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1 Problem 1

Solution by Hand:

1.

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\boxed{A^T = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}}$$

$$A^2 = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3\cdot 3+(0)(-1)+2\cdot 1 = 11 & -3-1+0 = -4 & 6-2-2 = 2 \\ 0+0+2 = 2 & 0+1+0 = 1 & 0+2-2 = 0 \\ 1+0+1 = 2 & -1+0+0 = -1 & 2+0+1 = 3 \end{pmatrix}$$

$$\Rightarrow \boxed{A^2 = \begin{pmatrix} 11 & -4 & 2 \\ 2 & 1 & 0 \\ 2 & -1 & 3 \end{pmatrix}}$$

$$A^TA = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} (3,0,1)\begin{pmatrix}3\\0\\1\end{pmatrix}, & (3,0,1)\begin{pmatrix}-1\\1\\0\end{pmatrix}, & (3,0,1)\begin{pmatrix}2\\2\\-1\end{pmatrix} \\ (-1,1,0)\begin{pmatrix}3\\0\\1\end{pmatrix}, & (-1,1,0)\begin{pmatrix}-1\\1\\0\end{pmatrix}, & (-1,1,0)\begin{pmatrix}2\\2\\-1\end{pmatrix} \\ (2,2,-1)\begin{pmatrix}3\\0\\1\end{pmatrix}, & (2,2,-1)\begin{pmatrix}-1\\1\\0\end{pmatrix}, & (2,2,-1)\begin{pmatrix}2\\2\\-1\end{pmatrix} \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} 10, & -3, & 5 \\ -3, & 2, & 0 \\ 5, & 0, & 9 \end{pmatrix}}$$

Python Solution:

[84]:
```python
A = np.array([[3, -1, 2], [0, 1, 2], [1, 0, -1]]) # matrix A for problem 1
print("A transpose: \n", A.T) # A^T
print("A squared: \n", A @ A) # A^2
print("A transpose times A: \n", A.T @ A) # (A^T)*A
```

```
A transpose:
 [[ 3  0  1]
 [-1  1  0]
 [ 2  2 -1]]
A squared:
 [[11 -4  2]
 [ 2  1  0]
 [ 2 -1  3]]
A transpose times A:
 [[10 -3  5]
 [-3  2  0]
 [ 5  0  9]]
```

2 Problem 2

[87]:
```python
x = np.array([3, 1, 4, 1, 5]) # vector x for problem 2
print("The Eucledean norm of vector x is: ", np.linalg.norm(x)) # answer
```

```
The Eucledean norm of vector x is:  7.211102550927978
```

3 Problem 3

3

$$\begin{pmatrix} 2 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 4 \end{pmatrix}$$

$$\Downarrow$$

$$2x_0 - 3x_1 + x_2 = 1$$
$$0x_0 + 2x_1 + 3x_2 = 7$$
$$x_0 + 0x_1 + x_2 = 4$$

[88]:
```python
A = np.array([[2, 3, 1], [0, 2, 3], [0, 1, 1]]) # matrix A for problem 3
b = np.array([1, 7, 4]) # vector b for problem 3
```

4 Problem 4

```
[89]: x = npla.solve(A, b) # solution vector x solves Ax = b
      print("The solution to Ax = b is: ", x) # problem 4 answer
```

```
The solution to Ax = b is:  [-6.5  5.  -1. ]
```

```
[92]: print(np.array_equal(b, A @ x)) # to confirm our answer
```

```
True
```

5 Problem 5

(5) $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

gives no solution for $Ax = b$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \Rightarrow \begin{matrix} 0x_0 + 0x_1 = 1 \\ 0x_0 + x_1 = 0 \end{matrix} \Rightarrow \begin{matrix} \boxed{0 = 1} \\ x_1 = 0 \end{matrix} \Rightarrow \text{contradiction}$$

There's a contradiction that arises from trying to solve $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ because the top row can't be multiplied by anything to result in 1.

6 Problem 6

(6) $A = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ give more than one solution to $Ax = b$

$$\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \Rightarrow \begin{matrix} 4x_1 = 4 \\ 0 = 0 \end{matrix} \Rightarrow x_1 = 1,\ x_0 \in \mathbb{R}$$

Two solutions: $(x_0 = 0, x_1 = 1)$ and $(x_0 = 10, x_1 = 1)$

7 Problem 7

No. When given a 2x2 matrix A and a 2x1 vector b, the solution to $Ax = b$ represents the points of intersection between the two lines (the solutions to the system of two linear equations). Two lines can run parallel without intersecting, they can intersect at a single point, or they can lie directly on top of each other. There's no scenario where two straight lines intersect at exactly two points.

8 Problem 8

(8) $A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$ $\qquad \left| \begin{pmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{pmatrix} \right| = 0$

$A - \lambda I = 0 \Rightarrow$

$\Rightarrow \lambda^2 - 8\lambda + 15 = 0$

$(\lambda - 3)(\lambda - 5) = 0$

$\lambda = 3 \text{ or } 5$

$\underline{\lambda = 3}$ $\quad Ax = \lambda x$

$\Rightarrow \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 3x_0 \\ 3x_1 \end{pmatrix}$

$\Rightarrow \begin{matrix} 4x_0 - x_1 = 3x_0 \\ -x_0 + 4x_1 = 3x_1 \end{matrix} \Rightarrow x_0 - x_1 = 0 \Rightarrow x_0 = x_1$

$\Rightarrow x_0 = x_1 = \pm 1 \Rightarrow \boxed{\lambda = 3 \text{ has } x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\underline{\lambda = 5}$

$\Rightarrow \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 5x_0 \\ 5x_1 \end{pmatrix}$

$\Rightarrow \left. \begin{matrix} 4x_0 - x_1 = 5x_0 \\ 4x_1 - x_0 = 5x_1 \end{matrix} \right\} x_0 = -x_1$

$\Rightarrow \boxed{\lambda = 5 \text{ has } x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

Answer: We found two eigenvalues, 3 and 5. The eigenvector x for eigenvalue 3 is $(1, 1)$ and the eigenvector x for eigenvalue 5 is $(1, -1)$.

9 Problems 9-13

(9) $f(x) = 7x^3 - 2x^2 + 4x - 5$

$\Rightarrow \boxed{f'(x) = 21x^2 - 4x + 4}$

(10) $z = xe^{y/2}$

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= e^{y/2} \\ \frac{\partial z}{\partial y} &= \frac{x}{2} e^{y/2} \end{aligned}}$$

(11) $f'(x) = x^2 + \sin x$

$\Rightarrow f(x) = \int x^2 + \sin x \, dx$

$\Rightarrow \boxed{f(x) = \frac{x^3}{3} - \cos x + C}$

(12) $h = 1280t - 16t^2$

max height: $1280 - 32t = 0$

$\Rightarrow \boxed{t = 40 \text{ seconds}}$

hits ground $(40)(2) = \boxed{80 \text{ seconds}}$ after fire

(13) $\frac{dy}{dx} = xy$, $x=0$, $y=1$

$\Rightarrow \frac{dy}{y} = x\,dx \Rightarrow \int \frac{1}{y}\,dy = \int x\,dx$

$\Rightarrow \ln y = \frac{x^2}{2}$

$\Rightarrow y = Ce^{x^2/2}$

We know $1 = Ce^0 \Rightarrow C = 1$

$\Rightarrow \boxed{y = e^{x^2/2}}$