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Assignment 3, Due Tuesday Feb 20

As you know, there's an exam coming up covering complex numbers. I won't cover logs on the exam, but I will cover things like complex powers, geometric representations, and trig functions.

1 A bit of programming - spend 10 minutes on this before Friday the 16th

Start from the Google Colab notebook we developed in class
`https://colab.research.google.com/drive/14KRuTQ0W3lphr3ycC7qjFoi5m-CB0Bci#scrollTo=Cl0SPSuivA0v`
make a new cell, and define your own function called "airy2". That function should be the second non-trivial (i.e. not always zero) solution to Airy's equation.

2 Solving differential equations

Using a table like the one in Boas §12.2, solve problem 12.1.1

3 Looking at the Legendre equation: part 1

In this assignment, as in many of the future assignments, we will investigate a particular topic in depth, rather than solving several separate problems. This week, we focus on the Legendre polynomials, which have broad applicability in mathematical physics, especially in the modeling of spherically symmetric systems.

The text of the following problems is taken (with some small changes) from Boyce and DiPrima, Chapter 5, section 3.

The following problems deal with the Legendre equation:

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \tag{1}$$

Following the convention of choosing a fundamental set of solutions such that

$$y_1(x) = 1 + b_2(x-x_0)^2 + ...$$
$$y_2(x) = (x-x_0) + c_2(x-x_0)^3 + ...$$
$$b_2 + c_2 = a_2$$

(Note that these series have already included the fact that one will be even and one will be odd, a fact that you'll show below.)

Two solutions of the Legendre equation for $|x| < 1$ are

$$
\begin{aligned}
y_1(x) &= 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 \\
&\quad + \sum_{m=3}^{\infty}(-1)^m \frac{\alpha\cdots(\alpha-2m+2)(\alpha+1)\cdots(\alpha+2m-1)}{(2m)!}x^{2m}, \\
y_2(x) &= x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5 \\
&\quad + \sum_{m=3}^{\infty} \frac{(\alpha-1)\cdots(\alpha-2m+1)(\alpha+2)\cdots(\alpha+2m)}{(2m+1)!}x^{2m+1}
\end{aligned}
$$

Problem 1 Write out the first 4 terms for y_1 and y_2.

Problem 2 Show that, if α is zero or a positive even integer $2n$, the series solution y_1 reduces to a polynomial of degree $2n$ containing only even powers of x. Find the polynomials corresponding to $\alpha = 0, 2, 4$. Similarly, show that if α is a positive odd integer $2n+1$, the series solution y_2 reduces to a polynomial of degree $2n+1$ containing only odd powers of x. Find the polynomials corresponding to $\alpha = 1, 3, 5$.

Problem 3 The Legendre polynomial $P_n(x)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ that also satisfies the condition $P_n(1) = 1$.

(a) Using the results of Problem 2, find the first five Legendre polynomials, $P_0(x), ..., P_5(x)$.

(b) Plot the graphs of $P_0(x), ..., P_5(x)$ in the range for which we've demonstrated convergence, $|x| \leq 1$. The end goal is to do this in a Python notebook. That may be too much programming, though. If the Python is a barrier, you can check your answers with Wolfram Alpha (e.g. go to wolframalpha.com and type "plot 0.5*(3x^2-1) from -1 to 1" in the box), or via Desmos. Then we can do the Python work in class.!

(c) Find the zeros of $P_0(x), ..., P_5(x)$.