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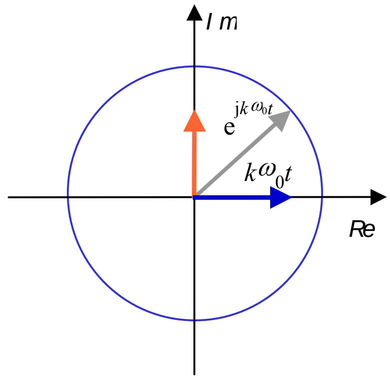


FOURIER ANALYSIS & SYNTHESIS

Jean Baptiste Joseph Fourier (1768-1830), intorduced the concept of Fourier series in his major work on the mathematical theory of heat conduction, *The Analytic Theory of Heat*. He established the partial differential equation governing heat diffusion and solved it using an infinite series of trigonometric (sine and cosine) functions.

It is interesting to note that while studying at Ecole Normale in Paris in 1794, Fourier was taught by Lagrange, Laplace and Monge. Fourier also accompanied Napoleon Bonaparte as the sientific adviser in the invaion of Egypt in 1798. While in Cairo, Fourier helped found the Cairo Institute and was one of the twelve members of the mathematics division, the others included Monge, Malus and Napoleon Bonaparte. Fourier was also in charge of collating the scientific and literary discoveries made during the time in Egypt.

The Fourier method of description of a signal in terms of a combination of elementary trigonometric functions had a profound effect on the way signals are viewed, analysed and processed. In communication and signal processing such fundmental concepts as frequency spectrum and bandwidth result from the Fourier repesentation of signals.

The Fourier analysis method is the most extensively applied signal-processing tool. The Fourier transform of a signal lends itself to easy interpretation and manipulation, and leads to the concept of frequency analysis and banwidth. Furthermore, even some biological systems, such as the human auditory system, perform some form of frequency analysis of the input signals.

This chapter begins with an introduction to the complex Fourier series and Fourier transform, and then considers the discrete Fourier transform, the fast Fourier transform, the 2-D Fourier transform and the discrete cosine transform. Important engineering issues such as the trade-off between the time and frequency resolutions, problems with finite data length, windowing and spectral leakage are considered. The applications of the Fourier transform in telecommunication and signal processing include filtering, correlation, music processing, signal coding, signal synthesis, feature extraction for pattern identification as in speech or image recognition, spectral analysis and radar signal processing.

2.1 Introduction

The primary objective of signal transformation is to express a signal in terms of a combination of a set of simple *elementary* signals, known as the *basis functions*. The transform's output should lend itself to convenient analysis, interpretation, modification and synthesis.

In Fourier transform the basic elementary signals are a set of sinusoidal signals (sines and cosines) with various periods of repetition giving rise to the concept of *frequency*. Many indespesible concepts in communication and signal processing theory such as the concepts of bandwidth, power spectrum and filtering result from the Fourier description of signals.

In Fourier analysis a signal is decomposed into its constituent sinusoidal vibrations. The amplitudes and phases of the sinusoids of various frequencies form the frequency spectrum of the signal. In inverse Fourier transform a signal can be synthesised by adding up its constituent frequencies.

It turns out that many signals that we encounter in daily life - such as speech, car engine noise and music - are generated by some form of vibrations and have a periodic or quasi-periodic structure. Furthermore, the cochlea in the human hearing system performs a kind of vibration analysis of the input audio signals. Therefore the concept of frequency analysis is not a purely mathematical abstraction in that some biological systems including humans and many species of animals have evolved sensory mechanisms that make use of the frequency analysis concept.

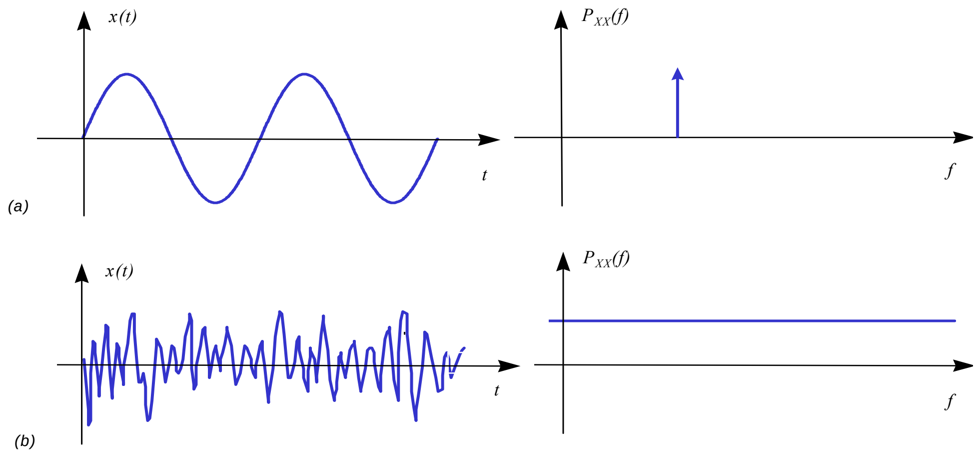


Figure 2.1 The concentration or spread of the power of a signal in frequency indicates the correlated or random character of a signal: (a) a predictable signal, (b) a random signal.

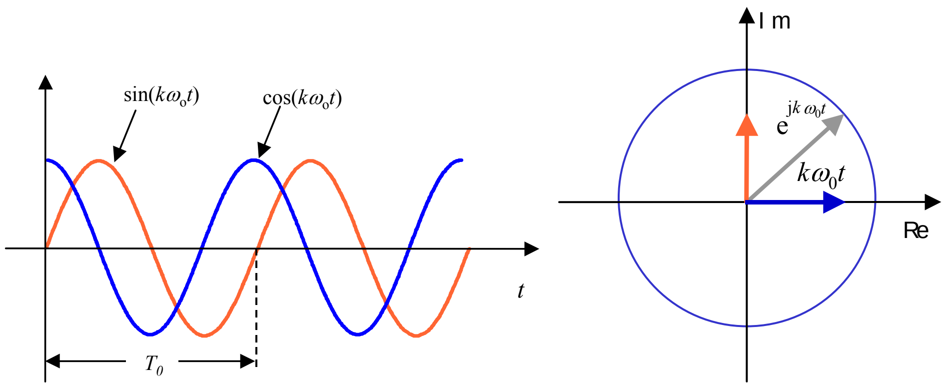
The power of the Fourier transform in signal analysis and pattern recognition lies in its simplicity and its ability to reveal spectral structures that can be used to characterise a signal. This is illustrated in Figure 2.1 for the two extreme cases of a sine wave and a purely random signal. For a periodic signal, such as a sine wave or a train of pulses, the signal power is concentrated in extremely narrow band(s) of frequencies indicating the existence of a periodic structure and the predictable character of the signal. In the case of a pure sine wave as shown in Figure 2.1.a the signal power is concentrated in just one frequency. For a purely random signal as shown in Figure2.1.b the average signal power is spread equally in the frequency domain indicating the lack of a predictable structure in the signal.

Notation: In this chapter the symbols t and m denote continuous and discrete time variables, and f and k denote continuous and discrete frequency variables respectively. The variable ω=2πf denotes the angular frequency in units of rad/s, it is used interchangeably (within a scaling of factor of 2π) with the frequency variable f in units of Hz.

A Note on Comparison of Fourier Series and Taylor series

Taylor series describes a function x *locally* in terms of its differentials or derivatives of the function at x. In contract, as shown next, Fourier series describes a function in terms of sine waves and cosine waves whose coefficients are calculated *globally* that is over the whole length of the signal.

2.2 Fourier Series: Representation of Periodic Signals



(a) (b)

Figure 2.2 - Fourier basis functions: (a) real and imaginary components of a complex sinusoid, (b) vector representation of a complex exponential. If the cosine is considered as the in-phase component then the sine is the quadrature component.

A periodic signal can be decribed in terms of a of series of harmonically related (i.e. integer multiples of a fundamenrtal frequency) sine and coie waves.

The following three sinusoidal functions form the *basis functions* for the Fourier analysis

$$x_1(t) = \cos\omega_0 t \tag{2.1}$$

$$x_2(t) = \sin\omega_0 t \tag{2.2}$$

$$x_3(t) = \cos\omega_0 t + j\sin\omega_0 t = e^{j\omega_0 t} \tag{2.3}$$

A cosine function is an even function with respect to the vertical axis at time t=0 and a sine function is an odd function. A weighted combination of a sine and a cosine at angular frequency ω_0 can model any phase of a sinusoidal signal component of $x(t)$ at that frequency. Figure 2.2.a shows the sine and the cosine components of the complex exponential (cisoidal) signal of Eq. (2.3), and Figure 2.2.b shows a vector representation of the complex exponential in a complex plane with real (*Re*) and imaginary (*Im*) dimensions. The Fourier basis functions are periodic with an angular frequency of ω_0 rad/s and a period of $T_0=2\pi/\omega_0=1/F_0$ seconds, where F_0 is the frequency in Hz.

The following properties make the sinusoids an ideal choice as the elementary building block basis functions for signal analysis and synthesis:

(i) Orthogonality; two sinusoidal functions of *different* frequencies have the following orthogonal property:

$$\int_{-\infty}^{\infty} \sin(\omega_1 t)\sin(\omega_2 t)\,dt = -\frac{1}{2}\int_{-\infty}^{\infty} \cos(\omega_1 + \omega_2)t\,dt + \frac{1}{2}\int_{-\infty}^{\infty} \cos(\omega_1 - \omega_2)t\,dt = 0 \tag{2.4}$$

For sinusoids the integration interval can be taken over one period (i.e T=2π/(ω_1+ω_2) and T=2π/(ω_1-ω_2). Similar equations can be derived for the product of cosines, or sine and cosine, of different frequencies. Orthogonality implies that the sinusoidal basis functions are independent and can be processed independently. For example in a graphic equaliser we can change the relative amplitudes of one set of frequencies, such as the bass, without affecting other frequencies, and in music coding the signals in different frequency bands are coded independently and allocated different number of bits.

(ii) The sine and cosine components of $e^{-j\omega_0 t}$ have only a relative phase difference of π/2 or equivalently a relative time delay of a quarter of one period i.e. T_0/4. This allows the decomposition of a signal in terms of orthogonal cosine (in-phase) and sine (quadrature) components.

(iii) Sinusoidal functions are infinitely differentiable. This is a useful property, as most signal analysis, synthesis and processing methods require the signals to be differentiable.

(iv) A useful consequence of transforms, such as the Fourier and the Laplace transforms, is that differential analysis on the time domain signal become simple algebraic operations on the transformed signal.

Associated with the complex exponential function $e^{\mathrm{j}\omega_0 t}$ is a set of harmonically related complex exponential of the form

$$[1, e^{\pm \mathrm{j}\omega_0 t}, e^{\pm \mathrm{j}2\omega_0 t}, e^{\pm \mathrm{j}3\omega_0 t}, \ldots\,] \tag{2.5}$$

The set of exponential signals in Eq. (2.5) are periodic with a fundamental frequency ω_0=2π/T_0=2πF_0 where T_0 is the period and F_0 is the fundamental frequency. These signals form the set of *basis functions* for the Fourier series analysis. Any linear combination of these signals of the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \tag{2.6}$$

is also a periodic signal with a period of T_0. Conversely any periodic signal $x(t)$ can be synthesised from a linear combination of harmonically related exponentials.

The Fourier series representation of a periodic signal are given by the following synthesis and analysis equations:

The complex-valued coefficient c_k conveys the amplitude (a measure of the strength) and the phase (or time delay) of the frequency content of the

Synthesis equation $$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad k = \cdots -1,0,1,\cdots \tag{2.7}$$

Analysis equation $$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t}\,dt \quad k = \cdots -1,0,1,\cdots \tag{2.8}$$

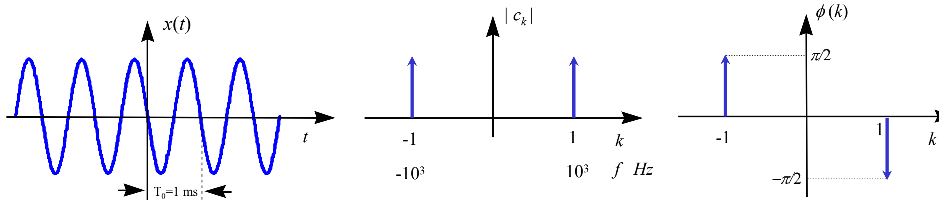
signal at $k\omega_0$ Hz. Note from the Fourier analysis Equation (2.8), that the coefficient c_k *may be interpreted as a measure of the correlation of the signal* $x(t)$ *and the complex exponential* $e^{-jk\omega_0 t}$.

The representation of a signal in the form of Eq. (2.7) as the sum of its constituent harmonics is referred to as the *complex Fourier series* representation. The set of complex coefficients … c_{-1}, c_0, c_1, … are known as the *frequency spectrum* of the signal.

Eq. (2.7) can be used as a synthesizer (as in a music synthesizers) to generate a signal as a weighted combination of its elementary frequencies. Note from Eqs. (2.7) and (2.8) that the complex exponentials that form a periodic signal occur only at discrete frequencies which are integer multiple harmonics of the fundamental frequency ω_0. Therefore the spectrum of a periodic signal, with a period of T_0, is *discrete* in frequency with *discrete spectral lines* spaced at integer multiples of $\omega_0=2\pi/T_0$.

Example 2.1 Given the Fourier series synthesis Eq. (2.7), obtain the frequency analysis Eq. (2.8).

Solution: To obtain c_n, the coefficient of the n^{th} harmonic, multiply both sides of Eq. (2.7) by $e^{-jn\omega_0 t}$ and integrate over one period to obtain



(a) (b)

Figure 2.3 – (a) A sine wave, (b) its magnitude spectrum, (c) its phase spectrum.

$$\int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t}\,dt = \int_{-T_0/2}^{T_0/2} \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-jn\omega_0 t}\,dt = \sum_{k=-\infty}^{\infty} c_k \int_{-T_0/2}^{T_0/2} e^{j(k-n)\omega_0 t}\,dt \quad (2.9)$$

From the orthogonality principle the integral of the product of two complex sinusoids of different frequency is zero. Hence, for $k \neq n$ the integral over one period of $e^{j(k-n)\omega_0 t}$ in the r.h.s of Eq. (2.9) is zero. For $k=n$, $e^{j(k-n)\omega_0 t} = e^0=1$ and its integral is equal to T_0 and the r.h.s of Eq. (2.9) is equal to $c_n T_0$. Hence

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jn\omega_0 t}\,dt \quad (2.10)$$

Example 2.2 Find the frequency spectrum of a 1 kHz sinewave shown in Figure2.2.a

$$x(t) = \sin(2000\pi t) \qquad -\infty < t < \infty \quad (2.11)$$

Solution A: The Fourier synthesis Eq. (2.7) can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk2000\pi t} = \cdots + c_{-1}e^{-j2000\pi t} + c_0 + c_1 e^{j2000\pi t} + \cdots \quad (2.12)$$

now the sine wave can be expressed as

$$x(t) = \sin(2000\pi t) = \frac{1}{2j}e^{j2000\pi t} - \frac{1}{2j}e^{-j2000\pi t} \quad (2.13)$$

Equating the coefficients of Eqs. (2.12) and (2.13) yields

$$c_1 = \frac{1}{2j},\ c_{-1} = -\frac{1}{2j} \ \text{ and } \ c_{k\neq\pm1} = 0 \quad (2.14)$$

Figure 2.2.b shows the magnitude and phase spectrum of the sinewave, where the spectral lines c_1 and c_{-1} correspond to the 1 kHz and −1 kHz frequencies respectively.

Solution B: Substituting $\sin(2000\pi t) = \frac{1}{2j}e^{j2000\pi t} - \frac{1}{2j}e^{-j2000\pi t}$ in the Fourier analysis Eq. (2.8) yields

$$
\begin{aligned}
c_k &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\left(\frac{1}{2j}e^{j2000\pi t} - \frac{1}{2j}e^{-j2000\pi t}\right)e^{-jk2000\pi t}\,dt \\
&= \frac{1}{2jT_0}\int_{-T_0/2}^{T_0/2} e^{j(1-k)2000\pi t}\,dt - \frac{1}{2jT_0}\int_{-T_0/2}^{T_0/2} e^{-j(1+k)2000\pi t}\,dt \qquad (2.15)
\end{aligned}
$$

Since sine and cosine functions are positive-valued over one half a period and odd symmetric (equal and negative) over the other half, it follows that Eq. (2.15) is zero unless k=1 or k=−1. Hence

$$
c_1 = \frac{1}{2j} \quad \text{and } c_{-1} = -\frac{1}{2j} \quad \text{and} \quad c_{k\neq\pm1} = 0 \qquad (2.16)
$$

Example 2.3 Find the frequency spectrum of a periodic train of pulses, shown in Figure 2.4, with an amplitude of 1.0, a fundamental frequency of 100 Hz and a pulse 'on' duration of 3 milliseconds.

Solution: The pulse period $T_0=1/F_0=0.01$ s, and the angular frequency $\omega_0=2\pi F_0=200\pi$ rad/s. Substituting the pulse signal in the Fourier analysis Eq. (2.8) gives

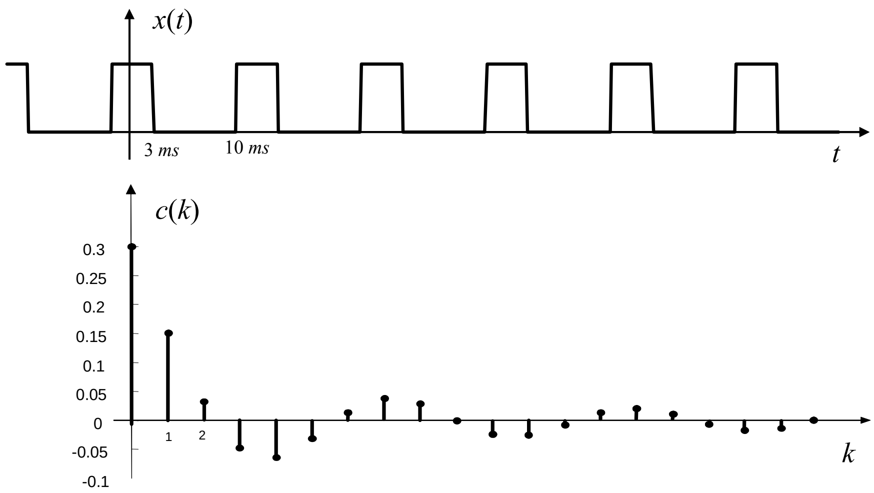


Figure 2.4 - A rectangular pulse train and its discrete frequency ‘line’ spectrum (only positive frequencies shown).

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,e^{-jk\omega_0 t}\,dt = \frac{1}{0.01}\int_{-0.0015}^{0.0015} e^{-jk200\pi t}\,dt$$
$$= \left.\frac{e^{-jk200\pi t}}{-j2\pi k}\right|_{t=-0.0015}^{t=0.0015} = \frac{e^{j0.3\pi k} - e^{-j0.3\pi k}}{j2\pi k} = \frac{\sin(0.3\pi k)}{\pi k} \tag{2.17}$$

For k=0 as c_0=sin(0)/0 is undefined, differentiate the numerator and denominator of Eq. (2.17) w.r.t. to the variable k (strictly this can be done for a continuous variable k i.e. when the period T_0 tends to infinity) to obtain

$$c_0 = \frac{0.3\pi\cos(0.3\pi\times 0)}{\pi} = 0.3 \tag{2.18}$$

Example 2.4 In Example 2.3, write the formula for synthesising the signal up to the N^{th} harmonic, and plot a few examples for increasing number of harmonics.

Solution: The equation for the synthesis of a signal upto the N^{th} harmonic content is given by

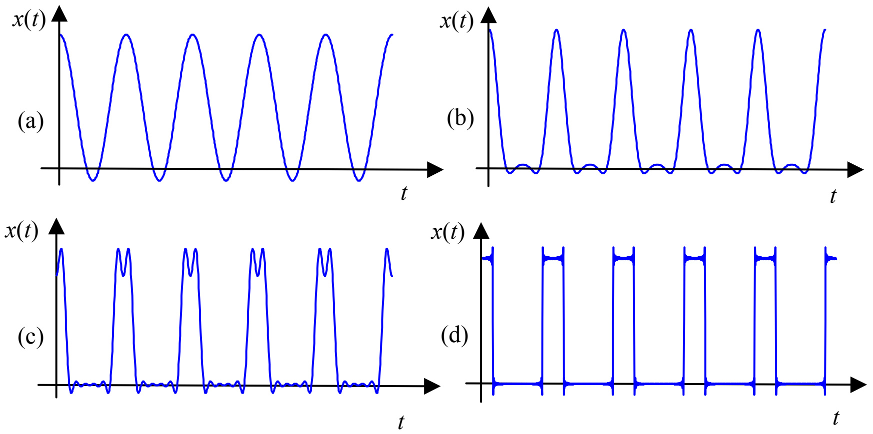


Figure 2.5 Illustration of the Fourier synthesis of a periodic pulse train, and the Gibbs phenomenon, with the increasing number of harmonics in the Fourier synthesis: (a) N=1, (b) N=3, (c) N=6, and (d) N=100.

$$
\begin{aligned}
x(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t} &= c_0 + \sum_{k=1}^{N} c_k e^{jk\omega_0 t} + \sum_{k=1}^{N} c_{-k} e^{-jk\omega_0 t} \\
&= c_0 + \sum_{k=1}^{N} [\mathrm{Re}(c_k) + j\,\mathrm{Im}(c_k)][\cos(k\omega_0 t) + j\sin(k\omega_0 t)] \\
&\quad + \sum_{k=1}^{N} [\mathrm{Re}(c_k) - j\,\mathrm{Im}(c_k)][\cos(k\omega_0 t) - j\sin(k\omega_0 t)] \\
&= c_0 + \sum_{k=1}^{N} [2\,\mathrm{Re}(c_k)\cos(k\omega_0 t) - 2\,\mathrm{Im}(c_k)\sin(k\omega_0 t)]
\end{aligned}
\tag{2.19}
$$

The following MatLab code generates a synthesised pulse train composed of N harmonics. In this example there are 5 cycles in an array of 1000 samples. Figure 2.5 shows the waveform for the number of harmonics equal to; 1,3,6, and 100.

The Sound of Fourier Series Harmonics

```
function FourierHarmonicSynthesis()
% Generates and displays, in time and frequency, the harmonics of
% periodic train of pulses, the signal is also played back through speakers.
% At each iteration an additional harmonic is added to the signal.
```

3.2.1 Fourier Synthesis at Discontinuity: Gibbs Phenomenon

The sinusoidal basis functions of the Fourier transform are smooth and infinitely differentiable. In the vicinity of a discontinuity the Fourier synthesis of a signal exhibits ripples as shown in the Figure 2.2. The peak amplitude of the ripples does not decrease as the number of harmonics used in the signal synthesis increases. This behaviour is known as the *Gibbs phenomenon*. For a discontinuity of unit height, the partial sum of the harmonics exhibits a maximum value of 1.09 (i.e. an overshoot of 9%) irrespective of the number of harmonics used in the Fourier series. As the number of harmonics used in the signal synthesis increases, the ripples become compressed toward the discontinuity but the peak amplitude of the ripples remains constant.

2.3 Fourier Transform: Representation of Non-periodic Signals

The Fourier representation of non-periodic signals can be obtained by considering a non-periodic signal as a special case of a periodic signal with an infinite period. If the period of a signal is infinite, then the signal does not repeat itself and hence it is non-periodic.

The Fourier series representation of periodic signals consist of harmonically related sinusoidal signals with a discrete spectra, where the

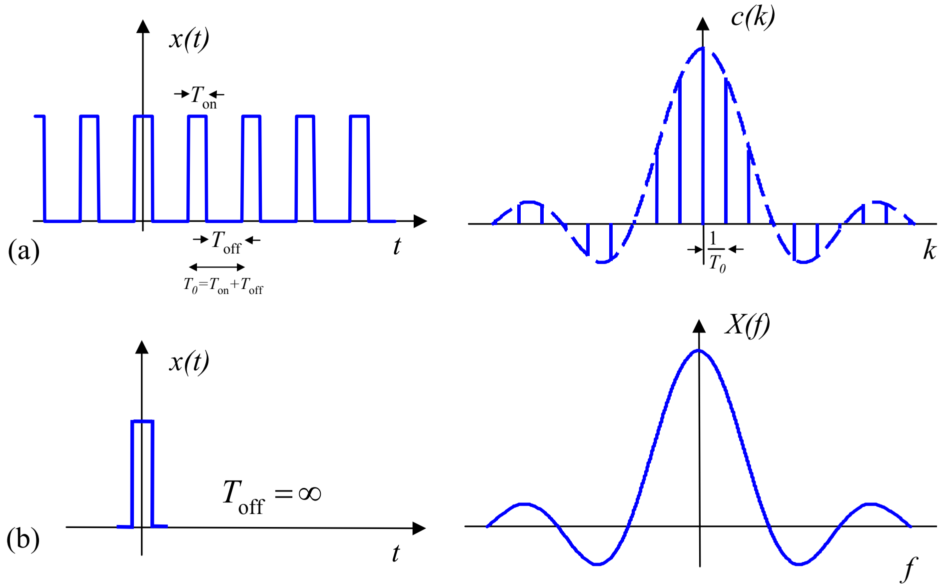


Figure 2.6 – (a) A periodic pulse train and its line spectrum, (b) a single pulse from the periodic train in (a) with an imagined 'off' duration of infinity; its spectrum is the envelope of the spectrum of the periodic signal in (a).

spectral lines are spaced at integer multiples of the fundamental frequency F_0. Now consider the discrete spectra of a periodic signal with a period of T_0, as shown in Figure 2.6.a. As the period T_0 increases, the fundamental frequency $F_0=1/T_0$ decreases, and successive spectral lines become more closely spaced. In the limit, as the period tends to infinity (i.e. as the signal becomes non-periodic) the discrete spectral lines merge and form a continuous spectrum.

Therefore, the Fourier equations for a non-periodic signal, (known as the Fourier transform), must reflect the fact that the frequency spectrum of a non-periodic signal is a continuous function of frequency. Hence, to obtain the Fourier transform relations the discrete-frequency variables and the discrete summation operations in the Fourier series Eqs. (2.7) and (2.8) are replaced by their continuous-frequency counterparts. That is the discrete summation sign Σ is replaced by the continuous summation integral sign $\int$, the discrete harmonics of the fundamental frequency kF_0 is replaced by the continuous frequency variable f, and the discrete frequency spectrum c_k is replaced by a continuous frequency spectrum say $X(f)$.

The Fourier synthesis and analysis equations for non-periodic signals, known as the *Fourier transform pair*, are given by

Fourier Transform (Analysis) Equation

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt \tag{2.20}$$

Inverse Fourier Transform (Synthesis) Equation

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df \tag{2.21}$$

Note from Eq. (2.20), that $X(f)$ *may be interpreted as a measure of the correlation of the signal* $x(t)$ *and the complex sinusoid* $e^{-j2\pi ft}$.

The condition for the existence (i.e. computability) of the Fourier transform integral of a signal $x(t)$ is that the signal must have finite energy, that is

$$Signal\ Energy = \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \qquad (2.22)$$

Example 2.5 *Derivation of the inverse Fourier transform.* Given the Fourier transform Eq. (2.20) derive the inverse Fourier transform Eq. (2.21).

Solution: The inverse Fourier transform can be obtained from the Fourier series equation. Consider the Fourier series analysis Eq. (2.8) for a periodic signal and the Fourier transform Eq. (2.20) for its non-periodic version (consisting of one period of the signal only). Comparing these equations reproduced below

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi k F_0 t} dt \qquad X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \qquad (2.23)$$

and substituting $X(kF_0)$ for the integral of the Fourier series, we have

$$c_k = \frac{1}{T_0} X(kF_0) \quad as\ T_0 \to \infty \qquad (2.24)$$

where $F_0=1/T_0$. Using Eq. (2.24), the Fourier synthesis Eq. (2.7) for a periodic signal can be rewritten as

$$x(t) = \sum_{k=-\infty}^{\infty} X(kF_0) e^{j2\pi k F_0 t} \Delta F \qquad (2.25)$$

where $\Delta F=1/T_0=F_0$ is the frequency spacing between successive spectral lines of the spectrum of a periodic signal as shown in Figure 2.6. Now as the period T_0 tends to infinity, $\Delta F=1/T_0$ tends towards zero, then the discrete frequency variables kF_0 and the discrete frequency interval ΔF should be replaced by a continuous frequency variables f and df and the discrete summation sign is replaced by the continuous integral sign. Thus Eq. (2.25) becomes

$$\underset{T_0 \to \infty}{x(t)} \Rightarrow \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \qquad (2.26)$$

Example 2.6 - The spectrum of an Impulse Function

Consider the unit-area pulse $p(t)$ shown in Figure2.7.a. As the pulse width Δ tends to zero the pulse tends to an impulse. The impulse function shown in Figure2.7.b is defined as a unit-area pulse with an infinitesimal time width as

$$\delta(t) = \lim_{\Delta \to 0} p(t) = \begin{cases} 1/\Delta & |t| \le \Delta/2 \\ 0 & |t| > \Delta/2 \end{cases} \quad (2.27)$$

All signals, discrete or continuous, can be considered as sum of a finite or infinite number of weighted and shifted impulses.

It is easy to see that the integral of the impulse function is given by

$$\int_{-\infty}^{\infty} \delta(t)dt = \Delta \times \frac{1}{\Delta} = 1 \quad (2.28)$$

The important **sampling property of the impulse function** is defined as

$$\int_{-\infty}^{\infty} x(t)\delta(t-T)dt = x(T) \quad (2.29)$$

The Fourier transform of the impulse function is obtained as

$$\Delta(f) = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}dt = e^0 \int_{-\infty}^{\infty} \delta(t)\,dt = 1 \quad (2.30)$$

The impulse function is often used as a *test function* to obtain the impulse response of a system. This is because as expressed by Eq(2.30) and shown in Figure2.7.c *an impulse is a spectrally rich signal containing all frequencies in equal amounts*. Furthermore any signal can be constructed as a combination of shifted and weighted impulses.

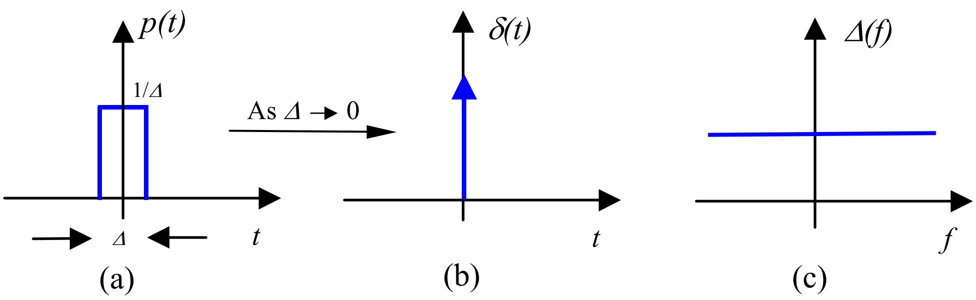


Figure 2.7 (a) A unit-area pulse, (b) the pulse becomes an impulse as $\Delta \to 0$, (c) the spectrum of the impulse function.

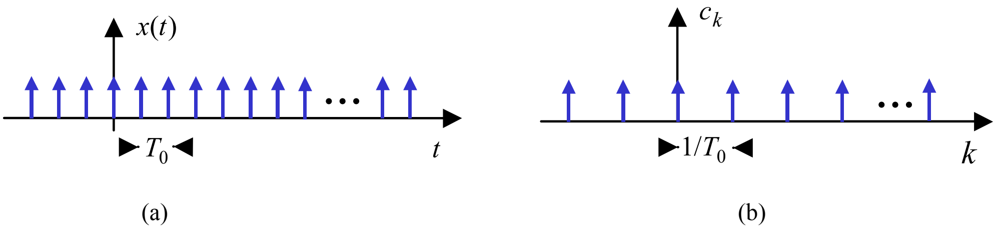


Figure 2.8 A periodic impulse train $x(t)$ and its Fourier series spectrum c_k.

Example 2.7 The Spectrum of A (Sampling) Train of Impulses. Find the spectrum of a 10 kHz periodic impulse train $x(t)$ shown in Figure 2.8.a with an amplitude of A=1 mV and expressed as

$$x(t) = \sum_{m=-\infty}^{\infty} A \times \underset{\Delta \to 0}{limit}\ p(t - mT_0) \times \Delta = \sum_{m=-\infty}^{\infty} A\ \delta(t - mT_0) \qquad (2.31)$$

where $p(t)$ is a unit-area pulse with width Δ as shown in Figure 2.7.a

Solution: $T_0=1/F_0=0.1$ ms and A=1 mV. For the time interval of one period $-T_0/2<t<T_0/2$ $x(t)=A\delta(t)$, hence we have

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} A\delta(t) e^{-jk\omega_0 t}\, dt = \frac{A}{T_0} = 10\ \ V \qquad (2.32)$$

As shown in Figure 2.8.b the spectrum of a periodic impulse train in time is a periodic impulse train in frequency.

Example 2.8 - The Spectrum of a Rectangular Pulse: Sinc Function

The rectangular pulse is a particularly important signal in digital signal analysis and digital communication. A rectangular pulse is inherent whenever a signal is segmented and processed frame by fame, in that each frame can be viewed as the result of multiplication of the signal and a rectangular window. Furthermore, the spectrum of a rectangular pulse can be used to calculate the bandwidth required by pulse radar systems or by digital communication systems that transmit modulated pulses to represent binary numbers.

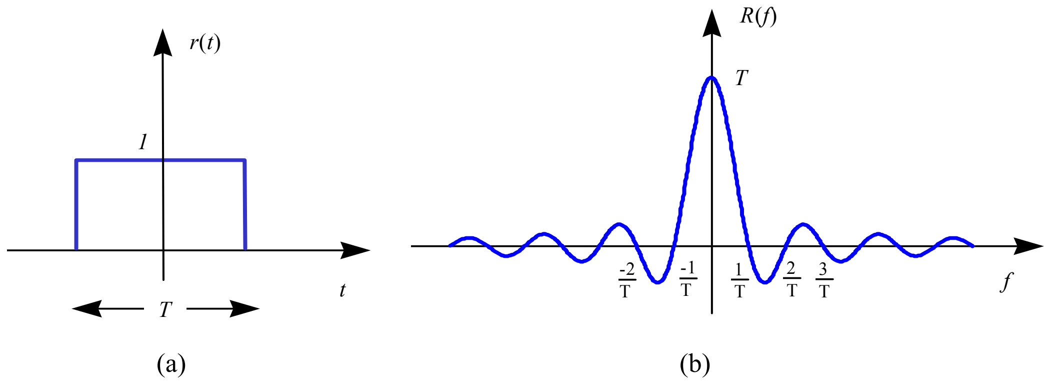


Figure 2.9 (a) A rectangular pulse, (b) its spectrum is a sinc function.

The Fourier transform of a rectangular pulse Figure 2.9.a of duration T seconds is obtained as

$$\begin{aligned} R(f) &= \int_{-\infty}^{\infty} r(t)e^{-j2\pi ft}\,dt = \int_{-T/2}^{T/2} 1.e^{-j2\pi ft}\,dt \\ &= \frac{e^{j2\pi fT/2} - e^{-j2\pi fT/2}}{j2\pi f} = T\frac{\sin(\pi fT)}{\pi fT} = T\,\mathrm{sinc}(fT) \end{aligned} \tag{2.33}$$

where sinc(x)= sin(πx)/πx.

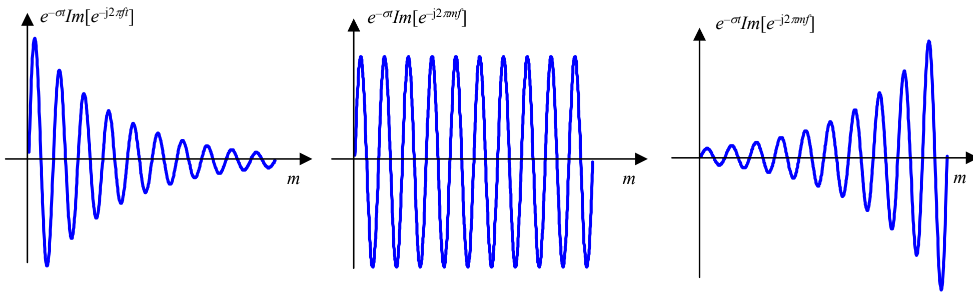
Figure 2.9.b shows the spectrum of the rectangular pulse. Note that most of the pulse energy is concentrated in the *main lobe* within a bandwidth of 2/*Pulse_Duration* i.e. *BW*=2/*T*. However, there is pulse energy in the side lobes that may interfere with other electronic devices operating at the side lobe frequencies.

Matlab code for drawing the spectrum of a rectangular pulse.
function Sinc( )

2.2.1 Comparison of Laplace and Fourier Transforms

As already mentioned Laplace was Fourier's teacher; perhaps this makes the relation between Fourier and Laplace transroms more interesting as it may be a case of the student outshining the teacher.

The Fourier transform assumes that the signal is stationary, this implies that the frequency content of the signal (the number of frequency components and their average magnitude and phase) does not change over time. Hence the signal is transformed into a combination of stationary sine



$\sigma > 0$ $\sigma = 0$ $\sigma < 0$

Figure 2.10 – The basis functions of Laplace transform.

(and cosine) waves of various time-invariant frequencies, magnitudes and phase. In contrast the Laplace transform can model a non-stationary signal as a combination of rising, steady and decaying sine waves.

The Laplace transform is particularly useful in solving linear ordinary differential equations as it can transform relatively difficult differential equations into relatively simple algebraic equations. The Laplace transform of $x(t)$ is given by the integral

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt \tag{2.34}$$

where the complex variable $s=\sigma+j\omega$. Note that $e^{-\sigma t}e^{-j\omega t}$ is a comlex sinusoidal signal $e^{-j\omega t}$ with an envelop of $e^{-\sigma t}$. The inverse Laplace transform is defined by

$$x(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} X(s)e^{st}\,ds \tag{2.35}$$

where σ_1 is a vertical contour in the complex plane chosen so that all singularities of $X(s)$ are to the left of it. The elementary functions (kernels) for the Laplace transform are damped or growing sinusoids of the form $e^{-st} = e^{-\sigma t}e^{-j\omega t}$ as shown in Figure 2.10. These basis functions are particularly suitable for transient signal analysis. In contrast, the Fourier basis functions are steady complex exponential, $e^{-j\omega t}$, of time-invariant amplitudes and phase, suitable for steady state or time-invariant signal analysis.

The Laplace transform is a one-sided transform with the lower limit of integration at $t = 0$, whereas the Fourier transform Eq. (2.21) is a two-sided transform with the lower limit of integration at $t = -\infty$. However for a one-sided signal, which is zero-valued for $t < 0$, the limits of integration for the Laplace and the Fourier transforms are identical. In that case if the variable s in the Laplace transform is replaced with the frequency variable $j2\pi f$ then the Laplace integral becomes the Fourier integral. Hence for a one-sided signal, *the Fourier transform is a special case of the Laplace transform* corresponding to $s=j2\pi f$ and $\sigma=0$. Also for a two-sided signal the bilateral Laplace transform (with limits of integration between $\pm\infty$) becomes Fourier transform when the variable s is replaced by $j2\pi f$. The relation between the Fourier and the Laplace transforms are discussed further in Chapter 4 on Z-transform. Note that Laplace transform can accommodate the initial state of a signal whereas the Fourier transform assumes that the signal is in a stationary steady state.

2.2.2 Properties of the Fourier Transform

There are a number of Fourier Transform properties that provide further insight into the transform and are useful in reducing the complexity of the application of Fourier transforms and its inverse transforms. These are:

Linearity

The Fourier transform is a linear operation, this mean the principle of superposition applies. Hence the Fourier transform of the weighted sum of two signals $x(t)$ and $y(t)$

$$z(t) = ax(t)+by(t) \tag{2.36}$$

is the weighted sum of their Fourier transforms $X(f)$ and $Y(f)$, i.e.

$$\begin{aligned} Z(f) &= \int_{-\infty}^{\infty} z(t)e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} [ax(t)+by(t)]e^{-j2\pi ft}\,dt = a\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt + b\int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}\,dt \\ &= aX(f)+bY(f) \end{aligned} \tag{2.37}$$

Symmetry

This property states that if the time domain signal $x(t)$ is real-valued (in practice this is often the case) then

$$X(f)=X^*(-f) \tag{2.38}$$

where the superscript asterisk * denotes the complex conjugate operation. Equation follows from the fact that if in then complex variable $e^{-j2\pi f}$ the frequency variable f is replaced by $-f$ and the result conjugated we will have $(e^{-j2\pi(-f)})^{*}= e^{-j2\pi f}$.

From Eq. (2.38) it follows that $Re\{X(f)\}$ is an even function of f and $Im\{X(f)\}$ is an odd function of f. Similarly the magnitude of $X(f)$ is an even function and the phase angle is an odd function.

Time Shifting and Frequency Modulation (FM)

Let $X(f)=\mathcal{F}[x(t)]$ be the Fourier transform of $x(t)$. If the time domain signal $x(t)$ is delayed by an amount T_0, the effect on its spectrum $X(f)$ is a phase change of $e^{-j2\pi T_0 f}$ as

$$\mathcal{F}\left[x(t-T_0)\right]=\int_{-\infty}^{\infty} x(t-T_0)e^{-j2\pi ft}\,dt = e^{-j2\pi fT_0}\int_{-\infty}^{\infty} x(t-T_0)e^{-j2\pi f(t-T_0)}\,dt$$

$$= e^{-j2\pi fT_0}\int_{-\infty}^{\infty} x(t')e^{-j2\pi ft'}\,dt' = e^{-j2\pi fT_0}X(f) \tag{2.39}$$

where $t'=t-T_0$.

Conversely, if $X(f)$ is shifted by an amount F_0, the effect on inverse Fourier transform is

$$\mathcal{F}^{-1}\left[X(f-F_0)\right]=\int_{-\infty}^{\infty} X(f-F_0)e^{j2\pi ft}\,dt = e^{-j2\pi F_0 t}\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df = e^{j2\pi F_0 t}x(t) \tag{2.40}$$

Note that the modulation Eq. (2.40) states that multiplying a signal $x(t)$ by $e^{j2\pi F_0 t}$ translates the spectrum of $x(t)$ onto the frequency channel F_0, this is the frequency translation principle in telecommunication.

Differentiation and Integration

As with the Laplace transform, through Fourier transform the mathematical operations of differentiation and integration can be transformed into simple algebraic operations. Let $x(t)$ be a continuous time signal with Fourier transform $X(f)$.

$$x(t)=\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df \qquad (2.41)$$

Differentiating both sides of the Fourier transform Eq. (2.41) we obtain

$$\frac{dx(t)}{dt}=\int_{-\infty}^{\infty} X(f)\frac{d\,e^{j2\pi ft}}{dt}\,df=\int_{-\infty}^{\infty} \underbrace{j2\pi f\,X(f)}_{\substack{Fourier Transform\\ of\ dx(t)/dt}}\ e^{j2\pi ft}\,df \qquad (2.42)$$

That is *multiplication* of $X(f)$ by the factor j2π*f* is equivalent to differentiation of $x(t)$ in time. Similarly division of $X(f)$ by j2π*f* is equivalent to integration of the function of time *x*(*t)*

$$\int_{-\infty}^{t} x(\tau)d\tau=\int_{-\infty}^{t}\int_{-\infty}^{\infty} X(f)e^{-j2\pi f\tau}\,df d\tau=\int_{-\infty}^{\infty} X(f)\left(\int_{-\infty}^{t} e^{-j2\pi f\tau}\,d\tau\right)df=\frac{1}{j2\pi f}X(f)+\pi X(0)\delta(f) \qquad (2.43)$$

Where the impulse term on the right-hand side reflects the average value that can result from the integration.

Time and Frequency Scaling

If *x*(*t*) and *X*(*f*) are Fourier transform pairs then compression of time scale results in expansion of bandwidth and vice versa. This inverse relationship between time scale and bandwidth may be expressed as

$$x(\alpha t)\xleftrightarrow{\mathcal{F}}\frac{1}{\alpha}X\left(\frac{f}{\alpha}\right) \qquad (2.44)$$

Equation (2.44) can be proved as

$$\int_{-\infty}^{\infty} x(\alpha t)e^{-j2\pi ft}\,dt=\int_{-\infty}^{\infty} x(\alpha t)\,e^{-j2\pi\frac{f}{\alpha}\alpha t}\,dt=\frac{1}{\alpha}\int_{-\infty}^{\infty} x(t')\,e^{-j2\pi\frac{f}{\alpha}t'}\,dt'=\frac{1}{\alpha}X\left(\frac{f}{\alpha}\right) \qquad (2.45)$$

where $t'=\alpha t$. For example try to say something very slowly, then $\alpha>1$, your voice spectrum will be compressed and you may sound like a slowed down tape or compact disc, you can do the reverse and the spectrum would be expanded and your voice shifts to higher frequencies. This property is further illustrated in section 2.9.4 in Figure 2.27xx.

Convolution

The convolution integral of two signals $x(t)$ and $h(t)$ is defined as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{2.46}$$

The convolution integral is also written as

$$y(t) = x(t) * h(t) \tag{2.47}$$

where asterisk * denotes the convolution operation. The convolution integral is used to obtain the time-domain response of linear systems to arbitrary inputs as will be discussed in later sections.

The Convolution Property of the Fourier Transform

It can be shown that convolution of two signals in the time domain corresponds to multiplication of the signals in the frequency domain, and conversely multiplication in the time domain corresponds to convolution in the frequency domain. To derive the convolutional property of the Fourier transform take the Fourier transform of the convolution of the signals $x(t)$ and $h(t)$ as

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right)e^{-j2\pi ft}dt &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi f(t-\tau)}dt\right)x(\tau)e^{-j2\pi f\tau}d\tau \\ &= X(f)H(f)\end{aligned} \tag{2.48}$$

Duality

Comparing the Fourier transform and the inverse Fourier transform relations we observe a symmetric relation between them. In fact the main difference between Eqs. (2.20) and (2.21) is a negative sign in the exponent of $e^{-j2\pi ft}$ in Eq (2.21). This symmetry leads to a property of Fourier transform known as the duality principle and stated as

$$\begin{aligned} x(t) &\xleftrightarrow{\mathcal{F}} X(f) \\ X(t) &\xleftrightarrow{\mathcal{F}} x(f) \end{aligned} \tag{2.49}$$

As illustrated in Figure2.11 the Fourier transform of a rectangular function of time $r(t)$ has the form of a sinc pulse function of frequency $\mathrm{sinc}(f)$. From

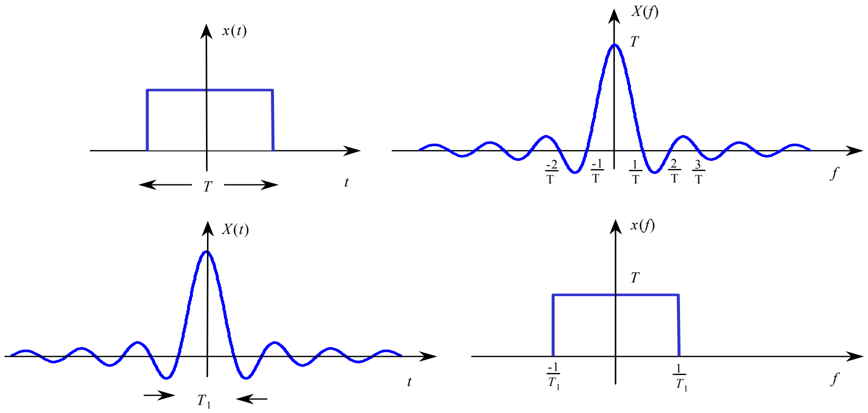


Figure 2.11 Illustration of the principle of duality.

the duality principle the Fourier transform of a sinc function of time sinc(t) is a rectangular function of frequency $R(f)$.

Parseval's Theorem: Energy Relations in Time and Frequency

Parseval's relation states that the energy of a signal can be computed by integrating the squared magnitude of the signal either over the time domain or over the frequency domain. If $x(t)$ and $X(f)$ are a Fourier transform pair, then

$$Energy = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df \tag{2.50}$$

This expression referred to as Parseval's relation follows from a direct application of the Fourier transform. Note that every characteristic value of a signal can be computed either in time or in frequency.

Example 2.9 The Spectrum of a Finite Duration Signal

Obtain and sketch the frequency spectrum of the following finite duration signal.

$$x(t) = \sin(2\pi F_0 t) \quad -NT_0/2 \le t \le NT_0/2 \tag{2.51}$$

where $T_0=1/F_0$ is the period and $\omega_0 = 2\pi/T_0$.

Solution: Substitute for $x(t)$ and its limits in the Fourier transform Eq. (2.21)

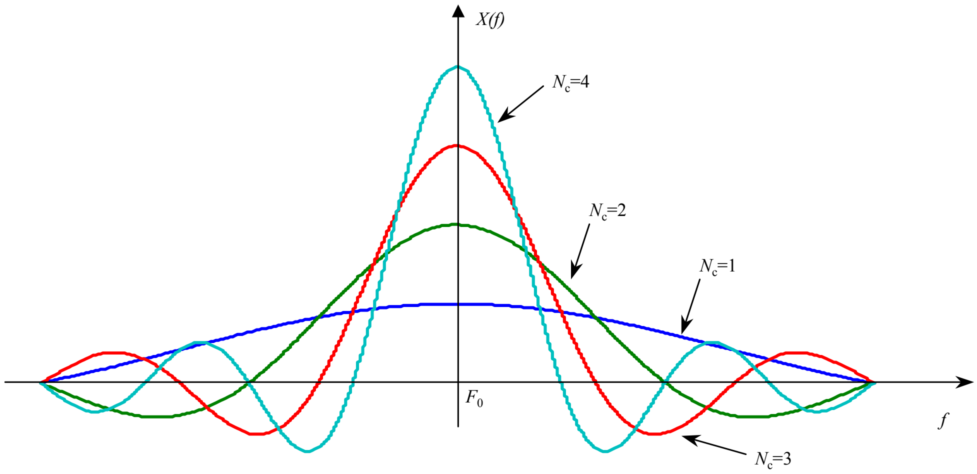


Figure 2.12 The spectrum of a finite duration sine wave with the increasing length of observation. N_c is the number of cycles in the observation window. Note that the width of the main lobe depends inversely on the duration of the signal.

$$X(f) = \int_{-NT_0/2}^{NT_0/2} \sin(2\pi F_0 t) e^{-j2\pi ft} \, dt \tag{2.52}$$

Substituting $\sin(2\pi F_0 t) = (e^{\mathrm{j}2\pi F_0 t} - e^{-\mathrm{j}2\pi F_0 t})/2\mathrm{j}$ in (2.52) gives

$$\begin{aligned} X(f) &= \int_{-NT_0/2}^{NT_0/2} \frac{e^{\mathrm{j}2\pi F_0 t} - e^{-\mathrm{j}2\pi F_0 t}}{2\mathrm{j}} e^{-\mathrm{j}2\pi ft} \, dt \\ &= \int_{-NT_0/2}^{NT_0/2} \frac{e^{-\mathrm{j}2\pi(f-F_0)t}}{2\mathrm{j}} dt - \int_{-NT_0/2}^{NT_0/2} \frac{e^{-\mathrm{j}2\pi(f+F_0)t}}{2\mathrm{j}} dt \end{aligned} \tag{2.53}$$

Evaluating the integrals yields

$$\begin{aligned} X(f) &= \frac{e^{-j\pi(f-F_0)NT_0} - e^{j\pi(f-F_0)NT_0}}{4\pi(f-F_0)} - \frac{e^{-j\pi(f+F_0)NT_0} - e^{j\pi(f+F_0)NT_0}}{4\pi(f+F_0)} \\ &= \frac{-j}{2\pi(f-F_0)} \sin\big(\pi(f-F_0)NT_0\big) + \frac{j}{2\pi(f+F_0)} \sin\big(\pi(f+F_0)NT_0\big) \end{aligned} \tag{2.54}$$

The signal spectrum can be expressed as the sum of two shifted (frequency-modulated) sinc functions as

$$X(f) = -0.5jNT_0 sinc\big((f - F_0)NT_0\big) + 0.5jNT_0\, sinc\big((f + F_0)NT_0\big) \quad (2.55)$$

Note that energy of a finite duration sinewave is spread in frequency across the main lobe and the side lobes of the sinc function. Figure 2.12 demonstrates that as the window length increases the energy becomes more concentrated in the main lobe and in the limit for an infinite duration window the spectrum tends to an impulse positioned at the frequency F_0.

Matlab code for drawing the spectrum of a finite duration sinwave.
% Sinewave Period T0, Number of Cycles in the window Nc, Number of samples of spectrum N

Plot Sine Spectrum( )

Example 2.10 *Calculation of the bandwidth for transmission of digital data at a rate of r_b bits per second.* In its simplest form binary data can be represented as a sequence of amplitude modulated pulses as

$$x(t) = \sum_{m=0}^{N-1} A(m)p(t - mT_b) \quad (2.56)$$

where $A(m)$ may be +1 or −1 and

$$p(t) = \begin{cases} 1 & |t| \le T_b/2 \\ 0 & |t| > 0 \end{cases} \quad (2.57)$$

T_b=1/r_b is pulse duration. The Fourier transform of $x(t)$ is given by

$$X(f) = \sum_{m=0}^{N-1} A(m)P(f)e^{-j2\pi mT_b f} = P(f)\sum_{m=0}^{N-1} A(m)e^{-j2\pi mT_b f} \quad (2.58)$$

The power spectrum of this signal is obtained as

$$E\big[X(f)X^*(f)\big] = E\left[\sum_{m=0}^{N-1} A(m)P(f)e^{-j2\pi mT_b f} \sum_{n=0}^{N-1} A(n)P^*(f)e^{j2\pi mT_b f}\right]$$
$$= |P(f)|^2 \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} E[A(m)A(n)]e^{-j2\pi(m-n)T_b f}$$
$$(2.59)$$

Now assuming that the data is uncorrelated we have

$$E[A(m)A(n)] = \begin{cases} 1 & m = n \\ 0 & m \ne n \end{cases} \quad (2.60)$$

Substituting Equations (2.60) in (2.59) we have

$$E[X(f)X^*(f)] = N|P(f)|^2 \tag{2.61}$$

From Equation (2.61) the bandwidth required from a sequence of pulses is basically the same as the bandwidth of a single pulse. From Figure (2.11) the bandwidth of the main lobe of spectrum of a rectangular pulse is $2r_b$ Hz.

$$B_{\mathrm{Rec}} = \frac{2}{T_p} = 2r_p \tag{2.62}$$

Note that the bandwidth required is twice the pulse rate r_p.

Example 2.11 Calculation of the bandwidth required for transmission of digital data at a rate of 1 mega pulses per second.

A data transmission rate of 1 mega pulses per second, with a pulse duration of 1 microsecond, requires a bandwidth of 2 mega Hz. Note that in *M*-array pulse modulation each pulse can signal $\log_2(M)$ bits. For example in 64-array modulation each pulse signals 6 bits and 1 mega pulses per second carries 6 mega bits per second.

2.2.3 Fourier Transform of a Sampled (Discrete-Time) Signal

A sampled signal $x(m)$ can be modelled as a sequence of time shifted and amplitude scaled discrete-time unit-impulse functions as

$$x(m) = \sum_{m=-\infty}^{\infty} x(t)\delta(t - mT_s) \tag{2.63}$$

where m is the discrete time variable and T_s is the sampling period. The Fourier transform of $x(m)$, a sampled version of a continuous signal $x(t)$, can be obtained from Eq. (2.21) as

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(t)\delta(t - mT_s)e^{-j2\pi ft}\,dt = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t)\delta(t - mT_s)e^{-j2\pi ft}\,dt \\ &= \sum_{m=-\infty}^{\infty} x(mT_s)e^{-j2\pi mfT_s} = \sum_{m=-\infty}^{\infty} x(mT_s)e^{-j2\pi mf/F_s} \end{aligned} \tag{2.64}$$

For convenience of notation and without loss of generality it is often assumed that sampling frequency $T_s = 1/F_s = 1$, hence

$$X_s(f) = \sum_{m=-\infty}^{\infty} x(m) e^{-j2\pi mf} \tag{2.65}$$

The inverse Fourier transform of a sampled signal is defined as

$$x(m) = \int_{-1/2}^{1/2} X_s(f) e^{j2\pi fm} df \tag{2.66}$$

where the period of integration is $-F_s/2 \geq f \geq F_s/2$ and it is assumed that the sampling frequency F_s=1. Note that $x(m)$ and $X_s(f)$ are equivalent in that they contain the same information in different domains. In particular, as expressed by the Parseval's theorem, the energy of the signal may be computed either in the time or in the frequency domain a

$$\textit{Signal Energy} = \sum_{m=-\infty}^{\infty} x^2(m) = \int_{-1/2}^{1/2} |X_s(f)|^2 \, df \tag{2.67}$$

Example 2.12 Show that the spectrum of a sampled signal is periodic with a period equal to the sampling frequency F_s.

Solution: substitute f+kF_s for the frequency variable f in Eq. (2.64)

$$X(f+kF_s) = \sum_{m=-\infty}^{\infty} x(mT_s) e^{-j2\pi m \frac{(f+kF_s)}{F_s}} = \sum_{m=-\infty}^{\infty} x(mT_s) e^{-j2\pi m \frac{f}{F_s}} \underbrace{e^{-j2\pi m \frac{kF_s}{F_s}}}_{=1} = X(f) \tag{2.68}$$

Figure 2.12.a shows the spectrum of a band-limited continuous-time signal. As shown in Figure 2.12.b after the signal is sampled its spectrum becomes periodic.

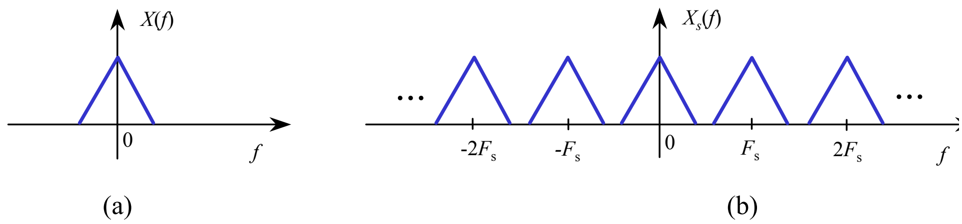


Figure 2.13 The spectrum of: (a) a continuous-time signal, and (b) its discrete-time sampled version.

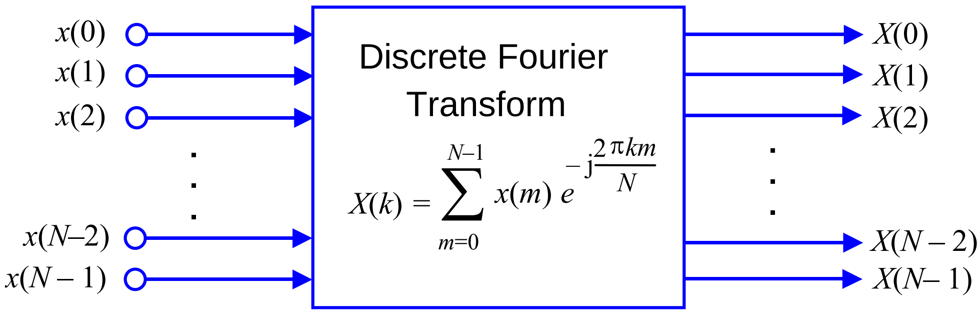


Figure 2.14 Illustration of the DFT as a parallel-input parallel-output signal processor.

2.4 Discrete Fourier Transform (DFT)

Discrete Fourier transform (DFT) is the Fourier transform adapted for digital signal processing. As illustrated in Figure 2.14 the input to DFT is N samples of a discrete signal $[x(0), \ldots, x(N\text{-}1)]$ and the output consists of N uniformly spaced samples $[X(0), \ldots, X(N\text{-}1)]$ of the frequency spectrum of the input.. When a non-periodic signal is sampled, its Fourier transform becomes periodic but *continuous* function of frequency, as shown in Eq. (2.68). The discrete Fourier transform is derived from sampling the Fourier transform of a discrete-time signal at N discrete-frequencies corresponding to integer multiples of the frequency sampling interval $2\pi/N$. The DFT is effectively the Fourier series of a sampled signal. For a finite duration discrete-time signal $x(m)$ of length N samples, the discrete Fourier transform (DFT) and its inverse (IDFT) are defined as

Discrete Fourier Transform (DFT) Analysis Equation

$$X(k) = \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi}{N}mk} \qquad k = 0, \ldots, N-1 \quad (2.69)$$

Inverse Discrete Fourier Transform (IDFT) Synthesis Equation

$$x(m) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}mk} \qquad m = 0, \ldots, N-1 \quad (2.70)$$

Note that the basis functions of a DFT are: $1, e^{-j\frac{2\pi}{N}}, e^{-j\frac{4\pi}{N}}, \ldots, e^{-j\frac{(N-1)\pi}{N}}$. The DFT equation can be written in the form of a linear system transformation, $\boldsymbol{X}=\boldsymbol{W}\boldsymbol{x}$, as the transformation of an input vector $\boldsymbol{x}=[x(0), x(1) \ldots x(N-1)]$ to an output vector $\boldsymbol{X}=[X(0)\, X(1) \ldots X(N-1)]$ as

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-2) \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{N}} & e^{-j\frac{4\pi}{N}} & \cdots & e^{-j\frac{2(N-2)\pi}{N}} & e^{-j\frac{2(N-1)\pi}{N}} \\ 1 & e^{-j\frac{4\pi}{N}} & e^{-j\frac{8\pi}{N}} & \cdots & e^{-j\frac{4(N-2)\pi}{N}} & e^{-j\frac{4(N-2)\pi}{N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{-j\frac{2(N-2)\pi}{N}} & e^{-j\frac{4(N-2)\pi}{N}} & \cdots & e^{-j\frac{2(N-2)^2\pi}{N}} & e^{-j\frac{2(N-2)(N-1)\pi}{N}} \\ 1 & e^{-j\frac{2(N-1)\pi}{N}} & e^{-j\frac{4(N-1)\pi}{N}} & \cdots & e^{-j\frac{2(N-1)(N-2)\pi}{N}} & e^{-j\frac{(N-1)^2\pi}{N}} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-2) \\ x(N-1) \end{bmatrix} \tag{2.71}$$

where the elements of the transformation matrix are $w_{km} = e^{-j\frac{2\pi}{N}km}$. The DFT spectrum consists of N uniformly spaced samples taken from one period (2π) of the continuous spectrum of the discrete time signal $x(m)$. At a sampling rate of F_s the discrete-frequency index k corresponds to kF_s/N Hz.

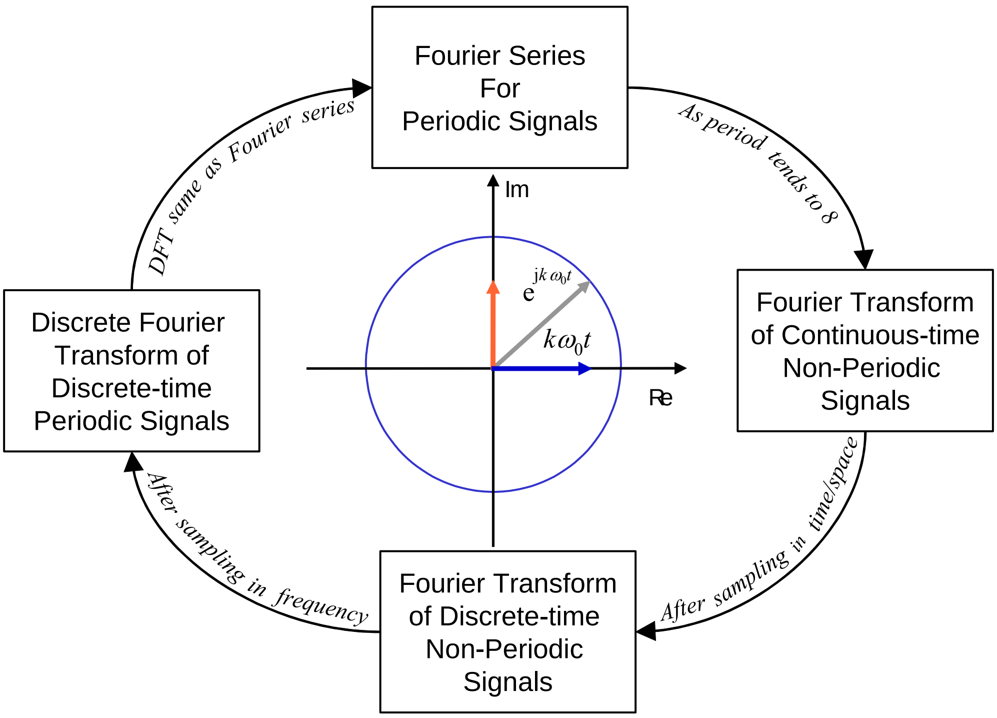


Figure 2.15 Illustration of the 'conceptual evolution' from Fourier series of a periodic signal to DFT of a sampled signal. The circle inside shows the in-phase and quadrature components of the basis function of Fourier transform.

Table 2.1 The form of input and outputs of different Fourier methods

	Input (time or space)		Output (frequency)	
	Periodic or Non-periodic	Continuous or discrete	Periodic or Non-periodic	Continuous or discrete
Fourier Series	*Periodic*	*either*		*discrete*
Fourier Transform	*Non-periodic*	*continuous*	*Non-periodic*	*continuous*
Fourier Transform of discrete-time signals	*Non-periodic*	*Discrete-time*	*Periodic*	*Continuous*
Discrete Fourier Transform	*Non-Periodic**	*Discrete-time*	*Periodic*	*Discrete*

A periodic signal has a discrete spectrum. Conversely any discrete frequency spectrum belongs to a periodic signal. Hence *the implicit assumption in the DFT theory, is that the input signal* $[x(0),\ldots, x(N\text{-}1)]$ *is periodic with a period equal to the observation window length of N samples*. Figure 2.15 illustrates the derivation of the Fourier transform and the DFT from the Fourier series as discussed in the preceding sections. Note that DFT is equivalent to the Fourier series. Table 2.1 shows the discrete/continuous, periodic/non-periodic form of the inputs and outputs of the Fourier transform methods.

Example 2.14: *Comparison of Fourier series and DFT.* The discrete Fourier transform (DFT) and the Fourier series are given by

$$X(k)=\sum_{m=0}^{N-1} x(m)e^{-j\frac{2\pi}{N}km} \qquad k = 0, \ldots, N-1 \tag{2.75}$$

$$c_k=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi kF_0t}\,dt \qquad k = 0, \ldots, N-1 \tag{2.76}$$

Note that the k^{th} DFT coefficient $X(k)$ equals the k^{th} Fourier series coefficient c_k if:

(1) $x(t)$ is sampled and replaced with $x(m)$ and the integral sign $\int$ is replaced with the discrete summation sign $\sum$.

(2) $x(m)$ is assumed to be periodic with a period of N samples.

It follows that the DFT of N samples of a signal $[x(0), \ldots, x(N\text{-}1)]$ is identical to Fourier series of periodic signal $\{\ldots [x(0), \ldots, x(N\text{-}1)], [x(0), \ldots, x(N\text{-}1)], [x(0), \ldots, x(N\text{-}1)], \ldots\}$ with a period of N samples.

Example 2.14- *Derivation of inverse discrete Fourier transform* The discrete Fourier transform (DFT) is given by

$$X(k)=\sum_{m=0}^{N-1}x(m)e^{-j\frac{2\pi}{N}km} \qquad k = 0, \ldots, N-1 \tag{2.72}$$

Multiply both sides of the DFT equation by $e^{j2\pi kn/N}$ and take the summation as

$$\begin{aligned}\sum_{k=0}^{N-1}X(k)\,e^{+j\frac{2\pi}{N}kn} &= \sum_{k=0}^{N-1}\sum_{m=0}^{N-1}x(m)e^{-j\frac{2\pi}{N}km}\,e^{+j\frac{2\pi}{N}kn} \\ &= \sum_{m=0}^{N-1}x(m)\underbrace{\sum_{k=0}^{N-1}e^{-j\frac{2\pi}{N}k(m-n)}}_{\begin{smallmatrix}N & if \ m=n \\ 0 & otherwise\end{smallmatrix}}\end{aligned} \tag{2.73}$$

Using the orthogonality principle, the inverse DFT can be derived as

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{-j\frac{2\pi}{N}kn} \qquad n = 0,\cdots,N-1 \tag{2.74}$$

2.4.1 Time and Frequency Resolutions: The Uncertainty Principle

Signals such as speech, music or image are composed of non-stationary — i.e. time-varying and/or space varying — events. For example speech is composed of a string of short-duration sounds called phonemes, and an image is composed of various objects. When using the DFT it is desirable to have a high enough time and space resolution in order to obtain the spectral characteristics of each individual elementary event or object in the input signal. However there is a fundamental trade-off between the length, i.e. the time or space resolution, of the input signal and the frequency resolution of the output spectrum. The DFT takes as the input a window of N uniformly spaced discrete-time samples [$x(0)$, $x(1)$, …, $x(N-1)$] with a total duration of $\Delta T=N.T_s$, and outputs N spectral samples [$X(0)$, $X(1)$, …, $X(N-1)$] spaced uniformly between zero Hz and the sampling frequency $F_s=1/T_s$ Hz. Hence the frequency resolution of the DFT spectrum Δf, i.e. the frequency space between successive frequency samples, is given by

$$\Delta f = \frac{1}{\Delta T} = \frac{1}{NT_s} = \frac{F_s}{N} \tag{2.77}$$

Note that the frequency resolution Δf and the time resolution ΔT are inversely proportional in that they can not both be simultaneously increased, in fact $\Delta T \Delta f = 1$. This is known as the uncertainty principle.

Example 2.15 A DFT is used in a DSP system for the analysis of an analog signal with a frequency content of up to 10 kHz. Calculate: (i) the minimum sampling rate F_s required, (ii) the number of samples required for the DFT to achieve a frequency resolution of 10 Hz at the minimum sampling rate and (iii) what is the time resolution.

Solution:

(i) Sampling rate > 2×10 kHz, say 22 kHz, and

(ii) $$\Delta f = \frac{F_s}{N} \qquad 10 = \frac{22000}{N} \qquad N \geq 2200 \tag{2.78}$$

(iii) $$\Delta T = \frac{1}{\Delta f} = 100\ ms$$

Example 2.16 Write a MATLAB program to explore the spectral resolution of a signal consisting of two sinewaves, closely spaced in frequency, with the varying length of the observation window.

Solution:

In the following program the two sinwaves have frequencies of 100 Hz and 110 Hz, and the sampling rate is 1 kHz. We experiment with two time windows of length N_1=1024 with a theoretical frequency resolution of Δf=1000/1024=0.98 Hz, and N_2=64 with a theoretical frequency resolution Δf=1000/64=12.7 Hz.

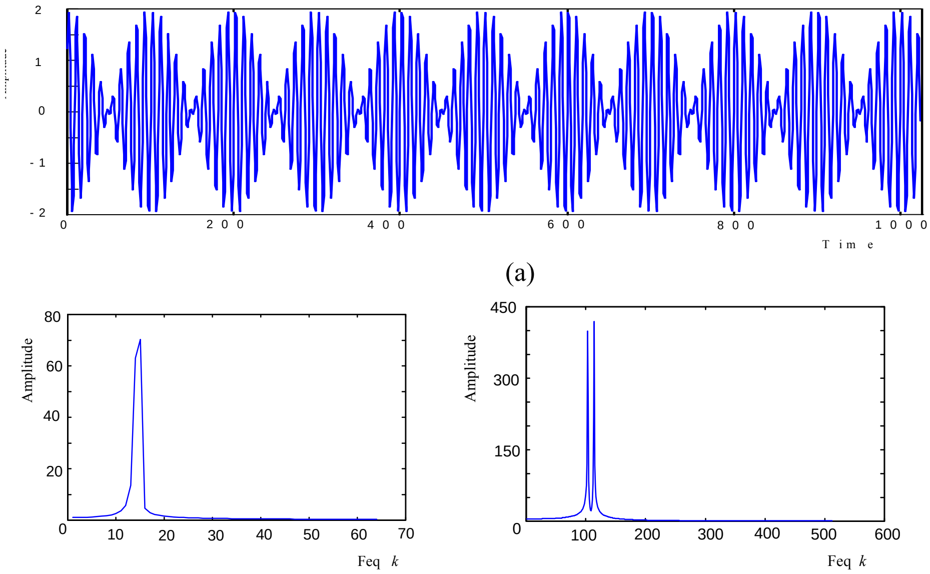
```
Fs=1000; F1=100; F2=110;

N=1:1024; N1=1024; N2=64;

x1=sin(2*pi*F1*N/Fs); x2=sin(2*pi*F2*N/Fs); y=x1+x2;

Y1=abs(fft(y(1:N1)));Y2=abs(fft(y(1:N2)));

figure(1); plot(y); figure(2); plot(Y1(1:N1/2)); figure(3); plot(Y2(1:N2/2));
```



(a)

(b) (c)

Figure 2.16 Illustration of time and frequency resolutions: (a) sum of two sinewaves with 10 Hz difference in their frequencies, (b) the spectrum of a segment of 64 samples from demonstrating insufficient frequency resolution to separate the sinewaves, (c) the spectrum of a segment of 1024 samples has sufficient resolution to show the two sinewaves.

Figure 2.16 shows a plot of the spectrum of two closely spaced sinewaves for different DFT windwo length and hence resolutions.

2.4.2 The Effect of Finite Length Data on DFT (Windowing)

In signal processing the DFT is usually applied to a relatively short segment of a signal of the order of a few tens of ms. This is because usually either we have a short length signal or when we have a long signal, due to various considerations such as stationarity of a signal or the allowable system delay, the DFT can only handle one segment at a time. A short segment of N samples of a signal, or a slice of N samples from a signal, is equivalent to multiplying the signal by a unit-amplitude rectangular pulse window of N samples. Therefore an N-sample segment of a signal $x(m)$ is equivalent to

$$x_w(m) = w(m)x(m) \tag{2.79}$$

where $w(m)$ is a rectangular pulse of N samples duration given is

$$w(m)=\begin{cases}1 & 0\le m \le N-1\\ 0 & otherwise\end{cases} \tag{2.80}$$

Multiplying two signals in time is equivalent to the convolution of their frequency spectra. Thus the spectrum of a short segment of a signal is convolved with the spectrum of a rectangular pulse as

$$X_w(k)=W(k)*X(k) \tag{2.81}$$

The result of this convolution is some spreading of the signal energy in the frequency domain as illustrated in the next example.

Example 2.17 Find the DFT of a rectangular window given by

$$w(m)=\begin{cases}1 & 0\le m\le N-1\\ 0 & otherwise\end{cases} \tag{2.82}$$

Solution: Taking the DFT of $w(m)$, and using the convergence formula for the partial sum of a geometric series, described in the appendix A, we have

$$W(k)=\sum_{m=0}^{N-1} w(m)e^{-j\frac{2\pi}{N}mk}=\frac{1-e^{-j2\pi k}}{1-e^{-j\frac{2\pi}{N}k}}=e^{-j\frac{(N-1)}{N}\pi k}\frac{\sin(\pi k)}{\sin(\pi k/N)} \tag{2.83}$$

Note that for the integer values of k, $w(k)$ is zero except for k=0. There are N-1 zero-crossings uniformly spread along 0 to F_s the sampling frequency. The discrete-time rectangular window has a sinc-shaped spectrum with a main-lobe bandwidth of $2F_s/N$.

Example 2.18 Find the spectrum of an N-sample segment of a complex sinewave with a fundamental frequency $F_0=1/T_0$.

Solution: Taking the DFT of $x(m)=e^{-j2\pi F_0 m}$ we have

$$\begin{aligned}X(k)&=\sum_{m=0}^{N-1} e^{-j2\pi F_0 m}e^{-j\frac{2\pi}{N}mk}=\sum_{m=0}^{N-1} e^{-j2\pi(F_0-\frac{k}{N})m}\\&=\frac{1-e^{-j2\pi(NF_0-k)}}{1-e^{-j2\pi(NF_0-k)/N}}=e^{-j\frac{(N-1)}{N}\pi(NF_0-k)}\frac{\sin(\pi(NF_0-k))}{\sin(\pi(NF_0-k)/N)}\end{aligned} \tag{2.84}$$

Note that for integer values of k, $X(k)$ is zero at all samples but one k=0.

2.4.3 End-Point Effects in DFT; Spectral Energy Leakage and Windowing

A window that rolls down smoothly is used to mitigate the effect spectral leakage due to the discontinuities at the end of a segment of a signal. In DFT the input signal is assumed to be periodic, with a period equal to the length of the observation window of N samples. For a sinusoidal signal when there is an integer number of cycles within the observation window, as in Figure 2.17.a, then the assumed periodic waveform is the same as an infinite duration pure sinusoid. However, if the observation window contains a non-integer number of cycles of a sinusoid then the assumed periodic waveform will not be a pure sine wave and will have end-point discontinuities. The spectrum of the signal then differs from the spectrum of sinewave as illustrated in Figure 2.17.b. The overall effects of finite length window and end-point discontinuities are:

1. The spectral energy which could have been concentrated at a single point is spread over a large band of frequencies.
2. A smaller amplitude signal located in frequency near a larger amplitude signal, may be interfered with, or even obscured, by one of the larger spectral side-lobes.

The end-point problems may be alleviated using a window that gently drops to zero. One such window is a raised cosine window of the form

$$w(m) = \begin{cases} \alpha - (1-\alpha)\cos\dfrac{2\pi m}{N} & 0 \le m \le N-1 \\ 0 & otherwise \end{cases} \tag{2.85}$$

For α=0.5 we have the Hanning window also known as the raised cosine window

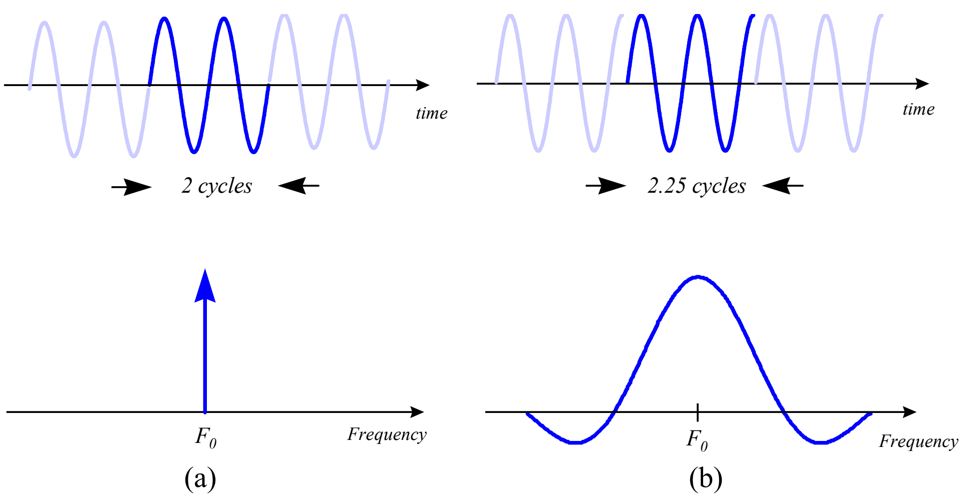


Figure 2.17 The DFT spectrum of exp(j2πfm) : (a) an integer number of cycles with in the N-sample analysis window, (b) a non-integer number of cycles in the window.

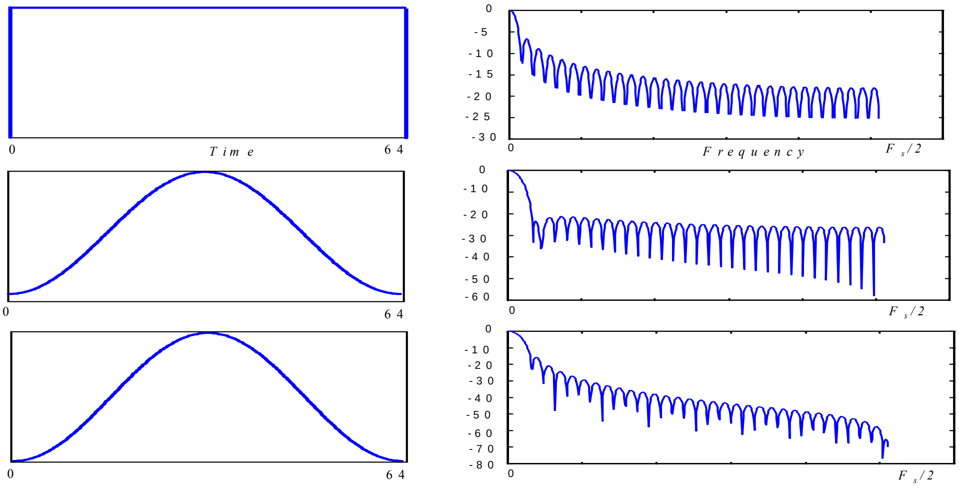


Figure 2.18 (a) Rectangular window frequency response, (b) Hamming window frequency response, (c) Hanning window frequency response.

$$w_{Han}(m) = 0.5 – 0.5\cos\frac{2\pi m}{N} \qquad 0 \le m \le N - 1 \qquad (2.86)$$

For α=0.54 we have the Hamming window

$$w_{Ham}(m) = 0.54 - 0.46\cos\frac{2\pi m}{N} \qquad 0 \le m \le N - 1 \qquad (2.87)$$

The main difference between various windows is the trade-off between bandwidth of the main lobe and the amplitude of side-lobes. Figure 2.18 shows the rectangular window, hamming window and hanning window and their respective spectra.

2.4.4 Spectral Smoothing: Interpolation Across Frequency

The spectrum of a short length signal can be interpolated to obtain a smoother looking spectrum. Interpolation of the frequency spectrum $X(k)$ is achieved by *zero-padding* of the time domain signal $x(m)$. Consider a signal of length N samples $[x(0), . . ., x(N–1)]$. Increase the signal length from N *to* $2N$ samples by padding N zeros to obtain the padded sequence $[x(0), . . ., x(N–1), 0, . . ., 0]$. The DFT of the padded signal is given by

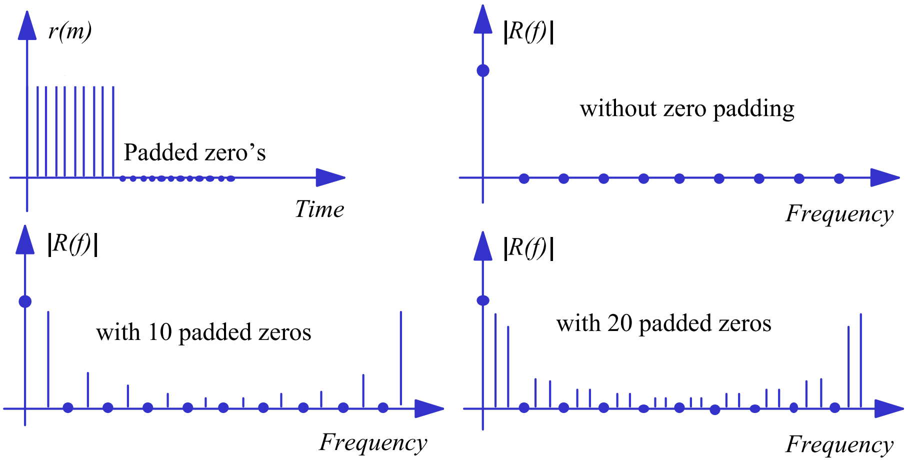


Figure 2.19 Illustration of the interpolating effect, in the frequency domain, of zero padding a signal in the time domain.

$$\begin{aligned} X(k) &= \sum_{m=0}^{2N-1} x(m) e^{-j\frac{2\pi}{2N}mk} \\ &= \sum_{m=0}^{N-1} x(m) e^{-j\frac{\pi}{N}mk} \end{aligned} \qquad k = 0, \ldots, 2N-1 \qquad (2.88)$$

The spectrum of the zero-padded signal, Eq. (2.88), is composed of $2N$ spectral samples; N of which, [$X(0)$, $X(2)$, $X(4)$, $X(6)$, . . . $X(2N–2)$] are the same as those that would be obtained from a DFT of the original N samples, and the other N samples [$X(1)$, $X(3)$, $X(5)$, $X(6)$, . . . $X(2N–1)$] are interpolated spectral lines that result from zero-padding. Note that zero padding does not increase the spectral resolution, it merely has an *interpolating or smoothing* effect in the frequency domain, as illustrated in Figure 2.19.

2.5 Short-Time Fourier Transform

In Fourier transform it is assumed that the signal is stationary, this implies that the signal statistics, such as the mean, the power, and the power spectrum, are time-invariant. Most real life signals such as speech, music, image and noise are non-stationary in that their amplitude, power, spectral composition and other features changes continuously with time. To apply Fourier transform to non-stationary signals, the signal is divided into appropriately short-time windows, such that within each window the signal may be assumed to be time-invariant. The Fourier transform applied to the

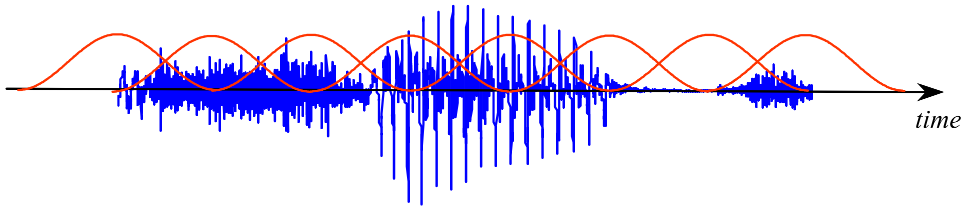


Figure 2.20 - Segmentation of speech using Hamming window for STFT.

short signal segment within each window is known as the short-time Fourier Transform (STFT). Figure 2.19 illustrates the segmentation of a speech signal into a sequence of overlapping, hamming windowed, short segments. The choice of window length is a compromise between the time resolution and the frequency resolution. For audio signals a time window of about 25 ms, corresponding to a frequency resolution of 40 Hz is normally adopted.

2.6 Fast Fourier Transform (FFT)

In this section we consider computationally fast implementation of the discrete Fourier transform. The discrete Fourier transform Eq. (2.60) can be rewritten as

$$X(k) = \sum_{m=0}^{N-1} x(m) w_N^{mk} \qquad k = 0, \ldots, N-1 \tag{2.89}$$

Where $w_N = e^{-j2\pi/N}$. The DFT Eq. (2.88) can be expressed in a matrix transformation form as

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{(N-1)} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{(N-1)} & w^{2(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{2.90}$$

In a compact form the DFT Eq. (2.74) can be written as

$$\boldsymbol{X} = \boldsymbol{W}_N \boldsymbol{x} \tag{2.91}$$

where the output vector $\boldsymbol{X}$ is the Fourier transform of the input sequence $\boldsymbol{x}$. The inverse DFT in a matrix notation can be written as

$$\boldsymbol{x} = \boldsymbol{W}_N^{-1} \boldsymbol{X} \tag{2.92}$$

Note that the individual elements of the inverse DFT matrix W_N^{-1} are the inverse of the individual elements of the DFT matrix $\boldsymbol{W_N}$. From Eq. (2.90), the direct calculation of the Fourier transform requires $N(N-1)$ multiplication and a similar number of additions.

Algorithms that reduce the computational complexity of the discrete Fourier transform are known as the fast Fourier transforms (FFT) methods. The FFT methods utilise the periodic and symmetric properties of the Fourier basis function $w_N = e^{-j2\pi/N}$ to avoid redundant calculations. Specifically the FFT methods utilise the followings:

1. The periodicity property of $w_N^{mk} = w_N^{(m+N)k} = w_N^{m(k+N)}$.
2. The complex conjugate symmetry property $w_N^{-mk} = \left(w_N^{mk}\right)^* = w_N^{(N-m)k}$.

In the following we consider two basic forms of FFT algorithms; decimation-in-time FFT and decimation-in-frequency FFT.

2.6.1 Decimation-in-Time FFT

Decimation-in-time FFT divides a sequence of input samples into a set of smaller sequences, and consequently smaller DFTs. Consider the DFT of the sequence $[x(0), x(1), x(2), x(3), ..., x(N-2), x(N-1)]$

$$X(k) = \sum_{m=0}^{N-1} x(m) w_N^{mk} \qquad k = 0, \ldots, N-1 \tag{2.93}$$

Divide the input sample sequence into two sub-sequences with even and odd numbered discrete-time indices such as $[x(0), x(2), ..., x(N-2)]$ and $[x(1), x(3), ..., x(N-1)]$. The DFT Eq. (2.93) can be rearranged and rewritten as

$$X(k) = \sum_{m \in even} x(m) w_N^{mk} + \sum_{m \in odd} x(m) w_N^{mk} \tag{2.94}$$

$$X(k) = \sum_{m=0}^{N/2-1} x(2m) w_N^{2mk} + \sum_{m=0}^{N/2-1} x(2m+1) w_N^{(2m+1)k} \tag{2.95}$$

$$X(k) = \sum_{m=0}^{N/2-1} x(2m) \left(w_N^2\right)^{mk} + w_N^k \sum_{m=0}^{N/2-1} x(2m+1) \left(w_N^2\right)^{mk} \tag{2.96}$$

Now $w_N = e^{-j2\pi/N}$ hence

$$w_N^2 = \left(e^{-j2\pi/N}\right)^2 = e^{-j2\pi/(N/2)} = w_{N/2} \tag{2.97}$$

Using Eq. (2.95), the DFT Eq. (2.04) can be written as

$$X(k) = \sum_{m=0}^{N/2-1} x(2m) w_{N/2}^{mk} + w_N^k \sum_{m=0}^{N/2-1} x(2m+1) w_{N/2}^{mk} \tag{2.98}$$

It can be seen that the DFT algorithm of Eq. (2.98) is composed of 2 DFTs of size $N/2$. Thus the computational requirement has been halved from $N(N-1)$ operations to approximately $2(N/2)^2$ operations. For an N, a power of 2, this process of division of a DFT into two smaller sized DFTs can be iterated $\log_2 N-1 = \log_2(N/2)$ times to arrive at DFTs of size 2, with $\boldsymbol{W}_2$ defined as

$$\boldsymbol{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.99}$$

From Eq. (2.99) the DFT of size 2 can be obtained by a single addition and a single subtraction and requires no multiplication. Figure 2.21 shows the flow graph for the computation of an elementary 2-point DFT called a butterfly.

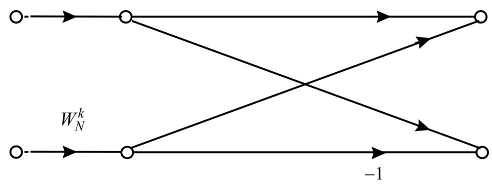


Figure 2.21 - Flow graph of a basic 2-point DFT butterfly.

Each stage of the decomposition requires $N/2$ multiplication; therefore the total number of complex multiplications required are $(N/2)\log_2(N/2)$. The total number of complex additions is $N\log_2 N$.

For example for N=256 the computation load for the direct DFT is

256×255=65280 multiplications

256×255=65280 additions

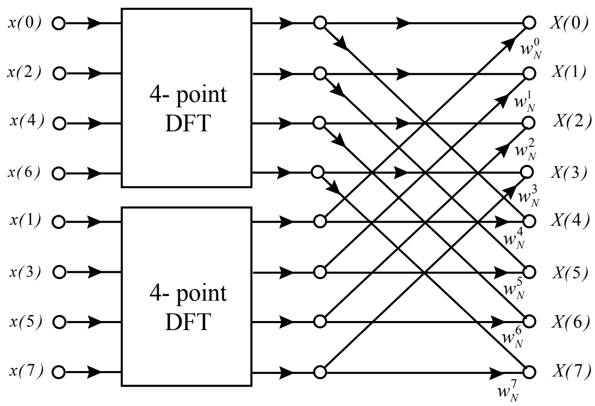
Using the FFT method the computational load is reduced considerably to

$128 \times \log_2 128 = 896$ multiplications

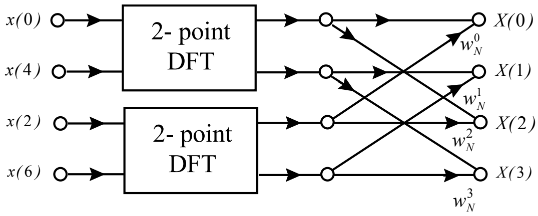
$256 \times \log_2 256 = 2048$ additions

Example 2.19 In this example we consider the decimation-in-time FFT implementation of the DFT of an 8-point sequence [$x(0)$, $x(1)$, $x(2)$, ..., $x(8)$].

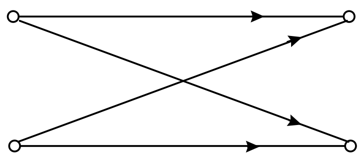
Solution: In the signal flow diagram of Figure2.22.a the 8-point DFT is divided into two 4-point DFTs. In Figure 2.22.b each 4-point DFT is further divided and represented as the combination of two two-point DFTs. Figure 2.22.c shows the signal flow diagram for the computation of a two point DFT. The FFT diagram of Figure 2.22.d is obtained by subsituting back the basic butterfly unit in the 2 point-DFT of 2.22.b and then substituting the result in the 4-point DFT blocks of figure 2.22.a.



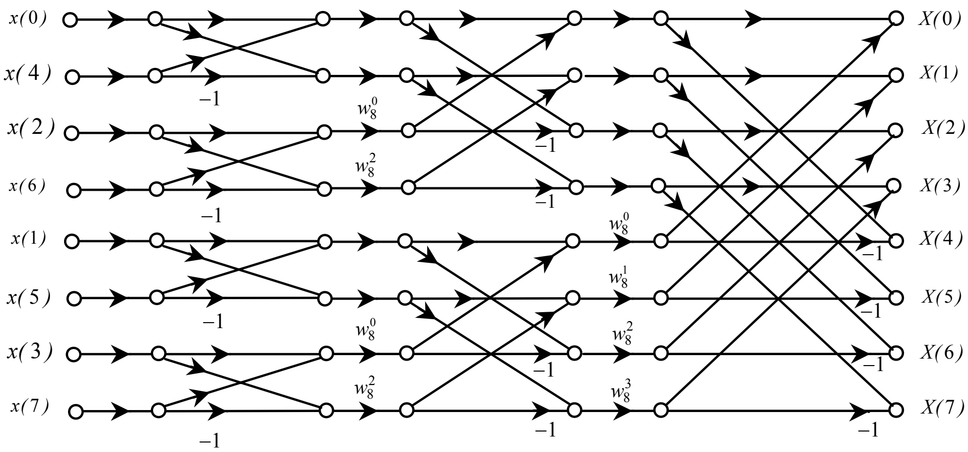
(a)



(b)



(c)



(d)

Figure 2.22 - Illustration of decimation-in-time FFT for an eight point sequence: (a) 8- point DFT described in terms of two 4-point DFTs, (b) 4-point DFT described in terms of two 2-point DFTs, (c) basic butterfly for computation of a 2-point DFT, (d) An 8-point FFT.

2.6.2 Bit Reversal

Consider the binary-index representation of the input and output sequences of the 8-point DFT of Figure 2.21.d:

$x(000)$	$X(000)$
$x(100)$	$X(001)$
$x(010)$	$X(010)$
$x(110)$	$X(011)$
$x(001)$	$X(100)$
$x(101)$	$X(101)$
x(011)	$X(110)$
$x(111)$	$X(111)$

It can be seen that the time index of the input signal $x(m)$ in the binary format is a bit reversed version of the naturally ordered sequence. For example $x(100)$ i.e. $x(4)$ takes the place of x(001) i.e. $x(1)$ and $x(110)$ i.e. $x(5)$ takes the place of $x(011)$ i.e. $x(3)$ and so on.

2.6.3 Inverse Fast Fourier Transform (IFFT)

The main difference between the Fourier transform and the inverse Fourier transform is the sign of the exponent of the complex exponential basis function: i.e. a minus sign for the Fourier transform $e^{-j2\pi/N}$, and a plus sign for the inverse Fourier transform $e^{j2\pi/N}$. Therefore the IFFT system can be obtained by replacing the variable w by w^{-1} and scaling the input sequence by a factor of N as expressed in Eq. (2.70)

2.6.4 Decimation-in-Frequency FFT

Decimation-in-time FFT divides a sequence of samples into a set of smaller sequences, and consequently smaller discrete Fourier transformations. Instead of dividing the time sequence, we can divide the output sequence leading to decimation-in-frequency FFT methods. Consider the DFT of the sequence [x(0), x(1), x(2), x(3), ..., x(N–2), x(N–1)]

$$X(k)=\sum_{m=0}^{N-1} x(m)\,w^{mk} \qquad k = 0, \ldots, N-1 \tag{2.100}$$

Divide the output sample sequence into two sample sequences with even and odd numbered discrete-time indices as [X(0), X(2), ..., X(N–2)] and [X(1), X(3), ..., X(N–1)]. For the even-number frequency samples of the DFT Eq. (2.100) can be rewritten as

$$X(2k)=\sum_{m=0}^{N/2-1} x(m)\,w_N^{2mk} + \sum_{m=N/2}^{N-1} x(m)\,w_N^{2mk} \tag{2.101}$$

$$X(2k)=\sum_{m=0}^{N/2-1} x(m)\,w_N^{2mk} + \sum_{m=0}^{N/2-1} x(m+N/2)\,w_N^{2(m+N/2)k} \tag{2.102}$$

Since w_N^{2mk} is periodic with a period of N we have

$$w_N^{2(m+N/2)k} = w_N^{2mk}\,w_N^{Nk} = w_N^{2mk} \tag{2.103}$$

and $w_N^2 = w_{N/2}$, Eq. (2.100) can be written as

$$X(2k)=\sum_{m=0}^{N/2-1} [x(m)+x(m+N/2)]\,w_{N/2}^{mk} \tag{2.104}$$

Eq. (2.102) is an N/2-point DFT. Similarly, for the odd-number frequency samples the DFT Eq. (2.98) can be rewritten as

$$X(2k+1) = \sum_{m=0}^{N/2-1} x(m) w_N^{m(2k+1)} + \sum_{m=N/2}^{N-1} x(m) w_N^{m(2k+1)} \tag{2.105}$$

$$X(2k+1) = \sum_{m=0}^{N/2-1} x(m) w_N^{m(2k+1)} + \sum_{m=0}^{N/2-1} x(m+N/2) w_N^{(m+N/2)(2k+1)} \tag{2.106}$$

Since w_N^{2km} is periodic with a period of *N* we have

$$w_N^{(m+N/2)(2k+1)} = w_N^{m(2k+1)} \underbrace{w_N^{kN}}_{=1} \underbrace{w_N^{N/2}}_{=-1} = -w_N^{m(2k+1)} \tag{2.107}$$

Therefore

$$X(2k+1) = \sum_{m=0}^{N/2-1} [x(m) - x(m+N/2)] w_N^{m(2k+1)} \tag{2.108}$$

or

$$X(2k+1) = \sum_{m=0}^{N/2-1} [x(m) - x(m+N/2)] w_{N/2}^{km} w_N^{m} \tag{2.109}$$

Eq. (2.107) is an *N*/2-point DFT of the sequence $[x(m) - x(m+N/2)]w_N^m$. Thus as before the *N*-point DFT can be implemented as the sum of 2 *N*/2-point DFTs.

Example 2.20 In this example we consider the decimation-in-frequency FFT of an 8-point sequence [*x*(0), *x*(1), *x*(2), ..., *x*(8)].

Solution: In the signal flow diagram of Figure2.22.a the 8-point DFT is divided into two 4-point DFTs. In Figure 2.22.b each 4-point DFT is further divided and represented as the combination of two two-point DFTs. Figure 2.22.c shows the signal flow diagram for the computation of a 2-point DFT. The FFT diagram of Figure 2.22.d is obtained by substituting back the basic butterfly unit in the 2 point-DFT of 2.22.b and then substituting the result in the 4-point DFT blocks of figure 2.22.a.

2.

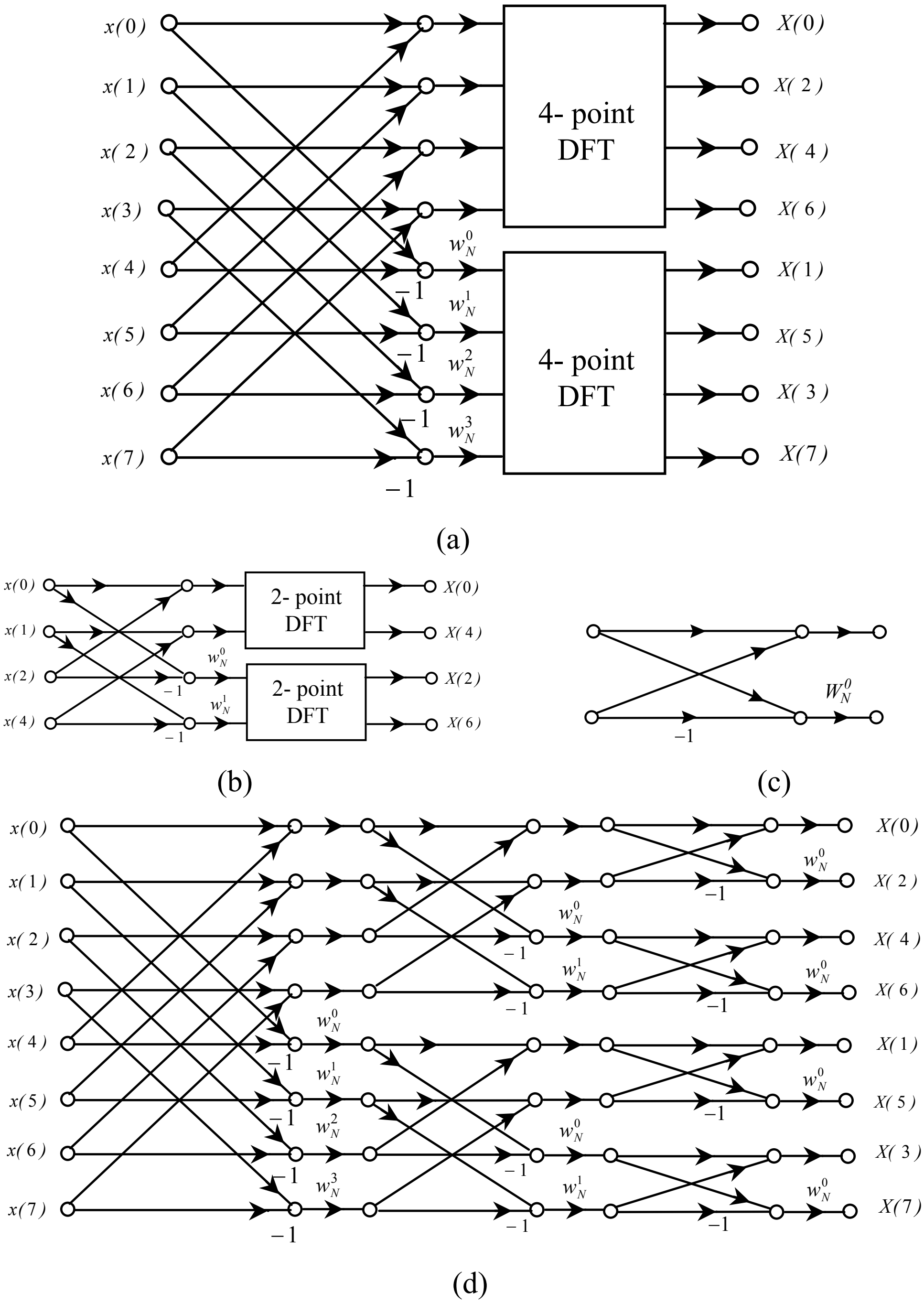


Figure 2.23- Illustration of decimation-in-frequency FFT for an 8-point sequence: (a) an 8- point DFT described in terms of two 4-point DFTs, (b) a 4-point DFT described in terms of two 2-point DFTs, (c) the basic butterfly for computation of a 2-point DFT, (d) an 8-point FFT.

2.7 2-D Discrete Fourier Transform (2-D DFT)

For a two-dimensional signal such as an image, the discrete Fourier transform is essentially two one-dimensional DFTs, consisting of the DFT of the columns of the input signal matrix followed by the DFT of the rows of the matrix or vice-versa. The 2-D DFT is defined as

$$X(k_1,k_2)=\sum_{m_1=0}^{N_1-1}\sum_{m_2=0}^{N_2-1}x(m_1,m_2)\,e^{-j\frac{2\pi}{N_1}m_1k_1}\,e^{-j\frac{2\pi}{N_2}m_2k_2} \qquad k_1=0,\ldots,N_1-1$$

$$k_2=0,\ldots,N_2-1 \qquad (2.110)$$

and, the 2-D inverse discrete Fourier transform (2D-IDFT) is given by

$$x(m_1,m_2)=\frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}X(k_1,k_2)e^{j\frac{2\pi}{N_1}m_1k_1}\,e^{j\frac{2\pi}{N_2}m_2k_2} \qquad m_1=0,\ldots,N_1-1$$

$$m_2=0,\ldots,N_2-1 \qquad (2.111)$$

The 2-D DFT Eq. (2.110) requires $N_1(N_1-1)N_2(N_2-1)$ multiplication and a similar number of additions. For example the 2-D DFT for an image block of size 8×8 pixels involves 3136 multiplications. Using the fast Fourier transform (FFT), of the rows and columns, the number of multiplications required is reduced considerably to $(N_1/2)\log_2(N_1/2)(N_2/2)\log_2(N_2/2)=64$ multiplications.

2.8 Discrete Cosine Transform (DCT)

The Fourier transform expresses a signal in terms of a combination of complex exponentials composed of cosine (real part) and sine (imaginary part) waveforms. A signal can be expressed purely in terms of cosine or sine basis functions leading to cosine transform and sine transform. For many processes, the cosine transform is a good approximation to the data-dependent optimal Karhunen-Loeve Transforms. The cosine transform is extensively used in speech and image compression and for feature extraction. The discrete cosine transform is given by

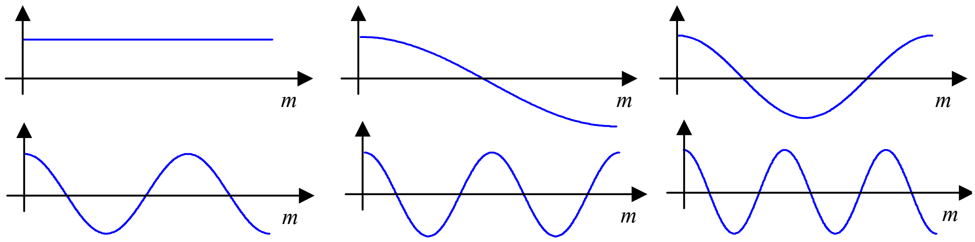


Figure 2.24 The first six basis functions of the discrete cosine transform.

$$X(k) = 2\sum_{n=0}^{N-1} x(m)\cos\frac{\pi k(2m+1)}{2N} \qquad 0 \le k \le N-1 \tag{2.112}$$

$$x(m) = \frac{1}{N}\sum_{k=0}^{N-1} w(k)X(k)\cos\frac{\pi k(2m+1)}{2N} \qquad 0 \le k \le N-1 \tag{2.113}$$

where the coefficient $w(k)$ is given by

$$w(k) = \begin{cases} \frac{1}{2} & k = 0 \\ 1 & k = 1,\cdots,N-1 \end{cases} \tag{2.114}$$

Figure 2.24 shows the first six basis functions of the discrete cosine transform. A useful property of DCT is its ability to compress and concentrate most of the energy of a signal into a relatively few low frequency coefficients, as shown in the example 2.17.

A modified form of the DCT, which is a unitary orthonormal transform, is given by

$$X(k) = \sqrt{\frac{2}{N}}\, w(k)\sum_{n=0}^{N-1} x(m)\cos\frac{\pi k(2m+1)}{2N} \qquad 0 \le k \le N-1 \tag{2.115}$$

$$x(m) = \sqrt{\frac{2}{N}}\sum_{k=0}^{N-1} w(k)X(k)\cos\frac{\pi k(2m+1)}{2N} \qquad 0 \le k \le N-1 \tag{2.116}$$

where the coefficient $w(k)$ is given by

$$w(k) = \begin{cases} \frac{1}{\sqrt{2}} & k = 0 \\ 1 & k = 1,\cdots N-1 \end{cases} \tag{2.117}$$

Example 2.21 Compare the DCT and the DFT of a discrete-time impulse function.

The discrete-time impulse function is given by

$$\delta(m)=\begin{cases}1 & m=0\\ 0 & m\neq 0\end{cases} \tag{2.118}$$

$$\Delta^{DCT}(k)=2\sum_{n=0}^{N-1}\delta(m)\cos\frac{\pi k(2m+1)}{2N}=2\cos\frac{\pi k}{2N} \qquad 0\le k\le N-1 \tag{2.119}$$

The Fourier transform of impulse function

$$\Delta^{DFT}(k)=\sum_{m=0}^{N-1}\delta(m)e^{-j\frac{2\pi}{N}km}=1 \qquad 0\le k\le 1 \tag{2.120}$$

Figure 2.25 shows the DFT spectrum and the DCT spectrum of an impulse function.

Example 2.22 Comparison of DFT and DCT of an impulse function. The Matlab program for this exercise is

```
x=zeros(1,1000);
x(1)=1;
xdct=dct(x);
xfft=abs(fft(x));
```

The plots in Figure 2.25 show that DCT

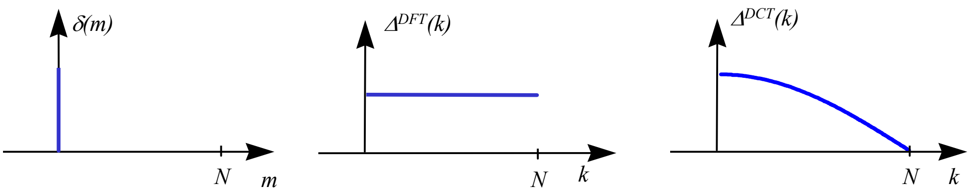


Figure 2.25 A digital impulse signal and its DFT and DCT spectra.

2.9 Some Applications of The Fourier Transform

The Fourier signal analysis/synthesis method is used in numerous applications in science engineering and data analysis. In this section we consider several application of the Fourier transform.

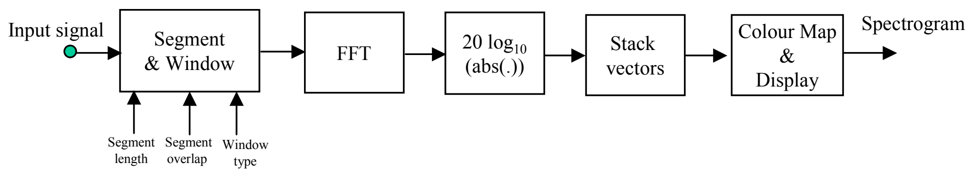


Figure 2.26 The block diagram illustration of spectrogram.

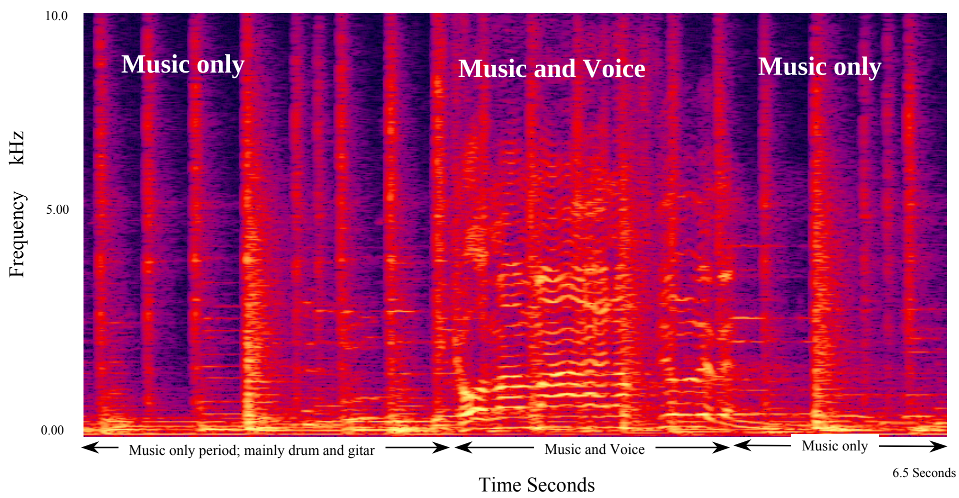


Figure 2.27 The spectrogram of music and voice.

2.9.1 Spectrogram

The spectrogram is a plot of the variation of the short time magnitude (or power) spectrum of a signal with time. A shown in Figure 2.26, the signal is divided into overlapping windowed segments of appropriately short duration (about 25 ms for audio signals), each segment is tranformed with an FFT, and the magnitude frequency vectors are stacked and plotted with the vertical axis representing the frequency and the horizontal axis the time. The magnitude values are colour coded with black colour representing the lowest value and a light colour representing the largest value.

The spectrogram provides a powerful description of the time variation of the spectrum of a time-varying signal. This is demonstrated in Figure 2.27, which shows the spectrogram of voice and music. In the music-only periods the quasi-periodic instances of hitting of the drum are clearly visible. Also the periods where voice is present can be clearly distinguished from the pitch structure of the vocal cord excitation. In fact looking at histogram, and relying on the pitch structure of speech, it is possible to manually segment the signal into music-only and voice-plus-music periods with ease and with a high degree of accuracy.

2.9.2 Signal Interpolation

From the time-frequency duality principle, just as zero-padding of a signal in time (or space) leads to interpolation of its frequency spectrum (see section 2.4.3), zero-padding in frequency followed by inverse transform provides interpolation of a signal in time (or space).

Consider a signal of length N samples $[x(0), x(1) \ldots, x(N-1)]$ together with its DFT $[X(0), X(1), \ldots X(N-1)]$. Increase the length of signal spectrum from N *to* $2N$ samples by padding N zeros to obtain the padded sequence $X_{\text{padded}} = [X(0), \ldots X((N+1)/2), 0, 0 \ldots, 0, [X((N+1)/2), X(1), \ldots X(N-1)]$. Note the padding is done so as to preserve the symmetric property of the Fourier transform of a real-valued signal. The inverse DFT of the padded signal gives an interpolated signal

$$x_i(m) = \frac{1}{2N}\sum_{k=0}^{2N-1} X_{padded}(k)e^{\mathrm{j}\frac{2\pi}{2N}mk} \qquad m = 0, \ldots, 2N-1 \quad (2.121)$$

Noting that the padded zero-valued spectral components have no effect on the sum product of Eq(2.119) we may write Eq (2.119) as

$$x_i(m) = \frac{1}{2N}\sum_{k=0}^{N-1} X(k)e^{\mathrm{j}\frac{2\pi}{2N}mk} \qquad m = 0, \ldots, 2N-1 \qquad (2.122)$$

The $2N$-point signal given by Eq (2.12) consist of the N original samples plus N interpolated samples. For interpolation by a factor of M Equation (2.120) may be modified as

$$x(m) = \frac{1}{MN}\sum_{k=0}^{N-1} X(k)e^{\mathrm{j}\frac{2\pi}{MN}mk} \qquad m = 0, \ldots, MN-1 \qquad (2.123)$$

2.9.3 Digital Filter Design

Given the required spectrum of a digital filter $H_d(f)$, the inverse Fourier transform can be applied to obtain the impulse response $h_d(m)$, and hence the coefficients, of an FIR filter implementation as

$$h_d(m) = \int_{-F_s/2}^{F_s/2} H_d(f)\, e^{j2\pi fm}\, df \tag{2.124}$$

Where F_s is the sampling rate. In chapter 4 we consider the methods for digital filter design in some detail. The following example, and the MatLab program, illustrate the design of a digital audio bass filter using the inverse discrete Fourier transform equation.

2.9.4 Digital BASS for Audio Systems

A digital Bass filter boosts the low frequency part of the signal spectrum. The required frequency response of a typical bass booster is show in Figure (2.28-a). The inverse DFT is used to obtain the impulse response corresponding to the specified frequency response. The inverse frequency response has in theory infinite duration. A truncated version of the impulse response and its frequency spectrum are shown in Figs. (2.28-a) and (2.28-b). Note that the frequency response of the filter obtained through IDFT is a good approximation of the specified frequency response.

```
% Digital Bass Filter
clear;
FiltrLnth=201;BoostAmpl=10; SamplFreq=16000; FreqResol=10;
HalfSamplFreq=SamplFreq/(2*FreqResol);
BoostFreq=320/FreqResol; BoostFreqEnd=3800/FreqResol;
BoostAmpl=10^(BoostAmpl/20);
BoostSlopUp=(BoostAmpl-1)/(BoostFreq-1);
BoostSlopDown=-(BoostAmpl-1)/(BoostFreqEnd-BoostFreq);

i=1:BoostFreq;
H(i)=1+(i-1)*BoostSlopUp;
i=BoostFreq+1:BoostFreqEnd;
H(i)=BoostAmpl+(i-BoostFreq)*BoostSlopDown;

i=BoostFreqEnd:HalfSamplFreq;
H(i)=1;
```

```
figure(1); plot(20*log10(H))

i=1:HalfSamplFreq
H(HalfSamplFreq+i)=conj(H(HalfSamplFreq+1-i));
h=ifft(H);
a((FiltrLnth+1)/2)=real(h(1));
HlfFiltrLnth=fix(FiltrLnth/2);

i=1:HlfFiltrLnth
a(i)=real(h(HlfFiltrLnth+2-i));
a(FiltrLnth+1-i)=a(i);
figure(2); plot(a);
a=[a zeros(1, 1024-FiltrLnth)];
y=20*log10(abs(fft(a)));
figure(3); plot(y(1:512));
```

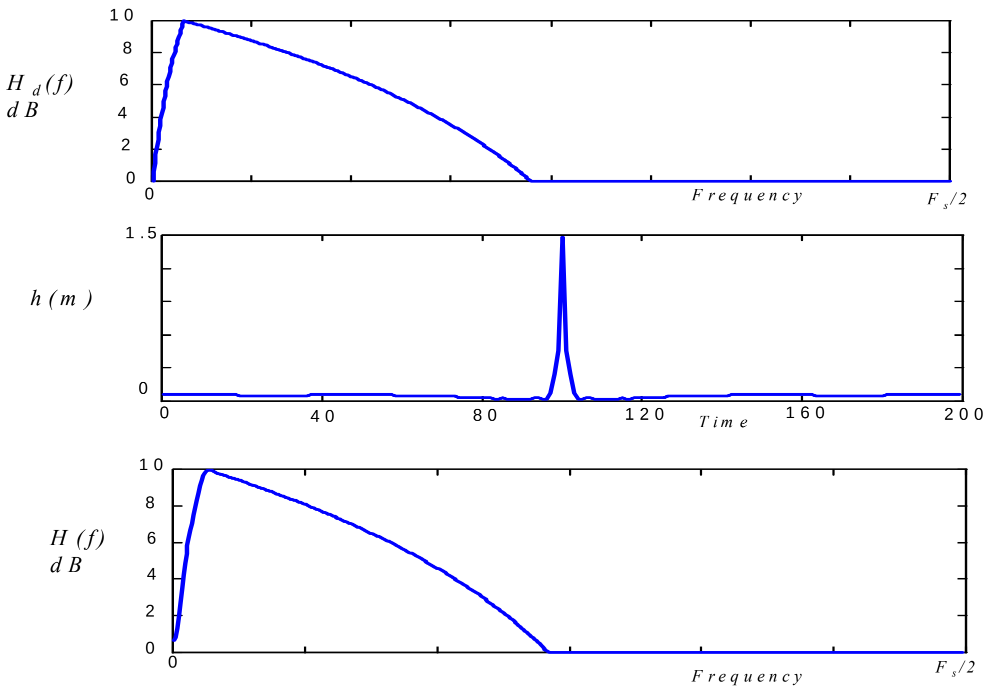


Figure 2.28 (a) The specified frequency response of a digital bass filter, (b) truncated inverse DFT of the frequency response specified in (a), and (c) the frequency response of the truncated impulse response.

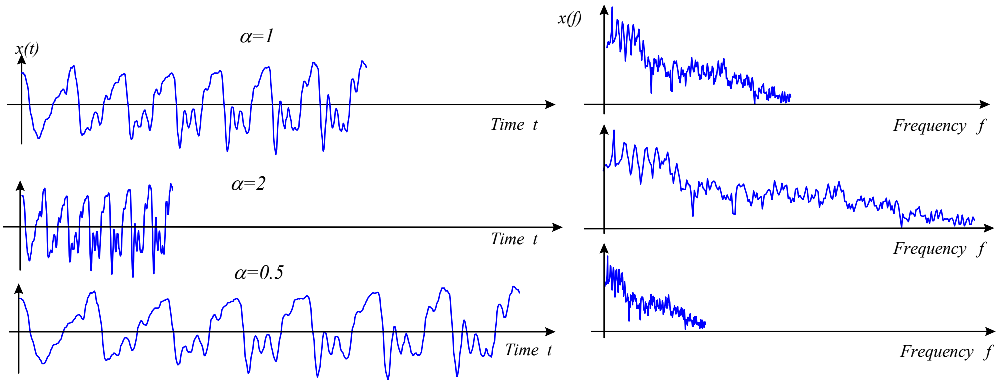


Figure 2.29 Illustration of a signal and its spectrum played back at twice and half the recording rate.

2.9.5 Voice and Pitch Conversion

Another example of the application of Fourier transform is in pitch conversion. A simple method to change the pitch and voice characteristics is frequency warping. This follows time-frequency scaling relationship of the Fourier transform pair:

$$x(\alpha t) \xleftrightarrow{\mathcal{F}} \frac{1}{\alpha} X\left(\frac{f}{\alpha}\right) \tag{2.125}$$

Figure 2.29 shows a segment of a speech signal and its spectrum for three values of α equal to: α =1 i.e. the original signal; α =2 corresponds to a compression in the time domain by a factor of 2:1 or equivalently an expansion in the frequency domain by a factor of 1:2; and α= 0.5 corresponds to an expansion in the time domain by a factor of 1:2 or equivalently a compression in the frequency domain by a factor of 2:1. A familiar example of Eq. (2.123) is when an audio signal is played back at a speed slower or faster than the recording speed. For α >1 the speech spectrum moves to the lower frequencies and the voice assumes a more male and gravel sounding quality, whereas for $\alpha < 1$ the voice shifts to the higher frequencies and the voice becomes more feminine and higher pitched.

2.9.6 Radar Signal Processing: Doppler Frequency Shift

Figure 2.30 shows a simple sketch of a radar system which can be used to estimate the range and speed of an object such as a moving car or a flying aeroplane. A radar consists of a transciever (Transmitter/Receiver), that generates and transmits sinusoidal pulses at microwave frequencies. The signal travels with the speed of light and is reflected from any object in its path. The analysis of the received echo provides such information as range, speed, and acceleration. The received signal has the form

$$x(t) = A(t)\cos[\omega_0(t - 2r(t)/c)] \tag{2.126}$$

where $A(t)$ the time-varying amplitude of the reflected wave depends on the position and characteristics of the target, $r(t)$ is the time-varying distance of the object from the radar, $2r(t)/c$ is the round trip delay and the constant c is the velocity of light .

The time-varying distance of the object can be expanded in a Taylor's series form as

$$r(t) = r_0 + \dot{r}t + \frac{1}{2!}\ddot{r}t^2 + \frac{1}{3!}\dddot{r}t^3 + \cdots \tag{2.127}$$

where r_0 is the distance, $\dot{r}$ is the velocity, $\ddot{r}$ is the acceleration etc. Approximating $r(t)$ with the first two terms of the Taylor's series expansion we have

$$r(t) \approx r_0 + \dot{r}t \tag{2.128}$$

substituting Eq. (2.126) in Eq. (2.124) in yields

$$x(t) = A(t)\cos[(\omega_0 - 2\dot{r}\omega_0/c)t - 2\omega_0 r_0/c] \tag{2.129}$$

Note that the frequency of reflected wave is shifted by an amount of

$$\omega_d = 2\dot{r}\omega_0/c \tag{2.130}$$

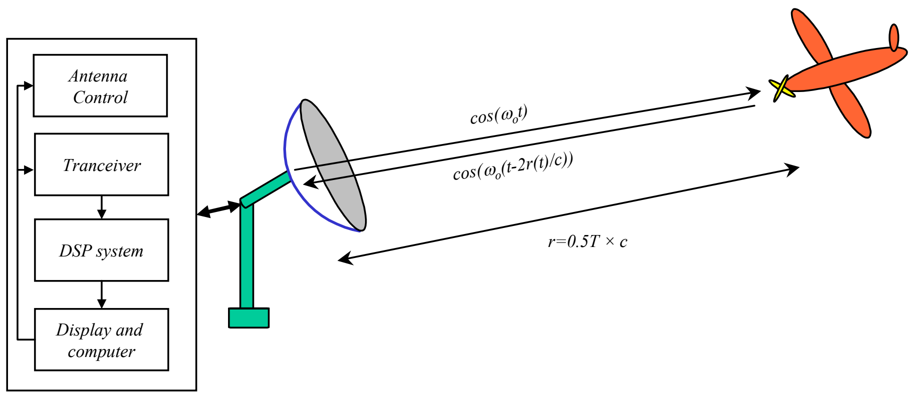


Figure 2.30 Illustration of a radar system.

This shift in frequency is known as the Doppler frequency. If the object is moving towards the radar, then the distance $r(t)$ is decreasing with time, $\dot{r}$ is negative and an increase in the frequency is observed. Conversely if the object is moving away from the radar then the distance $r(t)$ is increasing, $\dot{r}$ is positive and a decrease in the frequency is observed. Thus the frequency analysis of the reflected signal can reveal information on the direction and speed of the object. The distance r_0 is given by

$$r_0 = 0.5\,T \times c \tag{2.131}$$

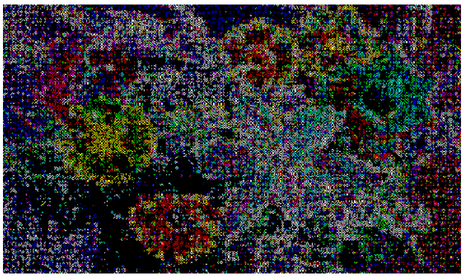
where T is the round-trip time for the signal to hit the object and arrive back at the radar and c is the velocity of light.

2.9.7 Image Compression in Frequency Domain

The discrete cosine transform is widely used in compression of still pictures and video. The DCT has the property that it compresses most of the signal energy into a relatively few low frequency coefficients. The higher frequency coefficients can be discarded with little visible effect on the perceived quality of an image. This is illustrated in the following example. Figure 2.31.a shows a flower picture. The image was segmented into subblocks of 8×8=64 pixels (picture elements). Each block was transformed to frequency using DCT. For each block the lowest 5x5=25 coefficients were retained and other coefficients were discarded, this corresponds to a compression ratio of 2.6:1 or some 61% compression. The compressed image reconstructed via inverse DCT is shown in Figure 2.31.b. The error image Figure 2.31.c

is obtained by subtracting the original and the compressed image. Table 2.31.d shows the root mean squared value of the DCT coefficients and illustrates that in the DCT domain the energy is mostly concentrated in the lower coefficients (i.e. in the top left corner of the table).

(a) *(b)*



254	24	11	7	4	3	2	1
23	12	8	5	3	2	2	1
11	8	6	4	3	2	1	1
6	5	4	3	2	2	1	1
4	3	3	2	2	1	1	1
3	2	2	2	1	1	1	1
2	2	1	1	1	1	1	0
1	1	1	1	1	1	0	0

(c) *(d)*

Figure 2.31 (a) A flower picture, (b) the picture reconstructed after discarding 60% of DCT coefficients, (c) the error, (d) the r.m.s value of DCT coefficients to the nearest integer, averaged over the picture.

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Exercises

2.1 (i) Write the basis functions for complex Fourier series, (ii) state three useful properties of the Fourier basis functions and (iii) using the orthogonality principle derive the inverse DFT.

2.2 Explain why the Fourier transform has become the most applied tool in signal analysis and state three applications of the Fourier transform.

2.3 Draw a block diagram illustration of a spectrogram and label the main signal processing stage in each block. State the main parameters and choices regarding time-frequency resolution and windowing in spectrogram. State your choice of sampling rate, DFT time window length and the resulting time and frequency resolution for analysis of
(ii) A speech signal with a bandwidth of 4 kHz.
(iii) Music signal with a bandwidth of 20 kHz.

2.4

(i) A segment of N samples of a signal is padded with $3N$ zeros. Derive the DFT equation for the zero-padded signal. Explain how the frequency resolution of the DFT of a signal changes by zero-padding.

(ii) Assuming that a segment of 1000 samples of a signal is zero padded with 3000 extra zeros. Calculate the time resolution, the actual frequency resolution and the interpolated apparent frequency resolution. Assume the signal is sampled at 44100 Hz.

(iii) Write a mathematical description of an impulse function and derive its Fourier transform and hence sketch its frequency spectrum.

2.5 Explain why the rectangular pulse and its frequency spectrum are so important in communication signal processing. Obtain and sketch the spectrum of the rectangular pulse given by

$$x(t) = \begin{cases} 1 & |t| \le T/2 \\ 0 & otherwise \end{cases}$$

Find an expression relating the bandwidth of the main lobe of the spectrum to the pulse duration. Calculate the percentage of the pulse power concentrated in the main lobe of the pulse spectrum.

Obtain the Fourier transform of a rectangular pulse of duration 1 microsecond and hence the bandwidth required to support data transmission at a rate of 1 megabit per second.
Find the main bandwidth of the spectrum of the rectangular pulse in terms of the pulse duration *T*.

3.6 State the minimum number of samples per cycle required to convert a continuous-time sine wave to a discrete-time sine wave.

Hence state the Nyquist-Shannon sampling theorem for sampling a continuous-time signal with a maximum frequency content of B Hz.

Calculate the bit rate per second and the bandwidth required to transmit a digital music signal sampled at a rate of 44100 samples per second and with each sample represented with a 16-bit word.

2.7 Express the DFT equation in terms of a combination of a cosine transform and a sine transform. State the relation between DFT and complex Fourier series.

i) What is the physical interpretation of the magnitude and phase of X(k)?
ii) Assuming a sampling rate of F_s=10 kHz, and a DFT length of N=256, what is the actual frequency value corresponding to the discrete frequency k?
iii) Obtain the length N of the input signal of the DFT to yield a frequency resolution of 40 Hz at a sampling rate of 8000 samples per second. What is the time resolution of the DFT?

2.8 Obtain the DFT of the following sequences

1) $\boldsymbol{x}$ =[3 1], and
2) $\boldsymbol{x_p}$=[3 1 0 0].

Quantify the improvements in the actual frequency resolution and the apparent frequency resolution of a DFT if N signal samples are padded by N zeros.

2.9 Find an expression for and sketch the frequency spectrum of a sine wave of angular frequency $\omega_0=2\pi/T_0$. Obtain an expression for the energy and power of this sinewave.
Obtain and sketch the frequency spectrum the following signal

$$x(t) = 10\sin(1000\pi t) + 5\sin(3000\pi t) + 2\sin(4000\pi t)$$

Calculate the total signal power and the power at each individual frequency, and hence deduce the Parseval's theorem.

2.10 Find the frequency spectrum of a periodic train of pulses with a period of 10 kHz and a pulse 'on' duration of 0.04 milliseconds. Write the formula for synthesising the signal up to the 10^{th} harmonic, and plot a few examples for the increasing number of harmonics.

2.11 Find the spectrum of a shifted impulse function defined as

$$\delta(t-T) = \underset{\Delta\to 0}{\text{limit}}\; p(t) = \begin{cases} 1/\Delta & |t-T| \le \Delta/2 \\ 0 & |t-T| > \Delta/2 \end{cases}$$

and hence deduce the relationship between the time-delay and the phase of a signal.

2.12 Find the Fourier transform of a burst of N cycles of a sinewave of period T_0 seconds. You can model a burst of sinewave as an infinite duration signal multiplied by a rectangular window, and then employ the convolutional property of the Fourier transform for the product of two signals. Sketch the spectrum of the signal and show the effect of increasing the length of the signal in time on its frequency spectrum.

3.13 Find and sketch the frequency spectrum of the following signals

(a) $x(t) = \delta(t)$

(b) $x(t) = \delta(t-k) + \delta(t+k)$

(c) $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT_s)$

2.14 Assume that a signal x(t) has a bandlimited spectrum X(f) with the highest frequency content of B Hz, and that x(t) is sampled periodically with a sampling interval of T_s seconds.

(a) Show that the frequency spectrum of the sampled signal $x_s(t)$ is periodic with a period of $F_s=1/T_s$ Hz, and
(b) hence deduce, and state, the Nyquist theorem.

2.15 Show that a time-delayed signal $x(m–m_0)$ can be modelled as the convolution of $x(m)$ and a delayed delta function $\delta(m–m_0)$. Hence, using the Fourier transform, show that a time delay of m_0 is equivalent to a phase shift of $e^{-j2\pi fm_0}$.

2.16 Derive the discrete Fourier transform (DFT) and the inverse discrete Fourier transform (IDFT) equations. State the fundamental assumption in derivation of the DFT equations.
Define the terme frequency resolution and time resolution and write the expression for the frequency resolution of the DFT.

2.17 Write the DFTs of the sequence {x(0), x(1), x(2), x(3)} and its zero-padded version {x(0), x(1), x(2), x(3), 0, 0, 0, 0}. Discuss the frequency-interpolating property of the zero-padding process as indicated by the DFT of the zero-padded sequence.

2.18 The Fourier transform of a sampled signal is given by

$$X(f)=\sum_{m=0}^{N-1} x(m)e^{-j2\pi fm}$$

Using the above equation, prove that the spectrum of a sampled signal is periodic, and hence state the sampling theorem.
A sinusoidal signal of frequency 10 kHz, $\sin(2\pi 10^4 t)$, is sampled, incorrectly, at a rate of 15 kHz. Sketch the frequency components of the sampled signal in the range of 0 to 30 kHz.

2.19 An analog signal x(t) with the spectrum X(f) sampled originally at a rate of 10 kHz is to be converted to a sampling rate of 40 kHz. In preparation for digital interpolation, the sampled signal is zero-inserted by a factor of 3 zeros in-between every two samples.
Derive an expression for the spectrum of the zero-inserted signal in terms of the original spectrum X(f).
Sketch the spectrum of the zero-inserted signal, and hence state the cut-off frequency of the interpolation filter.

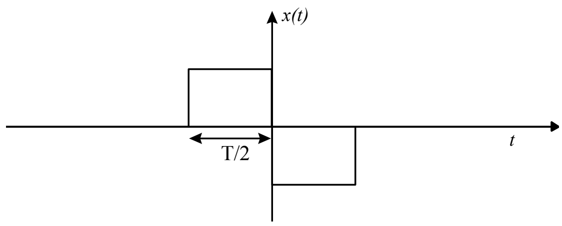
2.20 (a) The Discrete Fourier transform (DFT) equation is given by

$$X(k)=\sum_{m=0}^{N-1} x(m)e^{-j\frac{2\pi}{N}mk}$$

Explain the effects of finite length data on the DFT spectrum of the signal, and suggest a solution.
Calculate the DFT of the sequence [1, 1] and its zero-padded version [1, 1, 0, 0]. Discuss the interpolating property of the zero-padding operation as indicated by the DFT of the zero-padded sequence.

2.21 A signal is sampled at a rate of 20,000 samples per second. Calculate the frequency resolution of the DFT, in units of Hz, when
(i) a signal window of 500 samples is used as input to the DFT,
(ii) the number of signal samples is increased from 500 to 1000.
(iii) the number of signal samples 500 is augmented (padded) with 500 zeros.

2.22 Write the equations for, and explain the relations between, the Fourier, and the Laplace transforms.
Using the Fourier transform integral, and the superposition and time delay properties, obtain and sketch the spectrum of the pulse, shown in the figure below, of duration $T=10^{-6}$ seconds.
Calculate the bandwidth of the main lobe of the spectrum of the pulse where most of the pulse energy is concentrated.



2.23 A DFT is used as part of a digital signal processing system for the analyse of an analog signal with significant frequency content of up to 1 MHz. Calculate
(i) the minimum sampling rate F_s required, and
(ii) the number of samples in the DFT window required to achieve a frequency resolution of 2 kHz at the minimum sampling rate.
Explain the effects of finite length data on the DFT spectrum of a signal, and suggest a solution.

Matlab code for drawing the spectrum of a finite duration sinwave.

```
% Sinewave Period T0, Number of Cycles in the window Nc, Number of samples of
spectrum N
T0=.001; F0=1/T0; Nc=4 ; N=2000;
x=zeros(Nc,N) ;
for Nc=1:4;
  for f=1:N
    if ((f-F0)~=0)
      x(Nc,f)=Nc*T0*sin(pi*(f-F0)*T0*Nc)/(pi*(f-F0)*T0*Nc);
     else
    x(Nc,f)=Nc*T0;
   end
  end
  x(Nc,1000)=Nc*T0;
end  plot(x');
```