VECTOR-VALUED FUNCTIONS OF A VECTOR VARIABLE

- Let X and Y be sets. A function
$$f : X \longrightarrow Y$$
is a rule that associates a value $f(x) \in Y$ to every element $x \in X$.

- Single variable calculus:
$$f : X \longrightarrow \mathbb{R}, \quad \text{where} \quad X \subseteq \mathbb{R}.$$
Typically, X is an interval or a union of such.

- Multivariable calculus:
$$\mathbf{f} : X \longrightarrow \mathbb{R}^m, \quad \text{where} \quad X \subseteq \mathbb{R}^n.$$
Here,
$$\mathbb{R}^n := \{\mathbf{x} = (x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}.$$

- $\mathbf{f}$ itself can be viewed as a vector:
$$f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})),$$
where
$$f_i(\mathbf{x}) := i\text{-th component of } f(\mathbf{x}), \quad 1 \leq i \leq m.$$
Note that f_i is a real-valued (scalar-valued) function of the vector variable $\mathbf{x} \in \mathbb{R}^n$.
$$f_i : \mathbb{R}^n \longrightarrow \mathbb{R}.$$
We call f_i the i-th component function of $\mathbf{f}$ and we write
$$\mathbf{f} = (f_1, \ldots, f_m).$$

- Scalar-valued functions of a vector variable:

  (1) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f(\mathbf{x}) := x_1 - x_2$.

  (2) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f(x, y) := e^{-\pi(x^2+y^2)}$.

  (3) Define $f : \mathbb{R}^n \backslash \{\mathbf{0}\} \longrightarrow \mathbb{R}, \quad f(\mathbf{x}) := \dfrac{1}{\|\mathbf{x}\|^2}$. ($X \backslash Y := \{x \in X : x \notin Y\}$)

- Natural domains: If $\mathbf{f}$ is defined via formula without explicit reference to a domain $X \subseteq \mathbb{R}^n$, we take X — the natural domain of $\mathbf{f}$ — to be the largest subset of $\mathbb{R}^n$ on which the formula defining $\mathbf{f}$ makes sense.

  (1) $g(t) = \left(\dfrac{1}{t^2 - 1}, \dfrac{1}{t^3 - 1}\right)$, $X = \mathbb{R} \setminus \{-1, 1\}$.

  (2) $f(x, y) = \log(y - x)$, $X = \{(x, y) \in \mathbb{R}^2 : y > x\}$.

(3) $\theta(x,y,z) = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$, $X = \mathbb{R}^3 \setminus \{(0,0,0)\}$.