1 Runge-Kutta Methods

Runge-Kutta methods solve an ODE $\dot{x} = f(x)$, where x is some vector. The idea is to find coefficients b_i and a_{ij} such that

$$x_{n+1} = x_n + \sum_{i=1}^{s} b_i k_i.$$

k_i is a vector of approximate derivatives $k_i \approx f(x_n)$, such that the weighted sum gives a very good approximation to the true answer. The k_i are defined by

$$k_i = hf\left(x_n + \sum_{j=1}^{s} a_{ij} k_j\right).$$

The coefficients of a RK scheme can be put into a Butcher tableau. It is a convention to take $k_1 = f(x_n)$, so there is no top row.

$$\begin{array}{ccc} a_{11} & \cdots & a_{1s} \\ a_{21} & \cdots & a_{2s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{ss} \\ \hline b_1 & \cdots & b_s \end{array}$$

As an example, suppose we have a two-step method. Expanding both sides of the discrete approximation in time, we get

$$x_n + hf(x_n) + \frac{1}{2}h^2 f'(x_n)f(x_n) + O(h)^3 = x_n + b_1 k_1 + b_2 k_2.$$

Comparing terms of order h gives

$$b_1 + b_2 = 1.$$

Comparing terms of order h^2 gives

$$b_1(a_{11} + a_{12}) + b_2(a_{21} + a_{22}) = \frac{1}{2}.$$

Here are some solutions.

Huen's Method

$$\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hline 1/2 & 1/2 \end{array}$$

Midpoint method

$$\begin{array}{cc} 0 & 0 \\ 1/2 & 0 \\ \hline 0 & 1 \end{array}$$

Trapezoid Rule

$$\begin{array}{c|cc} 0 & 0 \\ 1/2 & 1/2 \\ \hline 1/2 & 1/2 \end{array}$$

2 Symplectic Runge-Kutta

Hamilton's equations of motion conserve a symplectic form $\omega = dq^i \wedge dp_i$. If an integrator preserves such a form *exactly*, then the integrator is called a symplectic integrator. Let ω_n be the 2-form at different timesteps.