1. (10 points) Show that $\lim_{x \to 0} \sin(1/x)$ does not exist.

2. (25 points) Thomae's function, also known as the ruler function, is defined as

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \ (x \text{ is rational}), \text{ with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ coprime}, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Since every rational number has a unique representation with coprime $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, the function is well-defined. Note that $q = 1$ is the only number in $\mathbb{N}$ that is coprime to $p = 0$. Prove the following properties of this function.

(a) f is **periodic** with period **1**: $f(x + n) = f(x)$ for all integer n and all real x.

(b) f is **discontinuous** at all rational numbers.

(c) f is **continuous** at all irrational numbers.

(d) f is **nowhere differentiable**.

(e) f is Riemann **integrable** on any interval and the integral evaluates to 0 over any set. (Prove without using the Lebesgue criterion for Riemann integrability.)

> The Lebesgue criterion for Riemann integrability states that a bounded function is Riemann integrable if and only if the set of all discontinuities has measure zero. Every countable subset of the real numbers—such as the rational numbers—has measure zero, so the discussion above shows that Thomae's function is Riemann integrable on any interval. The integral is equal to 0 over any set because the function is equal to zero *almost everywhere*.

3. (10 points) Let $\mathscr{R}$ denote the set of all Riemann integrable functions. If two functions $f \in \mathscr{R}$ and $g \in \mathscr{R}$, then the composition $f \circ g \in \mathscr{R}$? If true, prove it; if false, give a counterexample.

4. (10 points) Show that $\dfrac{\sin x_2 - \sin x_1}{x_2 - x_1}$ does **NOT** attain neither maximum nor minimum values for two distinct arbitrary real numbers x_1 and x_2.

5. (10 points) Show that $\cosh(\sinh x) < \sinh(\cosh x)$ for any real number x.

6. (25 points) Evaluate the following integrals.

(a) $\displaystyle\int \frac{1}{x^2 + 1}\, dx$

(b) $\displaystyle\int \frac{1}{ax^2 + bx + c}\, dx \quad \left(b^2 - 4ac < 0\right)$

(c) $\displaystyle\int \sqrt{\tan x}\, dx$

7. (Bonus) Prove or disprove that in the sense of the Riemann integral, if a function f is differentiable on the interval $[a, b]$, $\displaystyle\int_a^b f'(x)\, dx = f(b) - f(a)$.