Math 367 – Tutorial #4

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1. Define

$$\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \mathbf{f}(x, y) = \begin{pmatrix} x \sin(xy) \\ x \cos(xy) \end{pmatrix}$$

and

$$\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \mathbf{g}(u, v) = \begin{pmatrix} u^3 + 3u^2 v - v^3 + u^2 - v^2 \\ u^3 + v^3 - 2u^2 \end{pmatrix}$$

Compute $D(\mathbf{g} \circ \mathbf{f})(1, 0)$.

Solution:

$$\begin{aligned} \mathbf{f}(1,0) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ D\mathbf{f} &= \begin{pmatrix} \sin(xy) + xy\cos(xy) & x^2\cos(xy) \\ \cos(xy) - xy\sin(xy) & -x^2\sin(xy) \end{pmatrix} \\ D\mathbf{f}(1,0) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} D\mathbf{g} &= \begin{pmatrix} 3u^2 + 6uv + 2u & 3u^2 - 3v^2 - 3v \\ 3u^2 - 4u & 3v^2 \end{pmatrix} \\ D\mathbf{g}(\mathbf{f}(1,0)) &= D\mathbf{g}(0,1) \\ &= \begin{pmatrix} 0 & -6 \\ -4 & 3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} D(\mathbf{f} \circ \mathbf{g})(1,0) &= D\mathbf{g}(\mathbf{f}(1,0)) D\mathbf{f}(1,0) \\ &= \begin{pmatrix} 0 & -6 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -6 & 0 \\ 3 & 0 \end{pmatrix} \end{aligned}$$

2. The system of equations

$$\begin{aligned} w^2 + x^2 + y^2 + z^2 &= 4 \\ w + 2x + 3y + 4z &= 10 \end{aligned}$$

Defines y and z as functions of w and x in a neighborhood of $(1,1,1,1)$. Find the partial derivatives of y and z with respect to x and y at $(1,1,1,1)$.

Solution: Differentiate the system with respect to w, holding x fixed:

$$\begin{aligned} 2w + 2yy_w + 2zz_w &= 0 \\ 1 + 3y_w + 4z_w &= 0 \end{aligned}$$

Evaluating at $(1,1,1,1)$ and rearranging slightly, this becomes

$$\begin{aligned} 2y_w + 2z_w &= -2 \\ 3y_w + 4z_w &= -1 \end{aligned}$$

In matrix form:

$$\begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} y_w \\ z_w \end{pmatrix} = -\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{†}$$

At this point, you could solve for y_w and z_w, but we'll hold off.

Now differentiate the system with respect to x, holding w fixed:

$$\begin{aligned} 2x + 2yy_x + 2zz_x &= 0 \\ 2 + 3y_x + 4z_x &= 0 \end{aligned}$$

Evaluating at $(1,1,1,1)$ and rearranging slightly, this becomes

$$\begin{aligned} 2y_x + 2z_x &= -2 \\ 3y_x + 4z_x &= -2 \end{aligned}$$

In matrix form:

$$\begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} y_x \\ z_x \end{pmatrix} = -\begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{‡}$$

Combine (†) and (‡):

$$\begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} y_w & y_x \\ z_w & z_x \end{pmatrix} = -\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$$

Therefore,

$$\begin{aligned} \begin{pmatrix} y_w & y_x \\ z_w & z_x \end{pmatrix} &= -\begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}. \end{aligned}$$

Here's another take:

Solution: Define

$$\mathbf{f}\begin{pmatrix} w \\ x \end{pmatrix} = \begin{pmatrix} w \\ x \\ y(w,x) \\ z(w,x) \end{pmatrix}, \qquad \mathbf{g}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} w^2 + x^2 + y^2 + z^2 - 4 \\ w + 2x + 3y + 4z - 10 \end{pmatrix}$$

$$Df\begin{pmatrix} w \\ x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ y_w & y_x \\ z_w & z_x \end{pmatrix}$$

$$D\mathbf{g}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2w & 2x & 2y & 2z \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

By the chain rule,

$$\begin{aligned}
D(\mathbf{g}\circ\mathbf{f})\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= D\mathbf{g}\left(\mathbf{f}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) D\mathbf{f}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= D\mathbf{g}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} D\mathbf{f}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ y_w & y_x \\ z_w & z_x \end{pmatrix} \\
&= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} y_w & y_x \\ z_w & z_x \end{pmatrix} \\
&= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} y_w & y_x \\ z_w & z_x \end{pmatrix}
\end{aligned}$$

Since $\mathbf{g} = \mathbf{0}$ defines y and z implicitly as functions of x and y,

$$\mathbf{g}\circ\mathbf{f} = \mathbf{0}.$$

Therefore,

$$D(\mathbf{g}\circ\mathbf{f}) = \mathbf{0}.$$

Therefore,

$$\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} y_w & y_x \\ z_w & z_x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Rearranging, we get

$$\begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} y_w & y_x \\ z_w & z_x \end{pmatrix} = -\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$$

Solving, we get

$$\begin{aligned}
\begin{pmatrix} y_w & y_x \\ z_w & z_x \end{pmatrix} &= -\begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}.
\end{aligned}$$

3. Suppose

$$z = z(x, y), \qquad x = e^s \cos t, \qquad y = e^s \sin t.$$

Show that

$$z_{ss} + z_{tt} = (x^2 + y^2)(z_{xx} + z_{yy})$$

Solution: Note that

$$x_s = x, \qquad x_t = -y, \qquad y_s = y, \qquad y_t = x.$$

Using these relations, together with the chain rule, we get

$$\begin{aligned} z_s &= z_x x_s + z_y y_s \\ &= x z_x + y z_y \\ z_{ss} &= x_s z_x + x(z_{xx} x_s + z_{xy} y_s) + y_s z_y + y(z_{xy} x_s + z_{yy} y_s) \\ &= x z_x + y z_y + x^2 z_{xx} + y^2 z_{yy} + 2xy z_{xy} \end{aligned}$$

and

$$\begin{aligned} z_t &= z_x x_t + z_y y_t \\ &= -y z_x + x z_y \\ z_{tt} &= -y_t z_x - y(z_{xx} x_t + z_{xy} y_t) + x_t z_y + x(z_{yx} x_t + z_{yy} y_t) \\ &= -x z_x - y z_y + y^2 z_{xx} + x^2 z_{yy} - 2xy z_{xy}. \end{aligned}$$

Therefore,

$$z_{ss} + z_{tt} = (x^2 + y^2)(z_{xx} + z_{yy}).$$

4. (a) Suppose $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$u_x = v_y, \qquad u_y = -v_x \tag{†}$$

Show that u and v are both harmonic, i.e., that they satisfy Laplace's equation:

$$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0$$

Solution: We show that u is harmonic:

$$\begin{aligned} u_{xx} &= (u_x)_x = (v_y)_x = v_{xy} \\ u_{yy} &= (u_y)_y = (-v_x)_y = -v_{xy} \end{aligned}$$

Therefore,

$$u_{xx} + u_{yy} = 0.$$

The proof that v is harmonic is similar.

(b) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function. Show that if u and v satisty (†), then

$$g(x, y) = f(u(x, y), v(x, y))$$

is harmonic.

Solution: Using by the chain rule and relations $(*)$,

$$\begin{aligned}
g_x &= f_u u_x + f_v v_x \\
&= f_u u_x - f_v u_y \\
g_{xx} &= (f_{uu} u_x + f_{uv} v_x) u_x + f_u u_{xx} - (f_{vu} u_x + f_{vv} v_x) u_y - f_v u_{yx} \\
&= (f_{uu} u_x - f_{uv} u_y) u_x + f_u u_{xx} - (f_{vu} u_x - f_{vv} u_y) u_y - f_v u_{yx} \\
&= f_{uu} u_x^2 + f_{vv} u_y^2 - 2 f_{uv} u_x u_y + f_u u_{xx} - f_v u_{xy} \\
g_y &= f_u u_y + f_v v_y \\
&= f_u u_y + f_v u_x \\
g_{yy} &= (f_{uu} u_y + f_{uv} v_y) u_y + f_u u_{yy} + (f_{vu} u_y + f_{vv} v_y) v_y + f_v u_{xy} \\
&= (f_{uu} u_y + f_{uv} u_x) u_y + f_u u_{yy} + (f_{vu} u_y + f_{vv} u_x) u_x + f_v u_{xy} \\
&= f_{uu} u_y^2 + f_{vv} u_x^2 + 2 f_{uv} u_x u_y + f_u u_{yy} + f_v u_{xy} \\
g_{xx} + g_{yy} &= (f_{uu} + f_{vv})(u_x^2 + u_y^2) + f_u (u_{xx} + u_{yy})
\end{aligned}$$

Since f is harmonic by hypothesis,

$$f_{uu} + f_{vv} = 0.$$

Since u is harmonic by (a),

$$u_{xx} + u_{yy} = 0.$$

Therefore,

$$g_{xx} + g_{yy} = 0,$$

i.e., g is harmonic.