Math 367 – Tutorial #2

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1. Find equations for the tangent plane and the normal line...
   (a) ...to $z = e^{2x^2+3y}$ at $(x, y) = (3, -6)$;

Solution:

$$\begin{aligned} z(3,-6) &= 1 \\ z_x &= 4xe^{2x^2+3y} \\ z_x(3,-6) &= 12 \\ z_y &= 3e^{2x^2+3y} \\ z_y(3,-6) &= 3 \end{aligned}$$

The tangent plane to $z = e^{2x^2+3y}$ at $(x, y) = (3, -6)$ has equation

$$\begin{aligned} L(x,y) &= z(3,-6) + z_x(3,-6)(x-3) + z_y(3,-6)(y-(-6)) \\ &= 1 + 12(x-3) + 3(y+6). \end{aligned}$$

The normal line to $z = e^{2x^2+3y}$ at $(x, y) = (3, -6)$ passes through $(3, -6, 1)$ and has direction vector

$$(-\nabla z, 1) = (-12, -3, 1).$$

Hence, it has parametric equations

$$(x, y, z) = (3, -6, 1) + t(-12, -3, 1).$$

   (b) ...to $y = \sqrt{x_1^2 + 3x_2^2 + x_3^2}$ at $\mathbf{x} = (6, 2, 1)$.

Solution:

$$\begin{aligned}
y(6,2,1) &= 7 \\
y_{x_1} &= \frac{2x_1}{y} \\
y_{x_1}(6,2,1) &= \frac{12}{7} \\
y_{x_2} &= \frac{6x_2}{y} \\
y_{x_2}(6,2,1) &= \frac{12}{7} \\
y_{x_3} &= \frac{2x_3}{y} \\
y_{x_3}(6,2,1) &= \frac{2}{7}
\end{aligned}$$

The tangent plane to to $y = \sqrt{x_1^2 + 3x_2^2 + x_3^2}$ at $\mathbf{x} = (6,2,1)$ has equation

$$\begin{aligned}
L(\mathbf{x}) &= y(6,2,1) + y_{x_1}(6,2,1)(x_1 - 6) + y_{x_2}(6,2,1)(x_2 - 2) + y_{x_3}(6,2,1)(x_3 - 1) \\
&= 7 + \frac{12}{7}(x_1 - 6) + \frac{12}{7}(x_2 - 2) + \frac{2}{7}(x_3 - 1)
\end{aligned}$$

The normal line to $y = \sqrt{x_1^2 + 3x_2^2 + x_3^2}$ at $\mathbf{x} = (6,2,1)$ passes through $(6,2,1,7)$ and has direction vector

$$(-\nabla y, 1) = \left(-\frac{12}{7}, -\frac{12}{7}, -\frac{2}{7}, 1\right).$$

Hence, it has parametric equations

$$\mathbf{x} = (6,2,1,7) + t\left(-\frac{12}{7}, -\frac{12}{7}, -\frac{2}{7}, 1\right).$$

2. Use linear approximation to estimate...
   (a) $f(3.05, -6.1)$, where $f(x,y) = e^{2x^2+3y}$;

Solution:

$$\begin{aligned}
f(3.05, -6.1) &\approx L(3.05, -6.1) \\
&= 1 + 12(3.1 - 3.05) + 3(-6.1 + 6) \\
&= 1.3
\end{aligned}$$

$$\left|\frac{f(\mathbf{x}) - L(\mathbf{x})}{f(\mathbf{x})}\right| \times 100\% \approx \frac{1.3566 - 1.3}{1.3566} \times 100\% \approx 4\%$$

   (b) $f(6.1, 1.9, 0.9)$, where $f(\mathbf{x}) = \sqrt{x_1^2 + 3x_2^2 + x_3^2}$.

Solution:

$$\begin{aligned} f(6.1, 1.9, 0.9) &\approx L(6.1, 1.9, 0.9) \\ &= 7 + \frac{12}{7}(6.1 - 6) + \frac{12}{7}(1.9 - 2) + \frac{2}{7}(0.9 - 1) \\ &\approx 6.9714 \end{aligned}$$

$$\left| \frac{f(\mathbf{x}) - L(\mathbf{x})}{f(\mathbf{x})} \right| \times 100\% \approx \frac{6.9892 - 6.9714}{6.9892} \times 100\% \approx 0.25\%$$

What is the relative error percentage in your approximation?

$$\text{relative error percentage} = \left| \frac{f(\mathbf{x}) - L(\mathbf{x})}{f(\mathbf{x})} \right| \times 100\%$$

Note that you computed the relevant linear approximations (tangent planes) in problem 1.

3. Without doing any calculations, find an equations of the normal line and tangent plane to the sphere S_r of radius r,

$$S : x^2 + y^2 + z^2 = r,$$

at an arbitrary point $(a, b, c) \in S$.

Solution: By the geometry of S, the vector (a, b, c) is normal to S at (a, b, c). Therefore, the normal line N to S at (a, b, c) has equation

$$(x, y, z) = (a, b, c) + t(a, b, c)$$

Note that N passes through $(0, 0, 0)$ – take $t = -1$. Therefore,

$$(x, y, z) = t(a, b, c)$$

is also an equation of N.

The tangent plane to S at (a, b, c) is the plane through (a, b, c) normal to (a, b, c). As such, it has equation

$$a(x - a) + b(y - b) + c(z - c) = 0.$$

4. The surface S defined by

$$-x^2 - y^2 + z^2 = 1$$

is a two-sheeted hyperboloid. Find equations of the tangent planes to S at the points $(2, 2, \pm 3)$.

Solution: We compute z_x and z_y by differentiating implicitly:

$$\begin{aligned}
-2x + 2zz_x &= 0 \\
z_x &= \frac{x}{z} \\
z_x(2,2,\pm 3) &= \pm\frac{2}{3} \\
-yx + 2zz_y &= 0 \\
z_y &= \frac{y}{z} \\
z_y(2,2,\pm 3) &= \pm\frac{2}{3}
\end{aligned}$$

The tangent plane to S at $(2,2,3)$ is

$$z = 3 + \frac{2}{3}(x-2) + \frac{2}{3}(y-2).$$

The tangent plane to S at $(2,2,-3)$ is

$$z = -3 - \frac{2}{3}(x-2) - \frac{2}{3}(y-2).$$