1 Solve the differential equation $(y')^2 + 1 = ky^2$ for a constant k.

Solution. $y = \dfrac{1}{\sqrt{k}}$ is a constant solution. Solving for y' gives

$$y' = \frac{dy}{dx} = \pm\sqrt{ky^2 - 1}.$$

Notice that this equation is autonomous and separable. For $dy/dx = \sqrt{ky^2-1}$,

$$\begin{aligned}
\frac{dy}{\sqrt{ky^2-1}} &= dx \\
\frac{\ln\left(\sqrt{k}y + \sqrt{ky^2-1}\right)}{\sqrt{k}} &= x + c \qquad \text{(for some constant } c\text{)} \\
\sqrt{k}y + \sqrt{ky^2-1} &= e^{\sqrt{k}(x+c)} \\
\left(\sqrt{k}y - e^{\sqrt{k}(x+c)}\right)^2 &= \left(\sqrt{ky^2-1}\right)^2 \\
ky^2 - 2\sqrt{k}ye^{\sqrt{k}(x+c)} + e^{2\sqrt{k}(x+c)} &= ky^2 - 1 \\
\therefore y &= \frac{e^{\sqrt{k}(x+c)} + e^{-\sqrt{k}(x+c)}}{2\sqrt{k}}.
\end{aligned}$$

For $dy/dx = -\sqrt{ky^2-1}$,

$$y = \frac{e^{\sqrt{k}(-x+c)} + e^{-\sqrt{k}(-x+c)}}{2\sqrt{k}} = \frac{e^{\sqrt{k}(x-c)} + e^{-\sqrt{k}(x-c)}}{2\sqrt{k}}.$$

Therefore, the general solution is

$$y = \frac{e^{\sqrt{k}(x+c)} + e^{-\sqrt{k}(x+c)}}{2\sqrt{k}}.$$

□

2 Solve the differential equation $x^2y'' + 5xy' + 4y = x\ln x$.

Solution. Recognize the associated homogeneous equation as a Cauchy-Euler equation. Try $y = x^m$ to get the auxiliary equation $m(m-1)+5m+4 = (m+2)^2 = 0$. Since $m_1 = m_2 = -2$ is a repeated root, the complementary solution is $y_c(x) = c_1x^{-2} + c_2x^{-2}\ln x$.

Using the method of variation of parameters, we rewrite the equation in the standard form

$$y'' + \frac{5}{x}y' + \frac{4}{x^2}y = \frac{\ln x}{x},$$

and seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = u_1(x)x^{-2} + u_2(x)x^{-2}\ln x,$$

where

$$u_1'(x) = \frac{W_1}{W} = \frac{\begin{vmatrix} 0 & x^{-2}\ln x \\ \dfrac{\ln x}{x} & x^{-3}(1-2\ln x) \end{vmatrix}}{\begin{vmatrix} x^{-2} & x^{-2}\ln x \\ -2x^{-3} & x^{-3}(1-2\ln x) \end{vmatrix}} = -x^2(\ln x)^2,$$

$$u_2'(x) = \frac{W_2}{W} = \frac{\begin{vmatrix} x^{-2} & 0 \\ -2x^{-3} & \dfrac{\ln x}{x} \end{vmatrix}}{\begin{vmatrix} x^{-2} & x^{-2}\ln x \\ -2x^{-3} & x^{-3}(1-2\ln x) \end{vmatrix}} = x^2\ln x.$$

Using integration by parts, we have

$$\begin{aligned} u_1(x) &= -\int x^2(\ln x)^2\,dx \\ &= -\frac{1}{3}x^3(\ln x)^2 + \frac{2}{3}\int x^3\ln x\,dx \\ &= -\frac{1}{3}x^3(\ln x)^2 + \frac{2}{9}x^3\ln x - \frac{2}{9}\int x^4\,dx \\ &= -\frac{1}{3}x^3(\ln x)^2 + \frac{2}{9}x^3\ln x - \frac{2}{27}x^3 + C_1, \\ u_2(x) &= \int x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C_2. \end{aligned}$$

The choice $C_1 = C_2 = 0$ gives a particular solution

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = \frac{1}{9}x\ln x - \frac{2}{27}x.$$

Therefore, the general solution is

$$y(x) = y_c(x) + y_p(x) = c_1x^{-2} + c_2x^{-2}\ln x + \frac{1}{9}x\ln x - \frac{2}{27}x. \qquad \square$$

3(a) Use the Laplace transform to evaluate the Dirichlet integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

Solution. Let $F(s) = \mathscr{L}\left\{\frac{\sin t}{t}\right\}$. Then $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = F(0)$. Since $\mathscr{L}\{tf(t)\} = -\frac{d}{ds}\mathscr{L}\{f(t)\}$,

$$F'(s) = \frac{d}{ds}\mathscr{L}\left\{\frac{\sin t}{t}\right\} = -\mathscr{L}\{\sin t\} = -\frac{1}{1+s^2}.$$

Integrating both sides, we obtain

$$F(s) - F(a) = \int_a^s F'(x)\,dx = \int_a^s \frac{-1}{1+x^2}\,dx. = \cot^{-1} s - \cot^{-1} a.$$

Taking limits where a tends to infinity on both sides, we have

$$F(s) - \lim_{a\to\infty} F(a) = \cot^{-1} s - \lim_{a\to\infty} \cot^{-1} a,$$
$$\therefore F(s) = \cot^{-1} s.$$

Therefore,

$$\int_0^\infty \frac{\sin x}{x}\,dx = F(0) = \frac{\pi}{2}.$$

□

3(b) Solve the initial-value problem

$$y'' + 3y' + 2y = \frac{\sin x}{x}, \quad y(0) = -\frac{\pi}{6}, \quad y'(0) = 1.$$

Solution. Let $Y(s) = \mathscr{L}\{y(t)\}$. Taking Laplace transforms on both sides, we have

$$\left(s^2 + 3s + 2\right) Y(s) - (s+3)y(0) - y'(0) = \cot^{-1} s.$$

Solving for $Y(s)$,

$$\begin{aligned} Y(s) &= \frac{\cot^{-1} s - \pi(s+3) + 6}{6(s+1)(s+2)} \\ &= \frac{1}{6}\mathscr{L}\left\{\frac{\sin t}{t}\right\}\left(\mathscr{L}\left\{e^{-t}\right\} - \mathscr{L}\left\{e^{-2t}\right\}\right) - \frac{\pi}{6}\left(\frac{2}{s+1} - \frac{1}{s+2}\right) + \frac{1}{s+1} - \frac{1}{s+2}. \end{aligned}$$

Therefore,

$$y(t) = \mathscr{L}^{-1}\left\{Y(s)\right\} = \left(1 - \frac{\pi}{3}\right)e^{-t} + \left(\frac{\pi}{6} - 1\right)e^{-2t} + \frac{1}{6}\int_0^t \frac{\left(e^{-\tau} - e^{-2\tau}\right)\sin\tau}{\tau}\, d\tau.$$

$\square$

4 Evaluate $\mathscr{L}\left\{\sqrt{t}\right\}$.

Solution. One definition of the gamma function $\Gamma(z)$ is given by the improper integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \quad \mathrm{Re}(z) > 0.$$

Note that $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(n) = (n-1)!$ for any positive integer n. For $\alpha > -1$,

$$\begin{aligned}
\Gamma(\alpha+1) &= \int_0^\infty u^\alpha e^{-u}\,du \\
&= \int_0^\infty (st)^\alpha e^{-st} s\,dt \qquad (u = st,\ s > 0) \\
&= s^{\alpha+1}\int_0^\infty t^\alpha e^{-st}\,dt \\
&= s^{\alpha+1}\mathscr{L}\left\{t^\alpha\right\} \\
\therefore \mathscr{L}\left\{t^\alpha\right\} &= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.
\end{aligned}$$

It is known that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, so $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$. Therefore,

$$\mathscr{L}\left\{\sqrt{t}\right\} = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}, \quad s > 0.$$

□

5 Let y_1, y_2, $\cdots$, y_n be a fundamental set of solutions of the homogeneous linear nth-order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval I. For a general linear nth-order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x),$$

find a particular solution y_p on I using the method of variation of parameters.

Solution. Using the method of variation of parameters, we seek a solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x),$$

where the u_k', $k \in \{1, 2, \cdots, n\}$, are determined by the n equations

$$\sum_{k=1}^{n} u_k' \frac{d^j y_k}{dx^j} = \begin{cases} 0 & \text{if } j \in \{0, 1, \ldots, n-2\}, \\ f(x) & \text{if } j = n-1. \end{cases}$$

Cramer's rule gives

$$u_k' = \frac{W_k}{W}, \quad k \in \{1, 2, \cdots, n\},$$

where W is the Wronskian of y_1, y_2, $\ldots$, y_n and W_k is the determinant obtained by replacing the kth column of the Wronskian by the column $(0, 0, \ldots, f(x))$. That is,

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_k & \cdots & y_n \\ y_1' & y_2' & \cdots & y_k' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_k^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix},$$

$$W_k = \begin{vmatrix} y_1 & y_2 & \cdots & 0 & \cdots & y_n \\ y_1' & y_2' & \cdots & 0 & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & f(x) & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

□

6 Solve

$$\mathbf{X}' = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix} \mathbf{X}.$$

Find and classify all critical points of the system. Draw a vector field of the system to describe the geometric behavior of the solution depending on initial values.

Solution. We first find the eigenvalues and eigenvectors of the matrix of coefficients. From the characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & -4 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 = 0,$$

we see that the eigenvalues are $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. To find an eigenvector corresponding to λ_1, we solve the system of equations

$$\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $k_1 = 2ik_2$, the choice $k_2 = 1$ gives the following eigenvector $\mathbf{K}_1 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$.

Recall that linearly independent solutions of the homogeneous system whose coefficient matrix has a complex eigenvalue $\lambda_1 = \alpha + i\beta$ with a corresponding eigenvector $\mathbf{K}_1$ are

$$\begin{pmatrix} \mathbf{X}_1(t) \\ \mathbf{X}_2(t) \end{pmatrix} = \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix} \begin{pmatrix} \mathrm{Re}(\mathbf{K}_1) \\ \mathrm{Im}(\mathbf{K}_1) \end{pmatrix} e^{\alpha t}.$$

Therefore, the general solution is

$$\mathbf{X} = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 = c_1 \begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix} e^t + c_2 \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix} e^t.$$

The only critical point of the homogeneous system is the origin $(0,0)$. The trace of the coefficient matrix is $\tau = 2$, and its determinant is $\Delta = 5$. Since $\tau^2 - 4\Delta < 0$ and $\alpha = 1 > 0$, the critical point is an **unstable spiral point**. The right figure shows vector field of the system. □