Lecture 05:
Image Mappings / Homographies

Image Mappings
Overview

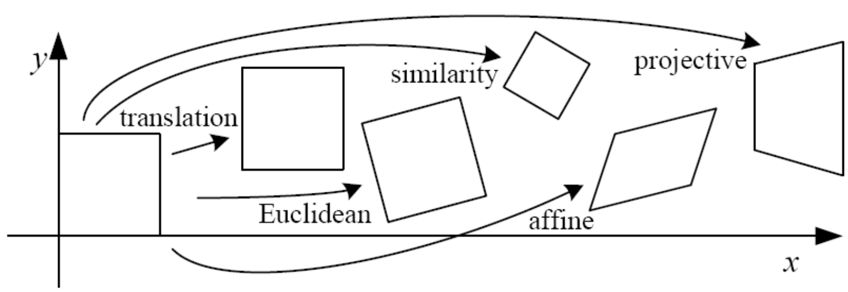
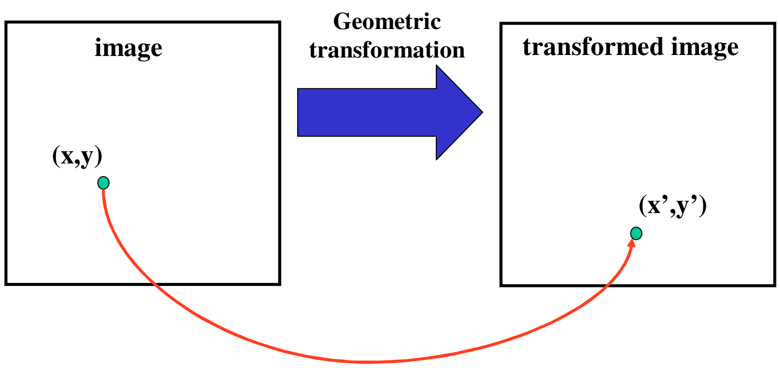


FIGURE 1. Basic set of 2D planar transformations

picture from R.Szeliski

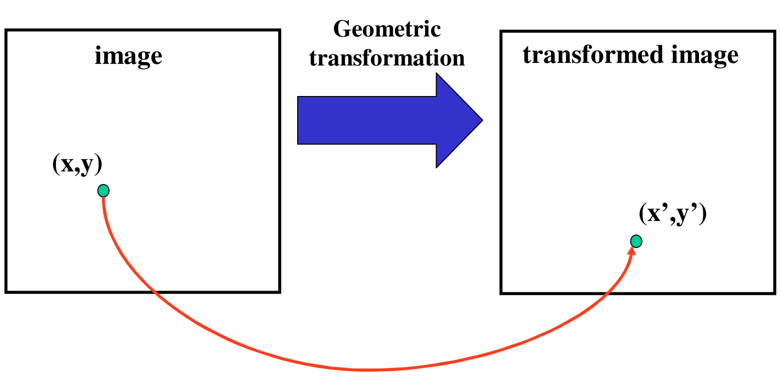
Geometric Image Mappings



$$x' = f(x, y, \{parameters\})$$
$$y' = g(x, y, \{parameters\})$$

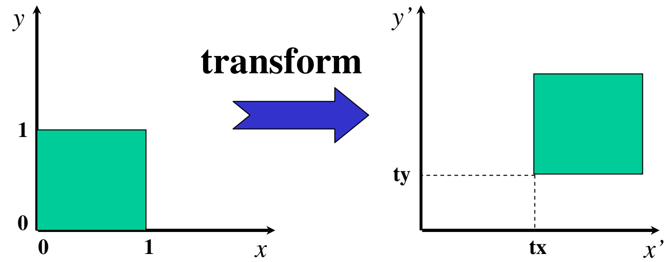
Linear Transformations

(Can be written as matrices)



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \left(M(params) \right) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translation



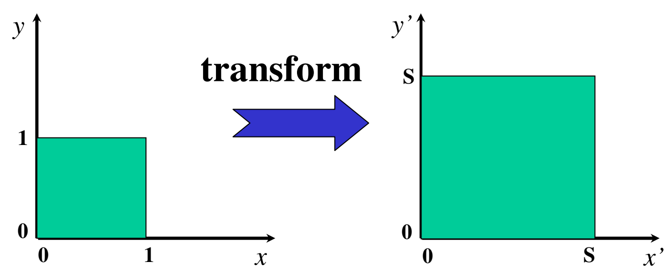
$$x' = x + t_x$$
$$y' = y + t_y$$

equations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

matrix form

Scale



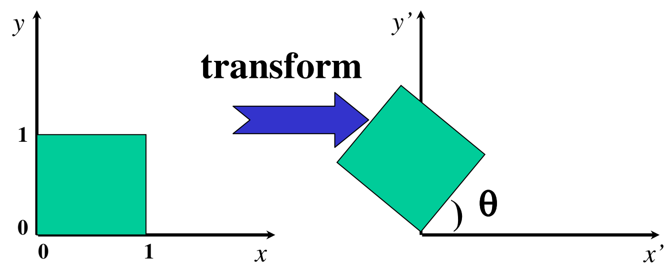
$$x' = s\, x_i$$
$$y' = s\, y_i$$

equations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

matrix form

Rotation

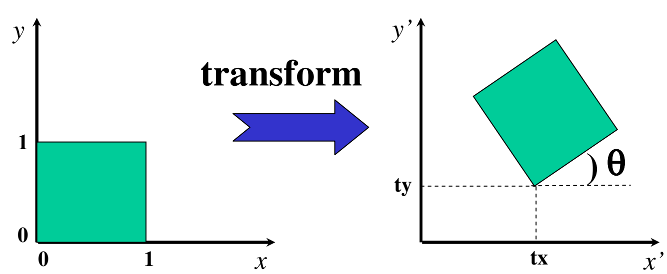


$$\begin{aligned} x' &= x_i \cos\theta - y_i \sin\theta \\ y' &= x_i \sin\theta + y_i \cos\theta \end{aligned}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

equations **matrix form**

Euclidean (Rigid)



$$\begin{aligned} x' &= x_i \cos\theta - y_i \sin\theta + t_x \\ y' &= x_i \sin\theta + y_i \cos\theta + t_y \end{aligned}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

equations **matrix form**

Partitioned Matrices

A *partitioned matrix*, or a *block matrix*, is a matrix M that has been constructed from other smaller matrices. These smaller matrices are called *blocks* or *sub-matrices* of M.

For instance, if we partition the below 5×5 matrix as follows

$$L = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 9 & 9 & 9 \\ 2 & 3 & 9 & 9 & 9 \\ 2 & 3 & 9 & 9 & 9 \end{array}\right),$$

then we can define the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, C = \begin{pmatrix} 2 & 3 \\ 2 & 3 \\ 2 & 3 \end{pmatrix}, D = \begin{pmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{pmatrix}$$

and write L as

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{or } L = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).$$

http://planetmath.org/encyclopedia/PartitionedMatrix.html

Partitioned Matrices

$$\left[\begin{array}{c} x' \\ y' \\ \hline 1 \end{array}\right] = \left[\begin{array}{cc|c} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} x \\ y \\ \hline 1 \end{array}\right]$$

$$\begin{bmatrix} \overset{\text{2x1}}{p'} \\ \overset{\text{1x1}}{1} \end{bmatrix} = \begin{bmatrix} \overset{\text{2x2}}{R} & \overset{\text{2x1}}{t} \\ \overset{\text{1x2}}{0} & \overset{\text{1x1}}{1} \end{bmatrix} \begin{bmatrix} \overset{\text{2x1}}{p} \\ \overset{\text{1x1}}{1} \end{bmatrix}$$ **matrix form**

$$p' = Rp + t$$ **equation form**

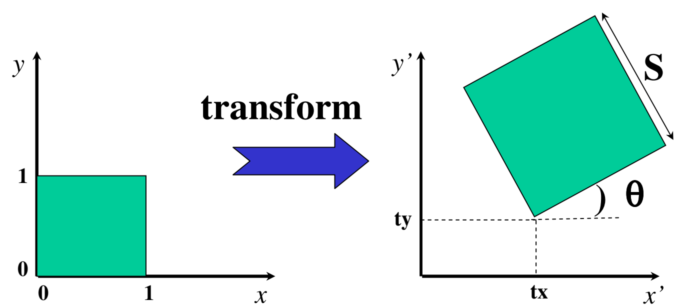
Another Example (from last time)

$$\begin{pmatrix} X \\ Y \\ Z \\ \hline 1 \end{pmatrix} = \left(\begin{array}{ccc|c} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} U \\ V \\ W \\ \hline 1 \end{pmatrix}$$

$$\begin{pmatrix} \overset{3x1}{\mathbf{P_C}} \\ \overset{1x1}{1} \end{pmatrix} = \begin{pmatrix} \overset{3x3}{\mathbf{R}} & \overset{3x1}{\mathbf{T}} \\ \overset{1x3}{0} & \overset{1x1}{1} \end{pmatrix} \begin{pmatrix} \overset{3x1}{\mathbf{P_W}} \\ \overset{1x1}{1} \end{pmatrix}$$

$$\mathbf{P_C = R\, P_W + T}$$

Similarity (scaled Euclidean)



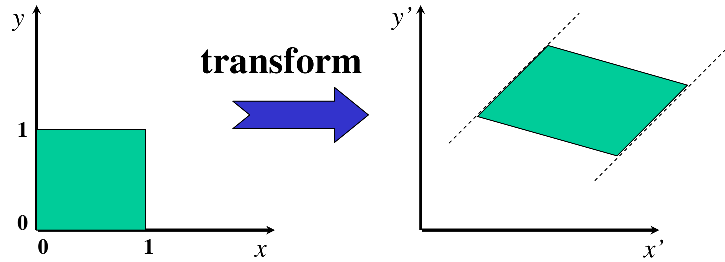
$$p' = sRp + t$$

equations

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = \begin{bmatrix} sR & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

matrix form

Affine



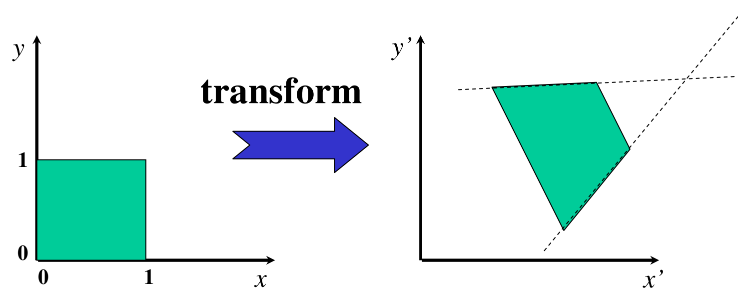
$$p' = Ap + b$$

equations

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

matrix form

Projective



$$p' = \frac{Ap + b}{c^T p + 1}$$

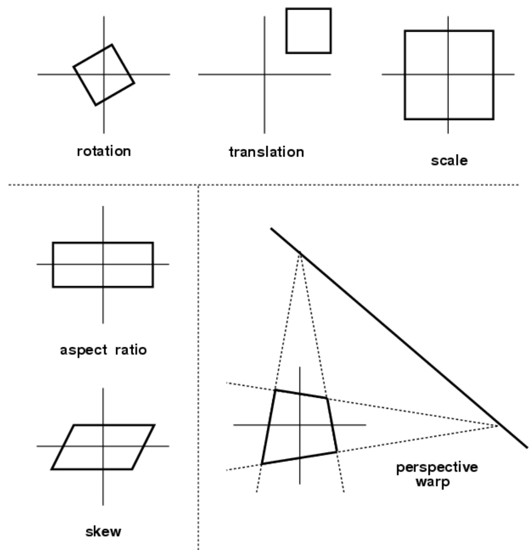
equations

Note!

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} \sim \begin{bmatrix} A & b \\ c^T & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

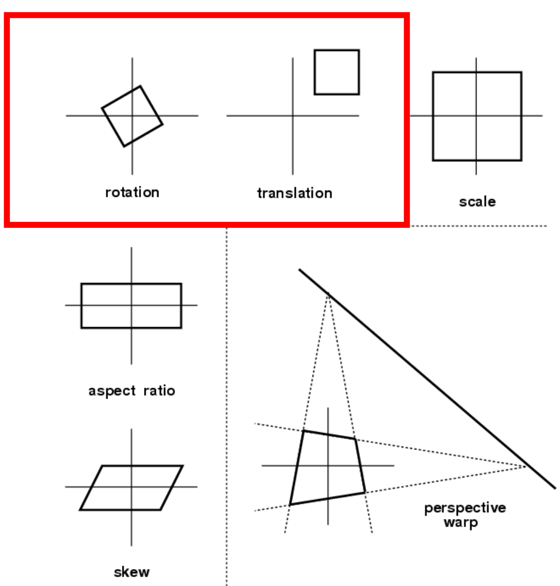
matrix form

Summary of 2D Transformations



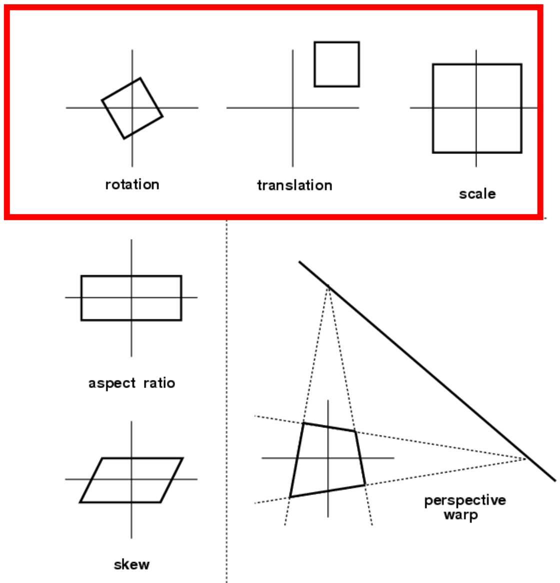
Summary of 2D Transformations

Euclidean



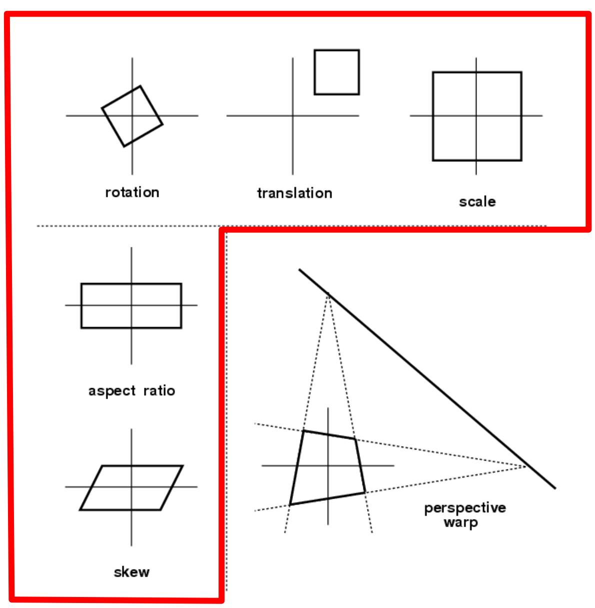
Summary of 2D Transformations

Similarity



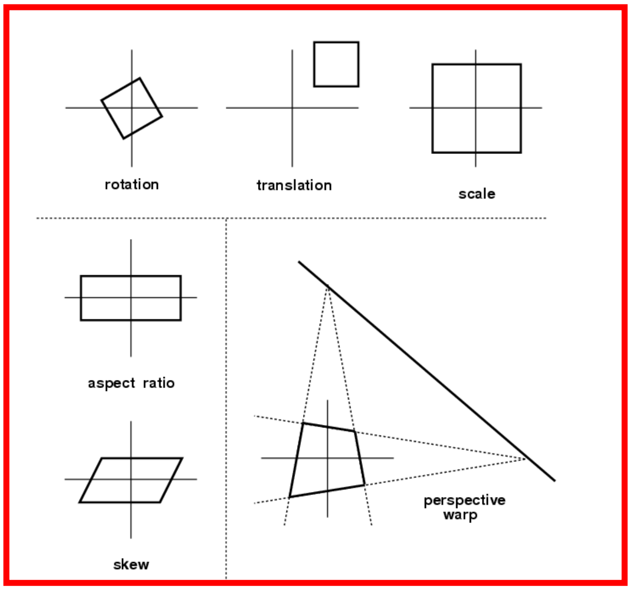
Summary of 2D Transformations

Affine

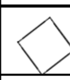
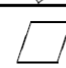


Summary of 2D Transformations

Projective



Summary of 2D Transformations

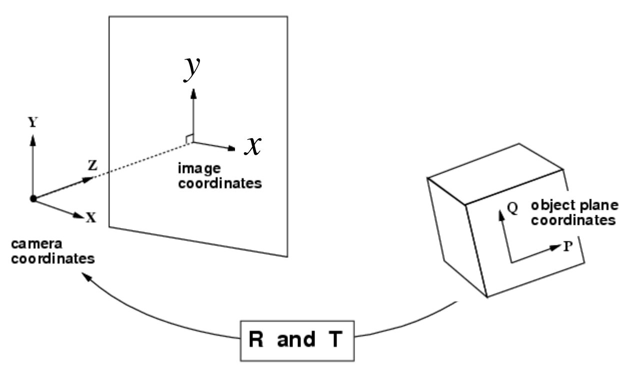
Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\begin{bmatrix} I \mid t \end{bmatrix}_{2\times 3}$	2	orientation $+\cdots$	
rigid (Euclidean)	$\begin{bmatrix} R \mid t \end{bmatrix}_{2\times 3}$	3	lengths $+\cdots$	
similarity	$\begin{bmatrix} sR \mid t \end{bmatrix}_{2\times 3}$	4	angles $+\cdots$	
affine	$\begin{bmatrix} A \end{bmatrix}_{2\times 3}$	6	parallelism $+\cdots$	
projective	$\begin{bmatrix} H \end{bmatrix}_{3\times 3}$	8	straight lines	

from R.Szeliski

Why do we care?

Planar Homographies

Motivation:
Points on Planar Surface



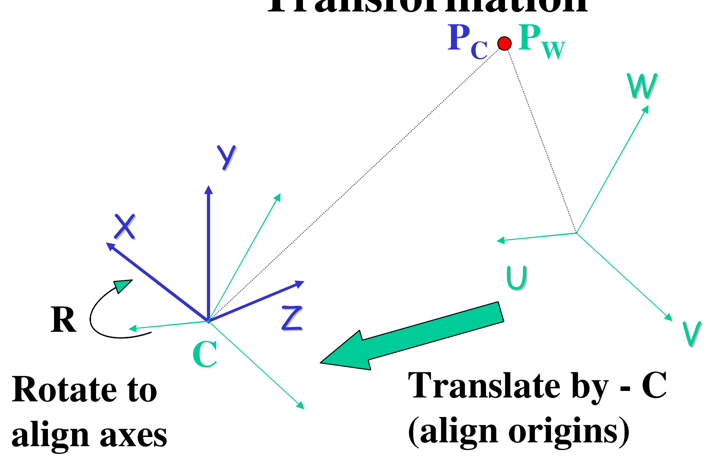
- 3d-to-2d mapping reduces to a 2d-to-2d mapping
- it becomes invertible!!!

Review : Forward Projection

World Coords		Camera Coords		Film Coords		Pixel Coords
U V W	M_{ext} →	X Y Z	M_{proj} →	x y	M_{aff} →	u v
U V W		X Y Z	M_{int} →			u v
U V W	M →					u v

$$\begin{matrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{31} & m_{33} & m_{34} \end{matrix}$$

Review: World to Camera Transformation



$$P_C = R\,(P_W - C)$$
$$= R\,P_W + T$$

Review: Perspective Matrix Equation

(Camera Coordinates)

$$x = f\frac{X}{Z}$$

$$y = f\frac{Y}{Z}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$p = M_{\text{proj}} \cdot P_C$$

Review: Film to Pixel Coords

In general, just think of this as a 2D affine transformation from film coords (x,y) to pixel coordinates (u,v):

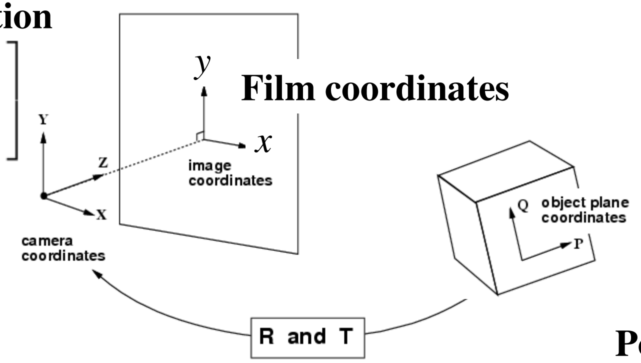
$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{M_{aff}}} \underbrace{\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{M_{proj}}} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\mathbf{u = M_{int}\, P_C = M_{aff}\, M_{proj}\, P_C}$$

Projection of Points on Planar Surface

Perspective projection

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} p \\ q \\ 0 \\ 1 \end{bmatrix}$$

Point on plane

Rotation + Translation

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection of Planar Points

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ \boxed{0} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

Projection of Planar Points (cont)

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} fr_{11} & fr_{12} & ft_x \\ fr_{21} & fr_{22} & ft_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

Homography H (planar projective transformation)

Projection of Planar Points (cont)

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

Homography H (planar projective transformation)

Important: For planar surfaces, 3D to 2D perspective projection reduces to a 2D to 2D transformation.

Important^2: This transformation is INVERTIBLE!

Special Case : Frontal Plane

What if the planar surface is perpendicular to the optic axis (Z axis of camera coord system)?

Then world rotation matrix simplies:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Special Case: Frontal Plane

So the homography for a frontal plane simplies:

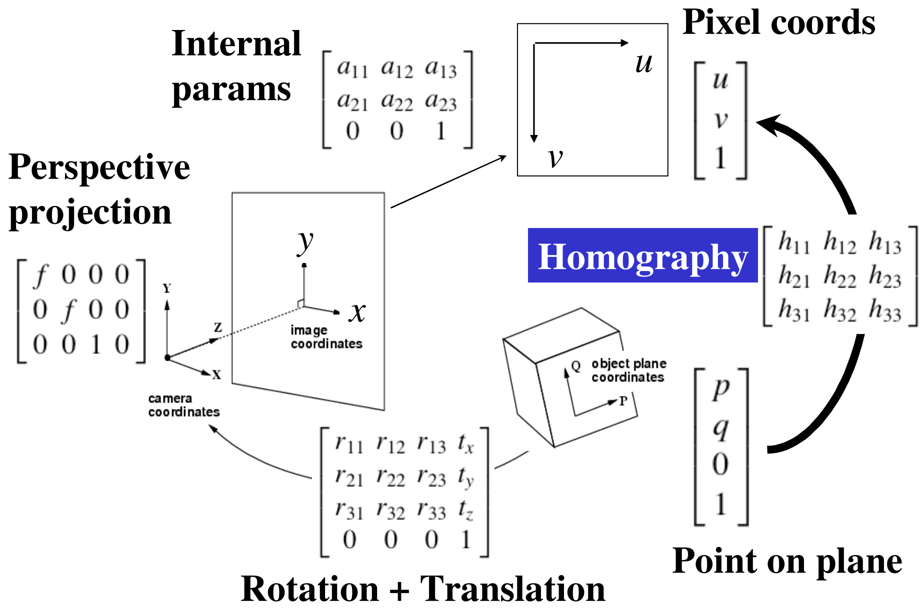
$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} fr_{11} & fr_{12} & ft_x \\ fr_{21} & fr_{22} & ft_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

$$\Downarrow$$

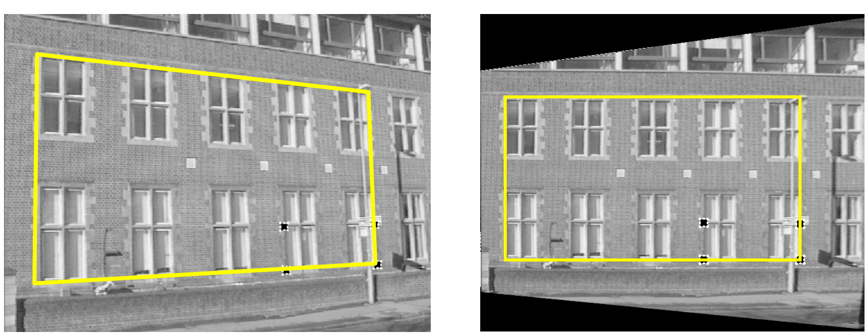
$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f\cos\theta & -f\sin\theta & ft_x \\ f\sin\theta & f\cos\theta & ft_y \\ 0 & 0 & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

Similarity Transformation!

Summary: Planar Projection



Applying Homographies to Remove Perspective Distortion

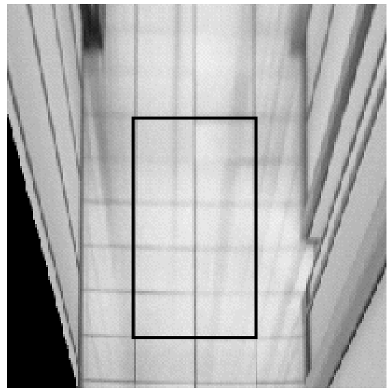


from Hartley & Zisserman

4 point correspondences suffice for
the planar building facade

Yin, MST
Spring, 2012

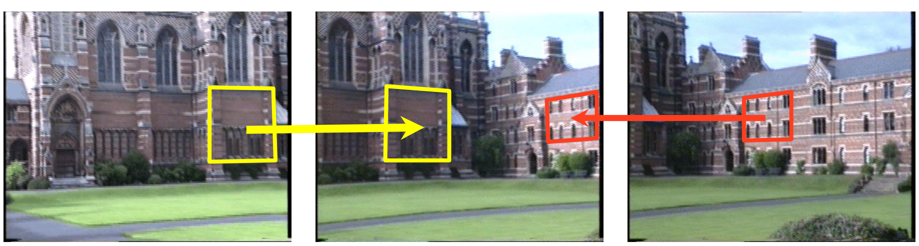
Homographies for Bird's-eye Views



from Hartley & Zisserman

Yin, MST
Spring, 2012

Homographies for Mosaicing



from Hartley & Zisserman

Two Practical Issues

How to estimate the homography given four or more point correspondences (least squares / RANSAC : next week)

How to (un)warp image pixel values to produce a new picture (next week)