Padé Approximants

M. Grady Saunders
mgs8033@rit.edu

1. Definition

A Padé approximant $P_{M,N}$ is a generalization of a truncated Taylor series approximant T_L. That is,

$$\begin{aligned} T_L(x) &= \sum_{\ell=0}^{L} c_\ell x^\ell \\ \to P_{M,N}(x) &= \frac{\sum_{m=0}^{M} a_m x^m}{\sum_{n=1}^{N} b_n x^n} \end{aligned}$$

where $L = M + N$ and $b_0 = 1$ by convention. Equating $T_L = P_{M,N}$ yields the coefficient relations

$$a_m = \sum_{k=0}^{\min(m,N)} b_k c_{m-k}. \tag{1}$$

2. Determination

For $M \leq N$, expanding equation 1 forms a system of N linear equations in N unknowns,

$$\left[\gamma_{M+i-j}\right]_{\substack{i\in[0,N)\\ j\in[0,N)}} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} = -\begin{bmatrix} c_{M+1} \\ c_{M+2} \\ \vdots \\ c_{M+N} \end{bmatrix} \tag{2}$$

where

$$\gamma_k = \begin{cases} c_k & k \geq 0 \\ 0 & k < 0 \end{cases}$$

which determines the b_n from the c_ℓ. Then, substituting back into equation 1 determines the a_m.

For $M > N$, expanding equation 1 forms a system of $M + 1$ linear equations in $M + 1$ unknowns,

$$\left[\gamma_{i,j}\right]_{\substack{i\in[0,M+1)\\ j\in[0,M+1)}} \begin{bmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_M \end{bmatrix} = -\begin{bmatrix} c_N \\ c_{N+1} \\ \vdots \\ c_{N+M} \end{bmatrix} \tag{3}$$

where

$$\gamma_{i,j} = \begin{cases} -\delta_{i,j} & j < M - N + 1 \\ c_{M+i-j} & j \geq M - N + 1 \end{cases}$$

$$\omega_k = \begin{cases} a_{k+N} & k < M - N + 1 \\ b_{k-M+N} & k \geq M - N + 1 \end{cases}$$

which determines the ω_k, being the a_m for $m = N, N + 1, \ldots, M$ and the b_n for all n. Then, substituting back into equation 1 determines the remaining a_m.