

# The Two-Body Problem

## Abstract

In my short essay on Kepler's laws of planetary motion and Newton's law of universal gravitation, the trajectory of one massive object near another was shown to be a conic section. However, the frame of reference was a coordinate system centered at one of the masses (typically the heavier). Unfortunately, since even a heavier object will exhibit some acceleration toward the lighter one, such a frame cannot be regarded as "inertial," meaning that Newton's laws are not entirely valid (without the introduction of "fictitious forces"). The present treatment of eliminates this objection; however, relativistic effects are still not considered.

## Two-Body Problem Relative to Inertial Frame

We consider the motions of two massive objects—which we'll call **planets**—relative to an inertial frame of reference, subject **only** to the mutual forces of gravitation. For convenience, we'll refer to the bodies as **planets**. Thus, we have a planet of mass  $m_1$  whose position vector is  $\vec{x}_1$  as well as a planet of mass  $m_2$  with position vector  $\vec{x}_2$ . In terms of  $\vec{x}_1$  and  $\vec{x}_2$ , the center of mass is located with position vector

$$\vec{R} = \left( \frac{1}{M} \right) (m_1 \vec{x}_1 + m_2 \vec{x}_2),$$

where  $M = m_1 + m_2$ .

The force applied to each of the planets has, by Newton's Law of Universal Gravitation, magnitude

$$F = \frac{Gm_1m_2}{r^2},$$

where  $r = |\vec{x}_1 - \vec{x}_2|$  is the distance between the planets, and where  $G$  is the universal gravitational constant, whose measured value in SI units is approximately

$$G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2.$$

Since these forces are opposite each other, Newton's Second Law immediately implies that

$$(m_1 + m_2) \frac{d^2}{dt^2} \vec{R} = m_1 \frac{d^2}{dt^2} \vec{x}_1 + m_2 \frac{d^2}{dt^2} \vec{x}_2 = \vec{0}.$$

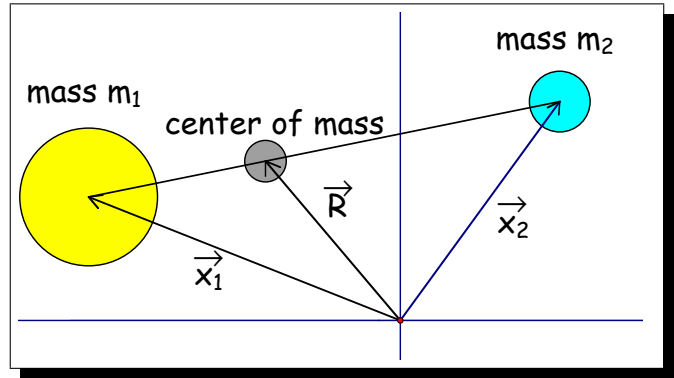


Figure 1: Two-Body Diagram

This already says that the center of mass of the two-body system exhibits no acceleration relative to the inertial frame of reference. Therefore the moving center of gravity can itself be taken as an inertial frame of reference.

## Equations Relative to Center of Mass

Since a non-rotating coordinate system having origin at the center of mass of our two-body system is an inertial frame, we shall ignore the movement of the center of mass and concentrate only on the movements of the planets relative to their common center of mass. Given this, we'll re-define the position vector  $\vec{R}$  to be that pointing from the center of mass to Planet 1 (having mass  $m_1$ ). Knowing how Planet 1 moves about the center of mass will dictate Planet 2's motion as well, since Planet 2's position vector is just  $-\frac{m_2}{m_1} \vec{R}$ . This configuration is depicted in Figure 2. The task, then, is

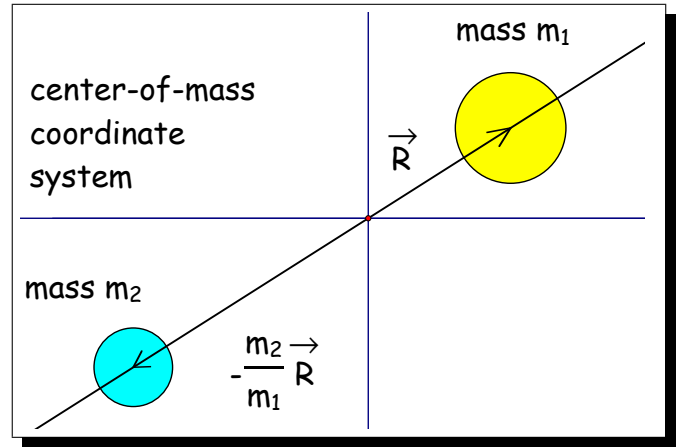


Figure 2: Center-of-mass frame

to compute how the position vector  $\vec{R}$  of Planet 1 evolves with time. Rather than letting  $r$  be the distance between the two planets, we let  $r$  be the length of the position vector  $\vec{R}$ . With this understanding, the distance between the two planets is simply  $\left(\frac{m_1 + m_2}{m_2}\right) r$ . Using Newton's law of universal gravitation, and denoting by  $\vec{r}$  the unit vector in the direction of  $\vec{R}$ , we immediately obtain

$$m_1 \frac{d^2}{dt^2} \vec{R} = - \frac{G m_1 m_2}{\left(\frac{m_1 + m_2}{m_2}\right)^2 r^2} \vec{r}.$$

This gives the acceleration of Planet 1 in this frame of reference:

$$\vec{a} = \frac{d^2}{dt^2} \vec{R} = - \frac{G m_2^3}{(m_1 + m_2)^2 r^2} \vec{r}.$$

Now comes the math! Let's gather together all the constants into a single

constant by defining

$$K = \frac{Gm_2^3}{(m_1 + m_2)^2},$$

leaving us with the task of solving the second-order differential equation

$$\vec{a} = \frac{d^2}{dt^2} \vec{R} = \frac{K}{r^2}. \quad (1)$$

To deal with the above differential equation, we let  $\vec{r}$  be the unit vector in the direction of  $\vec{R}$  and let  $\vec{s}$  be the unit vector in the direction of  $\frac{d}{dt} \vec{r}$ .

From  $\vec{r} \cdot \vec{r} = 1$ , we get

$$0 = \frac{d}{dt} (\vec{r} \cdot \vec{r}) = 2 \vec{r} \cdot \frac{d}{dt} \vec{r},$$

which implies that  $\vec{r} \cdot \vec{s} = 0$ .

Next, from  $\vec{r} = (\cos \theta, \sin \theta)$ , get

$$\frac{d}{dt} \vec{r} = (-\sin \theta, \cos \theta) \frac{d\theta}{dt} = \frac{d\theta}{dt} \vec{s}.$$

Likewise, it follows that

$$\frac{d}{dt} \vec{s} = -\frac{d\theta}{dt} \vec{r},$$

from which we obtain the velocity vector for Planet 1:

$$\vec{v} = \frac{d}{dt} \vec{R} = \frac{d}{dt} (r \cdot \vec{r}) = \frac{dr}{dt} \vec{r} + r \left( \frac{d}{dt} \vec{r} \right) = \frac{dr}{dt} \vec{r} + r \frac{d\theta}{dt} \vec{s}. \quad (2)$$

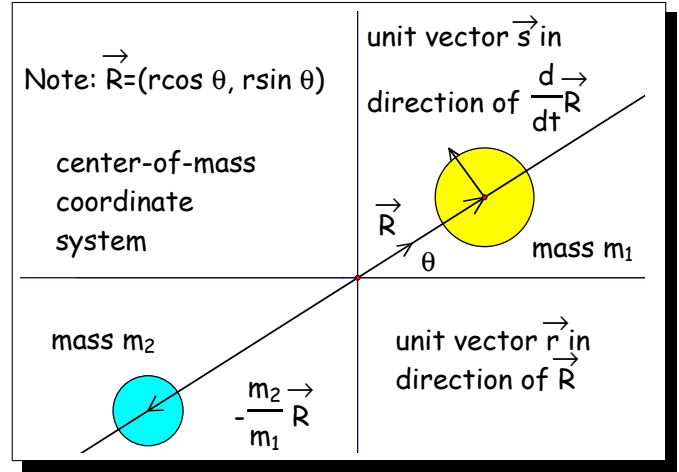


Figure 3: Polar coordinates

The acceleration is the time derivative of Equation (2):

$$\begin{aligned}
\vec{a} &= \frac{d}{dt} \vec{v} = \frac{d}{dt} \left( \frac{dr}{dt} \vec{r} + r \frac{d\theta}{dt} \vec{s} \right) \\
&= \frac{d^2r}{dt^2} \vec{r} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \vec{s} + r \frac{d^2\theta}{dt^2} \vec{s} - r \left( \frac{d\theta}{dt} \right)^2 \vec{r} \\
&= \left( \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right) \vec{r} + \left( 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right) \vec{s}
\end{aligned}$$

However, recalling from Equation (1) that  $\vec{a} = -\frac{K}{r^2} \vec{r}$ , we extract two equations of interest:

$$\frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 = -\frac{K}{r^2} \quad (3)$$

and

$$2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} = 0. \quad (4)$$

We can immediate extract useful information from Equation (4), namely that

$$\frac{d}{dt} \left( r^2 \frac{d\theta}{dt} \right) = 2r \frac{dr}{dt} \frac{d\theta}{dt} + r^2 \frac{d^2\theta}{dt^2} = 0.$$

This means that the quantity  $L = r^2 \frac{d\theta}{dt}$  **is a constant**.

Next, define the variable

$$u = \frac{L^2}{Kr}.$$

and that

$$\frac{d\theta}{dt} = \frac{L}{r^2} = \frac{K^2 u^2}{L^3}.$$

We have, using the Chain Rule, that

$$\frac{dr}{dt} = \frac{d}{d\theta} \left( \frac{L^2}{Ku} \right) \frac{d\theta}{dt} = -\frac{L^2}{Ku^2} \frac{d\theta}{dt} \frac{du}{d\theta} = -\frac{K}{L} \frac{du}{d\theta}.$$

Differentiate again and obtain

$$\frac{d^2r}{dt^2} = \frac{d}{d\theta} \left( \frac{dr}{dt} \right) \frac{d\theta}{dt} = -\frac{K}{L} \frac{d^2u}{d\theta^2} \frac{d\theta}{dt} = -\frac{K^3 u^2}{L^4} \frac{d^2u}{d\theta^2}$$

Substitute into Equation (3) and get

$$-\frac{K^3 u^2}{L^4} \frac{d^2 u}{d\theta^2} - \left( \frac{L^2}{Ku} \right) \left( \frac{K^2 u^2}{L^3} \right)^2 = -\frac{K^3 u^2}{L^4}.$$

Dividing by the common factor of  $-\frac{K^3 u^2}{L^4}$  results in the inhomogeneous second-order differential equation:

$$\frac{d^2 u}{d\theta^2} + u = 1.$$

The general solution of this has the form

$$u = u(\theta) = 1 + e \cos(\theta - \theta_0),$$

where  $e$  and  $\theta_0$  are constants of integration which can be determined from the initial conditions. In turn, we now know that the radial distance of Planet 1 from the center of mass is given, in terms of  $\theta$ , by

$$r = \frac{L^2}{Ku} = \frac{L^2}{K(1 + e \cos(\theta - \theta_0))}, \quad (5)$$

where the constants  $K$  and  $L$  are

$$K = \frac{Gm_2^3}{(m_1 + m_2)^2}, \quad \text{and} \quad L = r^2 \frac{d\theta}{dt}.$$

## The Center-of-Mass Inertial Frame and the Eccentricity

The two-body problem will have in all four initial conditions: the two initial positions of the planets and the two initial velocities. Correspondingly, there are four constants of integration which occur: the initial position and velocity of the center of mass relative to the original inertial frame, and the constants  $e$  and  $\theta_0$  appearing in Equation (5). The motion of the center of mass is easily determined. If  $\vec{x}_1(0)$  and  $\vec{x}_2(0)$  are the initial positions of Planets 1 and 2, respectively, then the initial position of the center of mass is

$$\vec{R}(0) = \frac{1}{m_1 + m_2} (m_1 \vec{x}_1(0) + m_2 \vec{x}_2(0)).$$

Likewise, if  $\vec{v}_1(0)$  and  $\vec{v}_2(0)$  are the initial velocities of Planets 1 and 2, then the combined system has total momentum (which is conserved)

$$\vec{p} = m_1 \vec{v}_1(0) + m_2 \vec{v}_2(0),$$

and so the center of mass will move with constant velocity

$$\vec{v} = \frac{1}{m_1 + m_2} \vec{p}.$$

This means that relative to the moving frame, the initial velocities of Planets 1 and 2 are

$$\vec{u}_1(0) = \vec{v}_1 - \vec{v} \text{ and } \vec{u}_2(0) = \vec{v}_2 - \vec{v},$$

respectively. Finally, we choose the  $x$ -axis of our moving frame so as to pass through the center of mass of Planet 1 at time  $t = 0$ . All of this is indicated in Figure 4 above.

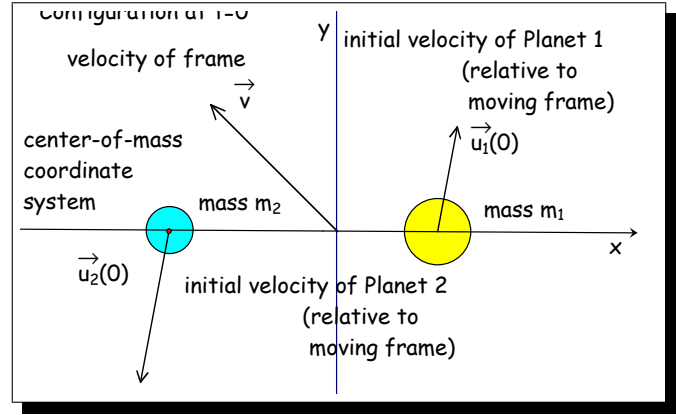


Figure 4: Initial conditions relative to moving frame

For the remainder of the section we shall be operating exclusively in the moving inertial center-of-mass frame. Referring to Figure 4 we let  $r_0$  be the initial distance of Planet 1 from the center of mass. Next we let  $\alpha$  be the angle that  $\vec{u}_1$  makes with the positive  $x$ -axis of the moving frame. Since at time  $t = 0$  we have  $\theta = 0$ , Equation 5 at time  $t = 0$  now reads

$$r_0 = \frac{L^2}{K(1 + e \cos \theta_0)}. \quad (6)$$

Equation (6) now can be written as

$$e \cos \theta_0 = \frac{L^2}{K r_0} - 1. \quad (7)$$

Next we take the time derivative of Equation (5) and get:

$$\frac{dr}{dt} = \frac{L^2 e \sin(\theta - \theta_0)}{K[1 + e \cos(\theta - \theta_0)]^2} \frac{d\theta}{dt};$$

at  $t = 0$  this simplifies to

$$u_0 \cos \alpha = -\frac{K r_0^2 e \sin \theta_0}{L^2} \times \frac{L}{r_0^2} = -\frac{K e \sin \theta_0}{L},$$

where  $u_0$  is the magnitude of the vector  $\vec{u}_1(0)$ . Therefore,

$$e \sin \theta_0 = -\frac{Lu_0 \cos \alpha}{K} \quad (8)$$

Squaring and adding Equations (7) and (8) leads to

$$e^2 = \left( \frac{L^2}{Kr_0} - 1 \right)^2 + \frac{L^2 u_0^2 \cos^2 \alpha}{K^2} \quad (9)$$

Equation (9) can be written purely in terms of the initial data once we realize that the constant  $L$  can be expressed as

$$L = r^2 \frac{d\theta}{dt} = r_0 u_0 \sin \alpha,$$

giving rise to

$$e^2 = \left( \frac{r_0 u_0^2 \sin^2 \alpha}{K} - 1 \right)^2 + \frac{r_0^2 u_0^4 \cos^2 \alpha}{K^2}.$$

Since  $K$  is the constant

$$K = \frac{Gm_1^2 m_2}{(m_1 + m_2)^2},$$

we see that the eccentricity of the orbit of Planet 1 (and hence of Planet 2) about the moving center of mass can be expressed purely in terms of the initial data  $r_0, u_0$ , the angle  $\alpha$ , the masses  $m_1$  and  $m_2$  and the gravitational constant  $G$ .

## One Planet's Orbit Relative to the Other

In the above, we have shown that each member planet of a two-body system will orbit about the system's center of mass in either an ellipse, a parabola, or a hyperbola, depending on the eccentricity. A question that remains is whether one planet's orbit about the other will likewise be a conic section. If we can show this, then Kepler's First Law will be validated, namely that the planets in the solar system orbit about the sun in elliptical orbits.

The demonstration is actually easier than one might think! Figure 5 depicts both the center-of-mass coordinate system, with planet 1's orbit about the center of mass, as well as the moving coordinate system centered at Planet 1. We are to show that, in polar coordinates based at Planet 1, the orbit of Planet 2 is also a conic section with the same eccentricity. If  $r$  is the polar distance from the center of mass to Planet 1, then in terms of the radial angle  $\theta$ , we have already shown that

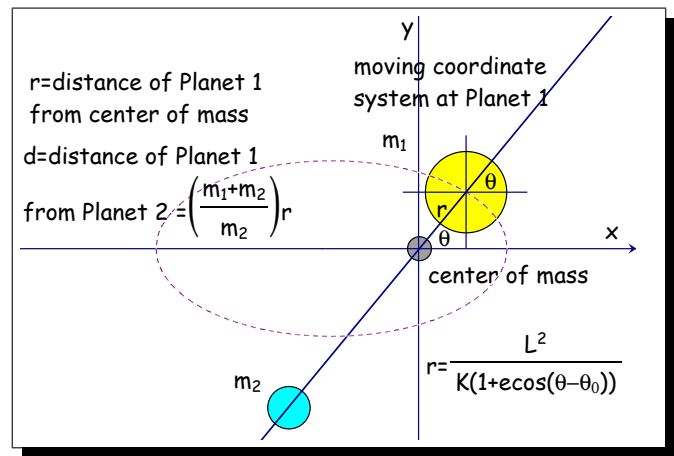


Figure 5: Coordinate System at Planet 1

$$r = \frac{L^2}{K(1 + e \cos(\theta - \theta_0))}.$$

Since the distance  $d$  between Planets 1 and 2 is given by

$$d = \frac{(m_1 + m_2)r}{m_2},$$

it follows that the **polar distance**  $s$  from Planet 1 to Planet 2 must be (note the minus sign!):

$$s = -\frac{L^2(m_1 + m_2)}{Km_2(1 + e \cos(\theta - \theta_0))},$$

which also describes a conic section with the same eccentricity  $e$ .