Physics 360/Math 360   TF 1:00 - 2:20 PM, Room: CST 225

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Assignment 4, Due Thursday Mar 10, end of day.

Legendre Problem 3 The Legendre polynomial $P_n(x)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ that also satisfies the condition $P_n(1) = 1$.

(a) Using the results of Problem 2, find the first five Legendre polynomials, $P_0(x), ..., P_5(x)$.

(b) Plot the graphs of $P_0(x), ..., P_5(x)$ in the range for which we've demonstrated convergence, $|x| \leq 1$. You can check your answers with Wolfram Alpha (e.g. go to wolframalpha.com and type "plot 0.5*(3x^2-1) from -1 to 1" in the box), but you need to print out solutions here using Python + matplotlib. For this, please refer to the in-class notebooks and ask your instructor/classmates for help early and often!

(c) Find the zeros of $P_0(x), ..., P_5(x)$.

Legendre Problem 4 The Legendre polynomials play an important role in mathematical physics. For example, solving the potential equation (Laplace's equation) in spherical coordinates, we encounter the equation

$$\frac{d^2F(\phi)}{d\phi^2} + \cot\phi \frac{dF(\phi)}{d\phi} + n(n+1)F(\phi) = 0, \qquad 0 < \phi < \pi$$

Show that the change of variables $x = \cos\phi$ leads to the Legendre equation with $\alpha = n$ for $y = f(x) = F(\cos^{-1}(x))$

Hint: you may need to use the fact that

$$\sin(\arccos(x)) = \sqrt{1-x^2}; \qquad \cot(\arccos(x)) = \frac{x}{\sqrt{1-x^2}}$$

Legendre Problem 5 Show that the Legendre equation can also be written as

$$[(1-x^2)y']' = -\alpha(\alpha+1)y$$

It then follows that

$$[(1-x^2)P_n'(x)]' = -n(n+1)P_n(x) \tag{1}$$

and

$$[(1-x^2)P_m'(x)]' = -m(m+1)P_m(x). \tag{2}$$

By multiplying (1) by $P_m(x)$ and (2) by $P_n(x)$, **integrating by parts**, and then subtracting one equation from the other, show that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \qquad \text{if} \quad n \neq m \tag{3}$$

This property (3) of the Legendre polynomials is known as the orthogonality property. If $m = n$, it can be shown that the value of the integral in (3) is $2/(2n+1)$.

Given a polynomial f of degree n, it is possible to express f as a linear combination of $P_0, P_1, ..., P_n$:

$$f(x) = \sum_{k=0}^{n} a_k P_k(x) \tag{4}$$

Note that, since the $n+1$ polynomials $P_0, ..., P_n$ are linearly independent, and the degree of P_k is k, any polynomial of degree n can be expressed as (4).
Using the result of Problem 7, you can show that

$$a_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x) dx$$

but you don't have to! You're done!