
Instructor: Michael Lerner, CST 213, Phone: 765-983-1784

Assignment 4, Due as Tuesday Mar 5

1 Looking at the Legendre equation: part 2

Problem 4 The Legendre polynomials play an important role in mathematical physics. For example, solving the potential equation (Laplace's equation) in spherical coordinates, we encounter the equation

$$\frac{d^2F(\phi)}{d\phi^2} + \cot\phi \frac{dF(\phi)}{d\phi} + n(n+1)F(\phi) = 0, \qquad 0 < \phi < \pi$$

Show that the change of variables $x = \cos\phi$ leads to the Legendre equation with $\alpha = n$ for $y = f(x) = F(\cos^{-1}(x))$

Hint: you may need to use the fact that

$$\sin(\arccos(x)) = \sqrt{1-x^2}; \qquad \cot(\arccos(x)) = \frac{x}{\sqrt{1-x^2}}$$

Problem 5 Show that the Legendre equation can also be written as

$$[(1-x^2)y']' = -\alpha(\alpha+1)y$$

It then follows that

$$[(1-x^2)P_n'(x)]' = -n(n+1)P_n(x) \tag{1}$$

and

$$[(1-x^2)P_m'(x)]' = -m(m+1)P_m(x). \tag{2}$$

By multiplying (1) by $P_m(x)$ and (2) by $P_n(x)$, **integrating by parts**, and then subtracting one equation from the other, show that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \qquad \text{if} \quad n \neq m \tag{3}$$

This property (3) of the Legendre polynomials is known as the orthogonality property. If $m = n$, it can be shown that the value of the integral in (3) is $2/(2n+1)$.

Given a polynomial f of degree n, it is possible to express f as a linear combination of $P_0, P_1, ..., P_n$:

$$f(x) = \sum_{k=0}^{n} a_k P_k(x) \tag{4}$$

Note that, since the $n+1$ polynomials $P_0, ..., P_n$ are linearly independent, and the degree of P_k is k, any polynomial of degree n can be expressed as (4).
Using the result of Problem 7, you can show that

$$a_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x) dx$$

but you don't have to! You're done!

2 Boas §7.2 Wave Review

Make sure you understand the following problems
1, 6, 7, 17, 21.
(You do not have to turn them in.)

3 Boas §7.4 Average Value

For your reference, these are Boas §7.4 problems #3, 4, 10

3.1

Find the average value of the function on the given interval. You may use equation 4.8 if it applies. It's well worth your time to make a quick sketch of the function, as you may be able to quickly see the average value. Especially when it's zero. If you find yourself spending more than 5 minutes on any one of these, please post to Piazza asking for hints, and then move on to the next one.

$$\sin x + 2\sin 2x + 3\sin 3x \qquad \text{on} \quad (0, 2\pi) \tag{5}$$

3.2

$$1 - e^{-x} \qquad \text{on} \quad (0, 1) \tag{6}$$

3.3

$$\cos x \qquad \text{on} \quad (0, 3\pi) \tag{7}$$

4 Boas §7.5 Fourier Series

4.1

Problem §7.5.9 (also graph the sum of the first four non-zero terms using Python in addition to solving)

4.2

Problem 12.

4.3

For extra credit, you may do problem 13.