TOPOLOGICAL VECTOR SPACES COHERENT WITH C^p-ARCS

MATTHEW GREGORY KRUPA
THE UNIVERSITY OF COLORADO, BOULDER
BOULDER, CO 80303

ABSTRACT. We call a space *coherent* with a collection of continuous maps going into it if it has the final topology with respect to these maps. We investigate coherence with various classes of C^p-curves to obtain the following differential geometric characterizations: (1) a Hausdorff locally convex topological vector space (HLCTVS) X is coherent with its C^p-arcs for some $p \geq 2$ iff $\dim X \leq 1$, (2) for $d > 1$, a d-dimensional manifold is coherent with its C^p-arcs iff $p = 0$ or 1, (3) a normable space is finite-dimensional iff it is coherent with its C^1-arcs, which we conjecture extends to all Fréchet spaces, (4) a C^p-manifold ($p > 0$) has non-empty boundary iff it is coherent with its C^1-embeddings of open intervals, and (5) a first countable Hausdorff space is locally path-connected iff it is coherent with its C^0-paths. A class of parameterizations of smoothly embedded intervals is defined and used to characterize pointwise continuity of maps from C^∞-manifolds into spaces, leading to a conjecture a strengthening of the Boman theorem. We show that those HLCTVSs coherent with their C^0-arcs form a subcategory of HLCTVS that lies in-between sequential and Fréchet-Urysohn spaces, which are also discussed. Some competing definitions of a "sequentially quotient map" are discussed, counterexamples are provided showing that they are not always interchangeable, and sufficient conditions for their equivalence are given. We show that $\mathbb{R}^\infty$ is coherent with its C^1-arcs while $\mathbb{R}^\mathbb{N}$ is not and we illustrate how this is due $\mathbb{R}^\mathbb{N}$'s weak topology by constructing a paradoxical smooth topological embedding of $[-1,1]$ into $[-1,1]^\mathbb{N}$ that maps 0 to this cube's center and everything else to points outside of $]-1,1[^\mathbb{N}$.

Contents

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Part 1. Introduction

That one may gain useful insight into a space by investigating its interactions with its curves has been demonstrated by the Boman theorem (**[3]**) and in the well-known reference Kriegl-Michor **[12]**. In contrast to **[12]**, which describes how various Hausdorff locally convex topological vector spaces (HLCTVSs) relate to their smooth curves, we will be interested in, among other things, how manifolds and various classes of topological vector spaces (TVSs) relate to their C^p-arcs for various $p \in \{0, 1, \ldots, \infty\}$ in terms of their coherence (def. 5) or non-coherence with these maps, that is, in terms of whether or not it is possible to uniquely identify the space's original topology upon being given only this collection of maps and their domains' topologies.

Now of course, a map $f : \mathbb{R} \to Y$ is continuous at 0 if and only if it is continuous from the left and from the right at 0 or equivalently, using coordinate free terminology, if and only if along every smooth embedding $\gamma : ([0, \infty[, 0) \to (\mathbb{R}, 0)$, the map $f \circ \gamma : [0, \infty[\to Y$ is continuous at 0. We show that this *fails* to generalize to maps on $\mathbb{R}^d$ for $d > 1$; that is, it is false that a map $f : \left(M, m^0\right) \to \left(Y, y^0\right)$ from a smooth manifold M of dimension $d > 1$ into a space Y is continuous at m^0 if and only if $f \circ \gamma : [0, \infty[\to Y$ is continuous at 0 whenever $\gamma : ([0, \infty[, 0) \to \left(M, m^0\right)$ is a smooth topological embedding such that $\gamma'(t) \neq \mathbf{0} \iff t \geq 0$. We find two ways to change this statement to make it true; the first is by replacing "$\gamma'(t) \neq \mathbf{0} \iff t \geq 0$" with "$\gamma'(t) \neq \mathbf{0} \iff t > 0$" while the second is by replacing "smooth" with "once continuously differentiable." Oddly, however, if one was to replace "smooth" with "p-times continuously differentiable" for some $p \geq 2$, then the statement would again be false; it is not clear why this distinction happens between $p = 1$ and $p = 2$ rather than, say, between $p = 0$ and $p = 1$ or between $p < \infty$ and $p = \infty$ or why there should even exist values of p for which this distinction must be made at all, but it may indicate the existence of a deeper and more subtle relationship between $\mathbb{R}^d$'s topology (for $d > 1$) and differentiability of order > 1 than what is described in this paper.

Isolating the properties of the curves from the two instances for which the above statement is true leads us to define weak C^p-almost arcs (def. 9.1), which are those curves γ that are one point C^p-extensions of smooth parameterizations of a smoothly embedded submanifold that is diffeomorphic to an interval (i.e a smoothly embedded interval), which we'll call *the smooth submanifold associated with* γ. The veracity of the above claim for weak smooth almost-arcs, stated in an equivalent form in corollary 0.3, is a consequence of theorem 9.7, which is itself a consequence of theorem 0.1 while the claim made for C^1-arcs is a consequence of theorem 0.2, which is itself a consequence of theorem 10.4

Theorem 0.1. Smooth manifolds with corners are coherent with their smooth almost arcs.

Theorem 0.2. Smooth manifolds with corners are coherent with their C^1-arcs.

Note that although the graph G of $y = \sin(1/x)$ for $x > 0$ is a smoothly embedded interval in $\mathbb{R}^2$, there is clearly no smooth surjective parameterization $]a, b[\to G$ of G that admits even a one-sided continuous extension. In contrast, by its very definition, the smooth submanifold associated with a weak C^p-almost arc will not encounter this issue and so they can be viewed as a class of particularly well-behaved smoothly embedded intervals and since the continuous extension is unique when it exists, each weak C^p-almost arc may consequently be viewed as a well-behaved parameterization of a well-behaved smoothly embedded interval. Corollary 0.3 thus states that to check a map's continuity at a point, we need only to check the continuity of compositions of f with such well-behaved parameterizations.

Corollary 0.3 (Smooth characterization of continuity at a point)**.** A map $f : (M, m^0) \to (Y, y^0)$ from a smooth manifold with corners into a space is continuous at m^0 if and only if for all smooth embeddings $\gamma :]0, \infty[\to M \smallsetminus \{m^0\}$, if extending $\gamma :]0, \infty[\to M$ to $[0, \infty[$ by sending 0 to m^0 results in a smooth map then extending $f \circ \gamma :]0, \infty[\to Y$ to $[0, \infty[$ by sending 0 to y^0 results in a continuous map.

Compare corollary 0.3 to the following clearly equivalent formulation of the well-known Boman theorem (**[3]**).

> A map $f : M \to Y$ between smooth manifolds is smooth if and only if for all $m^0 \in M$ and all smooth curves $\gamma : \mathbb{R} \smallsetminus \{0\} \to M \smallsetminus \{m^0\}$, if extending $\gamma : \mathbb{R} \smallsetminus \{0\} \to M$ to $\mathbb{R}$ by sending 0 to m^0 results in a smooth curve then so does extending $f \circ \gamma : \mathbb{R} \smallsetminus \{0\} \to Y$ to $\mathbb{R}$ by sending 0 to $f(m^0)$.

This suggests the following conjecture which, if true, represents a strengthening of the Boman theorem.

Conjecture 0.4. A map $f : M \to Y$ between smooth manifolds is smooth if and only if for all $m^0 \in M$ and all smooth embeddings $\gamma : \mathbb{R} \smallsetminus \{0\} \to M \smallsetminus \{m^0\}$, if extending $\gamma : \mathbb{R} \smallsetminus \{0\} \to M$ to $\mathbb{R}$ by sending 0 to m^0 results in a smooth curve then so does extending $f \circ \gamma : \mathbb{R} \smallsetminus \{0\} \to Y$ to $\mathbb{R}$ by sending 0 to $f(m^0)$.

In light of the fact that $\mathbb{R}^d$ is not coherent with its C^p-embeddings of intervals for all $d > 1$ and $p > 1$, it may be necessary to weaken conjecture 0.4, which is a rather strong claim, by replacing "smooth curve" with "C^p-curve ($p \in \{0, 1, \ldots, \infty\}$)."

Returning to C^1-arcs, theorem 10.4 produces theorem 0.2, which gives a characterization of continuity at a point that is even more interesting than corollary 0.3 since it characterizes continuity at a point in terms of smooth parameterizations of smoothly embedded intervals that are so well-behaved that not only does each one admit a C^1-extension to a closed interval, but this extension ends up parameterizing a C^1-embedded submanifold. In contrast, if $\dim M > 1$ then a consequence of M not being coherent with its C^2-arcs is that for any given $p > 1$, we can not replace "C^1-embedding" with "C^p-embedding" in corollary 0.5's statement.

Corollary 0.5 (C^1-arc characterization of continuity at a point)**.** A map $f : (M, m^0) \to (Y, y^0)$ from a smooth manifold with corners into a space is continuous at m^0 if and only if for all smooth embeddings $\gamma :]0, \infty[\to M \smallsetminus \{m^0\}$, if extending $\gamma :]0, \infty[\to M$ to $[0, \infty[$ by sending 0 to m^0 results in a C^1-embedding then extending $f \circ \gamma :]0, \infty[\to Y$ to $[0, \infty[$ by sending 0 to y^0 results in a continuous map.

Unlike corollary 0.3, we can restate corollary 0.5 more provocatively as corollary 0.6 by referring *only* to embedded submanifolds of M, where by a *C^p-embedded interval* we mean a C^p-submanifold that is C^p-isomorphic to an interval. Note that for any given $p > 1$, if $\dim M > 1$ then replacing "C^1-embedded interval" with "C^p-embedded interval" in corollary 0.6 will result in a false statement.

Corollary 0.6 (C^1-embedded intervals that one-point extend smooth submanifolds characterize continuity)**.** A map $f : (M, m^0) \to (Y, y^0)$ from a smooth manifold with corners into a space is continuous at m^0 if and only if for all smoothly embedded intervals S in $M \smallsetminus \{m^0\}$, if $S \cup \{m^0\}$ is a C^1-embedded interval then f's restriction to this C^1-submanifold is continuous.

Another consequence of theorem 10.4 is the following theorem, which may be viewed as making rigorous the naive notion that "manifolds are related to manifolds with (non-empty) boundary in the same way that $]0, \infty[$ is related to $[0, \infty[$."

Theorem 0.7. A C^k-manifold with corners ($k = 1, 2, \ldots, \infty$) is a C^k-manifold if and only if it's coherent with its C^1-embedding of open intervals.

Our investigation of coherence with various classes of C^p-arcs is also applied to TVSs and we are led first to the discovery of a new category of HLCTVSs, which are those that are coherent with their C^0-arcs, that sits in-between sequential HLCTVSs and Fréchet-Urysohn HLCTVSs. In particular, we prove:

Theorem 0.8. If a HLCTVS is Fréchet-Urysohn then it is coherent with its arcs.

The converse of theorem 0.8 is false since theorem 11.1 shows that the non-Fréchet-Urysohn HLCTVS $\mathbb{R}^\infty$ is coherent with its arcs. Since metrizable spaces are Fréchet-Urysohn spaces, we obtain corollary 0.9.

Corollary 0.9. Every metrizable LCTVS is coherent with its C^0-arcs.

Studying coherence of TVSs with C^p-arcs for $p \geq 1$ leads to theorem 0.10 and then immediately to theorem 0.11, which may be seen as a "differential geometric characterization" of finite-dimensionality for normable spaces.

Theorem 0.10. Let $p \in \{1, 2, \ldots, \infty\}$, X be a HTVS, and $\mathcal{C}$ be a collection of C^p-curves in X whose derivatives never vanish. If X is coherent with $\mathcal{C}$ then every closed and bounded subset of X is sequentially compact.

Theorem 0.11. A normable TVS is finite-dimensional if and only if it's coherent with its C^1-arcs, in which case it is coherent with the set of all its weak C^1-almost arcs.

Since there always exists a (potentially non-linear) homeomorphism between any two infinite-dimensional separable Fréchet spaces **[1]**, one is naturally led to conjecture 0.12.

Conjecture 0.12. A Fréchet space is finite-dimensional if and only if it's coherent with its C^1-arcs.

Remark 0.13. It's straightforward to verify that this conjecture holds if and only if it holds for all separable Fréchet spaces.

In addition, we show that the non-metrizable sequential HLVTVS $\mathbb{R}^\infty$ is coherent with its C^1-arcs while its Fréchet continuous dual space $\mathbb{R}^{\mathbb{N}}$ is not. To help understand why infinite-dimensionality and coherence with C^1-arcs are incompatible for such a large class of spaces, we construct in example 11.8 a paradoxical smooth topological embedding of $\mathbb{R}$ into $\mathbb{R}^{\mathbb{N}}$.

If we remove the requirement that our curves be topological embeddings then we are led to the following characterization of local path-connectedness, where note that when applying this theorem, lemma 5.7 and proposition 3.7 and may aid in checking the coherence condition.

Theorem 0.14. A first-countable Hausdorff space is locally path-connected if and only if it's coherent with its continuous paths.

In the same sense that the statement "a set d is finite if and only if $\mathbb{R}^d$ is locally compact" can be seen a statement providing a "topological characterization of when a set is finite," the following corollary 0.15, which follows from corollary 5.11 together with theorems 10.4 and 8.1, can be viewed as the statement closest to a "differential geometric characterization of when the a set is finite." To best of the author's knowledge, there is not other statement which, while using only elementary concepts, admits such an interpretation.

Corollary 0.15. For any set d,

(1) d is finite if and only if $\mathbb{R}^d$ is coherent with its C^1-arcs, and
(2) d is empty or a singleton set if and only if $\mathbb{R}^d$ is coherent with its C^p-arcs for some/all $p \in \{2, 3, \ldots, \infty\}$.

1. Notation and Terminology We will denote the set of all (resp. all positive, all non-negative) integers by $\mathbb{Z}$ (resp. $\mathbb{N}$, $\mathbb{Z}^{\geq 0}$). By *increasing* (resp. *decreasing*) we mean strictly increasing (resp. strictly decreasing) and by *monotone* (resp. *strictly monotone*) we mean either non-increasing or non-decreasing (resp. either increasing or decreasing). By $(x_l)_{l=1}^\infty \subseteq X$ (resp. $(x_i)_{i \in I} \subseteq X$), which we may abbreviate by $x_\bullet$, we mean a sequence (resp. an I-indexed tuple or I-directed net) in X where x_i is called *the* i^{th}*-component of* $x_\bullet$. We will say that a sequence, net, or tuple $x_\bullet$ is *injective* if whenever i and j are distinct indices then $x_i \neq x_j$ and that it *has infinite range* or is *infinite-ranged* if $\{x_i : i \text{ is an index }\}$ is infinite. By "$x_\bullet \to x$ is injective in X" we mean that $x_\bullet$ is injective, no component of $x_\bullet$ is equal to x, and that $x_\bullet \to x$ in X; the meanings of "let $x_\bullet \to x$ be injective in X" and "suppose $x_\bullet \to x$ has an injective subsequence in X" should be clear.

We will identify singleton sets with their contents so that in particular, if s is any object and d is a set then $\{s\}^d$ denotes both the singleton set of all maps $d \to \{s\}$ as well as *the constantly s d-tuple* $(s)_{l\in d}$. If $d \in \mathbb{N}$ then $\{s\}^d$ will denote the d-tuple $\{s\}^{\{1,\dots,d\}} = (s,\dots,s)$. Consequently, we may use $\{0\}^d$, rather than $\mathbf{0}$, to denote the zero of $\mathbb{R}^d$. Given objects s and t and $d, e \in \mathbb{Z}^{\geq 0}$ with $d+e>0$, we will denote the $(d+e)$-tuple whose first d-coordinates are s and whose last e-coordinates are t by either $\left(\{s\}^d, \{t\}^e\right)$ or $\{s\}^d \times \{t\}^e$.

By Id_X we mean the identity morphism of an object X and if X is a set and $S \subseteq X$ then In_S^X or simply In_S will denote the natural inclusion. For $a, b \in \mathbb{R}$, $[a,b[$ (resp. $]a,b[$, etc.) will denote the set of all $x \in \mathbb{R}$ such that $a \leq x < b$ (resp. $a < x < b$, etc.). By a *space*, (resp. *TVS*, *LCTVS*, *HTVS*, *HLCTVS*) we mean a topological space (resp. topological vector space, locally convex TVS, Hausdorff TVS, Hausdorff LCTVS). The continuous dual space of a TVS X will be denoted by X' and we say that X' *separates points on* X if for all non-zero $x \in X$ there exists some $\lambda \in X'$ such that $\lambda(x) \neq 0$. A subset R of a TVS X is said to *absorb* another subset S if there exists some $r > 0$ such that $S \subseteq rR$ and a subset is called *bounded* in X if every neighborhood of zero in X absorbs it. If X is a space then by τ_X we mean X's topology and for any subset S of a space X, $\tau_X\big|_S$ denotes the subspace topology that S inherits from (X, τ_X) while $\mathrm{Cl}_X(S)$ or $\overline{S}$ denotes its closure in X. By a *neighborhood* of a point x in a space X we mean *any* set containing x in its topological interior.

Part 2. Basic Notions

2. C^p-Curves and C^p-Manifolds Modeled on HTVSs All intervals are assumed to be non-degenerate. A *curve* in a space X is a C^0 (i.e. continuous) map $\gamma : J \to X$ from an interval $J \subseteq \mathbb{R}$ while a *path in* X is a curve in X whose domain is compact.

Now suppose that X is a HTVS, γ is a curve in X, and $t \in \operatorname{Dom}\gamma$. We say that γ is *differentiable at* t if the limit

$$\lim_{\substack{h\to 0\\ t+h\in\operatorname{Dom}(\gamma)}} \frac{1}{h}\left[\gamma(t+h) - \gamma(t)\right]$$

exists, in which case we'll denote it by $\gamma'(t)$ or $\gamma^{(1)}(t)$. We say that γ is *differentiable* if its derivative exists at every point of its domain and that it is *continuously differentiable* or C^1 if it is differentiable and the map $\gamma' : J \to X$ is continuous. For all $p \in \{1, 2, \dots, \infty\}$, we define γ *is* C^p, γ *is p-times continuously differentiable*, and γ *is smooth* by their usual recursive definitions. Say that γ is *non-vanishing* or 0-*vanishing at* t if $\gamma'(t) \neq \mathbf{0}$. For any $k, p \in \{1, 2, \dots, \infty\}$ with $k \leq p$, say that a C^p curve γ *vanishes k-times at t* if $\gamma^{(h)}(t) = \mathbf{0}$ for all integers $1 \leq h \leq k$ where if in addition $k = \sup\{h \in \mathbb{N} : \gamma \text{ vanishes } k\text{-times at } t\}$ then we'll say that γ is *p-vanishing at c*. If $p \in \{0, 1, \dots, \infty\}$ then call a C^p-curve a *C^p-embedding* if it is a topological embedding where if $p \geq 1$ then we also require that its first derivative never vanish. By a *C^p-arc in X* we mean a path in X that is a C^p-embedding and by an *arc in X* we mean a C^0-arc in X.

Observations 2.1. Suppose that $\Lambda : X \to Y$ is a continuous linear operator between HTVSs, γ is a curve in X, and $p \in \{0, 1, \dots, \infty\}$. If γ is C^1, $\gamma'(t_0) \neq \mathbf{0}$, and X' separates points on X then there exists some some $\delta > 0$ such that that γ is a C^1-embedding on $[t_0 - \delta, t_0 + \delta] \cap \operatorname{Dom}\gamma$: Pick any continuous linear functional $\lambda : X \to \mathbb{R}$ such that $0 \neq \lambda(\gamma'(t_0)) = (\lambda \circ \gamma)'(0)$, where since $\lambda \circ \gamma$ is a C^1 $\mathbb{R}$-valued curve, there exists a $\delta > 0$ such that $\lambda \circ \gamma\big|_{[t_0-\delta, t_0+\delta]} : [t_0 - \delta, t_0 + \delta] \cap \operatorname{Dom}\gamma \to \mathbb{R}$ is a C^1-embedding, which implies that $\gamma\big|_{[t_0-\delta,t_0+\delta]}$ is a C^1-embedding.

If $p \in \{0, 1, \dots, \infty\}$, $\mathcal{C}$ is a collection of HTVSs, and $F : X \to Y$ is a map between two HTVSs, then unlike when both X and Y are finite-dimensional, if one or both of these TVSs is infinite-dimensions then one finds in the literature multiple competing and non-interchangeable definitions of "F is p-times continuously differentiable." Consequently, there are various completing definitions of a C^p-manifold modeled on TVSs in

$\mathcal{C}$ and it may not be clear to which of these definitions this paper's results apply to. Rather than arbitrarily picking one of definitions, we will now describe the types of definitions of C^p-manifolds modeled on TVSs in $\mathcal{C}$ that this paper's results are compatible with.

We suppose that we have established some notion of "p-times continuous differentiability in $\mathcal{C}$" (e.g. as in **[9]**, **[12]**, **[13]**, or **[14]**) such that (1) for curves, this notion agrees with the above definitions of C^p, (2) the composition of C^p-maps is again C^p, and (3) continuous linear maps and inclusions between open subsets are C^p. By a *C^p-isomorphism* we mean a homeomorphism that is C^p with a C^p inverse. By a *C^p-chart in $\mathcal{C}$ on a set M* we mean an injection $\varphi : U \to X$ from a subset of $U \subseteq M$ onto a non-empty open subset of a space X where $X \in \mathcal{C}$ and if $\psi : V \to Y$ is another such chart then we say that *ψ is C^p-consistent with φ* if $\varphi \circ \psi^{-1}\big|_{\psi(U \cap V)} : \psi(U \cap V) \to \varphi(U \cap V)$ is a C^p-isomorphism between open subsets. Now the definitions of a *C^p-atlas in $\mathcal{C}$ on M*, a *C^p-manifold modeled on TVSs in $\mathcal{C}$*, and of *the topology on M by induced a C^p-atlas* are defined in the usual way (e.g. as in **[9]** or **[14]**). One may also define *C^p-manifolds with boundary modeled on TVSs in $\mathcal{C}$* by directly generalizing the definitions found in **[14]**. By a *C^p-manifold* we mean a finite-dimensional Hausdorff manifold (without boundary) with a C^p-atlas. There will be no need in this paper to assume that manifolds are connected or second countable.

Remark 2.2. Given TVSs X and Y, $U \in \operatorname{Open}(X)$, $x \in U$, $v \in X$, and a continuous map $F : U \to Y$, there is near universal agreement that the directional derivative of a F at x in the direction v should be defined as $\lim_{h \to 0} \dfrac{F(x + hv) - F(x)}{h}$, which is just the derivative at 0 of the Y-valued curve $h \mapsto F(x + hv)$. So in this sense, almost all of the most common notions of differentiation are built upon the theory of differentiation of TVS-valued curves, which is the reason why of all possible classes of maps into a given TVS that we could investigate, we will choose to focus on sets of curves.

3. Fréchet-Urysohn and Sequential Spaces For any subset S of a space X, *the sequential closure of S in X* is

$$\operatorname{SeqCl}_X(S) := \left\{ x \in X \,:\, \text{there exists some sequence } (s_l)_{l=1}^{\infty} \subseteq S \text{ such that } s_\bullet \to x \text{ in } X \right\}$$

which is a subset of $\operatorname{Cl}_X(S)$. The set S is *sequentially closed in X* if $S = \operatorname{SeqCl}_X(S)$ while S is *sequentially open in X* if $X \setminus S$ is sequentially closed in X, or equivalently if whenever $s \in S$ and $(x_l)_{l=1}^{\infty} \subseteq X$ converges to s in X then there is some $L \in \mathbb{N}$ such that $l \geq L \implies x_l \in S$.

Notation 3.1. Denote the set of all sequentially open subsets of (X, τ_X) by $\operatorname{SeqOpen}(X, \tau_X)$.

Although every open (resp. closed) subset of X is necessarily sequentially open (resp. sequentially closed), the converse is not necessarily true so we call those spaces for which the converse does hold *sequential spaces*. That is, (X, τ_X) is sequential if $\tau_X = \operatorname{SeqOpen}(X, \tau_X)$ or equivalently, if it has the following universal property.

Universal Property 3.2. X is sequential if and only if for all spaces Y, a map $f : X \to Y$ is continuous if and only if whenever $x_\bullet = (x_l)_{l=1}^{\infty} \to x$ in X then $f(x_\bullet) \to f(x)$ in Y.

The following proposition is straightforward to verify.

Proposition 3.3. $\operatorname{SeqOpen}(X, \tau_X)$ is a topology on X, finer than τ_X, making X into a sequential space that has the same convergent sequences and limits as (X, τ_X) (i.e. for any $x \in X$ and sequence $x_\bullet \subseteq X$, $x_\bullet \to x$ in $(X, \operatorname{SeqOpen}(X, \tau_X))$ if and only if $x_\bullet \to x$ in (X, τ_X)). Furthermore, if τ is any topology on X with the same convergent sequences and limits as (X, τ_X) then $\operatorname{SeqOpen}(X, \tau_X)$ is finer than τ and (X, τ_X) has the same sequentially open subsets as (X, τ_X). This implies, in particular, that if $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous then so is $f : (X, \operatorname{SeqOpen}(X, \tau_X)) \to (Y, \operatorname{SeqOpen}(Y, \tau_Y))$.

Example 3.4 (**[8]**)**.** The prime spectrum of a commutative Noetherian ring (with the Zariski topology) is sequential.

Example 3.5. If d is an uncountable set and X is a T_1-space with at least two distinct points $x_0 \neq x_1$ then X^d is not a sequential space: Let $S = \{\chi_I : I \text{ is a countable subset of } d\}$ where for any $I \subseteq d$, $\chi_I : d \to X$ is defined by $\chi_I(i) = x_1$ if $i \in I$ and $\chi_I(i) = x_0$ otherwise. Since a countable union of countable sets is countable, S is a sequentially closed subset of X^d. But S is not closed in X^d for if we order $\mathcal{F}$, the set of all finite subsets of d, by subset inclusion then the net $\mathcal{F} \to S$ defined by $I \mapsto \chi_I$ converges in X^d to $(x_1)_{i \in d} \in X^d \setminus S$.

Even if X is sequential, it's still possible that there are subsets $S \subseteq X$ such that $\mathrm{Cl}_X(S) \neq \mathrm{SeqCl}_X(S)$ where if X is a sequential space then this inequality holds if and only if $\mathrm{SeqCl}_X(S) \neq \mathrm{SeqCl}_X(\mathrm{SeqCl}_X(S))$. We call X a *Fréchet-Urysohn space* if $\mathrm{Cl}_X(S) = \mathrm{SeqCl}_X(S)$ for all $S \subseteq X$, which can be shown to be equivalent to all subspaces of X being sequential spaces. Sequential (resp. Fréchet-Urysohn) spaces can be viewed as exactly those spaces X where for all $S \subseteq X$, knowledge of which sequences in S converge to which point(s) of X is sufficient to determine whether or not S is closed in X (resp. to determine S's closure in X). Every first-countable space is clearly Fréchet-Urysohn although there are Fréchet-Urysohn spaces that are not first-countable (**[10,** ex. 1**]**). In particular, the following non-metrizable separable HLCTVS is sequential but not Fréchet-Urysohn.

Example 3.6. Let $\mathbb{R}^\infty = \left\{(x_1, x_2 \ldots) \in \mathbb{R}^{\mathbb{N}} : \text{ all but finitely many } x_i \text{ are } 0\right\}$ and for all $m \leq n$ in $\mathbb{N}$, define $\mathrm{In}_m^n : \mathbb{R}^m \to \mathbb{R}^n$ and $\mathrm{In}^m : \mathbb{R}^n \to \mathbb{R}^\infty$ by $\mathrm{In}_m^n(x) = x \times \{0\}^{n-m}$ and $\mathrm{In}^n(x) = x \times \{0\}^{\mathbb{N}}$, respectively. Give $\mathbb{R}^\infty$ the final topology induced by the insertions $\mathrm{In}^\bullet = (\mathrm{In}^n)_{n \in \mathbb{N}}$ (rather than $\mathbb{R}^{\mathbb{N}}$'s subspace topology), which makes $(\mathbb{R}^\infty, \mathrm{In}^\bullet)$ into an inductive limit of $((\mathbb{R}^n)_{n \in \mathbb{N}}, \mathrm{In}_m^n, \mathbb{N})$ in the category of HLCTVSs. Recall that each $\mathrm{In}^n : \mathbb{R}^n \to \mathbb{R}^\infty$ is a linear embedding, so that (as usual) we will henceforth identify every $\mathbb{R}^n$ with its image under In^n and thus identify $\mathbb{R}^n \subseteq \mathbb{R}^{n+1} \subseteq \mathbb{R}^\infty$ and $\mathbf{0} = \{0\}^n = \{0\}^{n+1}$. Using this identification, recall that a subset $S \subseteq \mathbb{R}^\infty$ is open (resp. closed) in $\mathbb{R}^\infty$ if and only if $S \cap \mathbb{R}^n$ is open (resp. closed) in $\mathbb{R}^n$ for all $n \in \mathbb{N}$ and that a sequence in $\mathbb{R}^\infty$ converges if and only if there is some $n \in \mathbb{N}$ such that $\mathbb{R}^n$ contains this sequence and it converges in $\mathbb{R}^n$. We will now show (the presumably already known fact) that $\mathbb{R}^\infty$ is a sequential space that is not Fréchet-Urysohn.

Proof. If $S \subseteq \mathbb{R}^\infty$ is sequentially closed in $\mathbb{R}^\infty$ then the same is true of every $S \cap \mathbb{R}^n$ in $\mathbb{R}^n$, which is then necessarily a closed subset of the sequential space $\mathbb{R}^n$. Thus S is closed in $\mathbb{R}^\infty$, which shows that $\mathbb{R}^\infty$ is sequential. To see that $\mathbb{R}^\infty$ is not Fréchet-Urysohn, let S be the complement of $\bigcup_{n=1}^{\infty} B_n$ in $\mathbb{R}^\infty$, where each B_n is the open ball in $\mathbb{R}^n$ of radius $1/n$ (in the Euclidean norm) centered at the origin. It is easy to see that every non-zero element of $\mathbb{R}^\infty$ is in S's sequential closure from which it follows that $\mathrm{Cl}_{\mathbb{R}^\infty}(S) = \mathbb{R}^\infty$. However, $\mathbf{0} \notin \mathrm{SeqCl}_{\mathbb{R}^\infty}(S)$ since any sequence in S that converges to $\mathbf{0}$ would be contained in $\mathbb{R}^n$ for some $n \in \mathbb{N}$, which would imply that it converge to $\mathbf{0}$ in $\mathbb{R}^n$ while simultaneously being contained in S, a set that is disjoint from the neighborhood B_n of $\mathbf{0}$. Thus $\mathrm{SeqCl}_{\mathbb{R}^\infty}(S) = \mathbb{R}^\infty \setminus \{\mathbf{0}\} \neq \mathrm{Cl}_{\mathbb{R}^\infty}(S)$. ∎

One benefit of knowing that a space is Fréchet-Urysohn, rather than merely sequential, is that it guarantees (3) of proposition 3.7, which can be useful for diagonal arguments. Proposition 3.7 partially generalizes lemma 3.3 of **[2]**, which proved (1) $\implies$ (3) under the additional assumption that the space is a HTVS.

Proposition 3.7. If (X, τ_X) is a Hausdorff sequential space then the following are equivalent:

(1) X is a Fréchet-Urysohn space,
(2) If $(x_l)_{l=1}^\infty \to x_0$ in X has infinite range and for each $l \in \mathbb{N}$, $\left(x_l^i\right)_{i=1}^\infty \to x_l$ in X then there exist $\iota, \lambda : \mathbb{N} \to \mathbb{N}$ with λ increasing such that $\left(x_{\lambda(n)}^{\iota(n)}\right)_{n=1}^\infty \to x_0$.
(3) Statement (2) with the additional requirement that ι be increasing.
(4) Statement (3) with the additional requirement that $\iota > \lambda$.

Proof. (4) $\implies$ (3) $\implies$ (2) are immediate. To prove (2) $\implies$ (4), apply (2) with the given $(x_l)_{l=1}^\infty$ but for each $l \in \mathbb{N}$, using $\left(x_l^{l+i}\right)_{i=1}^\infty$ instead of $\left(x_l^i\right)_{i=1}^\infty$ in order to obtain $\iota', \lambda' : \mathbb{N} \to \mathbb{N}$ such that λ' is increasing and $\left(x_{\lambda'(n)}^{\lambda'(n)+\iota'(n)}\right)_{n=1}^\infty \to x_0$. Pick $(n_k)_{k=1}^\infty \subseteq \mathbb{N}$ increasing such that $(\iota'(n_k) + \lambda'(n_k))_{k=1}^\infty$ is increasing and define $\lambda(k) = \lambda'(n_k)$ and $\iota(k) = \iota'(n_k) + \lambda'(n_k)$.

(2) $\implies$ (1): Let $S \subseteq X$, let $T = \operatorname{SeqCl}_X(S)$, and suppose that $x_0 \in \operatorname{SeqCl}_X(T)$ but $x_0 \notin T$. Then we may pick an injective sequence $(x_l)_{l=1}^{\infty}$ in T that converges to x_0 in X. For all $l \in \mathbb{N}$, there exists a sequence $(x_l^i)_{i=1}^{\infty}$ in S that converges to x_l in X. By assumption, there exist $\iota, \lambda : \mathbb{N} \to \mathbb{N}$ with λ increasing such that $\left(x_{\lambda(n)}^{\iota(n)}\right)_{n=1}^{\infty} \to x_0$ which shows that $x_0 \in \operatorname{SeqCl}_X(S) = T$, a contradiction. Thus $\operatorname{SeqCl}_X(T) = T$ so that X being a sequential space now implies that T is closed in X, as desired.

(1) $\implies$ (2): Suppose that X is a Fréchet-Urysohn, $x_\bullet = (x_l)_{l=1}^{\infty} \to x_0$ in X has an injective subsequence in X, and for all $l \in \mathbb{N}$, $\left(x_l^i\right)_{i=1}^{\infty} \to x_l$ in X. We may assume without loss of generality that $x_\bullet \to x_0$ is injective in X. Since X is Hausdorff, we may inductively pick $(l_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ increasing and sequences of open sets $(U_n)_{n=1}^{\infty}$ and $(V_n)_{n=1}^{\infty}$ such that all $(U_n)_{n=1}^{\infty}$ are pairwise disjoint and for all $n \in \mathbb{N}$, U_n is a neighborhood of x_{l_n}, $V_{n+1} \subseteq V_n$, and V_n is a neighborhood of x disjoint from $U_1, \ldots, U_n$ containing $U_{n+1}, U_{n+2}, \ldots$ such that $l \geq l_{n+1}$ implies $x_l \in V_n$. For each $n \in \mathbb{N}$, pick i_n such that $i \geq i_n$ implies $x_{l_n}^i \in U_n$. Replacing $(x_l)_{l=1}^{\infty}$ with $(x_{l_n})_{n=1}^{\infty}$ and replacing each $\left(x_{l_n}^i\right)_{i=1}^{\infty}$ with $\left(x_{l_n}^i\right)_{i=i_n}^{\infty}$, we may assume that for all $n \in \mathbb{N}$, x_n and each x_n^i belongs to U_n and that if $l > n$ then x_l and all x_l^i belong to V_n.

Let $S = \left\{x_l^i : i, l \in \mathbb{N}\right\}$ and observe that since $x_0 \in \operatorname{Cl}_X(S) = \operatorname{SeqCl}_X(S)$, we may pick a sequence $(s_n)_{n=1}^{\infty} \subseteq S$ that converges to x_0 in X. For all $n \in \mathbb{N}$, pick any $l_n, i_n \in \mathbb{N}$ such that $s_n = x_{l_n}^{i_n}$. Observe that $(l_n)_{n=1}^{\infty}$ is unbounded since if $L > l_n$ for all $n \in \mathbb{N}$ then all $s_n = x_{l_n}^{i_n}$ would be contained in $U_1 \cup \cdots \cup U_L$, a set that is disjoint from the neighborhood V_{L+1} of x_0, which contradicts the fact that $s_\bullet$ converges to x_0. So pick $(n_k)_{k=1}^{\infty}$ increasing such that $(l_{n_k})_{k=1}^{\infty}$ is increasing and let $\lambda(k) = l_{n_k}$ and $\iota(k) = i_{n_k}$. Since $\left(x_{\lambda(n)}^{\iota(n)}\right)_{n=1}^{\infty}$ is a subsequence of the convergent sequence $\left(x_{l_n}^{i_n}\right)_{n=1}^{\infty}$, it converges to x_0 in X. ■

4. Sequentially Quotient Mappings

Definition 4.1. Let $f : X \to Y$ be a map between spaces and let $y_\bullet = (y_i)_{i \in I}$ be a net in Y. If $y_\bullet$ is convergent in Y (i.e. there exists some $y \in Y$ such that $y_\bullet \to y$ in Y) then by an *f-lift of $y_\bullet$* we mean an I-directed convergent net $(x_i)_{i \in I}$ in X such that $f(x_i) = y_i$ for all $i \in I$. If there exists a net $x_\bullet = (x_i)_{i \in I}$ in X that is an f-lift of a convergent net $y_\bullet$ then we'll say that *f lifts $y_\bullet$ to $x_\bullet$*, that *f can lift $y_\bullet$*, that *$y_\bullet$ has an f-lift*, and that *$y_\bullet$ is f-liftable.* When we write *$(x_i)_{i \in I} \to x$ is an f-lift of $(y_i)_{i \in I} \to y$* then we mean that (1) $y_\bullet \to y$ in Y, (2) $f(x) = y$, and (3) $x_\bullet$ is an f-lift of $y_\bullet$ that converges to x in X. If we say that *$y_\bullet \to y$ has an f-lift* then we mean that there exists some $x_\bullet = (x_i)_{i \in I} \subseteq X$ and $x \in X$ such that $x_\bullet \to x$ is an f-lift of $y_\bullet \to y$.

Remark 4.2. Given $y_\bullet \to y$ in Y and a continuous surjection $f : X \to Y$, it is emphasized that in general, the statement "$y_\bullet$ has an f-lift" is not interchangeable with the statement "$y_\bullet \to y$ has an f-lift."

Sequentially quotient mappings were introduced in **[5]**.

Definition 4.3. A surjection $f : (X, \tau_X) \to (Y, \tau_Y)$ is *sequentially quotient* (**[4]**) if $f : (X, \operatorname{SeqOpen}(X, \tau_X)) \to (Y, \operatorname{SeqOpen}(Y, \tau_Y))$ is a quotient map. Say that $f : X \to Y$ is *sequentially quotient onto its image* if $f : X \to \operatorname{Im} f$ is sequentially quotient.

Warning 4.4. Many authors (e.g. **[7]**, **[18]**) define sequentially quotient mappings as those continuous surjections that satisfy (3) of proposition 4.5 rather than the original definition 4.3 and although these definitions are equivalent when the codomain is Hausdorff, this is not necessarily the case otherwise. The remainder of this subsection, which is not necessary for the rest of this paper, is dedicated to clarifying the relationships between competing definitions of "sequentially quotient" and while the last two sentences of proposition 4.5 are well-known observations, the rest of this subsection on sequentially quotient mappings is original.

Proposition 4.5. Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous surjection and consider the following statements:

(1) Whenever $y_\bullet = (y_l)_{l=1}^{\infty} \to y$ in Y then there exists a subsequence $y_{l_\bullet} = (y_{l_k})_{k=1}^{\infty}$ such that $y_{l_\bullet} \to y$ has an f-lift.

(2) f is sequentially quotient.
(3) Every convergent sequence in Y has an f-liftable subsequence.

Then (1) $\implies$ (2) and (3). If points in Y are sequentially closed then (2) $\implies$ (3) while if convergent sequences in Y have unique limits then (1) – (3) are all equivalent. If X is sequentially compact then (3) holds. If X is sequential then f is a quotient map if and only if Y is sequential and f is sequentially quotient.

Proof. That (1) implies both (2) and (3) is immediate, as is the fact that (3) holds when X is sequentially compact. If f is a quotient map and X is sequential then for any $S \subseteq Y$ such that $f^{-1}(S)$ is sequentially open, $f^{-1}(S)$ is open in X so that S is open, and thus sequentially open, in Y, which together with proposition 3.3 shows that f is sequentially quotient whence Y's sequentialness follows immediately.

(2) $\implies$ (3): Assume that points in Y are sequentially closed and that f is sequentially quotient. Suppose $y_\bullet$ is a convergent sequence in Y that has no f-liftable subsequence. If $y_\bullet$ has an infinite constant subsequence then we're done so assume without loss of generality that $y_\bullet$ is injective. Replacing $y_\bullet$ with a subsequence if necessary, we may assume without loss of generality that $y_\bullet$ has a limit that is not contained in $I := \{y_l : l \in \mathbb{N}\}$. Let $x_\bullet$ be a sequence in $f^{-1}(I)$ converging in X to $x \in X$. Suppose for the sake of contradiction that $f(x_\bullet)$ was infinite, let $\iota(1) = 1$, and let $\lambda(1) \in \mathbb{N}$ be such that $f(x_{\iota(1)}) = y_{\lambda(1)}$. Proceeding inductively on $n \geq 2$, note that since $f(x_\bullet)$ is infinite so is $\{f(x_i) : i > \iota(n-1)\} \cap \{y_l : l > \lambda(n-1)\}$, which permits us to pick $\iota(n) > \iota(n-1)$ and $\lambda(n) > \lambda(n-1)$ such that $f(x_{\iota(n)}) = y_{\lambda(n)}$. Since $f(x_{\iota(n)}) = y_{\lambda(n)}$ for all $n \in \mathbb{N}$ and $x_{\iota(\bullet)} \to x$, we obtain a contradiction. Since $f(x_\bullet)$ is a finite subset of I, we may pick a subsequence $x_{l_\bullet} = (x_{l_n})_{n=1}^\infty$ of $x_\bullet$ such that for some $L \in \mathbb{N}$, $f(x_{l_n}) = y_L$ for all $n \in \mathbb{N}$. Since $x_{l_\bullet} \to x$ in X, f is continuous, and $\{y_L\}$ is sequentially closed, it follows that $f(x) \in \operatorname{SeqCl}_Y(\{y_L\}) = \{y_L\} \subseteq I$. Thus $x \in f^{-1}(I)$ so that we've thus shown that $\operatorname{SeqCl}_X(f^{-1}(I)) \subseteq f^{-1}(I)$. Since $f^{-1}(I)$ is sequentially closed in X and $f : (X, \operatorname{SeqOpen}(X, \tau_X)) \to (Y, \operatorname{SeqOpen}(Y, \tau_Y))$ is a quotient map, I is sequentially closed in Y. But recall that $y_\bullet$ has a limit that is not contained in I, which gives us a contradiction.

Now suppose that convergent sequences in Y have unique limits, which immediately gives us (3) $\implies$ (1). Since our assumption implies that points are sequentially closed in Y, (2) $\implies$ (3) holds so that (1) – (3) are equivalent. ∎

Example 4.6 furnishes a sequentially quotient map that fails to satisfy (3) of proposition 4.5, despite its domain being a manifold and its codomain being sequential.

Example 4.6. For all $n \in \mathbb{Z}$, let $X_n = \mathbb{R} \times \{n\}$, give $X := \bigsqcup_{n \in \mathbb{Z}} X_n$ the disjoint union topology, let Y be the set $]0, \infty[\cup \{(0, n) : n \in \mathbb{Z}\}$, let $f : X \to Y$ be the surjection defined by

$$(z, n) \longmapsto \begin{cases} (0, n-1) & \text{if } z < 0 \\ (0, n) & \text{if } z = 0 \\ z & \text{if } z > 0 \end{cases}$$

and give Y the quotient topology induced by f. Since X is a manifold, Y is sequential and f is sequentially quotient. Observe that if $V \subseteq Y$ is an open set containing some $(0, n) = f(0, n)$ then $X_n \cap f^{-1}(V)$ is a neighborhood of $(0, n)$ in $\mathbb{R} \times \{n\}$ so that we may pick some negative real z such that $(z, n) \in f^{-1}(V)$, which implies that $(0, n-1) = f(z, n) \in f(f^{-1}(V)) = V$. Proceeding by induction, it follows that if an open set $V \subseteq Y$ contains $(0, n)$ then it contains $\{(0, l) : l \in \mathbb{Z}, l \leq n\}$, which implies that if a sequence of integers $l_\bullet = (l_k)_{k=1}^\infty$ diverges to $-\infty$ then for every $n \in \mathbb{Z}$, $(0, l_\bullet) := ((0, l_k))_{k=1}^\infty \to (0, n)$ in Y. However, if $l_\bullet \to -\infty$ then the convergent sequence $(0, l_\bullet)$ cannot have an f-lift $x_\bullet = ((z_k, n_k))_{k=1}^\infty$ for otherwise let $(z, n) = \lim x_\bullet$ and pick $N \in \mathbb{N}$ such that if $k \geq N$ then $x_k = (z_k, n_k) \in X_n$ so that $(0, l_k) = f(z_k, n_k) = f(z_k, n)$, which forces all these l_k to equal either n or $n-1$, a contradiction. In particular, although $((0, -l))_{l=1}^\infty$ converges in Y, it has no f-liftable subsequence. ∎

It is straightforward to verify that the map f in the next example satisfies (3), but not (2) (and hence also not (1)), of proposition 4.5.

Example 4.7. Give $Z := (\mathbb{R} \times \{0\}) \sqcup (\mathbb{R} \times \{1\})$ the disjoint union topology, give $X := (\mathbb{R} \times \{0\}) \cup \{(0,1)\}$ the resulting subspace topology, let $\pi : Z \to \mathbb{R} \cup \{(0,1)\}$ be the usual quotient map making $Y := \mathbb{R} \cup \{(0,1)\}$ into the line with two origins, where $\pi(0,1) = (0,1)$ and $\pi(z,i) = z$ otherwise, and let $f = \pi\big|_X : X \to Y$.

5. Coherence Suppose X is a set and $\mathcal{F}$ is a collection of X-valued maps where for each $f \in \mathcal{F}$, f's domain, denoted by $\operatorname{Dom} f$, has a topology τ_f. Then *the final topology $\tau_{\mathcal{F}}$ on X induced by $\mathcal{F}$* is the finest topology on X making all $f : (\operatorname{Dom} f, \tau_f) \to (X, \tau_{\mathcal{F}})$ continuous. This topology's open (resp. closed) sets are characterized by: a subset S of X is open (resp. closed) in $(X, \tau_{\mathcal{F}})$ if and only if for all $f \in \mathcal{F}$, $f^{-1}(S)$ is open (resp. closed) in $(\operatorname{Dom} f, \tau_f)$. If τ_X is a topology on X then say that τ_X (or (X, τ_X) or simply X if τ_X is understood) *is coherent with* $\mathcal{F}$ if $\tau_X = \tau_{\mathcal{F}}$.

Observations 5.1. Let (X, τ_X) and (Y, τ_Y) be spaces, let $\mathcal{F}$ and $\mathcal{G}$ be two collections of continuous maps into (X, τ_X), and suppose that τ_X is coherent with $\mathcal{F}$.

- If $\mathcal{F} \subseteq \mathcal{G}$ then τ_X is coherent with $\mathcal{G}$.
- A surjection $\pi : X \to Y$ is a quotient map if and only if τ_Y is coherent with $\pi \circ \mathcal{F} := \{\pi \circ f : f \in \mathcal{F}\}$.
- For every $f \in \mathcal{F}$, let $\mathcal{E}_f$ be a collection of continuous maps into $(\operatorname{Dom} f, \tau_{\operatorname{Dom} f})$. If $\tau_{\operatorname{Dom} f}$ is coherent with $\mathcal{E}_f$ for each $f \in \mathcal{F}$ then τ_X is coherent with
$$\bigcup_{f \in \mathcal{F}} \{f \circ e : e \in \mathcal{E}_f\}$$

Remark 5.2. Suppose $\mathcal{G}$ is a collection of continuous maps into (X, τ_X) with respect to which τ_X is coherent. In light of observation 5.1, it's reasonable to try to replace $\mathcal{G}$ with a subset $\mathcal{F}$ that τ_X is again coherent with but that may in some sense be "nicer to work with" than $\mathcal{G}$. Since we are primarily interested with how the topologies of HTVSs relate to their C^p-curves (via coherence) for various $p \in \{0, 1, \ldots, \infty\}$, "nicer" in this paper will mean having more derivatives, ideally with the first derivative not vanishing, which due to observation 2.1, leads us to further desire coherence with curves that are C^p-embeddings for the largest possible p. The rest of this paper is largely dedicated to finding such sets of curves since, for instance, even knowing that a space is coherent with a set of C^1-arcs could potentially allows one to use the tools of analysis and differential geometry to help prove that some particular subset is open or closed.

Example 5.3. If $\mathcal{O} \subseteq \tau_X$ is a cover of X and if every $O \in \mathcal{O}$ is given its subspace topology $\tau_X\big|_O$ then τ_X is coherent with the set of inclusion maps $\operatorname{In}_O^X : O \to X$ as O varies over $\mathcal{O}$.

Example and Definition 5.4. Call X a *k-space* and say that it is *compactly generated* if τ_X is coherent with the set of all inclusion maps $\operatorname{In}_K^X : \left(K, \tau_X\big|_K\right) \to X$ as K varies over all compact subspaces of X.

5.1. Relationship to the Original Definition of Coherence The following lemma, which applies to both examples 5.3 and 5.4, is readily verified.

Lemma 5.5. If each $f \in \mathcal{F}$ is a quotient map onto its image in (X, τ) (i.e. $f : (\operatorname{Dom} f, \tau_f) \to \left(\operatorname{Im} f, \tau\big|_{\operatorname{Im} f}\right)$ is a quotient map) then the following are equivalent:

(1) τ is coherent with $\mathcal{F}$.
(2) A subset $S \subseteq X$ is open in (X, τ) $\iff$ for all $f \in \mathcal{F}$, $S \cap \operatorname{Im} f$ is open in $\left(\operatorname{Im} f, \tau\big|_{\operatorname{Im} f}\right)$.
(3) A subset $S \subseteq X$ is closed in (X, τ) $\iff$ for all $f \in \mathcal{F}$, $S \cap \operatorname{Im} f$ is closed in $\left(\operatorname{Im} f, \tau\big|_{\operatorname{Im} f}\right)$.

If $\mathcal{A}$ is a collection of subsets of X then it is well-established terminology to say that a topology τ_X on X is *coherent with* $\mathcal{A}$ if a subset S of X is closed in (X, τ_X) if and only if for all $A \in \mathcal{A}$, $S \cap A$ is closed in $\left(A, \tau_X\big|_A\right)$. Lemma 5.5 shows that this established definition of coherence is equivalent to τ_X being coherent with set of inclusion maps $\left\{\operatorname{In}_A^X : A \in \mathcal{A}\right\}$ where each $A \in \mathcal{A}$ is given the subspace topology from (X, τ_X). In particular, this shows that the definitions of coherence, weak topology, and k-space given in this paper are equivalent to their usual definitions (e.g. as found in **[6]** or **[17]**) and this is also the reason for this paper's decision to reuse the word "coherent" by extending the word's usual meaning from collections of subsets to collections of maps in the manner defined above. Observe that if τ_X and $\mathcal{F}$ are as in lemma 5.5 then this terminology allows us to restate that lemma's conclusion as: τ_X is coherent with (the set of all maps in) $\mathcal{F}$ if and only if τ_X is coherent with the set of all images of maps in $\mathcal{F}$.

If we say that τ_X *is coherent with its paths* (resp. *arcs*) then we mean that τ_X is coherent with the set of all paths (resp. arcs) in X, which if τ_X is Hausdorff, is equivalent to τ_X being coherent with the set of all images of paths (resp. arcs) in X.

5.2. Coherence and Sequences

Proposition 5.6. Let $\mathcal{F}$ be a collection of continuous maps into (X, τ_X) whose domains are Fréchet-Urysohn spaces. If τ_X is coherent with $\mathcal{F}$ then X is a sequential space.

Proof. Let $S \subseteq X$ be sequentially closed, let $f : D \to X$ be in $\mathcal{F}$, and let $d \in \operatorname{Cl}_D\big(f^{-1}(S)\big)$. Let $(d_l)_{l=1}^{\infty} \subseteq f^{-1}(S)$ converge to d. Since f is continuous, $f(d_\bullet) \to f(d)$ in X so that $f(d) \in \operatorname{SeqCl}_X(S) = S$, which shows that $\operatorname{Cl}_D\big(f^{-1}(S)\big) = f^{-1}(S)$ is closed in D. ∎

The next lemma is a fundamentally important tool for this paper that underpins most of its main results.

Lemma 5.7. Let (X, τ_X) be Hausdorff, let $\mathcal{C}$ be a collection of continuous maps in X, and let $(\star)$ denote the following statement:

$(\star)$: whenever $x^\bullet = \big(x^l\big)_{l=1}^{\infty} \subseteq X$ is an infinite-ranged sequence converging to x in X then there exists some $\gamma \in \mathcal{C}$ and some γ-liftable subsequence $\big(x^{l_k}\big)_{k=1}^{\infty}$ of $x^\bullet$ such that $\big(x^{l_k}\big)_{k=1}^{\infty} \to x$ is injective in X.

If X is Fréchet-Urysohn and $(\star)$ holds then τ_X is coherent with $\mathcal{C}$. If τ_X is coherent with $\mathcal{C}$ and if every map in $\mathcal{C}$ is that is not sequentially quotient onto its image has a Fréchet-Urysohn domain, then $(\star)$ holds.

Remark 5.8. It will be clear from the proof that we may replace $(\star)$ with: "whenever $x^\bullet = (x^l)_{l=1}^{\infty} \to x$ is injective in X then there exists some $\gamma \in \mathcal{C}$ and some γ-liftable subsequence of $x^\bullet$."

Convention 5.9. If $\gamma \in \mathcal{C}$ is a curve, then we will pick the γ-liftable subsequence $\big(x^{l_k}\big)_{k=1}^{\infty}$ in $(\star)$ so that it has a monotone convergent γ-lift.

Proof. Assume first that τ_X is coherent with $\mathcal{C}$ and that whenever a map in $\mathcal{C}$ is not sequentially quotient onto its image then it has a Fréchet-Urysohn domain. Let $x^\bullet = \big(x^l\big)_{l=1}^{\infty} \subseteq X$ be an infinite-ranged sequence converging in X to $x \in X$ and observe that it suffices to prove $(\star)$'s conclusion under the additional assumption that $x^\bullet \to x$ is injective in X. Let $S = \big\{x^l : l \in \mathbb{N}\big\}$ and note that $x \notin S$. We will assume that such a γ and subsequence does not exist and obtain a contradiction by concluding that S is closed in X. Let $\gamma \in \mathcal{C}$ and note that since τ_X is coherent with $\mathcal{C}$ it's enough to show that $\gamma^{-1}(S)$ is closed. If $S \cap \operatorname{Im}\gamma$ is finite then $S \cap \operatorname{Im}\gamma$ is compact so $\gamma^{-1}(S)$ is closed. So assume that $S \cap \operatorname{Im}\gamma$ is infinite and consists of the subsequence $(x^{n_k})_{k=1}^{\infty}$.

Suppose first that $\gamma : \operatorname{Dom}\gamma \to \operatorname{Im}\gamma$ is sequentially quotient. If $x \in \operatorname{Im}\gamma$ then since $(x^{n_k})_{k=1}^{\infty} \to x$ in $\operatorname{Im}\gamma$ and $\gamma : \operatorname{Dom}\gamma \to \operatorname{Im}\gamma$ is sequentially quotient, we can pick a subsequences of $(x^{n_k})_{k=1}^{\infty}$ that we had been assumed not to exist. Thus $x \notin \operatorname{Im}\gamma$ so that since $S \cup \{x\}$ is closed in X, $\gamma^{-1}(S) = \gamma^{-1}(S \cup \{x\})$ is closed, as desired. So we may henceforth assume that γ's domain is Fréchet-Urysohn.

Suppose that $\gamma^{-1}(S)$ is not closed and let $t_0 \in \overline{\gamma^{-1}(S)} \setminus \gamma^{-1}(S)$. Since $\operatorname{Dom}\gamma$ is Fréchet-Urysohn and $t_0 \in \overline{\gamma^{-1}(S)}$, there exists some sequence $(t_j)_{j=1}^{\infty}$ in $\gamma^{-1}(S)$ converging to t_0. Let $j_1 = 1$ and let l_1 be the unique integer such that $\gamma(t_{j_1}) = x^{l_1}$. Having picked $j_1 < \cdots < j_k$ and $l_1 < \cdots < l_k$ such that $\gamma(t_{j_1}) = x^{l_1}, \ldots, \gamma(t_{j_k}) = x^{l_k}$, let $j_{k+1} > j_k$ be such that for all $j \geq j_{k+1}$, t_j belongs to the open neighborhood $\gamma^{-1}\big(X \setminus \{x^l : 1 \leq l \leq l_k\}\big)$ of t_0 and then let l_{k+1} be the unique index such that $\gamma(t_{j_{k+1}}) = x^{l_{k+1}}$. Since $(t_{j_k})_{k=1}^{\infty} \to t_0$ and X is Hausdorff, $\gamma(t_0) = \lim\limits_{k\to\infty} \gamma(t_{j_k}) = \lim\limits_{k\to\infty} x^{l_k} = x$, a contradiction that finishes the proof.

Now assume that X is Fréchet-Urysohn and that $(\star)$ holds. Let $S \subseteq X$ be such that $\gamma^{-1}(S)$ is closed for all $\gamma \in \mathcal{C}$. Let $x \in \mathrm{Cl}_X(S)$ and suppose for the sake of contradiction that $x \notin S$. Since X is Fréchet-Urysohn we may pick a sequence $\left(x^l\right)_{l=1}^{\infty} \subseteq S$ converging to x where since $x \notin S$, this sequence has infinite range. By assumption, there is some $\gamma \in \mathcal{C}$ and some γ-liftable subsequence $\left(x^{l_k}\right)_{k=1}^{\infty}$ of $\left(x^l\right)_{l=1}^{\infty}$. Let $(t_k)_{k=1}^{\infty} \to t_0$ be a γ-lift of $\left(x^{l_k}\right)_{k=1}^{\infty} \to x$. Since $\gamma(t_k) = x^{l_k} \in S$ for all $k \in \mathbb{N}$, this implies that $t_0 \in \overline{\gamma^{-1}(S)}$, where this set is just $\gamma^{-1}(S)$ so that $\gamma(t_0) \in S$. Since X is Hausdorff it follows that $x = \gamma(t_0) \in S$. ∎

Corollary 5.10. Suppose (X, τ_X) is a Hausdorff Fréchet-Urysohn space and $\mathcal{C}$ is a collection of continuous maps in X where each map either has a Fréchet-Urysohn domain or is otherwise sequentially quotient onto its image. Then τ_X is coherent with $\mathcal{C}$ if and only if $(\star)$ from lemma 5.7 holds.

Corollary 5.11. If $X = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x^i \geq 0, \ldots, x^n \geq 0\right\}$ (where $1 \leq i \leq n$) then X is *not* coherent with any set of C^1-embeddings into X whose domains are all open intervals.

Corollary 5.12. Let J be an interval and $\mathcal{A}$ be a cover of J by intervals such that for all $x \in J$, if x does does not belong to the interior of an interval in $\mathcal{A}$ then there exist intervals L (resp. R) in $\mathcal{A}$ containing x as a left (resp. right) endpoint. Then J is coherent with $\mathcal{A}$. In particular, J is coherent with $\{[a,b] : a, b \in J, a < b\}$.

The following lemma follows easily from corollary 5.12. Its last conclusion essentially states that if a space is coherent with a set of curves $\mathcal{C}$ then it will also be coherent with the set of curves that results if one replaces each non-path curve in $\mathcal{C}$ with a set of restrictions of this curve to compact intervals that cover its domain.

Lemma 5.13. Let (X, τ_X) be a space, $\mathcal{C}$ be a set continuous maps in X, and for every $\gamma \in \mathcal{C}$ let $\mathcal{A}_\gamma$ denote a collection of subsets of $\mathrm{Dom}\,\gamma$. Let

$$\mathcal{P} = \bigcup_{\gamma \in \mathcal{C}} \left\{\gamma\big|_A : A \in \mathcal{A}_\gamma\right\}.$$

If τ_X is coherent with $\mathcal{P}$ then τ_X is coherent with $\mathcal{C}$. If τ_X is coherent with $\mathcal{C}$ and the domain of every $\gamma \in \mathcal{C}$ is coherent with $\mathcal{A}_\gamma$ then τ_X is coherent with $\mathcal{P}$.

In particular, if $\mathcal{C}$ is a set of curves and if for all $\gamma \in \mathcal{C}$ we let $\mathcal{A}_\gamma = \{\mathrm{Dom}\,\gamma\}$ whenever γ is a path and let $\mathcal{A}_\gamma = \{[a,b] : a, b \in \mathrm{Dom}\,\gamma, a < b\}$ otherwise, then τ_X being coherent with $\mathcal{C}$ implies that it is coherent with $\mathcal{P}$.

Remark 5.14. Theorem 0.2 and corollary 5.11 show that the last conclusion could fail if we were to restrict paths to open intervals instead of restricting non-path curves to compact intervals.

Corollary 5.15. Let $k \in \{0, 1, \ldots, \infty\}$ and $0 \leq p \leq k$. If S is a subset of a C^k-manifold with corners that is coherent with the set of all S-valued C^p-curves (resp. C^p-embeddings) whose domains are open intervals then S is coherent with the set of all S-valued C^p-paths (resp. C^p-embeddings) with domain $[0,1]$.

Corollary 5.16. If a space (X, τ_X) is coherent with a set $\mathcal{P}$ of continuous maps into X then τ_X is coherent with any set of continuous extensions into X. In particular, if (X, τ_X) is coherent with a set of paths and if each path has a continuous extension to an open interval, then τ_X is coherent with these extensions.

Theorem 10.4 and remark 5.2 motivate the definitions of the following sets of curves.

Corollary 5.17. Let M be a smooth manifold (without boundary) modeled on HTVSs and let $p \in \{0, 1, \ldots, \infty\}$. For any interval I, let $\mathcal{A}_I$ denote the set of all C^p-embeddings $I \to M$ that are smooth everywhere except for possibly at a single point. Let $\mathcal{A}^*_{[0,1]}$ denote those maps in $\mathcal{A}_{[0,1]}$ that are smooth on $]0,1]$. If M is coherent with $\mathcal{A}^*_{[0,1]}$ or if there is any interval I such that M is coherent with $\mathcal{A}_I$ then M is coherent with $\mathcal{A}^*_{[0,1]}$ and with $\mathcal{A}_J$ and for all intervals J.

Proof. This follows easily from lemmata 5.7 and 5.13 and the fact that every C^p-embedding of $[0,1]$ into M has an extension to an open interval containing $[0,1]$. ∎

Part 3. Fréchet-Urysohn HLCTVSs are Coherent with Arcs

Lemma 5.18. Let X be a HLCTVS and suppose $\left(x^l\right)_{l=1}^{\infty} \to x^0$ in X has infinite range. Then there exists an increasing sequence $(l_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ and an arc $\gamma : ([-1,1],0) \to (X,x^0)$ such that $\gamma\left(\frac{1}{k}\right) = x^{l_k}$ for all $k \in \mathbb{N}$.

Remark 5.19. The idea of the construction of γ is similar to how it would be constructed in a $\mathbb{R}^d$ $(d \in \mathbb{N})$ except that the necessity of guaranteeing that γ be both injective and pass through infinitely many $x^{\bullet}$'s complicates the construction. It's due to these requirements that to construct γ, it may not be enough to simply construct a continuous curve passing through these points and then evoke the fact the image of a path is arc-wise connected. Indeed, it is not even clear that such a curve even necessarily exists without local convexity.

Proof. Assume without loss of generality that $x^0 = \mathbf{0}$ and that $x^{\bullet} \to \mathbf{0}$ is injective in X. If there exists a subsequence of $\left(x^l\right)_{l=0}^{\infty}$ that is contained in a finite-dimensional affine subspace then the conclusion is obvious so assume that no such subsequence exists. For any $S \subseteq X$ let $\operatorname{Aff}(S)$ (resp. $\operatorname{co}(S)$) denote the affine span (resp. convex hull) of S in X.

Let $l_1 = 1$ and pick $l_2 > l_1$ be such that $\mathbf{0} \notin \operatorname{Aff}\left(x^{l_1}, x^{l_2}\right)$. Suppose we have increasing integers $l_1, \ldots, l_k$ such that $\mathbf{0} \notin \operatorname{Aff}\left(x^{l_1}, \ldots x^{l_h}\right)$ and for all $h = 2, \ldots, k$, $x^h \notin \operatorname{Aff}\left(x^{l_1}, \ldots, x^{l_{h-1}}\right)$. Observe that if $x \in X$ is such that $\mathbf{0} \in \operatorname{Aff}\left(x, x^{l_1}, \ldots, x^{l_k}\right)$ then the fact that $\mathbf{0} \notin \operatorname{Aff}\left(x^{l_1}, \ldots, x^{l_k}\right)$ implies that x belongs to $\operatorname{Span}\left(x^{l_1}, \ldots, x^{l_k}\right)$. Since no infinite subsequence of $\left(x^l\right)_{l=0}^{\infty}$ is contained in any finite-dimensional affine subspace, this implies that there exists some $L > l_k$ such that $l \leq L \implies \mathbf{0} \notin \operatorname{Aff}\left(x^{l_1}, \ldots, x^{l_k}, x^l\right)$. Pick a non-empty balanced open set U_{k+1} such that $U_{k+1} \cap \operatorname{Aff}\left(x^{l_1}, \ldots, x^{l_k}\right) = \varnothing$ and let $l_{k+1} \geq L$ be such that $l \geq l_{k+1} \implies x^l \notin U_{k+1}$. Observe that $l \geq l_{k+1} \implies x^l \notin \operatorname{Aff}\left(x^{l_1}, \ldots, x^{l_k}, x^l\right)$, which completes the construction.

Define $\gamma_k : \left[\frac{1}{k+1}, \frac{1}{k}\right] \to X$ by $\gamma_k(t) = x^{l_{k+1}} + \left[\frac{t - \frac{1}{k+1}}{\frac{1}{k} - \frac{1}{k+1}}\right]\left(x^{l_{k+1}} - x^{l_k}\right)$. Note that that for all $k \in \mathbb{N}$, $\operatorname{co}\left(x^{l_{k+1}}, x^{l_k}\right) \cap \operatorname{Aff}\left(x^{l_1}, \ldots, x^{l_k}\right) = \left\{x^{l_k}\right\}$ and $\mathbf{0} \notin \operatorname{co}\left(x^{l_{k+1}}, x^{l_k}\right)$ so that $\operatorname{Im}\gamma_k \cap \operatorname{Im}\gamma_{k-1} = \left\{x^{l_k}\right\}$, and $\operatorname{Im}\gamma_k \cap \operatorname{Im}\gamma_{k+1} = \left\{x^{l_{k+1}}\right\}$, and for any $h \in \mathbb{N}$, if $|h - k| > 2$ then γ_h and γ_k have disjoint images that do not contain $\mathbf{0}$. Define $\gamma : [0,1] \to X$ by $\gamma(0) = \mathbf{0}$ and $\gamma = \gamma_k$ on $\left[\frac{1}{k+1}, \frac{1}{k}\right]$ and observe that γ is injective and that $\gamma\big|_{]0,1]}$ is continuous. To see that γ is continuous at $\mathbf{0}$, let U be any convex neighborhood of $\mathbf{0}$ in X and pick $L \in \mathbb{N}$ such that $k \geq L$ implies $x^{l_k} \in U$. Observe that for any $k \geq l$, $\gamma\left(\left[\frac{1}{k+1}, \frac{1}{k}\right]\right) = \operatorname{Im}\gamma_k = \operatorname{co}\left(x^{l_{k+1}}, x^{l_k}\right) \subseteq U$ so that $\gamma\left(\left]0, \frac{1}{k}\right]\right) \subseteq U$. Now since $\operatorname{co}(\operatorname{Im}\gamma)$ is not all of X (since $\dim X > 3$) we may pick $x \neq \mathbf{0}$ in X such that $\{rx : r \geq 0\} \cap \operatorname{Im}\gamma = \varnothing$ and now extend γ to $[-1,1]$ by defining $\gamma(t) = -tx$ for $t \in [-1,0]$, where this extension is necessarily injective and continuous and thus a topological embedding. ∎

Proposition 5.6 implies that HTVSs coherent with C^0-arcs are necessarily sequential. We now prove theorem 0.8, which together with proposition 5.6 shows that in the category of HLCTVSs, the class of spaces coherent with their arcs lies in-between the class of Fréchet-Urysohn spaces and the class of sequential spaces.

Proof of Theorem 0.8. Let $S \subseteq X$ be a non-empty subset of the Fréchet-Urysohn HLCTVS X such that for all arcs γ in X, $S \cap \operatorname{Im}\gamma$ is closed in $\operatorname{Im}\gamma$. Let $x \in \operatorname{Cl}_X(S)$ be a non-isolated point and pick a sequence $\left(x^l\right)_{l=1}^{\infty}$ in S converging to x. Let $(l_k)_{k=1}^{\infty}$ and $\gamma : ([0,1],0) \to (X,x)$ be as in lemma 5.18 and note that since each $\gamma\left(\frac{1}{k}\right)$ belongs to $S \cap \operatorname{Im}\gamma$, $x = \gamma(0)$ belongs to $\operatorname{Cl}_{\operatorname{Im}\gamma}(S \cap \operatorname{Im}\gamma)$. But since $S \cap \operatorname{Im}\gamma$ is by assumption a closed subset of $\operatorname{Im}\gamma$, it follows that $x \in S \cap \operatorname{Im}\gamma \subseteq S$. ∎

Part 4. HTVSs and Coherence with C^p-Embeddings of Intervals

6. Coherence with C^p-arcs $(p \geq 1)$

Proposition 6.1. Let X be a HTVS, let Y be a closed vector subspace of X whose continuous dual space Y' separates points on Y, and let $p \in \{1, 2, \ldots, \infty\}$. If X is coherent with a set $\mathcal{C}$ of C^p-arcs in X and if there exists a continuous projection $\rho : X \to Y$ onto Y, then Y is coherent with its C^p-arcs

Proof. If $\dim Y = 0$ or 1 then Y is trivially coherent with its C^p-arcs so assume that $\dim M > 1$. Let $S \subseteq Y$ be a subset such that for all C^p-arcs γ in Y, $\gamma^{-1}(S)$ is closed in γ's domain. Let $\gamma \in \mathcal{C}$ and suppose for the sake of contradiction that $\gamma^{-1}(S)$ is not closed in γ's domain. Pick $t_0 \in \overline{\gamma^{-1}(S)} \setminus \gamma^{-1}(S)$ and let $(t_l)_{l=1}^{\infty}$ be a sequence in $\gamma^{-1}(S)$ converging to t_0. By continuity of γ, $\gamma(t_0) \in \overline{Y \cap \operatorname{Im} \gamma} \subseteq \overline{Y} = Y$, which implies that $\gamma'(t_0) = \lim\limits_{l \to \infty} \dfrac{\gamma(t_l) - \gamma(t_0)}{t_l - t_0}$ also belongs to the closed space Y. In particular, this show that $(\rho \circ \gamma)'(t_0) = \rho(\gamma'(t_0)) = \gamma'(t_0)$ does not vanish. Since Y's continuous dual space separates points on Y, we may pick an interval $I \subseteq J$ containing t_0 and some $N \in \mathbb{N}$ such that $\eta = \rho \circ \gamma\big|_I : I \to Y$ is a C^p-arc and $(t_{l+N})_{l=1}^{\infty} \subseteq I$. By assumption, $\eta^{-1}(S)$ is closed in η's domain domain so that $t_0 \in \eta^{-1}(S)$, which implies that $t_0 \in I \cap \gamma^{-1}(S) \subseteq \gamma^{-1}(S)$, a contradiction. Hence, $\gamma^{-1}(S)$ is closed and since γ was an arbitrary C^p-arc in X, it follows that S is closed in X and thus closed in Y, as desired. ∎

If X is a TVS and Y is a vector subspace of X then recall (**[16,** p. 22**]**) that Y is said to be *complemented in X* if there exists a vector subspace Z of X such that the continuous linear map $\Lambda : Y \times Z \to X$ defined by $\Lambda(y, z) = y + z$ is an isomorphism of TVSs. In this case, X will be a direct sum of Y and Z (in the category of TVSs) and $\Lambda^{-1} = (\pi_Y, \pi_Z) : X \to Y \times Z$ will be continuous, where π_Y and π_Z are projections onto Y and Z, respectively. In particular, both projections will be continuous and if X is Hausdorff then both $Y = \ker \pi_Z$ and $Z = \ker \pi_Y$ will be closed in X so that by applying proposition 6.1 we obtain the following corollary.

Corollary 6.2. If X is a TVS with a continuous dual space that separates points on X and if $p \geq 1$ then X is coherent its C^p-arcs if and only if the same is true of every closed complement vector subspace of X.

Lemma 6.3. Let $(b_l)_{l=1}^{\infty}$ be a bounded sequence in a HTVS X, $(c_l)_{l=1}^{\infty}$ non-zero reals such that $\lim\limits_{l \to \infty} |c_l| = \infty$, $\gamma : (J, t_0) \to (X, \mathbf{0})$ a C^1-curve with non-zero derivative at t_0, and $(t_l)_{l=1}^{\infty} \subseteq J$ a sequence converging to t_0 such that $\gamma(t_l) = \frac{b_l}{c_l}$ for all $l \in \mathbb{N}$. Then $((t_l - t_0) c_l)_{l=1}^{\infty}$is bounded. Furthermore, if $(l_k)_{k=1}^{\infty}$ is increasing and $(c_{l_k}(t_{l_k} - t_0))_{k=1}^{\infty}$ is convergent then $\lim\limits_{k \to \infty} b_{l_k} = \gamma'(t_0) \lim\limits_{k \to \infty} c_{l_k}(t_{l_k} - t_0)$ exists.

Proof. Suppose not and pick an increasing sequence $(l_i)_{i=1}^{\infty}$ of positive integers such that $(c_{l_i})_{i=1}^{\infty}$ and $(t_{l_i})_{i=1}^{\infty}$ are monotone and $((t_{l_i} - t_0) c_{l_i})_{i=1}^{\infty}$ is monotone and divergent to either ∞ or $-\infty$. Let U and V be balanced neighborhoods of $\mathbf{0}$ in X such that $V + V \subseteq U$ and let $v = \gamma'(t_0)$. Let $N_0 \in \mathbb{N}$ be such that $i \geq N_0$ implies $\frac{\gamma(t_{l_i}) - \gamma(t_0)}{t_{l_i} - t_0} - v = \frac{b_{l_i}}{c_{l_i}(t_{l_i} - t_0)} - v \in V$ and observe that since V is balanced, we have $v \in \frac{b_{l_i}}{c_{l_i}(t_{l_i} - t_0)} + V$. Since $(b_l)_{l=1}^{\infty}$ is bounded and $((t_{l_i} - t_0) c_{l_i})_{i=1}^{\infty}$ is divergent to $\pm\infty$, the sequence $\left(\frac{b_{l_i}}{(t_{l_i} - t_0) c_{l_i}}\right)_{i=1}^{\infty}$ converges to $\mathbf{0}$ so pick an integer $N \geq N_0$ such that $i \geq N$ implies $\frac{1}{(t_{l_i} - t_0) c_{l_i}} b_{l_i} \in V$. In particular, $v \in \frac{b_{l_N}}{c_{l_N}(t_{l_N} - t_0)} + V \subseteq V + V \subseteq U$ and since U was an arbitrary neighborhood of $\mathbf{0}$, this gives us the contradiction $v = \mathbf{0}$.

Now suppose that $(l_k)_{k=1}^{\infty}$ is increasing and $(c_{l_k}(t_{l_k} - t_0))_{k=1}^{\infty}$ is convergent. Since $\gamma'(t_0) := \lim\limits_{k \to \infty} \dfrac{b_{l_k}}{c_{l_k}(t_{l_k} - t_0)}$ exists, the continuity of scalar multiplication implies that

$$\lim_{k \to \infty} b_{l_k} = \lim_{k \to \infty} \left[c_{l_k}(t_{l_k} - t_0) \cdot \frac{b_{l_k}}{c_{l_k}(t_{l_k} - t_0)} \right] = \left[\lim_{k \to \infty} c_{l_k}(t_{l_k} - t_0) \right] \cdot \left[\lim_{k \to \infty} \frac{b_{l_k}}{c_{l_k}(t_{l_k} - t_0)} \right]$$

also exists, as desired. ∎

Proof of Theorem 0.10. Let B be a closed and bounded subset of X and suppose that $(b_l)_{l=1}^{\infty} \subseteq B$ is a sequence. Since B is bounded we have $\lim\limits_{l \to \infty} \dfrac{b_l}{l} = \mathbf{0}$ so by lemma 5.7 there exists a C^1-arc γ, an increasing sequence $(l_k)_{k=1}^{\infty} \subseteq \mathbb{N}$, and a sequence $(t_k)_{k=0}^{\infty} \subseteq \operatorname{Dom} \gamma$ with $(t_k)_{k=1}^{\infty}$ monotone converging to t_0 such that $\gamma(t_0) = \mathbf{0}$ and $\gamma(t_k) = \frac{b_{l_k}}{l_k}$ for all $k \in \mathbb{N}$. Lemma 6.3 now implies that $b_\bullet$ has a convergent subsequence. ∎

Proof of Theorem 0.11. This follows immediately from theorems 0.10 and 10.4 and the fact that a HTVS is finite-dimensional if and only if it contains a non-empty compact neighborhood. ∎

Corollary 6.4. A HTVS that contains a closed complemented infinite-dimensional normable subspace is not coherent with any set C^1-curves with non-vanishing first derivatives.

Proof. Apply theorem 0.11 and proposition 6.1. ∎

Corollary 6.5. Let $p \in \{1, 2, \ldots, \infty\}$. In the category of C^p-manifolds with boundary modeled on normable TVSs, C^p-manifolds are exactly those objects that are coherent with their C^1-embeddings of $\mathbb{R}$.

Proof. This follows immediately from corollary 5.11 and theorems 0.11 and 10.4. ∎

7. Non-coherence with C^p-Embeddings ($p > 1$) of Intervals In this section, we prove theorem 7.4, which show that the only HLCTVSs that are coherent with a set of C^p-embeddings for $p \geq 2$ are $\{0\}$ and the field $\mathbb{R}$. Consequently, we will subsequently be primarily interested in studying spaces that are coherent with their C^1-arcs.

Remark 7.1. An interesting consequence of theorem 7.4 can essentially be summarized as follows: for almost all Hausdorff manifolds modeled on TVSs of dimension at least 2 (that one is likely to care about), there are "not enough C^2-arcs" into it to be able to reconstruct its topology from this set of curves alone. We will see later that the situation is sometimes different for C^1-arcs and that in particular, all finite dimensional $\mathbb{R}^d$ are coherent with their C^1-arcs.

Intuitively, this may be viewed as suggesting the (albeit vague) notion that "the fundamental difference" between the topologies of $\mathbb{R}^d$ for $d \leq 1$ and $\mathbb{R}^d$ for $2 \leq d < \infty$ lies in the difference between their C^1-arcs and C^2-arcs (it is curious that this distinction occurs between C^1-arcs and C^2-arcs rather than say, between C^0-arcs and C^1-arcs or between C^p-arcs with $p < \infty$ and C^∞-arcs). We will also see later that no infinite dimensional $\mathbb{R}^d$ or infinite dimensional normed space is coherent with its C^1-arcs, which may similarly be viewed as suggesting that "the fundamental difference" between the topologies of these spaces and the topologies of finite dimensional spaces lies in the difference between their C^0-arcs and C^1-arcs.

The next lemma follows immediately from the inverse function theorem.

Lemma 7.2. Let $\gamma = (x, y) : J \to \mathbb{R}^2$ be a C^p-curve ($p \in \{1, 2, \ldots, \infty\}$) where J contains 0. If $x'(0) \neq 0$ then there exists some $\epsilon > 0$ such that $x\big|_{[-\epsilon,\epsilon]} : J \cap [-\epsilon, \epsilon] \to \mathbb{R}$ is a C^p-isomorphism onto its image and the map $y \circ \left(x\big|_{[-\epsilon,\epsilon]}^{-1} \right) : \operatorname{Im}\left(x\big|_{[-\epsilon,\epsilon]} \right) \to \mathbb{R}$ is C^p.

Example 7.3. For all $l \in \mathbb{N}$, let $x_l = \frac{1}{l}$ and let $y_l = x_l^{3/2}$. Then there does *not* exist any C^2-curve $\gamma : (J, 0) \to \left(\mathbb{R}^2, \{0\}^2\right)$ with $\gamma'(0) \neq \{0\}^2$ such that $\gamma(t_k) = (x_{l_k}, y_{l_k})$ for some monotone sequence $(t_k)_{k=1}^\infty$ in J converging to 0 and some increasing sequence $(l_k)_{k=1}^\infty \subseteq \mathbb{N}$.

Proof. Suppose for the sake of contradiction that a such a curve $\gamma = (x, y) : (J, 0) \to \left(\mathbb{R}^2, \{0\}^2\right)$ and sequences $(t_k)_{k=1}^\infty$ and $(l_k)_{k=1}^\infty$ did exist. If necessary, we may replace γ, J, and $(t_k)_{k=1}^\infty$ with, respectively, $t \mapsto \gamma(-t)$, $-J$, and $(-t_k)_{k=1}^\infty$ so as to assume without loss of generality that $(t_k)_{k=1}^\infty$ is decreasing. Since $\gamma'(0)$ does not vanish, we may by lemma 7.2 pick $\epsilon > 0$ such that $\epsilon \in J$ and at least one coordinate of $\gamma\big|_{[0,\epsilon]} : [0, \epsilon] \to \mathbb{R}^2$ is a C^1-isomorphism onto its image in $\mathbb{R}$. Pick $k_0 \in \mathbb{N}$ be such that $k \geq k_0$ implies $t_k \in [0, \epsilon[$. Replacing γ, $(t_k)_{k=1}^\infty$, and $(l_k)_{k=1}^\infty$ with $\gamma\big|_{[0,\epsilon]}$, $(t_{k_0+k})_{k=1}^\infty$, and $(l_{k_0+k})_{k=1}^\infty$, respectively, we may assume without loss of generality that $\operatorname{Dom} \gamma = [0, \epsilon]$ and that at least one of γ's coordinates is a C^1-isomorphism onto its image.

If y was a C^1-isomorphism onto its image so that the map $G : \operatorname{Im} y \to \mathbb{R}$ defined by $G(r) = x(y^{-1}(r))$ is C^1 and $G(y_{l_k}) = x_{l_k}$ for all $k \in \mathbb{N}$. But then

$$G'(0) = \lim_{k\to\infty} \frac{G(y_{l_k})}{y_{l_k}} = \lim_{k\to\infty} \frac{x_{l_k}}{y_{l_k}} = \lim_{k\to\infty} l_k^{1/2} = \infty$$

gives a contradiction. Thus y is a not a C^1-isomorphism onto its image, which implies that x is a C^1-isomorphism onto its image.

So define $F : \operatorname{Im} x \to \mathbb{R}$ by $F(r) = y(x^{-1}(r))$ and observe that since $F(x_{l_k}) = y_{l_k}$ for all $k \in \mathbb{N}$ we have

$$F'(0) = \lim_{k\to\infty} \frac{y_{l_k}}{x_{l_k}} = \lim_{k\to\infty} x_{l_k}^{1/2} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{y_{l_k}}{x_{l_k}^2} = \lim_{k\to\infty} l_k^{1/2} = \infty$$

Since x^{-1} and y are C^2, F is twice differentiable at 0 so Taylor's theorem implies that $F^{(2)}(0) = 2 \lim_{k\to\infty} \frac{y_{l_k}}{x_{l_k}^2}$ exists, which gives a contradiction. ∎

In contrast to coherence with C^1-arcs, we now show that the only HLCTVSs that are coherent with a set of C^p-embeddings of intervals for some $p > 1$, are $\mathbb{R}^1$ and $\mathbb{R}^0$.

Theorem 7.4. Let X be a HTVS and $p \in \{2, 3, \ldots, \infty\}$. If X's continuous dual space has dimension 2 or more, then X is not coherent with any set C^p-embeddings of intervals.

Proof. Clearly, that $\dim X' \geq 2$ implies that there exists a closed 2-dimensional vector subspace Y of X and a continuous projection map $\rho : X \to Y$ onto Y. Now if X were coherent with a set of C^p-embeddings then by proposition 6.1, Y would be coherent with its C^p-arcs but since Y is linearly isomorphic to $\mathbb{R}^2$ this would imply that $\mathbb{R}^2$ is coherent with its C^p-arcs, which would contradict example 7.3. ∎

The following corollaries follow from theorem 7.4, lemma 5.7, and corollary 5.12.

Corollary 7.5. If $p \in \{2, 3, \ldots, \infty\}$ and k is a set of cardinality strictly greater than 1, then $\mathbb{R}^k$ is not coherent with any set of C^p-embeddings of intervals into $\mathbb{R}^k$.

Corollary 7.6. If $p \in \{2, 3, \ldots, \infty\}$ then no manifold modeled on a HLCTVS of dimension 2 or more can be coherent with any set of C^p-embeddings of intervals into it.

Corollary 7.7. If X is a HTVS whose continuous dual space separates points on X and if $S \subseteq X$ is convex then S is contained in a 1-dimensional affine subspace of X if and only if S is coherent with a set of S-valued C^p-curves in X for some/all $p \in \{2, 3, \ldots, \infty\}$.

8. Non-coherence of $\mathbb{R}^d$ (d infinite) with C^p-Arcs ($p \geq 1$)

Theorem 8.1. If $p \in \{1, 2, \ldots, \infty\}$ and d is an infinite set then the topology of $\mathbb{R}^d$ is *not* coherent with any set of C^p-embeddings of intervals.

Proof. Since every C^p-embedding is a C^1-embedding, by observation 5.1 it suffices to prove that $\mathbb{R}^d$ is not coherent with the set of all C^1-embeddings of intervals. So suppose for the sake of contradiction that $\mathbb{R}^d$ was coherent with a set of C^1-embeddings of intervals. Let $\mathbb{N}$ denote an arbitrary countable subset of d and let $c = d \setminus \mathbb{N}$ so that we may write $\mathbb{R}^d = \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^c$. For all $l \in \mathbb{N}$, let $m^l = (m_i^l)_{i \in d}$ be the indicator function:

$$m_i^l = \begin{cases} 1 & \text{if } i = l \\ 0 & \text{otherwise} \end{cases}$$

By lemma 5.7 there exists an increasing sequence of integers $(l_k)_{k=1}^{\infty}$, a C^1-embedding $\gamma = (\gamma_i)_{i\in d} : J \to \mathbb{R}^d$ of an interval J, and a sequence $(t_k)_{k=0}^{\infty} \subseteq J$ such that $(t_k)_{k=1}^{\infty}$ monotone converging to t_0, $\gamma(t_0) = \{0\}^d$, and $\gamma(t_k) = m^{l_k}$ for all $k \in \mathbb{N}$. For any $i \in d$ and $k \in \mathbb{N}$, if $i \in c$ then $\gamma_i(t_k) = m_i^{l_k} = 0$ while if $i \in \mathbb{N}$ then $\gamma_i(t_k) = m_i^{l_k} = 0$ for any $k \in \mathbb{N}$ such that $k > i$; either way we have

$$\gamma_i'(t_0) = \lim_{k\to\infty} \frac{0}{t_{l_k} - t_0} = 0$$

But then $\gamma'(t_0) = (\gamma_i'(t_0))_{i\in d} = \{0\}^d$, contradicting the fact that γ is a C^1-embedding. ■

9. Coherence of $\mathbb{R}^d$ ($d < \infty$) with Smooth Almost Arcs Although a smooth manifolds of dimension greater than 1 is never coherent with their smooth arcs, if one nevertheless desire a class of smooth curves with respect to which all smooth manifolds are coherent and that are as "similar" a possible to smooth arcs, which leads us to our next definition of a smooth almost arc. Alternatively, one may desire a class of C^0-arcs with nowhere vanishing derivatives in which, as we will later show, one may instead use the class of C^1-arcs on $[0,1]$ that are smooth arcs on $]0,1]$. This class of curves also arises naturally when investigating, for some given $p \in \{0,1,\ldots,\infty\}$, the question of when it is possible for a smooth embedding of $]0,\infty[\cong \mathbb{R}$, the most primitive and fundamental building block of manifolds, to be extended to a C^p curve on $[0,\infty[$.

Definition 9.1. Let $p \in \{1,2,\ldots,\infty\}$ and let M be a HLCTVS or a smooth manifold. By a *weak C^0-almost arc (on I) (in M)* we mean a C^p-topological embedding $\gamma : I \to M$ of a non-degenerate interval I for which there exists some point $c \in I$ such that $\gamma\big|_{I\smallsetminus\{c\}} : I \smallsetminus \{c\} \to M$ is a smooth embedding. By a *weak C^p-almost arc* we mean C^p-curve that is also a weak C^0-almost arc where if we call γ a *weak C^p-almost arc at c* then either γ is a smooth arc or else c is the unique point at which γ' vanishes, called *γ's vanishing point*, where in the latter case we'll call γ a *weak C^p-almost arc vanishing at c* and a *vanishing weak C^p-almost arc (at c)*. If a weak smooth almost arc γ is a smooth embedding then we'll call γ a C^p *0-almost arc* and say that γ is a *smooth 0-almost arc at c* for all $c \in \operatorname{Dom}\gamma$. For any $v \in \{1,\ldots,\infty\}$, call γ a *C^p v-almost arc (vanishing) at c* if γ is a weak C^p almost arc at c and γ is v-vanishing at c. By a *C^p-almost arc* (resp. *at c*) we mean C^p v-almost arc (resp. *at c*) for some $v \in \{0,\infty\}$ where if $v = \infty$ then we'll also call it a *C^p-almost arc vanishing at c* and a *vanishing C^p-almost arc (resp. at c)*. ■

Lemma 9.2. Let $\alpha : \mathbb{R} \to [0,1]$ be a smooth non-decreasing function such that $\alpha^{-1}(0) =]-\infty,0]$, $\alpha^{-1}(1) = [1,\infty[$, and $\alpha' > 0$ on $]0,1[$. Let $\beta = \alpha'$ and for all $n \in \mathbb{N}$, let $M_n = \sup\left|\operatorname{Im}\alpha^{(n)}\right|$, $S_n = \max\{1, M_1, \ldots, M_n\}$, and $\beta_n : \left[\frac{1}{n+1}, \frac{1}{n}\right] \to \mathbb{R}$ be $\beta_n(t) = n(n+1)\beta\left(n(n+1)\left(t - \frac{1}{n+1}\right)\right)$. Let $x_\bullet$ be a sequence in $\mathbb{R}$ and define $\gamma :]0,1] \to \mathbb{R}$ on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ by $\gamma(t) = (x_n - x_{n+1})\beta_n(t)$. Then the map $f :]0,1] \to \mathbb{R}$ defined by $f(t) = x_1 - \int_t^1 \gamma(t)dt$ is a smooth map such that $f\left(\frac{1}{n}\right) = x_n$ for all $n \in \mathbb{N}$. If in addition for all $n \in \mathbb{N}$, $|x_n - x_{n+1}| \le \epsilon_n := \frac{1}{n(n(n+1))^{n+1}S_n}$ then the map $]-\infty,1] \to \mathbb{R}$ that extends f and is identically 0 on $]-\infty,0]$, is smooth.

Proof. Note that for all $n \in \mathbb{N}$, $\int_{1/(n+1)}^{1/n} \gamma(t)dt = x_n - x_{n+1}$ so that $f(1) = x_1$ implies that $f\left(\frac{1}{n}\right) = x_n$ for all $n \in \mathbb{N}$. For positive integers $n \ge k$ and $t \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$, observe that

$$\left|\gamma^{(k)}(t)\right| = |x_n - x_{n+1}|\,(n(n+1))^{k+1}\,\beta^{(k)}\left(n(n+1)\left(t - \frac{1}{n+1}\right)\right) \le |x_n - x_{n+1}|\,(n(n+1))^{n+1}\,S_n$$

so that $|x_n - x_{n+1}| \le \epsilon_n$ implies that $\left|\gamma^{(k)}(t)\right| \le \frac{1}{n}$, which proves the γ's, and thus f's, constantly 0 extension to $]-\infty,1]$ is smooth. ■

Lemma 9.3. Let $\alpha : \mathbb{R} \to [0,1]$, $\beta = \alpha'$, $\beta_\bullet$, and $\epsilon_\bullet$ be as in lemma 9.2. If $x_\bullet \subseteq \mathbb{R}$ is a strictly decreasing sequence such that $x_n \le \epsilon_n$ and $|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n|$ for all $n \in \mathbb{N}$, then there exists a smooth almost arc $f : ([0,1],0) \to (\mathbb{R},0)$ vanishing at 0 such that $f\left(\frac{1}{n}\right) = x_n$ for all $n \in \mathbb{N}$.

Proof. Using lemma 9.2 twice (with the same α) gives us smooth non-decreasing functions $F, G :]-\infty,1] \to \mathbb{R}$ that are both 0 on $]-\infty,0]$ and that satisfy $F\left(\frac{1}{n}\right) = x_n$ and $G\left(\frac{1}{n}\right) = x_{n+1}$ for all $n \in \mathbb{N}$ where since we're using the same α, it becomes easy to see that $G < F$ and $(F-G)' \ge 0$ on $]0,1]$. Let $\varphi :]-\infty,1] \to \mathbb{R}$ be any smooth function such that $\varphi^{-1}(0) =]-\infty,0]$, $\varphi' > 0$ on $]0,1]$, and $0 < \varphi < 1$ on $]0,1]$. Note that since $F - G$ and φ are positive on $]0,1]$, the same is true of their product $\delta := \varphi \cdot (F-G) :]-\infty,1] \to \mathbb{R}$ while

$0 < \varphi < 1$ on $]0,1]$ implies that $0 < \delta < G$ on $]0,1]$. Since $(F-G)' \geq 0$ while $G, \varphi, \varphi' > 0$ on $]0,1]$, the product rule gives us $\delta' > 0$ on $]0,1$. For all $n \in \mathbb{N}$, let $y_n = x_n - (G+\delta)\left(\frac{1}{n}\right) = \left(1 - \varphi\left(\frac{1}{n}\right)\right)(x_n - x_{n+1})$ where observe that $x_{n+1} < y_n < x_n$ gives us $|y_{n+1} - y_n| \leq 2\epsilon_n$, which allows us to apply lemma 9.2 to obtain a smooth non-decreasing function $H :]-\infty, 1] \to \mathbb{R}$ such that $H = 0$ on $]-\infty, 0]$ and $H\left(\frac{1}{n}\right) = y_n$ for all $n \in \mathbb{N}$. The desired smooth map f is $H + G + \delta$ where $f' > 0$ on $]0,1]$ since $(H+G)' \geq 0$ and $\delta' > 0$ on $]0,1]$. ∎

Proposition 9.4. Let $d \in \mathbb{N}$ and let $x^\bullet = (x_1^\bullet, \ldots, x_d^\bullet) \subseteq \mathbb{R}^d$ be an infinite sequence converging to x in $\mathbb{R}^d$. There exists some subsequence $x^{l_\bullet}$ of $x^\bullet$ and some smooth almost arc $\gamma : ([0,1], 0) \to (\mathbb{R}^d, x)$ vanishing at 0 such that $x^{l_\bullet} \to x$ is injective in $\mathbb{R}^d$ and $\gamma\left(\frac{1}{n}\right) = x^{l_n}$ for all $n \in \mathbb{N}$.

Proof. Assume without loss of generality $x = \{0\}^d$, $x_1^\bullet$ strictly decreasing, and for all $h = 2, \ldots, d$ either $x_h^\bullet$ is constantly 0 or otherwise it is strictly decreasing. For all $h \in \{1, \ldots, d\}$ such that $x_h^\bullet$ is constantly 0, let $\gamma_h : [0,1] \to \mathbb{R}$ be the constant 0 function. Let $\alpha : \mathbb{R} \to [0,1]$, $\beta = \alpha'$, $\beta_\bullet$, and $\epsilon_\bullet$ be as in lemma 9.2. For each $h \in \{1, \ldots, d\}$ such that $x_h^\bullet$ is not constant, find an increasing sequence $(n_l)_{l=1}^\infty \subseteq \mathbb{N}$ such that $x^{n_l} \leq \epsilon_l$ and $x_h^{n_{l+1}} \leq x_h^{n_l}/2$ for all $l \in \mathbb{N}$ and then replace $x^\bullet$ with $x^{l_\bullet}$. This allows us to assume that for all $h \in \{1, \ldots, d\}$ such that $x_h^\bullet$ is not constant, $x_h^\bullet$ satisfies the hypotheses of lemma 9.3 which gives us a smooth almost arc $\gamma_h : [0,1] \to \mathbb{R}$ vanishing at 0 such that $\gamma_h\left(\frac{1}{n}\right) = x_h^n$ for all $n \in \mathbb{N}$. We have thus constructed the desired smooth almost arc $\gamma = (\gamma_1, \ldots, \gamma_d) : ([0,1], 0) \to \left(\mathbb{R}^d, \{0\}^d\right)$. ∎

Proof of Theorem 0.1. Lemma 5.7 shows that coherence is a local property so we may assume without loss of generality that the manifold is $\mathbb{R}^d$ for some $d \in \mathbb{Z}^{\geq 0}$. If $d = 0$ then the result follows vacuously so we may assume that $d > 0$. The conclusion now follows by applying lemma 5.7 with proposition 9.4. ∎

9.1. Application: Discontinuities of Maps on Manifolds Proving that a map $f : M \to Y$ from a smooth d-dimensional manifold into a space Y is continuous at some point $m^0 \in M$ is often done by going into coordinates and then imagining that it had instead been a map $\mathbb{R}^d$ the whole time. We show how theorem 0.1 allows us to go further than this and to imagine that our map is in fact defined *entirely* on the positive part of the x-axis of $\mathbb{R}^d$ that can be smoothly extended, where we begin to make this concept rigorous (see theorem 9.7's statement) with the following definition.

Definition 9.5. If M is a smooth manifold of dimension $d \in \mathbb{N}$ and (U, φ) is a smooth chart on M then by *the x-axis (coordinate) of φ* we mean the embedding $x_\varphi : I \to U$ defined by $x_\varphi(t) = \varphi^{-1}\left(t \times \{0\}^{d-1}\right)$, where $I := \left\{t \in \mathbb{R} : \left(t, \{0\}^{d-1}\right) \in \operatorname{Im} \varphi\right\}$. If $0 \notin I$ and $m \in M \setminus U$, then by *the canonical extension to m of φ's x-axis* we mean the (potentially discontinuous) map $\widehat{x_\varphi} : I \cup \{0\} \to M$ that extends x_φ and sends 0 to m, where we'll say that *φ's x-axis can be canonically and smoothly extended to m* if $\widehat{x_\varphi}$ is smooth and $0 \in \operatorname{Cl}_{\mathbb{R}}(I)$.

Remark 9.6. The next theorem essentially shows that the only way that a map $f : M \to Y$ on a smooth manifold fails to be continuous at a point m^0 is if there is some smooth chart (U, φ) "adjacent to m^0" (i.e. m^0 is contained in U's topological boundary) such that even though it is possible to smoothly extend this chart's x-axis x_φ to m^0, f's restriction to this extended x-axis is not a continuous extension of f's restriction to the original axis. Observe also that if one is studying why this map fails to be continuous at m^0, then this theorem allows one to reduce this question down to a question about the continuity of a function of one real variable.

Theorem 9.7. Let M be a smooth manifold of dimensional $d \in \mathbb{N}$, $m^0 \in M$, and $f : M \to Y$ be a map into a space. Then f is discontinuous at m^0 if and only if there is a surjective smooth chart $\varphi : U \to]0, \infty[\times \mathbb{R}^{d-1}$ on $M \setminus \{m^0\}$ such that the canonical extension $\widehat{x_\varphi} : [0, \infty[\to M$ to m^0 of φ's x-axis is smooth but $f \circ \widehat{x_\varphi} : [0, \infty[\to Y$ is discontinuous at 0.

Proof. Suppose that f is discontinuous at m^0 where since continuity at m^0 is a local property, we may assume without loss of generality that $M = \mathbb{R}^d$ and $m^0 = \{0\}^d$. Since M is Fréchet-Urysohn, this is only possible if there exists a sequence $m^\bullet \subseteq M$ such that $m^\bullet \to m^0$ is injective in M but $f(m^\bullet)$ diverges. Let

$m^{l_\bullet}$ and $\gamma : ([0,1],0) \to \left(\mathbb{R}^d, \{0\}^d\right)$ be as in proposition 9.4 and then replace $m^\bullet$ with $m^{l_\bullet}$ so that we may assume that $\gamma\left(\frac{1}{n}\right) = m^n$ for all $n \in \mathbb{N}$. Since $S := \gamma\left(]0,1[\right)$ is a smoothly embedded submanifold of M, there exists (**[11]**) a tubular neighborhood $T \in \operatorname{Open}(M)$ of S, say with smooth projection $\pi : T \to S$. Since S is contractible, there is exists a smooth global trivialization $\tau = (\pi, \tau_2) : T \to S \times \mathbb{R}^{d-1}$ of $\pi : T \to S$. Observe that the map $T \to]0,1[\times\mathbb{R}^{d-1}$ defined by $m \mapsto \left(\gamma^{-1}(\pi(m)), \tau_2(m)\right)$ is a smooth chart on M since $\gamma\big|_{]0,1[} :]0,1[\to S$ is a diffeomorphism. Since $]0,1[$ is diffeomorphic to $]0,\infty[$, the conclusion follows. ∎

Remark 9.8. One may formulate (in the obvious way) an analogue of theorem 9.7's statement in terms of C^1-arcs on $[0,1]$ that are smooth arcs on $]0,1]$ (instead of smooth almost arcs vanishing at 0) that can then be proven by using lemma 10.3 and theorem 10.4 in place of proposition 9.4 in the above proof.

Remark 9.9. Corollary 0.3 can clearly be reformulated as:

> A map $f : \left(M, m^0\right) \to \left(Y, y^0\right)$ from a smooth manifold M to a space Y is continuous at m^0 if and only whenever $\gamma : (]-\infty,0],0) \to \left(M, m^0\right)$ is a smooth almost arc at 0 then the net $f \circ \gamma\big|_{]-\infty,0[} :]-\infty,0[\to Y$ converges to y^0 in Y (where $]-\infty,0[$ is directed by its usual order).

This can be rephrased in a way that makes it underlying geometric interpretation apparent, where the terminology is defined in the obvious way (i.e. via its correspondence with the above characterization):

> A map $f : \left(M, m^0\right) \to \left(Y, y^0\right)$ from a manifold M to a space Y is continuous at m^0 if and only if whenever $m \in M$ *smoothly* tends towards m^0 along a smoothly embedded interval in $M \setminus \{m^0\}$ then $f(m)$ tends towards y^0 in Y.

This is the reason for corollary's description as a "smooth characterization of continuity at a point" and also suggests the following definition of "smoothly convergent."

Definition 9.10. Let $m^\bullet$ be a net in a smooth manifold M and let $m^0 \in M$. Say that $m^\bullet$ *converges smoothly (along a smooth arc) (in M) to* m^0 if there exists a smooth almost arc $\gamma : ([0,\infty[,0) \to \left(M, m^0\right)$ such that $m^\bullet \subseteq \operatorname{Im}\gamma$ and $\gamma^{-1}(m^\bullet) \to 0$.

Remark 9.11. Using this definition, corollary 0.3 becomes:

> A map $f : \left(M, m^0\right) \to \left(Y, y^0\right)$ from a smooth manifold M to a space Y is continuous at m^0 if and only if it sends sequences converging smoothly to m^0 in M to sequences converging to y^0 in Y.

Theorem 9.12 (Characterization of non-removable discontinuities of a map on a manifold)**.** Let M be a smooth manifold of dimension $d \in \mathbb{N}$, $m^0 \in M$, Y be a Hausdorff space, and $f : M \to Y$ be a map that is discontinuous at m^0. Then m^0 is a non-removable discontinuity of f if and only if there exist surjective smooth charts $\varphi : U \to]0,\infty[\times\mathbb{R}^{d-1}$ and $\psi : V \to]0,\infty[\times\mathbb{R}^{d-1}$ on $M \setminus \{m^0\}$ with x-axes x_φ and x_ψ that can both be canonically and smoothly extended to m^0 such that the closures of U and V in $M \setminus \{m^0\}$ are disjoint and both of the limits $\lim\limits_{\substack{m \to m^0 \\ m \in \operatorname{Im} x_\varphi}} f(m)$ and $\lim\limits_{\substack{m \to m^0 \\ m \in \operatorname{Im} x_\psi}} f(m)$ exist but are distinct.

Proof. The only part of this corollary that doesn't follow from theorem 9.7 is that we can choose U and V to have disjoint closures in $M \setminus \{m^0\}$ so let φ and ψ be as described in this corollary's statement except without the guarantee that the closures of U and V in $M \setminus \{m^0\}$ are disjoint. From $\lim\limits_{m \to m^0} f\big|_{\operatorname{Im} x_\varphi} \neq \lim\limits_{m \to m^0} f\big|_{\operatorname{Im} x_\psi}$ we can conclude that there is some $\epsilon > 0$ such that $x_\varphi(]0,\epsilon]) \cap x_\psi(]0,\epsilon]) = \varnothing$. By replacing φ and ψ with their restrictions to $\varphi^{-1}\left(]0,\epsilon[\times\mathbb{R}^{d-1}\right)$ and $\psi^{-1}\left(]0,\epsilon[\times\mathbb{R}^{d-1}\right)$ and then composing with a diffeomorphism $]0,\epsilon[\times\mathbb{R}^{d-1} \to]0,\infty[\times\mathbb{R}^{d-1}$, we may henceforth assume without loss of generality that $\operatorname{Im} x_\varphi \cap \operatorname{Im} x_\psi = \varnothing$. Let $X =]0,\infty[\times\{0\}^{d-1}$. Since $S := \varphi(U \cap \operatorname{Im} x_\psi)$ is a smooth submanifold of $\operatorname{Im}\varphi =]0,\infty[\times\mathbb{R}^{d-1}$ that is disjoint from X, we can find a tubular neighborhood T of X in $\operatorname{Im}\varphi$ whose closure is disjoint from S, where this implies that $\operatorname{Cl}_M(\varphi(T)) \cap \operatorname{Im} x_\psi = \varnothing$. Since $\psi(V \setminus \operatorname{Cl}_M(U))$ is an open neighborhood of X in $\operatorname{Im}\psi$ we may find a tubular neighborhood O of $\operatorname{Im} x_\psi$ in $\psi(V \setminus \operatorname{Cl}_M(U))$ whose closure is disjoint from the closed (in $\operatorname{Im}\psi$) subset $\psi(V \cap \operatorname{Cl}_M(U))$. As in the proof of theorem 9.7, we may use these tubular neighborhood to construct smooth surjective charts $\Phi : \varphi(T) \to]0,\infty[\times\mathbb{R}^{d-1}$ and $\Psi : S \to]0,\infty[\times\mathbb{R}^{d-1}$ (indeed, we'll even have $\operatorname{Im} x_\Phi = \operatorname{Im} x_\varphi$ and $\operatorname{Im} x_\Psi = \operatorname{Im} x_\psi$). ∎

10. Coherence of $\mathbb{R}^d$ ($d < \infty$) with C^1-Embeddings of Intervals

Lemma 10.1. Suppose that $(x_l)_{l=1}^\infty$, $(y_l)_{l=1}^\infty$, and $(\epsilon_l)_{l=1}^\infty$ are sequences of reals converging to 0 with $(\epsilon_l)_{l=1}^\infty$ positive, $(x_l)_{l=1}^\infty$ positive and decreasing, and $|y_{l+1} - y_l| < \epsilon_l |x_{l+1} - x_l|$ for all $l \in \mathbb{N}$. Then there exists a C^1-function $f : ([0, x_1], 0) \to (\mathbb{R}, 0)$ such that $f'(0) = 0$, f is smooth on $]0, x_1]$, and $f(x_l) = y_l$ for all $l \in \mathbb{N}$. Furthermore, if $(y_l)_{l=1}^\infty$ is decreasing (resp. increasing, non-increasing, non-decreasing) then f is increasing (resp. decreasing, non-decreasing, non-increasing).

Proof. For each $l \in \mathbb{N}$, since $|y_{l+1} - y_l| < \epsilon_l |x_{l+1} - x_l|$ we may find a smooth function $\beta_l : [0, x_1] \to \mathbb{R}$ such that $\beta_l^{-1}(\mathbb{R} \setminus \{0\}) =]x_{l+1}, x_l[$ and $y_l - y_{l+1} = \int_{x_{l+1}}^{x_l} \beta_l(t)dt$, where if $y_l - y_{l+1} \geq 0$ (resp. $y_l - y_{l+1} \leq 0$) then $\beta_l \geq 0$ (resp. $\beta_l \leq 0$). Define $\beta : [0, x_1] \to \mathbb{R}$ by $\beta(0) = 0$ and $\beta\big|_{[x_{l+1}, x_l]} = \beta_l\big|_{[x_{l+1}, x_l]}$ for all $l \in \mathbb{N}$, where β is clearly well-defined and also smooth on $]0, x_1]$. Observe that if $(y_l)_{l=1}^\infty$ is monotone then β is always either non-negative or non-positive where if $(y_l)_{l=1}^\infty$ is strictly monotone then all of β's zeros in $]0, x_1]$ are also isolated. Note that for all $l \in \mathbb{N}$, $\sup\limits_{x \in [0, x_l]} |\beta(x)| \leq \sup\limits_{k \geq l} \epsilon_k$ so that the assumption that $\lim\limits_{l\to\infty} \epsilon_l = 0$ implies that β is continuous at 0 and hence continuous everywhere.

Let $f : [0, x_1] \to \mathbb{R}$ be the C^1 function defined by $f(x) = y_1 + \int_{x_1}^x \beta(t)dt$. Observe that $f(x_1) = y_1$ and for any $k \geq 2$,

$$f(x_k) = y_1 - \int_{x_2}^{x_1} \beta_1(t)dt - \cdots - \int_{x_{k-1}}^{x_k} \beta_l(t)dt = y_1 - (y_1 - y_2) - \cdots - (y_{k-1} - y_k) = y_k$$

Since f is continuous, $f(0) = \lim\limits_{l\to\infty} f(x_l) = \lim\limits_{l\to\infty} y_l = 0$. If $(y_l)_{l=1}^\infty$ is decreasing (resp. increasing) then since $\beta \geq 0$ (resp. $\beta \leq 0$) and β has isolated zeros in $]0, 1]$, $\int_{x_1}^x \beta(t)dt$ is an increasing (resp. decreasing) function of x so that the same is true of f. Similarly, if $(y_l)_{l=1}^\infty$ is non-increasing (resp. non-decreasing) then $\beta \geq 0$ (resp. $\beta \leq 0$) so that f is non-decreasing (resp. non-increasing). ∎

Lemma 10.2. Suppose that $x_\bullet = (x_l)_{l=1}^\infty$, $y_\bullet = (y_l)_{l=1}^\infty$, and $\epsilon_\bullet = (\epsilon_l)_{l=1}^\infty$ are sequences of reals converging to 0 with $(\epsilon_l)_{l=1}^\infty$ positive, $(x_l)_{l=1}^\infty$ never zero, and $\left(\frac{y_l}{x_l}\right)_{l=1}^\infty$ converging to 0. Then there exists some increasing $\iota : \mathbb{N} \to \mathbb{N}$ such that $\left|y_{\iota(k)} - y_{\iota(l)}\right| < \epsilon_k \left|x_{\iota(k)} - x_{\iota(l)}\right|$ for all $k < l$ in $\mathbb{N}$, $\left(x_{\iota(l)}\right)_{l=1}^\infty$ is strictly monotone, and $\left(y_{\iota(l)}\right)_{l=1}^\infty$ is either constantly 0 or otherwise strictly monotone.

Proof. Observe that if $(\hat{\epsilon}_l)_{l=1}^\infty$ is a sequence of positive reals such that $\hat{\epsilon}_l \leq \epsilon_l$ for all $l \in \mathbb{N}$ then the desired conclusion follows if we prove this lemma with $(\hat{\epsilon}_l)_{l=1}^\infty$ in place of $\epsilon_\bullet$, so we may assume without loss of generality that $\epsilon_1 < 1$ and $\sum\limits_{k=l+1}^\infty \epsilon_k < \epsilon_l/2$ for all $l \in \mathbb{N}$. If there is an infinite subsequence of $y_\bullet$ consisting entirely of zeros then we're done, so assume otherwise and find increasing $(i_l)_{l=1}^\infty$ such that $(y_{i_l})_{l=1}^\infty$ is strictly monotone. By replacing $x_\bullet$, $y_\bullet$, and $\epsilon_\bullet$ with $(x_{i_l})_{l=1}^\infty$, $(y_{i_l})_{l=1}^\infty$, and $(\epsilon_{i_l})_{l=1}^\infty$ we may henceforth assume without loss of generality that $y_\bullet$ is strictly monotone. Similarly, we may assume that $x_\bullet$ is strictly monotone. Since the inequality $|y_i - y_j| < \epsilon_l |x_i - x_j|$ holds if and only if $|y_i - y_j| < \epsilon_l |(-x_i) - (-x_j)|$ holds, by replacing $x_\bullet$ with $(-x_l)_{l=1}^\infty$ if necessary, we may assume without loss of generality that all x_l are positive.

Pick $\iota(1) \in \mathbb{N}$ such that $l \geq \iota(1)$ implies $\left|\frac{y_l}{x_l}\right| < \epsilon_3/2$. Suppose we've picked increasing integers $0 < \iota(1), \ldots, \iota(n)$, where $n \geq 1$, such that

(1) for all $k = 1, \ldots, n$, if $l \geq \iota(k)$ then $\left|\frac{y_l}{x_l}\right| < \epsilon_{k+2}/2$,
(2) for all $k \in \mathbb{Z}$ and l, if $1 \leq k < l \leq n$ then $\left|\frac{y_{\iota(k)} - y_{\iota(l)}}{x_{\iota(k)} - x_{\iota(l)}}\right| < (\epsilon_l + \cdots + \epsilon_{k+1})/2$.

Pick $\iota(n+1) > \iota(n)$ such that for all $l \geq \iota(n+1)$, both of $\left|\frac{y_l}{x_l}\right|$ and $\left|\frac{y_{\iota(n)}-y_l}{x_{\iota(n)}-x_l} - \frac{y_{\iota(n)}}{x_{\iota(n)}}\right|$ are strictly less than $\epsilon_{n+3}/2$. Note that

$$\left|\frac{y_{\iota(n+1)} - y_{\iota(n)}}{x_{\iota(n+1)} - x_{\iota(n)}}\right| \leq \left|\frac{y_{\iota(n+1)} - y_{\iota(n)}}{x_{\iota(n+1)} - x_{\iota(n)}} - \frac{y_{\iota(n)}}{x_{\iota(n)}}\right| + \left|\frac{y_{\iota(n)}}{x_{\iota(n)}}\right| < \epsilon_{n+3}/2 + \epsilon_{n+2}/2 < \epsilon_{n+1}/2$$

which is (2) with $(n, n+1, n+1)$ in place of (k, l, n). If $n = 1$ then this completes the inductive step so that we may henceforth assume that $n > 1$.

If $1 \leq k \leq l < n+1$ with $k < n-1$ then $\left|y_{\iota(l+1)} - y_{\iota(l)}\right| < (\epsilon_{l+1}/2)\left|x_{\iota(l+1)} - x_{\iota(l)}\right| < (\epsilon_{n+1}/2)\left|x_{\iota(n+1)} - x_{\iota(k)}\right|$ so that $\left|y_{\iota(n+1)} - y_{\iota(k)}\right| \leq \left|y_{\iota(n+1)} - y_{\iota(n)}\right| + \cdots + \left|y_{\iota(k+1)} - y_{\iota(k)}\right| < (\epsilon_{n+1}/2 + \cdots + \epsilon_{k+1}/2)\left|x_{\iota(n+1)} - x_{\iota(k)}\right|$, which proves (2) and completes the inductive construction. Observe that (2) together with the fact that $\sum\limits_{l=k+2}^{\infty} \epsilon_l < \epsilon_{k+1}/2$ for all $k \in \mathbb{N}$, implies that for all $1 \leq k < l$, $\left|\frac{y_{\iota(k)}-y_{\iota(l)}}{x_{\iota(k)}-x_{\iota(l)}}\right|$ is bounded above by $(\epsilon_l + \cdots + \epsilon_{k+1})/2 < \epsilon_{k+1} < \epsilon_k$. ∎

Lemma 2 of the preprint **[15]** appears to be false, since it would allow one to conclude that through any sequence $(x_l)_{l=1}^{\infty}$ of non-zero points in $\mathbb{R}^n$ that converges to zero and for which there exists a $v \in \mathbb{R}^n$ such that

$$\lim_{l\to\infty} \frac{d\left(x_l, \mathbb{R}^{\geq 0} v\right)}{\|x_l\|_2} = 0$$

there exists a smooth curve with nowhere vanishing derivative, whose range contains infinitely many points of $(x_l)_{l=1}^{\infty}$. But as we have seen in example 7.3, such a curve, even if it's merely required to be C^2 rather than smooth, does may fail to exist if $n > 1$. However, we will now show in theorem 10.4 that if we reduce the requirement of smoothness to C^1 then such a curve will necessarily exist.

Despite lemma 2 of **[15]** being false for dimensions greater than 1, elements of this lemma's attempted proof were headed in the right direction and although the author proved the below statements independently, there are nevertheless several commonalities between the attempted proofs in **[15]** and the author's proof of theorem 10.4 below. Since the author cannot guarantee that these commonalities are not the result of having read **[15]** prior to attempting the independent proof of theorem 10.4, for full disclosure and honesty, the author has encapsulated all ideas that are common to both **[15]** and the proof of theorem 10.4 below in the following lemma 10.3, which is in fact actually a generalization of lemma 2 of **[15]** with a proof that includes details omitted form **[15]**.

Lemma 10.3. For all $d \in \mathbb{Z}^{\geq 0}$ and all $p \in \{0, 1, \ldots, \infty\}$ let $\star(p,d)$ denote the following statement:

$\star(p,d)$: Whenever $m_\bullet = \left(m^l\right)_{l=1}^{\infty}$ is an infinite-ranged sequence in $\mathbb{R}^d$ converging to m^0 then there exists a C^p-embedding $\gamma : ([0,\epsilon], 0) \to \left(\mathbb{R}^d, m^0\right)$, where $\epsilon > 0$, that can lift some subsequence of $m_\bullet$ to a monotone injective sequence in $[0,\epsilon]$.

Then for all p, both $\star(p,0)$ and $\star(p,1)$ are true, if $\star(p,2)$ is true then $\star(p,d)$ is true for all $d \geq 2$, and to prove $\star(p,2)$ it suffices to prove it for those sequences $\left(m^l\right)_{l=0}^{\infty} = \left(x^l, y^l\right)_{l=0}^{\infty}$ in $\mathbb{R}^2$ for which $x^0 = 0 = y^0$, both $\left(x^l\right)_{l=1}^{\infty}$ and $\left(\frac{y^l}{x^l}\right)_{l=1}^{\infty}$ are increasing to 0, and $\left(y^l\right)_{l=1}^{\infty}$ is decreasing to 0. Furthermore, this remains true if we add to statement $\star(p,d)$ the condition:

(S): γ is a smooth embedding on $]0,\epsilon]$.

Proof. If $d = 0$ then this is vacuously true while $d = 1$ is obvious. So assume that $\star(p,2)$ is true, let $d > 2$, and for each $l \in \mathbb{N}$ write $m^l = \left(m_1^l, \ldots, m_d^l\right)$ and let $m_{\geq 2}^l = \left(m_2^l, \ldots, m_d^l\right)$. Observe that if $h : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism and $\left(m^{l_k}\right)_{k=1}^{\infty}$ is any infinite-ranged subsequence of $\left(m^l\right)_{l=0}^{\infty}$ then it suffices to prove the theorem with $\left(h\left(m^{l_k}\right)\right)_{k=1}^{\infty}$ in place of $\left(m^l\right)_{l=0}^{\infty}$ so that in particular, we may assume without loss of generality that (1) $m^0 = \{0\}^d$, (2) all $(m_i^l)_{l=1}^{\infty}$ are non-negative and non-increasing, (3) $(m_1^l)_{l=1}^{\infty}$ is decreasing, and (4) for all $i \in \mathbb{N}$ such that $2 \leq i \leq d$, either $(m_1^l)_{l=1}^{\infty}$ is constant or else it's decreasing. If $\left(m_{\geq 2}^l\right)_{l=1}^{\infty}$

contains an infinite constant subsequence $\left(m_{\geq 2}^{l_k}\right)_{l=1}^{\infty}$ then each $m_{\geq 2}^{l_k} = \{0\}^{d-1}$ so let $t_k = m_1^{l_k}$ for all $k \in \mathbb{N}$ and let $\gamma := \left(\mathrm{Id}_{\mathbb{R}}, \{0\}^{d-1}\right) : \mathbb{R} \to \mathbb{R}^d$. Thus we may assume that $\left(m_{\geq 2}^{l}\right)_{l=1}^{\infty}$ is injective.

Proceeding by induction, suppose that the theorem has been proved for all dimensions less than d. Pick $\delta > 0$, a C^p-embedding $\beta : ([0, \delta], 0) \to \left(\mathbb{R}^{d-1}, \{0\}^{d-1}\right)$, a sequence $(s_p)_{p=1}^{\infty} \subseteq]0, \delta[$ decreasing to 0, and an increasing sequence $(q_p)_{p=1}^{\infty} \subseteq \mathbb{N}$ such that $\beta(s_p) = m_{\geq 2}^{q_p}$ for all $p \in \mathbb{N}$, where if (S) had been assumed then we also assume that β satisfies this additional condition. Now pick $\epsilon_0 > 0$, a C^p-embedding $\alpha = (\alpha_1, \alpha_2) : ([0, \epsilon_0], 0) \to \left(\mathbb{R} \times D_\delta, \{0\}^2\right)$, a decreasing sequence $(t_k)_{k=1}^{\infty}$ in $]0, \epsilon_0[$ converging to 0, and an increasing sequence of integers $(p_k)_{k=1}^{\infty}$ such that $\alpha(t_k) = \left(m_1^{q_{p_k}}, s_{p_k}\right)$ for all $k \in \mathbb{N}$, where if (S) had been assumed then we also assume that α satisfies this additional condition. Define $\eta = (\alpha_1, \beta \circ \alpha_2) : [0, \epsilon_0] \to \mathbb{R}^d$ and observe that η is a C^p-map and an injection whose derivative (if $p \geq 1$) vanishes nowhere and that will satisfies (S) if (S) had been assumed.

Now suppose $d = 2$ and that we want to prove $\star(p, 2)$. For all $l \in \mathbb{Z}^{\geq 0}$ we may now write $m^l = (x_l, y_l)$. Observe that both $(x_l)_{l=1}^{\infty}$ and $(y_l)_{l=1}^{\infty}$ are decreasing. If $\left(\frac{y_l}{x_l}\right)_{l=1}^{\infty}$ does not contain a convergent subsequence then it contains a subsequence that diverges to infinity so switch all x_l's and y_l's and assume without loss of generality that $\left(\frac{y_l}{x_l}\right)_{l=1}^{\infty}$ monotone converges to a non-negative s. If there exists a subsequence $(x_{l_k}, y_{l_k})_{k=1}^{\infty}$ and a polynomial $p(x)$ such that $y_{l_k} = p(x_{l_k})$ for all $k \in \mathbb{N}$ then let $\gamma(x) = (x, p(x))$ and we're done, so assume without loss of generality that no such subsequence and polynomial exist. In particular, this assumption implies that $\left(\frac{y_l}{x_l}\right)_{l=1}^{\infty}$ does not contain any infinite constant subsequence so we may further assume without loss of generality that $\left(\frac{y_l}{x_l}\right)_{l=1}^{\infty}$ is strictly monotone which, in particular, implies that no (x_l, y_l) lies on the line $x \to (x, sx)$. If after rotating the points (x_l, y_l) by $-\arctan(s)$, selecting strictly monotone subsequences, reflecting them across the x-axis if necessary (to have infinitely many positive y_l's), and then replacing $(x_l, y_l)_{l=1}^{\infty}$ with this new sequence, we may assume without loss of generality $\left(\frac{y_l}{x_l}\right)_{l=1}^{\infty}$ is strictly monotone decreasing to $s = 0$. Since $(x, y) \mapsto (-x, y)$ is a diffeomorphism we may assume without loss of generality that both $\left(x^l\right)_{l=1}^{\infty}$ and $\left(\frac{y^l}{x^l}\right)_{l=1}^{\infty}$ are increasing to 0. ∎

Theorem 10.4. Let $d \in \mathbb{Z}^{\geq 0}$, let $\mathcal{A}_I$ and $\mathcal{A}_{[0,1]}^{*}$ be defined as in corollary 5.17 for $p = 1$ and $M = \mathbb{R}^d$. Then $\mathbb{R}^d$ is coherent with $\mathcal{A}_{[0,1]}^{*}$ and with $\mathcal{A}_I$ for all intervals I. If $d > 0$ and if we instead had $M = \mathbb{R}^{d-1} \times \mathbb{R}^{\geq 0}$ then M would be coherent with $\mathcal{A}_{[0,1]}^{*}$, $\mathcal{A}_{[0,1]}$, $\mathcal{A}_{[0,1[}$, but not with $\mathcal{A}_{]0,1[}$.

Proof. Corollary 5.17 shows that it suffices to show that $\mathbb{R}^d$ is coherent with $\mathcal{A}_{[0,1]}^{*}$ while lemma 10.3 shows that it suffices to prove this statement for $d = 2$. So let $\left\{m^l = \left(x^l, y^l\right) : l \in \mathbb{N}\right\}$ be an infinite-ranged sequence in $\mathbb{R}^2$. By lemma 5.7, it suffices to find

(1) a C^1-embedding $\gamma : ([0, 1], 0) \to \left(M, \{0\}^d\right)$ that is smooth on $]0, 1]$,
(2) a decreasing sequence $(t_k)_{k=1}^{\infty}$ in $]0, 1]$ converging to 0,
(3) an increasing sequence of integers $(l_k)_{k=1}^{\infty}$

such that $\gamma(t_k) = m^k$ for all $k \in \mathbb{N}$ and $\gamma\big|_{D \setminus \{0\}}$ is a smooth embedding. By lemma 10.3, it suffices to prove this statement under the assumption that $m^0 = (0, 0)$, both $\left(x^l\right)_{l=1}^{\infty}$ and $\left(\frac{y^l}{x^l}\right)_{l=1}^{\infty}$ are decreasing to 0, and $\left(y^l\right)_{l=1}^{\infty}$ is decreasing to 0 while lemma 10.2, we may also assume without loss of generality that $|y_k - y_l| < \frac{1}{l}|x_k - x_l|$ for all $k, l \in \mathbb{N}$ with $k > l$. We obtain the desired curve by reparameterizing the curve constructed in lemma 5.7 by a linear transformation.

If $d > 0$ and we instead had $M = \mathbb{R}^{d-1} \times \mathbb{R}^{\geq 0}$ then the above proof goes through unchanged, except that the non-empty manifold boundary of M would, as shown in corollary 5.11, now prevent us from concluding that M is coherent with $\mathcal{A}_{]0,1[}$. The proof fails to generalize since we may now no longer extend any $\gamma : ([0, 1], 0) \to \left(M, \{0\}^d\right)$ satisfying the above properties (or any of its reparameterizations) to an open interval containing its domain. ∎

Theorem 0.2 follows immediately while corollary 0.15 follows with the help of theorems 10.4 and 8.1, and corollary 0.7 follows with with the aid from corollary 5.11 and theorem 10.4.

11. Coherence of $\mathbb{R}^\infty$ with C^1-Embeddings of Intervals The following theorem follows almost immediately from theorem 10.4 and the properties of $\mathbb{R}^\infty$ mentioned in example 3.6. It shows, in particular, there there are non-finite dimensional HLCTVSs that are coherent with their C^1-arcs so that the characterization in theorem 0.11 can not be extended to all HLCTVSs.

Theorem 11.1. Let $\mathcal{A}_I$ and $\mathcal{A}^*_{[0,1]}$ be defined as in corollary 5.17 for $p = 1$ and $M = \mathbb{R}^\infty$. Then $\mathbb{R}^\infty$ is coherent with $\mathcal{A}^*_{[0,1]}$ and with $\mathcal{A}_I$ for all intervals I.

Proof. As in the proof of theorem 10.4, it suffices to prove that $\mathbb{R}^\infty$ is coherent with each of $\mathcal{A}^*_{[0,1]}$. Let $S \subseteq \mathbb{R}^\infty$ be such that $\operatorname{Im}\gamma \cap S$ is closed in γ for all $\gamma \in \mathcal{A}^*_{[0,1]}$ and suppose that S was not closed in $\mathbb{R}^\infty$. So there exists some $n \in \mathbb{N}$ such that $S_n = S \cap \mathbb{R}^\infty$ is not closed in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is coherent with its C^1-embeddings of $[0,1]$ by theorem 10.4, the only way that S_n may fail to be closed in $\mathbb{R}^n$ is if there exists some C^1-embedding $\gamma_n : [0,1] \to \mathbb{R}^n$ in $\mathcal{C}^*_{[0,1]}$ such that $\operatorname{Im}(\gamma_n) \cap S_n$ is not closed in $\operatorname{Im}(\gamma_n)$. Since $\mathbb{R}^n$ is a vector subspace $\mathbb{R}^\infty$, when is γ_n considered as a map into $\mathbb{R}^\infty$ it will still be a C^1-embedding that is smooth on $]0,1]$ so that by applying our assumption about such maps into $\mathbb{R}^\infty$ we can conclude that $\operatorname{Im}(\gamma_n) \cap S$ is closed in γ_{n}, a contradiction. ■

Corollary 11.2. $\mathbb{R}^\infty$ is coherent with its C^1-arcs.

Corollary 11.3. $\mathbb{R}^\infty$ contains no closed complemented infinite-dimensional normable vector subspace.

Proof. Apply corollary 6.4. ■

Part 5. Characterization of Local Path-Connectedness

The following definition will allow us to simultaneously apply the subsequent lemma 11.5 to various notions of path-connectedness such as C^p-path connectedness ($p \in \{0, 1, \ldots, \infty\}$), piecewise C^p-path connectedness, C^p-arc connectedness, etc.

Definition 11.4. Let X be a space and let $\mathcal{C}$ be a collection of maps in X. If U is a subset of X, $x, y \in U$ and $n \in \mathbb{N}$ then by a *simple $\mathcal{C}$-chain of length n in U from x to y* we mean a sequence $\gamma_1, \ldots, \gamma_n$ of U-valued maps in $\mathcal{C}$ where $x \in \operatorname{Im}\gamma_1$, $y \in \operatorname{Im}\gamma_n$, and whenever $i, j \in \{1, \ldots, n\}$ are such that $|i - j| = 1$ then $\operatorname{Im}\gamma_i \cap \operatorname{Im}\gamma_j \neq \varnothing$. By a *simple $\mathcal{C}$-chain in U from x to y* we mean a simple $\mathcal{C}$-chain of any length in U from x to y.

We will say that U *is piecewise $\mathcal{C}$-connected (resp. U is $\mathcal{C}$-connected)* if for all distinct $x, y \in U$ there exists some simple $\mathcal{C}$-chain in U from x to y (resp. that has length 1). If either (1) x is isolated, or else (2) x is non-isolated and every neighborhood of x in X contains some neighborhood of x that is piecewise $\mathcal{C}$-connected (resp. is $\mathcal{C}$-connected) then we'll say that X is *neighborhood locally piecewise $\mathcal{C}$-connected* (resp. is *neighborhood locally $\mathcal{C}$-connected) at x* where if this neighborhood can always be chosen to be open in X then we'll remove the word "neighborhood." If we don't mention a point in the above definition then we mean that it's true at every point of X. ■

Surprisingly, despite the generality of definition 11.4, lemma 11.5's mild requirements on X will nevertheless always allow us to conclude that the space in question is locally piecewise $\mathcal{C}$-connected and neighborhood locally $\mathcal{C}$-connected.

Lemma 11.5. Let X be a Hausdorff Fréchet-Urysohn space and let $\mathcal{C}$ be a collection of continuous maps in X such that every non-isolated $x \in X$ is contained in the image of some $\gamma \in \mathcal{C}$, $\mathcal{C}$ satisfies condition ($\star$) in lemma 5.7, and the following presheaf-like condition is satisfied:

($\star\star$): for all $\gamma \in \mathcal{C}$, $t \in \operatorname{Dom}\gamma$, and neighborhoods N of t in $\operatorname{Dom}\gamma$, there exists some neighborhood W of t contained in N such that $\gamma\big|_W \in \mathcal{C}$.

For all $n \in \mathbb{N}$, $x \in X$, and neighborhoods U of x in X, let $C_n(x,U)$ denote the set of all $y \in U$ for which there exists a simple $\mathcal{C}$-chain of length n in U from x to y and let $C(x,U) = \bigcup\limits_{n=1}^{\infty} C_n(x,U)$. Then for all non-isolated $x \in X$ and neighborhoods U of x in X,

(1) $C_n(x,U)$ is a neighborhood of x in X for all $n \in \mathbb{N}$,
(2) if U is open in X then $C(x,U)$ is an open and closed subset of U containing x, which shows, in particular, that X is locally piecewise $\mathcal{C}$-connected at x,
(3) if $C_1(x,U) = C_2(x,U)$ then $C(x,U) = C_n(x,U)$ for all $n \in \mathbb{N}$,
(4) if all U-valued $\gamma \in \mathcal{C}$ have a connected domain then all $C_n(x,U)$'s are connected and $C(x,U)$ is the connected component of U containing x,
(5) $C(x,U) = C(y,U)$ for any $y \in C(x,U)$,
(6) $C_n(x,U) \subseteq C_{n+1}(x,U)$ for all $n \in \mathbb{N}$,
(7) if $y \in C_n(x,U)$ and $z \in C_k(y,U)$ then $x \in C_n(y,U)$ and $z \in C_{n+k}(x,U)$, and
(8) if U is open in X and for all $m \in U$ and $y,z \in C_1(m,U)$ there exists some U-valued $\gamma \in \mathcal{C}$ such that $m,y,z \in \operatorname{Im}\gamma$ then $C_1(x,U) = C(x,U)$ is open and closed in U.

Proof. Let $x \in X$ and let U be a neighborhood of x in X.

(6) - (7): Note that if $(\gamma_1,\ldots,\gamma_n)$ is a simple $\mathcal{C}$-chain in U from x to some point $y \in U$, then $(\gamma_1,\ldots,\gamma_n,\gamma_n)$ is trivially a simple $\mathcal{C}$-chain of length $n+1$, which shows that $C_n(x,U) \subseteq C_{n+1}(x,U)$ for all $n \in \mathcal{N}$. And if there is a simple $\mathcal{C}$-chain $(\eta_1,\ldots,\eta_k)$ of length k in U from y to some $z \in U$ then $(\gamma_1,\ldots,\gamma_n,\gamma_n,\eta_1,\ldots,\eta_k)$ simple $\mathcal{C}$-chain $(\eta_1,\ldots,\eta_k)$ of length $k+n$ in U from y so that $z \in C_{n+k}(x,U)$.

(1): Note that our assumptions imply that $C_1(x,U)$ contains x. Suppose that $C_1(x,U)$ was not a neighborhood of x in X. Then $x \in \operatorname{Cl}_U(U \setminus C_1(x,U))$ where since U is Fréchet-Urysohn, there exists some sequence $\left(x^l\right)_{l=1}^{\infty}$ of distinct points in $U \setminus C_1(x,U)$ that converge to x. By condition ($\star$), there exists some X-valued $\gamma \in \mathcal{C}$ and some γ-liftable subsequence of $x^\bullet$ so that by replacing $x^\bullet$ with this subsequence, we may assume without loss of generality that $x^\bullet$ is γ-liftable. If $(t_i)_{i=1}^{\infty} \to t$ is a γ-lift of $(x^i)_{i=1}^{\infty} \to x$ then by ($\star\star$) we may pick some neighborhood W of t contained in $\gamma^{-1}(U)$ such that $\gamma\big|_W \in \mathcal{C}$. Pick $N \in \mathbb{N}$ such that $t_N \in W$ and observe that $\gamma\big|_W(t_N) = x^N$, which contradicts the fact that $x^N \in U \setminus C_1(x,U)$. Thus $C_1(x,U)$ is a neighborhood of x in X that is contained in U and now (1) follows from (6).

(2) and (5): Clearly, for any $n \in \mathbb{N}$, if $y \in C_n(x,U)$ and $z \in C_1(y,U)$ then $z \in C_{n+1}(x,U)$ so that $y \in C_1(y,U) \subseteq C_{n+1}(x,U) \subseteq C(x,U)$, where $C_n(y,U)$ is a neighborhood of y in X. This shows that $C(x,U)$ is a neighborhood in X of each of its points and that $C(y,U) \subseteq C(x,U)$ for any $y \in C(x,U)$. With n and y as above, note that if $z \in C_k(x,U)$ for some $k \in \mathbb{N}$ then since $x \in C_n(y,U)$ we have that $z \in C_{k+n}(y,U) \subseteq C(y,U)$, which gives the reverse inclusion $C(x,U) \subseteq C(y,U)$ and proves (5).

Now suppose that $y \in U$ belongs to the closure of $C(x,U)$. Since X is Fréchet-Urysohn, we may pick a sequence $\left(x^l\right)_{l=1}^{\infty}$ in $C(x,U)$ in converging to y. By condition ($\star$), there exists some X-valued $\gamma \in \mathcal{C}$ and some γ-liftable subsequence of $\left(x^l\right)_{l=1}^{\infty}$ where as before, we may assume without loss of generality that $\left(x^l\right)_{l=1}^{\infty}$ is γ-liftable. Let $(t_i)_{i\in\mathbb{N}} \to t$ is an γ-lift of $(x^i)_{i\in\mathbb{N}} \to y$ and using ($\star\star$), pick some neighborhood W of t contained in $\gamma^{-1}(U)$. Pick $k \in \mathcal{P}$ such that $t_k \in W$ and let $n \in \mathbb{N}$ such that $\gamma\big|_W((t_k) = x^k \in C_n(x,U)$ where note that this, together with the fact that both x and x^k are in the image of $\gamma\big|_W$, implies that $x \in C_{n+1}(x,U)$. Thus $x \in C_{n+1}(x,U) \subseteq C(x,U)$, as desired.

(3) is proved by a straightforward induction argument.

(4): Note that $C_1(x,U)$ is just the union of all images of all U-valued maps in $\mathcal{C}$ whose images contain x so that if all U valued maps in $\mathcal{C}$ had a connected domain then $C_1(x,U)$ would be connected and from this observation, it is easy to see that one may inductively prove that all $C_n(x,U)$, and thus $C(x,U)$, are be connected. That $C(x,U)$ is the connected component of U containing x now follows from (2).

(8): If $m \in \mathrm{Cl}_U(C_1(x,U))$ then since $C_1(m,U)$ is a neighborhood of m in X there is some $z \in C_1(x,U) \cap C_1(m,U)$ so our assumption gives us a U-valued $\gamma \in \mathcal{C}$ such that $x, m, z \in \operatorname{Im}\gamma$, which implies that $m \in C_1(x,U)$. Our assumption clearly implies that $C_1(x,U) = C_2(x,U)$ so we may apply (2) and (3) to obtain the rest of (8). ■

Corollary 11.6. Let M be a C^p-manifold modeled on HTVSs where $0 \leq p \leq \infty$, let $S \subseteq M$ be a Fréchet-Urysohn subspace, let $0 \leq k \leq p$, and denote the set of all C^k-paths into S by $\mathcal{P}_k$. If S is coherent with $\mathcal{P}_k$ then S is locally path-connected, neighborhood locally $\mathcal{P}_k$-connected, and locally piecewise $\mathcal{P}_k$-connected

Warning 11.7. Observe that corollary 11.6 does *not* claim that S is locally $\mathcal{P}_k$-connected.

Proof of Theorem 0.14. Let X be a first-countable Hausdorff space. If X is coherent with its paths then it's locally path-connected by lemma 11.5 so suppose that X is locally-path-connected and let $S \subseteq X$ be such that for all paths γ in X, $\gamma^{-1}(S)$ is closed in γ's domain. Let $x \in \overline{S}$ and let $(U_l)_{l=1}^{\infty}$ be a countable decreasing neighborhood basis of x consisting of path-connected open sets and $(x_l)_{l=1}^{\infty}$ be a sequence of points in S such that for all $k, l \in \mathbb{N}$, if $l \geq k$ then $x_l \in U_k$. For all $l \in \mathbb{N}$, define γ on $\left[\frac{1}{l+1}, \frac{1}{-l}\right]$ to be a path in U_l from $\gamma(1/(l+1)) = x_{l+1}$ to $\gamma(1/l) = x_l$ and then let $\gamma(0) = x$. Clearly, $\gamma : [0,1] \to X$ is continuous on $]0,1]$ and since $[0, 1/l] \subseteq \gamma^{-1}(U_l)$ for all $l \in \mathbb{N}$, γ is also continuous at 0. Since $\gamma^{-1}(S)$ is closed in $[0,1]$ and $\{1, 1/2, \ldots\} \subseteq \gamma^{-1}(S)$, it follows that $0 \in \gamma^{-1}(S)$ so that $x = \gamma(0) \in S$, as desired. ■

Part 6. A Paradoxical Curve

The proof that $\mathbb{R}^{\mathbb{N}}$ is not coherent with C^1-arcs took advantage of the topological "rigidity" caused by having a non-vanishing derivative. In contrast, the following example illustrates how allowing the derivative of a curve to vanish at just one point allows for significantly "poorer" (or, depending on the perspective, "freer") behavior, even if the curve is a smooth topological embedding.

Example 11.8. Using $d := \mathbb{N}$, there exists a smooth almost arc $\gamma : ([-1,1], 0, 1) \to \left([-1,1]^d, \{0\}^d, \{1\}^d\right)$ vanishing at 0 such that $\gamma^{-1}\left(]-1,1[^d\right) = \{0\}$ and for all $t \in [0,1]$, $\gamma(-t) = -\gamma(t)$ and $\gamma(t) \in\,]-1,1]^d$.

Remarks 11.9.

(a) This curve from $[-1,1]$ into $[-1,1]^d$ is paradoxical since it sends 0 to the cube's center and the rest of its domain into "the surface of this cube" (i.e. into $[-1,1]^d \setminus\,]-1,1[^d$), where of course if d had instead been finite then such a curve could not exist. In addition, γ starts at the cube's "bottom-left and back-most" or "most negative corner" (i.e. $\gamma(-1) = (-1,-1,\ldots)$) and ends at the exact opposite corner. The paradox is resolved by recalling that $]-1,1[^{\mathbb{N}}$, which although it is convex, balanced, and absorbs $[-1,1]^{\mathbb{N}}$ (but is not absorbing in $\mathbb{R}^{\mathbb{N}}$), is nevertheless not an open subset of $[-1,1]^{\mathbb{N}}$.

(b) Except for the pathological property mentioned in (a), the curve γ is otherwise relatively "well-behaved" in the following ways (which serve to make γ even more pathological): it is smooth, a topological embedding, "symmetric about the origin" (i.e. an odd map), and $\gamma'(t)$ does not vanish whenever $\gamma(t)$ belongs to this "surface." This implies, in particular, that γ's restriction to $[-1,0[\, \cup\,]0,1]$ smoothly embeds these intervals into this cube's "surface" so that in this sense, everywhere outside of $\gamma(0)$, γ's image may be considered as being "as familiar and well-behaved as $[-1,0[\, \cup\,]0,1]$."

(c) By γ being smooth we mean that it is smooth as a curve into the TVS $\mathbb{R}^d$.

(d) It will be clear from γ's construction that had we been given $(\epsilon_l)_{l \in d} \in\,]0,1]^d$ then we could have constructed γ so that $\gamma\big|_{]0,1]}$ takes values in $\prod_{l \in d}\,]-\epsilon_l, 1]$ that are outside of $]-1,1[^d$, which consequently implies that $\gamma\big|_{[-1,0[}$ takes values in $\prod_{l \in d} [-1, \epsilon_l[$ that are outside of $]-1,1[^d$. The geometric meaning of this additional property can be made clearer by consider what it would mean if d had been (a set of cardinality) 2 or 3. This remark also applies to example 11.10.

Proof. Let $\alpha_1 : \left(\left[\frac{1}{3}, 1\right], \frac{1}{3}, 1\right) \to \left([0,1]^2, \{0\}^2, \{1\}^2\right)$ be any smooth arc such that $\alpha_1\left(\left[\frac{1}{3}, \frac{1}{2}\right]\right) \subseteq 0 \times [0,1]$. Suppose that $k \geq 1$ and that for all $i = 1, \ldots, k$ we have constructed a smooth arc $\alpha_i : \left(\left[\frac{1}{2i+1}, 1\right], \frac{1}{2i+1}\right) \to \left(]-1,1]^{2i}, \{0\}^{2i}\right)$ such that

(1) $\alpha_i\left(\left[\frac{1}{2i+1}, \frac{1}{2i}\right]\right) \subseteq \{0\}^{2i-1} \times [0,1]$,
(2) if $1 \leq h \leq i$ then $\alpha_i\big|_{\left[\frac{1}{2h+1}, 1\right]} = \left(\alpha_h, \{1\}^{2(i-h)}\right)$

Let $\alpha_{k+1}^R = (\alpha_k, 1, 1) : \left[\frac{1}{2k+1}, 1\right] \to]-1,1]^{2(k+1)}$ and observe that α_{k+1}^R is a smooth arc. Since $\alpha_{k+1}^R\big|_{\left[\frac{1}{2k+1}, \frac{1}{2k}\right]}$ is a smooth embedding with image into the one dimensional space $\{0\}^{2k-1} \times [0,1] \times \{1\}^2$ with $\alpha_{k+1}^R\left(\frac{1}{2k+1}\right) = \{0\}^{2k} \times \{1\}^2$, it's clear that we may pick a smooth arc

$$\alpha_{k+1}^L : \left[\frac{1}{2(k+1)+1}, \frac{1}{2k+1}\right] \to \{0\}^{2k+1} \times]-1,1] \times [0,1]^2$$

mapping $\frac{1}{2k+1}$ and $\frac{1}{2(k+1)+1}$ to $\{0\}^{2k} \times \{1\}^2$ and $\{0\}^{2(k+1)}$, respectively, such that $\alpha_{k+1}^L\left(\left[\frac{1}{2(k+1)+1}, \frac{1}{2(k+1)}\right]\right) \subseteq \{0\}^{2(k+1)-1} \times [0,1]$ and the map

$$\alpha_{k+1} = \alpha_{k+1}^L \cup \alpha_{k+1}^R : \left[\frac{1}{2(k+1)+1}, 1\right] \to]-1,1]^{2(k+1)}$$

defined by concatenating α_{k+1}^L and α_{k+1}^R, is a smooth arc. It is clear that α_{k+1} satisfies (1) and (2) so that this completes our inductive construction. Observe that (1) and (2) imply that for all $1 \leq i < l$, the first $(2i-1)$-coordinates of $\alpha_l\big|_{\left[\frac{1}{2l+1}, \frac{1}{2i}\right]}$ are identically 0.

For all $k, l \in \mathbb{N}$ with $k \leq l$, let $\mu_{kl} : [-1,1]^{2l} \to [-1,1]^{2k}$ and $\mu_k : [-1,1]^{\mathbb{N}} \to [-1,1]^{2k}$ both denote the canonical projections onto the first $2k$ coordinates. Clearly, if $1 \leq k \leq l$ then by construction, $\alpha_k = \mu_{kl} \circ \alpha_l\big|_{\operatorname{Dom} \alpha_k}$ so for any fixed $k \in \mathbb{N}$ and $t \in]0,1]$, the value $\mu_{kl} \circ \alpha_l(t)$ is independent of the choice of l, where $l \geq k$ is an integer such that $t \in \left[\frac{1}{2l+1}, 1\right]$. This implies that for all $k \in \mathbb{N}$, we may construct a well-defined map

$$\alpha^k : \left[\frac{1}{2k+1}, 1\right] \to]-1,1]^{\mathbb{N}}$$

where α^k is defined coordinate-wise as follows: for each integer $l \geq k$, the first $2l$-coordinates of α^k are $\alpha_l\big|_{\left[\frac{1}{2l+1}, 1\right]}$ (for the reader familiar with limits, observe that the curves $\left(\mu_{kl} \circ \alpha_l\big|_{\operatorname{Dom} \alpha_k}\right)_{l=k}^{\infty}$ form a cone from $\operatorname{Dom} \alpha_k = \left[\frac{1}{2k+1}, 1\right]$ into the projective system $\left([-1,1]^{2l}, \mu_{ln}, \{k, k+1, \ldots\}\right)$ and α^k is simply the limit of this cone). It's clear from α^k's definition that it is a smooth (since each of its coordinates is smooth) and it's even a smooth arc since its first $2k$-coordinates are α_k, which is a smooth arc. Observe α^k is never $\{0\}^{\mathbb{N}}$ since its first $2(k+1)$-coordinates, which are $\alpha_{k+1}\big|_{\left[\frac{1}{2(k+1)+1}, 1\right]}$, are never $\{0\}^{2(k+1)}$ and furthermore, note that the first $(2k-1)$-coordinates of $\alpha^k\big|_{\left]0, \frac{1}{2k}\right]}$ are identically 0 since the first $(2k-1)$-coordinates of $\alpha_l\big|_{\left[\frac{1}{2l+1}, \frac{1}{2k}\right]}$ are identically 0 for all $l > k$.

Observe that for any $1 \leq k \leq l$, $\alpha^l = \alpha^k$ on $\operatorname{Dom}(\alpha^k)$ so that we may define the map

$$\alpha := \bigcup_{k=1}^{\infty} \alpha^k :]0,1] \to]-1,1]^{\mathbb{N}}$$

which is a smooth embedding that is nowhere $\{0\}^{\mathbb{N}}$. Now define $\gamma^R : [0,1] \to]-1,1]^{\mathbb{N}}$ by $\gamma^R\big|_{]0,1]} = \alpha$ and $\gamma^R(0) = \{0\}^{\mathbb{N}}$ and note that it is injective. For any $k \in \mathbb{N}$, observe that $\mu_k \circ \gamma^R = 0$ on $\left[0, \frac{1}{2(k+1)}\right]$ since the first $(2(k+1)-1)$-coordinates of $\alpha^{k+1}\big|_{\left]0, \frac{1}{2(k+1)}\right]}$ are identically 0, which implies that all of γ's coordinates are smooth and that all derivatives at 0 of each of γ^R's coordinates vanish. Thus γ^R is a smooth map such that $\left(\gamma^R\right)^{(p)}(0) = \{0\}^{\mathbb{N}}$ for all $p \in \mathbb{Z}^{\geq 0}$.

Define $\gamma^L : [-1,0] \to [-1,1[^{\mathbb{N}}$ by $\gamma(t) = -\gamma(-t)$ and observe that γ is a smooth topological embedding whose derivative does not vanish on $[-1,0[$ and all of whose derivatives vanish at 0. Let $\gamma = \gamma^L \cup \gamma^R : [-1,1] \to [-1,1]^{\mathbb{N}}$ so that γ is a smooth map whose derivative does not vanish on $[-1,0] \smallsetminus \{0\}$ and all of whose derivatives vanish at 0. By construction, for all $0 < t \leq 1$ there exists some coordinate of $\gamma^R(t)$ that is equal to 1 so that $\gamma^R(t)$, and hence also $\gamma^L(-t)$, does not belong to $]0,1[^{\mathbb{N}}$. Thus $\gamma^R(]0,1]) \subseteq]-1,1]^{\mathbb{N}} \smallsetminus]-1,1[^{\mathbb{N}}$ and $\gamma^L([-1,0[) \subseteq [-1,1[^{\mathbb{N}} \smallsetminus]-1,1[^{\mathbb{N}}$, which in turn implies that $\gamma^L([-1,0[) \cap \gamma^R(]0,1]) = \varnothing$ so that the continuous map γ is injective and thus a topological embedding. ∎

Example 11.10. For any infinite set d, there exists a smooth topological embedding $\gamma : ([-1,1],0) \to \left([-1,1]^d, \{0\}^d\right)$ having all of the properties listed in example 11.8.

Proof. Write d as a disjoint of $d = \bigsqcup_{i \in I} d^i$ where every d^i is a countably infinite subset of d enumerated by $\left(d^i_l\right)_{l=1}^{\infty}$. Let $\hat{\gamma} = (\hat{\gamma}_l)_{l=1}^{\infty} : [-1,1] \to [-1,1]^{\mathbb{N}}$ denote the curve from example 11.8 and for all $i \in I$ and $l \in \mathbb{N}$ let $\gamma_{d^i_l} = \hat{\gamma}_l$. Then $\gamma = (\gamma_e)_{e \in d} : [-1,1] \to [-1,1]^d$ is the desired curve. ∎

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