 1/x^2 - 1$. By Archimedes' principle, choose $k_0 \in \mathbb{N}$ such that

$$\frac{1}{\epsilon} - 1 < k_0 c.$$

For $k \geq k_0$, we have

$$\begin{aligned} x^{2k} \leq x^{2k_0} = \frac{1}{(1+c)^{k_0}} \leq \frac{1}{1 + k_0 c} < \frac{1}{k_0 c} < \frac{\epsilon}{1-\epsilon} &= \frac{1}{1-\epsilon} - 1 \\ 1 + x^{2k} &< \frac{1}{1-\epsilon} \\ \frac{1}{1 + x^{2k}} &> 1 - \epsilon \\ |f_k(x) - f_0(x)| = \frac{x^{2k}}{1 + x^{2k}} &= 1 - \frac{1}{1 + x^{2k}} < \epsilon. \end{aligned}$$

Therefore, $\lim_{k \to \infty} f_k = f_0$ [pointwise].

b) We claim, again, that the convergence cannot be uniform. To prove this claim, fix any ϵ such that $0 < \epsilon < 1/4$ and form the [uniform] neighborhood $N_\infty(f_0; \epsilon)$. Since $f(x) = f(-x)$, it suffices to consider $x \in [0, 1]$. Fix any $k \in \mathbb{N}$. The unique point $a_k \in [0, 1]$ such that $f_k(a_k) = \epsilon$ is

$$a_k = \left(\frac{\epsilon}{1 - \epsilon} \right)^{1/(2k)}.$$

Likewise, the unique point $b_k \in [0, 1]$ such that $f_k(b_k) = 1/2 - \epsilon$ is

$$b_k = \left(\frac{1 - 2\epsilon}{1 + 2\epsilon} \right)^{1/(2k)}.$$

Note that $0 < a_k < b_k < 1$. Furthermore, for any $x \in (a_k, b_k)$, since f_k is strictly increasing,

$$\epsilon < f_k(x) < \frac{1}{2} - \epsilon.$$

These observations show that the function f_k fails to lie in the neighborhood $N_\infty(f_0; \epsilon)$. We get the same result for $x \in [-1, 0]$. Since k was initially chosen to be arbitrary, it follows that, although $\{f_k\}$ converges pointwise to f_0, the convergence cannot be uniform.

c) For $x \in [-1,1]$ and $k \in \mathbb{N}$,

$$0 \le f_k(x) = \frac{x^{2k}}{1+x^{2k}} \le x^{2k}$$
$$0 \le \int_{-1}^{1} f_k(x)\,dx \le \int_{-1}^{1} x^{2k}\,dx = \frac{2}{2k+1}.$$

$\lim_{k\to\infty} \frac{2}{2k+1} = 0$. By the Squeeze Play,

$$\lim_{k\to\infty} \int_{-1}^{1} f_k(x)\,dx = 0.$$

d) Obviously, $\int_{-1}^{1} f_0(x)\,dx = 0$. Therefore,

$$\lim_{k\to\infty} \int_{-1}^{1} f_k(x)\,dx = \int_{-1}^{1} f_0(x)\,dx.$$

■

6.96 Consider the following six statements:

p_1: f is continuous on $[a,b]$.

p_2: f is uniformly continuous on $[a,b]$.

p_3: f is differentiable on $[a,b]$.

p_4: f has an antiderivative on $[a,b]$.

p_5: f is in $R[a,b]$.

p_6: f is the indefinite integral of some g in $R[a,b]$.

There are 30 possible sentences of the form $p_i \to p_j$ with $i \neq j$. Which are theorems; which are not? Complete the following table, indicating your answers T or F, for true or false. For example, the statement $p_1 \to p_2$ is a theorem as indicated in the table.

Solution. There are 14 true statements. $p_3, p_6 \implies (p_1 \iff p_2) \implies p_4, p_5$.

	p_1	p_2	p_3	p_4	p_5	p_6
p_1		T	F	T	T	F
p_2	T		F	T	T	F
p_3	T	T		T	T	F
p_4	F	F	F		F	F
p_5	F	F	F	F		F
p_6	T	T	F	T	T	

Theorems:

- $p_1 \implies p_2$ by Theorem 3.4.2 and $p_1 \impliedby p_2$ by Exercise 3.67.
- $p_1 \implies p_4$ and $p_6 \implies p_1$ by Theorem 6.3.3.

- $p_1 \implies p_5$ by Theorem 6.2.7.
- $p_3 \implies p_1$ by Theorem 4.1.3.

Counterexamples:

- $p_1 \not\Longrightarrow p_3$: The Weierstrass function on any compact interval.
- $p_1 \not\Longrightarrow p_6$: The function of Exercise 5.11 on $[-1,1]$. It is continuous on $[-1,1]$, but of unbounded variation. By Theorem 6.3.3, f must be of bounded variation to satisfy p_6.
- $p_3 \not\Longrightarrow p_6$: Not every derivative is Riemann integrable. See Exercise 6.63.
- $p_4 \not\Longrightarrow p_1$ and $p_5 \not\Longrightarrow p_4$[1]: The floor function on $[0,2]$.
- $p_4 \not\Longrightarrow p_5$: The Volterra's function on $[0,1]$.
- $p_5 \not\Longrightarrow p_1$: The Thomae's function on $[0,1]$. See Example 2 in Section 6.2. ■

[1]Derivatives cannot have any jump discontinuities by Darboux's theorem. See Exercise 4.34.

Chapter 7

The Riemann-Stieltjes Integral

7.22 Let g be strictly monotone increasing on $[a,b]$. For f_1 and f_2 in $C([a,b])$, define

$$\langle f_1, f_2\rangle = \int_a^b f_1(x) f_2(x)\, dg(x).$$

a) Prove that $\langle\,\cdot\,,\,\cdot\,\rangle$ is an inner product on $C([a,b])$.

b) For f in $C([a,b])$, define

$$\|f\|_2 = \sqrt{\langle f, f\rangle} = \left[\int_a^b f^2(x)\, dg(x)\right]^{1/2}.$$

Prove the Cauchy-Schwarz inequality,

$$|\langle f_1, f_2\rangle| \le \|f_1\|_2 \|f_2\|_2,$$

for all f_1, f_2 in $C([a,b])$.

c) Prove that $\|\cdot\|_2$ is a norm on $C([a,b])$.

d) Use the norm in part (c) to define a metric d_2 on $C([a,b])$ as follows: $d_2(f_1, f_2) = \|f_1 - f_2\|_2$. Prove that d_2 is a metric on $C([a,b])$.

Solution.

a) $\langle\,\cdot\,,\,\cdot\,\rangle$ has the following properties:

(i) It is *additive* in both its variables. For f_1, f_2 and f_3 in $C([a,b])$,

$$\begin{aligned}
\langle f_1 + f_2, f_3\rangle &= \int_a^b [f_1(x) + f_2(x)] f_3(x)\, dg(x) \\
&= \int_a^b f_1(x) f_3(x)\, dg(x) + \int_a^b f_2(x) f_3(x)\, dg(x) \\
&= \langle f_1, f_3\rangle + \langle f_2, f_3\rangle. \\
\langle f_1, f_2 + f_3\rangle &= \int_a^b f_1(x)[f_2(x) + f_3(x)]\, dg(x) \\
&= \int_a^b f_1(x) f_2(x)\, dg(x) + \int_a^b f_1(x) f_3(x)\, dg(x)
\end{aligned}$$

$$= \langle f_1, f_2 \rangle + \langle f_1, f_3 \rangle.$$

(ii) It is *symmetric.*

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x) f_2(x)\, dg(x) = \int_a^b f_2(x) f_1(x)\, dg(x) = \langle f_2, f_1 \rangle.$$

(iii) It is *homogeneous* in both its variables.

$$\langle c_1 f_1, c_2 f_2 \rangle = \int_a^b c_1 f_1(x) c_2 f_2(x)\, dg(x) = c_1 c_2 \int_a^b f_1(x) f_2(x)\, dg(x) = c_1 c_2 \langle f_1, f_2 \rangle.$$

b) For $t \in \mathbb{R}$, $f_1, f_2 \in C([a,b])$, compute the inner product of $tf_1 + f_2 \in C([a,b])$ with itself. Using the properties of the inner product, we have

$$\begin{aligned} 0 \le \|tf_1 + f_2\|_2^2 &= \langle tf_1 + f_2, tf_1 + f_2 \rangle \\ &= \langle tf_1, tf_1 \rangle + \langle tf_1, f_2 \rangle + \langle f_2, tf_1 \rangle + \langle f_2, f_2 \rangle \\ &= t^2 \langle f_1, f_1 \rangle + 2t \langle f_1, f_2 \rangle + \langle f_2, f_2 \rangle \\ &= \|f_1\|_2^2 t^2 + 2\langle f_1, f_2 \rangle t + \|f_2\|_2^2 \end{aligned}$$

for all $t \in \mathbb{R}$. As a consequence, the discriminant must be nonpositive:

$$\frac{\Delta}{4} = \langle f_1, f_2 \rangle^2 - \|f_1\|_2^2 \|f_2\|_2^2 \le 0.$$

Therefore, $|\langle f_1, f_2 \rangle| \le \|f_1\|_2 \|f_2\|_2$.

c) $\| \cdot \|_2$ has the following properties:

(i) *Positive Definiteness*

$$\|f\|_2 = \sqrt{\langle f, f \rangle} = \left[\int_a^b f^2(x)\, dg(x) \right]^{1/2} \ge 0; \quad \|f\|_2 = 0 \Longleftarrow f = 0.$$

($\|f\|_2 = 0 \implies f = 0$) Suppose there were to exist a point x_0 in $[a,b]$ where $f^2(x_0) > 0$. By Theorem 3.4.3, there exists $m > 0$ and a neighborhood $N(x_0)$ such that $0 < m \le f^2(x)$ for all x in $[a,b] \cap N(x_0)$. Let the interval $[a,b] \cap N(x_0)$ have endpoints c and d with $c < d$. By Theorem 7.3.7, f is integrable on $[c,d]$. Since $f^2 \ge m$ on $[c,d]$,

$$\int_c^d f^2(x)\, dg(x) \ge m[g(d) - g(c)] > 0.$$

But $f^2 \ge 0$ on all of $[a,b]$. Therefore

$$\begin{aligned} \int_a^b f^2(x)\, dg(x) &= \int_a^c f^2(x)\, dg(x) + \int_c^d f^2(x)\, dg(x) + \int_d^b f^2(x)\, dg(x) \\ &\ge \int_c^d f^2(x)\, dg(x) > 0, \end{aligned}$$

contradicting the hypothesis that $\int_a^b f^2(x)\,dg(x) = 0$. Therefore there can exist no point in $[a, b]$ where f^2 is not zero. We conclude that f is identically 0 on $[a, b]$.

(ii) *Absolute Homogeneity*

$$\|cf\|_2 = \sqrt{\langle cf, cf\rangle} = \left[\int_a^b c^2 f^2(x)\,dg(x)\right]^{1/2} = |c|\left[\int_a^b f^2(x)\,dg(x)\right]^{1/2} = |c|\|f\|_2.$$

(iii) *Subadditivity*

$$\begin{aligned}\|f_1 + f_2\|_2^2 &= \|f_1\|_2^2 + 2\langle f_1, f_2\rangle + \|f_2\|_2^2 \\ &\le \|f_1\|_2^2 + 2\|f_1\|_2\|f_2\|_2 + \|f_2\|_2^2 \\ &= (\|f_1\|_2 + \|f_2\|_2)^2 \\ \therefore \|f_1 + f_2\|_2 &\le \|f_1\|_2 + \|f_2\|_2.\end{aligned}$$

d) d_2 has the following properties:

i) *Positive Definiteness*

$$\begin{aligned}&d_2(f_1, f_2) = \|f_1 - f_2\|_2 = \left[\int_a^b [f_1(x) - f_2(x)]^2\,dg(x)\right]^{1/2} \ge 0; \\ &d_2(f_1, f_2) = 0 \iff f_1 = f_2.\end{aligned}$$

ii) *Symmetry*

$$d_2(f_1, f_2) = \|f_1 - f_2\|_2 = \|f_2 - f_1\|_2 = d_2(f_2, f_1).$$

iii) *The Triangle Inequality*

$$\begin{aligned}d_2(f_1, f_3) = \|f_1 - f_3\|_2 &= \|(f_1 - f_2) + (f_2 - f_3)\|_2 \\ &\le \|f_1 - f_2\|_2 + \|f_2 - f_3\|_2 \\ &= d_2(f_1, f_2) + d_2(f_2, f_3).\end{aligned}$$

■

Chapter 8

Differential Calculus in $\mathbb{R}^n$

8.2 For $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^2$, let $f(\mathbf{x}) = x_1 x_2/\|\mathbf{x}\|$. Let $f(\mathbf{0}) = 0$. Show that the graph of f does not have a tangent plane at $\mathbf{c} = \mathbf{0}$.

Solution. We first compute its partial derivatives at $\mathbf{c}$.

$$D_1 f(\mathbf{c}) = \lim_{h\to 0} \frac{f(\mathbf{c}+h\mathbf{e}_1) - f(\mathbf{c})}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$
$$D_2 f(\mathbf{c}) = \lim_{h\to 0} \frac{f(\mathbf{c}+h\mathbf{e}_2) - f(\mathbf{c})}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Note that any plane containing the z-axis cannot be tangent to the graph of f at $\mathbf{c}$. By Theorem 8.1.6, it suffices to show that f is not differentiable at $\mathbf{c}$. We check whether the limit exists and evaluates to 0.

$$\begin{aligned}\lim_{\mathbf{t}\to\mathbf{0}} \frac{f(\mathbf{c}+\mathbf{t}) - f(\mathbf{c}) - df(\mathbf{c};\mathbf{t})}{\|\mathbf{t}\|} &= \lim_{\mathbf{t}\to\mathbf{0}} \frac{t_1 t_2}{\|\mathbf{t}\|^2} \\ &= \lim_{\mathbf{t}\to\mathbf{0}} \frac{t_1 t_2}{t_1^2 + t_2^2}.\end{aligned}$$

The limit does not exist by Example 3 in Section 3.1. Therefore, the graph of f does not have a tangent plane at $\mathbf{c}$. ■

8.13 Let g be a differentiable function on $\mathbb{R}^2$. For $\mathbf{t} = (t_1, t_2)$ in $\mathbb{R}^2$, let $x_1 = f_1(\mathbf{t}) = t_1 - t_2$ and $x_2 = f_2(\mathbf{t}) = t_2 - t_1$. Let $\mathbf{f} = (f_1, f_2)$ and $h(\mathbf{t}) = (g \circ \mathbf{f})(\mathbf{t})$. Show that

$$D_1 h(\mathbf{t}) + D_2 h(\mathbf{t}) = 0$$

for all $\mathbf{t}$ in $\mathbb{R}^2$.

Solution. By the chain rule, we have

$$\begin{aligned}\begin{bmatrix} D_1 h(\mathbf{t}) & D_2 h(\mathbf{t}) \end{bmatrix} &= \begin{bmatrix} D_1 g(\mathbf{f}(\mathbf{t})) & D_2 g(\mathbf{f}(\mathbf{t})) \end{bmatrix} \begin{bmatrix} D_1 f_1(\mathbf{t}) & D_2 f_1(\mathbf{t}) \\ D_1 f_2(\mathbf{t}) & D_2 f_2(\mathbf{t}) \end{bmatrix} \\ &= \begin{bmatrix} D_1 g(\mathbf{f}(\mathbf{t})) & D_2 g(\mathbf{f}(\mathbf{t})) \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\end{aligned}$$

for all $\mathbf{t}$ in $\mathbb{R}^2$. Therefore,

$$\begin{aligned} D_1h(\mathbf{t}) + D_2h(\mathbf{t}) &= \begin{bmatrix} D_1h(\mathbf{t}) & D_2h(\mathbf{t}) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} D_1g(\mathbf{f}(\mathbf{t})) & D_2g(\mathbf{f}(\mathbf{t})) \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} D_1g(\mathbf{f}(\mathbf{t})) & D_2g(\mathbf{f}(\mathbf{t})) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= 0. \end{aligned}$$

■

8.16 Suppose that g is a differentiable real-valued function on $\mathbb{R}^2$. For $\mathbf{x}$ in $\mathbb{R}^2$, let $y_1 = f_1(\mathbf{x}) = x_1^2 - x_2^2$ and $y_2 = f_2(\mathbf{x}) = x_2^2 - x_1^2$. Let $\mathbf{f} = (f_1, f_2)$ and $h = g \circ \mathbf{f}$. Prove that $x_2D_1h(\mathbf{x}) + x_1D_2h(\mathbf{x}) = 0$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

Solution. By the chain rule, we have

$$\begin{aligned} \begin{bmatrix} D_1h(\mathbf{x}) & D_2h(\mathbf{x}) \end{bmatrix} &= \begin{bmatrix} D_1g(\mathbf{f}(\mathbf{x})) & D_2g(\mathbf{f}(\mathbf{x})) \end{bmatrix} \begin{bmatrix} D_1f_1(\mathbf{x}) & D_2f_1(\mathbf{x}) \\ D_1f_2(\mathbf{x}) & D_2f_2(\mathbf{x}) \end{bmatrix} \\ &= \begin{bmatrix} D_1g(\mathbf{f}(\mathbf{x})) & D_2g(\mathbf{f}(\mathbf{x})) \end{bmatrix} \begin{bmatrix} 2x_1 & -2x_2 \\ -2x_1 & 2x_2 \end{bmatrix} \end{aligned}$$

for all $\mathbf{x}$ in $\mathbb{R}^2$. Therefore,

$$\begin{aligned} x_2D_1h(\mathbf{x}) + x_1D_2h(\mathbf{x}) &= \begin{bmatrix} D_1h(\mathbf{x}) & D_2h(\mathbf{x}) \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \\ &= \begin{bmatrix} D_1g(\mathbf{f}(\mathbf{x})) & D_2g(\mathbf{f}(\mathbf{x})) \end{bmatrix} \begin{bmatrix} 2x_1 & -2x_2 \\ -2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \\ &= \begin{bmatrix} D_1g(\mathbf{f}(\mathbf{x})) & D_2g(\mathbf{f}(\mathbf{x})) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= 0. \end{aligned}$$

■

8.47 Suppose that f is a real-valued differentiable function defined on an open set U in $\mathbb{R}^n$. Suppose also that

$$x_1D_1f(\mathbf{x}) + x_2D_2f(\mathbf{x}) + \cdots + x_nD_nf(\mathbf{x}) = pf(\mathbf{x}), \tag{8.41}$$

for some constant p in $\mathbb{R}$ and all $\mathbf{x}$ in U. Fix $\mathbf{x}$ in U. Show that there exists an interval $I = (t_1, t_2)$ in $\mathbb{R}$, with $0 < t_1 < 1 < t_2$, such that, for t in I, $f(t\mathbf{x}) = t^pf(\mathbf{x})$. (*Hint:* For t in I, define $g(t) = f(t\mathbf{x})$. Use (8.41) to show that $tg'(t) = pg(t)$. Deduce that $t^{-p}g(t)$ is constant.)

Solution. Since U is open in $\mathbb{R}^n$ and $\mathbf{x} \in U$, there exists some neighborhood $N(\mathbf{x})$ contained in U. Then there must be $t_1 \in (0, 1)$ and $t_2 \in (1, \infty)$ such that both $t_1\mathbf{x}$ and $t_2\mathbf{x}$ are in $N(\mathbf{x})$.

Let $g(t) = f(t\mathbf{x})$ for $t \in I = (t_1, t_2)$. By the chain rule, we have

$$g'(t) = \begin{bmatrix} D_1 f(t\mathbf{x}) & \cdots & D_n f(t\mathbf{x}) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{j=1}^{n} x_j D_j f(t\mathbf{x}).$$

By (8.41),

$$tg'(t) = \sum_{j=1}^{n} t x_j D_j f(t\mathbf{x}) = pf(t\mathbf{x}) = pg(t).$$

If $p = 0$, then $g'(t) = 0$. Assume $p \neq 0$. Multiplying both sides by t^{-p-1},

$$t^{-p} g'(t) - pt^{-p-1} g(t) = \frac{d}{dt}\left[t^{-p} g(t)\right] = 0.$$

Hence, we deduce that $t^{-p} g(t) = C$ for some constant C of t. Since $1 \in I$, $g(1) = f(\mathbf{x}) = C$. Therefore, $f(t\mathbf{x}) = t^p f(\mathbf{x})$ for $t \in I$. ■

8.49 Suppose that f, $D_1 f$, and $D_2 f$ are continuously differentiable on $\mathbb{R}^2$. Suppose also that f is positively homogeneous of degree p. Prove that, for all $\mathbf{x}$ in $\mathbb{R}^2$,

$$x_1^2 D_{11} f(\mathbf{x}) + 2x_1 x_2 D_{12} f(\mathbf{x}) + x_2^2 D_{22} f(\mathbf{x}) = p(p-1) f(\mathbf{x}).$$

Solution. By Exercise 8.46, since f is positively homogeneous of degree p,

$$\begin{bmatrix} D_1 f(\mathbf{x}) & D_2 f(\mathbf{x}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = pf(\mathbf{x}).$$

By Exercise 8.48, $D_1 f$ and $D_2 f$ are also positively homogeneous of degree $p-1$, so

$$\begin{bmatrix} D_{11} f(\mathbf{x}) & D_{21} f(\mathbf{x}) \\ D_{12} f(\mathbf{x}) & D_{22} f(\mathbf{x}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (p-1) \begin{bmatrix} D_1 f(\mathbf{x}) \\ D_2 f(\mathbf{x}) \end{bmatrix}.$$

Then we have

$$p(p-1) f(\mathbf{x}) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} D_{11} f(\mathbf{x}) & D_{12} f(\mathbf{x}) \\ D_{21} f(\mathbf{x}) & D_{22} f(\mathbf{x}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$f \in C^{(2)}$ implies $D_{12} f(\mathbf{x}) = D_{21} f(\mathbf{x})$ by Theorem 8.6.1. Therefore,

$$x_1^2 D_{11} f(\mathbf{x}) + 2x_1 x_2 D_{12} f(\mathbf{x}) + x_2^2 D_{22} f(\mathbf{x}) = p(p-1) f(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^2$. ■

Chapter 9

Vector-Valued Functions

9.12 Let $f_1,\ f_2,\ \cdots,\ f_n$ be continuously differentiable, real-valued functions on $\mathbb{R}$. Let $\mathbf{g} = (g_1, g_2, \cdots, g_n)$ be a continuously differentiable function mapping $\mathbb{R}^n$ into $\mathbb{R}^n$. For any point $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ in $\mathbb{R}^n$ and for each $j \in \{1, 2, \cdots, n\}$, define $h_j(\mathbf{x}) = g_j(f_1(x_1), f_2(x_2), \cdots, f_n(x_n))$. Let $\mathbf{h} = (h_1, h_2, \cdots, h_n)$. Prove that

$$J_{\mathbf{h}}(\mathbf{x}) = J_{\mathbf{g}}(f_1(x_1), f_2(x_2), \cdots, f_n(x_n)) \prod_{j=1}^{n} f_j'(x_j).$$

Solution. Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be

$$\mathbf{F}(\mathbf{x}) = (f_1(x_1), f_2(x_2), \cdots f_n(x_n)) = \begin{bmatrix} f_1(x_1) & f_2(x_2) & \cdots & f_n(x_n) \end{bmatrix}^T.$$

By Theorem 8.3.2, $\mathbf{F}$ is continuously differentiable. By the chain rule, since $h_i = g_i \circ \mathbf{F}$,

$$\begin{bmatrix} D_1 h_i(\mathbf{x}) \\ D_2 h_i(\mathbf{x}) \\ \vdots \\ D_n h_i(\mathbf{x}) \end{bmatrix}^T = \begin{bmatrix} D_1 g_i(\mathbf{F}(\mathbf{x})) \\ D_2 g_i(\mathbf{F}(\mathbf{x})) \\ \vdots \\ D_n g_i(\mathbf{F}(\mathbf{x})) \end{bmatrix}^T \begin{bmatrix} f_1'(x_1) & 0 & \cdots & 0 \\ 0 & f_2'(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_n'(x_n) \end{bmatrix} = \begin{bmatrix} D_1 g_i(\mathbf{F}(\mathbf{x})) f_1'(x_1) \\ D_2 g_i(\mathbf{F}(\mathbf{x})) f_2'(x_2) \\ \vdots \\ D_n g_i(\mathbf{F}(\mathbf{x})) f_n'(x_n) \end{bmatrix}^T.$$

for each $i, j \in \{1, 2, \cdots, n\}$. Therefore,

$$\begin{aligned} J_{\mathbf{h}}(\mathbf{x}) &= \begin{vmatrix} D_1 h_1(\mathbf{x}) & D_2 h_1(\mathbf{x}) & \cdots & D_n h_1(\mathbf{x}) \\ D_1 h_2(\mathbf{x}) & D_2 h_2(\mathbf{x}) & \cdots & D_n h_2(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 h_n(\mathbf{x}) & D_2 h_n(\mathbf{x}) & \cdots & D_n h_n(\mathbf{x}) \end{vmatrix} \\ &= \begin{vmatrix} D_1 g_1(\mathbf{F}(\mathbf{x})) f_1'(x_1) & D_2 g_1(\mathbf{F}(\mathbf{x})) f_2'(x_2) & \cdots & D_n g_1(\mathbf{F}(\mathbf{x})) f_n'(x_n) \\ D_1 g_2(\mathbf{F}(\mathbf{x})) f_1'(x_1) & D_2 g_2(\mathbf{F}(\mathbf{x})) f_2'(x_2) & \cdots & D_n g_2(\mathbf{F}(\mathbf{x})) f_n'(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 g_n(\mathbf{F}(\mathbf{x})) f_1'(x_1) & D_2 g_n(\mathbf{F}(\mathbf{x})) f_2'(x_2) & \cdots & D_n g_n(\mathbf{F}(\mathbf{x})) f_n'(x_n) \end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{vmatrix} D_1g_1(\mathbf{F}(\mathbf{x})) & D_2g_1(\mathbf{F}(\mathbf{x})) & \cdots & D_ng_1(\mathbf{F}(\mathbf{x})) \\ D_1g_2(\mathbf{F}(\mathbf{x})) & D_2g_2(\mathbf{F}(\mathbf{x})) & \cdots & D_ng_2(\mathbf{F}(\mathbf{x})) \\ \vdots & \vdots & \ddots & \vdots \\ D_1g_n(\mathbf{F}(\mathbf{x})) & D_2g_n(\mathbf{F}(\mathbf{x})) & \cdots & D_ng_n(\mathbf{F}(\mathbf{x})) \end{vmatrix} \prod_{j=1}^{n} f_j'(x_j) \\
&= J_{\mathbf{g}}(f_1(x_1), f_2(x_2), \cdots, f_n(x_n)) \prod_{j=1}^{n} f_j'(x_j).
\end{aligned}
$$

■

Chapter 10

Multiple Integrals

10.18 Use an appropriate change of variables to evaluate the integral $\iint_T (1+x^2+y^2)^{-2}\,dA$, where T is the triangle with vertices $(0,0)$, $(2,0)$, and $(0,\sqrt{3})$.

Solution. We have

$$T = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x \geq 0,\ y \geq 0,\ \frac{x}{2} + \frac{y}{\sqrt{3}} \leq 1 \right\}.$$

Define $\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} h_1(r,\theta) \\ h_2(r,\theta) \end{bmatrix} = \begin{bmatrix} 2r\cos\theta \\ \sqrt{3}r\sin\theta \end{bmatrix}.$$

Then

$$J_{\mathbf{h}}(r,\theta) = \begin{vmatrix} \dfrac{\partial h_1}{\partial r} & \dfrac{\partial h_1}{\partial \theta} \\ \dfrac{\partial h_2}{\partial r} & \dfrac{\partial h_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2\cos\theta & -2r\sin\theta \\ \sqrt{3}\sin\theta & \sqrt{3}r\cos\theta \end{vmatrix} = 2\sqrt{3}r.$$

Therefore, by Theorem 10.3.3,

$$\begin{aligned}
\iint_T (1+x^2+y^2)^{-2}\,dA &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\theta+\cos\theta}} |J_{\mathbf{h}}(r,\theta)| \left[1 + (2r\cos\theta)^2 + (\sqrt{3}r\sin\theta)^2\right]^{-2} drd\theta \\
&= 2\sqrt{3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\theta+\cos\theta}} \frac{r}{\left[(3+\cos^2\theta)r^2+1\right]^2}\, drd\theta \\
&= \sqrt{3} \int_0^{\frac{\pi}{2}} \frac{1}{3+\cos^2\theta} \left[-\frac{1}{(3+\cos^2\theta)r^2+1}\right]_0^{\frac{1}{\sin\theta+\cos\theta}} d\theta \\
&= \sqrt{3} \int_0^{\frac{\pi}{2}} \frac{1}{3+\cos^2\theta} \left[1 - \frac{(\sin\theta+\cos\theta)^2}{(3+\cos^2\theta)+(\sin\theta+\cos\theta)^2}\right] d\theta \\
&= \sqrt{3} \int_0^{\frac{\pi}{2}} \frac{1}{4+2\sin\theta\cos\theta+\cos^2\theta}\, d\theta \\
&= \sqrt{3} \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{4\sec^2\theta+2\tan\theta+1}\, d\theta \\
&= \sqrt{3} \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{4\tan^2\theta+2\tan\theta+5}\, d\theta \\
&= \sqrt{3} \int_0^{\infty} \frac{1}{4x^2+2x+5}\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{3}\left[\frac{2}{2\sqrt{19}}\tan^{-1}\left(\frac{8x+2}{2\sqrt{19}}\right)\right]_0^\infty \\
&= \sqrt{\frac{3}{19}}\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{\sqrt{19}}\right).
\end{aligned}
$$

■

Remark. Recall that

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) + C$$

if $b^2 - 4ac < 0$.

Chapter 11

Infinite Series

11.10 For which values of p in $\mathbb{R}$ does $\sum_{j=2}^{\infty} 1/(\ln j)^p$ converge?

Solution 1. Let $a_j = 1/(\ln j)^p$.

(i) If $p \leq 0$, then $\lim\limits_{j\to\infty} a_j \neq 0$, so the series diverges by the term test.

(ii) Assume $p > 0$. Let $f(x) = 1/(\ln x)^p$ for $x \in [2, \infty)$. Note that f is a positive, continuous, and nonincreasing function. We have
$$\int_2^{\infty} \frac{1}{(\ln x)^p}\, dx = \int_{\ln 2}^{\infty} \frac{e^t}{t^p}\, dt = \infty$$
since $e^t/t^p \to \infty$ as $t \to \infty$. By the integral test, the series diverges.

Therefore, the series diverges regardless of the value of p. ■

Solution 2. Recall that, for any $\epsilon > 0$,
$$\lim_{j\to\infty} \frac{\ln j}{j^{\epsilon}} = \lim_{j\to\infty} \frac{1/j}{\epsilon j^{\epsilon-1}} = \frac{1}{\epsilon} \lim_{j\to\infty} \frac{1}{j^{\epsilon}} = 0.$$
Hence, for any $p > 0$,
$$\lim_{j\to\infty} \frac{1/(j^{p\epsilon})}{1/(\ln j)^p} = \lim_{j\to\infty} \left(\frac{\ln j}{j^{\epsilon}}\right)^p = \left(\lim_{j\to\infty} \frac{\ln j}{j^{\epsilon}}\right)^p = 0.$$
The series $\sum_{j=2}^{\infty} 1/(j^{p\epsilon})$ diverges if $p\epsilon \leq 1$ or $p \leq 1/\epsilon$. We can make $1/\epsilon$ arbitrarily big. Therefore, the series always diverges. ■

11.19 For constants a and b such that $0 < a < b$, consider the series
$$1 + a + ab + a^2b + a^2b^2 + a^3b^2 + a^3b^3 + a^4b^3 + \cdots.$$

a) Show that, if $b < 1$, then the ratio test proves that the series converges. Show, however, if $a < 1 < b$, then the ratio test fails to resolve the question.

b) Suppose that $a < 1 < b$. Apply the root test to show that the series converges if $ab < 1$ and diverges if $ab > 1$. Explain separately why the series diverges if $ab = 1$. (This shows that the root test truly is stronger than the ratio test.)

Solution. Let $x_1 = 1$ and

$$x_{j+1} = \begin{cases} ax_j & \text{if } j \text{ is odd,} \\ bx_j & \text{if } j \text{ is even.} \end{cases}$$

a) Assume $b < 1$. By the ratio test (Theorem 11.2.6),

$$\limsup \frac{x_{j+1}}{x_j} = b < 1.$$

However, if $a < 1 < b$, then

$$\liminf \frac{x_{j+1}}{x_j} = a < 1 < b = \limsup \frac{x_{j+1}}{x_j},$$

so the ratio test is inconclusive.

b) Observe that

$$a^{\frac{j-1}{2}} b^{\frac{j-3}{2}} \leq x_j = a^{\left\lceil \frac{j-1}{2} \right\rceil} b^{\left\lfloor \frac{j-1}{2} \right\rfloor} \leq a^{\frac{j+1}{2}} b^{\frac{j-1}{2}}$$

for each $j \in \mathbb{N}$. Thus we know that

$$\lim_{j\to\infty} (x_j)^{1/j} = \sqrt{ab}.$$

By the root test, the series converges if $ab < 1$ and diverges if $ab > 1$. If $ab = 1$, then

$$x_j = \frac{a+1}{2} + (-1)^j \frac{a-1}{2},$$

so the series diverges by the term test.

■

11.26 Prove that, for each of the series $\sum_{j=1}^\infty \left[(2j)!/(2^j j!)^2\right]^2$ and $\sum_{j=2}^\infty 1/\left[j(\ln j)^2\right]$, Raabe's test (Exercise 11.22) gives a value of $L = 1$. Use other methods to show that the first series diverges and the second converges, thus showing that Raabe's test truly gives no information when $L = 1$.

Solution. Let $a_j = \left[(2j)!/(2^j j!)^2\right]^2$. Then we have

$$\begin{aligned} L &= \lim_{j\to\infty} j\left(1 - \frac{a_{j+1}}{a_j}\right) = \lim_{j\to\infty} j\left(1 - \left[\frac{(2j+2)!}{(2j)!}\right]^2 \left[\frac{2^j j!}{2^{j+1}(j+1)!}\right]^4\right) \\ &= \lim_{j\to\infty} j\left[1 - \frac{(2j+2)^2(2j+1)^2}{2^4(j+1)^4}\right] = \lim_{j\to\infty} j\left[\frac{(2j+2)^2 - (2j+1)^2}{(2j+2)^2}\right] \\ &= \lim_{j\to\infty} \frac{4j^2+3j}{4j^2+8j+4} = 1. \end{aligned}$$

Observe that

$$a_j = \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\dots\cdot\frac{2j-1}{2j}\right)^2 \geq \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\dots\cdot\frac{2j-2}{2j-1}\cdot\frac{2j-1}{2j} = \frac{1}{4j}$$

for each $j \in \mathbb{N}$. $\sum_{j=1}^\infty 1/(4j)$ is a divergent p-series. By the comparison test, the series diverges.

Let $b_j = 1/\left[j(\ln j)^2\right]$. Let $f(x) = x(\ln x)^2$. By the mean value theorem,

$$\begin{aligned} j\left(1 - \frac{b_{j+1}}{b_j}\right) &= j\left[1 - \frac{f(j)}{f(j+1)}\right] = j \cdot \frac{f(j+1) - f(j)}{f(j+1)} \\ &= \frac{jf'(c)}{f(j+1)} = \frac{j(\ln c)^2 + 2j\ln c}{(j+1)[\ln(j+1)]^2} \end{aligned}$$

for some $c \in (j, j+1)$. Thus we have the inequality

$$\frac{j(\ln j)^2 + 2j\ln j}{(j+1)[\ln(j+1)]^2} \leq j\left(1 - \frac{b_{j+1}}{b_j}\right) \leq \frac{j[\ln(j+1)]^2 + 2j\ln(j+1)}{(j+1)[\ln(j+1)]^2}$$

for each $j \in \mathbb{N}$. By the Squeeze Play, $L = 1$.

Note that $1/f$ is a positive, continuous, and nonincreasing function on $[2, \infty)$. Because

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \int_{\ln 2}^\infty \frac{1}{t^2}\,dt = \lim_{k\to\infty}\left(\frac{1}{\ln 2} - \frac{1}{\ln k}\right) = \frac{1}{\ln 2},$$

it follows that the series converges by the integral test. ■

Chapter 12

Series of Functions

12.6 For k in $\mathbb{N}$ and x in $[0,\infty)$, define

$$f_k(x) = \frac{xe^{-x/k}}{k}.$$

a) Find the pointwise limit f of $\{f_k\}$. Show that the convergence is not uniform on $[0,\infty)$. However, given any $b>0$, show that $\{f_k\}$ converges uniformly to f on $[0,b]$.

b) For each $b>0$ and k in $\mathbb{N}$, compute $\int_0^b f_k(x)\,dx$.

i) Compute $\lim_{k\to\infty}\int_0^b f_k(x)\,dx$.

ii) Compute $\lim_{b\to\infty}\int_0^b f_k(x)\,dx$.

Hence, determine whether

$$\lim_{b\to\infty}\lim_{k\to\infty}\int_0^b f_k(x)\,dx = \lim_{k\to\infty}\lim_{b\to\infty}\int_0^b f_k(x)\,dx.$$

Solution.

a) We claim that $f(x) = \lim_{k\to\infty} f_k(x) = 0$. Let $\epsilon>0$ be given. Choose $k_0(x)\in\mathbb{N}$ such that $x/k_0(x)<\epsilon$. Then

$$|f_k(x) - f(x)| = \frac{x}{k}e^{-\frac{x}{k}} \le \frac{x}{k_0(x)}\cdot 1 < \epsilon$$

whenever $k \ge k_0(x)$. The convergence is not uniform, however, because $f_k(k) = 1/e$ for each $k\in\mathbb{N}$. Therefore, $f_k \notin N_\infty(f;1/e)$ on $[0,\infty)$.

Restrict the domain to $[0,b]$ for any $b>0$. Let $\epsilon>0$ be given. Choose $k_0\in\mathbb{N}$ such that $k_0 > b/\epsilon$. Then, for $k\ge k_0$,

$$\|f_k(x) - f(x)\|_\infty = \left\|\frac{x}{k}e^{-\frac{x}{k}}\right\|_\infty \le \frac{\|x\|_\infty}{k_0}\cdot 1 < \frac{b}{b/\epsilon} = \epsilon,$$

so $f_k\in N_\infty(f;\epsilon)$ on $[0,b]$. Therefore, $\{f_k\}$ converges uniformly to f on $[0,b]$.

b) For each $b>0$ and k in $\mathbb{N}$,

$$\int_0^b f_k(x)\,dx = \int_0^b \frac{x}{k}e^{-\frac{x}{k}}\,dx = k\int_0^{\frac{b}{k}} te^{-t}\,dt = k\left[-(t+1)e^{-t}\right]_0^{b/k} = k-(b+k)e^{-b/k}.$$

i) Let $g(x) = e^{-bx}$. Then $g'(x) = -be^{-bx}$.

$$\lim_{k\to\infty} k\left(1 - e^{-b/k}\right) = -\lim_{k\to\infty} \frac{g(1/k) - g(0)}{1/k} = -g'(0) = b.$$

Therefore,

$$\lim_{k\to\infty} \int_0^b f_k(x)\,dx = \lim_{k\to\infty} \left[k\left(1 - e^{-b/k}\right) - be^{-b/k}\right] = b - b = 0.$$

ii) By L'Hôpital's rule,

$$\lim_{b\to\infty} (b+k)e^{-b/k} = \lim_{b\to\infty} \frac{b+k}{e^{b/k}} = \lim_{b\to\infty} \frac{k}{e^{b/k}} = 0.$$

Therefore,

$$\lim_{b\to\infty} \int_0^b f_k(x)\,dx = k.$$

Hence,

$$0 = \lim_{b\to\infty} \lim_{k\to\infty} \int_0^b f_k(x)\,dx \neq \lim_{k\to\infty} \lim_{b\to\infty} \int_0^b f_k(x)\,dx = \infty.$$

■

12.16 For x in $\mathbb{R}$, consider the series

$$\sum_{j=1}^{\infty} \frac{\cos(2j-1)x}{2j(2j-1)}.$$

a) Find all x where the series converges.

b) For which x is the convergence absolute? For which x is the convergence conditional?

c) Where is the convergence uniform?

Solution. Let $f_j = \dfrac{\cos(2j-1)x}{2j(2j-1)}$. Then we have that

$$\|f_j\|_\infty \leq \frac{1}{2j(2j-1)} = \frac{1}{2j-1} - \frac{1}{2j}$$

and that

$$\sum_{j=1}^{\infty} \left(\frac{1}{2j-1} - \frac{1}{2j}\right) = \ln 2.$$

By Weierstrass's M-test (Theorem 12.3.1), the series converges uniformly and absolutely on $\mathbb{R}$. ■