1 Show that $\lim_{x\to 0} \sin(1/x)$ does not exist.

10 pts

Solution. Suppose, to the contrary, that there exists an $L \in \mathbb{R}$ such that $L = \lim_{x\to 0} \sin(1/x)$. Let $\epsilon = 1/2$; then there exists a $\delta > 0$ such that $|\sin(1/x) - L| < \epsilon = 1/2$. According to the Archimedean property of the real numbers, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{2n\pi + \pi/2} < \frac{1}{2n\pi} < \delta.$$

If $x = \dfrac{1}{2n\pi}$, $|\sin(1/x) - L| = |L| < 1/2$. If $x = \dfrac{1}{2n\pi + \pi/2}$, $|\sin(1/x) - L| = |1 - L| < 1/2$. There is no real number L satisfying both $|L| < 1/2$ and $|1 - L| < 1/2$, so our assumption leads to a contradiction. Therefore, $\lim_{x\to 0} \sin(1/x)$ does not exist. □

2 Thomae's function, also known as the ruler function, is defined as

25 pts

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ (x is rational), with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ coprime,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Since every rational number has a unique representation with coprime $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, the function is well-defined. Note that $q = 1$ is the only number in $\mathbb{N}$ that is coprime to $p = 0$. Prove the following properties of this function.

(a) f is **periodic** with period **1**: $f(x+n) = f(x)$ for all integer n and all real x.

2 pts

Solution. For all $x \in \mathbb{R} \setminus \mathbb{Q}$, $f(x+n) = f(x) = 0$, since $(x+n) \in \mathbb{R} \setminus \mathbb{Q}$ implies $x \in \mathbb{R} \setminus \mathbb{Q}$. For all $x \in \mathbb{Q}$, there exists a unique pair of coprime integers $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $x = p/q$. Consider $x + n = (p + nq)/q$. If d divides p and q, it divides $p + nq$ and p. Conversely, if d divides $p + nq$ and q, it divides $(p + nq) - nq = q$ and p. Hence, $\gcd(p + nq, q) = \gcd(p, q) = 1$. Therefore, $f(x + n) = 1/q = f(x)$. □

(b) f is **discontinuous** at all rational numbers.

4 pts

Solution. Assume an arbitrary rational $c = p/q$, with a unique pair of coprime integers $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then $f(c) = 1/q$. Pick $\epsilon = 1/(2q)$. No matter how small $\delta > 0$ is taken, there is an irrational number $x \in (c - \delta,\, c + \delta)$ which implies

$$|f(x) - f(c)| = \left|0 - \frac{1}{q}\right| = \frac{1}{q} > \frac{1}{2q} = \epsilon.$$

Therefore, f is discontinuous on $\mathbb{Q}$. □

(c) f is **continuous** at all irrational numbers.

6 pts

Solution. Since f is periodic with period 1, and $0 \in \mathbb{Q}$, it suffices to check all irrational points in the interval $[0,\, 1]$. Assume an arbitrary irrational $c \in [0,\, 1] \setminus \mathbb{Q}$. Then $f(c) = 0$. Let $\epsilon > 0$ be given. According to the Archimedean property of the real numbers, there exists $k \in \mathbb{N}$ such that $1/k < \epsilon$. Define a set S as

$$S = \left\{ r \in [0,\, 1] \cap \mathbb{Q} \;\middle|\; f(r) \geq \frac{1}{k} \right\}.$$

There exist only finitely many rational numbers (reduced to lowest terms) in I having denominator not greater than k, so the set S is finite. Take $\delta = \min\{|c - r| : r \in S\}$. Then,

$$|x - c| < \delta \implies |f(x) - f(c)| \leq \frac{1}{k} < \epsilon.$$

Therefore, f is continuous on $\mathbb{R} \setminus \mathbb{Q}$. □

(d) f is **nowhere differentiable**.

5 pts

Solution. For rational numbers, this follows from discontinuity. Let $x \in \mathbb{R} \setminus \mathbb{Q}$, and let $\delta > 0$ be given. By Archimedes' principle, choose $q \in \mathbb{N}$ such that $1/q < \delta$. Also, there exists $p \in \mathbb{Z}$ such that $p - 1 \leq qx < p$. Then $x < r = p/q \leq x + 1/q$ implies $r \in N(x, \delta) \cap \mathbb{Q}$ and $f(r) \geq 1/q$. We have

$$\left| \frac{f(r) - f(x)}{r - x} \right| \geq 1,$$

but for $y \in N(x, \delta) \setminus \mathbb{Q}$,

$$\left| \frac{f(y) - f(x)}{y - x} \right| = 0.$$

Therefore, the limit $\lim_{t \to x} \dfrac{f(t) - f(x)}{t - x}$ does not exist, so f is not differentiable on $\mathbb{R} \setminus \mathbb{Q}$. □

(e) f is Riemann **integrable** on any interval and the integral evaluates to 0 over any set.

8 pts

Solution. We will check Riemann integrability of Thomae's function over an arbitrary interval $[a,\, b]$. Let $\epsilon > 0$ be given. According to Archimedean property of the real numbers, there exists $k \in \mathbb{N}$ such that $\dfrac{1}{k} < \dfrac{\epsilon}{2(b-a)}$. Define a set S as

$$S = \left\{ r \in \mathbb{Q} \cap [a,\, b] \;\middle|\; f(r) \geq \frac{1}{k} \right\}.$$

There exist only finitely many rational numbers (reduced to lowest terms) in $[a,\, b]$ having denominator not greater than N, so the set S is finite. Let $\{y_1,\, y_2,\, \cdots,\, y_n\}$ be an enumeration of points in S such that $0 < y_1 < y_2 < \cdots < y_n = 1$. Choose partition points $x_0,\, x_1,\, x_2,\, \cdots,\, x_{2n}$ such that

$$0 = x_0 < x_1 < y_1 < x_2 < x_3 < y_2 < x_4 < \cdots < x_{2n-1} < y_n = x_{2n} = 1$$

and such that $y_j - \dfrac{\epsilon}{4n} < x_{2j-1} < y_j \leq x_{2j} < y_j + \dfrac{\epsilon}{4n}$ so that $\displaystyle\sum_{j=1}^{n} \Delta x_{2j} < \sum_{j=1}^{n} \frac{\epsilon}{2n} < \frac{\epsilon}{2}$.

$$\begin{aligned}
U(f,\, P) - L(f,\, P) &= \sum_{j=1}^{n} \sup f([x_{2j-1}, x_{2j}]) \Delta x_{2j} + \sum_{j=0}^{n-1} \sup f([x_{2j}, x_{2j+1}]) \Delta x_{2j+1} \\
&< \sum_{j=1}^{n} \Delta x_{2j} + \frac{\epsilon}{2(b-a)} \sum_{j=0}^{n-1} \Delta x_{2j+1} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2(b-a)} \times (b-a) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

For any refinement P of P_0, $U(f, P) - L(f, P) = U(f, P) \leq U(f, P_0) < \epsilon$. Therefore, f is Riemann integrable on $[a, b]$. Since $L(f, P)$ is always zero, $\displaystyle\int_a^b f(x)\, dx = 0$. □

3 Let $\mathscr{R}$ denote the set of all Riemann integrable functions. If two functions $f \in \mathscr{R}$ and
10 pts $g \in \mathscr{R}$, then the composition $f \circ g \in \mathscr{R}$? If true, prove it; if false, give a counterexample.

Solution. **False.** Let

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \le 0 \end{cases}$$

and

$$g(x) = \begin{cases} 1/q & \text{if } x = p/q \ (x \text{ is rational}), \text{ with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ coprime}, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

These functions f and g are both Riemann integrable, but the composition $f \circ g$ is the Dirichlet function

$$(f \circ g)(x) = \begin{cases} 1 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}, \end{cases}$$

which is **NOT** Riemann integrable. Therefore, a composition of two Riemann integrable functions is not necessarily Riemann integrable. $\square$

4 Show that $\dfrac{\sin x_2 - \sin x_1}{x_2 - x_1}$ does **NOT** attain neither maximum nor minimum values for two distinct arbitrary real numbers x_1 and x_2.

10 pts

Solution. By the mean value theorem, there exists a $c \in \mathbb{R}$ strictly between x_1 and x_2 such that

$$\left|\frac{\sin x_2 - \sin x_1}{x_2 - x_1}\right| = |\cos c| \leq 1.$$

Let $x_1 = 0$. Taking limits where $x_2 = x$ approaches to 0, we have

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

1 is the only possible maximum value. Assume, for the sake of contradiction, there exist two distinct real numbers x_1 and x_2 such that

$$\begin{aligned}
\frac{\sin x_2 - \sin x_1}{x_2 - x_1} &= 1, \\
\sin x_2 - \sin x_1 &= x_2 - x_1, \\
\sin x_1 - x_1 &= \sin x_2 - x_2.
\end{aligned}$$

Let $f(x) = \sin x - x$. To check if f is injective, one evaluates its derivative $f'(x) = \cos x - 1 \leq 0$. f is strictly decreasing because $f'(x) < 0$ almost everywhere. f is injective, so there exist no such two distinct real numbers. Thus, $\dfrac{\sin x_2 - \sin x_1}{x_2 - x_1}$ does **NOT** attain a maximum value.

Similarly, one can also prove that it does **NOT** attain a minimum value either. $\square$

5 Show that $\cosh(\sinh x) < \sinh(\cosh x)$ for any real number x.

10 pts

Solution. The inequality trivially holds for $x = 0$ because

$$\cosh(\sinh 0) = \cosh 0 = 1 < \sinh 1 = \sinh(\cosh 0).$$

Plugging $-x$ to x gives the same inequality, so it is enough to show that the inequality holds for any positive number x. It follows from

$$\begin{aligned} e^y - 1 &= \int_0^y e^t\,dt \\ &> \int_0^y (1+t)\,dt \\ &> \int_0^y \left(1 + \frac{t}{\sqrt{1+t^2}}\right)dt \\ &= y + \sqrt{1+y^2} - 1 \end{aligned}$$

that $e^y > y + \sqrt{1+y^2}$ for any positive number y. By the mean value theorem, there exists a $c \in \left(y,\, \sqrt{1+y^2}\right)$ such that

$$\begin{aligned} \sinh\sqrt{1+y^2} - \sinh y &= \left(\sqrt{1+y^2} - y\right)\cosh c \\ &= \frac{\cosh c}{\sqrt{1+y^2} + y} \\ &> e^{-y}\cosh y \\ &\geq e^{-y} \\ &= \cosh y - \sinh y. \end{aligned}$$

Therefore, $\cosh y < \sinh\sqrt{1+y^2}$. Letting $y = \sinh x$ gives $\cosh(\sinh x) < \sinh(\cosh x)$. $\square$

6 Evaluate the following integrals.

25 pts

(a) $\displaystyle\int \frac{1}{x^2+1}\,dx$

3 pts

Solution. Using the substitution $x = \tan t$, we have

$$\int \frac{1}{x^2+1}\,dx = \int \frac{1}{\sec^2 t}\sec^2 t\,dt = t + C = \tan^{-1} x + C.$$

□

(b) $\displaystyle\int \frac{1}{ax^2+bx+c}\,dx \qquad (b^2 - 4ac < 0)$

7 pts

Solution. From the condition $b^2 - 4ac < 0$, we assume $a \neq 0$ and $c \neq 0$.
Let $p = \dfrac{b}{2a}$ and $q = \dfrac{4ac-b^2}{4a^2}$, then

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a}\int \frac{1}{\left(x+\dfrac{b}{2a}\right)^2 + \dfrac{4ac-b^2}{4a^2}}\,dx = \frac{1}{a}\int \frac{1}{(x+p)^2+q}\,dx.$$

Using the substitution $x + p = \sqrt{q}\tan t$, we have

$$\begin{aligned}
\frac{1}{a}\int \frac{1}{(x+p)^2+q}\,dx &= \frac{1}{a}\int \frac{1}{q\sec^2 t}\sqrt{q}\sec^2 t\,dt \\
&= \frac{t}{a\sqrt{q}} + C \\
&= \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{x+p}{\sqrt{q}}\right) + C \\
&= \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) + C.
\end{aligned}$$

□

(c) $\displaystyle\int \sqrt{\tan x}\, dx$

15 pts

Solution 1. Using the substitution $\tan x = t^2$, we have $dx = \dfrac{2t}{t^4+1}dt$, and

$$\begin{aligned}
\int \sqrt{\tan x}\, dx &= 2\int \frac{t^2}{t^4+1}\, dt \\
&= 2\int \frac{t^2}{\left(t^2-\sqrt{2}t+1\right)\left(t^2+\sqrt{2}t+1\right)}\, dt \\
&= \frac{1}{\sqrt{2}}\int \left(\frac{t}{t^2-\sqrt{2}t+1} - \frac{t}{t^2+\sqrt{2}t+1}\right)\, dt.
\end{aligned}$$

Now solving: $\displaystyle\int \frac{t}{t^2-\sqrt{2}t+1}\, dt$

$$\begin{aligned}
\int \frac{t}{t^2-\sqrt{2}t+1}\, dt &= \int \frac{\frac{1}{2}\left(2t-\sqrt{2}\right)+\frac{1}{\sqrt{2}}}{t^2-\sqrt{2}t+1}\, dt \\
&= \frac{1}{2}\int \frac{2t-\sqrt{2}}{t^2-\sqrt{2}t+1}\, dt + \frac{1}{\sqrt{2}}\int \frac{1}{t^2-\sqrt{2}t+1}\, dt \\
&= \frac{1}{2}\ln\left(t^2-\sqrt{2}t+1\right) + \tan^{-1}\left(\sqrt{2}t-1\right) + C_1.
\end{aligned}$$

Now solving: $\displaystyle\int \frac{t}{t^2+\sqrt{2}t+1}\, dt$

$$\begin{aligned}
\int \frac{t}{t^2+\sqrt{2}t+1}\, dt &= \int \frac{\frac{1}{2}\left(2t+\sqrt{2}\right)-\frac{1}{\sqrt{2}}}{t^2+\sqrt{2}t+1}\, dt \\
&= \frac{1}{2}\int \frac{2t+\sqrt{2}}{t^2+\sqrt{2}t+1}\, dt - \frac{1}{\sqrt{2}}\int \frac{1}{t^2+\sqrt{2}t+1}\, dt \\
&= \frac{1}{2}\ln\left(t^2+\sqrt{2}t+1\right) - \tan^{-1}\left(\sqrt{2}t+1\right) + C_2.
\end{aligned}$$

Plugging in solved integrals, we have

$$\begin{aligned}
\int \sqrt{\tan x}\, dx =& \frac{1}{2\sqrt{2}}\ln\left(\frac{\tan x-\sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\right) \\
&+ \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\sqrt{2\tan x}+1\right) + \tan^{-1}\left(\sqrt{2\tan x}-1\right)\right] + C.
\end{aligned}$$

□

Solution 2. Let $I = \displaystyle\int \sqrt{\tan x}\, dx$, $J = \displaystyle\int \sqrt{\cot x}\, dx$. Using the substitution $\tan x = t^2$, we have $dx = \dfrac{2t}{t^4+1} dt$, and

$$\begin{aligned}
I + J &= \int \left(t + \frac{1}{t}\right) \frac{2t}{t^4+1}\, dt \\
&= 2\int \frac{1 + \dfrac{1}{t^2}}{\left(t - \dfrac{1}{t}\right)^2 + 2}\, dt \\
&= 2\int \frac{\sqrt{2}\sec^2 u}{2\sec^2 u}\, du \qquad \left(t - \frac{1}{t} = \sqrt{2}\tan u\right) \\
&= \sqrt{2}u + C_1 \\
&= \sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}}\right) + C_1.
\end{aligned}$$

Also,

$$\begin{aligned}
I - J &= \int \left(t - \frac{1}{t}\right) \frac{2t}{t^4+1}\, dt \\
&= 2\int \frac{1 - \dfrac{1}{t^2}}{\left(t + \dfrac{1}{t}\right)^2 - 2}\, dt \\
&= 2 \cdot \frac{1}{2\sqrt{2}} \int \left(\frac{1}{u - \sqrt{2}} - \frac{1}{u + \sqrt{2}}\right) du \qquad \left(t + \frac{1}{t} = u \geq 2\right) \\
&= \frac{1}{\sqrt{2}} \ln\left(\frac{u - \sqrt{2}}{u + \sqrt{2}}\right) + C_2 \\
&= \frac{1}{\sqrt{2}} \ln\left(\frac{\sqrt{\tan x} + \sqrt{\cot x} - \sqrt{2}}{\sqrt{\tan x} + \sqrt{\cot x} + \sqrt{2}}\right) + C_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I &= \frac{(I+J) + (I-J)}{2}, \\
\therefore \int \sqrt{\tan x}\, dx &= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + \frac{1}{2\sqrt{2}} \ln\left(\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right) + C.
\end{aligned}$$

□

Solution 3. Let $I = \displaystyle\int \sqrt{\tan x}\, dx$, $J = \displaystyle\int \sqrt{\cot x}\, dx$. For x such that both $\sin x$ and $\cos x$ are positive,

$$\begin{aligned}
J + I &= \int \frac{\cos x + \sin x}{\sqrt{\cos x \sin x}}\, dx \\
&= \int \frac{\cos x + \sin x}{\sqrt{\dfrac{1 - (\sin x - \cos x)^2}{2}}}\, dx \\
&= \sqrt{2} \int \frac{1}{\sqrt{1 - t^2}}\, dt \qquad (\sin x - \cos x = t) \\
&= \sqrt{2} \sin^{-1} t + C_1 \\
&= \sqrt{2} \sin^{-1} (\sin x - \cos x) + C_1.
\end{aligned}$$

Also,

$$\begin{aligned}
J - I &= \int \frac{\cos x - \sin x}{\sqrt{\cos x \sin x}}\, dx \\
&= \int \frac{\cos x - \sin x}{\sqrt{\dfrac{(\sin x + \cos x)^2 - 1}{2}}}\, dx \\
&= \sqrt{2} \int \frac{1}{\sqrt{t^2 - 1}}\, dt \qquad (\sin x + \cos x = t) \\
&= \sqrt{2} \cosh^{-1} t + C_2 \\
&= \sqrt{2} \cosh^{-1} (\sin x + \cos x) + C_2.
\end{aligned}$$

For x such that both $\sin x$ and $\cos x$ are negative, $\sqrt{\sin x}\sqrt{\cos x} = -\sqrt{\sin x \cos x}$. Therefore,

$$\begin{aligned}
I &= \frac{(J + I) - (J - I)}{2}, \\
\therefore &\int \sqrt{\tan x}\, dx \\
&= C + \begin{cases} \dfrac{1}{\sqrt{2}} \sin^{-1}(\sin x - \cos x) - \dfrac{1}{\sqrt{2}} \cosh^{-1}(\sin x + \cos x) & \text{if } \sin x > 0 \text{ and } \cos x > 0, \\ -\dfrac{1}{\sqrt{2}} \sin^{-1}(\sin x - \cos x) - \dfrac{1}{\sqrt{2}} \cosh^{-1}(-\sin x - \cos x) & \text{if } \sin x < 0 \text{ and } \cos x < 0. \end{cases}
\end{aligned}$$

□