# Math 367 – Tutorial #1

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September 14-16, 2021

1. Evaluate $z_x$ and $z_y$, where:
   (a) $z = xe^{x^2+y^2}$

   **Solution:**

   $$z_x = e^{x^2+y^2} + xe^{x^2+y^2}(2x),$$
   $$z_y = xe^{x^2+y^2}(2y)$$

   (b) $z = \cos(e^{x^2y^3})$

   **Solution:**

   $$z_x = -\sin(e^{x^2y^3})e^{x^2y^3}(2xy^3)$$
   $$z_y = -\sin(e^{x^2y^3})e^{x^2y^3}(3x^2y^2)$$

   (c) $xy^2 + yz^2 + xyz = 1$

   **Solution:** Differentiate both sides with respect to $x$, remembering that $z$ is a functino of $x$:
   $$y^2 + 2yzz_x + yz + xyz_x = 0$$
   Now solve for $z_x$:
   $$z_x = -\frac{y^2}{2yz + xy}$$
   Differentiate both sides with respect to $y$, remembering that $z$ is a functino of $y$:
   $$2xy + z^2 + 2yzz_y + xz + xyz_y = 0$$
   Now solve for $z_y$:
   $$z_t = -\frac{2xy + z^2 + xz}{2yz + xy}$$

(d) $z = x^y$

**Solution:** To find $z_x$, just use the power rule:

$$z_x = yx^{y-1}$$

To find $z_y$, take logarithms of both sides of $z = x^y$:

$$\ln z = y \ln x$$

Differentiate both with respect to $y$, remebering that $z$ is a functino of $y$:

$$\frac{1}{z} z_y = \ln x$$

Solve for $z_y$:

$$z_y = z \ln x = x^y \ln x$$

Alternatively, you could just look up the derivative of $f(x) = a^x$. The above calculation is just a derivation of that formula.

2. Let

$$f(\mathbf{x}) = \|\mathbf{x}\|^r, \quad \mathbf{x} \in \mathbb{R}^n.$$

Show that

$$\nabla f(\mathbf{x}) = r\|\mathbf{x}\|^{r-2}\mathbf{x}.$$

**Solution:** We have:

$$\begin{aligned} f_{x_i}(\mathbf{x}) &= \frac{\partial}{\partial x_i} \left(x_1^2 + \cdots x_n^2\right)^{r/2} \\ &= \frac{r}{2} \left(x_1^2 + \cdots x_n^2\right)^{(r-2)/2} 2x_i \\ &= r\|x\|^{r-2} x_i \end{aligned}$$

Therefore,

$$\begin{aligned} \nabla f(\mathbf{x}) &= (r\|x\|^{r-2}x_1, \ldots, r\|x\|^{r-2}x_n) \\ &= r\|x\|^{r-2}(x_1, \ldots, x_n) \\ &= r\|x\|^{r-2}\mathbf{x}. \end{aligned}$$

3. Let $y = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$. Show that

$$\nabla y^r = r y^{r-1} \nabla y.$$

**Solution:** In class, we noted the following form of the chain rule, where $g$ is a real-valued function of a single variable:

$$\nabla g(f(\mathbf{x})) = g'(f(\mathbf{x})) \nabla f(\mathbf{x})$$

If $g(t) = t^r$, then $g'(t) = rt^{r-1}$ and

$$\nabla y^r = \nabla g(y) = g'(y) \nabla y = r y^{r-1} \nabla y.$$

Alternatively, you can work from first principles:

$$\frac{\partial}{\partial x_i} f(\mathbf{x})^r = r f(\mathbf{x})^{r-1} f_{x_i}(\mathbf{x})$$

Therefore,

$$\begin{aligned} \nabla f(\mathbf{x})^r &= (r f(\mathbf{x})^{r-1} f_{x_1}(\mathbf{x}), \ldots, r f(\mathbf{x})^{r-1} f_{x_n}(\mathbf{x})) \\ &= r f(x)^{r-1} (f_{x_1}(\mathbf{x}), \ldots, f_{x_n}(\mathbf{x})) \\ &= r f(x)^{r-1} \nabla f(\mathbf{x}) \end{aligned}$$

Equivalently,

$$\nabla y^r = r y^{r-1} \nabla y.$$