(1) Subspaces of $\mathbb{R}^n$
   (a) definition of a subspace of $\mathbb{R}^n$
   (b) nullspace, eigenspace, orthogonal complement
   (c) linear combinations and spans
   (d) column space of a matrix, row space of a matrix
   (e) the pivot columns of a matrix span its column space
   (f) sums of subspaces, spanning sets of sums
(2) Linear independence
   (a) Linear dependence and independence
   (b) $n+1$ vectors in $\mathbb{R}^n$ are linearly dependent.
   (c) The pivot columns of a matrix are linearly independent.
   (d) orthogonal sets — definition and linear independence
   (e) linearly independent subspaces, direct sums of subspaces
   (f) Two subspaces with zero intersection are linearly independent. In particular, W and $W^{\perp}$ are linearly independent.
   (g) Eigenspaces for distinct eigenvectors are linearly independent.
(3) Basis and dimension
   (a) definition of a basis of a subspace
   (b) Every subspace of $\mathbb{R}^n$ has a basis (containing a given linearly independent subset, contained in a given spanning set).
   (c) Any two bases of a subspace of $\mathbb{R}^n$ have the same size.
   (d) coordinates vectors, change of basis matrices
   (e) definition of dimension
   (f) orthonormal sets, orthonormal bases
   (g) bases of W and $W^{\perp}$, taken together, form a basis of $\mathbb{R}^n$.
   (h) rank of a matrix, row rank equals column rank, rank-nullity theorem
   (i) the four subspaces, $N(A^T) = C(A)^{\perp}$, $N(A) = C(A^T)^{\perp}$
(4) Linear transformations
   (a) definition, kernel, image
   (b) the matrix of an endomorphism with respect to a basis, identifications of kernel and image with nullspace and column space, respectively
   (c) defining a linear transformation by giving its values on a basis
   (d) the matrix of a transformation subordinate to a direct sum decomposition
   (e) similarity and diagonalizability
   (f) projections and their diagonalizability
   (g) the least-squares solutions
(5) Spectral theory
   (a) the spectral theorem, assuming n real eigenvalues, counted with multiplicity
   (b) the matrix of a transformation with respect to bases of its domain and codomain
   (c) the singular value decomposition (SVD), connection with the four subspaces
   (d) applications of the SVD: pseudoinverse, low-rank approximations, principal component analysis, etc.
(6) Abstract vector spaces

(a) axioms, examples (subspaces of $\mathbb{R}^n$, polynomials, matrices, functions from a set to a vector space)
(b) subspaces, function spaces, spaces of solutions to linear ODEs
(c) finite-dimensional vector spaces, bases
(d) linear transformations, isomorphism, a vector-space of dimension n is isomorphic to $\mathbb{R}^n$.
(e) inner product spaces
(f) orthogonal projection onto finite-dimensional subspaces
(g) Fourier series

1. Subspaces of $\mathbb{R}^n$

- Definition: A *subspace of* $\mathbb{R}^n$ is a subset U of $\mathbb{R}^n$ that is *closed under addition*, i.e.,
$$\text{if } u_1 \in U \text{ and } u_2 \in U \text{ then } u_1 + u_2 \in U,$$
and *closed under scalar multiplication*, i.e.,
$$\text{if } t \in \mathbb{R} \text{ and } u \in U \text{ then } tu \in U.$$
- Example: Let
$$U = \{u \in \mathbb{R}^3 : u_1 - 2u_2 = 3u_3\}$$
. Then U is a subspace of $\mathbb{R}^3$.
- Definition: The nullspace $N(A)$ of a matrix $A \in \mathbb{R}^{m\times n}$ is the set of all vectors $x \in \mathbb{R}^n$ such that $Ax = 0$:
$$N(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$
- Theorem: $N(A)$ is a subspace of $\mathbb{R}^n$
- $\{u \in \mathbb{R}^3 : u_1 - 2u_2 = 3u_3\} = N\left(\begin{bmatrix}1 & -2 & -3\end{bmatrix}\right)$
- Prove that $\{x \in \mathbb{R}^n : Ax = b\}$ is a subspace of $\mathbb{R}^n$ if and only if $b = 0$.
- Definition: Let $\lambda \in \mathbb{R}$. The λ-eigenspace $E_\lambda(A)$ of a matrix $A \in \mathbb{R}^{n\times n}$ is the set of all λ-eigenvectors of A:
$$E_\lambda(A) = \{x \in \mathbb{R}^n : Ax = \lambda x\}.$$
- Corollary: $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.
- Proof: $E_\lambda(A) = N(\lambda I - A)$.
- Definition: Let S be a subset of $\mathbb{R}^n$. The *orthogonal complement of* S written $S^\perp$, is the set of vectors orthogonal to all elements of S:
$$S^\perp = \{v \in \mathbb{R}^n : u \cdot v = 0 \text{ for all } u \in S\}$$
- Theorem: Suppose $S = \{v_1, \ldots, v_m\}$. Then $S^\perp$ is a subspace of $\mathbb{R}^n$.
- Proof: $S^\perp = N\left(\begin{bmatrix} v_1^t \\ \vdots \\ v_m^t \end{bmatrix}\right)$.
- Definition: Let S be a subset of $\mathbb{R}^n$. A *linear combination of elements of* S is a finite sum of the of the form
$$t_1u_1 + \cdots + t_ku_k,$$

where $t_j \in \mathbb{R}$ and $u_j \in S$.

- Theorem: A subset U of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if and only if U is *closed under linear combinations*, i.e., if and only if every linear combination of elements of U is, itself, an element of U.
- Definition: Let S be a subset of $\mathbb{R}^n$. The *span of* S, written $\langle S\rangle$, is the set of all linear combinations of elements of S.
- Theorem: $\langle S\rangle$ is a subset of $\mathbb{R}^n$.
- Proof: It suffices to show that $\langle S\rangle$ is closed under linear combinations, i.e.:

  a linear combination of linear combinations of elements of S
  is a linear combination of elements of S.
- Exercise: Prove that $u \in \langle v_1, \ldots, v_k\rangle$ if and only if
$$\langle u, v_1, \ldots, v_k\rangle = \langle v_1, \ldots, v_k\rangle.$$
- Theorem: $\langle S\rangle$ is the smallest subspace of $\mathbb{R}^n$ containing S, i.e.,

  if U is a subset of $\mathbb{R}^n$ and $S \subseteq U$ then $\langle S\rangle \subseteq U$.
- Corollary: A subset U of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if and only if $U = \langle U\rangle$.
- Exercise: Suppose $\langle S_1\rangle = \langle S_2\rangle$. Must S_1 equal S_2?
- Definition: Let S be a subset of $\mathbb{R}^n$ and let U be a subspace of $\mathbb{R}^n$. We say that S *spans* U or that S *is a spanning set of* U if $U = \langle S\rangle$.
- Find vectors $v_1, \ldots, v_k$ such that $\{v_1, \ldots, v_k\}$ spans $N(A)$, where $A =$ ....
- Find a finite spanning set for $E_\lambda(A)$, where $A =$ ... and $\lambda =$ ....
- Find a vector v_1 such $\{v_1\}$ spans $\left\{\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right\}^\perp$.
- Find a vector v_1 such $\{v_1\}$ spans $\left\{\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}^\perp$
- Can you find two vectors v_1 and v_2 such $\{v_1, v_2\}$ spans $\left\{\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\right\}^\perp$. Two *unit vectors*? Two *orthogonal* vectors? Two orthogonal unit vectors? A single vector?
- Let $A \in \mathbb{R}^{m\times n}$. The *column space of* A, written $C(A)$, is the set of vectors of the form Ax, where $x \in \mathbb{R}^n$:
$$C(A) = \left\{Ax : x \in \mathbb{R}^{m\times n}\right\} \subseteq \mathbb{R}^m$$
- Theorem: $C(A)$ is the span of the set of column vectors of A.
- Corollary: $C(A)$ is a subspace of $\mathbb{R}^m$.
- Definition: Let $A \in \mathbb{R}^{m\times n}$ and let U be a subspace of $\mathbb{R}^n$. The *image of* U *under* A, written AU, is the set of all matrix-vector products Au for $u \in U$:
$$AU = \{Au : u \in U\} \subseteq \mathbb{R}^m.$$

- Note that $C(A) = A\mathbb{R}^n$.
- Theorem: AU is a subspace of $\mathbb{R}^m$.
- Corollary: $C(A)$ is a subspace of $\mathbb{R}^m$.
- Definition: The j-th column of A is a *pivot column* if the reduced row echelon form of A has a leading 1 in column j.
- Theorem: The pivot columns of A span $C(A)$.
- Exercise: Suppose $AU_1 = AU_2$. Does it follow that $U_1 = U_2$?
- Definition: Let $A \in \mathbb{R}^{m\times n}$ and let V be a subspace of $\mathbb{R}^m$. The *inverse image of U under A*, written $A^{-1}V$, is the set of all vectors $x \in \mathbb{R}^n$ whose matrix-vector product with A belongs to V.
$$A^{-1}V = \{x \in \mathbb{R}^n : Ax \in V\} \subseteq \mathbb{R}^n.$$
- The inverse image $A^{-1}V$ is defined even if A^{-1} does not exist. However:
- Theorem: If A is invertible, then
$$A^{-1}V = \{A^{-1}v : v \in V\}.$$
- Note that $N(A) = A^{-1}\{0\}$
- Theorem: $A^{-1}V$ is a subspace of $\mathbb{R}^n$.
- Corollary: $N(A)$ is a subspace of $\mathbb{R}^n$.
- Suppose $A^{-1}V_1 = A^{-1}V_2$. Does it follow that $V_1 = V_2$?
- Prove that $U \subseteq A^{-1}(AU)$ and that $A(A^{-1}V) \subseteq V$. Give counterexamples to show that equality does not hold, in general.
- Exercise: Suppose that $\{u_1, \dots, u_k\}$ spans U. Prove that $\{Au_1, \dots, Au_k\}$ spans AU.
- Exercise: Suppose that $\{v_1, \dots, v_k\}$ spans V and that $u_1, \dots, u_k \in \mathbb{R}^n$ are vectors such that $Au_1 = v_1, \dots, Au_k = v_k$. Must $\{u_1, \dots, u_k\}$ span $A^{-1}V$?
- Definition: Let U and V be subspaces of $\mathbb{R}^n$. The *sum of U and V*, written $U + V$ is the set of sums $u + v$ for $u \in U$ and $v \in V$:
$$U + V = \{u + v : u \in U,\ v \in V\} \subseteq \mathbb{R}^n.$$
- Theorem: $U + V$ is a subspace of $\mathbb{R}^n$.
- $\langle S\rangle + \langle T\rangle = \langle S \cup T\rangle$.
- Definition: Let $U_1, \dots, U_k$ be subspaces of $\mathbb{R}^n$. The *sum of $U_1, \dots, U_k$* is the set of sums $u_1 + \cdots + u_k$, where $u_j \in U_j$:
$$\sum_{j=1}^{k} U_j = U_1 + \cdots + U_k = \{u_1 + \cdots + u_k : u_1 \in U_1, \dots, u_k \in U_k\}.$$
- Theorem: $U_1 + \cdots + U_k$ is a subspace of $\mathbb{R}^n$.
- Exercise:$\langle S_1\rangle + \cdots + \langle S_k\rangle = \langle S_1 \cup \cdots \cup S_k\rangle$
- Exercise: $\langle u_1, \dots, u_k\rangle = \langle u_1\rangle + \cdots + \langle u_k\rangle$
- Exercise: Prove that $(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp$. Generalize to k subspaces.

- Prove that $A(U_1 + U_2) = AU_1 + AU_2$ and that $A^{-1}(V_1 + V_2) = A^{-1}V_1 + A^{-1}V_2$. Generalize to k summands.

2. Linear dependence and independence

- Definition: Let $v_1, \dots, v_k \in \mathbb{R}^n$. A *linear dependence relation among* $v_1, \dots, v_k$ is an identity of the form
$$t_1 v_1 + \cdots t_k v_k = 0,$$
where $t_1, \dots, t_k \in \mathbb{R}$. Such a relation is *trivial* if $t_1 = 0, \dots, t_k = 0$.
- A set S of vectors in $\mathbb{R}^n$ is *linearly independent* if the only linear dependence relation among elements of S is the trivial one. Otherwise, it's *linearly dependent*.
- Theorem: S is linearly independent if and only if every element of $\langle S \rangle$ can be written uniquely as a linear combination of elements of S.
- A set S of nonzero vectors in $\mathbb{R}^n$ is *orthogonal* if every pair of distinct vectors pair S are orthogonal, i.e.,
$$u \cdot v = 0 \text{ for all } u, v \in S \text{ with } u \neq v.$$
- Theorem: An orthogonal set is linearly independent.
- Theorem: Let $a_1, \dots, a_n \in \mathbb{R}^m$ be distinct vectors. Then $\{a_1, \dots, a_n\}$ is linearly independent if and only if the nullspace of the matrix
$$A := \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$$
is zero.
- Corollary: Suppose $u_1, \dots, u_{k+1}$ are distinct vectors in $\langle v_1, \dots, v_k \rangle$. Then $\{u_1, \dots, u_k\}$ is linearly dependent.
- Corollary: A set of $n+1$ vectors in $\mathbb{R}^n$ is linearly dependent.
- Theorem: Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then vectors $v_1, \dots, v_k \in \mathbb{R}^n$ are linearly independent if and only if the vectors $Av_1, \dots, Av_k$ are.
- Corollary: The pivot columns of a matrix are linearly independent.
- Let $A \in \mathbb{R}^{m \times n}$ and let $v_1 \dots, v_k \in \mathbb{R}^n$. If $Av_1, \dots, Av_k$ are linearly dependent (resp., independent), does it follow that $v_1, \dots, v_k$ are?
- Linearly independent sets are "minimal" spanning sets:
- Theorem: Let U be a subspace of $\mathbb{R}^n$. A spanning set S of U is linearly independent if and only if no proper subset of S spans U.
- Definition: Subspaces U_1 and U_2 of $\mathbb{R}^n$ are *linearly independent* if the pair (u_1, u_2) is linearly independent for all $u_1 \in U_1$ and all $u_2 \in U_2$.
- Theorem: U_1 and U_2 are linearly independent if and only if $U_1 \cap U_2 = \{0\}$.
- Corollary: Let λ_1 and λ_2 be distinct eigenvalues of A. Then the eigenspaces $E_{\lambda_1}(A)$ and $E_{\lambda_2}(A)$ are linearly independent.
- Definition: Subspaces U_1 and U_2 of $\mathbb{R}^n$ are orthogonal if every element of U_1 is orthogonal to every element of U_2, i.e.,

if $u_1 \cdot u_2 = 0$ for all $u_1 \in U_1$ and all $u_2 \in U_2$.

- Exercise: Let U_1 and U_2 be linearly independent subsets of $\mathbb{R}^n$ and let S_1 and S_2 be subsets of U_1 and U_2, respectively. Then $S_1 \cup S_2$ is linearly independent if and only if both S_1 and S_2 are linearly independent.
- Definition: Subspaces $U_1, \ldots, U_k$ of $\mathbb{R}^n$ are *linearly independent* if the sequence $(u_1, \ldots, u_k)$ is linearly independent for all $u_1 \in U_1, \ldots, u_k \in U_k$.
- Definition: Subspaces $U_1, \ldots, U_k$ of $\mathbb{R}^n$ are *orthogonal* if they are pairwise orthogonal.
- Exercise: Suppose $U_1, \ldots, U_j$ are orthogonal. Prove that U_i and $\sum_{j \neq i} U_j$ are orthogonal for all i.
- Theorem: The following are equivalent for subspaces $U_1, \ldots, U_k$ of $\mathbb{R}^n$:
  (1) $U_1, \ldots, U_k$ are linearly independent.
  (2) U_i and $\sum_{j \neq i} U_i$ are linearly independent for all i, i.e.,
  $$U_i \cap \sum_{j \neq i} U_i = \{0\},$$
  for all i.
  (3) Every element of $U_1 + \cdots U_k$ can be written uniquely in the form $u_1 + \cdots u_k$, where $u_1 \in U_1, \ldots, u_k \in U_k$.
- Theorem: Let $A \in \mathbb{R}^{n \times n}$ and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A. Then the eigenspaces $E_{\lambda_1}(A), \ldots, E_{\lambda_k}(A)$ are linearly independent.

3. Basis and dimension

- Definition: Let U be a nonzero subspace of $\mathbb{R}^n$. A set B of vectors in U is a *basis of* U if B is linearly independent and B spans U.
- Theorem/Example: The set of pivot columns of A is a basis of $C(A)$.
- Theorem: Let U be a nonzero subspace of $\mathbb{R}^n$. Then U has a basis B with at most n elements.
- Proof: Let $\mathcal{L}$ be the set of linearly independent subsets of U and let $\mathcal{N}$ be the set of sizes of elements of $\mathcal{L}$, i.e.,
  $$\mathcal{N} = \{|S| : S \in \mathcal{L}\}.$$
  Note that $\mathcal{N}$ is nonempty: Since U is nonzero, it contains a nonzero vector v. Since a set consisting of a single nonzero vector is linearly independent, $\{v\} \in \mathcal{L}$. Thus, $1 = |\{v\}| \in \mathcal{N}$. Since each $S \in \mathcal{L}$ is a linearly independent subset of $\mathbb{R}^n$, $|S| \leq n$ for all $S \in \mathcal{L}$. It follows that $\mathcal{N}$ is a nonempty subset of $\{1, \ldots, n\}$.

  Let d be the largest element of $\mathcal{N}$ and let $B \in \mathcal{L}$ be a linearly independent set with $|B| = d$. I claim, now, that B spans U. For suppose not. Then there is a nonzero vector $u \in U$ that does not belong to $\langle B \rangle$. But $0 \neq u \notin \langle B \rangle$ with B linearly independent implies $\{u\} \cup B$ is linearly independent. Thus, $\{u\} \cup B \in \mathcal{L}$. But, as $u \notin B$, $|\{u\} \cup B| = |B| + 1 > d$, contradicting the maximality of d. Therefore, B must span U. By its construction, B is linearly independent. Therefore, B is a basis of U.

Strengthen to get a basis containing a given linearly independent set.

- Every linearly independent subset of U is contained in a basis of U.
- Theorem: Let B be a minimal spanning set of U, i.e., a spanning subset none of whose proper subsets span U. Then B is a basis of U.
- Theorem: Let B_1 and B_2 be bases of a subspace U of $\mathbb{R}^n$. Then $|B_1| = |B_2|$.
- By ??, $|B_1| \leq |B_2|$ as B_1 is linearly independent and B_2 spans $\mathbb{R}^n$. Symmetrically, $|B_2| \leq |B_1|$ as B_2 is linearly independent and B_1 spans U. Therefore, $|B_1| = |B_2|$.
- Definition: Let U be a subspace of $\mathbb{R}^n$. The *dimension of* U, written $\dim U$, is the size of any basis of U.
- Examples: $\dim \mathbb{R}^n = n$. If $v \in \mathbb{R}^n$ is a nonzero vector, then $\dim\langle v \rangle = 1$. If S is a linearly independent subset of $\mathbb{R}^n$, then $\dim\langle S \rangle = |S|$.
- Theorem: If U_1 and U_2 are subspaces of $\mathbb{R}^n$ with $U_1 \subseteq U_2$, then $\dim U_1 \leq \dim U_2$, with equality if and only if $U_1 = U_2$.
- Corollary: If S is a linearly independent subset of U with $|S| = \dim U$, then S is a basis of U.
- Corollary: Any linearly independent subset of $\mathbb{R}^n$ with n elements is a basis of $\mathbb{R}^n$.
- Theorem: Let S be a subset of U. Then S is a basis of U if and only if any element of u can be written uniquely as a linear combination of elements of S.
- Definition: Let U be a d-dimensional subspace of $\mathbb{R}^n$ with ordered basis $B = (b_1, \ldots, b_n)$ and let $u \in U$. The *coordinate vector of u with respect to B*, written $[u]_B$, is the vector

  $$[u]_B = \begin{bmatrix} t_1 \\ \vdots \\ t_d \end{bmatrix}$$

  characterized by the identity

  $$u = t_1 b_1 + \cdots + t_d b_d,$$

  or, equivalently, by the identity

  $$\begin{bmatrix} b_1 & \cdots & b_d \end{bmatrix} [u]_B = u.$$

- Convention: If $B = (b_1, \ldots, b_d)$ is a sequence of vectors, we use the same symbol, B, for the matrix whose j-th column if v_j, $j = 1, \ldots, d$:

  $$B = \begin{bmatrix} b_1 & \cdots b_d \end{bmatrix}$$

- With this convention, $[u]_B$ is characterized by the identity

  $$B[u]_B = u.$$

- Theorem: Let B be a basis of $\mathbb{R}^n$. Then

  $$[u]_B = B^{-1} u.$$

- Definition: A set (resp., sequence) S of vectors in $\mathbb{R}^n$ is *orthonormal* if its elements (resp., terms) are pairwise orthogonal unit vectors, i.e., $B = \{b_1, \ldots, b_d\}$ (resp., $B = (b_1, \ldots, b_d)$) is orthonormal if
$$b_i \cdot b_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$
- Orthonormal sets/sequences are orthogonal and, thus, linearly independent.
- A sequence $B = (b_1, \ldots, B_d)$ is orthonormal if and only if
$$B^t B = I_d.$$
If $d = n$, this means that B is invertible and $B^{-1} = B^t$. In this case, the identity
$$BB^t = I_n$$
holds, as well.
- Definition: B is an orthonormal *basis of* U if it is both an orthonormal set and a basis of B.
- Theorem: Let $B = (b_1, \ldots, b_d)$ be an orthonormal basis of U. Then
$$[u]_B = \begin{bmatrix} b_1 \cdot u \\ \vdots \\ b_d \cdot u \end{bmatrix} = B^t u.$$
- Theorem: Every orthonormal set in $\mathbb{R}^n$ is contained in an orthonormal basis of $\mathbb{R}^n$.
- Theorem: Suppose $B = (b_1, \ldots, b_n)$ is an orthonormal basis of $\mathbb{R}^n$. Then
$$\langle b_1, \ldots, b_k \rangle^\perp = \langle b_{k+1}, \ldots, b_n \rangle.$$
- Corollary: Let U be a subspace of $\mathbb{R}^n$. Then
$$\dim U^\perp = n - \dim U.$$
- Theorem: Let B and C be bases of U. Then there is a unique matrix $[B]_C \in \mathbb{R}^{d \times d}$ such that
$$[B]_C [u]_B = [u]_C$$
for all $\in U$.
- Definition: The matrix $[C]_B$ is called the *change of basis matrix on* U *from* B *to* C.
- Theorem:
  (1) $[B]_B = I_d$
  (2) $[D]_C [C]_B = [D]_B$.
  (3) $[C]_B = [B]_C^{-1}$
- Definition: Let $A \in \mathbb{R}^{m \times n}$. The *rank of* A, written $r(A)$, is the dimension of its column space:
$$r(A) = \dim C(A).$$
The *nullity of* A, written $n(A)$, is the dimension of its nullspace:
$$n(A) = \dim N(A).$$

- Let $\gamma \in \mathbb{R}^{m\times m}$ be invertible and let $A \in \mathbb{R}^{m\times n}$. Then:
  (1) $R(\gamma A) = R(A)$,
  (2) $C(\gamma A) = \gamma C(A)$.
- Theorem: Let $\gamma \in \mathbb{R}^{n\times n}$ be invertible and let $v_1, \ldots, v_k \in \mathbb{R}^n$. Then
  $$\dim\langle \gamma v_1, \ldots, \gamma v_k\rangle = \dim\langle v_1, \ldots, v_k\rangle.$$
  In particular, if V is a subspace of $\mathbb{R}^n$, then
  $$\dim \gamma V = \dim V.$$
- Corollary: $r(\gamma A) = r(A)$.
- Theorem: (row rank = column rank) $\dim R(A) = \dim C(A)$.
- Proof: Let $\gamma \in \mathbb{R}^{m\times m}$ be an invertible matrix such that γA is in row echelon form. The nonzero rows of γA are linearly independent (why?) and, thus, form a basis for $R(\gamma A)$. Therefore, $\dim R(\gamma A)$ is the number of nonzero rows in γA. But the number of nonzero rows in γA equals the number of pivot columns of B (why?) which, in turn, equals the rank of γA. Thus,
  $$\dim R(\gamma A) = r(\gamma A).$$
  But $\dim R(A) = \dim R(\gamma A)$, by ??, and $r(\gamma A) = r(A)$, by ??. Therefore,
  $$\dim R(A) = r(A).$$
- $N(A) = C(A^T)^\perp$. $\dim N(A) = \dim C(A^T)^\perp = n - r(A^T) = n - r(A)$

4. Subspaces of $\mathbb{R}^n$

Let $\gamma \in \mathbb{R}^{m\times m}$ be an invertible matrix such that γA is in row echelon form. The nonzero rows of γA are linearly independent (why?) and, thus, form a basis for $R(\gamma A)$. Therefore, $\dim R(\gamma A)$ is the number of nonzero rows in γA. But the number of nonzero rows in γA equals the number of pivot columns of B (why?) which, in turn, equals the rank of γA.

Let $\gamma \in \mathbb{R}^{m\times m}$ be an invertible matrix such that γA is in row echelon form. The nonzero rows of γA are linearly independent (why?) and, thus, form a basis for $R(\gamma A)$. Therefore, $\dim R(\gamma A)$ is the number of nonzero rows in γA. But the number of nonzero rows in γA equals the number of pivot columns of B (why?) which, in turn, equals the rank of γA.