Math 367 – Tutorial #2

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1. Find the directional derivative of
$$f(x, y) = xe^y + ye^x$$
at $(0, 0)$ in the direction making an angle $\theta = \pi/6$ to the horizontal.

Solution: To find directional derivatives, we take dot products with the gradient vector.
$$\nabla f(x, y) = (e^y + ye^x, xe^y + e^x)$$
$$\nabla f(0, 0) = (1, 1)$$
The unit vector at an angle of $\pi/6$ to the horizontal is
$$\mathbf{u} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$
We have:
$$D_{\mathbf{u}} f(0, 0) = \nabla f(0, 0) \cdot \mathbf{u} = \frac{\sqrt{3} + 1}{2}$$

2. Find and classify the critical points of the function.
   (a) $f(x, y) = x^2 + xy + y^2 - 6x + 6$

Solution: To find the critical points, we solve $\nabla f = (0, 0)$.
$$\nabla f(x, y) = (2x + y - 6, x + 2y)$$
The system
$$2x + y - 6 = 0, \quad x + 2y = 0$$
has unique solution
$$(x, y) = (4, -2).$$
To classify the critical point at $(4, -2)$, we compute second partials:
$$f_{xx} = 2, \quad f_{xy} = 1, \quad f_{yy} = 2$$
The associated discriminant quantity is
$$D = f_{xx} f_{yy} - f_{xy}^2 = (2)(2) - 1 = 3 > 0.$$
Therefore, by the second derivative test, $f(x, y)$ has a local minimum at $(0, 0)$.

(b) $f(x,y) = x^3 + y^2 + 2xy - 4x - 3y + 5$

Solution: To find the critical points, we solve $\nabla f = (0,0)$.

$$\nabla f(x,y) = (3x^2 + 2y - 4, 2y + 2x - 3)$$

To solve the system

$$3x^2 + 2y - 4 = 0, \quad 2y + 2x - 3 = 0,$$

we solve the second equation for $2y$,

$$2y = 3 - 2x,$$

and substitute into the first:

$$\begin{aligned} 0 &= 3x^2 + (3 - 2x) - 4 \\ &= 3x^2 - 2x - 1 \\ &= (3x+1)(x-1) \end{aligned}$$

We get

$$x = -\frac{1}{3}, \quad 1$$

Computing the corresponding y-values, we get that the critical points of f are at

$$P = \left(-\frac{1}{3}, \frac{11}{6}\right), \quad Q = \left(1, \frac{1}{2}\right).$$

To classify these critical points, we compute second derivatives:

$$f_{xx}(x,y) = 6x, \quad f_{xy}(x,y) = 2, \quad f_{yy}(x,y) = 2.$$

As

$$D(P) = 6\left(-\frac{1}{3}\right)(2) - 2^2 = -8 < 0$$

f has a saddle point at P. Since

$$D(Q) = 6(1)(2) - 2^2 = 8 > 0$$

and $f_{xx}(Q) = 6 > 0$, f has a local minimum at Q.

(c) $f(x,y,z) = \frac{1}{2}(5x^2 + 11y^2 + 2z^2 + 16xy + 20xz - 4yz)$

Solution: Let's write f in matrix form:

$$f(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A = \begin{pmatrix} 5 & 8 & 10 \\ 8 & 11 & -2 \\ 10 & -2 & 2 \end{pmatrix}$$

We have:

$$\nabla f(x,y,z) = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Since A is invertible (its determinant is $-1458 \neq 0$),

$$\nabla f(x,y,z) = (0,0,0) \iff (x,y,z) = (0,0,0).$$

Thus, $(0, 0, 0)$ is the only critical point of f. The matrix A has eigenvalues -9, 9, and 18. Since there are both positive and negative numbers among these, we conclude that $(0, 0, 0)$ is a saddle point of f.