1(a) By using an "ϵ–δ type argument", prove that

$$\lim_{x\to\infty} \frac{1}{f(x)} = 0$$

if $\lim_{x\to\infty} f(x) = \infty$.

Solution. Let $\epsilon > 0$ be given. Since $\lim_{x\to\infty} f(x) = \infty$, there exists a positive number $M > 0$ such that

$$x > M \implies \frac{1}{\epsilon} < f(x) \le |f(x)| \implies \left|\frac{1}{f(x)}\right| < \epsilon.$$

□

1(b) Let $[x]$ denote the greatest integer smaller than or equal to x. Determine the limit $\lim_{x\to\infty} \frac{\sin x}{[x]}$.

Solution. Since $\lim_{x\to\infty} [x] = \infty$, by (a), $\lim_{x\to\infty} \frac{1}{[x]} = 0$. Note that

$$x \ge 1 \implies -1 \le \sin x \le 1 \implies -\frac{1}{[x]} \le \frac{\sin x}{[x]} \le \frac{1}{[x]}.$$

Taking limits where x tends to infinity, by the squeeze theorem, we have

$$-\lim_{x\to\infty} \frac{1}{[x]} \le \lim_{x\to\infty} \frac{\sin x}{[x]} \le \lim_{x\to\infty} \frac{1}{[x]},$$
$$\therefore \lim_{x\to\infty} \frac{\sin x}{[x]} = 0.$$

□

2 Which of the following statements are true and which are false? If true, prove it; if false, give a counterexample.

(a) If $f(x)$ is continuous at $x = a$, then so is $|f(x)|$.

Solution. True. Since $|x|$ is a continuous function, $\lim_{x\to a} |f(x)| = \left|\lim_{x\to a} f(x)\right| = |f(a)|$. $\square$

(b) If $|f(x)|$ is continuous at $x = a$, then so is $f(x)$.

Solution. False. If $f(x)$ is given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

$|f(x)| = 1$ is obviously continuous everywhere. However, $f(x)$ is continuous nowhere. $\square$

3 If $y^3 = x^2 + 2x$, use the implicit differentiation to find

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,2)} \quad \text{and} \quad \left.\frac{d^2y}{dx^2}\right|_{(x,y)=(2,2)}.$$

Solution. Using the implicit differentiation on both sides, we have

$$3y^2\frac{dy}{dx} = 2x + 2.$$

Using it again, we have

$$6y\left(\frac{dy}{dx}\right)^2 + 3y^2\frac{d^2y}{dx^2} = 2.$$

Plugging $(x, y) = (2, 2)$ to both equations, we have

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,2)} = \frac{1}{2} \quad \text{and} \quad \left.\frac{d^2y}{dx^2}\right|_{(x,y)=(2,2)} = -\frac{1}{12}.$$

□

4 If two resistors of R_1 and R_2 ohms are connected in parallel in an electric circuit to make a resistor of R ohms, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

If R_1 is decreasing at the rate of 1 ohm/sec and R_2 is increasing at the rate of 0.5 ohm/sec, at what rate is R changing when $R_1 = 75$ ohms and $R_2 = 50$ ohms?

Solution. Plugging $R_1 = 75$ ohms and $R_2 = 50$ ohms to the equation, we get $R = 30$ ohms. Differentiating both sides with respect to t, we have

$$\frac{1}{R^2}\frac{dR}{dt} = \frac{1}{R_1^2}\frac{dR_1}{dt} + \frac{1}{R_2^2}\frac{dR_2}{dt}.$$

Plugging all values we know to the equation, we have

$$\frac{dR}{dt} = \frac{1}{50} = 0.02 \text{ ohm/sec}.$$

□

5 Find an equation of the curve $y = f(x)$ whose slope at the point (x, y) is $\pi f(x)^\alpha \sin(\pi x)$ for a positive constant α satisfying $\alpha > 1$ if the curve is required to pass though the point $(1, 1)$.

Solution. The function f satisfies $f'(x) = \pi f(x)^\alpha \sin(\pi x)$.

$$\int_1^x f(t)^{-\alpha} f'(t)\, dt = \int_1^x \pi \sin(\pi t)\, dx.$$
$$\frac{f(x)^{1-\alpha} - 1}{1 - \alpha} = -\cos(\pi x) - 1.$$

Thus, the equation of the curve $y = f(x)$ is $y = [(\alpha - 1)\cos(\pi x) + \alpha]^{\frac{1}{1-\alpha}}$. □

6 Let $f(x) = x^{1/3}(x+3)^{2/3}$.

(a) Find the intervals on which f is increasing and decreasing.

Solution. Recall that a differentiable function f is increasing if and only if its derivative f' is positive, or zero at some points whose neighborhood takes positive derivatives. Since $f(x)^3 = x(x+3)^2$, differentiating both sides, we have

$$3f(x)^2 f'(x) = (x+3)^2 + 2x(x+3) = 3(x+1)(x+3),$$
$$f'(x) = \frac{x+1}{x^{2/3}(x+3)^{1/3}}.$$

Now it remains to find where f' is positive.

$$f'(x) > 0 \iff x \in (-\infty, -3) \cup (-1, \infty)$$

Hence, f is increasing on $(-\infty, -3) \cup (-1, \infty)$, and decreasing elsewhere. □

(b) Find the local maximum and local minimum values of f.

Solution. Recall that the only places where a function f can possibly have an extreme value are

- interior points where $f' = 0$,
- interior points where f' is undefined,
- endpoints of the domain of f.

Since f is defined on $\mathbb{R}$, it remains to check for only critical points. Considering how f changes in the neighborhood of each critical points, $x = -3$, $x = -1$, and $x = 0$,

- the local maximum value of f is $f(3) = 0$,
- and the local minimum value of f is $f(-1) = -2^{2/3}$.

□

(c) Find the intervals of concavity and the points of inflection.

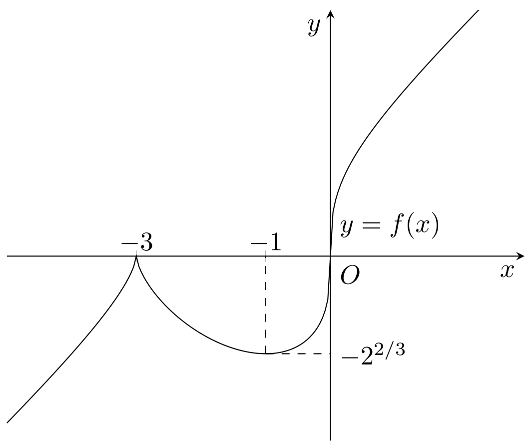
Solution. Recall that a twice differentiable function f is concave upward (or convex downward) if and only if its second derivative f'' is positive, or zero at some points whose neighborhood takes positive second derivatives. Since

$$f''(x) = -\frac{2}{x^{5/3}(x+3)^{4/3}} > 0 \iff x \in (0, \infty),$$

f is concave upward on $(0, \infty)$, and concave downward elsewhere.
Also, recall that the point of inflection of a twice differentiable function f is a point in the domain where the sign of its second derivative f'' changes. Therefore, the point of inflection is only $x = 0$, or more specifically, $(0, 0)$ (the origin). □

(d) Use the information from parts (a), (b) and (c) to draw the graph of f.

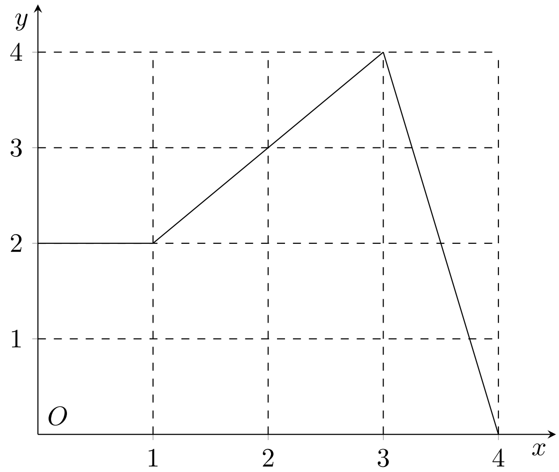
Solution.



□

7 Answer each of the following.

(a) Suppose that the function $y = f(x)$ is given by the graph and $F(x) = \int_0^x f(t)\,dt$.



Determine $f(2)$, $f'(2)$, $F(2)$, and $F'(2)$.

Solution. As presented in the graph, $f(2) = 3$, $f'(2) = 1$, and

$$F(2) = \int_0^2 f(x)\,dx = \int_0^1 2\,dx + \int_1^2 (x+1)\,dx = \frac{9}{2}.$$

By the fundamental theorem of calculus, $F'(2) = f(2) = 3$. $\square$

(b) Find $g'(x)$ if $g(x) = \int_2^{x^2} \frac{t}{t^3+1}\,dt$.

Solution. By the Leibniz's rule, $g'(x) = \frac{x^2}{x^6+1} \cdot 2x = \frac{2x^3}{x^6+1}$. $\square$

8 Find the area enclosed by the curve $x = y^2$ and the line $y = x - 2$.

Solution. Two graphs intersect at two points, $(1, -1)$ and $(4, 2)$. The area enclosed by them is given by

$$\int_{-1}^{2} (y + 2 - y^2)\, dy = \int_{-1}^{2} (y + 1)(2 - y)\, dy = \frac{[2 - (-1)]^3}{6} = \frac{9}{2}.$$

□

9 Compute each of the following integrals:

(a) $\displaystyle\int_0^4 (x^2+9)^{0.5} x\,dx$

Solution. Substitute $x^2+9=t$; then we have

$$\int_0^4 (x^2+9)^{0.5} x\,dx = \int_9^{25} t^{0.5}\,\frac{dt}{2} = \frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_9^{25} = \frac{98}{3}.$$

□

(b) $\displaystyle\int \cos^4 3x \sin 3x\,dx$

Solution. Substitute $\cos 3x = t$; then we have

$$\int \cos^4 3x \sin 3x\,dx = \int t^4\left(-\frac{dt}{3}\right) = -\frac{1}{3}\left(\frac{t^5}{5}\right) + C = -\frac{1}{15}\cos^5 3x + C.$$

□

(c) $\displaystyle\int \frac{x^2}{\sqrt{x^3+2}}\,dx$

Solution. Substitute $x^3+2=t$; then we have

$$\int \frac{x^2}{\sqrt{x^3+2}}\,dx = \int t^{-\frac{1}{2}}\,\frac{dt}{3} = \frac{1}{3}\left(2t^{\frac{1}{2}}\right) + C = \frac{2}{3}\sqrt{x^3+2} + C.$$

□

10 Find the area of the surface obtained by revolving the curve $y = \sqrt{1-x}$, $0 \le x \le 1$ about the x-axis.

Solution. The surface area of the revolution is given by

$$\int_0^1 2\pi y \sqrt{1+(y')^2}\,dx = 2\pi \int_0^1 \sqrt{1-x}\sqrt{\frac{5-4x}{4(1-x)}}\,dx = \pi \int_0^1 \sqrt{5-4x}\,dx.$$

Using the substitution $5x-4=t$, we have

$$\pi \int_5^1 \sqrt{t}\left(-\frac{dt}{4}\right) = \frac{\pi}{4}\int_1^5 \sqrt{t}\,dt = \frac{\pi}{4}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_1^5 = \frac{\pi\left(5\sqrt{5}-1\right)}{6}.$$

□

11 Find the center of mass of the plate covering the region defined by $0 \le x \le 1 - y^2$ if the density at the point (x, y) is $\delta = y^2$.

Solution. The mass of the plate M is given by

$$M = \int dm = \int_{-1}^{1} \delta x \, dy = \int_{-1}^{1} y^2 \left(1 - y^2\right) \, dy = 2 \int_0^1 \left(y^2 - y^4\right) \, dy = \frac{4}{15}.$$

The moment about the x-axis M_x is given by

$$M_x = \int \tilde{y} \, dm = \int_{-1}^{1} y \delta x \, dy = \int_{-1}^{1} y^3 \left(1 - y^2\right) \, dy = 0.$$

The moment about the y-axis M_y is given by

$$M_x = \int \tilde{x} \, dm = \int_{-1}^{1} \frac{x}{2} \delta x \, dy = \int_0^1 y^2 \left(1 - y^2\right)^2 \, dy = \frac{8}{105}.$$

So, the center of mass of the plate is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{2}{7}, 0\right).$$

$\square$