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Algebraic Models for Loop Spaces and Configuration Spaces

Masterarbeit

zur Erlangung des akademischen Grades
Master of Science (M. Sc.)

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eingereicht am: 6.3.2024

Abstract

String topology is the study algebraic structures on the homology of the free loop space and related spaces. We exhibit two constructions of the loop product and coproduct on a simply connected manifold M that allow us to prove that the cohomology of the free loop space has the structure of a co-Batalin-Vilkovisky-algebra. The first construction uses the Fulton-MacPherson compactification of the configuration space of n ordered points, which is a convenient context for the construction of the intersection product on M. The second construction is purely algebraic and shows that the intersection map in the definition of the loop product is algebraically a lift of the intersection product on M. Starting with a differential graded algebra model A of M, we generalize the Hochschild chain complex construction to construct explicit cochain complexes which model mapping spaces from graphs into M, such as the path space and the figure eight space of M. Our main tool is a theorem known as the pullback-pushout lemma, which shows how one can compute the cohomology of a pullback of spaces using the derived tensor product.

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1. Introduction

1.1. Background and Motivation

By a result due to Quillen [Qui69], the study of the homotopical properties of simply connected topological spaces is equivalent to the study of the homotopical properties of simply connected differential graded commutative algebras. Given a specific simply connected manifold M, one may construct other spaces from M, such as the free loop space $LM \coloneqq C^0(S^1, M)$ and the configuration spaces $\mathrm{Conf}_M(n)$ of n ordered pairwise distinct points in M. Given a differential graded algebra A which computes the cohomology of M, such as the de Rham algebra $\Omega^*(M)$ or the singular cochain algebra $C^*(M)$, we consider the problem of constructing cochain complexes which compute the cohomology of spaces such as LM and $\mathrm{Conf}_M(n)$. This problem appears in Adams' work on the based loop space ΩM and the cobar construction [Ada56] and Chen's work [Che77] on the bar construction and the free loop space LM. In [Jon87], Jones showed that there is an isomorphism between the cohomology of the loop space and the Hochschild cohomology of $C^*(M)$. In [LS08b], Lambrechts and Stanley showed that one can construct a model of the configuration spaces, if one starts with a differential graded algebra A which in addition has a chain level intersection coproduct, the cohomology version of the intersection product.

String Topology was initiated by Moira Chas and Dennis Sullivan in [CS99] and it is concerned with the study of algebraic operations on the homology of the free loop space LM of a closed oriented manifold M. They introduced the Chas-Sullivan product

$$H_*(LM) \otimes H_*(LM) \to H_{*-\dim M}(LM)$$

and the so called BV operator $\Delta\colon H_*(LM) \to H_{*+1}(LM)$. In [GH09], the loop coproduct is introduced, which is a degree $n-1$ operation on $H_*(LM, M)$, where $M \subseteq LM$ is embedded via constant loops.

Following [NRW22, Sec. 1], String Topology is at the intersection of several fields of mathematics, including the following:

- String Topology has an interpretation in Symplectic Geometry: the Floer homology of the cotangent bundle T^*M is isomorphic to $H_*(LM)$ and the loop product corresponds to the pair of pants product in Floer homology, cf. [AS10] and for a more general picture [CL09].

- The homology of LM and string operations can be used to study closed geodesics on Riemannian manifolds via infinite dimensional Morse theory of the energy functional. Gromoll and Meyer [GM69] showed for a closed simply connected manifold M, if the sequence of Betti numbers of LM is unbounded, then there exist infinitely many geometrically distinct periodic geodesics in M. More recently, the loop product has been used to study closed geodesics as well, see for instance [GH09].

- Families of higher operations can be defined on $H_*(LM)$. This leads to the study of structures which organize families of operations, such as properads and 2-dimensional field theories, cf. [God07], [CHV06, I.2, I.3].

The Chas-Sullivan product is constructed as follows. Consider the following maps:

$$LM \times LM \xleftarrow{i} LM \times_M LM \xrightarrow{c} LM$$

where $LM \times_M LM \subseteq LM \times LM$ is the space of pairs of loops with the same base point, i is the inclusion map and c is the composition of loops. One can construct a pullback map

$$i_!^{LM} \colon H_*(LM \times LM) \to H_{*-n}(LM \times_M LM)$$

The construction of $i_!^{LM}$ is involved and there are several approaches, for example using the Thom isomorphism [CHV06], [HW17] or via transversality [Cha05]. [CS99] interpret this map as follows: if a homology class a in $LM \times LM$ is represented by a manifold A which is transverse to the submanifold $LM \times_M LM \subseteq LM \times LM$, then

$$i_!^{LM}(a) = [A \cap LM \times_M LM].$$

The Chas-Sullivan product is then defined as the composition

$$c_* \circ i_!^{LM} \colon H_*(LM) \otimes H_*(LM) \cong H_*(LM \times LM) \to H_{*-\dim M}(LM)$$

Chas and Sullivan exhibit another operator on $H_*(LM)$. Denote by $\rho \colon S^1 \times LM \to LM$ the S^1-action on LM, given by $(t, \gamma) \mapsto \gamma(- + t)$. Let $[S^1] \in H_1(S^1)$ be the fundamental class of S^1. The BV operator on $H_*(LM)$ is the composition

$$\Delta \colon H_*(LM) \xrightarrow{[S^1]\times -} H_{*+1}(S^1 \times LM) \xrightarrow{\rho_*} H_{*+1}(LM)$$

Together with the loop product this forms a Batalin-Vilkovisky (BV) algebra (see Definition 1.1) on $H_*(LM)$.

1.2. Results and Outline

The main goal of this thesis is to give a clean presentation of the construction of the loop product and BV operator in [NW19] and to show that these operations yield a co-BV algebra on cohomology.

We work in cohomology throughout the thesis, this means that the loop product

$$H_*(LM) \otimes H_*(LM) \to H_{*-\dim M}(LM)$$

on homology turns into a coproduct

$$H^*(LM) \otimes H^*(LM) \leftarrow H^{*-\dim M}(LM)$$

on cohomology. Following [NW19] we occasionally write maps on cohomology from right to left. The homological BV algebra structure is turned into the structure of a co-BV algebra on cohomology.

Definition 1.1. Let V be a graded vector space.

1. A **Batalin-Vilkovisky (BV) algebra** on V is given by a degree 0 product $V \otimes V \to V$ and a degree -1 operator $\Delta\colon V \to V$, such that
   - the product is unital, associative and graded commutative
   - $\Delta^2 = 0$ and $\Delta(1) = 0$
   - The following seven-term identity holds:
   $$\begin{aligned} 0 =&\Delta(abc) - \Delta(ab)c - (-1)^{|a|}a\Delta(bc) - b(-1)^{(|a|+1)|b|}b\Delta(ac) \\ &+ \Delta(a)bc + (-1)^{|a|}a\Delta(b)c + (-1)^{|a|+|b|}ab\Delta(c) \end{aligned}$$
   Equivalently, the **Gerstenhaber bracket**
   $$(a, b) \coloneqq \Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b)$$
   satisfies the Leibniz identity
   $$(a, bc) = (a, b)c + (-1)^{|b||c|}(a, c)b$$
2. A **co-BV algebra** on V is given by a degree 0 coproduct map $\theta\colon V \to V \otimes V$ and a degree 1 operator $\Delta\colon V \to V$, such that
   - the coproduct θ is counital with counit ε, coassociative and graded cocommutative
   - $\Delta^2 = 0$ and $\varepsilon \circ \Delta = 0$
   - The cobracket
   $$\begin{aligned} &S\colon V \to V \otimes V \\ &S \coloneqq \theta \circ \Delta + (\Delta \otimes \mathrm{id} + \sigma \otimes \Delta)\theta \end{aligned}$$
   where $\sigma\colon V \to V, a \mapsto (-1)^{|a|}a$, satisfies the co-Leibniz identity
   $$(\theta \otimes \mathrm{id})S = (\mathrm{id} \otimes S)\theta + (\mathrm{id} \otimes \tau)(S \otimes id)\theta$$
   where $\tau\colon V \otimes V \to V \otimes V, v \otimes v' \mapsto (-1)^{|v||v'|}v' \otimes v$.

The **Hochschild homology** of a differential graded algebra A is the homology of the complex $CH_*(A, A) = B(A, A, A) \otimes_A A$, where $BC(A, A, A)$ is the two-sided bar construction, cf. Appendix B.1. If A is a Frobenius algebra (cf. Section 6.2), one can define a coproduct on $CH_*(A, A)$ by mapping

$$a_0 \otimes \cdots \otimes a_n \otimes a \mapsto \sum_{j=0}^{n} \sum \pm(a_0 \otimes \cdots \otimes a_j)a'_k \otimes (a_{j+1} \otimes \cdots \otimes a_n)_{a''_k}$$

where the coproduct of A is written as $\Delta(a) = \sum_k a'_k \otimes a''_k$. The Connes-Rhinehart differential on the Hochschild complex is given by

$$B(a_0 \otimes \ldots a_n \otimes a) = \sum_{j=0}^{n} \pm(a_j \otimes \cdots \otimes a_n \otimes a \otimes a_0 \otimes a_{j-1}) \otimes 1$$

One can show [Abb15] that these two operations induce the structure of a co-BV algebra on $H^*(A, A)$.

Definition 1.2. For a graded vector space V and $k \in \mathbb{Z}$, we denote by $V[k]$ the **shifted graded vector space** with $V[k]_n = V_{k+n}$. If V is a chain complex, then we define a differential on the **shifted complex** $V[k]$ by $d_{V[k]} = (-1)^k d_V$.

Let M be a simply connected closed oriented manifold. Using cohomology with coefficients in a field of characteristic 0, one can choose a Frobenius algebra A whose cohomology is the cohomology of M, such that the coproduct is the intersection coproduct $H^{*-n}(M) \to H^*(M) \otimes H^*(M)$. Our main goal is showing the following theorem:

Theorem 1. *The loop product (cohomology coproduct) and the co-BV operator form a co-BV algebra on $H^*(LM)[n]$. There are isomorphisms*

$$HH_*(A, A) \cong H^*(LM)$$

for $A = C^*(M)$ and $A = \Omega^*(M)$ which preserve the co-BV operations.

Naef and Willacher frame their results using the de Rham complex in [NW19], even on infinite dimensional spaces such as the path space PM and loop space LM. In Section 3 we explain the framework of diffeological spaces, which allows us to define de Rham forms on more general spaces than finite dimensional manifolds. The arguments in [NW19] are possible because the de Rham complex $\Omega^*(X)$ of a diffeological space X has the following properties:

- Its product is graded commutative on the chain level.
- It has umkehr maps on the chain level for projections $\Delta^n \times X \to X$, given by the fiber integral.
- It is quasi-isomorphic to $C^*(X)$ for nice spaces (though not in general, see Section 3.3).

The singular cochain complex almost satisfies these requirements as well, as the analogue of the fiber integral is the slant product, see Appendix A.5, but the product is only homotopy commutative. In the main part of this thesis, we develop the results of [NW19] on de Rham forms, as well as analogous results using the singular cochain complex.

Let G be a finite graph and M a simply connected closed manifold. Then there is a space associated to G which we also denote by G and we can consider the mapping space $\mathrm{Map}(G, M)$. Let A be a differential graded algebra which models M, such as $C^*(M)$ or $\Omega^*(M)$. In Section 4 we construct a chain complex $A^{\otimes G}$. We will show the following in Section 4.3:

Theorem 1.3. *1. The assignment $(A, G) \mapsto A^{\otimes G}$ extends to a contravariant functor in A which preserves quasi-isomorphisms and a covariant functor in G*

2. Let M be a closed simply connected manifold.

a) There is a natural quasi-isomorphism

$$I\colon \Omega^*(M)^{\otimes G} \to \Omega^*(\mathrm{Map}(G, M))$$

b) *In coefficients in a field of characteristic* 0*, there exists a natural zigzag of quasi-isomorphisms*

$$C^*(M)^{\otimes G} \to \ldots \leftarrow C^*(C^0(G,M))$$

This construction generalizes the isomorphism between the Hochschild cohomology and the loop space cohomology $HH_*(A,A) \cong H^*(LM)$ and the bar construction $\mathrm{B}(A,A,A) \simeq C^*(PM)$. The map I is a generalization of the iterated integral map due to Chen [Che77]. The main tool for this construction is the left derived tensor product $M \otimes^L_A N$ of two differential graded modules over a differential graded algebra A, which allows us to construct a cochain complex for pullbacks of spaces. More precisely, consider the following diagram of topological spaces.

$$\begin{tikzcd} E \times_B X \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

We will use the following well known result (cf. Theorem 4.13) for cohomology with coefficients in a field of characteristic 0:

Theorem 1.4. *Assume p is a Serre fibration with fiber F, the space E is path connected, B and X are simply connected, and either B or F is of finite rational type. Then the natural map*

$$C^*(X) \otimes^L_{C^*(B)} C^*(E) \to C^*(E \times_B X)$$

is a quasi-isomorphism.

The left derived tensor product can be computed by the two sided bar construction $B(M,A,N)$, which we recall in Appendix B.1 and Appendix B.3. Finally, we will show in Lemma 4.25 that the concatenation of paths is given by the deconcatenation coproduct on $\mathrm{B}(A,A,A)$.

In Section 5 we introduce a construction of the umkehr map $i_!^{LM} \colon H_*(LM \times LM) \to H_{*-n}(LM \times_M LM)$ due to [NW19], and the analogous construction of the cohomological counterpart $i^!_{LM} \colon H^{*-n}(LM \times_M LM) \to H^*(LM \times LM)$. This construction relies on the compactification of the configuration spaces of two points. We justify that this construction yields the same umkehr map as the classical construction using tubular neighborhoods and the Thom isomorphism. An additional result is the following theorem, cf Theorem 5.20, which shows that the umkehr map $i^!_{LM}$ can be constructed algebraically from the corresponding umkehr map $i^! \colon H^{*-n}(M) \to H^*(M \times M)$.

Theorem 1.5. *If M is simply connected, then the following diagram commutes in $D(Ch)$ and the vertical maps are quasi-isomorphisms:*

$$\begin{tikzcd} C^*(LM \times LM) & & C^*(LM \times_M LM)[-n] \arrow[ll, "i^!_{LM}"'] \\ C^*(LM \times LM) \otimes^L_{C^*(M \times M)} C^*(M \times M) \arrow[u, "\simeq"] & & C^*(LM \times LM) \otimes^L_{C^*(M \times M)} C^*(M)[-n] \arrow[ll, "\mathrm{id} \otimes i^!"'] \arrow[u, "\simeq"] \end{tikzcd}$$

The maps $i^!$ and $i^!_{LM}$ are induced by zigzags of chain maps, in Section 2.1 we recall that in the derived category of chain complexes one can interpret such zigzags as maps instead.

In Section 6 we give models of the loop operations which only depend on a differential graded algebra A which computes the cohomology of M. Recall that for an oriented closed manifold M, the intersection coproduct $H^{*-n}(M) \to H^*(M) \otimes H^*(M)$ together with the cup product, induces a Frobenius algebra structure on $H^*(M)$. In Section 6.2.2 we recall that by [LS08a], if M is additionally simply connected, there is a Frobenius algebra A which is quasi-isomorphic to $\Omega^*(M)$ and which induces the Frobenius algebra structure on $H^*(M)$. Using this chain level intersection map, we can construct chain level models for all the maps in the construction of the loop product in Section 5, using the results from Section 4 on the algebras $A^{\otimes G}$. This allows us to show that the isomorphism $HH_*(A, A) \cong H^*(LM)$ preserves the loop product (cohomology coproduct). A similar analysis shows that the BV operators are preserved. In particular one can conclude the co-BV relation on $H^*(LM)$ from an algebraic computation on the Hochschild complex.

2. Preliminaries

2.1. Derived Category of Chain Complexes

We will use the language of derived categories throughout this work, specifically in the case of derived chain complexes. The following is a very elementary introduction, readers versed in derived categories should skip this subsection. For a pedagogical introduction we refer to [Tho01]. For a formal treatment, see [GM13] or [Wei94].

The motivation for us to use derived categories is the following: We will regularly work with maps on the homology of chain complexes $\cdot \to \cdot \leftarrow \cdots \to \cdot$ which are induced by zigzags of chain maps, with the reverse maps being quasi-isomorphisms. Such a zigzag induces a map in homology, but by considering only the induced map on homology, we lose the information that these maps were originally induced by chain maps. Using derived categories allows us to consider such zigzags as morphisms and thus allows us to put them into commutative diagrams, as we would on homology. At the same time we retain the information that these maps were induced by chain maps to a greater degree.

Let $Ch = Ch(R)$ be the category of chain complexes of modules over a ring R. The derived category $D(Ch)$ of Ch is the localization of the category Ch at the collection of quasi-isomorphisms, in other words it is the category obtained from Ch by formally adjoining two-sided inverses to quasi-isomorphisms.

Theorem 2.1 (Universal Property of $D(Ch)$)**.** *There exists a category $D(Ch)$ and a functor $D\colon Ch \to D(Ch)$ which maps quasi-isomorphisms to isomorphisms, such that any functor $F\colon Ch \to C$ which maps quasi-isomorphisms to isomorphisms, factors uniquely through D.*

This is [GM13, Thm III.2.1].

Any two categories satisfying this property are equivalent and for our purposes we may use this as a definition.

Proposition 2.2. *(1) If a chain map $f\colon A \to B$ is a quasi-isomorphism and has a one-sided inverse g, then $D(f)D(g) = \mathrm{id}_{D(B)} \in D(Ch)$ and $D(g)D(f) = \mathrm{id}_{D(A)} \in D(Ch)$, i.e. the maps $D(f)$ and $D(g)$ are two-sided inverses in the derived category.*

(2) If $f, g\colon A \to B$ in Ch are chain homotopic, then $D(f) = D(g)$.

Proof. (1) is trivial. We show (2). Define a complex C, which as a graded vector space is $A \oplus A[1] \oplus A$, with differential $(a, a', a'') \mapsto (da + a', da', da'' + a')$. Let $\iota_0, \iota_1\colon A \to C$ be the inclusions into the first resp. third direct summand. Then ι_0 and ι_1 are both quasi-isomorphisms with the same one-sided inverse, namely the chain map $C \to A$ given by $(a, a', a'') \mapsto a - a''$. Hence $D(\iota_1) = D(\iota_2)$ by (1). Let P be a homotopy between f and g, then there is a chain map $\gamma_P\colon C \to B$ given by $f + P + g$. Now $\gamma_P\iota_1 = f$ and $\gamma_P\iota_2 = g$ and we conclude $D(f) = D(g)$. □

Remark 2.3. We often omit the explicit mention of the functor D. If we say that a map or a diagram in Ch has a certain property in $D(Ch)$, we mean that its image under the functor D has said property. Any zigzag of maps with quasi-isomorphisms in the reverse direction induces a map in $D(Ch)$, hence we may arrive at diagrams of which one cannot ask to commute at the chain level, but which can commute in $D(Ch)$.

Remark 2.4. By Theorem 2.1, the cohomology functor H^* factors through $D(Ch)$. Thus if two chain maps are equal in $D(Ch)$, they are also equal on cohomology and if two chain complexes are isomorphic in $D(Ch)$ then they have isomorphic cohomology. The converse is not necessarily true.

Example 2.5. Let $R = k$ be a field and C a chain complex, then the tensor product functor $- \otimes C\colon Ch \to Ch$ preserves quasi-isomorphisms, since C is degreewise free, (cf. the proof of Theorem B.2). This means that $D(- \otimes C)\colon Ch \to D(Ch)$ maps quasi-isomorphisms to isomorphisms, so by Theorem 2.1 it factors as $Ch \xrightarrow{D} D(Ch) \xrightarrow{-\otimes C} D(Ch)$ through a functor which we also denote by $- \otimes C$. Given morphisms $f, g\colon A \to B$ in Ch, which are equal in $D(Ch)$, we find that $f \otimes \mathrm{id}_C = g \otimes \mathrm{id}_C$ in $D(Ch)$.

2.2. Loop Product

Let M be an n-dimensional compact oriented manifold and denote by $LM = C^0(S^1, M)$ the free loop space of M. The **loop product** is a degree $-n$ product on the homology of the loop space

$$H_*(LM) \otimes H_*(LM) \to H_{*-n}(LM)$$

Denote by $LM \times_M LM$ the subspace of $LM \times LM$ of pairs (γ, γ') such that $\gamma(0) = \gamma'(0)$. Such loops can be composed to obtain a new loop. Denote by $c\colon LM \times_M LM \to LM$ the composition of loops, given by tracing out the first loop on the interval $[0, \frac{1}{2}]$ and the second loop on the interval $[\frac{1}{2}, 1]$. Denote by $i\colon LM \times_M LM \to LM$ the inclusion map.

Construction 2.6. The loop product is the composition of three maps

$$\bullet\colon H_*(LM)\otimes H_*(LM)\xrightarrow{\times} H_*(LM\times LM)\xrightarrow{i_!^{LM}} H_{*-n}(LM\times_M LM)\xrightarrow{c_*} H_{*-n}(LM)$$

where the map $\times$ is the Künneth isomorphism (we work with coefficients in a field) and the map $i_!^{LM}$ is an intersection map that will be explained in more detail in the following.

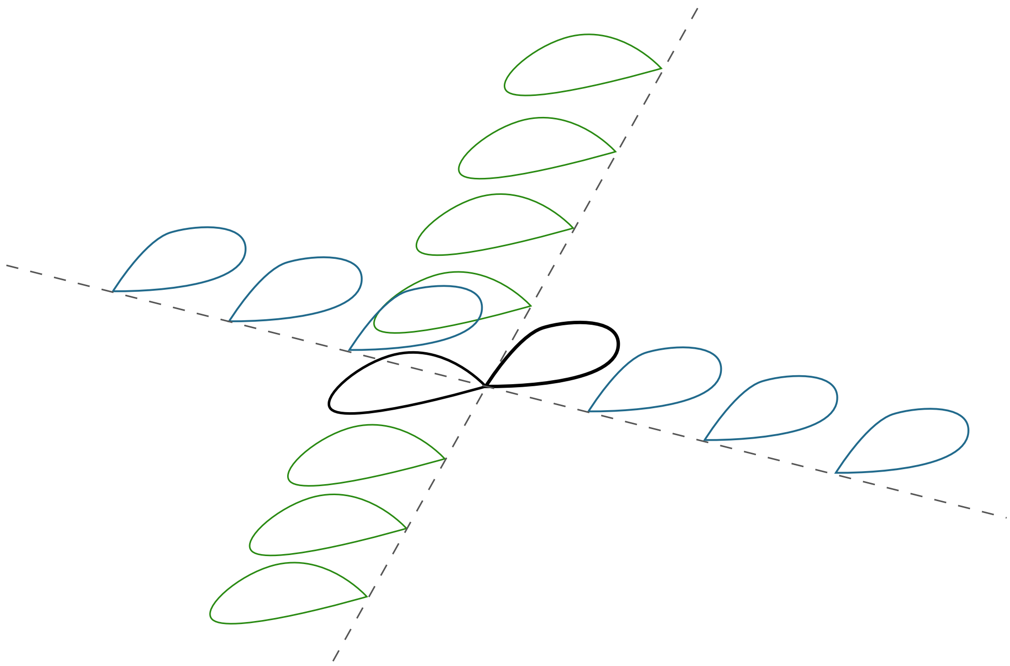


Figure 1: Depiction of the Loop Product. Two families of loops are depicted. To construct the loop product one takes the locations where the base points are equal and concatenates the loops there.

Remark 2.7. The intersection map $i_!^{LM}$ can be thought of as taking the intersection of a homology class in $LM \times LM$ with the space $LM \times_M LM$. This is how Chas and Sullivan originally define it in [CS99], assuming the homology class is represented by a submanifold of $LM \times LM$ which is transverse to $LM \times_M LM$. We will use a construction using the Thom isomorphism and tubular neighborhoods, following [CJ02]. See Appendix A.3 for a review of the Thom isomorphism. For our construction we will use that the Thom isomorphism of an oriented vector bundle is the umkehr map of the zero section, regarded as a map from the base space into the disk bundle. As this is an umkehr map, one can think of it as taking the intersection of a homology class with the boundary. Thus we need a vector bundle such that the inclusion of the zero section into the total space can be related to the inclusion $LM\times_M \to LM \times LM$.

To begin the construction of $i_!^{LM}$, consider the pullback diagram

$$\begin{tikzcd}
LM \times_M LM \arrow[r] \arrow[d, "ev"] & LM \times LM \arrow[d, "ev \times ev"] \\
M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

Choose a Riemannian metric on M. Denote by $\nu(\Delta)$ the normal bundle of the diagonal $\Delta\colon M \to M \times M$, then there is an isomorphism $TM \cong \nu(\Delta)$. Since M is compact, let $\varepsilon > 0$ be small enough such that the exponential map exp induces a diffeomorphism $D_\varepsilon M \to \nu(\Delta) \to N$, where $D_\varepsilon M$ is the subbundle of TM consisting of vectors of length $< \varepsilon$ and $N \subseteq M \times M$ is an open neighborhood of the diagonal $\Delta M \subseteq M \times M$.

The pullback bundle $ev^*(D_\varepsilon M)$ of $D_\varepsilon M$ along the evaluation map $LM \times_M LM \to M$ is a rank n bundle on $LM \times_M LM$. Since M is oriented, the pullback bundle is oriented. Let $LM \times LM|_N = (ev \times ev)^{-1}(N) \subseteq LM \times LM$. By [CJ02, p. 8] there is a homotopy equivalence between

$$ev^*(D_\varepsilon M) \simeq LM \times LM|_N \tag{1}$$

which maps the complement of the zero section to the complement of $LM \times_M LM \subseteq LM \times LM|_N$. For brevity we write $H_*(X, X \setminus A) = H_*(X|A)$. There is an isomorphism in homology

$$\eta\colon H_*(LM \times LM|LM \times_M LM) \xleftarrow{\cong} H_*(LM \times LM|_N, LM \times LM|_{N_0})$$
$$\xrightarrow{\cong} H_*(ev^*(TM), ev^*(TM)_0)$$

where the first map is an excision isomorphism and the second map is induced by the homotopy equivalence 1, $ev^*(TM)_0$ is the total space of the bundle minus the zero section.

Assume that TM is oriented, then the bundle $ev^*(TM)$ is oriented and thus has a Thom class $u_{LM \times_M LM} \in H^*(ev^*(TM), ev^*(TM)_0)$. One checks that this is the pullback of the Thom class $u \in u \in H^n(D_\varepsilon M, (D_\varepsilon M)_0) \cong H^n(TM, (TM)_0)$ of M along the map of pairs $(ev^*(TM), ev^*(TM)_0) \to (TM, (TM)_0)$. The Thom isomorphism ϕ of the bundle $ev^*(TM)$ is the map

$$\phi\colon H_*(ev^*(TM), ev^*(TM)_0) \xrightarrow{\frown u_{LM \times_M LM}} H_{*-n}(ev^*(TM)) \to H_{*-n}(LM \times_M LM).$$

Definition 2.8. The intersection map $i_!^{LM}$ is the following composition of maps:

$$i_!^{LM}\colon H_*(LM \times LM) \to H_*(LM \times LM \mid LM \times_M LM)$$
$$\xrightarrow{\eta} H_*(ev^*(TM), ev^*(TM)_0) \xrightarrow{\phi} H_{*-n}(LM \times_M LM)$$

Definition 2.9. Analogously, taking the cup product instead of the cap product for the Thom isomorphism, one constructs an intersection map

$$i^!\colon H^*(LM \times LM) \leftarrow H^{*-n}(LM \times_M LM)$$

$$i^!_{LM}\colon H^*(LM \times LM) \leftarrow H^*(LM \times LM \mid LM \times_M LM)$$
$$\xleftarrow{\eta} H^*(ev^*(TM), ev^*(TM)_0) \xleftarrow{\phi^*} H^{*-n}(LM \times_M LM)$$

on cohomology and this leads to a coproduct operation on cohomology

$$\Delta\colon H^*(LM) \otimes H_*(LM) \leftarrow H^{*-n}(LM)$$

defined as the composition

2.3. Loop Coproduct (Cohomology Product)

We now recall the definition of the loop coproduct: It is a map

$$H_*(LM, M) \to H_*(LM, M) \otimes H_*(LM, M)[n-1]$$

where $H_*(LM, M)$ denotes the homology relative to constant loops, via the inclusion $M \to LM$. Recall that $[n-1]$ denotes the shift of a graded vector space, where $V[k]_n = V_{k+n}$.

We denote by $\bigcirc_2$ the circle with two marked points, constructed by gluing two copies of the unit interval. Let $C^0(\bigcirc_2, M)$ be the space of paths $\gamma, \gamma' \in PM$ such that $\gamma(1) = \gamma'(0)$ and $\gamma(0) = \gamma'(1)$, i.e. the loop space with two marked points. There is an evaluation map $\mathrm{Map}(\bigcirc_2, M) \to M \times M$.

The **splitting map** s is the map of spaces

$$\begin{aligned} s\colon I \times LM &\to C^0(\bigcirc_2), \\ (t, \gamma) &\mapsto s \mapsto (s \mapsto \gamma(st), s \mapsto \gamma(t + s(1-t))) \end{aligned}$$

which splits a loop into the two parts defined on the intervals $[0, t]$ and $[t, 1]$ respectively and reparametrizes both parts to the interval $[0, 1]$. There are two embeddings $LM \to C^0(\bigcirc_2, M)$ by mapping a loop γ to the pair (γ, x) where x is the constant path at $x = \gamma(0)$. We denote the union of the images of these maps by $LM \amalg_M LM$. The splitting map is a map of pairs

$$s\colon (I \times LM, \partial I \times LM \cup I \times M) \to (C^0(\bigcirc_2, M), LM \amalg_M LM).$$

We obtain the following composition

$$\tilde{s}_*\colon C_*(LM, M) \xrightarrow{\times [I]} C_{*+1}(I \times LM, \partial I \times LM \cup I \times M) \xrightarrow{s_*} C_{*+1}(C^0(\bigcirc), LM \amalg_M LM)$$

where $[I]$ is a representative of the fundamental class of the pair $(I, \partial I)$.

Consider the pullback diagram:

$$\begin{tikzcd} LM \times_M LM \arrow[r, "j"] \arrow[d] & C^0(\bigcirc_2, M) \arrow[d] \\ M \arrow[r, "\Delta"] & M \times M \end{tikzcd}$$

One can again construct an umkehr map $j_!$ on homology. In fact, one can construct this relative to the subspace $LM \amalg_M LM$:

$$j_!\colon H_*(C^0(\bigcirc_2, M), LM \amalg_M LM) \to H_*(LM \times_M LM, LM \amalg_M LM)$$

Definition 2.10. The **Loop Coproduct** is the following composition on homology:

$$\begin{aligned} H_*(LM, M) &\xrightarrow{\tilde{s}_*} H_{*+1}(C^0(\bigcirc_2, M), LM \amalg_M LM) \\ &\xrightarrow{j_!} H_{*+1-n}(LM \times_M LM, LM \amalg_M LM) \\ &\to H_*(LM, M) \otimes H_*(LM, M)[1-n] \end{aligned}$$

where the last map is the Künneth isomorphism.

Definition 2.11. Analogously, one defines a product on homology

$$H_*(LM, M) \leftarrow H_*(LM, M) \otimes H_*(LM, M)[1-n].$$

3. Diffeological Spaces and Cohomology

The theory of diffeological spaces is a generalization of the theory of differentiable manifolds. It is a way to define differential forms on mapping spaces such as the path and the loop space of a manifold. These forms have the same algebraic operations of pullback, wedge product and exterior derivative as in the case of smooth manifolds. There is a de Rham map, which in nice cases, though not in general, provides a quasi-isomorphism to the singular cohomology. There is also a fiber integration map on de Rham forms, e.g. in the case of projections $M \times X \to X$ with closed oriented manifolds M and diffeological spaces X. In contrast to e.g. the singular cochain algebra, the de Rham algebra is commutative on the chain level, which allows us to apply the pushout pullback lemma 4.13.

In the theory of diffeological spaces, smoothness of a space X is encoded using so called plots, which are distinguished maps from open euclidean sets of arbitrary dimension. Plots then take the role of the smooth maps from euclidean sets into X, so in contrast to the charts of manifolds, plots are only smooth, not diffeomorphisms in the resulting theory. Since it is easier to exhibit smooth maps than diffeomorphisms, one can equip more general spaces than manifolds with the structure of a diffeological space. Individual diffeological spaces can have bad properties compared to manifolds, for example one can consider infinite dimensional spaces or arbitrary quotients of manifolds. On the other hand, the category of diffeological spaces has good properties: it is closed under products, coproducts, quotients, subsets and there is a diffeology on the mapping space between diffeological spaces.

There is a related notion of so called differentiable spaces, coined by Chen [Che73], where he uses convex euclidean sets instead of open euclidean sets to test smoothness. We work with diffeological spaces throughout this work.

We begin by outlining the definition of diffeological spaces and differential forms in Section 3.1. For a diffeological space with a compatible topology, we compare the smooth singular complex and the topological singular complex in Section 3.2. We then discuss how the cohomology of the de Rham complex of a diffeological space relates to the cohomology of the topological space in Section 3.3. In Section 3.4 we apply the theory to the case of mapping spaces such as PM and LM. In Section 3.5 we discuss the fiber integration of differential forms, which generalizes to the context of diffeological spaces in certain settings.

The reader may skip this section on first reading to come back later when questions arise.

3.1. Diffeological Spaces

We refer to [IZ13] for the material in this subsection.

Definition 3.1. Let X be a nonempty set, a **diffeology** $\mathcal{D} = \mathcal{D}_X$ on X is a set of so called **plots** $P\colon U \to X$ defined on open subsets $U \subseteq \mathbb{R}^n$, where n is not fixed, such that

1. every constant map $U \to \{x\} \subseteq X$ is a plot,
2. for every map $P\colon U \to X$, if for every point $r \in U$ there exists an open neighborhood V of r such that $P|_V$ is a plot, then P is also a plot,
3. for every plot $P\colon U \to X$ and every smooth map $F\colon V \to U$, the pullback map $P \circ F\colon V \to X$ is a plot.

A set together with a specified diffeology is called a **diffeological space**.

Example 3.2. (i) Any topological space can be equipped with its topological diffeology, in which the plots are all continuous maps $U \to X$.

(ii) Any smooth manifold M is a diffeological space, where the plots are the smooth maps $U \to M$ for every open $U \subseteq \mathbb{R}^n$ for all n.

(iii) Given two diffeological spaces X, Y, the cartesian product $X \times Y$ is a diffeological space, where a map $P\colon U \to X \times Y$ is a plot if and only if the projections $\pi_X \circ P\colon U \to X$ and $\pi_Y \circ P\colon U \to Y$ are plots of X resp. Y.

(iv) Given a diffeological space X and a subset A, one can define the diffeology on A by taking plots of A to be those plots of X which map into A.

Definition 3.3. Let X and X' be two diffeological spaces. A map $f\colon X \to X'$ is said to be **smooth**, if for each plot P of X, $f \circ P$ is a plot of X'. The set of smooth maps from X to X' is denoted $\mathcal{D}(X, X')$.

Proposition 3.4. *1. There is a diffeology on $\mathcal{D}(X, X')$ in which a map $P\colon U \to \mathcal{D}(X, X')$ is a plot if and only if for every plot $Q\colon V \to X$, the parametrization $U \times V \to X', (r, s) \mapsto P(r)(Q(s))$ is a plot of X'.*

2. With respect to this diffeology, the evaluation map $ev\colon X \times C^\infty(X, X') \to X'$ and composition map $\mathcal{D}(X, X') \times \mathcal{D}(X', X'') \to \mathcal{D}(X, X'')$ are smooth.

Definition 3.5. A **differential k-form** on a diffeological space $(X, \mathcal{D})$ is a $\mathcal{D}$-indexed collection α_P of differential forms $\alpha_P \in \Omega^k(U_P)$ for each plot $P: U_P \to X$, such that for every smooth map $f: U \to V$ of euclidean open sets, $f^*\alpha_P = \alpha_{fP}$. The collection of all differential k-forms on X is denoted $\Omega^*(X)$.

Example 3.6. If M is a smooth manifold with its smooth diffeology, the k-forms according to this definition are in bijection with the regular k-forms.

One can define the usual operations on differential forms such as pullback along smooth maps, exterior derivative and wedge product. For instance, the exterior derivative is defined via $d\alpha = \{d\alpha_P\}_P$. These satisfy the usual computation rules, in particular $\Omega^*(X)$ is a differential graded commutative algebra and its cohomology is called the de Rham cohomology of X.

Remark 3.7. While the definition of differential forms of diffeological spaces is evident, there are multiple definitions of tangent spaces and bundles for diffeological spaces, which in general are not equivalent, see for instance [CW15].

3.2. Singular and Smooth Cohomology

In this subsection we introduce the smooth singular cohomology, in preparation for the construction of the de Rham map in Section 3.3. We compare its cohomology with the singular cohomology. A reference is [Gür14].

Definition 3.8. A **topological diffeological space** X is a space which has both a topology and a diffeology, such that all plots of X are continuous.

Definition 3.9. For a topological diffeological space X,

1. its **singular simplicial set** $S_*(X)$ is the simplicial set which in degree n consists of all continuous maps $\Delta^n \to X$ for all X,
2. its **smooth simplicial set** $S_*^\infty(X)$ which consists of all smooth simplexes $\Delta^n \to X$ for all n. By a smooth simplex we mean that there is a smooth map from a neighborhood of $\Delta^n \subseteq \mathbb{R}^n$ to the diffeological space X.

Denote by $C_*(X)$ resp. $C_*^\infty(X)$ the corresponding chain complexes, and similarly $C^*(X)$ resp. $C^*_\infty(X)$ the associated cochain complexes.

The following lemma gives a sufficient condition for the smooth and continuous homology to be isomorphic.

Lemma 3.10. *Let X be a topological diffeological space such that for every continuous simplex $\sigma: \Delta^n \to X$ with smooth $(n-1)$-faces, $n \geq 1$, there exists a continuous homotopy $h: \Delta^n \times I \to X$ relative to the boundary $\partial\Delta^n$ from σ to a smooth n-simplex of X. Then the canonical maps*

$$C_*^\infty(X) \to C_*(X) \qquad C^*(X) \to C^*_\infty(X)$$

are chain homotopy equivalences.

Sketch of proof. This theorem is stated in [Che77]. To prove this one can proceed as in the case of manifolds, cf. [Par]: one constructs a converse map $C_*(X) \to C_*^\infty(X)$ inductively by mapping each continuous simplex to a fixed smooth approximation, such that the approximations are compatible at the boundary. The topological homotopies of the approximation maps then yield a chain homotopy which witnesses that the two chain maps are homotopy inverse. $\square$

Corollary 3.11. *For every smooth manifold M, the canonical maps*

$$C_*^\infty(X) \to C_*(X) \qquad C^*(X) \to C^*_\infty(X)$$

are chain homotopy equivalences.

Proof. One shows (cf. [Par]) the hypothesis of Lemma 3.10 using the Whitney approximation theorem, which we state in the following. $\square$

We state Whitney's approximation theorem as in [Lee03, Theorem 6.26].

Theorem 3.12 (Whitney's Approximation Theorem,)**.** *Let X be a smooth manifold with or without boundary, Y a smooth manifold without boundary, and let $f\colon X \to Y$ be a continuous map, smooth on a closed (possibly empty) subset $A \subseteq X$. Then f is homotopic relative to A to a smooth map $X \to Y$.*

3.3. De Rham Cohomology and De Rham Map

Following [Gür14], we construct the de Rham map of diffeological spaces similarly as in the case of smooth manifolds.

For any smooth simplex $\sigma\colon \Delta^n \to X$ and an n-form $\omega \in \Omega^n(X)$, one can define its integral $\int_\sigma \omega := \int_{\Delta^n} \sigma^*\omega$. The differential form $\sigma^*\omega \in \Omega^*(\Delta^n)$ is a differential form in the sense of smooth manifolds since σ extends smoothly near Δ^n, which implies that the integral is well-defined.

Definition 3.13. Let X be a diffeological space. The **de Rham comparison map** is the map

$$\begin{aligned} I_{dR}\colon \Omega^*(X) &\to C^*_\infty(X) \\ \omega &\mapsto \left(\sigma \mapsto \int_\sigma \omega = \int_{\Delta^n} \sigma^*\omega\right) \end{aligned}$$

for $\omega \in \Omega^n(X)$ and $\sigma \in S_n^\infty(X)$.

Proposition 3.14. *The de Rham comparison map I_{dR} of diffeological spaces is an injective chain map.*

Proof. Let $\sigma\colon \Delta^n \to X$ be a smooth singular simplex and $\omega \in \Omega^*(X)$. The form $\sigma^*\omega$ is a differential form in the classical sense on Δ^n and it is well known that the de Rham map on smooth manifolds is a chain map, hence $\int_{\Delta^n} d\sigma^*\omega = \int_{\partial\Delta^n} \sigma^*\omega$ and thus

$I_{dR}(d\omega)(\sigma) = \delta(I_d R(\omega)(\sigma))$, that is I_{dR} is a chain map in the case of diffeological spaces as well.

To show injectivity, let $\omega \in \Omega^n(X)$, such that $\int_\sigma \omega = 0$ for all smooth singular simplices σ. Let $P\colon U \to X$ be a plot of X, with $U \subseteq \mathbb{R}^m$ open, we want to show that $\omega_P = 0 \in \Omega^n(U)$. By hypothesis, the integral of ω_P over any simplex in U is 0. For $x \in U$, if $\omega_P|_x \neq 0$, then there exist some vectors $X_1, \ldots, X_n \in T_x U \cong \mathbb{R}^n$ such that $\omega_P|_x(X_1, \ldots, X_n) > 0$ and by continuity, these vectors extend to local vector fields such that ω evaluates positively in a small neighborhood of x. But then, for a simplex σ which is tangent to $X_1, \ldots, X_n$, the integral of ω is nonzero, which is a contradiction. $\square$

Remark 3.15. The de Rham map is not a quasi-isomorphism on general diffeological spaces. A counterexample is the orbit space of the $\mathbb{R}$-action on $S^1 \times S^1$ by translation by $t(1, \alpha)$ for some irrational $\alpha \in \mathbb{R}$, see [IZ13].

A smooth CW complex is the analogue of a CW complex in the category of diffeological spaces: it is a diffeological space X which is constructed as the colimit of a sequence of diffeological spaces $X_n \to X_{n+1}$ such that X_{n+1} is obtained from X_n by gluing copies of $(n+1)$-disks to X_n along smooth maps on the boundary of the disks.

Theorem 3.16. *If X is smoothly homotopy equivalent to a (not necessarily finite) smooth CW complex, then the de Rham map is a quasi-isomorphism.*

This is theorem 2.2.14 in [Gür14], to prove this one shows that the two functors Ω^* and C^*_∞ satisfy an analogue of the Eilenberg Steenrod axioms for diffeological spaces in place of topological spaces.

3.4. Differential Forms on Path Spaces of Manifolds

In this subsection we consider the diffeology on path spaces and similar constructions of a smooth manifold M. [NW19] do not explain the diffeological structure on the path space which one should use for their argument. We propose a diffeological structure on the path space that is suitable for our computations and discuss the relation between its de Rham cohomology and singular cohomology.

A first attempt for a diffeology on the path space PM of M is to use the mapping space diffeology of maps from $I \to M$, as in Proposition 3.4. The smooth maps from I to M in the diffeological sense are the smooth maps in the classical sense. In this sense, a map $U \to PM$ is a plot if the adjoint map $U \times I \to M$ is smooth. However we want to model the concatenation of paths $c\colon PM \times_M PM \to PM$, and in general the concatenation of two smooth paths is not smooth. Similarly we want to model the splitting map $s\colon I \times LM \to C^0(\bigcirc_2, M)$ from Section 2.3. Thus we need to equip the path and loop space with a diffeological structure such that these maps are smooth. We are unaware of a construction in the literature which provides this. The original construction by Chen [Che77] is in terms of the diffeological mapping space. The construction due to Gugenheim [Gug77], which uses piecewise smooth paths, does not yield a smooth splitting map.

We use piecewise smooth paths, following [Sta17].

Definition 3.17. We denote by $C^\infty_{ps}(I, M)$ the space of piecewise smooth paths in M. We denote by $C^\infty_{ps,b}(I, M)$ be the spaces of piecewise smooth paths in M resp. piecewise smooth paths with bounded derivatives in M, that is, each derivative is a bounded function on its domain of definition.

In [Sta17], these spaces are equipped with natural topologies and it is shown that there is a homotopy equivalence between $C^\infty_{ps,b}(I, M)$ and $C^0(I, M)$. Similarly, one can define a topology and diffeology on the loop space.

Remark 3.18. Following [Sta17], we warn that the circle action $LM \times S^1 \to LM$ and reparametrization maps $PM \to PM$ via piecewise smooth maps $I \to I$ do not induce continuous maps on the spaces of piecewise paths resp. loops with respect to the topology that we have chosen.

For a finite set $F \subseteq I$, denote by $C^\infty_F(I, M)$ the space of continuous maps which are smooth outside F, and denote by $C^\infty_{F,b}(I, M)$ the subspace of functions with bounded derivatives. The space $C^\infty_{ps,b}$ is the union of the subspaces $C^\infty_{F,b}(I, M)$.

Definition 3.19. The diffeology on $C^\infty_{ps}(I, M)$ is the colimit diffeology as a colimit of $C^\infty_F(I, M)$ and the diffeology on $C^\infty_{ps,b}(I, M)$ is the subspace diffeology.

Remark 3.20. In the following sections, the symbols LM and PM will be used both for the space of continuous maps and the space of piecewise smooth maps with bounded derivative. It will usually be clear from context which one is used: if we are working with the singular chain complex C^* then it is the former, if we are working with the diffeological de Rham complex Ω^* then it is the latter.

Let G be a graph, given by sets V, E and source and target maps $s, t \colon E \rightrightarrows V$ with finitely many edges and vertices. Denote by G also the topological space associated to G.

Definition 3.21. We denote by $C^0(G, M)$ the topological space of continuous maps from G to M, with the topology of uniform convergence. We denote by $\mathrm{Map}(G, M)$ the topological diffeological space of maps from G to M, which are continuous and piecewise smooth with bounded derivatives along the edges.

The results of Section 4 and Section 6 are stated using the de Rham complex in [NW19] and in these statements depend on the assumption that $\Omega^*(\mathrm{Map}(G, M))$ computes the singular cohomology of $C^0(G, M)$. These complexes are related through the sequence of chain maps.

(1) The de Rham comparison map

$$\Omega^*(\mathrm{Map}(G, M)) \to C^*_\infty(\mathrm{Map}(G, M))$$

(2) The comparison map between singular and smooth singular cochains

$$C^*_\infty(\mathrm{Map}(G, M)) \leftarrow C^*(\mathrm{Map}(G, M)$$

(3) The comparison map between the space of smooth maps $\mathrm{Map}(G, M)$ and continuous maps $C^0(G, M)$.

$$\Omega^*(\mathrm{Map}(G, M)) \to C^*_\infty(\mathrm{Map}(G, M)) \leftarrow C^*(\mathrm{Map}(G, M) \leftarrow C^*(C^0(G, M)) \qquad (2)$$

We will discuss when these maps are quasi-isomorphisms. We can prove the following theorem:

Theorem 3.22 ((1))**.** *1. If* $\mathrm{Map}(G, M)$ *is smoothly homotopy equivalent to a smooth CW complex, then the de Rham comparison map*

$$\Omega^*(\mathrm{Map}(G, M)) \to C^*_\infty(\mathrm{Map}(G, M))$$

is a quasi-isomorphism.

2. *The map* $C^*_\infty(\mathrm{Map}(G, M)) \leftarrow C^*(\mathrm{Map}(G, M)$ *is a quasi-isomorphism.*

3. *The comparison map between the cohomology of* $\mathrm{Map}(G, M)$ *and* $C^0(G, M)$ *is a quasi-isomorphism:*

$$C^*(\mathrm{Map}(G, M) \leftarrow C^*(C^0(G, M)).$$

We are unaware of a proof in the literature that the space $\mathrm{Map}(G, M)$ is homotopy equivalent to a smooth cw complex or that the de Rham comparison map is a quasi-isomorphism on these spaces. The assumption on the cohomologies is required to proceed with the arguments in [NW19], as we outline them in Section 4 and Section 6. We thus make the following assumption.

Assumption 3.23. *Let M be a (closed) smooth manifold and G be a finite graph, resp. its assotiated topological space. We assume that the de Rham comparison map*

$$\Omega^*(\mathrm{Map}(G, M)) \to C^*_\infty(\mathrm{Map}(G, M))$$

is a quasi-isomorphism.

Remark 3.24. Note that this assumption is only required for statements involving the de Rham complex. We develop parallel results for the singular cochain complex, these do not require this assumption.

(1) In general, the de Rham map

$$\Omega^*(X) \to C^*_\infty(X)$$

is not a quasi-isomorphism. Recall that the proof of the de Rham theorem in the case of smooth manifolds depends on structure of the manifold that is not available in the case of diffeological spaces, e.g. a locally finite covering with contractible spaces and contractible intersections. It appears to be folklore that the de Rham map is an isomorphism in the case of LM with its mapping space diffeology.

Recall from Theorem 3.16 that if X is homotopy equivalent to a smooth CW complex, then the de Rham theorem map is an isomorphism. In particular this is the first part of Theorem 3.22.

Remark 3.25. It is known due to Milnor [Mil59], that for spaces X and Y, where X is compact metrizable and Y is homotopy equivalent to a CW complex, the mapping space $C^0(X, Y)$ has the homotopy type of a CW complex. There are specific cell complexes for the homotopy type of the free loop space, see for example [RS18]. However these constructions yield topological CW complexes, not smooth CW complexes, and they work with spaces of continuous maps, not smooth maps.

(2) We consider the comparison map between the smooth and the diffeological structure on $\mathrm{Map}(G, M)$.

Lemma 3.26. *If M is a smooth manifold without boundary, then the maps*

$$C^*_\infty(\mathrm{Map}(G, M)) \to C_*(\mathrm{Map}(G, M) \qquad C^*_\infty(\mathrm{Map}(G, M)) \leftarrow C^*(\mathrm{Map}(G, M)$$

are quasi-isomorphisms.

Proof. We first show this for $C^\infty_{F,b}(I, M)$. We apply Lemma 3.10. Thus we have to show that for every continuous simplex $\sigma\colon \Delta^n \to C^\infty_{F,b}(I, M)$ with smooth $(n-1)$-faces, $n \geq 1$, there exists a continuous homotopy $h\colon \Delta^n \times I \to C^\infty_{F,b}(I, M)$ relative to the boundary $\partial\Delta^n$ from σ to a smooth n-simplex of X.

A continuous map $\sigma\colon \Delta^n \to C^\infty_{F,b}(I, M)$ corresponds to a continuous map $\bar{\sigma}\colon \Delta^n \times I \to M$. We assume that σ is smooth on $\partial\Delta^n$, hence $\bar{\sigma}$ is smooth on $\partial\Delta^n \times [t_i, t_{i+1}]$. Using the Whitney approximation theorem, this is homotopic relative $\partial\Delta^n \times [t_i, t_{i+1}] \cup \Delta^n \times \{t_i, t_{i+1}\}$ to a smooth map $\Delta^n \times [t_i, t_{i+1}] \to M$. This corresponds to a homotopy from σ to a smooth map $\Delta^n \to C^\infty_{F,b}(I, M)$, relative to the boundary. Thus we conclude that for $C^\infty_{F,b}(I, M)$, the singular and smooth singular homology are isomorphic.

We now consider the space $C^\infty_{ps,b}(I, M)$. Since the image of a singular simplex $\sigma\colon \Delta \to C^\infty_{ps,b}(I, M)$ is compact, one can show that there is some F such that σ maps into $C^\infty_{F,b}(I, M)$, see [Sta17, Prop. 5.6]. Thus, the singular chain complex of $C^\infty_{ps,b}(I, M)$ is a filtered colimit of the singular chain complexes $C_*(C^\infty_{F,b}(I, M))$, indexed by all finite $F \subseteq I$. Similarly the smooth singular chain complex of $C^\infty_{ps,b}(I, M)$ is a filtered colimit. Thus the comparison map $C^\infty_*(C^\infty_{ps,b}(I, M)) \to C_*(C^\infty_{ps,b}(I, M))$ is a filtered colimit of quasi-isomorphisms and thus a quasi-isomorphism. The statement on the cochain complex follows via universal coefficients.

For G instead of I, the same proof applies: One first works with maps which are smooth along a fixed partition of each edge, then with arbitrary piecewise smooth maps. □

(3) We consider the inclusion $C^\infty_{ps,b}(I, M) \subseteq C^0(I, M)$.

Lemma 3.27. *The inclusions $C^\infty(I, M) \subseteq C^\infty_{ps,b}(I, M) \subseteq C^0(I, M)$ are continuous and induce homotopy equivalences.*

See [HW17, Sec 1] or [Sta09, Thm 4.6]. From this we conclude the same for $\mathrm{Map}(G, M)$ and thus the third part of Theorem 3.22.

3.5. Fiber Integration

We first recall the definition of the fiber integral in the euclidean case, cf. [BT$^+$82]. We then define a fiber integration map for diffeological spaces F and X, which will be a map

$$\Omega^d(X \times F) \otimes C_n^\infty(F) \to \Omega^{d-n}(X).$$

Construction 3.28 (Euclidean Fiber Integral)**.** Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open. We use multi-index notation: for any $I = (i_1, \ldots, i_k)$ let $du_I = du_{i_1} \wedge \cdots \wedge du_{i_k}$ and $dv_J = dv_{j_1} \wedge \cdots \wedge dv_{j_l}$ for a multi-index J. Any differential form $\omega \in \Omega^{k+l}(U \times V)$ is a sum of forms

$$\omega = \sum_{I,J} f_{I,J} du_{i_1} \wedge \cdots \wedge du_{i_k} \wedge dv_{j_1} \wedge \cdots \wedge dv_{j_l}$$

where the sum is over all multi-indices I and J with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_l$, for unique smooth functions $f_{I,J} \in C^\infty(U \times V)$. We are only interested in those summands with $J = \{1, \ldots, m\}$, denote $f_I = f_{I,\{1,\ldots,m\}}$ and $dv = dv_1 \wedge \cdots \wedge dv_m$.

1. We say that the fiber integral of ω is well-defined if for all multi-indices I, the integral
$$\int_V f_I(u, v) dv$$
converges for all $u \in U$ and yields a smooth function $U \to \mathbb{R}$.

2. In this case, we define the **fiber integral** of $\omega \in \Omega^*(U \times V)$ to be the form $\int_V \omega \in \Omega^{*-\dim V}(U)$ given at $u \in U$ by
$$u \mapsto \left(\int_V \omega \right)_u = \sum_I du_I \left(\int_V f_I(u, v) dv \right).$$

Remark 3.29. For example, recall that a form $\omega \in \Omega^*(U \times V)$ is said to have compact vertical support if for all u, the set $\{v \in V \mid \omega|_{(u,v)} \neq 0\} \subseteq V$ is contained in a compact subset of V. Fiber integration then yields a map

$$\Omega^*_{cv}(U \times V) \to \Omega^{*-\dim V}(U),$$

where $\Omega^*_{cv}(U \times V)$ denotes the forms with compact support in V direction.

Remark 3.30. [BT$^+$82] and [GHV72] define fiber integration in the context of with oriented vector bundles $E \to B$ over finite dimensional manifolds, using only forms with compact vertical support. Our definition is slightly different: we only consider cartesian products where the base space and fiber are subsets of Euclidean spaces. We have also made the convergence assumption explicit instead of relying on compact vertical support. Nevertheless this is a standard construction.

Lemma 3.31. *1. If $f \colon U \to U'$ is smooth, then for $\omega \in \Omega^*(U' \times V)$, if both fiber integrals $\int_V \omega \in \Omega^*(U')$ and $\int_V (f \times \mathrm{id}_V)^* \omega \in \Omega^*(U)$ are well-defined, then:*

$$\int_V (f \times \mathrm{id}_V)^* \omega = f^* \int_V \omega$$

2. *If $g\colon V \to V'$ is a diffeomorphism, then for $\omega \in \Omega^*(U \times V')$*

$$\int_V (\mathrm{id} \times g)^* \omega = \pm \int_{V'} \omega$$

if both fiber integrals are well-defined, with positive sign if g is orientation preserving and negative sign otherwise.

3. *Let $p\colon U \times V \to U$ be the projection, then for $\omega_1 \in \Omega^*(U)$ and $\omega_2 \in \Omega^*(U \times V)$,*

$$\int_V p^* \omega_1 \wedge \omega_2 = \omega_1 \wedge \int_V \omega_2$$

if both fiber integrals are well-defined.

In some cases, one of these integrals is well-defined if the other is well-defined, but we will not use this so it is not stated explicitly.

Remark 3.32. If $K \subseteq V \subseteq \mathbb{R}^n$ is compact, we may define the fiber integral $\int_K \omega \in \Omega^*(U)$ of $\omega \in \Omega^*(U \times V)$ to be the form given at $u \in U$ by

$$u \mapsto \left(\int_K \omega\right)_u = \sum du_I \left(\int_K f_I(u,v)dv\right).$$

This is always well-defined: Since K is compact, ω is bounded in the K direction, hence the integral converges. Moreover it is smooth since for every $u \in U$ and every $p \geq 0$, there is an open set $u \in N \subseteq U$ such that ω and all its p-th derivatives are bounded on the strip $N \times K$.

We now define a more general fiber integral where $U \subseteq \mathbb{R}^n$ is replaced with a general diffeological space.

Construction 3.33 (Fiber integral over smooth singular chains)**.** Let F and X be diffeological spaces, $\sigma\colon \Delta^n \to F$ be a smooth simplex in F and $\omega \in \Omega^*(X \times F)$, then for each plot $P\colon U \to X$ of X we have $(P \times \sigma)^* \omega \in \Omega^*(U \times \Delta^n)$. Since σ extends to a small neighborhood $V \supseteq \Delta^n$, the form $(P \times \sigma)^* \omega$ extends to a form in $\Omega^*(U \times V)$. Since Δ^n is compact, the fiber integral $\int_{\Delta^n} (P \times \sigma)^* \omega$ is well-defined, c.f. Remark 3.32. These forms piece together to a well-defined form on X by part 1 of Lemma 3.31:

$$\int_\sigma \omega = \left\{ \int_{\Delta^n} (P \times \sigma)^* \omega \right\} \in \Omega^*(X).$$

This extends linearly to chains of smooth simplices and one obtains a linear map

$$\begin{aligned} \int \colon \Omega^d(X \times F) \otimes C_n^\infty(F) &\to \Omega^{d-n}(X) \\ \omega \otimes \alpha &\mapsto \int_\alpha \omega \end{aligned}$$

called the **fiber integral** over smooth singular chains.

Remark 3.34. With a similar construction, using a partition of unity, one can construct a fiber integral over a compact oriented manifold M or more generally a map $M \to F$.

The fiber integral map satisfies the following properties:

Proposition 3.35. *1. If $\sigma \in C_k^\infty(F), \tau \in C_l^\infty(X)$ and $\omega \in \Omega^{k+l}(X \times F)$, then*

$$\int_\tau \int_\sigma \omega = \int_{\tau\times\sigma} \omega \in \mathbb{R}$$

where $\tau \times \sigma = EZ(\tau, \sigma) \in C_{k+l}^\infty(X \times F)$ denotes the image of the Eilenberg-Zilber map.

2. If $f\colon F \to F', g\colon X \to X'$, then for $\alpha \in C_*^\infty(F), \omega \in \Omega^*(X' \times F')$,

$$\int_\alpha (g \times f)^* \omega = g^* \int_{f_*\alpha} \omega$$

3. Let $p\colon X \times F \to X$ be the projection, then for $\omega_1 \in \Omega^*(X), \omega_2 \in \Omega^*(X \times F)$ and $\alpha \in C_*^\infty(F)$:

$$\int_\alpha p^*(\omega_1) \wedge \omega_2 = \omega_1 \wedge \int_\alpha \omega_2$$

Proof. Part (2) and (3) follow directly from Lemma 3.31.

For part (1), using (2) we may take $F = \Delta^k, X = \Delta^l$ and σ and τ to be the identity without loss of generality. The Eilenberg-Zilber map gives a partition of the product $\Delta^k \times \Delta^l$ via smooth maps $\Delta^{k+l} \to \Delta^k \times \Delta^l$ with disjoint images except on a zero set. Hence both sides are equal to $\int_{\Delta^k\times\Delta^l} \omega$. □

Let $I_{dR}\colon \Omega^*(F) \to C^*_\infty(F)$ be the de Rham map as in Section 3.3. By the following theorem, fiber integration corresponds to the slant product Appendix A.5 in the smooth singular cochain complex under I_{dR}.

Theorem 3.36. *Let F be a smooth manifold and B a diffeological space, let $\alpha\colon \Delta^p \to F$ be a smooth simplex and $\omega \in \Omega^{p+q}(B \times F)$ a differential form. Then*

$$I_{dR}(\omega)/\alpha = I_{dR}\left(\int_\alpha \omega\right)$$

Proof. Let $\beta \in C_p^\infty(B)$, then

$$\langle I_{dR}(\omega)/\alpha, \beta\rangle = \langle I_{dR}(\omega), \alpha \times \beta\rangle = \int_{\alpha\times\beta} \omega = \int_\beta \int_\alpha \omega = \left\langle I_{dR}\left(\int_\alpha \omega\right), \beta\right\rangle$$

□

From this we can show Stokes' theorem for fiber integration over smooth singular chains with a simple formal argument:

Proposition 3.37. *Let X be a diffeological space, $\omega \in \Omega^{p+q}(F \times X)$ and $\alpha \in C_p^\infty(F)$, then*

$$d\left(\int_\alpha \omega\right) = \int_\alpha d\omega + (-1)^{\deg \omega - \deg \alpha} \int_{\partial\alpha} \omega.$$

Proof. From the formal properties of the slant product Appendix A.5 and 3.36, it follows that

$$\begin{aligned} I_{dR}\left(d\int_\alpha \omega\right) &= d(I_{dR}(\omega))/\alpha = dI_{dR}(\omega)/\alpha + (-1)^{\deg\omega - \deg\alpha} I_{dR}(\omega)/\partial\alpha \\ &= I_{dR}\left(\int_\alpha d\omega + (-1)^{\deg\omega-\deg\alpha}\int_{\partial\alpha}\omega\right) \end{aligned}$$

We conclude, since I_{dR} is injective (3.14). □

With this we can easily show that the de Rham cohomology of diffeological spaces is invariant with respect to smooth homotopy.

Theorem 3.38. *Let $f_0, f_1 \colon M' \to M$ be smooth maps of diffeological spaces. Then every smooth homotopy $F \colon I \times M' \to M$ from f_0 to f_1 induces a chain homotopy $H \colon \Omega^*(M) \to \Omega^{*-1}(M')$ such that*

$$dH + Hd = f_1^* - f_0^*$$

Proof. From the previous proposition,

$$d\int_I F^*\omega = \int_I F^* d\omega + (-1)^{\deg\omega - 1}(f_1^*\omega - f_0^*\omega)$$

hence we can take H to be $\omega \mapsto (-1)^{\deg\omega}\int_I F^*\omega$. □

4. Cochain Complex Models for Graph Mapping Spaces

We are interested in the cohomology of the path space PM of a manifold M and more generally the mapping spaces from a graph into M. Here, PM denotes the space of piecewise smooth paths in M, topologized by uniform convergence. More generally, for a finite graph G, the space $\mathrm{Map}(G, M)$ consists of maps from the associated topological space G to M which are piecewise smooth along the edges. This has the structure of a topological diffeological space (cf. Section 3.1) and we can define the complex of de Rham forms $\Omega^*(\mathrm{Map}(G, M))$. By the discussion in Section 3.4 and Assumption 3.23, we will assume that $\Omega^*(\mathrm{Map}(G, M))$ is quasi-isomorphic to $C^*(C^0(G, M))$, the singular cochain complex of the space of continuous maps from G to M.

Classically, a differential graded commutative algebra (DGCA) A is called a DGCA model of another DGCA A' if there exists a sequence of quasi-isomorphisms $A \to \ldots \leftarrow A'$ such that each map is a map of DGCAs. A DGCA model of a space X is defined to be a DGCA model of $A_{PL}(X)$, the DGCA of polynomial forms on X, cf. [FHT12, Ch. 10]. In this work we will call a model of a chain complex C any chain complex C' together with an isomorphism $C \simeq C'$ in the derived category of chain complexes $D(Ch(k))$. Similarly a model of a map $C \to C'$ is a commutative square in $D(Ch(k))$

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\simeq"] & C' \arrow[d, "\simeq"] \\ B \arrow[r] & B' \end{tikzcd}$$

where the horizontal maps are isomorphisms.

The goal of this section is to define a model of the mapping space $\mathrm{Map}(G, M)$, using only a dgca model A of M and the data of the graph. The basic building block is the two sided bar construction $B = \mathrm{B}(A, A, A)$, which is a model of the path space through the iterated integral map $I\colon \mathrm{B}(A, A, A) \to \Omega^*(PM)$. This is discussed in Section 4.1. In Section 4.3 we build the model $A^{\otimes G}$ of $\mathrm{Map}(G, M)$. This generalizes the bar construction for the path space and the Hochschild complex for the free loop space. This model is functorial in the graph, thus maps between mapping spaces that are induced by graph homomorphisms can be modeled as well. In Section 4.4 we discuss maps which are not given by graph homomorphisms, namely the concatenation of paths and the splitting of paths. These can nevertheless be modeled.

4.1. Bar Construction and Path Spaces

In this subsection we follow [NW19]. We introduce the bar construction $\mathrm{B}(A, A, A)$ of a DGCA A. We show that if $A = \Omega^*(M)$ for a manifold, then there exists a quasi-isomorphism $I\colon \mathrm{B}(A, A, A) \to \Omega^*(PM)$ called the iterated integral map.

Let A be a (unital) DGCA. Denote by $B = \mathrm{B}(A, A, A)$ the two sided non-normalized **bar construction**

$$\mathrm{B}(A, A, A) = A \otimes T(A[1]) \otimes A = \sum_{n \geq 0} A \otimes A[1]^{\otimes n} \otimes A$$

Its differential is $d_B = d_0 + (-1)^{\varepsilon_n - n} d_1$ where d_0 is given by the differential on A:

$$d_0(\omega_0, \omega_1, \dots, \omega_n, \omega_{n+1}) = \sum_{i=0}^{n+1} (-1)^{\varepsilon_{i-1}} \omega_0 \otimes \dots \otimes d\omega_i \otimes \dots \otimes \omega_{n+1}$$

with $\varepsilon_i = |\omega|_0 + \dots + |\omega|_i$ (and $\varepsilon_{-1} = 0$). The differential d_1 is defined using the multiplication on A:

$$d_1(\omega_0, \omega_1, \dots, \omega_n, \omega_{n+1}) = \sum_{i=0}^{n-1} (-1)^i \omega_0 \otimes \dots \otimes \omega_i \wedge \omega_{i+1} \otimes \dots \otimes \omega_n$$

For properties of the bar construction, see Appendix B.1.

The complexes $A^{\otimes n}$ form a simplicial chain complex, i.e. a simplicial object in the category of chain complexes with simplicial boundary maps induced by $\omega_0 \otimes \dots \otimes \omega_n \mapsto \omega_0 \otimes \dots \otimes \omega_i \omega_{i+1} \otimes \dots \otimes \omega_n$ and degeneracy maps $\omega_0 \otimes \dots \otimes \omega_n \mapsto \omega_0 \otimes \dots \otimes 1 \otimes \dots \otimes \omega_n$, inserting 1 at the i-th factor. The un-normalized bar construction $\mathrm{B}(A, A, A)$ is the total complex of this simplicial object, the differential d_1 is induced by the simplicial boundary maps and d_B is the induced differential on the total complex.

Definition 4.1. The **simplicial normalization** $\hat{\mathrm{B}}(A, A, A)$ of the bar construction is the quotient by the images of the degeneracy maps:

$$\hat{\mathrm{B}}(A, A, A) \coloneqq \bigoplus_{n \geq 0} A \otimes \bar{A}[1] \otimes A$$

where $\bar{A} = A/\langle 1 \rangle$ (as a quotient of vector spaces).

Remark 4.2. Chen [Che76], cf. [GJP91] uses a stronger normalization: For $f \in A^0$, let

$$S_i(f)\left(\omega_0(\omega_1|\dots|\omega_k)\omega_{k+1}\right) = \omega_0(\omega_1|\dots|\omega_{i-1}|f|\omega_i|\dots|\omega_k)\omega_{k+1} \in \mathrm{B}$$

and $R_i(F) = dS_i(f) - S_i(f)d$ where d is the differential of B. Let D be the subspace of B generated by $S_i(f)$ and $R_i(f)$. Chen uses the normalization B/D. We will only work with the simplicial normalization throughout.

Let now $A = \Omega^*(M)$ for a manifold M. We introduce the iterated integral map. The standard n-simplex Δ^n is the set $\{0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq 1\} \subseteq \mathbb{R}^n$. Let

$$\begin{aligned} ev_n\colon \qquad \Delta^n \times PM &\to M \times M^n \times M \\ ((t_1, \dots, t_n), \gamma) &\mapsto (\gamma(0), \gamma(t_1), \dots, \gamma(t_n), \gamma(1)) \end{aligned}$$

In Section 3.5 we introduced a fiber integral map which we apply to the map $\Delta^n \times PM \to PM$:

$$\int_{\Delta^n} \colon \Omega^*(\Delta^n \times PM) \to \Omega^{*-n}(PM).$$

Definition 4.3. The **iterated integral** on de Rham forms is the map

$$I\colon \mathrm{B}(\Omega^*(M), \Omega^*(M), \Omega^*(M)) \to \Omega^*(PM)$$
$$I(\omega_0 \otimes \dots \otimes \omega_{n+1}) = \int_{\Delta^n} ev_n^*(\pi_0^*\omega_0 \wedge \dots \wedge \pi_{n+1}^*\omega_{n+1})$$

It acts on each summand as the composition

$$\Omega^*(M) \otimes \Omega^*(M)^n \otimes \Omega^*(M)[n] \to \Omega^{*+n}(M^{n+2}) \xrightarrow{ev^*} \Omega^{*+n}(\Delta^n \times PM) \xrightarrow{\int_{\Delta^n}} \Omega^*(PM)$$

Remark 4.4. We give an informal intuition for the iterated integral map here. We think of cohomology classes of degree k as represented by codimension k subspaces[1] We interpret the pullback map as taking the preimage. The fiber integral map $\int_{\Delta^n} \colon \Omega^*(PM \times \Delta^n) \to \Omega^{*-n}(PM)$ is the umkehr map on cohomology of the projection $PM \times \Delta^n \to PM$ and we interpret this as taking the image of a subspace under the projection. Thus let $a_0, \dots, a_{n+1}$ be cochains in M represented by subspaces $S_0, \dots, S_{n+1}$ of M. We think of the image under ev^* as $ev^{-1}(S_0 \times \dots \times S_{n+1})$, which is the collection of all paths γ such that $\gamma(t_i) \in S_i$ for all i. Then we take the image of this collection under the map

[1]For example, in geometric cohomology [FMMS22], a cochain in a finite dimensional manifold M is represented by smooth proper map $W \to M$ from a manifold with corners, plus some additional data.

$\Delta^n \times PM \to PM$, which means forgetting the exact times t_i, though not the order of the S_i.

We can interpret the boundary operator of the bar construction in this picture: the space $I(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1})$ consists of paths that are in $S_i \cap S_{i+1}$ at t_i, but such a path is equivalently a path which is in S_i at t_i and at S_{i+1} at t_{i+1} with $t_i = t_{i+1}$, which is part of the boundary of $I(a_0, \dots, a_{n+1})$. This corresponds to the differential d_1 of the bar complex. A path is also in the boundary of $I(a_0, \dots, a_{n+1})$ if it is in ∂S_i at t_i for some i, this corresponds to the differential d_0.

Remark 4.5. There exist different sign conventions for the bar construction and the iterated integral map. In his original work, Chen uses a slightly different construction of the fiber integral map: the integral over Δ^n can be interpreted as the integral of t_1 over I, t_2 over $t_1 \le t_2 \le 1$ and so on, see the latter part of the proof of Lemma 4.6. This leads to different signs in each degree compared to our construction. Thus to make the iterated integral map a chain map, Chen's differential of the bar construction also differs by signs from our definition. In our sign convention, the differential of the bar construction is chosen such that our iterated integral map is a chain map without modification.

Lemma 4.6. *The iterated integral is a chain map $I\colon \mathrm{B}(A, A, A) \to \Omega^*(PM)$ and it descends to a chain map $I\colon \hat{\mathrm{B}}(A, A, A) \to \Omega^*(PM)$.*

Proof. We first show that the iterated integral is a chain map on $\mathrm{B}(A, A, A)$. We use the Stokes' theorem for the fiber integral (Proposition 3.37) $d \int_F \omega = \int_F d\omega + (-1)^{|\omega| - \dim F} \int_{\partial F} \omega$. We write $\varepsilon_i = |\omega|_0 + \dots + |\omega|_i$.

$$d\left(I\left(\omega_0 \otimes \dots \otimes \omega_{n+1}\right)\right) = \int_{\Delta^n} ev^* \left(d\left((\pi_0^*\omega_0) \wedge \dots \wedge (\pi_{n+1}^*\omega_{n+1})\right)\right) \\ + (-1)^{\varepsilon_n - n} \int_{\partial\Delta^n} ev^* \left((\pi_0^*\omega_0) \wedge \dots \wedge (\pi_{n+1}^*\omega_{n+1})\right) = \quad S_1 + S_2$$

The first summand can be computed as

$$\begin{aligned} S_1 =& \int_{\Delta^n} ev^* \left(d\left((\pi_0^*\omega_0) \wedge \dots \wedge (\pi_{n+1}^*\omega_{n+1})\right)\right) \\ =& \sum_{i=0}^{n+1} (-1)^{\varepsilon_{i-1}} \int_{\Delta^n} ev^* \left(\pi_0^*\omega_0 \wedge \dots \wedge \pi_i^* d\omega_i \wedge \dots \wedge \pi_{n+1}^*\omega_{n+1}\right) \\ =& \sum_{i=0}^{n+1} (-1)^{\varepsilon_{i-1}} I\left(\omega_0 \otimes \dots \otimes d\omega_i \otimes \dots \otimes \omega_{n+1}\right) \\ =& I(d_0(\omega_0 \otimes \dots \otimes \omega_{n+1})) \end{aligned}$$

For the second summand, the boundary $\partial\Delta^n$ splits into the faces $\partial_i\Delta^n$, positively oriented if i is even and negatively oriented otherwise. On the i-th face $\partial_i\Delta^n$, we have $t_i = t_{i+1}$ (with $t_0 = 0, t_{n+1} = 1$) so $\pi_i \circ ev = \pi_{i+1} \circ ev$.

$$
\begin{aligned}
S_2 =& (-1)^{\varepsilon_n - n} \int_{\partial \Delta^n} ev^* \left((\pi_0^* \omega_0) \wedge \cdots \wedge (\pi_{n+1}^* \omega_{n+1}) \right) \\
=& \sum_{i=0}^{n} (-1)^{\varepsilon_n - n + i} \int_{\partial_i \Delta^n} ev^* ((\pi_0^* \omega_0) \wedge \cdots \wedge (\pi_i^* (\omega_i \wedge \omega_{i+1})) \wedge \cdots \wedge (\pi_{n+1}^* \omega_{n+1}))
\end{aligned}
$$

We reparametrize through the maps $\Delta^{n-1} \to \partial_i \Delta^n, (t_0, \ldots, t_n) \mapsto (t_0, \ldots, t_i, t_i, \ldots, t_n)$.

$$
\begin{aligned}
=& \sum_{i=0}^{n} (-1)^{\varepsilon_n - (n-i)} \int_{\Delta^{n-1}} ev^* ((\pi_0^* \omega_0) \wedge \cdots \wedge (\pi_i^* (\omega_i \wedge \omega_{i+1})) \wedge \cdots \wedge (\pi_n^* \omega_{n+1})) \\
=& \sum_{i=0}^{n} (-1)^{\varepsilon_n - (n-i)} I \left(\omega_0 \otimes \cdots \otimes \omega_i \wedge \omega_{i+1} \otimes \cdots \otimes \omega_{n+1} \right) \\
=& \sum_{i=0}^{n} (-1)^{\varepsilon_n - n} I (\omega_0 \otimes \cdots \otimes \omega_i \omega_{i+1} \otimes \cdots \otimes \omega_{n+1}) \\
=& I ((-1)^{\varepsilon_n - n} d_1 (\omega_0 \otimes \cdots \otimes \omega_i \omega_{i+1} \otimes \cdots \otimes \omega_{n+1}))
\end{aligned}
$$

Thus I is a chain map.

It remains to check that I vanishes on elements $\omega_0 \otimes \cdots \otimes 1 \otimes \cdots \otimes \omega_{n+1}$. More generally, we can show that $I(\omega_0 \otimes \cdots \otimes \omega_n) = 0$ if ω_i is a 0-form for some $1 \leq i \leq n$. We write $ev_n \colon \Delta^n \times PM \to M^{n+2}$ as a product $ev_n = ev_n^0 \times \cdots \times ev_n^{n+1}$ where $ev_n^i = \pi_i \circ ev_n$. We can factor the projection $\Delta^n \times PM \to PM$ through $\Delta^n \times PM \to \Delta^{n-1} \times PM \to PM$ by forgetting t_i first. One can compute

$$
\begin{aligned}
\int_{\Delta^n} ev_n^* (\omega_0 \otimes \cdots \otimes \omega_n) &= \int_{\Delta^{n-1}} \int_{t_{i-1} \leq t \leq t_i} ev_n^* (\omega_0 \otimes \cdots \otimes \omega_n) \\
&= \pm \int_{\Delta^{n-1}} \Big(\pi_0^* (ev_n^i)^* \omega_0 \wedge \cdots \wedge \pi_{n+1}^* (ev_n^{(n+1)})^* \omega_{n+1} \wedge \\
&\qquad \wedge \int_{t_{i-1} \leq t \leq t_i} \pi_i^* (ev_n^i)^* \omega_i \Big)
\end{aligned}
$$

but the fiber integral over a 1-dimensional space is a degree -1 map, hence the inner integral vanishes if ω_i is a 0-form:

$$
\int_{t_{i-1} \leq t \leq t_i} \pi_i^* (ev_n^i)^* \omega_i = 0
$$

□

Remark 4.7. On the singular cochain complex $C^*(M)$, we can also form the bar construction $B(C^*(M), C^*(M), C^*(M))$, which is still a chain complex. By Theorem 3.36, the analogue of the fiber integral is the slant product and we can construct a map $\mathrm{B}(C^*(M), C^*(M), C^*(M)) \to C^*(PM)$ as in the de Rham case. But since $C^*(M)$ is not commutative, we cannot perform the computation for S_2 in the singular cochain complex, hence we cannot show that this is a chain map. However we can still construct a zigzag of quasi-isomorphisms $B(C^*(M), C^*(M), C^*(M)) \xrightarrow{\simeq} C^*(M) \xleftarrow{\simeq} C^*(PM)$. By item (1) of Proposition 4.8, this is a suitable replacement of the iterated integral map, at least in the derived category of chain complexes.

Let $const\colon M \to PM$ be the constant path map and let $ev_{0,1}\colon PM \to M \times M$ be the evaluation at 0 and 1. The following proposition gives models of these maps. Recall $A = \Omega^*(M)$ and $B = \mathrm{B}(A, A, A)$. Let $\alpha\colon B \to A$ be the map given on the direct summands of B by the product $A^{\otimes 2} \to A$ and the zero map $A^{\otimes n} \to A$ for $n > 2$. Let $\beta\colon A^{\otimes 2} \to B$ be the map given by the inclusion of $A^{\otimes 2}$ into the direct sum $\mathrm{B} = \bigoplus A^{\otimes n}$.

Proposition 4.8. *1. The following diagram commutes:*

$$\begin{tikzcd} B \arrow[r, "I"] \arrow[d, "\alpha"] & \Omega^*(PM) \arrow[d, "const^*"] \\ A \arrow[r, equal] & \Omega^*(M) \end{tikzcd} \tag{3}$$

2. The following diagram commutes:

$$\begin{tikzcd} A \otimes A \arrow[r] \arrow[d, "\beta"] & \Omega^*(M \times M) \arrow[d, "ev_{0,1}^*"] \\ B \arrow[r, "I"] & \Omega^*(PM) \end{tikzcd} \tag{4}$$

3. The iterated integral map $I\colon \mathrm{B}(A, A, A) \to \Omega^*(PM)$ is a quasi-isomorphism.

Proof. 1. Let $p\colon M \times \Delta^n \to M$ be the projection. There is a map $\Omega^*(M)^{\otimes k} \to \Omega^*(M)$ given by the $(k-1)$-fold wedge product, which we denote by $\wedge$. Consider the following commutative diagram:

$$\begin{tikzcd} \Omega^{*+n}(M)^{\otimes(n+2)} \arrow[r] \arrow[d, "\wedge"] & \Omega^{*+n}(M^{n+2}) \arrow[r, "ev^*"] \arrow[d, "\Delta"] & \Omega^{*+n}(PM \times \Delta^n) \arrow[r, "\int_{\Delta^n}"] \arrow[d, "const^* \otimes \mathrm{id}"] & \Omega^*(PM) \arrow[d, "const^*"] \\ \Omega^{*+n}(M) \arrow[r] & \Omega^{*+n}(M) \arrow[r, "p^*"] & \Omega^{*+n}(M \times \Delta^n) \arrow[r, "\int_{\Delta^n}"] & \Omega^*(M) \end{tikzcd}$$

The composition along the bottom row vanishes for $n > 0$ since

$$\int_{\Delta^n} p^*\omega = \omega \int_{\Delta^n} 1$$

which vanishes if $1 \in \Omega^0(M)$ is not an n-form. For $n = 0$, the space Δ^0 is a single point and the composition along the top row is the identity. Taking the direct sum over $n \geq 0$ shows that Diagram 3 commutes.

2. Consider the following commutative diagram:

$$\begin{tikzcd}
\Omega^*(PM) & \Omega^{*+n}(PM \times \Delta^n) \arrow[l, "\int_{\Delta^n}"'] & \Omega^{*+n}(M^{n+2}) \arrow[l, "ev^*"'] & \Omega^*(M)^{\otimes(n+2)}[n] \arrow[l] \\
\Omega^*(M \times M) \arrow[u] & \Omega^{*+n}(M \times M \times \Delta^n) \arrow[l, "\int_{\Delta^n}"'] \arrow[u] & \Omega^{*+n}(M \times M) \arrow[l] \arrow[u] & \Omega^*(M)^{\otimes 2}[n] \arrow[l] \arrow[u]
\end{tikzcd}$$

The commutativity of Diagram 4 follows from taking $n = 0$.

3. In Diagram 3, the bottom and right arrows are quasi-isomorphisms and the map $B \to A$ is a quasi-isomorphism by Theorem B.1. Hence I is a quasi-isomorphism.

□

Let $A_s = C^*(M)$ be the singular cochain algebra and $B_s = B(A_s, A_s, A_s)$ and denote by $I_s \colon B_c \to C^*(PM)$ the composition $B_s \to A_s \xleftarrow{\simeq} C^*(PM)$ in $D(Ch(k))$.

Proposition 4.9. *1. The following diagram commutes in $D(Ch(k))$:*

$$\begin{tikzcd}
B_s \arrow[r, "I_c"] \arrow[d, "\alpha"] & C^*(PM) \arrow[d, "const^*"] \\
C^*(M) \arrow[r, equal] & C^*(M)
\end{tikzcd} \tag{5}$$

2. The following diagram commutes in $D(Ch(k))$:

$$\begin{tikzcd}
C^*(M) \otimes C^*(M) \arrow[r] \arrow[d, "\beta"] & C^*(M \times M) \arrow[d, "ev_{0,1}^*"] \\
B_s \arrow[r, "I"] & C^*(PM)
\end{tikzcd} \tag{6}$$

Proof. The first part is by definition. For (2), we have the following diagram:

$$\begin{tikzcd}
C^*(M) \otimes C^*(M) \arrow[r] \arrow[d, "\beta"] & C^*(M) \otimes C^*(M) \arrow[d, "\smile"] & C^*(M \times M) \arrow[l] \arrow[d, "ev_{0,1}^*"] \\
B_s \arrow[r] & C^*(M) & C^*(PM) \arrow[l]
\end{tikzcd}$$

□

Note that all maps are maps of two-sided $C^*(M)$-modules, thus the proposition also holds in the derived category of $C^*(M)$ bimodules.

4.2. Pullback-Pushout Lemma / Eilenberg Moore Theorem

Consider the following pullback of topological spaces:

$$\begin{array}{ccc} E \times_B X & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array}$$

We want to describe an algebraic model of the pullback $E \times_B X$ using an algebraic model of the diagram $E \to B \leftarrow X$. The slogan is that the pullback of the spaces should be described by the pushout of the cochain algebras. However, since cohomology is invariant with respect to homotopy, but the pullback of spaces in general is not, we consider homotopy pullbacks instead of regular pushouts. Recall that if the map $E \to B$ is a fibration, then the pullback diagram above is in fact a homotopy pullback diagram, cf. Appendix C. Similarly, on the level of cochain algebras, one has to consider a homotopy invariant version of the pushout.

We begin with a statement involving specific algebraic models from rational homotopy theory, specifically Sullivan models. We then generalize this to hold in a more general setting. Let k be a field of characteristic 0.

Definition 4.10. Let A be a unital not necessarily commutative dg algebra over k and let M be a right A module and N be a left A module. Then the tensor product $M \otimes_A N$ is the cokernel of the difference of the chain maps

$$M \otimes A \otimes N \rightrightarrows M \otimes N$$

mapping $m \otimes a \otimes n$ to $ma \otimes n$ resp. $m \otimes an$.

Definition 4.11. A **relative Sullivan algebra** is a homomorphism of differential graded commutative algebras $(A, d) \to (A \otimes \Lambda V, D)$ for a graded vector space V such that V has a basis $\{v_\alpha \mid \alpha \in J\}$ where J is well-ordered, such that $Dv_\beta \in A \otimes \Lambda V_{<\beta}$ for all $\beta \in J$, where $V_{<\beta}$ is spanned by the v_α with $\alpha < \beta$.

If $A = K$ then an algebra $(\Lambda V, D)$ satisfying the above property is called a **Sullivan Algebra**.

For a space X, denote by $A_{PL}(X)$ the complex of **Sullivan's polynomial forms**, cf. [FHT12, Ch. 10]. This is a CDGA which is quasi-isomorphic to $C^*(X)$, and the assignment $X \mapsto A_{PL}(X)$ extends to a contravariant functor.

In [Hes07, 2.4] there is the following statement using Sullivan models:

Theorem 4.12. *Let $p\colon E \to B$ be a Serre fibration, where E is path connected and B is simply connected, with fiber F, let $f\colon X \to B$ be continuous. Suppose that B or F is of finite rational type. Let A_X and A_B be Sullivan algebras and $\iota\colon A_B \to A_E$ a relative Sullivan algebra and assume that there is a homotopy commutative diagram*

$$\begin{tikzcd}
A_X \arrow[d, "\simeq"] & A_B \arrow[l, "\phi"'] \arrow[r, "\iota"] \arrow[d, "\simeq"] & A_E \arrow[d, "\simeq"] \\
A_{PL}(X) & A_{PL}(B) \arrow[l, "A_{PL}(f)"'] \arrow[r, "A_{PL}(p)"] & A_{PL}(E)
\end{tikzcd}$$

with vertical quasi-isomorphisms, then the map

$$A_X \otimes_{A_B} A_E \to A_{PL}(E) \otimes_{A_{PL}(B)} A_{PL}(B) \to A_{PL}(E \times_B X)$$

is a quasi-isomorphism of CDGAs.

This is called the pullback-pushout lemma, because $A_X \otimes_{A_B} A_E$ is the pushout in the category of differential graded commutative algebras, and it yields an algebraic model of the pullback.

Recall from Appendix B.3 that the two sided bar construction is a construction of the derived tensor product $\otimes_A^L$, hence we write $M \otimes_A^L N = \mathrm{B}(M, A, N)$. In particular, the left derived tensor product preserves quasi-isomorphisms Theorem B.2, which the normal tensor product over A generally does not, hence $\otimes^L$ is more adapted to the computation of cohomology. Using this construction, we may transfer the preceding theorem to more general algebraic models and obtain the formulation of the pullback pushout lemma as in [NW19].

We assume that we have a pullback diagram as in Diagram 4.2 and a commutative diagram of differential graded algebras

$$\begin{tikzcd}
A_X \arrow[d, "\simeq"] & A_B \arrow[l, "\phi"'] \arrow[r, "\iota"] \arrow[d, "\simeq"] & A_E \arrow[d, "\simeq"] \\
A_{PL}(X) & A_{PL}(B) \arrow[l, "A_{PL}(f)"'] \arrow[r, "A_{PL}(p)"] & A_{PL}(E)
\end{tikzcd}$$

where the vertical maps are quasi-isomorphisms.

Theorem 4.13. *Assume p is a Serre fibration with fiber F, so that Diagram 4.2 is a homotopy pushout diagram, E is path connected, B and X are simply connected, and either B or F is of finite rational type. Then the natural map*

$$A_X \otimes_{A_B}^L A_E \to A_{PL}(E \times_B X)$$

is a quasi-isomorphism. In particular, the following map is a quasi-isomorphism

$$A_{PL}(E) \otimes_{A_{PL}(B)}^L A_{PL}(X) \to A_{PL}(E \times_B X).$$

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
A_{PL}(E) \otimes_{A_{PL}(B)}^L A_{PL}(X)) \arrow[r] \arrow[d, "\simeq"] & A_{PL}(E) \otimes_{A_{PL}(B)} A_{PL}(X) \arrow[r] \arrow[d] & A_{PL}(E \times_B X) \\
A_E \otimes_{A_B}^L A_X \arrow[r] & A_E \otimes_{A_B} A_X \arrow[ur] &
\end{tikzcd}$$

1. We first use as A_B, A_E and A_X the Sullivan models from Theorem 4.12 and deduce that the upper horizontal composition is a quasi-isomorphism. By [FHT12, Thm 14.3] such models exist (this is where we need the additional assumption on X). The map between the $\otimes^L$ terms is a quasi-isomorphism since it is induced by quasi-isomorphisms, see B.2. Since the map $A_B \to A_E$ is a relative Sullivan algebra, A_E is a cofibrant A_B-algebra. Thus by B.11, the map $A_E \otimes^L_{A_B} A_X \to A_E \otimes_{A_B} A_X$ is a quasi-isomorphism. The map $A_X \otimes_{A_B} A_E \to A_{PL}(E \times_B X)$ is a quasi-isomorphism by Theorem 4.12. This shows that the upper horizontal composition is a quasi-isomorphism.

2. Now for general A_B, A_E, A_X, we know that the upper horizontal composition and the left vertical composition are quasi-isomorphisms. This shows the theorem.

□

Remark 4.14. A closer inspection of the proof and [FHT12, Thm 14.3] reveals that in place of X being simply connected, it is enough that $H^1(X) = 0$.

Remark 4.15. One can replace A_{PL} in the statement of Theorem 4.13 with other differential graded algebras quasi-isomorphic to A_{PL}. For example, by [FHT12, 10.10] there is a chain of quasi-isomorphisms of cochain algebras connecting A_{PL} and the singular cochain complex C^*. Then the maps

$$A_{PL}(E) \otimes^L_{A_{PL}(B)} A_{PL}(X) \to A_{PL}(E \times_B X)$$
$$C^*(E) \otimes^L_{C^*(B)} C^*(X) \to C^*(E \times_B X)$$

are connected by a commutative zigzag of quasi-isomorphisms, so either is a quasi-isomorphism if and only if the other is.

Remark 4.16. Originally, this was shown in an analogous version for homology using cotensor products, as the Eilenberg-Moore spectral sequence, see [EM66]. There are also more general statements, working directly with C^* and not necessarily using field coefficients, see [Toe20, Appendix A].

4.3. Models for Mapping Spaces from Graphs

Let again M be a compact, connected and simply connected, oriented manifold. In this section we model mapping spaces $\mathrm{Map}(G, M)$ where G is (the geometric realization of) a graph, and maps between those spaces. This includes for example the loop space and the figure eight space of M. The main ingredient is the iterated integral map of the previous subsection and the pullback pushout lemma, Theorem 4.13. Our model is not stated in this generality in [NW19]. Note that if there is more than one edge, then our model is different from the model given by Patras and Thomas for mapping spaces from simplicial sets into spaces, see [PT03].

Let G be a graph given by ordered finite sets E of edges and V of vertices, with source and target maps $s, t\colon E \to V$. To this graph is associated a topological space which shall also be denoted by G. We construct a complex $A^{\otimes G}$ which is quasi-isomorphic to $\Omega^*(\mathrm{Map}(G, M))$. This is constructed using a copy of the bar construction for each edge and a copy of A for each vertex, which are assembled with a relative tensor product. Figure 2 depicts an example, using a graph with two nodes, two parallel edges and a lobe.

The space G is the pushout in the following diagram of spaces

$$\begin{tikzcd} E \times \{0,1\} \arrow[r, "s \amalg t"] \arrow[d] & V \arrow[d] \\ E \times I \arrow[r] & G \end{tikzcd}$$

Applying $\mathrm{Map}(-, M)$ induces a diagram as follows, which exhibits $\mathrm{Map}(G, M)$ as a pullback.

$$\begin{tikzcd} \mathrm{Map}(E \times \{0,1\}, M) & \mathrm{Map}(V, M) \arrow[l, "\phi"'] \\ \mathrm{Map}(E \times I, M) \arrow[u, "\psi"] & \mathrm{Map}(G, M) \arrow[l] \arrow[u] \end{tikzcd}$$

We begin by creating a model of the diagram $\mathrm{Map}(E \times I, M) \to \mathrm{Map}(E \times \{0,1\}, M) \leftarrow \mathrm{Map}(V, M)$. We then apply Theorem 4.13 to show that the pushout of the model diagram will be a model of the pullback of the spaces.

Let A be a DGCA. For any finite ordered set S, we can define the tensor product $A^{\otimes S} = \bigotimes_{s \in S} A$ and equip it with the usual tensor product chain complex and algebra structure. Given a map $f\colon S \to S'$ with $S = \{s_1, \dots, s_n\}$ and $S' = \{s'_1, \dots, s'_l\}$, we can construct a map $A^{\otimes S} \to A^{\otimes S'}$ using the algebra structure of A:

$$a_{s_1} \otimes \dots \otimes a_{s_n} \mapsto (-1)^{\varepsilon} \prod_{f(s)=s'_1} a_s \otimes \dots \otimes \prod_{f(s)=s'_l} a_s$$

where $(-1)^{\varepsilon}$ is the Koszul sign from reordering $a_{s_1} \otimes \dots \otimes a_{s_n}$ into the order on the right hand side.

Let $A = \Omega^*(M)$ and $B = \mathrm{B}(A, A, A)$. Consider the following maps

$$\begin{aligned} \Phi &\colon (A^{\otimes 2})^{\otimes E} \to B^{\otimes E} \\ \Psi &\colon (A^{\otimes 2})^{\otimes E} \to A^{\otimes V} \otimes A^{\otimes V} \to A^{\otimes V} \end{aligned}$$

The map Φ is the product of the inclusions $A^{\otimes 2} \to B$, given simply by

$$a_{e_1} \otimes a'_{e_1} \otimes \dots \otimes a_{e_n} \otimes a'_{e_n} \mapsto (a_{e_1} \otimes a'_{e_1}) \otimes \dots \otimes (a_{e_n} \otimes a'_{e_n}).$$

The map Ψ is induced by the map $s \amalg t\colon E \amalg E \to V \amalg V$ and the multiplication on A: it maps

$$a_{e_1} \otimes a'_{e_1} \otimes \dots \otimes a_{e_n} \otimes a'_{e_n} \mapsto (-1)^{\varepsilon} \prod_{s(e)=v_1} a_e \prod_{t(e)=v_1} a'_e \otimes \dots \otimes \prod_{s(e)=v_n} a_e \prod_{t(e)=v_n} a'_e$$

for $V = \{v_1, \dots, v_n\}$, where ε is the Koszul sign of permuting the factors $a_{e_1}, a'_{e_1}, \dots, a_{e_n}, a'_{e_n}$ into the sequence on the right hand side.

We can relate ϕ^*, ψ^* to Φ, Ψ through the following diagram:

$$\begin{tikzcd}
B^{\otimes E} \arrow[d, "I^{\otimes E}"] & (A^{\otimes 2})^{\otimes E} \arrow[l, "\Phi"'] \arrow[r, "\Psi"] \arrow[d] & A^{\otimes V} \arrow[d, equal] \\
\Omega^*(PM)^{\otimes E} \arrow[d] & \Omega^*(M \times M)^{\otimes E} \arrow[l] \arrow[r] \arrow[d] & \Omega^*(M)^{\otimes V} \arrow[d] \\
\Omega^*(\mathrm{Map}(I, M)^{\times E}) \arrow[d, equal] & \Omega^*(\mathrm{Map}(\{0,1\}, M)^{\times E}) \arrow[l, "\phi^*"'] \arrow[r, "\psi^*"] \arrow[d, equal] & \Omega^*(\mathrm{Map}(\{pt\}, M)^{\times V}) \arrow[d, equal] \\
\Omega^*(\mathrm{Map}(E \times I, M)) & \Omega^*(\mathrm{Map}(E \times \{0,1\}, M)) \arrow[l, "\phi^*"'] \arrow[r, "\psi^*"] & \Omega^*(\mathrm{Map}(V, M))
\end{tikzcd}$$

Thus Φ and Ψ are models of ϕ and ψ:

Lemma 4.17. *There is a commutative diagram where each vertical map is a quasi-isomorphism:*

$$\begin{tikzcd}
B^{\otimes E} \arrow[d] & (A^{\otimes 2})^{\otimes E} \arrow[l, "\Phi"'] \arrow[r, "\Psi"] \arrow[d] & A^{\otimes V} \arrow[d] \\
\Omega^*(\mathrm{Map}(E \times I, M)) & \Omega^*(\mathrm{Map}(E \times \{0,1\}, M)) \arrow[l, "\phi^*"'] \arrow[r, "\psi^*"] & \Omega^*(\mathrm{Map}(V, M))
\end{tikzcd}$$

Recall that the tensor product $M \otimes_A N$ of two differential graded modules M, N over a differential graded algebra A is the quotient of the tensor product $M \otimes N$ by the differential ideal generated by $ma \otimes n - m \otimes an$ for homogeneous elements $m \in M, a \in A, n \in N$.

Through the maps Φ, Ψ, there is an $(A^{\otimes 2})^{\otimes E}$-module structure on $B^{\otimes E}$ and $A^{\otimes V}$.

Definition 4.18. The graph complex $A^{\otimes G}$ of $\mathrm{Map}(G, M)$ is the complex

$$A^{\otimes G} = B^{\otimes E} \otimes_{(A^{\otimes 2})^{\otimes E}} A^{\otimes V}.$$

One can show that Φ and Ψ are maps of algebras and $A^{\otimes G}$ is the pushout in the category of differential graded commutative algebras. Thus $A^{\otimes G}$ is defined as the pushout of the model and we have an algebraic analogue of Diagram 4.3:

$$\begin{tikzcd}
(A^{\otimes 2})^{\otimes E} \arrow[r, "\Psi"] \arrow[d, "\Phi"] & A^{\otimes V} \arrow[d] \\
B^{\otimes E} \arrow[r] & A^{\otimes G}
\end{tikzcd}$$

An example of the graph complex is depicted in Fig. 2. We picture a homogeneous element of $A^{\otimes G}$ as the graph G decorated with elements of A resp. $T(A[1]) = \bigoplus_{n \geq 0} A[1]^{\otimes n}$ at the vertices resp. the edges. We illustrate the differential of $A^{\otimes G}$ as follows. The differential d is a sum $d = d_0 + d_1$ where d_0 is induced only by the differential of A. The

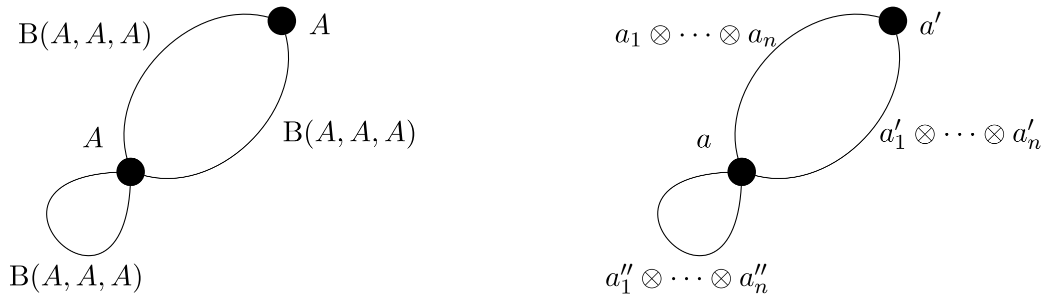
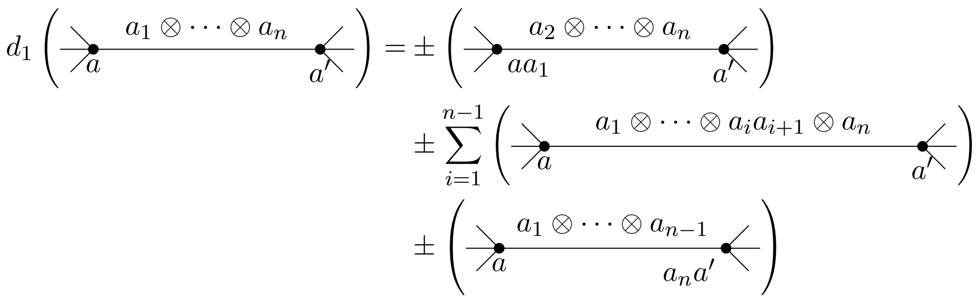


Figure 2: The above sketches the complex $A^{\otimes G}$ for a graph with two vertices and three edges. The left image depicts the complex, the right image depicts a generating element. As graded vector spaces, $B = A \otimes T(A[1]) \otimes A$ where $T(A[1]) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A[1]^{\otimes n}$. In this example, as graded vector spaces, $A^{\otimes G} = A^{\otimes 2} \otimes (T(A[1]))^{\otimes 3}$, where each factor corresponds to a vertex or an edge.

differential d_1 is defined using the product on A and the source and target maps of the graph. The following formula depicts the computation of the differential d_1: At any edge, one modifies the decorations of the graph as depicted. This is then summed over all edges, so that each edge generates $n+1$ summands.



Construction 4.19. For a finite ordered graph G and a manifold M, the **iterated integral map** for $\mathrm{Map}(G, M)$ is the composition

$$I_G\colon A^{\otimes G} \to \Omega^*(\mathrm{Map}(E \times I, M)) \otimes_{\Omega^*(\mathrm{Map}(E \times \{0,1\}, M))} \Omega^*(\mathrm{Map}(V, M)) \to \Omega^*(\mathrm{Map}(G, M))$$

Remark 4.20. We can explicitly compute the map

$$I_G\colon A^{\otimes G} \to \Omega^*(\mathrm{Map}(G, M)).$$

As graded vector spaces, $A^{\otimes G}$ is a direct sum of $\bigotimes_{e \in E} A[1]^{k_e} \oplus A^{\otimes V}$ over all $k_e \geq 0$ for $e \in E$: the outer A factors of the bar construction are identified with the A factors of the vertices by taking the tensor product relative to $(A^{\otimes 2})^{\otimes E}$. Write $s = \sum_{e \in E} k_e$. On each direct summand, I_G is given by

$$\bigotimes_{e \in E} A[1]^{\otimes k_e} \otimes A^{\otimes V} \to \Omega^{*+s}\left(M^{s+|V|}\right) \xrightarrow{ev^*} \Omega^{*+s}\left(\mathrm{Map}(G, M) \times \prod_{e \in E} \Delta^{k_e}\right) \xrightarrow{\int} \Omega^*(\mathrm{Map}(G, M))$$

where the evaluation map

$$ev = ev_{\{k_e\}} \colon \operatorname{Map}(G, M) \times \prod_{e \in E} \Delta^{k_e} \to M^{s+|V|}$$

evaluates each element of the mapping space at the times in Δ^{k_e} along the edge e and at each vertex v.

For example, if $G = \bigcirc_1 = S^1$ is the circle with one vertex and one edge, then $ev_n \colon LM \times \Delta^n \to M^n \times M$ is given by mapping $t_1, \ldots, t_n, \gamma \mapsto \gamma(t_1), \ldots, \gamma(t_n), \gamma(0)$.

To apply Theorem 4.13 we need to add the assumption that M is of finite type. Here a manifold is called of finite type if its cohomology is finite dimensional in each degree.

Theorem 4.21. *Let M be a simply connected manifold and of finite type. Then the iterated integral map*

$$I_G \colon A^{\otimes G} \to \Omega^*(\operatorname{Map}(G, M)).$$

is a quasi-isomorphism, hence $A^{\otimes G}$ is a model of $\operatorname{Map}(G, M)$.

Moreover, for a morphism of graphs $G \to G'$, there is a commutative diagram

$$\begin{tikzcd} A^{\otimes G} \arrow[r, "I_G"] \arrow[d] & \Omega^*(\operatorname{Map}(G, M)) \arrow[d] \\ A^{G'} \arrow[r, "I_{G'}"] & \Omega^*(\operatorname{Map}(G', M)) \end{tikzcd}$$

We use assumption 3.23 in this proof.

Proof. We apply the pullback-pushout theorem 4.13 to the pullback diagram of $\operatorname{Map}(G, M)$

$$\begin{tikzcd} \operatorname{Map}(E \times \{0,1\}, M) & \operatorname{Map}(V, M) \arrow[l, "\phi"'] \\ \operatorname{Map}(E \times I, M) \arrow[u, "\psi"] & \operatorname{Map}(G, M) \arrow[l] \arrow[u] \end{tikzcd}$$

All spaces here are connected and simply connected since M is. The map $\operatorname{Map}(E \times I, M) \to \operatorname{Map}(\{0,1\} \times E, M)$ is a Serre fibration (it is a product of the evaluation maps $PM \to M \times M$) hence the diagram is a homotopy pullback diagram. For the application of Theorem 4.13 we use the models from Lemma 4.17. By Remark 4.15 we may apply the theorem with Ω^* in place of A_{PL}, relying on Assumption 3.23.

Thus the composition

$$B^{\otimes E} \otimes^L_{(A^{\otimes 2})^{\otimes E}} A^{\otimes V} \to B^{\otimes E} \otimes_{(A^{\otimes 2})^{\otimes E}} A^{\otimes V} \to \Omega^*(M^G)$$

is quasi-isomorphism.

Recall from Lemma B.9 that B is a cofibrant $A^{\otimes 2}$ module, hence $B^{\otimes E}$ is a cofibrant $(A^{\otimes 2})^{\otimes E}$ module. By Corollary B.11, the first map in the composition is a quasi-isomorphism, and so the map

$$B^{\otimes E} \otimes_{(A^{\otimes 2})^{\otimes E}} A^{\otimes V} = A^{\otimes G} \to \Omega^*(\operatorname{Map}(G, M))$$

is also a quasi-isomorphism. $\square$

We can now easily give models for common spaces and maps in string topology. Let $\bigcirc_1$ be the graph with one vertex and one edge, thus $LM = \mathrm{Map}(\bigcirc_1, M)$. Further let 8 be the graph with one vertex and two edges, so $LM \times_M LM = \mathrm{Map}(8, M)$. Let $\bigcirc_2$ be the circle graph with two points and let $\bigcirc_1 \amalg \bigcirc_1$ be the coproduct of graphs (disjoint union) of two copies of $\bigcirc_1$.

Corollary 4.22. *The following maps are quasi-isomorphisms if M is simply connected and of finite type.*

$$B \otimes_{A^{\otimes 2}} A \xrightarrow{I_{\bigcirc_1}} \Omega^*(LM)$$

$$(B \otimes B) \otimes_{A^{\otimes 2} \otimes A^{\otimes 2}} A^{\otimes 2} \xrightarrow{I_{\bigcirc_2}} \Omega^*(Map(\bigcirc_2, M))$$

$$(B \otimes B) \otimes_{A^{\otimes 2} \otimes A^{\otimes 2}} A \xrightarrow{I_8} \Omega^*(Map(8, M))$$

$$(B \otimes_{A^{\otimes 2}} A) \otimes (B \otimes_{A^{\otimes 2}} A) \xrightarrow{I_{\bigcirc_1 \amalg \bigcirc_1}} \Omega^*(LM \times LM)$$

$$B \otimes_A B \to \Omega^*(PM \times_M PM)$$

The following diagram commutes:

$$\begin{tikzcd}
(B \otimes B) \otimes_{A^{\otimes 4}} A \arrow[r, "I_8"] & \Omega^*(\mathrm{Map}(8)) \\
(B \otimes_{A^{\otimes 2}} A) \otimes (B \otimes_{A^{\otimes 2}} A) \arrow[u] \arrow[r, "\bigcirc_1 \amalg \bigcirc_1"] & \Omega^*(LM \times LM) \arrow[u]
\end{tikzcd}$$

The diagram follows from the naturality of I with respect to the map $\bigcirc_1 \amalg \bigcirc_1 \to 8$.

We obtain similar results using the singular chain complex. Denote by $A_s = C^*(M)$ and $B_s = \mathrm{B}(A_s, A_s, A_s)$.

Lemma 4.23. *Let G be a graph as above. There is a commutative diagram where each vertical map is a quasi-isomorphism:*

$$\begin{tikzcd}
B_s^{\otimes E} \arrow[d, "\simeq"] & (A_s^{\otimes 2})^{\otimes E} \arrow[l] \arrow[r] \arrow[d, "\simeq"] & A_s^{\otimes V} \arrow[d, "\simeq"] \\
A_s^{\otimes E} & (A_s^{\otimes 2})^{\otimes E} \arrow[l] \arrow[r] & A_s^{\otimes V} \\
C^*(C^0(E \times I, M)) \arrow[u, "\simeq"] & C^*(C^0(E \times \{0,1\}, M)) \arrow[l] \arrow[r] \arrow[u, "\simeq"] & C^*(C^0(V, M)) \arrow[u, "\simeq"]
\end{tikzcd}$$

The quasi-isomorphisms $C^*(C^0(E\times I, M)) \to C^*(C^0(M))^{\otimes E}$ are a generalization of the quasi-isomorphism $C^*(PM) \to C^*(M)$.

We can construct a complex $A_s^{\otimes G} = B_s^{\otimes E} \otimes_{A_s^{\otimes 2E}} A_s^{\otimes V}$ as above. With a similar argument as in Theorem 4.21, without using Assumption 3.23, one can show the following:

Theorem 4.24. *There is a zigzag of quasi-isomorphisms which are maps of two-sided A_s-modules.*

$$A_s^{\otimes G} \xleftarrow{\simeq} \dots \xrightarrow{\simeq} C^*(C^0(G, M))$$

which is natural with respect to maps in G. In detail, this is

$$\begin{aligned} A_s^{\otimes G} &\xleftarrow{\simeq} B_s^{\otimes E} \otimes^L_{(A_s^{\otimes 2})^{\otimes E}} A_s^{\otimes V} \\ &\xrightarrow{\simeq} A_s^{\otimes E} \otimes^L_{(A_s^{\otimes 2})^{\otimes E}} A_s^{\otimes V} \\ &\xleftarrow{\simeq} C^*(C^0(E\times I, M)) \otimes^L_{C^*(C^0(E\times\{0,1\},M))} C^*(C^0(V, M)) \\ &\xrightarrow{\simeq} C^*(C^0(G, M)) \end{aligned}$$

In particular there are quasi-isomorphisms as in Corollary 4.22 using the singular cochain complex.

4.4. Concatenation and Splitting map

Concatenating and splitting up paths is not modeled by the theory of the previous subsection. We model them algebraically in the following.

The **concatenation of paths** is defined as the map

$$c\colon PM \times_m PM \to PM$$
$$c(\gamma_1, \gamma_2)(t) = \begin{cases} \gamma_1(\frac{1}{2}t) & \text{if } t \le \frac{1}{2} \\ \gamma_2(\frac{1}{2}t + \frac{1}{2}) & \text{if } t \ge \frac{1}{2} \end{cases}$$

This induces a concatenation of loops $c\colon LM \times_M LM \to LM$. We model the concatenation of paths by the **deconcatenation coproduct** on B

$$\begin{aligned} \bar{c}\colon B &\to B \otimes_A B \\ a \otimes a_1 \otimes \dots \otimes a_n \otimes a' &\mapsto \sum_{k+l=n} a \otimes a_1 \otimes \dots \otimes a_k \otimes 1 \otimes a_{k+1} \otimes \dots \otimes a_{k+l} \otimes a' \end{aligned}$$

The following lemma shows that $\bar{c}$ models the concatenation of paths c on cohomology and $\bar{c} \otimes \mathrm{id}\colon B \otimes_{A^{\otimes 2}} A \to (B \otimes B) \otimes_{A^{\otimes 2} \otimes A^{\otimes 2}} A$ models the concatenation of loops.

Lemma 4.25. *Let $c\colon PM \times_M PM \to PM$ be the concatenation of paths. The following diagrams commute:*

$$\begin{array}{ccc} B & \longrightarrow & \Omega^*(PM) \\ \downarrow{\scriptstyle \bar{c}} & & \downarrow{\scriptstyle c^*} \\ B \otimes_A B & \longrightarrow & \Omega^*(PM \times_M PM) \end{array} \tag{7}$$

$$\begin{tikzcd}
B \otimes_{A^{\otimes 2}} A \arrow[rr, "I_{\bigcirc_1}"] \arrow[d, "\bar{c}"'] & & \Omega^*(LM) \arrow[d, "c^*"'] \\
(B \otimes B) \otimes_{A^{\otimes 2} \otimes A^{\otimes 2}} A \arrow[rr, "I_8"] & & \Omega^*(LM \times_M LM)
\end{tikzcd} \tag{8}$$

Proof. This is stated in [NW19] without proof. We give a detailed proof here.

1. For Diagram 7: let $k + l = n$, consider the maps

$$d_{k,l} \colon \Delta^k \times \Delta^l \to \Delta^n$$
$$(t_1, \ldots, t_k), (t_{k+1}, \ldots, t_{k+l}) \mapsto \left(\frac{1}{2} t_1, \ldots, \frac{1}{2} t_k, \frac{1}{2} + \frac{1}{2} t_{k+1}, \ldots, \frac{1}{2} + \frac{1}{2} t_{k+l} \right)$$

so that the images of the $d_{k,l}$ cover Δ^n and are disjoint except at the boundary. Consider the following diagram with $k + l = n$.

$$\begin{tikzcd}
A^{\otimes n+2}[n] \arrow[r] \arrow[d] & \Omega^{*+n}(M^{n+2}) \arrow[r, "ev^*"] \arrow[d] & \Omega^{*+n}(PM \times \Delta^n) \arrow[r, "\int_{\Delta^n}"] \arrow[d, "(d_{k,l} \times c)^*"] & \Omega^*(PM) \arrow[d, "c^*"] \\
A^{\otimes k+l+3}[n] \arrow[r] & \Omega^{*+n}(M^{k+2} \times_M M^{l+2}) \arrow[r, "ev_{k,l}"] & \Omega^{*+n}(PM \times_M PM \times \Delta^k \times \Delta^l) \arrow[r, "\int_{\Delta^k \times \Delta^l}"] & \Omega^*(PM \times_M PM)
\end{tikzcd}$$

All squares commute except for the right, which commutes only after summing over all k, l with $k + l = n$:

$$\begin{aligned}
c^* \int_{\Delta^n} \omega &= c^* \sum_{k+l=n} \int_{\mathrm{im}(d_{k,l})} \omega \\
&= \sum_{k+l=n} \int_{\Delta^k \times \Delta^l} (c \times d_{k,l})^* \omega
\end{aligned}$$

The left hand side is the upper path of the square, the right hand side is the lower path of the square, up to commuting the factors. This concludes the proof for Diagram 7.

2. For Diagram 8: By applying the functor $- \otimes_{A^{\otimes 2}} A$ to Diagram 7, we obtain the diagram

$$\begin{tikzcd}
(B \otimes_A B) \otimes_{A^{\otimes 2}} A \arrow[r, "I_{\bigcirc_2}"] & \Omega^*(\bigcirc_2) \\
B \otimes_{A^{\otimes 2}} A \arrow[r, "I_{\bigcirc_1}"] \arrow[u] & \Omega^*(LM) \arrow[u]
\end{tikzcd}$$

The inclusion $C^0(\bigcirc_2, M) \to LM \times_M LM$ is modeled by the diagram

$$\begin{tikzcd}
(B \otimes_A B) \otimes_{A^3} A \arrow[r] & \Omega^*(LM \times_M LM) \\
(B \otimes_A B) \otimes_{A^{\otimes 2}} A \arrow[r] \arrow[u] & \Omega^*(\bigcirc_2) \arrow[u]
\end{tikzcd}$$

as in 4.21, via the map of graphs $\bigcirc_2 \to 8$ mapping the two points to the pinch point of the figure 8. Combining the two diagrams, we obtain the diagram as claimed.

□

The analogue for the simplicial cochain complex is the following

Proposition 4.26. *The following diagram commutes:*

$$\begin{tikzcd}
B_s \arrow[r, "\alpha"] \arrow[d, "\bar{c}"] & C^*(M) \arrow[d, equal] & C^*(PM) \arrow[l, "const^*"'] \arrow[d, "c^*"] \\
B_s \otimes_A B_s \arrow[r] & C^*(M) & C^*(PM \times_M PM) \arrow[l, "const^*"']
\end{tikzcd}$$

Recall that the **splitting map** is defined as

$$s \colon I \times PM \to PM \times_M PM$$
$$s(t, \gamma) = \Big(r(\gamma|_{[0,t]}), r(\gamma|_{[t,1]})\Big)$$

where r is the reparametrization map which maps a path defined on any interval to a path on the interval $[0, 1]$. We denote the induced map $I \times LM \to C^0(\bigcirc_2, M)$ by s as well.

We introduce a model of the interval $\mathcal{I} = \mathbb{R}(1-t) \oplus \mathbb{R}t \oplus \mathbb{R}dt \subseteq \Omega^*(I)$, the cochain complex of **Whitney forms on the interval**. There is a projection

$$\Omega^*(I) \to \mathcal{I}$$
$$f \mapsto f(0)(1-t) + f(1)t + \left(\int_I f\right) dt$$

which is a quasi-isomorphism with quasi-inverse given by the inclusion

$$\iota_{\mathcal{I}} \colon \mathcal{I} \subseteq \Omega^*(I).$$

Recall from Proposition 4.8 that $\alpha \colon B \to A$ is given the direct summands of B by the product for $n = 0$ and 0 for $n > 0$. Similarly we can define a map $\alpha' \colon B \otimes_A B \to A$ which is induced by the composition

$$B \otimes B \xrightarrow{\alpha \otimes \alpha} A \otimes A \to A,$$

where the second map is the product on A.

We model the splitting map s by the map

$$\begin{aligned}\bar{s}\colon B\otimes_A B &\to \mathcal{I}\otimes B\\ a_1\otimes\gamma_1\otimes a_2\otimes\gamma_2\otimes a_3 &\mapsto (1-t)\otimes\alpha(a_1\otimes\gamma_1\otimes a_2)1_A\otimes\gamma_2\otimes a_3\\ &\quad + t\otimes a_1\otimes\gamma_1\otimes 1_A\alpha(a_2\otimes\gamma_2\otimes a_3)\\ &\quad + (-1)^{|a_1\gamma_1 a_2|}dt\otimes a_1\otimes(\gamma_1\otimes a_2\otimes\gamma_2)\otimes a_3\end{aligned}$$

for $a_i\in A,\gamma_1\in A[1]^{\otimes k},\gamma_2\in A[1]^{\otimes l}$.

For spaces X,Y and forms $\omega_1\in\Omega^*(X),\omega_2\in\Omega^*(Y)$, we write $\omega_1\times\omega_2=p_1^*\omega_1\wedge p_2\omega_2\in\Omega^*(X\times Y)$, where $p_1\colon X\times Y\to X$ and $p_2\colon X\times Y\to Y$ are the projections. Similarly for $\omega_1\in C^*(X),\omega_2\in C^*(Y)$ we write $\omega_1\times\omega_2=p_1^*\omega_1\wedge p_2\omega_2\in C^*(X\times Y)$.

We first consider the singular cochain complex.

Proposition 4.27. *Let $A=C^*(M)$. The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{I}\otimes B_s \arrow[r, "\iota\times\alpha"] & C^*(I\times M) & C^*(I\times PM) \arrow[l, "(\mathrm{id}\times const)^*"'] \\
B_s\otimes_A B_s \arrow[u, "\bar{s}"] \arrow[r, "\alpha'"] & C^*(M) \arrow[u] & C^*(PM\times_M PM) \arrow[l, "const^*"] \arrow[u, "s^*"']
\end{tikzcd}$$

Thus s^* is modeled by $\bar{s}$ on the bar construction.

The following theorem is adapted from [NW19, Prop. 4.2]. It shows that the splitting map is modeled by the map $\bar{s}$.

Theorem 4.28. *Let $A=\Omega^*(M)$. The following diagrams commute in the derived category of chain complexes $D(Ch_{\mathbb{R}})$:*

$$\begin{tikzcd}
\mathcal{I}\otimes B \arrow[r, "\iota_{\mathcal{I}}\times I"] & \Omega^*(I\times PM) \\
B\otimes_A B \arrow[u, "\bar{s}"'] \arrow[r, "I"] & \Omega^*(PM\times_M PM) \arrow[u, "s^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}\otimes(B\otimes_{A^{\otimes 2}} A) \arrow[r, "\iota_{\mathcal{I}}\times I"] & \Omega^*(I\times LM) \\
(B\otimes_A B)\otimes_{A^{\otimes 2}} A \arrow[u] \arrow[r, "I"] & \Omega^*(\mathrm{Map}(\bigcirc_2, M)) \arrow[u, "s^*"']
\end{tikzcd}$$

All the horizontal maps are quasi-isomorphisms from Theorem 4.21.

Proof. The analogous version of Diagram 4.27 for the de Rham complex also commutes:

$$\begin{tikzcd}
\mathcal{I}\otimes B \arrow[r, "\iota\times\alpha"] & \Omega^*(I\times M) & \Omega^*(I\times PM) \arrow[l, "(\mathrm{id}\times const)^*"'] \\
B\otimes_A B \arrow[u, "\bar{s}"] \arrow[r, "\alpha'"] & \Omega^*(M) \arrow[u] & \Omega^*(PM\times_M PM) \arrow[l, "const^*"] \arrow[u, "s^*"']
\end{tikzcd}$$

Since the vertical composition is given by I in the derived category, this shows the first part.

The second diagram follows from the first diagram, by a similar argument as in the proof of Lemma 4.25. $\square$

Remark 4.29. In [NW19] it is stated without proof that one can obtain a chain level commutative diagram

$$\begin{tikzcd} \mathcal{I} \otimes B \arrow[r] & \mathcal{I} \otimes \Omega^*(PM) \\ B \otimes_A B \arrow[u, "\bar{s}"] \arrow[r, "I"] & \Omega^*(PM \times_M PM) \arrow[u, "s^*"'] \end{tikzcd}$$

We caution that if one attempts a similar proof as in Lemma 4.25, one is lead to taking the umkehr map of the map

$$\begin{aligned} r\colon \Delta^k \times I \times \Delta^l &\to \Delta^{n+1} \\ (s_1, \ldots, s_k), t, (t_1, \ldots, t_k) &\mapsto (ts_1, \ldots, ts_k, t, t + (1-t)t_1, \ldots, t + (1-t)t_l). \end{aligned}$$

for k and l with $k + l = n$. This map has finite fiber outside the boundary, hence its fiber integral is almost everywhere given by a finite sum. However we are restricted to differential forms which are defined everywhere, including the boundary. The statement on derived chain complexes is good enough for our purposes.

Remark 4.30. In [NW19] it is also stated that the wedge product on $\Omega^*(PM)$ corresponds to the shuffle product $\sqcup\!\sqcup$ on $\mathrm{B}(\Omega^*(M), \Omega^*(M), \Omega^*(M))$:

$$\begin{tikzcd} B \otimes B \arrow[r, "I \otimes I"] \arrow[d, "\sqcup\!\sqcup"] & \Omega^*(PM) \otimes \Omega^*(PM) \arrow[d, "\wedge"] \\ B \arrow[r, "I"] & \Omega^*(PM) \end{tikzcd}$$

This is shown in [Che73, 4.1].

5. String Topology Operations via Configuration Spaces

In this section we introduce the construction of the loop product and coproduct from [NW19]. Classically, the intersection maps in these operations are constructed via tubular neighborhoods and the Thom isomorphism. Naef and Willacher in [NW19] instead construct the intersection map using the compactified configuration space of two points $\overline{\mathrm{Conf}}_M(2)$. The main observation that allows this is that the homotopy cofiber of the map $\overline{\mathrm{Conf}}_M(2) \to M \times M$ is a version of the Thom space of M, hence one may factor the Thom isomorphism through it. We begin by recalling the compactification of configuration spaces over a manifold M in Section 5.1. In Section 5.2 we construct the intersection product on a closed manifold M using configuration spaces. We then give an analogous construction of the loop product in Section 5.3. We will show in Section 6.1 that on homology this construction is equal to the classical construction 2.2.

5.1. Fulton-MacPherson Compactification of Configuration Spaces

We begin by introducing the compactification of the configuration spaces. We follow [Sin04].

Definition 5.1. The ordered configuration space of n points in a space M is defined as

$$\mathrm{Conf}_M(n) = \{(x_1, \dots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

Let M be a closed manifold. We fix an embedding $M \hookrightarrow \mathbb{R}^D$. We construct the Fulton-MacPherson compactification of $\mathrm{Conf}_M(n)$ as follows:

Let $\underline{n} = \{1, \dots, n\}$. For $(i, j) \in \mathrm{Conf}_{\underline{n}}(2)$, let

$$\pi_{ij} \colon \mathrm{Conf}_M(n) \to S^{D-1}$$

be the map which sends a tuple $(x_1, \dots, x_n)$ to the unit vector in the direction $x_i - x_j$. For $(i, j, k) \in \mathrm{Conf}_{\underline{n}}(3)$, let

$$s_{ijk} \colon \mathrm{Conf}_M(n) \to (0, \infty), \qquad s_{ijk}(x) = |x_i - x_j| \,/\, |x_i - x_k| \,.$$

Let $\iota \colon \mathrm{Conf}_M(n) \to M^n$ be the embedding. Define the ambient space of the Fulton-MacPherson compactification

$$A_M(n) = M^n \times (S^{D-1})^{\mathrm{Conf}_{\underline{n}}(2)} \times [0, \infty]^{\mathrm{Conf}_{\underline{n}}(3)}.$$

We embed the configuration space into this ambient space by

$$\begin{aligned} &\alpha_{M,n} \colon \mathrm{Conf}_M(n) \to A_M(n) \\ &\alpha_{M,n} = \iota \times \prod_{(i,j) \in \mathrm{Conf}_{\underline{n}(2)}} \pi_{ij} \times \prod_{(i,j,k) \in \mathrm{Conf}_{\underline{n}}(3)} s_{ijk} \end{aligned}$$

Definition 5.2. The **Fulton-MacPherson-Axelrod-Singer compactification** of the ordered configuration space of n points in M is the closure of the image of the above map: $\overline{\mathrm{Conf}}_M(n) = cl_{A_M(n)}(im(\alpha_{M,n}))$.

In [Sin04] it is shown that if M is a closed manifold, then $\overline{\mathrm{Conf}}_M(n))$ is a compact manifold with corners, whose top stratum is homeomorphic to $\mathrm{Conf}_M(n)$. In particular, the compactification is homotopy equivalent to the original configuration space.

Remark 5.3. One can restore the original configuration of points in the configuration space from the image of the $\pi_{i,j}$ and $s_{i,j,k}$, up to translation and rotation in $\mathbb{R}^D$.

The additional elements in the closure correspond to the situation where parts of the tuple become “infinitesimally close” as seen from some of the other points, but still have well defined directions with respect to each other. For example in $\overline{\mathrm{Conf}}_M(3)$ we may have $x \in \overline{\mathrm{Conf}}_M(3)$ with $\pi_{1,2}(x) = \pi_{1,3}(x)$ and $s_{1,2,3}(x) = 1$, so if we try to restore the original configuration of points from this, the points x_2 and x_3 are equal. However as an element of the compactification this still has a well-defined $\pi_{2,3}(x)$, i.e. a well defined direction between x_2 and x_3. The relation of being infinitesimally close can be modeled via trees with n leaves and there exists a stratification on $\overline{\mathrm{Conf}}_M(n)$ where each stratum corresponds to a tree with n leaves, see [Sin04].

Example 5.4. Since we are largely interested in the case $n = 2$, we give an explicit description of $\overline{\mathrm{Conf}}_M(2)$. In the $n = 2$ case, the compactification $\overline{\mathrm{Conf}}_M(2)$ is a manifold with boundary. The interior is $\mathrm{Conf}_M(2)$. The boundary is homeomorphic to the unit tangent bundle UTM of M and we denote it by $UTM \subseteq \overline{\mathrm{Conf}}_M(2)$. Recall that we fixed an embedding $M \to \mathbb{R}^D$ and hence there is an embedding of the tangent spaces into $T_xM \to \mathbb{R}^D$ for each $x \in M$. A sequence (x'_n, x''_n) in $\mathrm{Conf}_M(2)$ converges to $X \in UT_xM$ if (x'_n) and (x''_n) converge to x and $x'_n - x''_n \in \mathbb{R}^D/\mathbb{R}_+$ converges to $X \in UT_xM \subseteq \mathbb{R}^D/\mathbb{R}_+$.

5.2. Intersection Map in M via Configuration Spaces

Let M be a closed oriented manifold. We consider a construction of the intersection map on the cohomology of a closed oriented manifold M as proposed by [NW19]. The intersection product is a map $H_*(M) \otimes H_*(M) \to H_{*-n}(M)$, or in cohomology, a map $H^*(M) \otimes H^*(M) \leftarrow H^{*-n}(M)$. For a review of the intersection product, see section A.2. We heavily make use of homotopy pushouts and homotopy cofibers in this subsection, for a review see Appendix C.

Recall that the homotopy cofiber of a map $f\colon X \to Y$ of spaces is constructed using the mapping cone $\mathrm{cone}(X \to Y) = X \times I \amalg Y/\sim$ where $\sim$ is generated by $(x, 0) \sim (x', 0)$ for all $x, x \in X$ and $(x, 1) \sim f(x)$.

Remark 5.5. Recall that the mapping cone $\mathrm{cone}(f)$ of a map $f\colon A \to B$ of cochain complexes[2] is the graded vector space $A \times B[1]$ with the differential $d(a, b) = (d_A a, f(a) - d_B(b))$. The sequence $A \to B \to \mathrm{cone}(f) \to A[1]$ induces a long exact sequence in cohomology. There is a natural chain map $\mathrm{cone}(f) \to B/f(A)[1]$ given by $(a, b) \mapsto b$ and another natural chain map $\ker(f) \to \mathrm{cone}(f)$ given by $a \mapsto (a, 0)$.

For an inclusion between spaces $f\colon X \hookrightarrow Y$, the map $f^*\colon C^*(Y) \to C^*(X)$ is surjective and the natural map $\ker f^* = C^*(Y, X) \to \mathrm{cone}(f^*)$ is a quasi-isomorphism. If f is a general continuous map, then f^* is in general not surjective and there is no long exact sequence in cohomology for $\ker f^*$.

Definition 5.6. For a continuous map $f\colon X \to Y$, we denote

$$C^*(Y, X) \coloneqq \mathrm{cone}(C^*(Y) \xrightarrow{\simeq} C^*(X)).$$

Remark 5.7. We also recall in Proposition C.12 that for a map between spaces $f\colon X \to Y$, there exists a zigzag of quasi-isomorphisms between the mapping cone of the chain complexes and the reduced chain complex of the mapping cone of spaces:

$$\tilde{C}_*(\mathrm{cofib}(f\colon X \to Y)) \xrightarrow{\simeq} \dots \xleftarrow{\simeq} \mathrm{cone}(C_*(X) \to C_*(Y))$$

We need to construct the following maps to obtain the intersection map:

[2] There is a distinction between the mapping cone of chain complexes and of cochain complexes; the grading is slightly different and the differential differs by signs. See Proposition C.12 for the definition of the mapping cone of chain complexes. In some references the mapping cone of cochain complexes is called the mapping cocone. We write cone for both.

(1) A quasi-isomorphism $C^*(M \times M, \overline{\mathrm{Conf}}_M(2)) \simeq C^*(M, UTM) = \mathrm{cone}(C^*(M) \to C^*(UTM))$

(2) A Thom map $C^{*-n}(M) \to C^*(M, UTM)$.

We begin with the quasi-isomorphism $C^*(M \times M, \overline{\mathrm{Conf}}_M(2)) \simeq C^*(M, UTM)$

(1) Recall that $\overline{\mathrm{Conf}}_M(2)$ is the compactified configuration space of two ordered points in M and by Example 5.4, this is a manifold with boundary with top stratum homeomorphic to $\mathrm{Conf}_M(2)$ and a codimension 1 stratum homeomorphic to the unit sphere bundle $UTM \subseteq TM$. The square

$$\begin{tikzcd} UTM \arrow[r] \arrow[d] & \overline{\mathrm{Conf}}_M(2) \arrow[d] \\ M \arrow[r, "\Delta"] & M \times M \end{tikzcd} \tag{9}$$

is a pushout diagram of topological spaces. In fact, the inclusion $UTM \hookrightarrow \overline{\mathrm{Conf}}_M(2)$ is a cofibration by Theorem C.7 and thus the diagram is a homotopy pushout diagram by Theorem C.8. Thus the homotopy cofibers of the vertical maps are homotopy equivalent due to the pasting law for homotopy pushouts Theorem C.10.

Lemma 5.8. *The horizontal maps in Diagram 9 induce a homotopy equivalence*

$$\mathrm{cofib}(UTM \to M) \xrightarrow{\simeq} \mathrm{cofib}(\overline{\mathrm{Conf}}_M(2)) \to M \times M).$$

The following map is a quasi-isomorphism

$$C^*(M \times M, \overline{\mathrm{Conf}}_M(2)) \xrightarrow{\simeq} C^*(M, UTM).$$

Proof. The first part is shown above. The map between the mapping cones is a quasi-isomorphism due to the zigzag from Remark 5.7. □

The cofibers are in fact homotopy equivalent to the Thom space DTM/UTM, where DTM is the unit disk bundle in TM: the inclusion $UTM \to DTM$ is a cofibration, so the homotopy cofiber is given by the quotient in this case.

(2) Since the maps $UTM \to M$ and $UTM \to DTM$ are homotopy equivalent, there is another homotopy equivalence

$$\mathrm{cofib}(UTM \to M) \simeq \mathrm{cofib}(UTM \to DTM)$$

The Thom isomorphism (cf. Appendix A.3) of the oriented vector bundle TM is given on the chain level by the composition

$$C^*(M) \to C^*(DTM) \to C^{*+n}(DTM, UTM).$$

The second map is given by taking the cup product with a representative $u \in C^n(DTM, UTM)$ of the Thom class, which yields a map $- \smile u\colon C^*(DTM) \to C^{*+n}(DTM, UTM)$. We denote by $u|_{DTM} \in C^n(DTM)$ and $u|_M \in C^n(M)$ the image of $u \in C^n(DTM, UTM)$ under the maps $C^n(DTM, UTM) \to C^n(DTM) \to C^n(M)$ (the second map is induced by the zero section).

Construction 5.9. The map $Th\colon C^*(M) \to C^{*+n}(M, UTM)$ is given by $c \mapsto (c \smile u|_M, 0)$

Since u vanishes in $C^*(UTM)$, the pullback of $u|_M$ along the map $UTM \to M$ vanishes. This is required to show that Th is a chain map.

The map Th fits into the following commutative diagram, which shows that Th is in fact related to the Thom isomorphism.

$$\begin{tikzcd}
 & C^{*+n}(DTM, UTM) & \\
C^*(DTM) \arrow[r] \arrow[ur, "- \smile u"] \arrow[d] & \mathrm{cone}(C^*(DTM) \to C^*(UTM))[n] \arrow[u] \arrow[d] & \\
C^*(M) \arrow[r, "Th"] & \mathrm{cone}(C^*(M) \to C^*(UTM))[n] \arrow[r, equal] & C^{*+n}(M, UTM)
\end{tikzcd} \tag{10}$$

The upper triangle states that we can lift the Thom map $- \smile u$ along the quasi-isomorphism $\mathrm{cone}(C^*(DTM) \to C^*(UTM)) \to C^*(DTM, UTM)$: The map

$$C^*(DTM) \to \mathrm{cone}(C^*(DTM) \to C^*(UTM))[n]$$

is given by $c \mapsto (c \smile u|_{DTM}, 0)$, this is a chain map since $u|_{DTM} \in C^*(DTM)$ vanishes in $C^*(UTM)$.

Remark 5.10. One can check that choosing $u \in C^*(DTM, UTM)$ to be a different representative of the Thom class yields a homotopic map: if $u - u' = d(v)$ then a homotopy is given by $c \mapsto (c \smile v, 0)$. Thus Th induces a well defined map in cohomology.

The class $u|_M$ represents the Euler class in $H^n(M)$. However $u|_M$ cannot be chosen arbitrarily inside its cohomology class and it does not vanish even for manifolds with vanishing Euler class since Th is a quasi-isomorphism.

Definition 5.11. The construction of the intersection coproduct according to [NW19] is the map of derived chain complexes $C^{*-n}(M) \to C^*(M) \otimes C^*(M)$, given as the composition of the sequence of maps in the derived category $D(Ch(k))$, read from right to left[3].

$$\begin{tikzcd}
C^*(M)\otimes C^*(M)\simeq C^*(M\times M) & C^*(M\times M,\overline{\mathrm{Conf}}_M(2)) \arrow[l] \arrow[d, "\cong"] & \\
& C^*(M,UTM) & C^{*-n}(M) \arrow[l, "Th"']
\end{tikzcd}$$

Since we are working with field coefficients, this induces a map

$$H^{*-n}(M)\to H^*(M)\otimes H^*(M).$$

The following statement is implicit in [NW19], but not explicitly stated and proven.

Theorem 5.12. *The intersection maps $i^!\colon C^{*-n}(M)\to C^*(M)\otimes C^*(M)$ defined as above and as in Construction A.13 are equal, as maps in $D(Ch(K))$.*

Proof. Fix a tubular neighborhood of the diagonal $\Delta M\subseteq M\times M$. Since the normal bundle in this setting is isomorphic to the tangent bundle, this means that there is a map $\eta\colon DTM\to M\times M$ which is a homeomorphism onto its image, taking the zero section to the diagonal and DT_0M to the complement of the diagonal. Consider the following diagram:

$$\begin{tikzcd}
& \overline{\mathrm{Conf}}_M(2) \arrow[rr, hook] \arrow[dd] & & UTM \arrow[dl] \arrow[dd] \\
\mathrm{Conf}_M(2) \arrow[ur] \arrow[dd] & & D_0TM \arrow[ll, "\eta"'] \arrow[dd] & \\
& M\times M \arrow[dl, equal] & & M \arrow[ll] \arrow[dl] \\
M\times M & & DTM \arrow[ll] &
\end{tikzcd}$$

Each face of the diagram commutes up to homotopy. This is easy to see for all except the top face. In fact all except the right and top face commute as maps of sets. We will show that the top face commutes up to homotopy in Lemma 5.13 after this proof.

The cube shaped diagram induces a diagram of homotopy cofibers

$$\begin{tikzcd}
\mathrm{cofib}\left(\overline{\mathrm{Conf}}_M(2)\to M\times M\right) & \mathrm{cofib}\left(UTM\to M\right) \arrow[l] \arrow[d] \\
\mathrm{cofib}\left(\mathrm{Conf}_M(2)\to M\times M\right) \arrow[u] & \mathrm{cofib}\left(D_0TM\to DTM\right) \arrow[l]
\end{tikzcd} \tag{11}$$

which also commutes up to homotopy and all maps involved are quasi-isomorphisms: the top map is a homotopy equivalence by Lemma 5.8, the vertical maps are homotopy

[3] Recall that following [NW19] we occasionally write maps in cohomology from right to left, to underline that e.g. a homological product becomes a cohomological coproduct.

equivalences because they are induced by homotopy equivalences and as a consequence the bottom map is also a quasi-isomorphism.

Thus we have a commutative diagram

$$\begin{tikzcd}
H^*(M \times M) & H^*(M \times M, \overline{\mathrm{Conf}}_M(2)) \arrow[l] \arrow[r, "\simeq"] \arrow[d, "\simeq"] & H^*(M, UTM) & \\
& H^*(M \times M, \mathrm{Conf}_M(2)) \arrow[ul] \arrow[r, "\simeq", "\eta^*"'] & H^*(DTM, D_0TM) \arrow[u, "\simeq"] & H^{*-n}(M) \arrow[l] \arrow[ul, "Th"']
\end{tikzcd}$$

The center square is justified by the above. The triangle on the right is a more concise version of Diagram 10.

The upper composition is the intersection map constructed in this subsection. The lower composition is the intersection map constructed via tubular neighborhoods in Construction A.13 (note $\mathrm{Conf}_M(2) = M \times M \setminus \Delta(M)$). □

From the previous proof, it remains to show that the top face of the cube commutes up to homotopy.

Lemma 5.13. *The following diagram commutes up to homotopy:*

$$\begin{tikzcd}
\overline{\mathrm{Conf}}_M(2) & UTM \arrow[l] \arrow[d] \\
\mathrm{Conf}_M(2) \arrow[u] & D_0TM \arrow[l]
\end{tikzcd}$$

Proof. Recall that the map $D_0TM \to \mathrm{Conf}_M(2)$ is given by the composition $TM \xrightarrow{\cong} \nu \to M \times M$ where ν is the normal bundle of $\Delta(M) \subseteq M \times M$ and the map $\nu \to M \times M$ is given by a tubular neighborhood of the diagonal.

We make a particular choice of this data, such that we can control the differential near the zero section. We choose a Riemannian metric g on M, which induces a Riemannian metric on $M \times M$, and let $\nu = \nu(\Delta(M))$ be the normal bundle of $\Delta(M) \subseteq M \times M$ with respect to this metric. There is an open subset $U \subseteq \nu$ which contains the zero section, such that the exponential map $\exp\colon U \to M \times M$ of $M \times M$ is a diffeomorphism onto its image (cf. [Lee06, Ex. 8-5]). We may assume that the tubular neighborhood of the diagonal is given by U and the exponential map, since the intersection map in Construction A.13 is independent of the choice of the tubular neighborhood.

There is an isomorphism of vector bundles $TM \cong \nu$ given by $X \mapsto (X, -X)$. The topological sphere and disk bundles UTM and DTM are chosen independent of g, hence we may embed them into TM so that they correspond to subsets of U under the isomorphism $TM \cong \nu$. We claim that the map

$$H\colon (0,1] \times UTM \to M \times M \setminus \Delta(M) \subseteq \overline{\mathrm{Conf}}_M(2), \qquad (t, X) \mapsto (\exp(tX), \exp(-X))$$

extends at $t = 0$ continuously via the identity $UTM \to UTM \subseteq \overline{\mathrm{Conf}}_M(2)$ and hence yields a homotopy between the maps of the top face. Note that for $t = 1$ this is the three-fold composition in the top face. At $t = 0$, we have to show that for a sequence $X_n \to X \in UTM$, $t_n \to 0$ the sequence $H(t_n, X_n)$ converges to $X \in UTM \subseteq \overline{\mathrm{Conf}}_M(2)$. The base points of $H(t_n, X_n) = (\exp(t_n X_n), \exp(-t_n X_n))$ converge to the base point of X in $\Delta(M)$. We write $H(t_n, X_n) = (H_1(t_n, X_n), H_2(t_n, X_n))$. Recall that we have chosen an embedding $M \subseteq \mathbb{R}^d$ to define the compactification $\overline{\mathrm{Conf}}_M(2)$. One can show that

$$\lim_{n\to\infty} \frac{1}{2t_n}(H_1(t_n, X_n) - H_2(t_n, X_n)) = \left.\frac{d}{dt}\right|_{t=0} \exp(tX) = X$$

hence $H_1(t_n, X_n) - H_2(t_n, X_n)$ converges in $\mathbb{R}^d/\mathbb{R}_+$ to X. By Example 5.4 this shows that $H(t_n, X_n)$ converges to X in $\overline{\mathrm{Conf}}_M(2)$. Thus H is a homotopy as required. $\square$

5.3. Loop Product (Cohomology Coproduct) using Configuration Spaces

Let M be a closed oriented manifold. Recall from Construction 2.6 that the loop product $H^*(LM) \otimes H^*(LM) \leftarrow H^{*-n}(LM)$ is a composition of three maps, one of which is the intersection map $i^!\colon H^*(LM \times_M LM) \to H^{*+n}(LM \times LM)$. Naef and Willacher propose in [NW19] an alternative construction of this intersection map, which we outline here.

The construction of this intersection map proceeds analogous to the construction in the previous subsection, "pulled back along the map $LM \times LM \to M \times M$". Recall that the notation $C^*(X, Y)$ for a map $f\colon X \to Y$ denotes the complex of the mapping cone of f^*, as in Definition 5.6. We will construct two maps.

(1) We will construct a lift of the quasi-isomorphism $C^*(M \times M, \overline{\mathrm{Conf}}_M(2)) \xrightarrow{\simeq} C^*(M, UTM)$ from Lemma 5.8, along the map $LM \times LM \to M \times M$. This will be a map $C^*(LM \times LM, LM \times LM|_{\overline{\mathrm{Conf}}_M(2)}) \xrightarrow{\simeq} C^*(LM \times LM|_M, LM \times LM|_{UTM})$, but we have yet to explain the spaces involved.

(2) We will construct a lift of the Thom map $C^{*-n}(M) \to C^*(M, UTM)$, this will be a map $C^{*-n}(LM \times LM|_M) \to C^*(LM \times LM|_M, LM \times LM|_{UTM})$.

We will then compose these maps to obtain the intersection map $i^!$.

(1) The free loop space LM is the pullback of the path space fibration $PM \to M \times M$ and the diagonal map $M \to M \times M$. As the first map is a fibration, this is in fact a homotopy pullback, cf. Theorem C.8.

More generally, consider a fibration $E \to M \times M$. We can pull back the diagram

$$\begin{array}{ccc} UTM & \longrightarrow & \overline{\mathrm{Conf}}_M(2) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\Delta} & M \times M \end{array} \tag{12}$$

along the fibration $E \to M \times M$ and obtain the diagram

$$\begin{array}{ccc} E|_{UTM} & \longrightarrow & E|_{\overline{\mathrm{Conf}}_M(2)} \\ \downarrow & & \downarrow \\ E|_M & \xrightarrow{\Delta} & E \end{array} \tag{13}$$

Proposition 5.14. *Diagram 13 is a homotopy pushout diagram of E.*

Proof. Recall that diagram 12 is a homotopy pushout diagram. The two diagrams assemble into a cube where the top face is diagram 13, the bottom face is diagram 12:

$$\begin{tikzcd}
 & E|_{UTM} \arrow[rr] \arrow[dl] \arrow[dd] & & E|_{\overline{\mathrm{Conf}}_M(2)} \arrow[dl] \arrow[dd] \\
E|_M \arrow[rr] \arrow[dd] & & E \arrow[dd] & \\
 & UTM \arrow[rr] \arrow[dl] & & \overline{\mathrm{Conf}}_M(2) \arrow[dl] \\
M \arrow[rr] & & M \times M &
\end{tikzcd} \tag{14}$$

The bottom face is a homotopy pushout diagram of $M \times M$. The vertical maps are pullbacks of fibrations, hence fibrations. Thus the sides of the cube are homotopy pullback diagrams. By Mather's cube theorem Theorem C.14, the top face Diagram 13 is homotopy pushout too. □

By applying Proposition 5.14 to the fibration $LM \times LM \to M \times M$, we obtain the following homotopy pushout diagram:

$$\begin{tikzcd}
LM \times LM|_{UTM} \arrow[r] \arrow[d] & LM \times LM|_{\overline{\mathrm{Conf}}_M(2)} \arrow[d] \\
LM \times LM|_M \arrow[r] & LM \times LM
\end{tikzcd} \tag{15}$$

where each term is the fiber product of $LM \times LM$ with the corresponding term in diagram 12 over $M \times M$. In detail, the spaces in the diagram can be described as follows:

- $LM \times LM|_M$ is the space of figure eights in M, i.e. two loops starting at the same point in M.
- $LM \times LM|_{UTM} = (LM \times LM) \times_{M \times M} UTM$ is the space of figure eights in M, together with a unit tangent vector at the node of the eight.
- $LM \times LM|_{\overline{\mathrm{Conf}}_M(2)}$ is a space which contains pairs of loops which need not necessarily intersect, but if they intersect at $t = 0$, then there is additionally have a unit tangent vector at the node.

Since the diagram is a homotopy pushout, the homotopy cofibers of the vertical maps are homotopy equivalent by Theorem C.10.

Definition 5.15. For brevity, we will use quotient notation for homotopy cofibers: For a map $f\colon A \to B$, denote

$$B/A = \frac{B}{A} \coloneqq \mathrm{cofib}(f\colon A \to B) = \mathrm{cone}(f\colon A \to B)$$

We have shown the following lemma:

Lemma 5.16. *There is a homotopy equivalence of cofibers:*

$$\frac{LM \times LM|_M}{LM \times LM|_{UTM}} \simeq \frac{LM \times LM}{LM \times LM|_{\overline{\mathrm{Conf}}_M(2)}}$$

This induces an isomorphism in $D(Ch(k))$ (using the notation from Definition 5.6):

$$C^*(LM \times LM|_M, LM \times LM|_{UTM}) \xrightarrow{\simeq} C^*(LM \times LM, LM \times LM|_{\overline{\mathrm{Conf}}_M(2)})$$

(2) In analogy to Construction 5.9, we construct a Thom map

$$Th_{LM} = Th\colon C^*(LM \times LM|_M) \to C^{*+n}(LM \times LM|_M, LM \times LM|_{UTM}),$$

working on the total spaces instead of the base spaces of the fibrations. Denote by $LM \times LM|_{DTM}$ the pullback of $LM \times LM \to M \times M$ along the composition of maps $DTM \to M \xrightarrow{\Delta} M \times M$. Let $u \in C^n(DTM, UTM)$ be a representative of the Thom class of M and denote its pullback through the map of pairs $(LM \times LM|_{DTM}, LM \times LM|_{UTM}) \to (DTM, UTM)$ by $u_{LM} \in C^n(LM \times LM|_{DTM}, LM \times LM|_{UTM})$. Further let $u_{LM}|_{DTM} \in C^n(LM \times LM|_{DTM})$ and $u_{LM}|_M \in C^n(LM \times LM|_M)$ be the images of u_{LM} along the maps $C^*(LM \times LM|_{DTM}, LM \times LM|_{UTM}) \to C^*(LM \times LM|_{DTM}) \to C^*(LM \times LM|_M)$ (the second map is induced by the zero section $M \to DTM$).

Construction 5.17. The map

$$Th_{LM} = Th\colon C^*(LM \times LM|_M) \to C^{*+n}(LM \times LM|_M, LM \times LM|_{UTM})$$

is given by $c \mapsto (c \smile u_{LM}|_M, 0)$.

The following lemma makes precise the statement that the map Th_{LM} is a lift of the corresponding map Th on M.

Lemma 5.18. *If M is simply connected, the following maps are quasi-isomorphisms*

$$C^*(LM \times LM) \otimes^L_{C^*(M \times M)} C^*(M) \xrightarrow{\simeq} C^*(LM \times LM|_M)$$
$$C^*(LM \times LM) \otimes^L_{C^*(M \times M)} C^*(UTM) \xrightarrow{\simeq} C^*(LM \times LM|_{UTM})$$

and the following diagram commutes in $D(Ch)$:

$$\begin{tikzcd}
C^*(LM \times LM|_M)[-n] \arrow[r, "Th_{LM}"] & C^*(LM \times LM|_M, LM \times LM|_{UTM}) \\
C^*(LM \times LM) \otimes^L_{C^*(M\times M)} C^*(M)[-n] \arrow[u, "\simeq"] \arrow[r, "\mathrm{id}\otimes Th"] & C^*(LM \times LM) \otimes^L_{C^*(M\times M)} C^*(M, M, UT)) \arrow[u, "\simeq"]
\end{tikzcd} \tag{16}$$

Proof. Since M is simply connected, so is UTM. Thus the maps are quasi-isomorphisms by Theorem 4.13.

The left vertical map factors through the underived tensor product $C^*(LM \times LM) \otimes_{C^*(M\times M)} C^*(M)[-n]$ and similarly for the right map. Thus it suffices to consider Diagram 16 with the left derived tensor product $\otimes^L$ replaced with the regular tensor product. We now chase an element $a \otimes b$ with $a \in C^*(LM \times LM)$ and $b \in C^*(M)$ through the diagram. The path through the upper left maps this to $(i^*(a) \smile ev^*(b) \smile u_{LM}|_M, 0)$ where $i\colon LM \times LM|_M \to LM \times LM$ is the inclusion and $ev\colon LM \times_M LM \to M$ is the evaluation map. The path through the lower right maps $a \otimes b$ to $(i^*(a) \smile (ev^*(b \smile u_M)), 0)$ and by construction, $ev^*(u_M) = u_{LM}|_M$. $\square$

Construction 5.19. We define the intersection map

$$i^!_{LM\times_M LM}\colon C^{*-n}(LM \times LM|_M) \to C^*(LM \times LM)$$

of the inclusion $i_{LM\times_M LM}\colon LM \times_M LM \to LM \times LM$ as the following composition read right to left:

$$\begin{tikzcd}
C^*(LM \times LM) & C^*(LM \times LM, LM \times LM|_{\overline{\mathrm{Conf}}_M(2)}) \arrow[l] \arrow[d, "\cong"] & \\
& C^*(LM \times LM|_M, LM \times LM|_{UTM}) & C^{*-n}(LM \times LM|_M) \arrow[l, "Th"']
\end{tikzcd} \tag{17}$$

The **loop product** is now the following map in $D(Ch(k))$:

$$C^*(LM) \otimes C^*(LM) \leftarrow C^*(LM \times LM) \xleftarrow{i^!} C^{*-n}(LM \times LM|_M) \xleftarrow{c^*} C^{*-n}(LM),$$

here the last map is induced by the concatenation map $c\colon LM \times LM|_M \to LM$.

The following theorem states that the intersection map $i^!_{LM}$ is a lift of the intersection map $i^!$ on M.

Theorem 5.20. *If M is simply connected, then the following diagram commutes in $D(Ch)$ and the vertical maps are quasi-isomorphisms:*

$$\begin{tikzcd}
C^*(LM \times LM) & & C^*(LM \times_M LM)[-n] \arrow[ll, "i^!_{LM}"'] \\
C^*(LM \times LM) \otimes^L_{C^*(M\times M)} C^*(M \times M) \arrow[u, "\simeq"'] & & C^*(LM \times LM) \otimes^L_{C^*(M\times M)} C^*(M)[-n] \arrow[ll, "\mathrm{id} \otimes i^!"'] \arrow[u, "\simeq"']
\end{tikzcd}$$

Proof. The diagram is commutative since we have closely mirrored the construction of the intersection product on M. We have already shown that the Thom maps on LM and M correspond to each other in Lemma 5.18. Similarly one considers the other two maps in Diagram 17, we omit the details here. □

With a similar argument one can show Theorem 5.20 for the classical construction of the intersection map $i^!_{LM \times LM}$ if M is simply connected. We omit the details here.

Corollary 5.21. *Let M be a closed oriented, simply connected manifold. The two constructions of the derived intersection map $i^!_{LM} \colon C^*(LM \times LM) \leftarrow C^*(LM \times_M LM)[-n]$ from Section 2.2 and Construction 5.19 are equal as maps in $D(Ch(\mathbb{R}))$. In particular they are the same on cohomology. Consequently, the same holds for the different definitions of the loop product (cohomology coproduct).*

5.4. Loop Coproduct (Cohomological Product)

We only sketch the construction of the loop coproduct according to [NW19, Ch. 3.3], the reader may seek details there. For brevity, we work with cohomology instead of derived chain complexes here. Again, only the construction of the intersection map differs from the construction in Section 2.3. Let M be a closed oriented manifold.

Recall that $\bigcirc_2$ is the space which consists of two copies of the unit interval I, glued together at the end so that one obtains a circle with two marked points. The space $C^0(\bigcirc_2, M)$ is the mapping space from $\bigcirc_2$ to M. Similarly we denote by 8 the space of two circles which are identified at a point and $C^0(8, M)$ the mapping space into M. The intersection map will be a map

$$j^! \colon H^{*+n}(C^0(\bigcirc_2, M), LM \amalg_M LM) \leftarrow H^*(C^0(8, M), LM \amalg_M LM)$$

which is the umkehr map of the inclusion $LM \times_M LM \to C^0(\bigcirc_2, M)$.

We apply 5.14 to the fibration $C^0(\bigcirc_2, M) \to M \times M$. This means that we consider the following cubical diagram where the maps on the top are pullbacks of the bottom face along $C^0(\bigcirc_2, M) \to M \times M$. The vertical maps are fibrations and since the bottom face is a homotopy pushout, so is the top face.

$$
\begin{tikzcd}
& C^0(\bigcirc_2, M)|_{UTM} \arrow[rr] \arrow[dl] \arrow[dd] & & C^0(\bigcirc_2, M)|_{\overline{\mathrm{Conf}}_M(2)} \arrow[dl] \arrow[dd] \\
C^0(\bigcirc_2, M)|_M \arrow[rr] \arrow[dd] & & C^0(\bigcirc_2, M) \arrow[dd] & \\
& UTM \arrow[rr] \arrow[dl] & & \overline{\mathrm{Conf}}_M(2) \arrow[dl] \\
M \arrow[rr] & & M \times M &
\end{tikzcd}
\tag{18}
$$

In the notation of [NW19], the top face is the diagram:

$$
\begin{tikzcd}
\mathrm{Map}'(8) \arrow[r] \arrow[d] & \mathrm{Map}'(\bigcirc_2) \arrow[d] \\
\mathrm{Map}(8) \arrow[r] & \mathrm{Map}(\bigcirc_2)
\end{tikzcd}
\tag{19}
$$

- where $\mathrm{Map}(8) = \mathrm{Map}(\bigcirc_2, M)|_M$ consists of figure eight loops in M,
- $\mathrm{Map}(\bigcirc_2)$ consists of loops with an additional marked point
- $\mathrm{Map}'(8) = \mathrm{Map}(\bigcirc_2, M)|_{UTM}$ consists of figure eights in M with a unit tangent vector at the node of the figure eight.

Again, there is a homotopy equivalence of cofibers

$$\mathrm{Map}(8)/\,\mathrm{Map}'(8) \simeq \mathrm{Map}(\bigcirc_2)/\,\mathrm{Map}'(\bigcirc_2)).$$

Denote $F = LM \amalg_M LM$, which is a space with a fibration $F \to M$, and let $F|_{UTM}$ be the pullback of F along the map $UTM \to M$. There is a homotopy equivalence of cofibers of cofibers

$$\frac{(\mathrm{Map}(\bigcirc_2)/\,\mathrm{Map}'(\bigcirc_2))}{(F/F|_{UTM})} \simeq \frac{(\mathrm{Map}(8)/\,\mathrm{Map}'(8))}{(F/F|_{UTM})}$$

where the fraction notation and / notation denote the homotopy cofiber.

Now we define $j^!$ to be the following composition, read right to left

$$
\begin{aligned}
j^! : \qquad & H^*(\mathrm{Map}(\bigcirc_2), F) \leftarrow H^*\left(\tfrac{\mathrm{Map}(\bigcirc_2)/\,\mathrm{Map}'(\bigcirc_2)}{F/F|_{UTM}}\right) \\
& \xleftarrow{\simeq} H^*\left(\tfrac{\mathrm{Map}(8)/\,\mathrm{Map}'(8)}{F/F|_{UTM}}\right) \leftarrow H^{*-n}(\mathrm{Map}(8), F)
\end{aligned}
$$

The last map is a Thom map which is constructed using the Thom isomorphism, similar as in the previous subsections.

As in Section 2.3 the **Loop Coproduct** is the following composition on cohomology:

$$\begin{aligned} H^*(LM, M) &\xleftarrow{\tilde{s}^*} H^{*+1}(\mathrm{Map}(\bigcirc_2), LM \amalg_M LM) \\ &\xleftarrow{j^!} H^{*+1-n}(LM \times_M LM, LM \amalg_M LM) \\ &\leftarrow H^*(LM, M) \otimes H_*(LM, M)[1-n] \end{aligned}$$

where the last map is the Künneth isomorphism.

6. Algebraic Models of String Topology Operations

As an application of the algebraic models of mapping spaces in Section 4 and the discussion of the loop product in Section 5.3, we are able to give an algebraic model of the loop product and the co-BV algebra structure on LM. In Section 6.1 we give a model of the loop product using Hochschild homology and show that the iterated integral map intertwines this with the loop product in the derived category. In Section 6.2 we explain how one can obtain a chain level model of the loop coproduct using a chain level model of the intersection product of M. In Section 6.3 we show that the quasi-isomorphism between the Hochschild complex and the cochain complex of the loop space preserves the co-BV operator as a map of derived chain complexes. In particular we show that the Chas-Sullivan operations satisfy the co-BV algebra relations on cohomology in the simply connected case.

6.1. Loop Product Model (Cohomology Coproduct)

We combine Theorem 5.20 and the models from Section 4 to give a model of the loop product. Let $A = C^*(M)$ and $B = \mathrm{B}(A, A, A)$ the bar construction of A. The following theorem shows that the loop product (coproduct on cohomology) is given on the Hochschild complex $CH_*(A) = \mathrm{B}(A, A, A) \otimes_{A^{\otimes 2}} A$ by the deconcatenation coproduct on B and the umkehr map of the diagonal $i^! \colon C^{*-n}(M) \to C^*(M \times M)$.

Theorem 6.1. *Let M be a closed oriented, simply connected manifold. The following diagram in $D(Ch)$ commutes:*

$$\begin{tikzcd} C^*(LM \times LM) & C^*(LM \times_M LM)[-n] \arrow[l, "i^!_{LM}"'] & C^*(LM)[-n] \arrow[l, "c"'] \\ B \otimes_{A^{\otimes 2}} A \otimes B \otimes_{A^{\otimes 2}} A \arrow[u, "\simeq"'] & B^{\otimes 2} \otimes_{A^{\otimes 4}} A[-n] \arrow[l, "\mathrm{id} \otimes i^!"'] \arrow[u, "\simeq"'] & B \otimes_{A^{\otimes 2}} A[-n] \arrow[l, "\bar{c}"'] \arrow[u, "\simeq"'] \end{tikzcd} \tag{20}$$

The upper horizontal composition is the loop product (cohomology coproduct). The vertical maps are induced by zigzags of quasi-isomorphisms in from Theorem 4.24. The lower horizontal composition is induced by the intersection coproduct on A and the deconcatenation coproduct on $B \to B \otimes_A B$ (see Lemma 4.25).

Proof. We can consider the two squares separately. For the left square, recall from Theorem 5.20 the commutative diagram

$$\begin{tikzcd}
C^*(LM \times LM) & & C^*(LM \times_M LM)[-n] \arrow[ll, "i^!_{LM}"'] \\
C^*(LM \times LM) \otimes^L_{C^*(M\times M)} C^*(M \times M) \arrow[u, "\simeq"'] & & C^*(LM \times LM) \otimes^L_{C^*(M\times M)} C^*(M)[-n] \arrow[ll, "\mathrm{id} \otimes i^!"'] \arrow[u, "\simeq"']
\end{tikzcd} \tag{21}$$

and the intersection map $i^! \colon C^*(M)[-n] \to C^*(M \times M)$ fits into the following diagram

$$\begin{tikzcd}
C^*(M \times M) \arrow[d, "\simeq"] & C^*(M)[-n] \arrow[l] \arrow[d, equal] \\
C^*(M) \otimes C^*(M) & C^*(M)[-n] \arrow[l]
\end{tikzcd} \tag{22}$$

In Theorem 4.24 we showed that there is a natural zigzag of quasi-isomorphisms of two sided A-modules

$$C^*(LM \times LM) \simeq \ldots \simeq (B \otimes B) \otimes_{A^{\otimes 4}} A^{\otimes 2}$$

This yields a commutative diagram

$$\begin{tikzcd}
C^*(LM \times LM) \otimes^L_{C^*(M\times M)} C^*(M) \otimes C^*(M) \arrow[d, "\simeq"] & C^*(LM \times LM) \otimes^L_{C^*(M\times M)} C^*(M)[-n] \arrow[l, "\mathrm{id} \otimes i^!"'] \arrow[d, "\simeq"] \\
\ldots & \ldots \arrow[l] \\
(B \otimes B) \otimes_{A^{\otimes 4}} A^{\otimes 2} \arrow[u, "\simeq"'] & (B \otimes B) \otimes_{A^{\otimes 4}} A[-n] \arrow[l, "\mathrm{id} \otimes i^!"] \arrow[u, "\simeq"']
\end{tikzcd} \tag{23}$$

This shows that the left square in Diagram 20 commutes. The right square is analogous. □

There is an analogous statement for the de Rham complex, there the vertical maps are honest chain maps, namely the iterated integral maps as in Theorem 4.21.

Theorem 6.2. *Let M be a closed oriented, simply connected manifold. The following diagram in $D(Ch)$ commutes:*

$$\begin{tikzcd}
\Omega^*(LM \times LM) & \Omega^*(LM \times_M LM)[-n] \arrow[l, "i^!_{LM}"'] & \Omega^*(LM)[-n] \arrow[l, "c"'] \\
B \otimes_{A^{\otimes 2}} A \otimes B \otimes_{A^{\otimes 2}} A \arrow[u, "\simeq"'] & B^{\otimes 2} \otimes_{A^{\otimes 4}} A[-n] \arrow[l, "\mathrm{id} \otimes i^!"'] \arrow[u, "\simeq"'] & B \otimes_{A^{\otimes 2}} A[-n] \arrow[l, "\bar{c}"'] \arrow[u, "\simeq"']
\end{tikzcd} \tag{24}$$

The upper horizontal composition is the loop product (cohomology coproduct). The vertical maps are induced by zigzags of quasi-isomorphisms in from Theorem 4.24. The lower horizontal composition is induced by the intersection coproduct on A and the deconcatenation coproduct on $B \to B \otimes_A B$ (see Lemma 4.25).

6.2. Frobenius Algebra Models

We would like to work with the models from the previous section as if they were implemented at the chain level. All the maps relating to A and $\mathrm{B}(A, A, A)$ exist at the chain level except for the intersection map $\Delta^!\colon A \to A \otimes A[n]$. By a theorem of Lambrechts and Stanley [LS08a], one can construct an algebra that is quasi-isomorphic to $C^*(M)$ and that has a coproduct on the chain level which is equal to $\Delta^!$ in $D(Ch)$.

6.2.1. Poincaré Duality Algebras and Frobenius Algebras

Let k be a field of any characteristic, let A be a graded algebra and let V be a graded vector space over k. We call a bilinear graded map $\mu\colon V \otimes V \to k[-n]$ a pairing of degree n. Denote by $V^\vee$ the graded dual of V.

Definition 6.3.

1. A graded module V is connected if $V^0 \cong k$ and simply connected if additionally $V^1 = 0$. It is of finite type if it has finite dimension in each degree.
2. A pairing $\mu\colon V \otimes V \to k[-n]$ is nondegenerate if $\mu(v, v') = 0$ for all $v' \in V$ implies $v = 0$.
3. A pairing is perfect if the map $V \to V^\vee$, $v \mapsto \mu(v, \dot{})$ is an isomorphism.

Note that any perfect pairing is nondegenerate. If V is not finite dimensional, the converse is not generally true.

Definition 6.4. A **Poincaré duality algebra** or Poincaré duality CDGA in dimension n is a connected CDGA A and a chain map $\varepsilon\colon A \to k[-n]$ of degree $-n$, such that the induced bilinear pairing

$$A \otimes A \to k[-n], a \otimes b \mapsto \varepsilon(ab)$$

is non-degenerate. The map ε is called the orientation map of A. A graded algebra is a Poincaré duality algebra if it is a Poincaré duality CDGA with differential zero.

The following is the main result of

Theorem 6.5 ([LS08a])**.** *Let k be a field of any characteristic and let A be a CDGA over k such that $H^*(A)$ is a simply connected oriented Poincaré duality algebra in dimension n of finite type. Then there exists a CDGA A' weakly equivalent to A such that A' is a simply connected Poincaré CDGA of finite type.*

Two CDGAs A and A' are weakly equivalent if there exists a zigzag of quasi-isomorphisms and morphisms of CDGAs $A \xrightarrow{\simeq} \ldots \xleftarrow{\simeq} A'$. In fact, an inspection of the proof in [LS08a] shows that in the theorem one can take this zigzag to be

$$A' \xleftarrow{\simeq} S \xrightarrow{\simeq} A$$

where S is a minimal Sullivan model of A.

Moreover, if the orientation on $H^*(A) \to k[-n]$ is induced by an orientation map $A \to k[-n]$ on A, one can define an orientation on S such that the morphisms in the zigzag preserve the orientation.

This construction is not functorial in A.

We show that if A is a finite dimensional Poincaré duality CDGA, then it has the additional structure of a coproduct, which together with the product on A forms a Frobenius algebra.

Construction 6.6. Assume that A is a finite dimensional Poincaré duality CDGA. Thus the nondegeneracy of the pairing $A^k \otimes A^{n-k} \to k$ implies that it is perfect, hence A is isomorphic to its degree shifted graded dual $A \cong A^\vee[-n]$, where $A^\vee = \mathrm{Hom}(A, k)$. Using this observation, we can construct the **coproduct** of the Poincaré duality algebra as the following degree n map:

$$\Delta\colon A \to A^\vee[-n] \to (A \otimes A)^\vee[-n] \cong A^\vee \otimes A^\vee[-n] \cong A \otimes A[n]$$

Here the second map is induced by the product on A and the third map is an isomorphism since A is finite dimensional.

Definition 6.7. A **differential graded Frobenius algebra** is a unital CDGA A together with a coproduct $\Delta\colon A \to A \otimes A$ of possibly nonzero degree, such that Δ is a coassociative cocommutative chain map, Δ is counital, and such that the Frobenius relation holds:

$$a\Delta(a') = \Delta(aa') = \Delta(a)a' \in A \otimes A$$

The Frobenius relation is equivalent to Δ being a map of A-modules and the multiplication being a map of A-comodules.

Proposition 6.8. *The coproduct constructed for a finite dimensional Poincaré duality CDGA makes A into a Frobenius algebra with counit ε given by the orientation.*

Since $\Delta(a) = a\Delta(1)$, the coproduct is completely determined by the "diagonal cocycle" $\Delta(1) = \Delta_A \in (A \otimes A)^{(n)}$. One can characterize this in terms of a basis of A: Given a homogeneous basis $\{a_i\}$ of A, there exists a dual basis $\{a_i^*\}$ with respect to the pairing, i.e. such that $\varepsilon(a_i a_j^*) = \delta_{ij}$ is given by the Kronecker symbol. Then $\Delta(1) = \sum_i (-1)^{|a_i|} a_i \otimes a_i^* \in A \otimes A$ independent of the choice of basis. Denote $e_A = m(\Delta_A) = \sum_i (-1)^{|a_i|} a_i a_i^*$ where m is the multiplication in A. Since $\varepsilon\colon A^n \to k$ is an isomorphism, denote $\mathrm{vol}_A \coloneqq \varepsilon^{-1}(1)$. Then $e_A = \chi(A)\,\mathrm{vol}_A$ or equivalently $\varepsilon(e_A) = \chi(A)$, where $\chi(A) = \sum (-1)^i \dim A_i$ is the Euler characteristic of A.

Proposition 6.9. *The Euler characteristic of A is*

$$\chi(A) \coloneqq \sum (-1)^i \dim A_i = \varepsilon(e_A) = \varepsilon(m(\Delta(1)))$$

6.2.2. Frobenius Algebra Models

We begin by stating the main result of this subsection.

Theorem 6.10. *1. If M is a closed oriented simply connected manifold, there exists a finite dimensional Frobenius algebra A which is quasi-isomorphic to $C^*(M)$ through a zigzag $A \leftarrow \cdots \rightarrow \Omega^*(M) \leftarrow \cdots \rightarrow C^*(M)$, such that this chain of quasi-isomorphisms takes the coproduct on A to the intersection coproduct $\Delta^!$ on $C^*(M)$, seen as maps in $D(Ch(\mathbb{R}))$.*

2. Moreover the graded dual $A^\vee$ is quasi-isomorphic to $C_*(M)$, and the coproduct on $A^\vee$ is identified with the Alexander-Whitney diagonal $C_*(M) \to C_*(M) \otimes C_*(M)$ and the product on $A^\vee$ is identified with the intersection product on $C_*(M)$ in $D(Ch(\mathbb{R}))$.

We will call such an A a **Frobenius algebra model** of M in this work, but this terminology is not standard. Given such a model of $C^*(M)$, we can replace $C^*(M)$ in Theorem 6.1 with A and obtain the following:

Theorem 6.11. *If M is a closed oriented simply connected manifold and A is a Frobenius algebra model of M, then in the derived category, the loop product is given under the isomorphism $C^*(LM) \simeq \mathrm{B}(A, A, A) \otimes_{A^{\otimes 2}} A$ of Theorem 6.1 by the chain map*

$$\begin{aligned} \mathrm{B}(A, A, A) \otimes_{A^{\otimes 2}} A &\to \mathrm{B}(A, A, A) \otimes_{A^{\otimes 2}} A \otimes \mathrm{B}(A, A, A) \otimes_{A^{\otimes 2}} A \\ a_0 \otimes \cdots \otimes a_{n+1} \otimes a &\mapsto \sum (-1)^{\deg a'(\deg a_{i+1} + \cdots + \deg a_{n+1} - (n-i-1))} \\ &\qquad (a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a') \otimes (1 \otimes a_{i+1} \otimes \cdots \otimes a_{n+1} \otimes a'') \end{aligned}$$

where we are using Sweedler's notation for the coproduct $\Delta a = \sum a' \otimes a''$. This is the tensor product of the intersection coproduct on A and the deconcatenation coproduct on $\mathrm{B}(A, A, A)$.

Proof of Theorem 6.10. 1. $\Omega^*(M)$ is a CDGA with orientation $\Omega^n(M) \to \mathbb{R}$ given by the integral $\int_M \omega$. By Theorem 6.5, $\Omega^*(M)$ is quasi-isomorphic to a finite dimensional simply connected Poincaré duality algebra A through a zigzag $A \leftarrow S \to \Omega^*(M)$. In particular A is a Frobenius algebra. We need to show that the coproduct coincides with the coproduct on $\Omega^*(M)$.

Consider the zigzag

$$A \xleftarrow{\simeq} S \xrightarrow{\simeq} \Omega^*(M) \xrightarrow{I_{DR}} C^*_\infty(M) \leftarrow C^*(M). \tag{25}$$

We show that all the maps in zigzag 25 preserve the product and pairing up to homotopy. The first two maps preserve the product and orientation map by the remarks after Theorem 6.5. The de Rham map I_{DR} preserves the product up to homotopy (in fact it is a map of A_∞ algebras, cf. [Gug77]) and the map $C^*_\infty(M) \leftarrow C^*(M)$ preserves the cup product on the chain level. The orientation map on $C^*(M)$ is given by pairing with a representative of the fundamental class, different representatives yield homotopic orientations. I_{dR} preserves the orientation map on the chain level and $C^*_\infty(M) \leftarrow C^*(M)$ preserves it on the chain level if one chooses the same (smooth) representative of the fundamental class in both instances.

The coproduct of A by definition fits into the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\cong"] & A \otimes A[n] \arrow[d, "\cong"] \\
A^\vee[-n] \arrow[r] & A^\vee \otimes A^\vee[-n]
\end{tikzcd} \tag{26}$$

The intersection coproduct is described by the following diagram

$$\begin{tikzcd}
C^*(M) \arrow[r, "\Delta^!"] \arrow[d, "\frown[M]"] & C^*(M) \otimes C^*(M)[n] \arrow[d, "\frown[M] \otimes \frown[M]"] \\
C_{-*}(M)[-n] \arrow[r] & C_{-*}(M) \otimes C_{-*}(M)[-n]
\end{tikzcd} \tag{27}$$

We will show that in $D(Ch)$, one can connect the diagrams 26 and 27 using the zigzag 25 and its graded dual.

Using the map $A \leftarrow S$ and the graded dual $A^\vee \to S^\vee$, we obtain a cube-shaped diagram which commutes in $D(Ch)$

$$\begin{tikzcd}
& A \otimes A[n] \arrow[dd, "\simeq"] & & S \otimes S[n] \arrow[ll, "\simeq"'] \arrow[dd, "\simeq"] \\
A \arrow[ur, "\Delta"] \arrow[dd, "\simeq"] & & S \arrow[ll, "\simeq"'] \arrow[ur] \arrow[dd, "\simeq"] & \\
& A^\vee \otimes A^\vee[-n] \arrow[rr, "\simeq"] & & S^\vee \otimes S^\vee \arrow[dl] \\
A^\vee[-n] \arrow[ur] \arrow[rr, "\simeq"] & & S^\vee[-n] &
\end{tikzcd}$$

This connects Diagram 26 to the Diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & S \otimes S[n] \arrow[d] \\
S^\vee[-n] \arrow[r] & S^\vee \otimes S^\vee[-n]
\end{tikzcd} \tag{28}$$

Proceeding along the zigzag 25, we obtain a sequence of cubes with commutative faces, connecting Diagram 26 to the diagram

$$\begin{array}{ccc} C^*(M) & \longrightarrow & C^*(M)\otimes C^*(M)[n] \\ \downarrow & & \downarrow \\ C^*(M)^\vee[-n] & \longrightarrow & C^*(M)^\vee \otimes C^*(M)^\vee[-n] \end{array} \tag{29}$$

Thus the zigzag 25 intertwines the coproduct on A with the top map in this diagram in $D(Ch)$. It remains to show that this map is the intersection map $\Delta^!$. We can relate this to the defining Diagram 27 of $\Delta^!$ through the double dual inclusion $C_*(M) \to C_*(M)^{\vee\vee} = C^*(M)^\vee$. Since $C_*(M)$ has finite dimensional cohomology, the double dual inclusion is a quasi-isomorphism, hence an isomorphism in $D(Ch)$ [4]. Thus we can exhibit one last cube which connects Diagram 29 to Diagram 27. This shows the claim on the coproducts.

2. We have already shown that $A^\vee \simeq C_*(M)$ and that this sequence of quasi-isomorphisms identifies the coproduct on $A^\vee$ with the diagonal on $C_*(M)$. The claim about the intersection product is proven with the same method as the claim on the intersection coproduct in (1).

□

6.3. The co-BV Algebra Structure

We have seen that the loop coproduct on $H^*(LM)$ is modeled by the coproduct on $\mathrm{B}(A, A, A) \otimes A$. To complete the discussion of the BV algebra structure on $H_*(LM)$, which corresponds to the co-BV structure on $H^*(LM)$, we consider the BV operator Δ. The BV operator on $H_*(LM)$ is given by mapping

$$\Delta\colon C_*(LM) \xrightarrow{[S^1]\times -} C_{*+1}(S^1 \times LM) \xrightarrow{\rho_*} C_{*+1}(LM)$$

where $[S^1]$ represents the fundamental class of S^1 and $\rho\colon S^1 \times LM \to LM$ is the translation action by $\rho(t,\alpha)(s) = \alpha(t+s)$. In cohomology, the co-BV operator is the degree -1 operator

$$\Delta^{BV}: \Omega^*(LM) \xrightarrow{\rho^*} \Omega^*(S^1 \times LM) \xrightarrow{\int_{S^1}} \Omega^{*-1}(LM)$$

.

The co-BV operator on the Hochschild complex $CH_*(A,A) = B \otimes_{A\otimes A} A$ is given by

$$\mathrm{B}(a_0, \dots, a_n) = \sum_{j=0}^{n} \pm a_j \otimes a_{j+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{j-1} \otimes 1$$

[4]In field coefficients we can show this as follows: any complex with finite dimensional cohomology is isomorphic to a direct sum of a finite dimensional complex with zero differential and an acyclic complex. In both of these cases the double dual inclusion is clearly a quasi-isomorphism, hence also for the direct sum.

Remark 6.12. We recall that in Remark 3.18 we cautioned that the map ρ is neither smooth nor continuous using our definition of the piecewise smooth path space. For the first part of Theorem 6.13 we optimistically assume that one can give a better definition of this space which fixes the issues from Remark 3.18. The statement on the singular cochain complex is true regardless of this discussion.

Theorem 6.13. *Let M be a closed oriented simply connected manifold.*

1. *The operators B on $B \otimes_{A^{\otimes 2}} A$ and Δ^{BV} on $\Omega^*(LM)$ on $B \otimes_{A \otimes A} A$, as maps of derived chain complexes, are intertwined by the iterated integral map $B \otimes_{A^{\otimes 2}} A \xrightarrow{\simeq} \Omega^*(LM)$.*
2. *Similarly the operators B and Δ^{BV} on $C^*(LM)$ are intertwined by the zigzag of quasi-isomorphisms $B \otimes_{A^{\otimes 2}} A \simeq C^*(LM)$.*

Proof. We only show this for Ω^*, for C^* the proof is similar: the fiber integral is replaced with the slant product with the fundamental class and the iterated integral map I is replaced by the zigzags of Theorem 4.24.

Following [NW19, 4.3], we represent the action ρ on LM using the splitting map $s\colon I \times LM \to \mathrm{Map}(\bigcirc_2, M)$ from Theorem 4.28. Translating a loop by t is the same as composing its restriction to $[t, 1]$ with its restriction to $[0, t]$. This shows that in the following diagram, the right part commutes up to homotopy (since the paths are reparametrized).

$$\begin{tikzcd}
LM & I \times LM \arrow[l] \arrow[r, "s"] \arrow[d] & \mathrm{Map}(\bigcirc_2, M) \arrow[r, "\tau"] & \mathrm{Map}(\bigcirc_2, M) \arrow[r, "c"] & LM \\
& S^1 \times LM \arrow[ul] \arrow[urrr, bend right=10, "\rho"'] & & &
\end{tikzcd}$$

Here τ exchanges the two parts of the circle and c is the concatenation map. The left part is the projection to LM. Using the models we have obtained so far, we can represent the upper horizontal composition on differential forms. Letting $\gamma \in B, a \in A$ and representing the deconcatenation coproduct on B as $\sum \gamma' \otimes \gamma''$, we obtain

$$\begin{tikzcd}
\Omega^*(I \times LM) & \Omega^*(\mathrm{Map}(\bigcirc_2, M)) \arrow[l, "s^*"'] & \Omega^*(\mathrm{Map}(\bigcirc_2, M)) \arrow[l, "\tau^*"'] & \Omega^*(LM) \arrow[l, "c^*"'] \\
\mathcal{I} \otimes B \otimes_{A^{\otimes 2}} A \arrow[u] & (B \otimes B) \otimes_{A^{\otimes 4}} A^{\otimes 2} \arrow[l] \arrow[u] & (B \otimes B) \otimes_{A^{\otimes 4}} A^{\otimes 2} \arrow[l] \arrow[u] & B \otimes_{A^{\otimes 2}} A \arrow[l, "\bar{c}"'] \arrow[u] \\
\pm dt(\gamma''|a|\gamma') \otimes 1 + \dots & \pm\gamma'' \otimes \gamma' \otimes \otimes 1 \otimes a \arrow[l, mapsto] & \pm\gamma' \otimes \gamma'' \otimes a \otimes 1 \arrow[l] & \gamma \otimes a \arrow[l]
\end{tikzcd}$$

We have used the model of c from Lemma 4.25, τ from Theorem 4.21 and s from Theorem 4.28. The image in $\mathcal{I} \otimes B \otimes_{A^{\otimes 2}} A$ contains two other summands, which contain t or $(1 - t)$ as factors in $\mathcal{I}$.

To conclude, we have to show that the following diagram commutes in the derived category, where $p\colon I \times LM \to S^1 \times LM$

$$\begin{tikzcd} \Omega^{*-1}(LM) & \Omega^*(I \times LM) \arrow[l, "\int_I"'] \\ & \Omega^*(S^1 \times LM) \arrow[u, "p^*"'] \arrow[ul, "\int_{S^1}"] \end{tikzcd} \tag{30}$$

If it commutes, then we can compute Δ^{BV} by taking the fiber integral $\int_I\colon \Omega^*(I \times LM) \to \Omega^{*-1}(LM)$ after the chain above, and this corresponds to projecting to the dt summand in $\mathcal{I} \otimes B \otimes_{A^{\otimes 2}} A$, which shows the claim.

To show that the diagram commutes, let $\mathcal{S} = k\langle 1\rangle \oplus k\langle dt\rangle$ be the two dimensional chain complex with two generators 1 and dt of degree 0 resp. 1 and differential 0. This is quasi-isomorphic to $\Omega^*(S^1)$ and we there is a quasi-isomorphism $\mathcal{S} \otimes \Omega^*(LM) \to \Omega^*(S^1 \times LM)$. Recall also the interval complex $\mathcal{I}$ in Theorem 4.28, which also has an isomorphism $\mathcal{I} \otimes \Omega^*(LM) \to \Omega^*(I \times LM)$. We can replace Diagram 30 with the diagram

$$\begin{tikzcd} \Omega^{*-1}(LM) & \mathcal{I} \otimes \Omega^*(LM) \arrow[l] \\ & \mathcal{S} \otimes \Omega^*(LM) \arrow[u] \arrow[ul] \end{tikzcd}$$

where the maps into $\Omega^*(LM)$ are given by projecting $1 \otimes \omega_1 + dt \otimes \omega'$ resp. $1 \otimes \omega_1 + t \otimes \omega_2 + dt \otimes \omega'$ to ω'. The pullback $\Omega^*(S^1 \times LM) \to \Omega^(\mathcal{S} \times LM)$ is modeled by the inclusion $\mathcal{S} \to \mathcal{I}$. This diagram commutes on the chain level. The quasi-isomorphisms from this into Diagram 30 give rise to three squares which commute on the chain level. This shows that Diagram 30 commutes in $D(Ch)$. ☐

Corollary 6.14. *If M is a simply connected closed oriented manifold, the loop coproduct and Δ^{BV} form a co-BV algebra on $H^*(LM)$.*

Proof. For any Frobenius algebra A, one can show algebraically that the operations Δ and B on the Hochschild complex $CH_*(A, A) = \mathrm{B}(A, A, A) \otimes_{A^{\otimes 2}} A$ satisfy the co-BV relations in $D(Ch)$, see theorem 5.4 in [Abb15] or theorem 1 of [Tra08].

n particular if we take A to be a Frobenius algebra model of $\Omega^*(M)$, then the homology of the Hochschild complex has the structure of a co-BV algebra. The zigzag $A \to \ldots \leftarrow C^*(M)$ induces zigzags $B(A, A, A) \otimes_{A^{\otimes 2}} \to \ldots \leftarrow B(C^*(M), C^*(M), C^*(M)) \otimes_{C^*(M)^{\otimes 2}} C^*(M)$ and we have shown that the latter is quasi-isomorphic to $C^*(LM)$. All of these maps preserve the co-BV operations on cohomology. Thus the co-BV relations hold in the cohomology of LM. ☐

6.4. Example: S^3

For S^3, the explicit Frobenius algebra model A is as graded vector spaces generated by an element 1 of degree 0 and e of degree $n = 3$, with zero differential. The product is determined by noting that 1 is the unit and the coproduct is given by $1 \mapsto 1 \otimes e + (-1)^n e \otimes 1$ and $e \mapsto e \otimes e$.

To compute the Hochschild complex, we can replace $\mathrm{B}(A, A, A)$ with the normalized bar construction

$$\hat{B} = \hat{\mathrm{B}}(A, A, A) = \bigoplus_{n \geq 0} A \otimes \bar{A}^{\otimes n} \otimes A$$

where $\bar{A} = A/\langle 1 \rangle$. Thus to compute the cohomology of LS^n for $n = 2, 3$ we can compute the cohomology of $\hat{B} \otimes_{A^{\otimes 2}} A$. This is generated as a graded vector space by the $(k+3)$-fold product $1 \otimes e \otimes \cdots \otimes e \otimes 1 \otimes e =: e^k e \in (A \otimes \bar{A}^k \otimes A) \otimes_{A^2} A$ of degree $k(n-1) + n$ and the $k+3$-fold product $1 \otimes e \otimes \cdots \otimes e \otimes 1 \otimes 1 =: e^k 1$ of degree $k(n-1)$ for $k \geq 1$. The differential maps

$$\begin{aligned} e^k e &\mapsto 0 \\ e^k 1 &\mapsto (-1)^{nk} e^k e + (-1)^k e^{k-1} e \end{aligned}$$

For odd n this is 0. Thus the cohomology of LS^3 has dimension 1 in the dimensions $(k-1)(n-1)$ and $(k-1)(n-1) + n$ for $k \geq 1$ and 0 in other degrees. The coproduct which corresponds to the loop product is given by

$$\begin{aligned} e^k e &\mapsto \sum_{l=0}^{k} \pm e^l e \otimes e^{k-l} e \\ e^k 1 &\mapsto \sum_{l=0}^{k} \pm e^l 1 \otimes e^{k-l} e \pm e^l e \otimes e^{k-l} 1 \end{aligned}$$

7. Further Developments

7.1. Loop Coproduct

Naef and Willacher perform in [NW19] a analysis similar as in Section 6.1 for the loop coproduct (cohomology product). Recall that this is a map $H^*(LM, M) \otimes H^*(LM, M) \to H^*(LM, M)$. If A is quasi-isomorphic to $\Omega^*(M)$, one can show that $\Omega^*(LM, M) = \mathrm{cone}(\Omega^*(LM) \to \Omega^*(M))$ is quasi-isomorphic to the reduced Hochschild complex

$$\bar{C}H_*(A, A) \coloneqq \bigoplus_{k \geq 1} (A[1])^{\otimes k} \otimes A = \mathrm{B}(A, A, A) \otimes_{A^{\otimes 2}} A/A$$

If A is a Frobenius algebra model of $\Omega^*(M)$, one can define a degree $n-1$ product on $\bar{C}H_*(A, A)$ by mapping $\alpha \in A[1]^{\otimes k}, x \in A, \beta \in A[1]^{\otimes l}, y \in A$ to

$$(\alpha \otimes x) \otimes (\beta \otimes y) \mapsto \sum \pm (\alpha \otimes (xy)_{(1)} \otimes \beta) \otimes (xy)_{(2)}$$

where in Sweedler's notation, the coproduct $\Delta(xy) = \sum (xy)_{(1)} \otimes (xy)_{(2)}$.

Theorem 7.1. *[NW19, Thm. 1.3] If A is a Frobenius algebra model of $\Omega^*(M)$, then there is a chain of quasi-isomorphisms $\bar{C}H_*(A, A) \simeq \Omega^*(LM, M)$ which intertwines the product on $H^*(LM, M)$ from Section 5.4 with the product on $H^*(\bar{C}H_*(A, A))$.*

The proof of this is very similar to the proof of Theorem 6.1, using the models of the splitting, concatenation and intersection maps that we have shown. Since $H^*(LM, M)$ is modeled by mapping cones from the outset, the construction of the intersection map in Section 5.4 involves taking mapping cones of mapping cones. For details of the proof we refer to [NW19, Sec. 6].

7.2. The Lambrechts-Stanley Model of the Configuration Spaces

Let M be a smooth closed oriented simply connected manifold. Recall that in Section 4.3 we constructed a model of the mapping spaces $\mathrm{Map}(G, M)$ from $\Omega^*(M)$. In this subsection we sketch models of the compactified configuration spaces $\overline{\mathrm{Conf}}_M(n)$ from Section 5.1. We begin with the model due to Lambrechts and Stanley [LS08b].

Let A be a Frobenius algebra model of M. Given a set of generators X with fixed degrees, by $S(X)$ the free graded commutative algebra generated by elements of X. Recall that $A^{\otimes r}$ has the structure of a differential graded commutative algebra and denote by $p_i^*\colon A \to A^{\otimes r}$ the map

$$a \mapsto 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$$

with a in the i-th position and similarly $p_{ij}^*\colon A \otimes A \to A^{\otimes r}$.

Definition 7.2. The graph complex of A for the compactified configuration space of r points is the quotient of differential graded commutative algebras

$$G_A(r) := A^{r\otimes} \otimes S(\omega_{ij} | 1 \leq i \neq j \leq r)/I.$$

where each ω_{ij} has degree $\dim M - 1$ and I the graded ideal generated by

- the relations $\omega_{ij}^2, \omega_{ji} = (-1)^{\dim M}\omega_{ij}$, and the Arnol'd relation $\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$.
- $p_i^*(a)\omega_{ij} = p_j^*(a)\omega_{ij}$ for $a \in A$ and $1 \leq i, j \leq r$.

Denote by d the differential on $G_A(r)$ which is the induced by the tensor product differential on $A^{r\otimes} \otimes S(\omega_{ij})_{1\leq i\neq j\leq r}$, which acts on each factor A as the differential of A and maps ω_{ij} to the diagonal class

$$d_{split}(\omega_{ij}) = p_{ij}^*(\Delta_A(1)).$$

Remark 7.3. This is called a graph complex for the following reason: One can interpret a product of the generators ω_{ij} in $S(\omega_{ij} | 1 \leq i \neq j \leq r)$ as a graph with r nodes and an edge between the nodes i, j for each factor ω_{ij} in the product. An element of $A^{r\otimes} \otimes S(\omega_{ij} | 1 \leq i \neq j \leq r)$ is thus a graph which is decorated with an element of A at each node. The differential of such a graph is given by (1) the differential of the decorations and (2) splitting up each edge and adding decorations given by the diagonal class $\Delta_A(1)$ to the corresponding nodes.

Theorem 7.4. *There is a quasi-isomorphism* $G_A(r) \to \overline{\mathrm{Conf}}_M(r)$.

There is a second construction which one can apply even if M is not simply connected, due to Campos and Willacher [CW16]. We only describe this informally, using the depiction via graphs as in Remark 7.3. The space $TwGra_M(r)$ is spanned by graphs with n vertices labeled from 1 to r, called "external" vertices and an arbitrary number k of indistinguishable "internal" vertices. Both types of vertices can be decorated by elements of $\bar{H}^*(M)$. Additionally we impose the condition that ever connected component of the graph contains an external vertex. The degree of internal vertices is $-\dim(M)$, the degree of edges is $\dim(M) - 1$ and each decoration has its own degree as an element of $\bar{H}^*(M)$. The differential is given by two terms which correspond to (1) splitting up each edge and adding decorations given by the diagonal class $\Delta_{H^*(M)}(1)$ to the corresponding nodes and (2) contracting edges incident to at least on internal vertex. If in (1), a component without an external vertex is created, this is turned into a scalar according to a so called partition function.

Theorem 7.5. *There is a zigzag of quasi-isomorphisms* $TwGra_M(r) \xrightarrow{\simeq} \dots \xleftarrow{\simeq} C^*(\overline{\mathrm{Conf}}_M(r))$.

7.3. Cyclic Homology

The classifying space BG of a topological group G is the quotient of a weakly contractible space EG by a free proper action of G. In the case of $G = S^1$, one can take $BG = \mathbb{C}P^\infty$ and $EG = S^\infty$. Recall that there is an S^1-action $\rho\colon LM \times S^1 \to LM$ on LM. The homotopy orbit space of this action is $LM \times_{S^1} EG$, this is a version of the orbit space LM/S^1 which is better adapted to the study of homotopical properties.

Definition 7.6. The equivariant cohomology $H^*_{S^1}(LM)$ is the cohomology of the homotopy quotient space $LM \times_{S^1} ES^1$.

The cyclic group $\mathbb{Z}/n\mathbb{Z}$ acts on $A^{\otimes n}$ via the generator

$$\begin{aligned} t = t_{n-1}\colon A^{\otimes n} &\to A^{\otimes n} \\ a_1, \dots, a_n &\mapsto \pm a_n \otimes a_0 \otimes \dots \otimes a_{n-1} \end{aligned}$$

Let $N = \mathrm{id} + t + \dots + t^{n-1}$. One checks that t and thus N induce chain maps on the Hochschild complex $\mathrm{B} \otimes_{A^{\otimes 2}} A$.

Definition 7.7. The **cyclic complex** $Cyc(A)$ of A is the image of N in $\hat{C}(A)$, which is a subcomplex. Equivalently, it is the total complex of $A^{\otimes n}/(\mathrm{id} - t)$, with differential induced by the differential of the Hochschild complex.

The following is stated in [NW19].

Theorem 7.8. *Let M be a closed oriented simply connected smooth manifold and let A be a Frobenius algebra model of M. Let $\bar{A} = A/(1)$. Then there is a quasi-isomorphism*

$$HCyc(A) \cong \bar{H}^{-*}_{S^1}(LM). \tag{31}$$

A homological version of this is shown in [CEG10].

One can construct operations on $H^{-*}_{S^1}(LM)$ as follows. There is a map $LM \simeq LM \times ES^1 \to LM \times_{S^1} ES^1$ and one can construct an umkehr map $p^! \colon H_{*+1}(LM) \to H_*(LM \times_{S^1} ES^1)$. The string bracket (cohomology cobracket) is the operation

$$\bar{H}^*_{S^1}(LM) \xrightarrow{p^*} \bar{H}^*(LM) \to (\bar{H}^*(LM) \otimes \bar{H}^*(LM))[n] \xrightarrow{p^! \otimes p^!} (\bar{H}^*_{S^1}(LM) \otimes \bar{H}^*_{S^1}(LM))[n-2]$$

Similarly one can define the string cobracket (cohomology bracket) on cohomology, using the loop coproduct (cohomology product), which is a map

$$\bar{H}^*_{S^1}(LM) \otimes \bar{H}^*_{S^1}(LM) \to \bar{H}^*_{S^1}(LM)[n-2]$$

This construction yields a Lie bialgebra, cf. [CS99], [CS04].

Similarly one can construct operations on the cyclic complex, which yield the structure of a Lie bialgebra on homology.

Theorem 7.9. *The isomorphism 31 preserves the Lie bialgebra operations.*

Theorem 1.5 of [NW19] is the following:

Theorem 7.10. *If M is a closed oriented, not necessarily simply connected manifold, one can construct an algebra A over $\mathbb{R}$ which is quasi-isomorphic to $\Omega^*(M)$, such that there exists a map*

$$HCyc(\bar{A}) \to \bar{H}^{-*}_{S^1}(LM) \tag{32}$$

which preserves the Lie bialgebra operations.

This is shown using the explicit models of the compactified configuration space due to [CW16] outlined in Section 7.2.

A. Elements of Singular Cohomology

In this section, we state classical results of singular homology and cohomology for reference and comparison with the main part of the thesis. Specifically, we consider Poincaré duality, the intersection product, the Thom isomorphism and the slant product. All of these provide examples of umkehr maps and we are interested in them from this viewpoint.

We define umkehr maps via Poincaré duality in the case of closed oriented manifolds in Appendix A.1. A more general definitions of umkehr maps is given in [CK09], but we do not use it in this work. A special role is taken by the intersection product, which is the umkehr map of the diagonal. In Appendix A.2, we construct the intersection product and show some of its algebraic properties. The Thom isomorphism is a classical construction of an oriented vector bundle $E \to B$, which induces an isomorphism on homology $H_{*+n}(E, E_0) \xrightarrow{\simeq} H^*(B)$, where E_0 is E minus the zero section. In Appendix A.3 we show that it is the umkehr map of the zero section map $B \to (E, E_0)$, in the case where our definition of umkehr maps applies. There is another well known construction of the intersection product using the Thom isomorphism and tubular neighborhoods. In this construction, using tubular neighborhoods, the diagonal of a manifold is identified with the zero section of a bundle, so that the Thom isomorphism provides an umkehr map. In Appendix A.4 we show that this construction of the intersection product yields the same map. Finally, in Appendix A.5 we review formal properties of the slant product. The slant product can be used to construct umkehr maps $Y \times X \to Y$ if X has a fundamental class and it is a version of fiber integration in singular chains.

We use singular chains and cochains with $\mathbb{Z}$-coefficients in this section, unless stated otherwise.

A.1. Umkehr Maps via Poincaré Duality

One can use Poincaré duality to construct umkehr maps as maps in the derived category of chain complexes. We first recall the statement of Poincaré duality.

Let M be a smooth manifold of dimension n, possibly with boundary ∂M.

Definition A.1. A **homological orientation** of M is an assignment of a homology class $z_p \in H_n(M, M \setminus \{p\})$ for each $p \in M$, which is locally consistent in the sense that there exists a covering of M with open sets U diffeomorphic to $\mathbb{R}^n$, such that for all $p, q \in M$, the chain of isomorphisms $H^*n(M, M \setminus \{p\}) \cong H^*(U, U \setminus \{p\}) \cong H^*(U, U \setminus \{q\}) \cong H^*n(M, M \setminus \{q\})$ maps z_p to z_q.

This is from [Hat02, Subsec. 3.3]. One can show that a homological orientation with coefficients in $\mathbb{Z}$ or $\mathbb{Q}$ is corresponds bijectively to an orientation of M and the corresponding notions of orientability are equivalent.

Theorem A.2. *Let M be a smooth compact oriented manifold, possibly with boundary.*

*(a) There exists a unique **fundamental class** $[M] \in H_n(M, \partial M)$, such that for any $p \in M \setminus \partial M$, we have $[M]|_{(M, M \setminus \{p\})} = z_p \in H_n(M, M \setminus \{p\})$.*

(b) If $\partial M = A \cup B$ is the union of two compact $(n-1)$-dimensional manifolds with common boundary $\partial A = \partial B = A \cap B$, then the cap product with $[M]$ gives an isomorphism

$$H^k(M, A) \to H_{n-k}(M, B).$$

(c) In particular, there are isomorphisms

$$H^k(M) \to H_{n-k}(M, \partial M)$$
$$H^k(M, \partial M) \to H_{n-k}(M).$$

Parts (b) and (c) are [Hat02, Thm 3.43], part (a) is due to the remark just before the cited theorem, as a consequence of [Hat02, Thm 3.27].

Definition A.3. Let $f\colon M \to N$ be a map of smooth closed oriented manifolds with $m = \dim M$ and $n = \dim N$. Then there is a contravariant **umkehr map** on homology which we denote by $f_!\colon H_*(N) \to H_{*+m-n}(M)$ which is defined via Poincaré duality as the unique map that makes the following diagram commutative:

$$\begin{tikzcd} H_*(N) \arrow[r, "f_!"] & H_{*+m-n}(M) \\ H^{n-*}(N) \arrow[r, "f^*"] \arrow[u, "\cong"] & H^{n-*}(M) \arrow[u, "\cong"] \end{tikzcd} \tag{33}$$

More generally, the analogous diagram in the derived category of chain complexes $D(Ch(\mathbb{Z}))$ yields a derived umkehr map $f_!\colon C_*(N) \to C_{*+m-n}(M)$.

Similarly, the covariant **umkehr map** on cohomology $f^!\colon H^*(M) \to H^{*-m+n}(N)$ is defined via the diagram

$$\begin{tikzcd} H^*(M) \arrow[r, "f^!"] & H^{*-m+n}(N) \\ H_{m-*}(M) \arrow[r, "f_*"] \arrow[u, "\cong"] & H_{m-*}(N) \arrow[u, "\cong"] \end{tikzcd} \tag{34}$$

Remark A.4. We follow the convention from [Bre13] in using the lower shriek notation $f_!$ for the homological, contravariant umkehr map and $f^!$ for the cohomological, covariant umkehr map.

Remark A.5. There is the following intuition for umkehr maps: One can informally think of a homology class as represented by a subspace, and the map f_* takes the image, while $f^!$ can be thought of as taking the preimage. For example, the umkehr map $\Delta^!$ of the diagonal $\Delta\colon X \to X \times X$ can be thought of as the intersection with the diagonal and $\Delta_!$ is called the intersection product. While the construction of f_* on the singular chain complex $C_*(X)$ is easy, one cannot easily represent the preimage of a singular simplex in Y using singular simplices in X and in general there is some technical burden to the construction of $f^!$. Similarly in cohomology, if one thinks of a degree k cohomology class as represented by a subspace of codimension k, then one may think of f^* as taking the preimage, while the covariant $f_!$ takes the image along f.

Definition A.6. If $(M, \partial M = A \cup B)$ and $(N, \partial N = C \cup D)$ are as required for Poincaré duality in Theorem A.2 then a map $f\colon M \to N$ with $f(A) \subseteq C$ and $f(B) \subseteq D)$ gives rise to a **relative umkehr map**

$$\begin{tikzcd} H_*(N,C) \arrow[r, "f_!"] & H_{*+m-n}(M,A) \\ H^{n-*}(N,D) \arrow[u, "\cong"'] \arrow[r, "f^*"] & H^{n-*}(M,B) \arrow[u, "\cong"'] \end{tikzcd} \tag{35}$$

where $m = \dim M$ and $n = \dim N$.

If M is a closed oriented manifold, denote by μ_M its Poincaré duality pairing:

$$\begin{aligned} \mu_M\colon C^*(M) \otimes C^*(M) &\to k \\ a \otimes b &\mapsto \langle a \smile b, [M] \rangle \end{aligned}$$

Proposition A.7. *The umkehr map $f^!$ of a map $f\colon M \to N$ of closed oriented manifolds is adjoint to f^* with respect to the Poincaré duality pairings in the sense that*

$$\mu_N \circ (f^! \otimes \mathrm{id}) = \mu_M \circ (\mathrm{id} \otimes f^*)$$

Proof. This follows from a routine computation using properties of the cap product. $\square$

There is a similar statement on homology, using the intersection pairing.

A.2. Intersection Product

We use field coefficients in this section, so that $H_*(C_*(M) \otimes C_*(M)) \cong H_*(M) \otimes H_*(M)$.

Definition A.8. Let M be a closed oriented manifold of dimension n. The intersection product $\cdot\colon H_*(M) \otimes H_*(M) \to H_{*-n}(M)$ is the Poincaré dual of the cup product. This means it is the upper map in the following commutative square, where the vertical maps are given by Poincaré duality and the lower map is the cup product.

$$\begin{tikzcd} H_*(M) \otimes H_*(M) \arrow[r, "\cdot"] & H_{*-n}(M) \\ H^{m-*}(M) \otimes H^{m-*}(M) \arrow[u, "\simeq"'] \arrow[r, "\smile"] & H^{2m-*}(M) \arrow[u, "\simeq"'] \end{tikzcd} \tag{36}$$

More abstractly, we can regard it as a map in the derived category $D(Ch(k))$ given as the top map in the diagram

$$\begin{tikzcd} C_*(M) \otimes C_*(M) \arrow[r, "\cdot"] & C_{*-n}(M) \\ C^{m-*}(M) \otimes C^{m-*}(M) \arrow[u, "\simeq"'] \arrow[r, "\smile"] & C^{2m-*}(M) \arrow[u, "\simeq"'] \end{tikzcd} \tag{37}$$

Remark A.9. Since under the isomorphism $H_*(M \times M) \cong H_*(M) \otimes H_*(M)$, the cup product corresponds to the diagonal map, one can equivalently use the following diagram

$$\begin{array}{ccc} H_*(M \times M) & \xrightarrow{\Delta_!} & H_{*-n}(M) \\ \downarrow\cong & & \downarrow\cong \\ H^{2m-*}(M \times M) & \xrightarrow{\Delta^*} & H^{2m-*}(M) \end{array} \tag{38}$$

where the left vertical morphism is the Poincaré Duality of $M \times M$. Thus up to the isomorphism $H_*(M \times M) \cong H_*(M) \otimes H_*(M)$, the intersection product $\cdot$ is the umkehr map $\Delta_!$.

For some computation rules involving umkehr maps and the intersection product, we refer to [Bre13, Prop. 14.1].

There is a dual construction on cohomology, which yields a degree m coproduct map $H^{*-n}(M) \to H^*(M) \otimes H^*(M)$, given by

$$\begin{array}{ccc} C^*(M) & \xrightarrow{\Delta^!} & C^*(M) \otimes C^*(M)[m] \\ \downarrow\simeq & & \downarrow\simeq \\ C_{m-*}(M) & \xrightarrow{\Delta} & C_{-*}(M) \otimes C_{-*}(M)[m] \end{array} \tag{39}$$

A.3. Thom Isomorphism

We introduce the Thom isomorphism, following [MS74]. Let ξ be topological vector bundle $E \xrightarrow{\pi} B$ of rank n and let E_0 be E minus the zero section. Denote the fiber by $F = F_b = \pi^{-1}(b)$.

Definition A.10. We will call a family $\{u_b \in H^n(F_b, F_b \setminus \{0\})\}_{b \in B}$ a **cohomological orientation** of ξ if for every trivialization $E|_U \to U \times \mathbb{R}^n$ and all $b, b' \in U$, the isomorphism $H^n(F_b, F_b \setminus 0) \leftarrow H^n(E|_U, E|_U \setminus 0) \to H^n(F_{b'}, F_{b'} \setminus 0)$ maps u_b to $u_{b'}$.

If coefficients in a ring R are used, this is called a cohomological R-orientation. If ξ is oriented, one can assign canonically to every fiber $F = \pi^{-1}(b)$ a cohomology class $u_b = u_F \in H^n(F, F_0)$, such that the assignments are locally compatible. Thus an orientation of ξ induces a cohomological orientation with $\mathbb{Z}$-coefficients. Conversely a cohomological orientation with $\mathbb{Z}$-coefficients induces an orientation of ξ.

Theorem A.11. *Let ξ be an orientable rank n vector bundle with paracompact base space, with a selected cohomological orientation.*

*(a) There exists a unique cohomology class $u = u(\xi) \in H^n(E, E_0)$ such that $u|_F = u_F \in H^n(F, F_0)$. This is called the **Thom class**, fundamental class or orientation class of ξ.*

(b) The following map is an isomorphism

$$y \mapsto y \smile u, \quad H^*(E) \to H^{n+*}(E, E_0)$$

(c) The following map is an isomorphism

$$\eta \mapsto \eta \frown u, \quad H_{*+n}(E, E_0) \to H^*(E)$$

Definition A.12. The **Thom isomorphism** is one of the following compositions of isomorphisms

$$\Phi^*\colon H^*(B) \xrightarrow{\pi^*} H^*(E) \xrightarrow{\smile u} H^{*+n}(E, E_0)$$
$$\Phi_*\colon H_{*+n}(E, E_0) \xrightarrow{\frown u} H_*(E) \xrightarrow{\pi_*} H_*(B)$$

Proof. For a detailed proof of the theorem, see [MS74]. We give a quick sketch of the proof there. One first verifies that the theorem holds for trivial vector bundles, i.e. cross products. Using a Mayer-Vietoris argument one then shows that it holds on finite unions of trivialization domains, in particular on compact base spaces. For the general cases, one uses that homology is a colimit indexed by compact spaces. □

We remark that the Thom isomorphism is the umkehr map of the zero section in the following sense. The pair (E, E_0) is homotopy equivalent to the pair (DE, UE), where DE and UE are the unit disk and unit sphere bundle of E with respect to some bundle metric. The Thom isomorphism can thus be seen as a map $H_{*+n}(DE, UE) \to H_*(B)$. If B is a smooth closed manifold and E is an oriented bundle over B, then we have a definition of the umkehr map of the zero section $s\colon (B, \varnothing) \to (DE, UE)$ and one can show that this is in fact the Thom isomorphism. The Thom class is Poincaré dual to the zero section $s_*[B] \in H_n(DE, UE)$, cf. [Hut11, Lemma 4.1]. As the Thom isomorphism can be constructed in a more general setting, without the requirement of Poincaré duality, it is widely used to construct umkehr maps.

A.4. Intersection product via Tubular neighborhoods

Recall that the Thom isomorphism can be interpreted as an intersection with the zero section of a vector bundle. The intersection product is equivalent to taking the intersection with the diagonal in $M \times M$. Using tubular neighborhoods, one can express the diagonal as the zero section of a bundle and hence express the intersection product via the Thom isomorphism.

Construction A.13. Let $N \subseteq M \times M$ be a tubular neighborhood of the diagonal, so that N is isomorphic to the normal of the tangent bundle of the diagonal $\Delta(M) \subseteq M \times M$. Note that the normal bundle is isomorphic to the tangent bundle via $(X, Y) \mapsto (X, -Y)$. Denote by DTM the disk bundle of the tangent bundle TM and DT_0M the disk bundle minus the zero section. Similarly, N_0 is N minus M. Then we have the following sequence of maps

$$\begin{tikzcd}
H_*(M \times M) \arrow[r] & H_*(M \times M, M \times M \setminus \Delta) & H_*(N, N_0) \arrow[l, "\cong"'] \arrow[d, "\cong"] & \\
 & & H_*(DTM, DT_0M) \arrow[r, "\pi_*(\cdot \frown u)"] & H_{*-n}(M)
\end{tikzcd} \tag{40}$$

The second map is an excision isomorphism and the last map is the Thom isomorphism of the tangent bundle of M, i.e. the cap product with the Thom class $u = u(TM)$.

Theorem A.14. *The intersection product $\Delta_!$ as defined in Definition A.8 is equal to the map defined above*

Proof. 1. We first show that the intersection product $\Delta_!$ is equal to the following map, which intersects at the diagonal of $M \times M$:

$$H_*(M \times M) \xrightarrow{\frown[\Delta]} H_{*-n}(M \times M) \xrightarrow{p_{1*}} H_{*-n}(M)$$

The first map intersects with the diagonal, that is the intersection with the diagonal homology class $\Delta_*[M] \in H_n(M \times M)$, where the intersection product is defined via Poincaré Duality. The second map is the projection to the first factor of the product (the following calculation also shows that one could just as well use the second instead).

By the naturality property of the intersection product, one has for any $a \in H_*(M \times M)$ that

$$\begin{aligned} p_{1*}(a \frown \Delta_*([M])) &= p_{1*}\Delta_*(\Delta^!(a) \frown [M]) \\ &= p_{1*}\Delta_*(\Delta^!(a)) = \Delta^!(a) \end{aligned}$$

Since $\Delta^!$ is the intersection product, this shows the first claim.

2. Let $[\Delta] = \Delta_*[M] \in H_n(M \times M)$ be the fundamental class of the diagonal and $[\Delta]^* \in H^n(M \times M)$ its Poincaré dual. On the other hand, let $u \in H^n(DTM, DT_0M)$ be the Thom class of $DTM \to M$ and denote by $u_N \in H^n(N, N_0)$ and $u_\Delta \in H^n(M \times M, M \times M \backslash \Delta)$ be its images under the isomorphisms $H^*(DTM, DT_0M) \xrightarrow{\cong} H^*(N, N_0) \xleftarrow{\cong} H^*(M \times M, M \times M \setminus \Delta)$. In lemma 4.2 of [Hut11] it is shown that the image of u_Δ in $H^n(M \times M)$ is $[\Delta]^*$.

   Hence the intersection product is equal to $p_{1*}(\cdot \frown [\Delta]^*$.

$$H_*(M \times M) \xrightarrow{\frown u_\Delta^{M \times M}} H_{*-n}(M \times M) \xrightarrow{p_{1*}} H_{*-n}(M)$$

3. We now consider the following diagram, which translates the map from part 2 into the map defined via tubular neighborhoods. Denote by $u_N \in H_m(N, N_0)$ and $u_{\Delta M} \in H_m(M \times M, M \times M \setminus \Delta M)$ the image of u under the chain of maps $H_m(DTM, DT_0M) \cong H_m(N, N_0) \cong H_m(M \times M, M \times M \setminus \Delta M)$.

$$\begin{tikzcd}
H_*(M \times M) \arrow[r] \arrow[d, "\frown[\Delta]^*"] & H_*(M \times M, M \times M \setminus \Delta) \arrow[d, "\frown u_\Delta"] & H_*(N, N_0) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "\frown u_N"] & H_*(DTM, DT_0M) \arrow[d, "\frown u"] \\
H_{*-n}(M \times M) \arrow[r] \arrow[d, "p_{1*}"] & H_{*-n}(M \times M) \arrow[d, "p_{1*}"] & H_{*-n}(N) \arrow[l] \arrow[r] \arrow[d, "\cong"] & H_{*-n}(DTM) \arrow[d, "\cong"] \\
H_{*-n}(M) \arrow[r, "="] & H_{*-n}(M) & H_{*-n}(M) \arrow[l, "="'] \arrow[r, "="] & H_{*-n}(M)
\end{tikzcd}$$

The top squares commute due to the naturality of the cap product. The maps in the bottom squares are all either versions of the diagonal map, identity maps or the projection $H_*(M \times M) \to H_*(M)$, hence commutativity is easy to see. The left vertical maps give the intersection product from step 2 and the composition of the top row and right column give the map defined in A.4.

$\square$

Remark A.15. One can perform an analogous construction and argument for the degree n coproduct $\Delta^!$ in cohomology.

A.5. Slant Product

The slant product on cohomology is a map

$$/ \colon C^n(Y \times X) \otimes C_i(X) \to C^{n-i}(Y).$$

If X is a space with Poincaré duality and with fundamental class $[X]$, then the map $-/[X], C^*(X \times Y) \to C^*(Y)$ is the umkehr map of the projection $X \times Y \to Y$. In a de Rham setting the slant product is given via fiber integration, as is shown in subsection 3.5.

We review formal properties of this operation, following [Spa89, chapter 6.1], see there for proofs and details.

Definition A.16. The **slant product** on cohomology is the map

$$/ \colon C^n(Y \times X) \otimes C_i(X) \to C^{n-i}(Y)$$

which is defined by the relation

$$\langle c^*/c', c\rangle = \langle c^*, c' \times c)\rangle$$

for $c^* \in C^n(Y \times X), c' \in C_i(X), c \in C_{n-i}(Y)$ where $\times \colon C_*(Y) \otimes C_*(X) \to C_*(Y \times X)$ is the Eilenberg Zilber map.

Proposition A.17. *The slant product satisfies the following properties:*

1. $(c \times c') \,/\, \alpha = c\langle c', \alpha\rangle$ *for* $c \in H^*(Y), c' \in H^*(X), \alpha \in H_*(Y)$.
2. ***Boundary relation****: for* $c^* \in C^n(Y \times X)$ *and* $c \in C_i(X)$,
$$\delta(c^*/c) = \delta c^*/c + (-1)^{n-i} c^*/\partial c$$
3. ***Naturality****: Let* $f \colon X \to X', g \colon Y \to Y'$ *be continuous and* $u \in C^n(X' \times Y'), z \in C_q(Y)$*, then in* $C^{n-q}(X)$*, one has*
$$((f \times g)^* u)/z = f^*(u/g_*(z)).$$

There is a relative version

$$/ \colon H^n((Y,B) \times (X,A)) \otimes H_i(X,A) \to H^{n-i}(Y,B)$$

Remark A.18. There is a related operation on homology, also called the slant product, which is a map

$$\backslash \colon C^i(X) \otimes C_n(X \times Y) \to C_{n-i}(Y).$$

A reference is [Dol12, VII.11].

B. Elements of Homological Algebra

B.1. Two sided Bar Construction

Let K be a field, A be a unital differential graded algebra over K, M a right dg A-module and N a left dg A-module. The chain complex $M \otimes A^{\otimes k} \otimes N$ is a chain complex with the usual tensor product differential

$$\begin{aligned} d_0(m \otimes a_1 \otimes \cdots \otimes a_k \otimes n) =& dm \otimes a_1 \otimes \cdots \otimes a_k \otimes n + (-1)^{\varepsilon_1} m \otimes da_1 \otimes \cdots \otimes a_k \otimes n \\ &+ (-1)^{\varepsilon_2} m \otimes a_1 \otimes da_2 \otimes \cdots \otimes a_k \otimes n + \dots \\ &+ (-1)^{\varepsilon_{k+1}} m \otimes a_1 \otimes \cdots \otimes a_k \otimes dn \end{aligned}$$

where $\varepsilon_l = |m| + |a_1| + \cdots + |a_{l-1}|$. The collection of chain complexes $M \otimes A^{\otimes k} \otimes N$ forms a simplicial chain complex, i.e. a simplicial object in the category of chain complexes. This means it is an $\mathbb{N}$-indexed collection of chain complexes with so called face maps and degeneracy maps satisfying certain relations, cf. [ML13, VII.5]. The face maps are chain maps $d_k^i \colon M \otimes A^{\otimes k} \otimes N \to M \otimes A^{\otimes k-1} \otimes N$

$$d_k^i(m, a_1, \dots, a_k, n) = \begin{cases} ma_1 \otimes \cdots \otimes a_k \otimes n & \text{if } i = 0 \\ m \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes n & \\ m \otimes a_1 \otimes \cdots \otimes a_k n & \text{if } i = k \end{cases}$$

for $i = 0, \dots, k$ and the degeneracy maps are chain maps $s_k^i \colon M \otimes A^{\otimes k} \otimes N \to M \otimes A^{\otimes k+1} \otimes N$ for $i = 0, \dots, k$

$$s_k^i(m, a_1, \dots, a_k, n) = m \otimes a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes n.$$

The face maps combine to a degree 0 map

$$\begin{aligned} &d_1 \colon M \otimes A^{\otimes k} \otimes N \to M \otimes A^{\otimes (k-1)} \otimes N, \\ &d_1 = \sum_{i=0}^{k} (-1)^i d_k^i. \end{aligned}$$

Since this is induced by a simplicial object, it follows that $d_1 \circ d_1 = 0$ and since the face maps d_k^i are chain maps with respect to d_0, the two differentials commute: $d_1 \circ d_0 = d_0 \circ d_1$.

Denote by $\mathrm{B}(M,A,N)$ the two-sided bar construction, namely

$$\mathrm{B}(M,A,N) = \bigoplus_{n\geq 0} M \otimes A[1]^{\otimes n} \otimes N$$

and let $\hat{\mathrm{B}}(M,A,N) = \bigoplus_{n\geq 0} M \otimes \overline{A}[1]^{\otimes n} \otimes N$ be its (simplicial) normalization, where $\overline{A} = A/\langle 1\rangle$ as a quotient of vector spaces. The normalization is the quotient by the image of the degeneracy maps.

The bar construction $\mathrm{B}(M,A,N)$ is the totalization of a double complex and its differential is

$$d(m\otimes a_1 \otimes \cdots \otimes a_k \otimes n) = d_0(m\otimes a_1 \otimes \cdots \otimes a_k \otimes n) + (-1)^{\varepsilon} d_1(m\otimes a_1 \otimes \cdots \otimes a_k \otimes n)$$

with $\varepsilon = |m| + |a_1| + \cdots + |a_k| + |n| - k$.

Theorem B.1. *(a) The following sequence*

$$M \xleftarrow{d_1} M\otimes A \xleftarrow{d_1} M \otimes A^{\otimes 2} \xleftarrow{d_1} \ldots$$

is split exact, hence a resolution of M by free A-modules.

(b) The canonical map $\mathrm{B}(M,A,A) \to M$ given by the product map $M\otimes A \to M$ for $n = 0$ and 0 otherwise, is a quasi-isomorphism of chain complexes.

Proof. (a): Consider the map $b = s^i_{p-1}\colon M\otimes A^{\otimes p} \to M \otimes A^{\otimes (p+1)}$ which maps $m\otimes a_1 \otimes \cdots \otimes a_p \mapsto m \otimes a_1 \otimes \cdots \otimes a_n \otimes 1$. Then $d_1 b + b d_1 = \mathrm{id}_{M\otimes A^{\otimes p}}$. Hence the sequence above is indeed exact.

We denote by Tot the total complex of a double complex. The total complex $\mathrm{Tot}(M \leftarrow M\otimes A \leftarrow M \otimes A^{\otimes 2} \leftarrow \ldots)$ is acyclic: Let $a \in M\otimes A^{\otimes n}$ such that $da = 0$, then $dba = dba + bda = a + d_0 ba + b d_0 a$ hence in homology $[a] = -[(bd_0 + d_0 b)a] = \pm[(bd_0 + d_0 b)^n a] \in H^*(\mathrm{Tot}(M \leftarrow M\otimes A \leftarrow M\otimes A^{\otimes 2} \leftarrow \ldots))$. But b commutes with d_0, hence for $n \geq 2$ this vanishes.

There exists a short exact sequence of chain complexes

$$0 \to A \to \mathrm{Tot}(M \leftarrow M\otimes A \leftarrow \ldots) \to \mathrm{Tot}(0 \leftarrow M \otimes A \leftarrow \ldots) \to 0$$

Since the middle complex is acyclic, in the associated long exact sequence there is an isomorphism $H^{*-1}(\mathrm{Tot}(0 \leftarrow M\otimes A \leftarrow \ldots)) \cong H^*(M)$, and this is induced by the map $M\otimes A \to M$ on chains. $\square$

Let M, N, M', N' be A-modules and consider morphisms of A-modules $M \to M'$ and $N \to N'$, then there is a morphism of chain complexes $\mathrm{B}(M,A,N) \to \mathrm{B}(M',A,N')$. Recall that we use coefficients in a field.

Similarly we may change the differential graded algebra A. Let $A' \to A$ be a morphism of differential graded algebras. This makes the A-modules M and N into left resp. right A'-modules and we can take the bar construction $\mathrm{B}(M,A',N)$, moreover there is a natural chain map $\mathrm{B}(M,A',N) \to \mathrm{B}(M,A,N)$.

Theorem B.2. *Assume that A, A', M, M', N, N' are bounded below chain complexes.*

1. *If the maps $M \to M'$ and $N \to N'$ are quasi-isomorphisms, then the induced map $\mathrm{B}(M, A, N) \to \mathrm{B}(M', A, N')$ is a quasi-isomorphism.*
2. *If the map $A' \to A$ is a quasi-isomorphism, then so is the map $\mathrm{B}(M, A', N) \to \mathrm{B}(M, A, N)$.*

The proof relies on the following lemma:

Lemma B.3. *If $f: C \to C'$ is a quasi-isomorphism of chain complexes of K-vector spaces, then for any bounded below chain complex B, the map $f \otimes \mathrm{id}: C \otimes B \to C' \otimes B$ is a quasi-isomorphism.*

Proof. Since we are working with field coefficients, B is a direct sum of complexes of the form $\cdots \to 0 \to K \to 0 \to \ldots$ and $\cdots \to 0 \to K \xrightarrow{\sim} K \to 0 \to \ldots$. For any such complex the claim is easy to check. Since homology and direct sum commute, a direct sum of quasi-isomorphisms is again a quasi-isomorphism. □

Proof of Theorem B.2.

1. By the lemma, the induced maps

$$M \otimes A^{\otimes p} \otimes N \to M' \otimes A'^{\otimes p} \otimes N'$$

are quasi-isomorphisms.

Define a filtration on the bar construction via $F^p \,\mathrm{B}(M, A, N) = \bigoplus_{n \leq p} M \otimes A^n \otimes N$. Each $F^p \,\mathrm{B}(M, A, N)$ is a chain complex and there is an exact sequence of chain complexes

$$0 \to F^p \,\mathrm{B}(M, A, N) \to F^{p+1}(M, A, N) \to M \otimes A^{\otimes(p+1)} \otimes N \to 0$$

Using induction and the five lemma, we can show that for all p, the map $F^p \,\mathrm{B}(M, A, N) \to F^p \,\mathrm{B}(M', A, N')$ is a quasi-isomorphism. Finally,

$$\mathrm{B}(M, A, N) = \mathrm{colim}_{p \in \mathbb{N}} F^p \,\mathrm{B}(M, A, N)$$

as a colimit of chain complexes, and the map $\mathrm{B}(M, A, N) \to \mathrm{B}(M', A, N')$ is a filtered colimit of quasi-isomorphisms. Any filtered colimit of quasi-isomorphisms is again a quasi-isomorphism, as one can check using the explicit construction of filtered colimits.

2. As in (1), the maps $M \otimes A'^{\otimes p} \otimes N \to M \otimes A^{\otimes p} \otimes N$ are quasi-isomorphisms and we proceed as in the proof of (1).

□

B.2. The Model Category of DG-A-modules

Before discussing the bar construction as a derived tensor product, we recall the model structure on the category of DG A-modules. See [BMR13] for details. References for general model categories are [Hov07] and [Hir03]. A reader who is unfamiliar with model categories may choose to skip this section and only skim the theorems of Appendix B.3, taking the notion of cofibrant A-module as a black box.

First recall the model structure on the category of chain complexes over a field:

Theorem B.4. *There is a model structure on the category $Ch(K)$ of chain complexes of modules over a field, such that*

- *the weak equivalences are quasi-isomorphisms*
- *the cofibrations are degreewise monomorphisms*
- *the fibrations are degreewise epimorphisms.*

See e.g. [BMR13]. We warn that if K is not a field, the situation is more complicated.

Let A be a unital not necessarily commutative dg-algebra over a field K and $\mathrm{Mod}(A)$ the category of dg-modules over A.

There is an adjunction between the category Ch_R of chain complexes over a ring R and the category $\mathrm{Mod}(A)$ of differential graded modules over a differential graded algebra A over R:

$$F\colon Ch_R \rightleftarrows \mathrm{Mod}(A)\colon U \tag{41}$$

where the left adjoint F is the extension of scalars functor $X \to A \otimes_R X$ and the right adjoint is the forgetful functor, forgetting the action of A. One can transfer the model category structure of Ch_R to $\mathrm{Mod}(A)$ by taking weak equivalences and fibrations those maps whose images under U have the same property.

The following is part of theorem 3.3 in [BMR13].

Theorem B.5. *The category* $\mathrm{Mod}(A)$ *is a model category, where*

- *the fibrations are the degreewise surjective maps*
- *the weak equivalences are the quasi-isomorphisms*

In the adjunction 41, the functor F preserves cofibrations and acyclic cofibrations and the functor U preserves fibrations and acyclic fibrations.

Recall that an object X in a model category is cofibrant if the map $0 \to X$ is a cofibration.

Theorem B.6 ([BMR13], 9.12)**.** *A map $W \to Y$ of DG A-modules is a cofibration if and only if it is a monomorphism with cofibrant cokernel.*

B.3. The Bar Construction and the Derived Tensor Product

Let A be a unital, not necessarily commutative DG algebra over a field k.

Definition B.7. The **tensor product** $M \otimes_A N$ of a left A-module M and a right A-module N over A is the chain complex defined as the colimit of diagram

$$M \otimes A \otimes N \rightrightarrows M \otimes N$$

where the arrows take $m \otimes a \otimes n$ to $ma \otimes n$ and to $m \otimes an$ respectively. Thus it is the quotient of $M \otimes N$ by the image of the difference map.

This yields a functor

$$\otimes_A \colon \operatorname{Mod}(A)^{op} \times \operatorname{Mod}(A) \to Ch(K).$$

This functor does not preserve quasi-isomorphisms in general, but it does preserve quasi-isomorphisms between cofibrant objects by Theorem B.8. On the other hand, the bar construction $\mathrm{B}(M, A, N)$ preserves quasi-isomorphisms by Theorem B.2 and we will show that in Corollary B.11 that it is quasi-isomorphic to $M \otimes_A N$ if M or N are cofibrant. In the language of derived functors, by Theorem B.8 there exists a derived functor $D(\operatorname{Mod}(A)^{op}) \times D(\operatorname{Mod}(A)) \to D(Ch(K))$ of $\otimes_A$ and by Corollary B.11 this is constructed by the bar construction, thus we write $M \otimes_A^L N \coloneqq \mathrm{B}(M, A, N)$.

Theorem B.8. *If $P \to P'$ is a quasi-isomorphism between cofibrant left A-modules, then the induced map*

$$M \otimes_A P \to M \otimes_A P'$$

is a quasi-isomorphism of chain complexes. Similarly if $P \to P'$ is a quasi-isomorphism between right right A-modules, then the induced map $P' \otimes_A M \to P \otimes_A M$ is a cofibration.

We refer for instance to [Hin97, 3.2].

Lemma B.9. *The bar construction $\mathrm{B}(A, A, A)$ is cofibrant as a dg A^e-module. The bar construction $\mathrm{B}(A, A, M)$ is a cofibrant dg A-module.*

Proof. Consider again the resolution

$$A \xleftarrow{d_1} A^{\otimes 2} \xleftarrow{d_1} A^{\otimes 3} \xleftarrow{d_1} \dots$$

Each $A^{\otimes n}$ for $n \geq 2$ is cofibrant: recall that the functor $- \otimes A^e \colon Ch(K) \to \operatorname{Mod}(A^e)$ preserves cofibrations since the model structure on $\operatorname{Mod}(A^e)$ is transferred from $Ch(K)$. $A^{\otimes n}$ is isomorphic to $A^e \otimes A^{n-2}$ and in $Ch(K)$ every complex is cofibrant since K is a field. Similarly, each $A^{\otimes n} \otimes M$ is a cofibrant dg A-module.

We conclude with Theorem B.10. $\square$

Theorem B.10. *Let A be a differential graded algebra over a field, and let*

$$M_1 \xleftarrow{d_1} M_2 \xleftarrow{d_1} \dots$$

be an exact sequence of cofibrant DG A-modules. Then the total complex

$$\mathrm{Tot}(M) = \mathrm{Tot}(M_1 \xleftarrow{d_1} M_2 \xleftarrow{d_1} \dots)$$

is cofibrant.

Proof. Let $F^p\mathrm{Tot}(M) = \mathrm{Tot}(M_1 \leftarrow \dots \leftarrow M_p)$. Since each M_p is cofibrant, the inclusions $F^{p-1}\mathrm{Tot}(M) \to F^p\mathrm{Tot}(M)$ are cofibrations (B.6). Hence by induction each $F^p\mathrm{Tot}(M)$ is cofibrant. Thus the colimit $\mathrm{Tot}(M)$ is also cofibrant. $\square$

Corollary B.11. *If P is cofibrant, then the map*

$$\mathrm{B}(M, A, P) \to M \otimes_A P$$

is a quasi-isomorphism.

Proof. $\mathrm{B}(M, A, P) = M \otimes_A \mathrm{B}(A, A, P)$ and the map $\mathrm{B}(A, A, P) \to P$ is a quasi-isomorphism between cofibrant left A-modules. We conclude with Theorem B.8. $\square$

C. Homotopy Pushouts of Topological Spaces

Consider the diagram of topological spaces $X \xleftarrow{f} A \xrightarrow{g} Y$. The pushout $X \amalg_A Y$ of such a diagram is in general poorly behaved with respect to homotopy: in a diagram as follows, where the vertical maps are homotopy equivalences

$$\begin{tikzcd} X \arrow[d] & A \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & A' \arrow[l, "f'"'] \arrow[r, "g'"] & Y' \end{tikzcd}$$

the map of pushouts $X \amalg_A Y \to X' \amalg_{A'} Y'$ is not neccessarily a homotopy equivalence.

Example C.1. Let $*$ be the one point space, S^n the n-dimensional sphere and D^n the n-dimensional disk. Compare the diagram $* \leftarrow S^{n-1} \to D^n$, with pushout S^n and the homotopy equivalent diagram $* \leftarrow S^{n-1} \to *$, whose pushout is $*$, thus the pushouts are not homotopy equivalent.

The homotopy pushout is a version of the pushout which is homotopy invariant. In contrast to the pushout of topological spaces, a universal property for the homotopy pushout is difficult to formulate, at least in the language of categories. We present two approaches, following [MP11, 1.2]. The first is via a direct construction, called the double mapping cylinder, which is a standard construction of the homotopy pushout. One may then define when a diagram is a homotopy pushout diagram by comparing to the double mapping cylinder.

A second approach is to identify certain diagrams that are “general enough”, so that the normal pushout is also a homotopy pushout.

C.1. The Standard Homotopy Pushout

Definition C.2. Let $X \xleftarrow{f} W \xrightarrow{g} Y$ be given. The **double mapping cylinder** of f, g is the quotient space

$$M(f,g) = \frac{X \amalg (W \times I) \amalg Y}{f(w) \sim (w,0) \quad (w,1) \sim g(w)}$$

There are obvious inclusion maps $i_X \colon X \to M(f,g), i_Y \colon Y \to M(f,g)$. There is a canonical homotopy $\psi = \psi_{f,g} \colon i_X f \simeq i_Y g$ given by $\psi_t(w) = (w,t)$ for $w \in W, t \in I$.

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\psi}{\Longrightarrow}"] & Y \arrow[d, "i_Y"] \\ X \arrow[r, "i_X"] & M(f,g) \end{tikzcd} \tag{42}$$

We will call this square the **standard homotopy pushout** of f and g.

Remark C.3. Suppose we are given another square with homotopy

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{F}{\Longrightarrow}"] & Y \arrow[d, "k"] \\ X \arrow[r, "h"] & Z \end{tikzcd} \tag{43}$$

then we get a comparison map

$$\begin{aligned} &\theta_F \colon M(f,g) \to Z \\ &\theta_F(x) := h(x), \quad \theta_F(y) := k(y), \quad \theta_F(w,t) := F(w,t) \end{aligned}$$

for $x \in X, Y \in Y, (w,t) \in W \times I$. This fits into a diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_Y"] \arrow[ddr, bend left, "k"] & \\ X \arrow[r, "i_X"] \arrow[drr, bend right, "h"'] & M(f,g) \arrow[dr, "\theta_F"] & \\ & & Z \end{tikzcd} \tag{44}$$

and we have the following equations

$$\theta_F \psi = F, \quad \theta_F i_X = h, \quad \theta_F i_Y = k$$

Note that the first equation is an equation of homotopies and is not pictured in the diagram.

Definition C.4. The Diagram 43 is called a **homotopy pushout diagram** if the natural map θ_F is a homotopy equivalence. In this case we will call Z a homotopy pushout of the diagram $X \xleftarrow{f} W \xrightarrow{g} Y$.

C.2. Cofibrations of Topological Spaces

We quickly recall cofibrations here. For a more in-depth introduction we refer to [MP11, Ch. 6].

Let $I = [0,1]$ and Y^I be the space of continuous maps $I \to Y$. Let $ev_0 \colon Y^I \to Y$ be the evaluation map $\gamma \mapsto \gamma(0)$.

Definition C.5. A map $i \colon A \to X$ is a **cofibration** if for all commutative squares of the form

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"] & Y^I \arrow[d, "ev_0"] \\ X \arrow[r, "f"] \arrow[ur, dashed, "\tilde{h}"] & Y \end{tikzcd} \tag{45}$$

there exists an $\tilde{h}$ which makes the two triangles commute.

Cofibrations of topological spaces are the quintessential example of cofibrations in the framework of model categories and as such satisfy all the formal properties known from model categories, which we will not recall here. References for general model categories are [Hov07] and [Hir03].

We give a criterion for recognizing cofibrations of topological spaces, taken from [MP11, Ch. 6.4].

Definition C.6. Let $A \subseteq X$ be a subspace of a space X. The pair (X, A) is called a **neighborhood deformation retract pair** (NDR pair) if there is a map $u \colon X \to I$ such that $u^{-1}(0) = A$ and a homotopy $h \colon X \times I \to X$ such that $h_0 = \mathrm{id}, h(a,t) = a$ for $a \in A, t \in I$ and $h(x,1) \in A$ if $u(x) < 1$.

Theorem C.7. *Let $A \subseteq X$ be a closed subspace of X. The inclusion $i \colon A \hookrightarrow X$ is a cofibration if and only if (X, A) is an NDR-pair.*

For example, using tubular neighborhoods and the above theorem one can show that any embedding of a smooth submanifold is a cofibration.

C.3. Homotopy Pushouts and Cofibrations

The following theorem shows that homotopy pushouts can be computed using cofibrations.

Theorem C.8. *Consider a diagram as follows, where all the vertical maps are homotopy-equivalences.*

$$\begin{tikzcd} X \arrow[d] & W \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & W' \arrow[l, "f'"'] \arrow[r, "g'"] & Y' \end{tikzcd}$$

1. *Assume that in the diagram both f and f' are cofibrations. Then the induced map of their pushouts (as topological spaces) $X \amalg_A Y \to X' \amalg_{W'} Y'$ is a homotopy equivalence.*

2. *If f is a cofibration, then the natural map $M(f,g) \to X \amalg_W Y$ is a homotopy equivalence.*

3. *The double mapping cones $M(f,g)$ and $M(f',g')$ are homotopy equivalent through the induced map.*

Part (2) shows that the homotopy pushout can be computed by the homotopy pushout if one of the maps is a cofibration. Part (3) shows that the homotopy pushout is homotopy invariant.

Proof. This is 2.1.3 and 2.1.4 in [MP11]. For the first part, see there. In the more general context of model categories, a model category is called proper if (1) holds.

For (2), factor $f\colon W \to X$ into $W \to M(f) \to X$, where $M(f)$ is the mapping cylinder: $M(f) = (([0,1] \times W) \amalg X)/(0,w) \sim f(w)$. The map $W \to M(f)$ is a cofibration and the map $M(f) \to X$ is a homotopy equivalence. Now $M(f,g)$ is the pushout of the cofibration $W \to M(f)$ and the map $W \to Y$ and we conclude the second part.

For (3), as before, $M(f,g)$ is the pushout of the cofibration $W \to M(f)$ and the map $W \to Y$. Similarly $M(f',g')$ is the pushout of the cofibration $W' \to M(f')$ and the map $W' \to Y'$. Hence by (1), the induced map $M(f,g) \to M(f',g')$ is a homotopy equivalence. $\square$

A dual version of the theorem holds for homotopy pullbacks with fibrations instead of cofibrations.

Definition C.9. The homotopy pushout of a map $f\colon A \to B$ together with the canonical map $A \to *$ where $*$ is the one point space is called the **homotopy cofiber** of f and denoted $\mathrm{cofib}(f)$.

By the preceding theorem, if f is a cofibration, then the cofiber is homotopy equivalent to the quotient B/A.

By the construction of the standard homotopy pushout, the homotopy cofiber of $f\colon A \to B$ is constructed by the **mapping cone** $\mathrm{cone}(f) = B \amalg (A \times I)/\sim$, where the equivalence relation $\sim$ is generated by $f(a) \sim (a,0)$ and $(a,1) \sim (a',1)$.

Theorem C.10 (Pasting Lemma)**.** 1. *Given a diagram of two commutative squares, if the right square is a (homotopy) pullback, then the total square is a (homotopy) pullback if and only if the left square is a (homotopy) pullback. Similarly if the left square is a (homotopy) pushout, then the total square is a (homotopy) pushout if and only if the right square is a (homotopy) pushout.*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd} \tag{46}$$

2. *For any homotopy pushout diagram, there is a homotopy equivalence between the homotopy cofibers of the vertical maps (and similarly of the horizontal maps).*

Proof. 1. The statement on regular (non-homotopy) pullbacks and pushouts holds in any category, see e.g. [ML13, p. 72, ex. 8].

In case all the vertical maps are cofibrations, the homotopy pushout is just the normal pushout by Theorem C.8. In the general case, one may factor any map as a cofibration followed by a homotopy equivalence, using the mapping cylinder, hence one may replace Diagram 46 by a homotopy equivalent diagram in which the vertical maps are cofibrations.

2. For the homotopy equivalence of cofibers, consider the diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array} \tag{47}$$

Assume that the left square is a homotopy pushout, then by (1), the right square is a homotopy pushout if and only if the total square is a homotopy pushout. By definition, this means that F is a homotopy cofiber of the map $B \to E$ if and only if it is a homotopy cofiber of the map $A \to D$.

□

We recall the relation between cofibers of spaces and cofibers (mapping cones) of chain complexes.

Definition C.11. The **mapping cone** $\mathrm{cone}(\phi)$ of a chain map $\phi\colon C \to D$ is the complex which, as a graded module is given by $C[-1] \oplus D$ with differential

$$(c, d) \mapsto (-\partial_C c, \phi(c) + \partial_D d).$$

The reduced homology $H_*(X)$ of a space X is the homology of the augmented complex $\tilde{C}_*(X)$, which is the complex $\cdots \to C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$ where the augmentation ε is given by mapping every point to 1.

Proposition C.12. *For any continuous map $f\colon A \to B$, the chain complexes $\tilde{C}_*(\mathrm{cofib}(f))$ and the mapping cone $\mathrm{cone}(C_*(A) \xrightarrow{f_*} C_*(B))$ are quasi-isomorphic. More precisely, there is a sequence of quasi-isomorphisms*

$$\mathrm{cone}(C_*(A) \xrightarrow{f} C_*(B)) \xrightarrow{\simeq} C_*(\mathrm{cofib}(f), \{\omega\}) \xleftarrow{\simeq} \tilde{C}_*(\mathrm{cofib}(f))$$

where $\omega \in \mathrm{cone}(f)$ is the point represented by any $(a, 1)$ for $a \in A$.

In particular, for an inclusion $\iota\colon A \subseteq B$, the complexes $\tilde{C}_*(\mathrm{cofib}(\iota))$ and $C_*(B, A)$ are quasi-isomorphic.

Proof. It is easy to see that the map $\tilde{C}_*(\mathrm{cone}(f)) \xrightarrow{\simeq} C_*(\mathrm{cone}(f), \{\omega\})$ is a quasi-isomorphism.

Let $E, F \subseteq \mathrm{cone}(f)$ be the open subsets corresponding to $A \times [0, \frac{2}{3})$ resp. $A \times (\frac{1}{3}, 1]$. Then F is homotopy equivalent to the end point $\omega \in \mathrm{cone}(f)$ which is represented by any element of $A \times \{1\} \subseteq A \times I$. The space E is homotopy equivalent to B and $E \cap F$ is homotopy equivalent to A. Taking the Mayer-Vietoris of $\mathrm{cone}(f) = E \cup F$ relative to $\{\omega\} \subseteq \mathrm{cone}(f)$ yields an exact sequence

$$\cdots \to H_{*+1}(\mathrm{cone}(f), \{\omega\}) \to H_*(A) \to H_*(B) \to H_*(\mathrm{cone}(f), \{\omega\}) \to \dots$$

On the other hand for the mapping cone of chain complexes $\mathrm{cone}(f_*) = \mathrm{cone}(C_*(A) \to C_*(B))$ there is a long exact sequence

$$\cdots \to H_{*+1}(\mathrm{cone}(f_*)) \to H_*(A) \to H_*(B) \to H_*(\mathrm{cone}(f_*)) \to \dots$$

We define a map of chain complexes $\mathrm{cone}(f_*) \to C_*(\mathrm{cone}(f), \{\omega\})$ as follows. Recall that as graded modules, $\mathrm{cone}(f_*) = C_*(B) \oplus C_*(A)[-1]$. On the $C_*(B)$ summand we use the map $B \to \mathrm{cone}(f)$. On the $C_*(A)[-1]$ summand we use the map $C_{*-1}(A) \to C_*(A \times I), \alpha \mapsto \alpha \times [I]$. This yields a chain map and by the five lemma it is a quasi-isomorphism.

Finally, for an inclusion $A \subseteq B$ there is a chain map $\mathrm{cone}(C_*(A) \to C_*(B)) \to C_*(B, A)$, given by mapping $(a, b) \mapsto b$. Using the long exact sequences for both, one concludes that this is a quasi-isomorphism. □

Remark C.13. An analogue of the theorem holds for cohomology. The analogue of the mapping cone in cohomology is the mapping cocone: Given a map of cochain complexes $\phi\colon C \to D$ this is $\mathrm{cocone}(\phi) = \mathrm{cone}(\phi) = C \times D[1]$ with differential $(c, d) \mapsto (\delta_C c, \phi(c) + \delta_D d)$. For an inclusion $A \subseteq X$, the natural map $C^*(X, A) \to \mathrm{cone}(C^*(X) \to C^*(A))$ is given by $c \mapsto (c|_X, 0)$, where $c|_X$ is the image of $c \in C^*(X, A)$ under the map $C^*(X, A) \to C^*(X)$.

We will require two more results on homotopy pushouts. We caution that the following two theorems do not hold in general model categories, even though they appear like formal results. The following theorem states essentially that the homotopy pullback of a homotopy pushout diagram is still a homotopy pushout diagram.

Theorem C.14 (Mather's Cube Theorem)**.** *Given a cube shaped commutative diagram, such that the bottom face is a homotopy pushout and the side faces are homotopy pullbacks, then the top face is a homotopy pushout.*

This is a theorem from [Mat76]. See also [Str11]

The following theorem states that the pullback of a cofibration along a fibration is again a cofibration.

Theorem C.15. *Given a pullback diagram of topological spaces*

$$\begin{tikzcd} E_A \arrow[r, "j"] \arrow[d, "q"] & E \arrow[d, "p"] \\ A \arrow[r, "i"] & B \end{tikzcd} \tag{48}$$

where i is a cofibration and p is a fibration, then j is a cofibration.

This is [Str11, 14.1].

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Berlin, den 06.03.2024 ....