Math 367 – Tutorial #4

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1. Define
$$\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \mathbf{f}(x, y) = \begin{pmatrix} x\sin(xy) \\ x\cos(xy) \end{pmatrix}$$
and
$$\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \mathbf{g}(u, v) = \begin{pmatrix} u^3 + 3u^2v - v^3 + u^2 - v^2 \\ u^3 + v^3 - 2u^2 \end{pmatrix}$$
Compute $D(\mathbf{g} \circ \mathbf{f})(1, 0)$.

2. The system of equations
$$w^2 + x^2 + y^2 + z^2 = 4$$
$$w + 2x + 3y + 4z = 10$$
Defines y and z as functions of w and x in a neighborhood of $(1, 1, 1, 1)$. Find the partial derivatives of y and z with respect to x and y at $(1, 1, 1, 1)$.
Here's another take:

3. Suppose
$$z = z(x, y), \qquad x = e^s \cos t, \qquad y = e^s \sin t.$$
Show that
$$z_{ss} + z_{tt} = (x^2 + y^2)(z_{xx} + z_{yy})$$

4. (a) Suppose $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy
$$u_x = v_y, \qquad u_y = -v_x \tag{†}$$
Show that u and v are both harmonic, i.e., that they satisfy Laplace's equation:
$$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0$$
(b) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function. Show that if u and v satisty (†), then
$$g(x, y) = f(u(x, y), v(x, y))$$
is harmonic.