1. Find U such that $UA = \mathrm{rref}(A)$

Notation: If B is the matrix you get by applying the elementary row operation O to A, we write

$$B = OA.$$

To find U: Row-reduce the *horizontal join*

$$\left(A \,\middle|\, I \right),$$

i.e., apply EROs $O_1, O_2, \ldots, O_n$ to $\left(A \,\middle|\, I \right)$ until it has the form $\left(\mathrm{rref}\, A \,\middle|\, U \right)$:

$$\left(A \,\middle|\, I \right) \xrightarrow{O_1} \left(O_1 A \,\middle|\, O_1 I \right) \xrightarrow{O_2} \left(O_2 O_1 A \,\middle|\, O_2 O_1 I \right) \xrightarrow{O_3} \cdots \xrightarrow{O_n} \left(\mathrm{rref}\, A \,\middle|\, U \right).$$

Then:

$$U \text{ is invertible and } UA = \mathrm{rref}\, A.$$

2. Finding the inverse of a matrix

Finding the inverse of an invertible matrix is a *special case* of the problem of finding an invertible U such that $UA = \mathrm{rref}\, A$:

$$\textbf{If } \mathrm{rref}\, A = I, \textbf{ then } UA = I.$$

Since U is invertible, so is A (why?) and

$$A^{-1} = U.$$

In particular:

If rref $A = I$ **then** A is invertible.

The *converse* of this implication is also true:

If A is invertible, **then** rref $A = I$.

Thus:

A is invertible **if and only if** rref $A = I$.

3. Why this works: Elementary matrices

Let $E(O)$ be the matrix you get by applying the ERO O to the identity matrix, I:

$$E(O) = OI.$$

$E(O)$ is called the elementary matrix (EM) of O. A matrix E is an EM if $E = E(O)$ for some ERO O.

EM fact 1: $E(O)A$ is the matrix you get by applying O to A.

$$E(O)A = OA.$$

As above, suppose

$$\left(\operatorname{rref} A \,\middle|\, U \right) = O_n \cdots O_2 O_1 \left(A \,\middle|\, I \right).$$

Set

$$E_i = E(O_i).$$

Since[1]

$$\begin{aligned} O_n \cdots O_2 O_1 \left(A \,\middle|\, I \right) &= \left(O_n \cdots O_2 O_1 A \,\middle|\, O_n \cdots O_2 O_1 I \right), \\ &= \left(E_n \cdots E_2 E_1 A \,\middle|\, E_n \cdots E_2 E_1 \right), \end{aligned}$$

we have:

$$U = E_n \cdots E_2 E_1 \quad \text{and} \quad UA = \operatorname{rref} A.$$

4. Writing a matrix as a product of elementary matrices

Suppose that A is invertible. Then, as we observed above, U is invertible and

$$A^{-1} = U.$$

Inverting both sides of this identity yields

$$A = U^{-1}$$

Since $U = E_n \cdots E_2 E_1$,

$$U^{-1} = E_1^{-1} E_2^{-1} \cdots E_n^{-1}.$$

Remember: the inverse of a product is a product of inverses, *in the reversed order*. Thus, A is a product of *inverses of* elementary matrices:

$$A = U^{-1} = E_1^{-1} E_2^{-1} \cdots E_n^{-1}.$$

[1]The first identity describes how EROs interact with horizontal joins of matrices. The second follows from the EM fact 1, relating EROs and EMs.

But:

EM fact 2: Inverses of EMs exist and are, themselves, EMs. More precisely, the inverse of the EM of an ERO is the EM of the inverse ERO:

$$E(O)^{-1} = E(O^{-1})$$

Perhaps we have not stated the following explicitly:

ERO fact: Every ERO is invertible.

Type	O	O^{-1}
1	swap rows i and j	swap rows i and j
2	multiply row i by c ($c \neq 0$)	multiply row i by $1/c$
3	add $k\times$(row i) to row j	add $-k\times$(row i) to row j

Thus,

$$A = E_1^{-1} E_2^{-1} \cdots E_n^{-1}$$

is an expression for A as a product of elementary matrices.