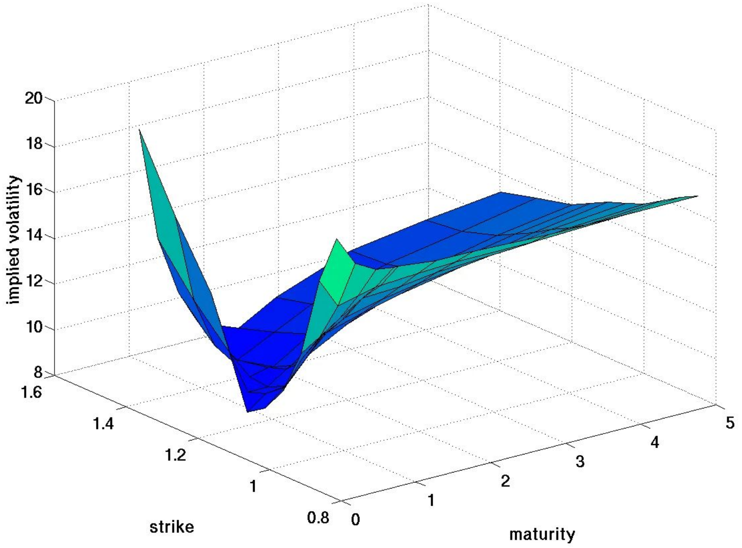
The complete Gaussian kernel in the multi-factor Heston model:
Option pricing and implied volatility applications

Authors: Chevychelov, Gorodov, Golubev, Makarenkov, Serebryakova

Group: 211

Motivation

The model that authors introduce is a generalization of the Black Scholes approach to option pricing. Heston model itself has a range of drawbacks, such as sufficient flexibility to capture the shape of the implied volatility surfaces etc. That is why a multi-factor extension of the Heston model has been proposed.



Introduction

The tasks of the paper:

- Show an approach to extract the Gaussian kernel behind the multi-factor Heston model, which allows a clear connection of the prices of European option contracts in the multi-factor Heston framework to the corresponding prices in the Black-Scholes model
- Illustrate how the option prices and implied volatility respond to changes in model parameters

Theoretical background

The multi-factor Heston model assumes the following stochastic volatility model:

$$dx_t = \left(r(t) - \frac{1}{2} \sum_{j=1}^{n} v_{j,t} \right) dt + \sum_{j=1}^{n} \sqrt{v_{j,t}} dZ_{j,t}, \quad t > 0, \qquad (1)$$

$$dv_{j,t} = \chi_j (v_j^* - v_{j,t}) dt + \gamma_j \sqrt{v_{j,t}} dW_{j,t}, \quad t > 0, \qquad (2)$$

where x_t denotes the log-price variable, $v_{1,t,}$ . . . $v_{n,t}$ is the corresponding variances, r(t) is the instantaneous risk-free rate (assumed to be known in advance), χ_j, $v*_j$, and γ_j are positive constants, and $Z_{j,t,}$ $W_{j,t,}$ j = 1, 2, . . ., n, are standard Wiener processes.

Applications of the model

Option pricing (explicit formulas for European vanilla call and put options)
With spot price S_0, maturity T and strike price E

Discount factor: $B(T) = e^{-\int_0^T r(s)ds}$

$$C(S_0, T, E) = B(T) \int_{\log E}^{+\infty} (e^{x'} - E) M(\log S_0, \underline{v}_0, 0, x', T) dx'$$

and

$$P(S_0, T, E) = B(T) \int_{-\infty}^{\log E} (E - e^{x'}) M(\log S_0, \underline{v}_0, 0, x', T) dx'$$

The accuracy of the approximation formulas

Table 2
Descriptive statistics for the relative errors of second- and third-order option price approximations evaluated on grid $\mathcal{M}$ in the case of the Heston model.

Second-order approximations in vols of vols ($C_{2,H}$, $P_{2,H}$)						
γ	mean$_C$	median$_C$	std$_C$	mean$_P$	median$_P$	std$_P$
0.01	2.7090e−9	0.000	8.7598e−8	2.3767e−9	0.000	7.1053e−8
0.05	3.3058e−7	0.000	9.9919e−7	2.9665e−7	0.000	9.1210e−7
0.15	8.6177e−6	2.9231e−6	1.9097e−5	8.0870e−6	2.8545e−6	1.6523e−5
0.25	3.9080e−5	8.7593e−6	8.5551e−5	3.6756e−6	9.2254e−6	7.4764e−5
0.5	2.8757e−4	6.0871e−5	6.1026e−4	2.7410e−4	6.5693e−5	5.5215e−4
0.8	1.0428e−3	2.3942e−4	2.1199e−3	1.0099e−3	2.4380e−4	1.9904e−3
2.0	1.0785e-2	3.4351e-3	1.8412e-2	1.0854e-2	3.3176e-3	1.8488e-2
Third-order approximations in vols of vols ($C_{3,H}$, $P_{3,H}$)						
γ	mean$_C$	median$_C$	std$_C$	mean$_P$	median$_P$	std$_P$
0.01	4.5346e−10	0.000	3.3345e−8	9.2518e−10	0.000	4.7762e−8
0.05	1.1567e−7	0.000	6.1056e−7	1.0622e−7	0.000	5.5577e−7
0.15	3.0780e−6	0.000	5.3205e−6	2.8741e−6	0.000	4.9957e−6
0.25	1.2798e−5	4.5284e−6	2.0294e−5	1.2180e−5	5.3118e−6	1.9446e−5
0.5	8.0037e−5	3.4110e−5	1.0768e−4	7.8271e−5	3.6047e−5	1.0436e−4
0.8	2.8491e−4	1.1447e−4	4.0639e−4	2.8161e−4	1.2737e−4	3.6937e−4
2.0	4.0807e−3	7.8427e−4	8.8272e−3	4.1534e−3	8.4030e−4	8.7858e−3

Further, authors studied the accuracy of the formulas in reproducing European option prices and their performance in terms of computational time.

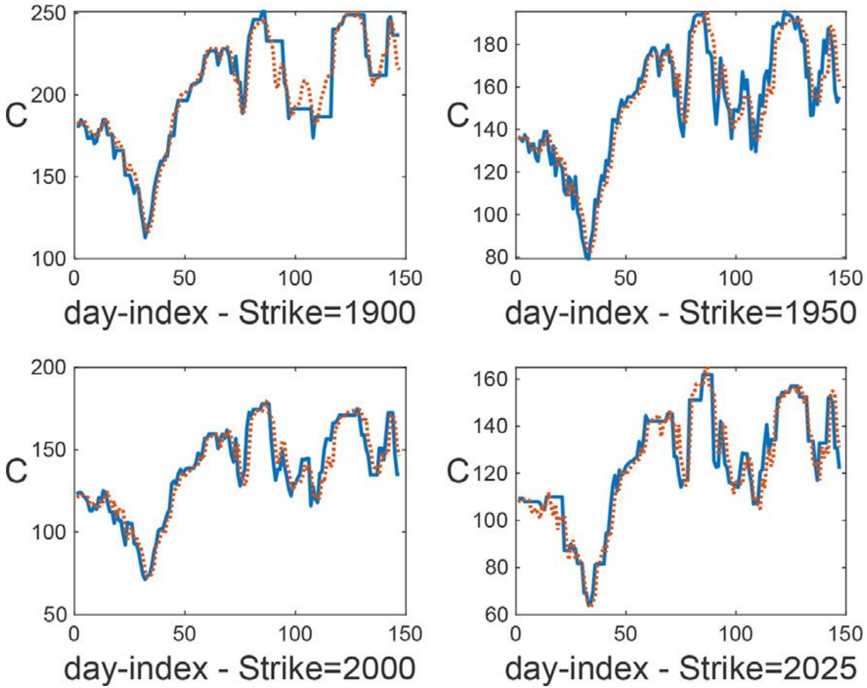
Empirical calibration study

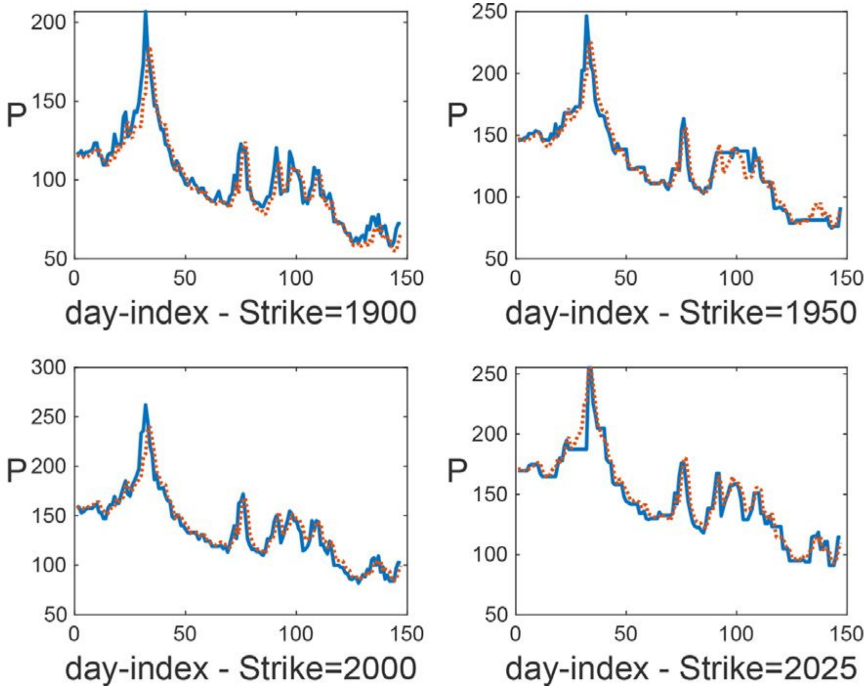
Here, authors provide empirical evidence that by using the second-order approximations for the implied volatility we can obtain “consistent” estimates of the Heston model parameters from both the call and put options.

$$\min_{\underline{\Theta}^C \in \mathcal{V}} \sum_{j=1}^{n_E} \left[\sigma_C^o(S_i, T_i, E_j) - \frac{\Sigma_{2,H}(S_i, T_i, E_j)}{\sqrt{T_i}} \right]^2 , \quad \rightarrow \quad \min_{\underline{\Theta}^P \in \mathcal{V}} \sum_{j=1}^{n_E} \left[\sigma_P^o(S_i, T_i, E_j) - \frac{\Sigma_{2,H}(S_i, T_i, E_j)}{\sqrt{T_i}} \right]^2 .$$

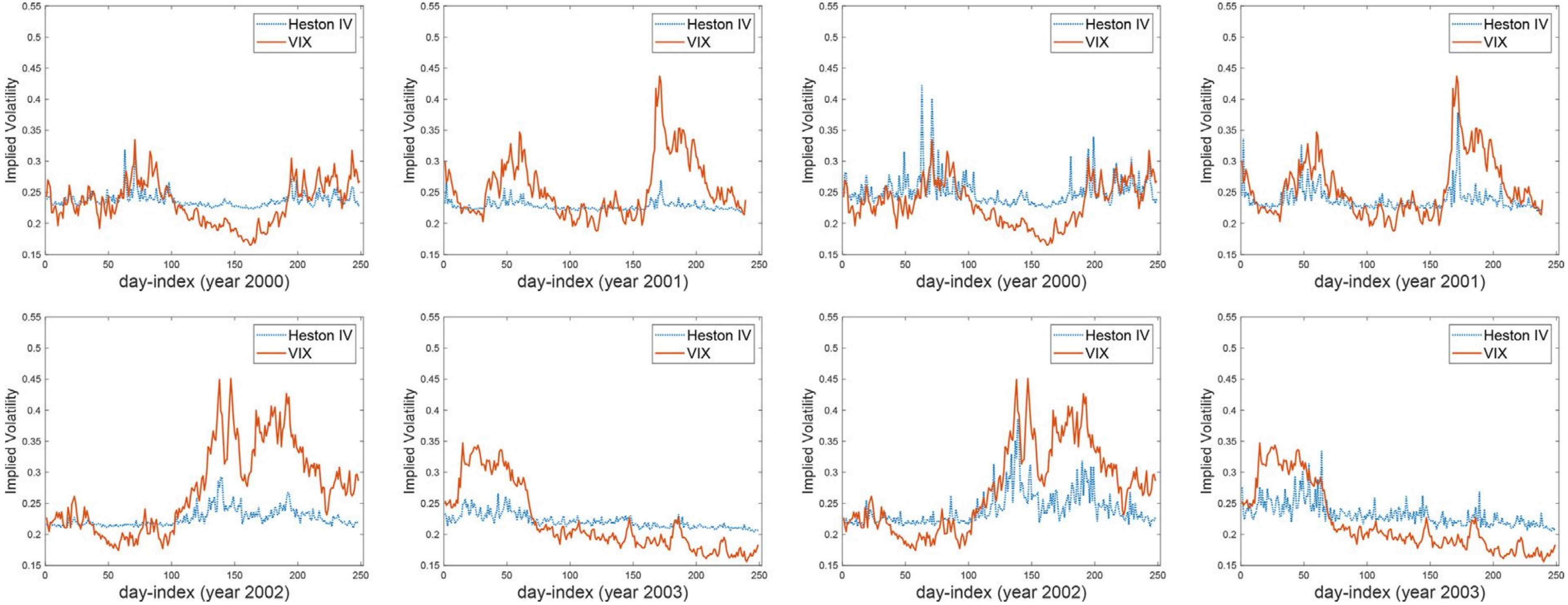
Use calibration procedure starting from the observed put prices $P^o(Si, Ti, Ej)$, where P^o is the observed value of the put option, i = 1, 2, . . ., n_T , and j = 1, 2, . . ., n_E , and solve the problem

Call and put


C
C
C
C
250
200
150
100
180
160
140
120
100
80
50
60
0
50
100
150
day-index - Strike=1900
day-index - Strike=1950
day-index - Strike=2000
day-index - Strike=2025


P
P
P
P
300
250
200
150
100
50
0
50
100
150
day-index - Strike=1900
day-index - Strike=1950
day-index - Strike=2000
day-index - Strike=2025

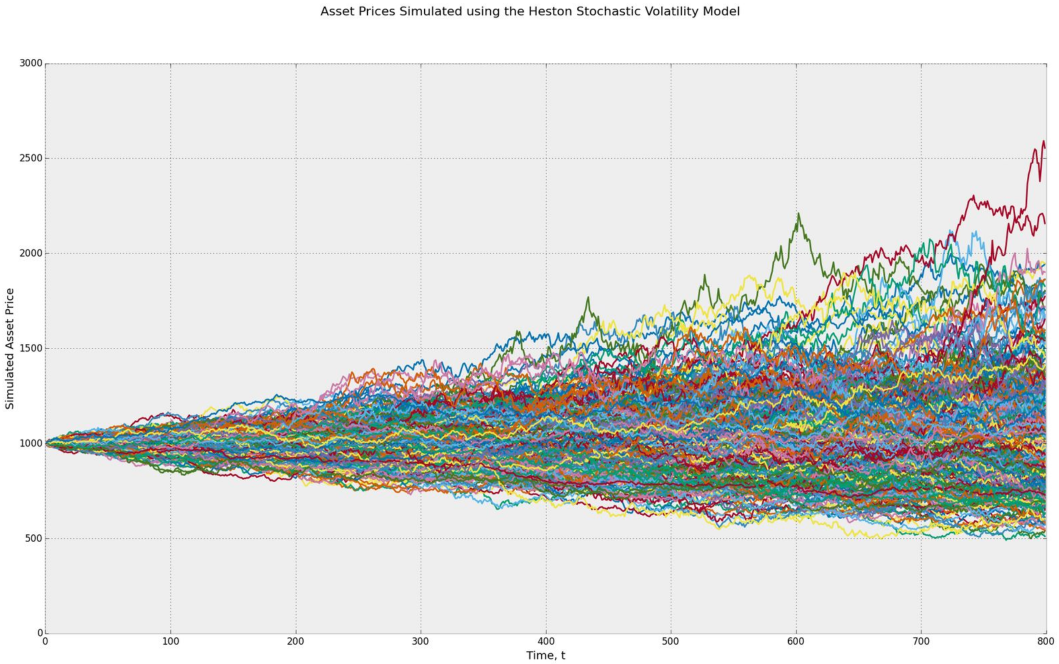
VIX and implied volatility from Heston model



Conclusions of the article

- A series of numerical exercises shows that formulas that were introduced are accurate, computationally efficient, and easy to calibrate
- Authors numerically demonstrate that approximations compare favourably with other pricing formulas available in the literature
- The results of this work, and in particular the decomposition of the option prices and implied volatility in terms of the Greeks of the options and higher-order risks, may have applications in other areas, such as portfolio management and asset allocation

Our experiment

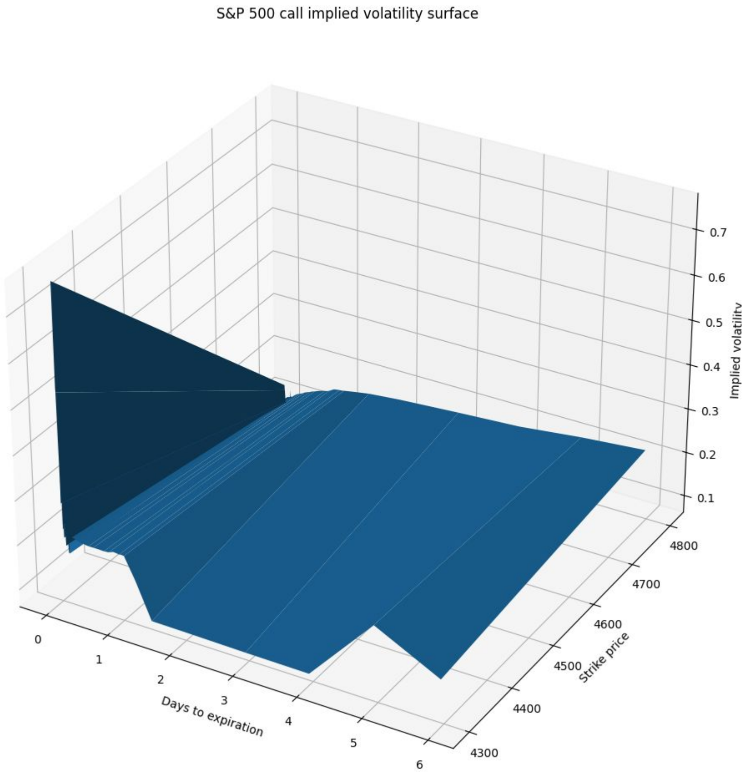


In our study, we also realized a version of Heston model, conducted several calculations on real data examples and tried to calibrate our model further as authors of the article proposed to do.

Receiving S&P 500 options data

As a source of data we chose yahoo finance ticker `^SPX`

Strike	Last Price	Bid	Ask	Change	% Change	Volume	Open Interest	Implied Volatility
4,050.00	716.87	710.30	711.00	-15.84	-2.16%	2	2	0.00%
4,100.00	616.00	661.70	662.40	0.00	-	-	1	0.00%
4,275.00	512.65	485.30	486.00	0.00	-	-	1	0.00%
4,300.00	463.81	464.20	465.10	+11.51	+2.54%	5	10	73.24%
4,350.00	412.07	410.30	411.00	+6.05	+1.49%	33	47	0.00%
4,360.00	401.53	404.20	405.10	+25.74	+6.85%	2	9	64.26%
4,375.00	385.10	388.60	389.50	0.00	-	10	12	54.30%

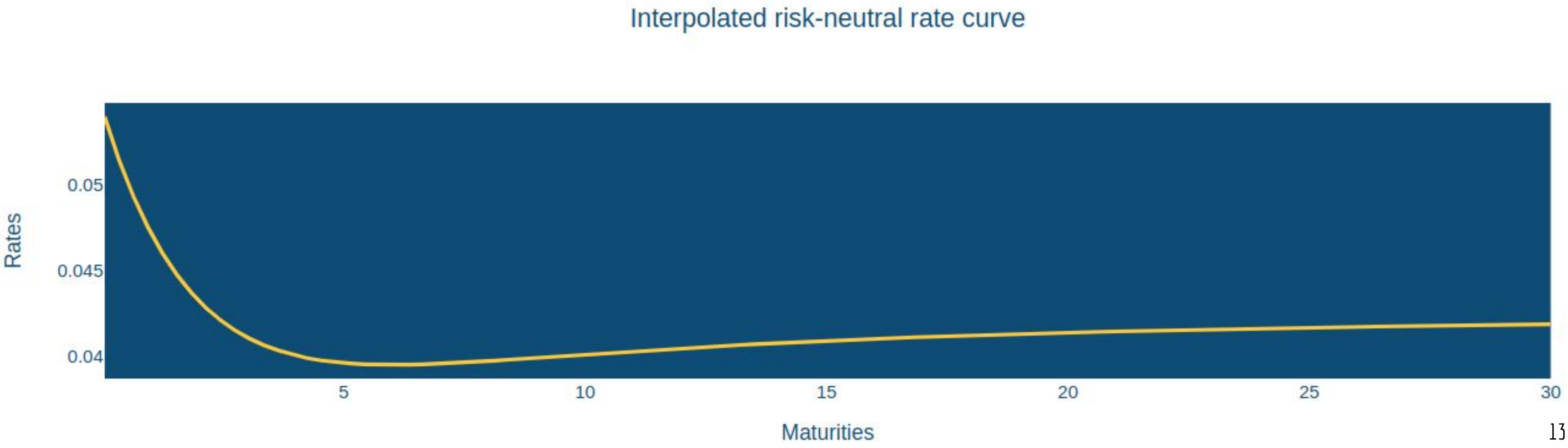


Receiving interest rates

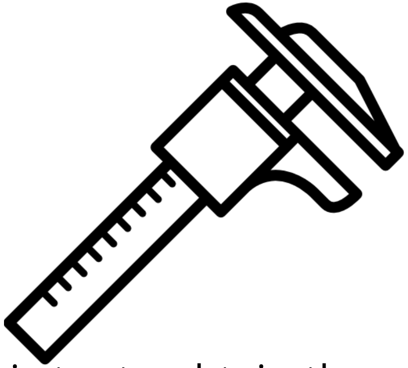
Curve was constructed from 4 tickers with US treasury bonds rates and interpolated

US Treasury Bonds Rates

Symbol	Name
^IRX	13 WEEK TREASURY BILL
^FVX	Treasury Yield 5 Years
^TNX	CBOE Interest Rate 10 Year T No
^TYX	Treasury Yield 30 Years



Calibration



Why Calibrate?

In practice, we actually don’t use the pricer to calculate prices. What we do is try to obtain the parameters for the Heston model based on the prices we observe in the markets. These parameters are used in pricing complex/exotic options. For very complex options, we can’t really find a closed form formula for pricing exotic options. What we do in this case is use the plain vanilla options to find parameters of the Heston model and use the parameters we obtain from the market to calculate the price of the exotic options using Monte Carlo methods.

Implementation on real data

In order to implement calibration algorithm we used Levenberg Marquardt algorithm to find the optimal value based on a given volatility surface.

Here the **residual = (MarketPrices - ModelPrices)/MarketPrices**

And the **objective function** to be minimized is **sum over squared residuals**.

Calibration is performed for current S&P 500 index price 4756$

Fit results

Fit Statistics

fitting method	leastsq
# function evals	592
# data points	3096
# variables	5
chi-square	2622794.86
reduced chi-square	848.526321
Akaike info crit.	20882.8754
Bayesian info crit.	20913.0647

Correlations (unreported values are < 0.100)

Parameter1	Parameter 2	Correlation
kappa	theta	-1.0000
sigma	kappa	+0.6510
sigma	theta	-0.6491
volvol	rho	-0.6145
sigma	volvol	-0.5943
sigma	rho	+0.4343

Parameters

name	value	standard error	relative error	initial value	min	max	vary
sigma	0.90807097	0.08225647	(9.06%)	0.5	0.01000000	10.0000000	True
kappa	0.01743955	0.15339303	(879.57%)	0.1	0.01000000	10.0000000	True
theta	9.20557555	79.2558290	(860.95%)	0.1	0.01000000	10.0000000	True
volvol	0.01000001	0.00267130	(26.71%)	0.1	0.01000000	10.0000000	True
rho	-0.96328966	0.01161177	(1.21%)	-1	-1.00000000	1.00000000	True