Modeling the Motion of a Ball Falling with Air Resistance Proportional to Its Velocity Squared

Maxwell Goldberg

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Contents

Abstract

Equations for the motion of an object experiencing air resistance that is proportional to its velocity squared can be solved using more complex integration techniques. This paper will derive these equations and explain the calculus used to derive them.

Setting Up the Equation

Imagine a ball with mass m released from rest and falling through the air. It will experience air resistance that can be modeled as being proportional to its velocity squared. If we examine a free body diagram of the ball, we will see that there is an F_w going down and an F_f going up.

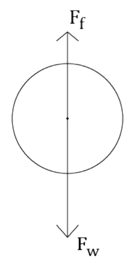

Figure 1: Free Body Diagram of the Ball

We know that $F_w = mg$ and $F_f = bv^2$, where b is a constant encompassing all of the constant factors affecting the air resistance. So, we can use Newton's Second Law to set up an equation. Because acceleration is the derivative of velocity, we can derive a differential equation to solve for an equation for the velocity.

$$\sum F_{net} = ma$$
$$F_f - F_w = ma$$
$$bv^2 - mg = ma$$
$$bv^2 - mg = m\frac{dv}{dt}$$
$$\frac{dt}{m} = \frac{dv}{bv^2 - mg}$$
$$\int \frac{dt}{m} = \int \frac{dv}{bv^2 - mg}$$

The integral on the left is easy to solve. However, the integral on the right requires some advanced integration techniques. Here are two potential ways to solve the integral.

Trigonometric Substitution

A trig sub is a type of u-sub that takes advantage of certain aspects of the integral. Here is a basic example of how trig sub works.

Example

$$\int \sqrt{1-x^2}\ dx$$

In an integral such as this, doing a u-sub seems hard because the derivative of $1-x^2$ is nowhere to be found in the expression. However, it can be approached a different way. If we say $x = \sin\theta$, this is still a valid substitution, but a smart one. It allows us to simplify the equation by using trig specific features, such as identities. This integral can be solved with the following three trig identities: $\sin^2\theta + \cos^2\theta = 1$, $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$, and $\sin 2\theta = 2\sin\theta\cos\theta$.

$$x = \sin\theta$$

$$dx = \cos\theta\ d\theta$$

$$\int \sqrt{1-\sin^2\theta}\cos\theta\ d\theta$$

$$\int \sqrt{\cos^2\theta}\cos\theta\ d\theta$$

$$\int \cos^2\theta\ d\theta$$

$$\frac{1}{2}\int (\cos 2\theta + 1)\ d\theta$$

$$\frac{1}{2}(\frac{\sin 2\theta}{2} + \theta) + C$$

$$\frac{1}{2}(\sin\theta\cos\theta) + \theta) + C$$

The final step is to put the expression back into terms of x. To do so, we must make a right triangle using the initial statement that $\sin\theta = \frac{x}{1}$ in order to derive expressions for the other trigonometric functions that have appeared in the expression. The triangle for this problem is shown in Figure 2 and is found by using the trigonometric functions and the Pythagorean Theorem.

$$\sin\theta = \frac{x}{1}$$

$$\frac{1}{2}(\sin\theta\cos\theta + \theta) + C$$

$$\frac{1}{2}(x\sqrt{1-x^2} + \arcsin x) + C$$

This is the correct answer. You can verify it if you take the derivative. Now, let's try solving the original equation with trig sub.

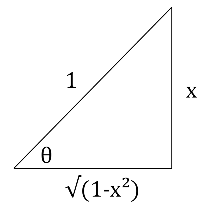

Figure 2: Triangle for Example Trig Sub

Problem

Notice that the denominator of the problem has $bv^2 - mg$. We will want to substitute v with a trig function that will allow us to simplify it with a trigonometric identity. It is of the general form $x^2 - 1$, which allows us to see more easily that this resembles the identity $\tan^2\theta = \sec^2\theta - 1$, so we should substitute v with $\sec x$.

Before we do this though, we need to account for the constants in the equation. Note that the identity is $\tan^2\theta = \sec^2\theta - 1$, not $\tan^2\theta = 2\sec^2\theta - 1$ or $\tan^2\theta = \sec^2\theta - 2$. The secant cannot have a multiple in front and the number being subtracted must be a 1. We can take care of this though. First, we can factor out a b and get $b(v^2 - \frac{mg}{b})$. Now, if we say $v = \sqrt{\frac{mg}{b}}\sec\theta$, then we can see that once it is squared, we can factor out a $\frac{mg}{b}$ and be left with $\sec^2\theta - 1$ in the denominator. Let's try this.

$$\int \frac{dv}{bv^2 - mg}$$

$$\frac{1}{b}\int \frac{dv}{v^2 - \frac{mg}{b}}$$

$$v = \sqrt{\frac{mg}{b}}\sec\theta$$

$$dv = \sqrt{\frac{mg}{b}}\sec\theta\tan\theta\; d\theta$$

$$\frac{1}{b}\int \frac{\sec\theta\tan\theta\sqrt{\frac{mg}{b}}}{\frac{mg}{b}(\sec^2\theta - 1)}\; d\theta$$

$$\frac{1}{b}\sqrt{\frac{b}{mg}}\int \frac{\sec\theta\tan\theta}{\tan^2\theta}\; d\theta$$

$$\frac{1}{\sqrt{bmg}}\int \frac{\sec\theta}{\tan\theta}\; d\theta$$

$$\frac{1}{\sqrt{bmg}} \int \csc\theta \, d\theta$$

$$-\frac{1}{\sqrt{bmg}} \ln|\csc\theta + \cot\theta| + C$$

Now, we must put the equation back into terms of v. Constructing the triangle shown in Figure 3 allows us to replace the $\csc\theta$ and $\cot\theta$, and further simplifying it allows us to get a final answer.

$$\sec\theta = \frac{v\sqrt{b}}{\sqrt{mg}}$$

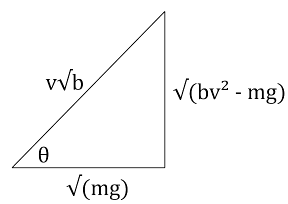

Figure 3: Triangle for Trig Sub for Velocity Integral

$$-\frac{1}{\sqrt{bmg}} \ln\left|\frac{v\sqrt{b}}{\sqrt{bv^2 - mg}} + \frac{\sqrt{mg}}{\sqrt{bv^2 - mg}}\right| + C$$

$$-\frac{1}{\sqrt{bmg}} \ln\left|\frac{v\sqrt{b} + \sqrt{mg}}{\sqrt{bv^2 - mg}}\right| + C$$

$$\frac{1}{\sqrt{bmg}} \left(\ln\left|\sqrt{bv^2 - mg}\right| - \ln\left|v\sqrt{b} + \sqrt{mg}\right|\right) + C$$

$$\frac{1}{\sqrt{bmg}} \left(\frac{1}{2}\ln\left|\left(v\sqrt{b} - \sqrt{mg}\right)\left(v\sqrt{b} + \sqrt{mg}\right)\right| - \ln\left|v\sqrt{b} + \sqrt{mg}\right|\right) + C$$

$$\frac{1}{\sqrt{bmg}} \left(\frac{1}{2}\ln\left|v\sqrt{b} - \sqrt{mg}\right| + \frac{1}{2}\ln\left|v\sqrt{b} + \sqrt{mg}\right| - \ln\left|v\sqrt{b} + \sqrt{mg}\right|\right) + C$$

$$\frac{1}{2\sqrt{bmg}} \left(\ln\left|v\sqrt{b} - \sqrt{mg}\right| - \ln\left|v\sqrt{b} + \sqrt{mg}\right|\right) + C$$

$$\frac{1}{2\sqrt{bmg}} \ln\left|\frac{v\sqrt{b} - \sqrt{mg}}{v\sqrt{b} + \sqrt{mg}}\right| + C$$

This is our final answer for the integral of the right side of the original equation according to trig substitution.

$$\int \frac{dv}{bv^2 - mg} = \frac{1}{2\sqrt{bmg}} \ln \left| \frac{v\sqrt{b} - \sqrt{mg}}{v\sqrt{b} + \sqrt{mg}} \right| + C$$

You could check this by taking the derivative to verify the answer, or you could check by doing the integral a second way.

Partial Fraction Decomposition

The trig sub way may have been hard to follow, so this way may be easier for you. Partial fraction decomposition isn't an integration method like trig sub. Rather it's a way to simplify the integrand. The aim of partial fraction decomp is to take a fraction that is hard to integrate and break it into several smaller fractions that are easy to integrate. Let's do an example.

Example

$$\int \frac{dx}{x^2 + 5x + 6}$$

First, let's look at only the integrand. We want to break it into several fractions whose denominators are the factors of the integrand's denominator. For example, the denominator factors to $(x + 3)(x + 2)$. So, we'll setup the following.

$$\frac{1}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2}$$

We want to split $\frac{1}{(x+3)(x+2)}$ into $\frac{A}{x+3} + \frac{B}{x+2}$, where A and B are the coefficients that make this true. Doing so will eliminate the pesky quadratic from the denominator and make it into a sum of two fractions with linear denominators, whose integrals are just ln. This means that if we can solve for the values for A and B, then we can solve this integral.

So, how do we find A and B? Well, first we will take our original equation and multiply both sides by $(x + 3)(x + 2)$.

$$1 = A(x + 2) + B(x + 3)$$

Now that we have this equation, we can easily get the values of A and B by substituting a value in for x. Note that this equation is true for *any value of* x, so we can pick any value of x to plug in. When we do, we will get a new equation in terms of A and B, making this into a problem of solving a system of equations.

Let's pick an easy value of x to plug in. One such value would $x = -2$ because the $A(x+2)$ would become 0, meaning that we could solve for B. Let's try it.

$$1 = A(-2 + 2) + B(-2 + 3)$$

$$B = 1$$

That was simple. Now let's pick $x = -3$ for similar logic. We could substitute 1 in for B if it helps us solve the equation, but in this case it doesn't matter.

$$1 = A(-3+2) + 1(-3+3)$$
$$A = -1$$

Now, we have our values for A and B.

$$\frac{1}{(x+3)(x+2)} = -\frac{1}{x+3} + \frac{1}{x+2}$$

Let's rewrite the integral and solve it.

$$\int \frac{dx}{x^2+5x+6} = \int \left(\frac{1}{x+2} - \frac{1}{x+3}\right) dx$$
$$\int \frac{dx}{x^2+5x+6} = \ln|x+2| - \ln|x+3| + C$$
$$\int \frac{dx}{x^2+5x+6} = \ln\left|\frac{x+2}{x+3}\right| + C$$

It is important to note that there are more rules for different cases of of partial fraction decomposition. One rule is that the numerator you're solving for must always be set up as being one degree lower than its denominator. For example, if the denominator is x^2+x+1, then the numerator must be $Ax+B$. However, those rules aren't required for the problem at hand, so we won't delve deep into them. Now, we can try to use partial decomp to solve the original integral.

Problem

First, let's decompose our original fraction and multiply both sides by $bv^2 - mg$ as we did in the example.

$$\frac{1}{bv^2 - mg} = \frac{A}{v\sqrt{b} - \sqrt{mg}} + \frac{B}{v\sqrt{b} + \sqrt{mg}}$$
$$1 = A\left(v\sqrt{b} + \sqrt{mg}\right) + B\left(v\sqrt{b} - \sqrt{mg}\right)$$

Now, we must pick values to plug in for x to solve for A and B. Here, it seems that plugging in $x = 0$ will cancel out some terms, so let's try it out.

$$1 = A\left(0\sqrt{b} + \sqrt{mg}\right) + B\left(0\sqrt{b} - \sqrt{mg}\right)$$
$$1 = A\sqrt{mg} - B\sqrt{mg}$$

$$\frac{1}{\sqrt{mg}} = A - B$$

$$A = \frac{1}{\sqrt{mg}} + B$$

We now have one equation to work with. For our second guess, there isn't another value of x that will cancel out terms, so let's pick number that will keep our equation simple, such as $x = 1$. In addition to plugging in $x = 1$, we can also substitute A with the above equation in order to solve for B.

$$1 = \left(\frac{1}{\sqrt{mg}} + B\right)\left(\sqrt{b} + \sqrt{mg}\right) + B\left(\sqrt{b} - \sqrt{mg}\right)$$

$$1 = \sqrt{\frac{b}{mg}} + 1 + B\sqrt{b} + B\sqrt{mg} + B\sqrt{b} - B\sqrt{C}$$

$$-\sqrt{\frac{b}{mg}} = 2B\sqrt{b}$$

$$B = -\frac{1}{2\sqrt{mg}}$$

Now that we have B, we can solve for A using the first equation.

$$A = \frac{1}{\sqrt{mg}} + B$$

$$A = \frac{1}{\sqrt{mg}} - \frac{1}{2\sqrt{mg}}$$

$$A = \frac{1}{2\sqrt{mg}}$$

Now, we have the following as our answer.

$$\frac{1}{bv^2 - mg} = \frac{\frac{1}{2\sqrt{mg}}}{v\sqrt{b} - \sqrt{mg}} - \frac{\frac{1}{2\sqrt{mg}}}{v\sqrt{b} + \sqrt{mg}}$$

Let's substitute this in for $\frac{1}{bv^2 - mg}$ and solve the integral.

$$\frac{1}{2\sqrt{mg}} \int \left(\frac{1}{v\sqrt{b} - \sqrt{mg}} - \frac{1}{v\sqrt{b} + \sqrt{mg}}\right) dv$$

$$\frac{1}{2\sqrt{mg}} \left(\frac{1}{\sqrt{b}} \ln\left|v\sqrt{b} - \sqrt{mg}\right| - \frac{1}{2\sqrt{b}} \ln\left|v\sqrt{b} + \sqrt{mg}\right|\right) + C$$

$$\frac{1}{2\sqrt{bmg}} \ln\left|\frac{v\sqrt{b} - \sqrt{mg}}{v\sqrt{b} + \sqrt{mg}}\right| + C$$

We have gotten the same integral as we did using trig sub but with partial fraction decomposition instead. The math checks out. Let's now move on and finish solving for the velocity equation of the ball.

Solving for the Velocity Equation

Our differential equation comes out to this.

$$\int \frac{dt}{m} = \int \frac{dv}{bv^2 - mg}$$

$$\frac{t}{m} = \frac{1}{2\sqrt{bmg}} \ln \left| \frac{v\sqrt{b} - \sqrt{mg}}{v\sqrt{b} + \sqrt{mg}} \right| + C$$

The first step is solving for C using our initial condition. We know that at time $t = 0$, the velocity is also 0 since the object started falling from rest. If we plug in these values, we will find that C must equal 0.

$$\frac{0}{m} = \frac{1}{2\sqrt{bmg}} \ln \left| \frac{0\sqrt{b} - \sqrt{mg}}{0\sqrt{b} + \sqrt{mg}} \right| + C$$

$$0 = \frac{1}{2\sqrt{bmg}} \ln 1 + C$$

$$C = 0$$

Now, we have to solve the equation for v. First, we will get rid of the ln.

$$\frac{t}{m} = \frac{1}{2\sqrt{bmg}} \ln \left| \frac{v\sqrt{b} - \sqrt{mg}}{v\sqrt{b} + \sqrt{mg}} \right|$$

$$\ln \left| \frac{v\sqrt{b} - \sqrt{mg}}{v\sqrt{b} + \sqrt{mg}} \right| = 2t\sqrt{\frac{bg}{m}}$$

$$\left| \frac{v\sqrt{b} - \sqrt{mg}}{v\sqrt{b} + \sqrt{mg}} \right| = e^{2t\sqrt{bg/m}}$$

$$\frac{v\sqrt{b} - \sqrt{mg}}{v\sqrt{b} + \sqrt{mg}} = \pm e^{2t\sqrt{bg/m}}$$

Before we continue, we must take care of the $\pm$ that resulted from solving for the absolute value signs. If we test our initial condition of $v(0) = 0$, we will find that the $\pm$ must be a negative sign in order to keep the equation true.

$$\frac{0\sqrt{b} - \sqrt{mg}}{0\sqrt{b} + \sqrt{mg}} = \pm e^{2(0)\sqrt{bg/m}}$$

$$-\frac{\sqrt{mg}}{\sqrt{mg}} = \pm e^0$$

$$-1 = \pm 1$$

$$-1 = -1$$

Now, we can continue to solve for v and finally get an equation for velocity.

$$\frac{v\sqrt{b}-\sqrt{mg}}{v\sqrt{b}+\sqrt{mg}} = -e^{2t\sqrt{bg/m}}$$

$$v\sqrt{b}-\sqrt{mg} = -ve^{2t\sqrt{bg/m}}\sqrt{b} - e^{2t\sqrt{bg/m}}\sqrt{mg}$$

$$v\sqrt{b}+ve^{2t\sqrt{bg/m}}\sqrt{b} = \sqrt{mg} - e^{2t\sqrt{bg/m}}\sqrt{mg}$$

$$v\sqrt{b}\left(1+e^{2t\sqrt{bg/m}}\right) = \sqrt{mg}\left(1-e^{2t\sqrt{bg/m}}\right)$$

$$v = \sqrt{\frac{mg}{b}}\,\frac{1-e^{2t\sqrt{bg/m}}}{1+e^{2t\sqrt{bg/m}}}$$

Now that we have the equation for velocity, we could differentiate it to find an equation for acceleration and integrate it to find an equation for position. However, doing that with this equation would be a pain. Instead, we can simplify the equation, but to do so, we must be familiar with the hyperbolic trigonometric functions.

Hyperbolic Trigonometric Functions

Definitions

Everyone knows the regular trigonometric functions. However, few know the hyperbolic trigonometric functions. The central difference between the two is that while the normal trig functions have to do with a unit circle, the hyperbolic trig functions have to do with a unit hyperbola. Their names are the same as the normal trig functions except with an h after them. For example, hyperbolic sine is $\sinh x$. We could derive the mathematical definitions of $\sinh x$ and $\cosh x$, but that will get us too off-topic. Instead, they are given below.

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

With these definitions, we can derive definitions for all of the other hyperbolic functions.

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

It should be noted that $\operatorname{csch} x$ and $\coth x$ have vertical asymptotes at $x = 0$, but other than that, none of the hyperbolic functions have restrictions on their domains. Now we have definitions for all of the hyperbolic functions.

Derivatives and Integrals

We can derive formulas for the derivatives and integrals for all of the hyperbolic functions by using their definitions. Let's start by working out the derivative and integral for $\sinh x$.

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$
$$\frac{d}{dx}\sinh x = \frac{e^x + e^{-x}}{2}$$
$$\frac{d}{dx}\sinh x = \cosh x$$

$$\int \sinh x = \int \frac{e^x - e^{-x}}{2}\,dx$$
$$\int \sinh x = \frac{e^x + e^{-x}}{2} + C$$
$$\int \sinh x = \cosh x + C$$

This shows us how the derivatives of $\sinh x$ and $\cosh x$ are cyclical just like $\sin x$ and $\cos x$, except without any sign changes. This can be used to derive the derivatives for the other four hyperbolic functions. Here's a list of them all.

$$\frac{d}{dx}\sinh x = \cosh x + C$$
$$\frac{d}{dx}\cosh x = \sinh x + C$$
$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x + C$$
$$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x + C$$
$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x + C$$
$$\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x + C$$

The integrals for hyperbolic functions are the reverse of the above derivatives. For example, $\int \operatorname{sech}^2 x = \tanh x + C$. Now that we have this information, we can continue with the problem.

Extra Knowledge

Similar to the normal trig functions, there are also hyperbolic identities and inverse hyperbolic functions. We will not be using them to help find the equations for position and acceleration. Instead, this is only for you to further your knowledge of the hyperbolic functions.

An example of an identity is $\cosh^2 x - \sinh^2 x = 1$. Let's try verifying it.

$$\begin{gathered}
\cosh^2 x - \sinh^2 x = 1 \\
\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1 \\
\frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = 1 \\
\frac{4}{4} = 1 \\
1 = 1
\end{gathered}$$

This is the hyperbolic analog of the Pythagorean Identity $\sin^2 x + \cos^2 x = 1$. There are also analogous identities for the other two Pythagorean Identities, $\tan^2 x + 1 = \sec^2 x$ and $\cot^2 x + 1 = \csc^2 x$.

$$\begin{gathered}
\tanh^2 x + \operatorname{sech}^2 x = 1 \\
\coth^2 x - \operatorname{csch}^2 x = 1
\end{gathered}$$

There are various other identities that resemble the normal trig identities. They are listed below, but you could always try verifying them if you'd like.

$$\begin{gathered}
\sinh 2x = 2 \sinh x \cosh x \\
\cosh 2x = \cosh^2 x + \sinh^2 x \\
\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \\
\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y
\end{gathered}$$

The inverse hyperbolic functions are not very common, so we will only go over the definitions. Here are the definitions of $\operatorname{arcsinh} x$, $\operatorname{arccosh} x$, and $\operatorname{arctanh} x$.

$$\begin{gathered}
\operatorname{arcsinh} x = \ln(x + \sqrt{x^2 + 1}) \\
\operatorname{arccosh} x = \ln(x + \sqrt{x^2 - 1}) \\
\operatorname{arctanh} x = \frac{1}{2} \ln(\frac{1 + x}{1 - x})
\end{gathered}$$

There are restrictions on the domains of all of the inverse hyperbolic functions except for $\operatorname{arcsinh} x$. There are differentiation and integration rules for these too, but this extra knowledge section isn't meant to be too long, so we won't discuss them.

Deriving Equations for Position and Acceleration

How Hyperbolic Functions Fit into the Problem

Before we solve for the position and acceleration equations, let's manipulate the equation for $\tanh x$ by getting rid of all of the negative exponents in order to make it more familiar.

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$$

Now, the velocity equation looks very similar to $\tanh x$. If we factor out a -1, then we'll see they're in the same form.

$$v = \sqrt{\frac{mg}{b}}\,\frac{1 - e^{2t\sqrt{bg/m}}}{1 + e^{2t\sqrt{bg/m}}}$$
$$v = -\sqrt{\frac{mg}{b}}\,\frac{e^{2t\sqrt{bg/m}} - 1}{e^{2t\sqrt{bg/m}} + 1}$$

This means that the velocity equation for the ball can be rewritten in terms of $\tanh x$. However, we must first deal with the $\sqrt{\frac{bg}{m}}$ in the exponent. In the equation for $\tanh x$, we can see that the x inside $\tanh x$ is equal to the $2x$ in the e^{2x} in the definition on the right. Therefore, we can conclude that if we were to pit the velocity equation in terms of $\tanh t$, then the $2t\sqrt{\frac{bg}{m}}$ would be $t\sqrt{\frac{bg}{m}}$ in the tanh. So, let's rewrite it as tanh.

$$v = -\sqrt{\frac{mg}{b}}\,\frac{e^{2t\sqrt{bg/m}} - 1}{e^{2t\sqrt{bg/m}} + 1}$$
$$v = -\sqrt{\frac{mg}{b}}\,\tanh\left(t\sqrt{\frac{bg}{m}}\right)$$

Now, we can take the derivative and integral of this equation much more easily.

Finding the Position and Acceleration Equations

Let's first find the acceleration equation by taking the derivative of the velocity equation.

$$\frac{dv}{dt} = -\sqrt{\frac{mg}{b}}\,\frac{d}{dt}\left[\tanh\left(t\sqrt{\frac{bg}{m}}\right)\right]$$

$$a = -g\,\mathrm{sech}^2\left(t\sqrt{\frac{bg}{m}}\right)$$

That was relatively easy. Now, how about position? The integral of the velocity equation isn't as straight forward, but it's easy with a u-sub.

$$\int v\;dt = -\sqrt{\frac{mg}{b}}\int \tanh\left(t\sqrt{\frac{bg}{m}}\right)dt$$

$$x = -\sqrt{\frac{mg}{b}}\int \frac{\sinh\left(t\sqrt{bg/m}\right)}{\cosh\left(t\sqrt{bg/m}\right)}dt$$

$$u = \cosh\left(t\sqrt{\frac{bg}{m}}\right)$$

$$du = \sqrt{\frac{bg}{m}}\;\sinh\left(\sqrt{\frac{bg}{m}}\right)\;dt$$

$$x = -\frac{m}{b}\int\frac{du}{u}$$

$$x = -\frac{m}{b}\ln|u| + C$$

$$x = -\frac{m}{b}\ln\left|\cosh\left(t\sqrt{\frac{bg}{m}}\right)\right| + C$$

If the position the ball is released from is treated as $x = 0$, then we can use this initial condition to solve for C.

$$0 = -\frac{m}{b}\ln\left|\cosh\left(0\sqrt{\frac{bg}{m}}\right)\right| + C$$

$$C = 0$$

So, our final position, velocity, and acceleration equations are the following.

$$x = -\frac{m}{b}\ln\left|\cosh\left(t\sqrt{\frac{bg}{m}}\right)\right|$$

$$v = -\sqrt{\frac{mg}{b}}\;\tanh\left(t\sqrt{\frac{bg}{m}}\right)$$

$$a = -g\,\mathrm{sech}^2\left(t\sqrt{\frac{bg}{m}}\right)$$

Conclusion

Numerically Verifying Our Answers

So far we have only verified these equations in a theoretical sense. However, what if we made a mistake or interpreted something incorrectly? Can we check this somehow? Scientists often require experimental evidence to verify theories, so we will get some type of "experimental" evidence to verify this result too. I have written a program in QuickBasic64 using numerical integration to confirm whether these equations are correct. Here is the output.

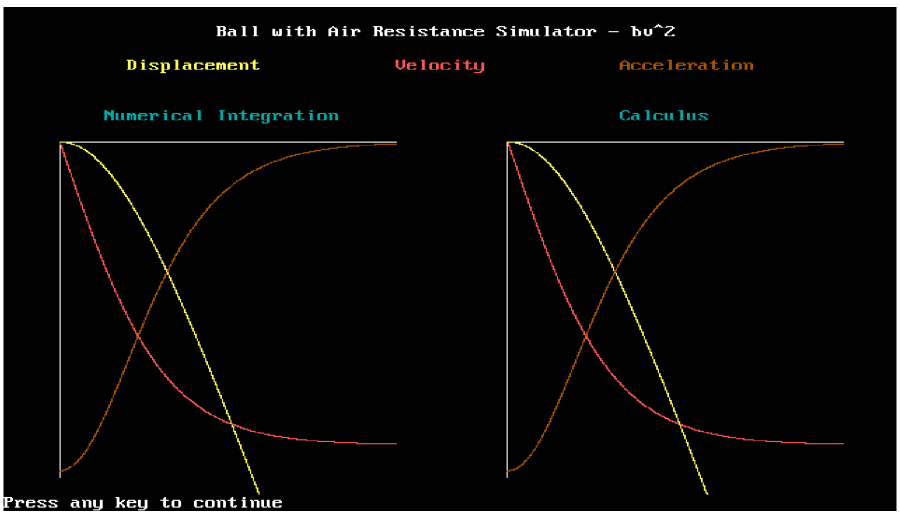

Figure 4: QB64 Simulation of the Problem

As you can see in Figure 4, the program plots the motion of the ball using numerical integration on the left and our equations on the right. They are practically identical, so we can confidently conclude that our equations are correct.

Closing Remarks

I hope this paper helped you to learn about new integration techniques and the hyperbolic functions. I also hope our conclusion gave you insight into how objects around us work in our everyday lives. It's a common theme in physics for seemingly simple or common occurrences to be extremely hard to understand, sometimes even impossible. However, we still must strive to understand the workings of these complex situations. Doing so will help us to better understand the world, which will lead us to create better inventions, simulations, and ideas that will benefit humanity as a whole.

List of Figures