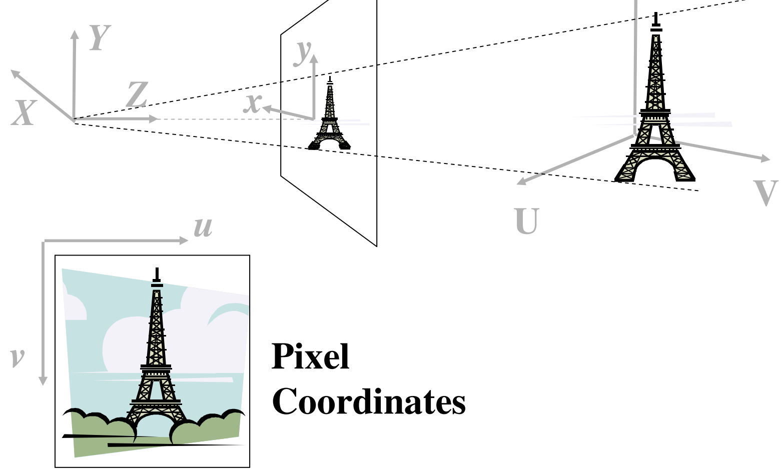
Teams

Teams	Projects
Stewart Boling, Nicolas Pereira	Face detection, identification and recognition from mobile devices
Michael Wisely, Mat Nuckolls	Undecided yet
Nate Eloe, Jacob Gardener	Undecided yet

Lecture 04:
Camera Projection (cont.)

Imaging Geometry

Camera Coordinates | **Image (film) Coordinates** | **World Coordinates**



Forward Projection

World Coords		Camera Coords		Film Coords		Pixel Coords
U, V, W	→	X, Y, Z	→	x, y	→	u, v

We want a mathematical model to describe how 3D World points get projected into 2D Pixel coordinates.

Our goal: describe this sequence of transformations by a big matrix equation!

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Spring 2012

Forward Projection

World Coords		Camera Coords		Film Coords		Pixel Coords
U, V, W	→	X, Y, Z	→	*x*, *y*	→	*u*, *v*

Rigid Transformation (rotation+translation) between world and camera coordinate systems

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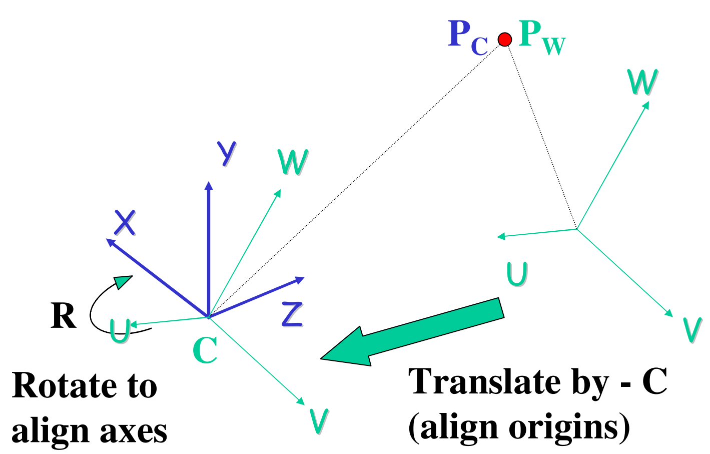
Spring 2012

World to Camera Transformation



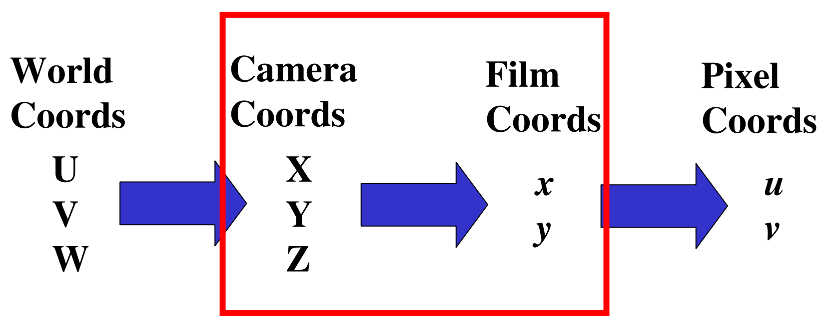
$$P_C = R\,(P_W - C)$$

Matrix Form, Homogeneous Coords

$$\mathbf{P_C = R\ (\ P_W - C\ )}$$

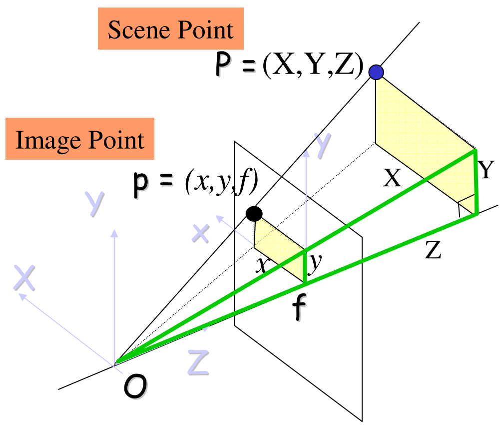
$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -c_y \\ 0 & 0 & 1 & -c_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

Forward Projection



3D-to-2D Projection
• perspective projection

Basic Perspective Projection



Perspective Projection Eqns

$$x = f\,\frac{X}{Z}$$

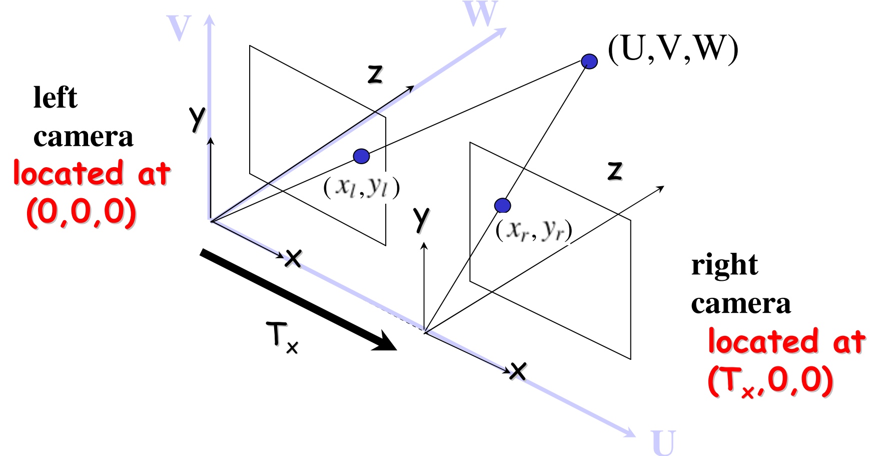
$$y = f\,\frac{Y}{Z}$$

Perspective Matrix Equation

(in Camera Coordinates)

$$x = f\,\frac{X}{Z} \qquad y = f\,\frac{Y}{Z} \quad \Longleftrightarrow \quad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Example: Simple Stereo System



Left camera located at world origin (0,0,0)
and camera axes aligned with world coord axes.

Simple Stereo, Left Camera

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

camera axes aligned with world axes

located at world position (0,0,0)

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

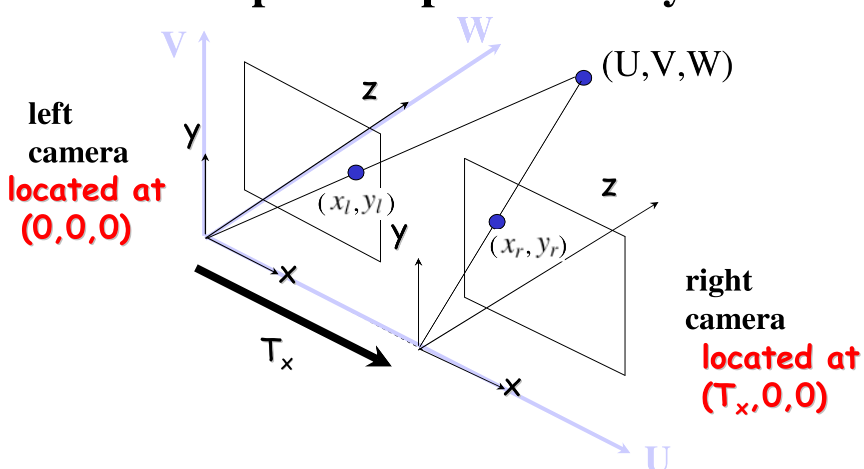
Simple Stereo Projection Equations

Left camera

$$\begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{U} \\ \mathrm{V} \\ \mathrm{W} \\ 1 \end{bmatrix}$$

$$x_l = f\,\frac{\mathrm{U}}{\mathrm{W}} \qquad y_l = f\,\frac{\mathrm{V}}{\mathrm{W}}$$

Example: Simple Stereo System



Right camera located at world location (Tx,0,0)
and camera axes aligned with world coord axes.

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Spring 2012

Simple Stereo, Right Camera

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \mathbf{-T_x} \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

camera axes aligned with world axes

located at world position (T_x,0,0)

$$= \begin{bmatrix} 1 & 0 & 0 & -T_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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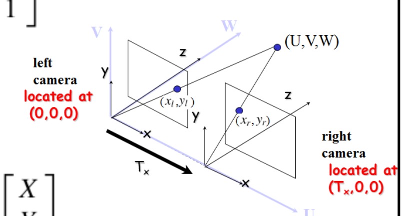
Simple Stereo Projection Equations

Left camera

$$\begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \\ 1 \end{bmatrix}$$

$$x_l = f\,\frac{U}{W} \qquad y_l = f\,\frac{V}{W}$$



Right camera

$$\begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -T_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$x_r = f\,\frac{U - T_x}{W} \qquad y_r = f\,\frac{V}{W}$$

Our Trick(s) to Figure out the Rotation

$$\begin{pmatrix}X\\Y\\Z\\1\end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0\\ r_{21} & r_{22} & r_{23} & 0\\ r_{31} & r_{32} & r_{33} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cancel{1} & 0 & 0 & \cancel{-a_x}\\ & & & \\ & & & \\ \cancel{0} & 0 & 0 & \cancel{1}\end{pmatrix}\begin{pmatrix}U\\V\\W\\1\end{pmatrix}$$

forget about this while thinking about rotations

$$\mathbf{P_C = R\ P_W}$$

This equation says how vectors in the world coordinate system (including the coordinate axes) get transformed into the camera coordinate system.

Figuring out Rotations

$$\begin{pmatrix}X\\Y\\Z\\1\end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0\\ r_{21} & r_{22} & r_{23} & 0\\ r_{31} & r_{32} & r_{33} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}U\\V\\W\\1\end{pmatrix} \qquad \mathbf{P_C = R\ P_W}$$

what if world U axis (1,0,0) corresponds to camera axis (a,b,c)?

$$\begin{pmatrix}\mathbf{a}\\\mathbf{b}\\\mathbf{c}\\1\end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0\\ r_{21} & r_{22} & r_{23} & 0\\ r_{31} & r_{32} & r_{33} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\\1\end{pmatrix} \Rightarrow \begin{pmatrix}\mathbf{a}\\\mathbf{b}\\\mathbf{c}\\1\end{pmatrix} = \begin{pmatrix} \mathbf{a} & r_{12} & r_{13} & 0\\ \mathbf{b} & r_{22} & r_{23} & 0\\ \mathbf{c} & r_{32} & r_{33} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\\1\end{pmatrix}$$

we can immediately write down the first column of R!

Figuring out Rotations

and likewise with world V axis and world W axis...

same axis in camera coords

axis in world coords

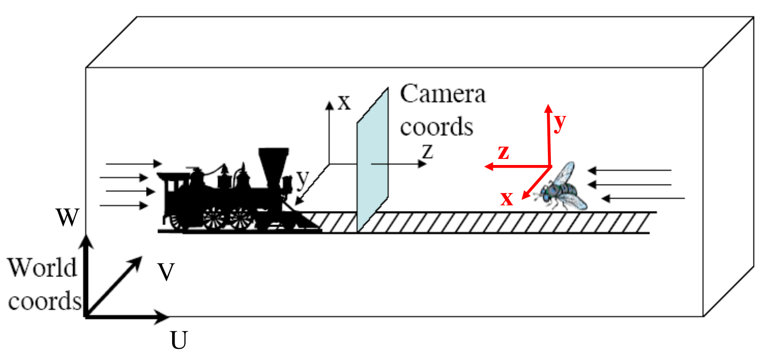
$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

world U axis (1,0,0)
in camera coords

world V axis (0,1,0)
in camera coords

world W axis (0,0,1)
in camera coords

Example



$$R_{train} \quad \begin{matrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{matrix} \qquad\qquad R_{fly} \quad \begin{matrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{matrix}$$

Figuring out Rotations

Alternative approach: sometimes it is easier to specify what camera X,Y,or Z axis is in world coordinates. Then do rearrange the equation as follows.

$$\mathbf{P_C = R\, P_W} \Rightarrow \mathbf{R^{-1}P_C = P_W} \overset{*}{\Rightarrow} \mathbf{R^{T}P_C = P_W}$$

$$\begin{pmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

* Rotation matrix is an orthogonal matrix.

Figuring out Rotations

$$\begin{pmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix} \qquad \mathbf{R^{T}P_C = P_W}$$

what if camera X axis (1,0,0) corresponds to world axis (a,b,c)?

$$\begin{pmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbf{a} & r_{21} & r_{31} & 0 \\ \mathbf{b} & r_{22} & r_{32} & 0 \\ \mathbf{c} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ 1 \end{pmatrix}$$

we can immediately write down the first column of R^T, *(which is the first row of R).*

Figuring out Rotations

and likewise with camera Y axis and camera Z axis...

same axis in camera coords

axis in world coords

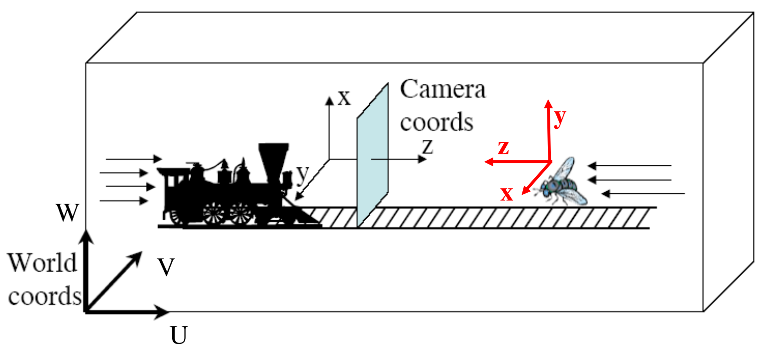
$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

camera X axis (1,0,0)
in world coords

camera Y axis (0,1,0)
in world coords

camera Z axis (0,0,1)
in world coords

Example



$$R_{train} \quad \begin{matrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{matrix} \qquad\qquad R_{fly} \quad \begin{matrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{matrix}$$

Note: External Parameters also often written as R,T

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -c_y \\ 0 & 0 & 1 & -c_z \\ 0 & 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

R (P_W - C)

= R P_W - R C

= R P_W + T

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

PUTTING IT ALL TOGETHER

3D to 2D Projection : Matrix Form

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -c_y \\ 0 & 0 & 1 & -c_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

2d film coordinates :

$u = x' / z'$

$v = y' / z'$

Short Summary

World Coords (U, V, W) → Camera Coords (X, Y, Z) → Film Coords (x, y) → Pixel Coords (u, v)

We now know how to transform 3D world coordinate points into camera coords, and then do perspective project to get 2D points in the film plane.

Intrinsic Camera Parameters

World Coords (U, V, W) → Camera Coords (X, Y, Z) → Film Coords (x, y) → Pixel Coords (u, v)

Affine Transformation

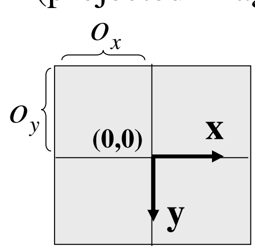
Intrinsic parameters

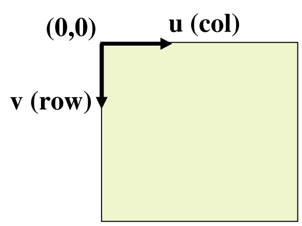
- Describes coordinate transformation between film coordinates (projected image) and pixel array
- Film cameras: scanning/digitization
- CCD cameras: grid of photosensors

Intrinsic parameters (offsets)

film plane
(projected image)

pixel array



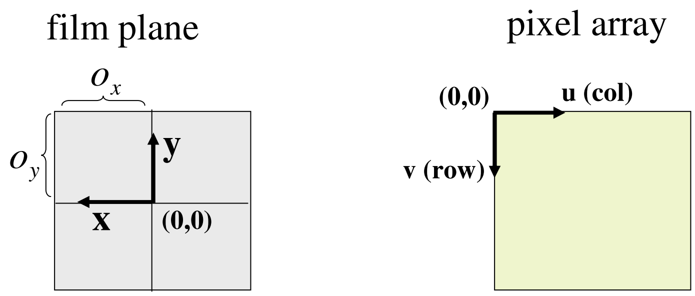


$$u = x + o_x = f\frac{X}{Z} + o_x \qquad v = y + o_y = f\frac{Y}{Z} + o_y$$

o_x and o_y called image center or principle point

Intrinsic parameters

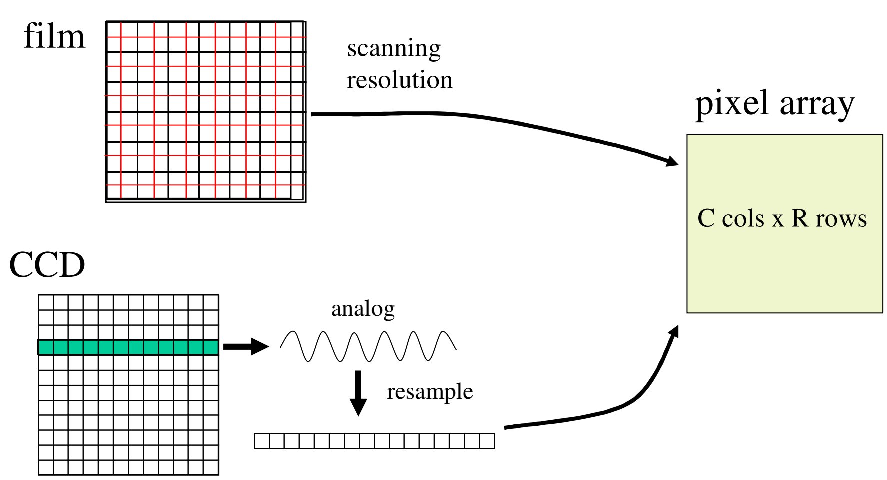
Sometimes one or more coordinate axes are flipped



$$u = -f\frac{X}{Z} + o_x \qquad v = -f\frac{Y}{Z} + o_y$$

Intrinsic parameters (scales)

sampling determines how many rows/cols in the image



Effective Scales: s_x and s_y

$$u = \frac{1}{s_x} f \frac{X}{Z} + o_x \qquad v = \frac{1}{s_y} f \frac{Y}{Z} + o_y$$

Note, since we have different scale factors in x and y, we don't necessarily have square pixels!

Aspect ratio is s_y / s_x

Perspective projection matrix

Adding the intrinsic parameters into the perspective projection matrix:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} f/s_x & 0 & o_x & 0 \\ 0 & f/s_y & o_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

To verify:

$$u = \frac{x'}{z'} \qquad v = \frac{y'}{z'}$$

$$u = \frac{1}{s_x} f \frac{X}{Z} + o_x \qquad v = \frac{1}{s_y} f \frac{Y}{Z} + o_y$$

Note

In general, I like to think of the conversion as a separate 2D affine transformation from film coords (x,y) to pixel coordinates (u,v):

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{M_{aff}}} \underbrace{\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{M_{proj}}} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\mathbf{u = M_{int}\, P_C = M_{aff}\, M_{proj}\, P_C}$$

Summary : Forward Projection

World Coords		Camera Coords		Film Coords		Pixel Coords
U V W	$\xrightarrow{M_{ext}}$	X Y Z	$\xrightarrow{M_{proj}}$	x y	$\xrightarrow{M_{aff}}$	u v
U V W	$\xrightarrow{M_{ext}}$	X Y Z	$\xrightarrow{M_{int}}$			u v
U V W	$\xrightarrow{M}$					u v

$$M = \begin{matrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{31} & m_{33} & m_{34} \end{matrix}$$

Summary: Projection Equation

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \sim \underbrace{\underbrace{\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Film plane to pixels} \;\; \mathbf{M_{aff}}} \underbrace{\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{Perspective projection} \;\; \mathbf{M_{proj}}}}_{\mathbf{M_{int}}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{World to camera} \;\; \mathbf{M_{ext}}}}_{\mathbf{M}} \begin{bmatrix} U \\ V \\ W \\ 1 \end{bmatrix}$$