1. Subspaces of $\mathbb{R}^n$

Definition 1.1. A *subspace of* $\mathbb{R}^n$ is a subset U of $\mathbb{R}^n$ that is *closed under addition*:

$$\text{if } u_1 \in U \text{ and } u_2 \in U \text{ then } u_1 + u_2 \in U,$$

and *closed under scalar multiplication*:

$$\text{if } t \in \mathbb{R} \text{ and } u \in U \text{ then } tu \in U.$$

Example 1.2. The set

$$U := \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^3 : u_1 - 2u_2 = 3u_3 \right\} \subseteq \mathbb{R}^3$$

is a subspace of $\mathbb{R}^3$.

Example 1.3. Lines, planes (graphically).

1.1. Nullspace.

Definition 1.4. Let $A \in \mathbb{R}^{m \times n}$. The *nullspace of* A, written $N(A)$, is the set of all vectors $x \in \mathbb{R}^n$ killed by A:

$$N(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$

Example 1.5. Let U be as in Example 1.2. Observe:

$$u_1 - 2u_2 = 3u_3$$

$$\text{if and only if} \qquad u_1 - 2u_2 - 3u_3 = 0$$

$$\text{if and only if} \qquad \begin{bmatrix} 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = [0].$$

Therefore,

$$U = N\left(\begin{bmatrix} 1 & -2 & -3 \end{bmatrix} \right)$$

Theorem 1.6. $N(A)$ *is a subspace of* $\mathbb{R}^n$.

Proof. Closure under addition: Let $u_1, u_2 \in N(A)$. Then $Au_1 = 0$ and $Au_2 = 0$, by definition of $N(A)$. We compute:

$$A(u_1 + u_2) = Au_1 + Au_2 = 0 + 0 = 0.$$

Therefore, by the defintion of $N(A)$, $u_1 + u_2 \in N(A)$.

Closure under scalar multiplication: Let $t \in \mathbb{R}$ and let $u \in N(A)$. Then $Au = 0$, by definition of $N(A)$. We compute:

$$A(tu) = tAu = t0 = 0.$$

Therefore, by the defintion of $N(A)$, $tu \in N(A)$. $\qquad\square$

Exercise 1.7. Prove that $\{x \in \mathbb{R}^n : Ax = b\}$ is a subspace of $\mathbb{R}^n$ if and only if $b = 0$.

Definition 1.8. Let $\lambda \in \mathbb{R}$ and let $A \in \mathbb{R}^{m \times n}$. The λ-*eigenspace of* A, written $E_\lambda(A)$, is the set of all λ-eigenvectors of A:

$$E_\lambda(A) = \{x \in \mathbb{R}^n : Ax = \lambda x\}.$$

Corollary: $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.

Proof. $E_\lambda(A) = N(\lambda I - A)$, and nullspaces are subspaces. $\qquad\square$

1.2. **Orthogonal complement.**

Definition 1.9. Let S be a subset of $\mathbb{R}^n$. The *orthogonal complement of S* written $S^\perp$, is the set of vectors orthogonal to all elements of S:

$$S^\perp = \{v \in \mathbb{R}^n : u \cdot v = 0 \text{ for all } u \in S\}$$

Theorem 1.10. *$S^\perp$ is a subspace of $\mathbb{R}^n$.*

Proof. Closure under addition: Let $u_1, u_2 \in S^\perp$ and $v \in S$. We must show that $(u_1 + u_2) \cdot v = 0$. By definition of $S^\perp$, $u_1 \cdot v = 0$ and $u_2 \cdot v = 0$. Therefore,

$$(u_1 + u_2) \cdot v = u_1 \cdot v + u_2 \cdot v = 0 + 0 = 0,$$

as was to be shown.

Closure under scalar multiplication: Let $t \in \mathbb{R}$, let $u \in S^\perp$, and let $v \in S$. We must show that $(tu) \cdot v = 0$. By definition of $S^\perp$, $u \cdot v = 0$. Therefore,

$$(tu) \cdot v = t(u \cdot v) = t(0) = 0,$$

as was to be shown.

$\qquad\square$

The orthogonal complement of a finite set of vectors is a nullspce:

Theorem 1.11. *Suppose $S = \{a_1, \ldots, a_k\}$ and let*

$$A = \begin{bmatrix} a_1 & \cdots a_k \end{bmatrix}.$$

Then

$$S^\perp = N(A^T).$$

Proof. Observe:

$$A^T x = \begin{bmatrix} a_1^T \\ \vdots \\ a_k^T \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ \vdots \\ a_k^T x \end{bmatrix} = \begin{bmatrix} a_1^T \cdot x \\ \vdots \\ a_k^T \cdot x \end{bmatrix}$$

Therefore, $A^T x = 0$ if and only if $a_j \cdot x = 0$ for all j, i.e., if and only if $x \in S^\perp$. Thus, $N(A^T) = S^\perp$. $\qquad\square$

We will see, later, that the orthogonal complement of *any* subset of $\mathbb{R}^n$ is a nullspace.

Example 1.12. Lines, planes

1.3. **Image and column space.**

Definition 1.13. Let $A \in \mathbb{R}^{m \times n}$ and let U be a subspace of $\mathbb{R}^n$. The *image of U under A*, written AU, is the set of all matrix-vector products Au for $u \in U$:

$$AU = \{Au : u \in U\} \subseteq \mathbb{R}^m.$$

Theorem 1.14. *AU is a subspace of $\mathbb{R}^m$.*

The image of $U = \mathbb{R}^n$ plays a special role and gets a special name.

Definition 1.15. Let $A \in \mathbb{R}^{m \times n}$. The image of $\mathbb{R}^n$ under A is called the *column space of A* and written $C(A)$:

$$C(A) = A\mathbb{R}^n = \{Ax : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

1.4. **Linear combinations.**

Definition 1.16. Let $u_1, \ldots, u_k \in \mathbb{R}^n$. A *linear combination of $u_1, \ldots, u_k$* is a vector of the form

$$t_1 u_1 + \cdots + t_k u_k,$$

where $t_j \in \mathbb{R}$.

Example 1.17. The sum $u_1 + u_2$ is a linear combination of u_1 and u_2. The scalar multiple tu is a linear combination of u.

Example 1.18. Let $a_1, \ldots, a_n \in \mathbb{R}^m$ and let $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$. Then the linear combinations of $a_1, \ldots, a_n$ are precisely the vectors of the form Ax, for $x \in \mathbb{R}^n$:

$$x_1 a_1 + \cdots + x_n a_n = Ax.$$

Thus, the column space of A is the set of linear combinations of the column vectors of A:

$$C(A) = \{x_1 a_1 + \cdots + x_n a_n : x_1, \ldots, x_n \in \mathbb{R}\}.$$

Exercise 1.19. Let S be a subset of $\mathbb{R}^n$. Prove the following statements.

(1) The sum of two linear combinations of elements of S is a linear combination of elements of S.
(2) A scalar multiple of a linear combination of elements of S is a linear combination of elements of S.
(3) A linear combination of linear combinations of elements of S is a linear combination of elements of S.

Theorem 1.20. *Let U be a subset of $\mathbb{R}^n$. Then U is a subspace of $\mathbb{R}^n$ if and only if U is closed under linear combinations, i.e., if and only if every linear combination of (finitely many) elements of U is, itself, an element of U.*

Corollary 1.21. *Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Then $C(A)$ is a subspace of $\mathbb{R}^m$.*

Example 1.22. Solution of a homogeneous system are linear combinations of *basic solutions*. (We'll give a more satisfying definition of basic solution later.)

1.5. **Span.**

Definition 1.23. Let S be a subset of $\mathbb{R}^n$. The *span of S*, written $\langle S \rangle$, is the set of all linear combinations of elements of S.

Theorem 1.24. $\langle S \rangle$ *is a subspace of* $\mathbb{R}^n$.

Proof. Do Exercise 1.19. □

Exercise 1.25. Prove that $\langle S \rangle$ is the smallest subspace of $\mathbb{R}^n$ containing S, i.e., that if U is a subspace of $\mathbb{R}^n$ and $S \subseteq U$ then $\langle S \rangle \subseteq U$. (Use Theorem 1.20.)

Exercise 1.26. Let S be a subset of $\mathbb{R}^n$. Prove that $S \subseteq \langle S \rangle$.

Theorem 1.27. *Let U be a subset of $\mathbb{R}^n$. Then U is a subspace of $\mathbb{R}^n$ if and only if $U = \langle U \rangle$.*

Proof. Suppose U is a subspace of $\mathbb{R}^n$. By Exercise 1.26, $U \subseteq \langle U \rangle$. By hypothesis, U is a subspace of $\mathbb{R}^n$; U obviously contains U. Therefore, by Exercise 1.26, $\langle U \rangle \subseteq U$. Having proved both inclusions, we conclude that $U = \langle U \rangle$.

Conversely, suppose $U = \langle U \rangle$. Then U is a subspace of $\mathbb{R}^n$ by Theorem 1.24 □

Exercise 1.28. Let S and T be subsets of $\mathbb{R}^n$. Prove that if $S \subseteq T$ then $\langle S \rangle \subseteq \langle T \rangle$. Is the converse true?

Theorem 1.29. *Let $u, v_1, \ldots, v_k \in \mathbb{R}^n$. Then $u \in \langle v_1, \ldots, v_k \rangle$ if and only if*

$$(1) \qquad \langle v_1, \ldots, v_k \rangle = \langle u, v_1, \ldots, v_k \rangle.$$

Proof. Suppose $u \in \langle v_1, \ldots, v_k \rangle$. We must prove identity (1). By Exercise 1.28,

$$\langle v_1, \ldots, v_k \rangle \subseteq \langle u, v_1, \ldots, v_k \rangle.$$

To prove the reverse inclusion, let $x \in \langle u, v_1, \ldots, v_k \rangle$. Then there are scalars $r, s_1, \ldots, s_k$ such that

$$(2) \qquad x = ru + s_1 v_1 + \cdots s_k v_k.$$

Since $u \in \langle v_1, \ldots, v_k \rangle$, by hypothesis, there are scalars $t_1, \ldots, t_k \in \mathbb{R}$ such that

$$(3) \qquad u = t_1 v_1 + \cdots t_k v_k.$$

Substituting (3) into (2), we get

$$x = r(t_1 v_1 + \cdots t_k v_k) + s_1 v_1 + \cdots s_k v_k$$
$$= (rt_1 + s_1)v_1 + \cdots (rt_k + s_k)v_k.$$

showing that x is a linear combination of $v_1, \ldots v_k$. Therefore, $x \in \langle v_1, \ldots, v_k \rangle$. Since $x \in \langle u, v_1, \ldots, v_k \rangle$ was chosen arbitrarily,

$$\langle u, v_1, \ldots, v_k \rangle \subseteq \langle v_1, \ldots, v_k \rangle,$$

completing the proof of (1).

Conversely, suppose that (1) holds. We must show that $u \in \langle v_1, \ldots, v_k \rangle$. But this is clear:

$$u \in \langle u, v_1, \ldots, v_k \rangle \qquad \qquad \text{by Exercise 1.26}$$
$$= \langle v_1, \ldots, v_k \rangle \qquad \qquad \text{by (1)} \qquad \qquad □$$

Definition 1.30. Let S be a subset of $\mathbb{R}^n$ and let U be a subspace of $\mathbb{R}^n$. We say that S *spans* U or that S *is a spanning set of* U if $U = \langle S \rangle$.

Exercise 1.31. Find vectors $v_1, \ldots, v_k$ such that $\{v_1, \ldots, v_k\}$ spans $N(A)$, where $A = \ldots$.

Exercise 1.32. Find a finite spanning set for $E_\lambda(A)$, where $A = \ldots$ and $\lambda = \ldots$.

Exercise 1.33. Find a vector v_1 such $\{v_1\}$ spans $\left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}^\perp$.

Exercise 1.34. Find a vector v_1 such $\{v_1\}$ spans $\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}^\perp$.

Exercise 1.35. Can you find two vectors v_1 and v_2 such $\{v_1, v_2\}$ spans $\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right\}^\perp$.

Two *unit vectors*? Two *orthogonal* vectors? Two orthogonal unit vectors? A single vector?

Exercise 1.36. Suppose that $\{u_1, \ldots, u_k\}$ spans U. Prove that $\{Au_1, \ldots, Au_k\}$ spans AU.

1.6. **Column space, again.**

Example 1.37. By Example 1.18, the column space of a matrix is the span of its column vectors:
$$C\left(\begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix} \right) = \langle a_1, \ldots, a_k \rangle.$$

Theorem 1.38. $C(A)$ *is spanned by its pivot columns.*

We prove this theorem with the help of two lemmas ("helper theorems").

Lemma 1.39. *Let* $v_1 \ldots, v_k \in \mathbb{R}^n$ *and let* $X \in \mathbb{R}^{n \times n}$.
$$\langle Xv_1, \ldots, Xv_k \rangle = X \langle v_1, \ldots, v_k \rangle.$$

Proof. Let
$$u \in \langle Xv_1, \ldots, Xv_k \rangle,$$
with the goal of showing that
$$u \in X \langle v_1, \ldots, v_k \rangle.$$
By definition of $\langle Xv_1, \ldots, Xv_k \rangle$, there are scalars $t_1, \ldots, t_k \in \mathbb{R}$ such that
$$u = t_1 Xv_1 + \cdots t_k Xv_k.$$
But $t_j Xv_j = X(t_j v_j)$, so
$$u = X(t_1 v_1) + \cdots X(t_k v_k).$$
Therefore, setting
$$v := t_1 v_1 + \cdots t_k v_k,$$
we have
$$u = X(t_1 v_1 + \cdots t_k v_k) = Xv.$$

Evidently, $v \in \langle v_1, \ldots, v_k \rangle$. Thus, $u = Xv$ with $v \in \langle v_1, \ldots, v_k \rangle$. Therefore,

$$u \in X \langle v_1, \ldots, v_k \rangle.$$

Since $u \in \langle Xv_1, \ldots, Xv_k \rangle$ was arbitrary, we conclude that

$$\langle Xv_1, \ldots, Xv_k \rangle \subseteq X \langle v_1, \ldots, v_k \rangle.$$

Conversely, Let

$$u \in X \langle v_1, \ldots, v_k \rangle,$$

with the goal of showing that

$$u \in \langle Xv_1, \ldots, Xv_k \rangle.$$

Then there is an element $v \in \langle v_1, \ldots, v_k \rangle$ such that $u = Xv$. As $v \in \langle v_1, \ldots, v_k \rangle$, there are scalars $t_1, \ldots, t_k$ such that

$$v = t_1 v_1 + \cdots t_k v_k.$$

Therefore,

$$
\begin{aligned}
u = Xv & \\
&= X(t_1 v_1 + \cdots + t_k v_k) \\
&= X(t_1 v_1) + \cdots X(t_k v_k) \\
&= t_1(Xv_1) + \cdots + t_k(Xv_k) \\
&\in \langle Xv_1, \ldots, Xv_k \rangle.
\end{aligned}
$$

Since $u \in X \langle v_1, \ldots, v_k \rangle$ was arbitrary, it follows that

$$X \langle v_1, \ldots, v_k \rangle \subseteq \langle Xv_1, \ldots, Xv_k \rangle.$$

Having proved the reverse inclusion above, statement (1) is proved.

$\square$

Lemma 1.40. *Theorem 1.38 holds when A is in reduced row echelon form.*

Proof. We must show that the nonpivot columns of A belong to the span of the pivot columns of A. Let $(1, j_i), \ldots, (r, j_r)$ be the positions of the leading ones of A, so that $a_{j_1}, \ldots, a_{j_r}$ are the pivot columns of A. Since A is in reduced row echelon form, a leading one of A is the only nonzero element in its column. Therefore, $a_{j_i} = e_i \in \mathbb{R}^m$. Thus, it suffices to show that the nonpivot columns of A belong to $\langle e_1, \ldots, e_r \rangle \subseteq \mathbb{R}^m$.

Suppose a_j is a nonpivot column of A. Suppose that q pivot columns of A lie to the left of a_j. Then $1 \leq q \leq r$. (Why?) As A is in reduced row echelon form, $a_{i,j} = 0$ for $i > q$. Thus,

$$
\begin{aligned}
a_j &= a_{1,j} e_1 + \cdots + a_{m,j} e_m && \text{(property of } \{e_1, \ldots, e_m\}) \\
&= a_{1,j} e_1 + \cdots + a_{q,j} e_q && \text{(as } a_{i,j} = 0 \text{ for } i > q) \\
&\in \langle e_1, \ldots, e_q \rangle && \text{(by definition of } \langle e_1, \ldots, e_q \rangle) \\
&\subseteq \langle e_1, \ldots, e_r \rangle && \text{(as } q \leq r),
\end{aligned}
$$

establishing the claim. $\square$

Proof of Theorem 1.38. Let $\gamma \in \mathbb{R}^{m \times m}$ be an invertible matrix such that $B = \gamma A$ is in reduced row echelon form. Write

$$B = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}.$$

Let $(1, j_i)$, ..., (r, j_r) be the positions of the leading ones of B, so that $a_{j_1}, \ldots, a_{j_r}$ are the pivot columns of B. Then, by Lemma ??,

$$\langle b_1, \ldots, b_n \rangle = \langle b_{j_1}, \ldots, b_{j_r} \rangle.$$

Write

$$A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}.$$

Since $B = \gamma A$, $b_j = \gamma a_j$ for all j. Substituting in to the above identity, we get

$$\langle \gamma a_1, \ldots, \gamma a_n \rangle = \langle \gamma a_{j_1}, \ldots, \gamma a_{j_r} \rangle.$$

By Lemma ??,

$$\langle \gamma a_1, \ldots, \gamma a_n \rangle = \gamma \langle a_1, \ldots, a_n \rangle$$

and

$$\langle \gamma a_{j_1}, \ldots, \gamma a_{j_r} \rangle = \gamma \langle a_{j_1}, \ldots, a_{j_r} \rangle.$$

Therefore,

$$\gamma \langle a_1, \ldots, a_n \rangle = \gamma \langle a_{j_1}, \ldots, a_{j_r} \rangle$$

Cancelling the γs (see Exercise 1.41), we get

$$\langle a_1, \ldots, a_n \rangle = \langle a_{j_1}, \ldots, a_{j_r} \rangle \qquad \square.$$

Exercise 1.41. Let U and V be subspaces of $\mathbb{R}^n$.

(1) Let $X \in \mathbb{R}^{m \times n}$. Prove: if $U \subseteq V$ then $XU \subseteq XV$.
(2) Show, by example, that $XU \subseteq XV$ need not imply $U \subseteq V$.
(3) Let $\gamma \in \mathbb{R}^{n \times n}$ be an invertible matrix. Prove: $U \subseteq V$ if and only if $\gamma U \subseteq \gamma V$.

1.7. **Sums of subspaces.**

Definition 1.42. Let U and V be subspaces of $\mathbb{R}^n$. The *sum of U and V*, written $U + V$ is the set of sums $u + v$ for $u \in U$ and $v \in V$:

$$U + V = \{u + v : u \in U,\ v \in V\} \subseteq \mathbb{R}^n.$$

Theorem 1.43. $U + V$ *is a subspace of* $\mathbb{R}^n$.

Theorem 1.44. $\langle S \rangle + \langle T \rangle = \langle S \cup T \rangle$.

Definition 1.45. Let $U_1, \ldots, U_k$ be subspaces of $\mathbb{R}^n$. The *sum of $U_1, \ldots, U_k$* is the set of sums $u_1 + \cdots + u_k$, where $u_j \in U_j$:

$$\sum_{j=1}^{k} U_j = U_1 + \cdots + U_k = \{u_1 + \cdots + u_k : u_1 \in U_1, \ldots, u_k \in U_k\}.$$

Theorem 1.46. $U_1 + \cdots + U_k$ *is a subspace of* $\mathbb{R}^n$.

Exercise 1.47. $\langle S_1 \rangle + \cdots + \langle S_k \rangle = \langle S_1 \cup \cdots \cup S_k \rangle$

Exercise 1.48. $\langle u_1, \ldots, u_k \rangle = \langle u_1 \rangle + \cdots + \langle u_k \rangle$

Exercise 1.49. Prove that $(U_1 + U_2)^{\perp} = U_1^{\perp} \cap U_2^{\perp}$. Generalize to k subspaces.

Exercise 1.50. Prove that $A(U_1 + U_2) = AU_1 + AU_2$ and that $A^{-1}(V_1 + V_2) = A^{-1}V_1 + A^{-1}V_2$. Generalize to k summands.

2. Linear dependence and independence

Definition 2.1. Let $v_1, \ldots, v_k \in \mathbb{R}^n$. A *linear dependence relation among* $v_1, \ldots, v_k$ is an identity of the form

$$t_1 v_1 + \cdots t_k v_k = 0,$$

where $t_1, \ldots, t_k \in \mathbb{R}$. Such a relation is *trivial* if $t_1 = 0, \ldots, t_k = 0$.

A set S of vectors in $\mathbb{R}^n$ is *linearly independent* if the only linear dependence relation among elements of S is the trivial one. Otherwise, it's *linearly dependent*.

Theorem 2.2. *S is linearly independent if and only if every element of $\langle S \rangle$ can be written uniquely as a linear combination of elements of S.*

Definition 2.3. A set S of vectors in $\mathbb{R}^n$ is *orthogonal* if every pair of distinct vectors pair S are orthogonal, i.e.,

$$u \cdot v = 0 \text{ for all } u, v \in S \text{ with } u \neq v.$$

Theorem 2.4. *Let S be an orthogonal set in $\mathbb{R}^n$ with $0 \notin S$. Then S is linearly independent.*

Theorem 2.5. *Let $a_1, \ldots, a_n \in \mathbb{R}^m$ be distinct vectors. Then $\{a_1, \ldots, a_n\}$ is linearly independent if and only if the nullspace of the matrix*

$$A := \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$$

is zero.

Corollary 2.6. *The columns vectors of an invertible matrix are linearly independent.*

Corollary 2.7. *Suppose $u_1, \ldots, u_{k+1}$ are distinct vectors in $\langle v_1, \ldots, v_k \rangle$. Then $\{u_1, \ldots, u_k\}$ is linearly dependent.*

Corollary 2.8. *A set of $n + 1$ vectors in $\mathbb{R}^n$ is linearly dependent.*

Theorem 2.9. *Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly independent if and only if the vectors $Av_1, \ldots, Av_k$ are.*

Corollary 2.10. *The pivot columns of a matrix are linearly independent.*

Exercise 2.11. Let $A \in \mathbb{R}^{m \times n}$ and let $v_1 \ldots, v_k \in \mathbb{R}^n$. If $Av_1, \ldots, Av_k$ are linearly dependent (resp., independent), does it follow that $v_1, \ldots, v_k$ are?

Linearly independent sets are "minimal" spanning sets:

Theorem 2.12. *Let U be a subspace of $\mathbb{R}^n$. A spanning set S of U is linearly independent if and only if no <u>proper</u> subset of S spans U.*

2.1. **Linear independent subspaces.**

2.1.1. *Two subspaces.*

Definition 2.13. Subspaces U_1 and U_2 of $\mathbb{R}^n$ are *linearly independent* if the pair (u_1, u_2) is linearly independent for all nonzero $u_1 \in U_1$ and all nonzero $u_2 \in U_2$.

Example 2.14. Let u_1 and u_2 be linearly independent vectors in $\mathbb{R}^n$. Then $\langle u_1 \rangle$ and $\langle u_2 \rangle$ are linearly independent. In other words, distinct lines are linearly independent.

To see this, let $x_1 \in \langle u_1 \rangle$ and let $x_2 \in \langle u_2 \rangle$. We need to show that x_1 and x_2 are linearly independent. So, suppose $s_1 x_1 + s_2 x_2 = 0$. We will show that $s_1 = 0$ and $s_2 = 0$. Since $x_1 \in \langle u_1 \rangle$, $x_1 = t_1 u_1$ for some nonero $t_1 \in \mathbb{R}$. Symmetrically, $x_2 = t_2 u_2$ for some nonzero $t_2 \in \mathbb{R}$. Substituting, we get

$$s_1 x_1 + s_2 x_2 = s_1 t_1 u_1 + s_2 t_2 u_2.$$

Since u_1 and u_2 are assumed linearly independent, we must have $s_1 t_1 = 0$ and $s_2 t_2 = 0$. Thus, $s_1 = 0$ and $s_2 = 0$ as t_1 and t_2 are nonzero.

Theorem 2.15. *U_1 and U_2 are linearly independent if and only if $U_1 \cap U_2 = \{0\}$.*

Example 2.16. Let

$$U = N\left(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\right), \quad V = N\left(\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\right).$$

Then

$$U \cap V = N\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\right) = \left\langle \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\rangle.$$

In particular $U \cap V \neq \{0\}$.

Exercise 2.17. Let u_1 and u_2 be linearly independent vectors in $\mathbb{R}^3$. Prove that $\{u_1\}^\perp$ and $\{u_2\}^\perp$ are linearly dependent. Give examples, to show that this result fails if $\mathbb{R}^3$ is replaced by $\mathbb{R}^n$ with $n > 3$.

Corollary 2.18. *Let λ_1 and λ_2 be distinct eigenvalues of A. Then the eigenspaces $E_{\lambda_1}(A)$ and $E_{\lambda_2}(A)$ are linearly independent.*

Proof. Do Exercise 2.19. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Exercise 2.19. Let λ_1 and λ_2 be distinct real numbers and let $A \in \mathbb{R}^{n \times n}$. Prove that

$$E_{\lambda_1}(A) \cap E_{\lambda_2}(A) = \{0\}.$$

Exercise 2.20. Let U_1 and U_2 be linearly independent subspaces of $\mathbb{R}^n$ and let S_1 and S_2 be subsets of U_1 and U_2, respectively. Then $S_1 \cup S_2$ is a linearly independent set if and only if both S_1 and S_2 are.

Definition 2.21. Subspaces U_1 and U_2 of $\mathbb{R}^n$ are orthogonal if every element of U_1 is orthogonal to every element of U_2, i.e.,

$$\text{if } u_1 \cdot u_2 = 0 \text{ for all } u_1 \in U_1 \text{ and all } u_2 \in U_2.$$

Example 2.22. Let U be a subspace of $\mathbb{R}^n$. Then U and $U^\perp$ are orthogonal.

Exercise 2.23. Let U_1 and U_2 are orthogonal subspaces of $\mathbb{R}^n$. Prove that U_1 and U_2 are linearly independent.

2.1.2. *k subspaces.*

Definition 2.24. Subspaces $U_1, \ldots, U_k$ of $\mathbb{R}^n$ are *linearly independent* if the sequence $(u_1, \ldots, u_k)$ is linearly independent for all $u_1 \in U_1, \ldots, u_k \in U_k$.

Theorem 2.25. *Let $A \in \mathbb{R}^{n \times n}$ and let Λ be the set of eigenvalues of A. Then the eigenspaces $E_\lambda(A)$, $\lambda \in \Lambda$, are linearly independent.*

Proof. Suppose not. Let Λ' be a subset of Λ, minimal with respect to the property that the eigenspaces $E_\lambda(A)$, $\lambda \in \Lambda'$, are linearly dependent. (This means: if Λ'' is a proper subset of Λ', then the $E_\lambda(A)$, $\lambda \in \Lambda''$, are linearly indepedent.) Note that $k \geq 2$. (Why?)

Suppose $\Lambda' = \{\lambda_1, \ldots, \lambda_k\}$ with the λ_j pairwise distinct. By the linear dependence of Λ', there are vectors $x_j \in E_{\lambda_j}(A)$ such that $\{x_1, \ldots, x_k\}$ is linearly dependent. Thus, there are scalars $t_1, \ldots, t_k$, not all zero, such that

(4)
$$0 = t_1 x_1 + \cdots t_k x_k.$$

In fact, by the minimality of Λ', $t_1, \ldots, t_k$ must *all* be nonzero. (Explain.) Multiply both sides of this identity by A:

$$0 = A(t_1 x_1 + \cdots t_k x_k) = t_1 A x_1 + \cdots + t_k A x_k.$$

Since x_j is a λ_j-eigenvector of A,

(5)
$$0 = t_1 \lambda_1 x_1 + \cdots t_k \lambda_k x_k.$$

Subtracing λ_k times (4) from (5) yields

$$\begin{aligned}
0 = 0 - \lambda_k(0) \\
= t_1 \lambda_1 x_1 + \cdots t_k \lambda_k x_k - \lambda_k(t_1 x_1 + \cdots t_k x_k) \\
= t_1(\lambda_1 - \lambda_k) + \cdots + t_{k-1}(\lambda_{k-1} - \lambda_k).
\end{aligned}$$

(This the right hand side of this identity makes sense as $k \geq 2$.) Since the λ_j are pairwise distinct, $\lambda_j - \lambda_k \neq 0$ for $j \leq k - 1$. The t_j being nonzero, $t_j(\lambda_j - \lambda_k) \neq 0$ for all $j \leq k - 1$. Thus,

$$0 = t_1(\lambda_1 - \lambda_k) + \cdots + t_{k-1}(\lambda_{k-1} - \lambda_k)$$

is a nontrivial linear depedence relation among the eigenvectors $x_1, \ldots, x_{k-1}$ and, therefore, the eigenspaces $E_\lambda(A)$, $\lambda \in \Lambda'' := \{\lambda_1, \ldots, \lambda_{k-1}\}$, are linear dependent. But Λ'' has fewer elements than Λ', contradicting the minimality of the latter. $\square$

Theorem 2.26. *Let $U_1, \ldots, U_k$ be pairwise orthogonal subspaces of $\mathbb{R}^n$. Then $U_1, \ldots, U_k$ are linearly independent.*

Proof. Let $u_j \in U_j$ be a nonzero vector and suppose that

$$t_1 u_1 + \cdots t_k u_k = 0.$$

We must show that $t_j = 0$ for all j. Suppose $1 \leq j \leq k$. Take the dot product of each side of the above identity with u_j:

$$t_1(u_1 \cdot u_j) + \cdots + t_j(u_j \cdot u_j) + \cdots t_k(u_k \cdot u_j) = 0 \cdot u_j = 0.$$

Since the $U_1, \ldots, U_k$ are pairwise orthogonal, $u_i \cdot u_j = 0$ if $i \neq j$. Therefore,

$$t_j(u_j \cdot u_j) = 0.$$

But $u_j \cdot u_j \neq 0$ as $u_j \neq 0$, so $t_j = 0$, as was to be shown. $\square$

Exercise 2.27. Suppose $U_1, \ldots, U_j$ are orthogonal. Prove that U_i and $\sum_{j \neq i} U_j$ are orthogonal for all i.

Theorem 2.28. *The following are equivalent for subspaces $U_1, \ldots, U_k$ of $\mathbb{R}^n$:*

(1) *$U_1, \ldots, U_k$ are linearly independent.*
(2) *U_i and $\sum_{j \neq i} U_i$ are linearly independent for all i, i.e.,*

$$U_i \cap \sum_{j \neq i} U_i = \{0\},$$

for all i.
(3) *Every element of $U_1 + \cdots U_k$ can be written uniquely in the form $u_1 + \cdots u_k$, where $u_1 \in U_1, \ldots, u_k \in U_k$.*

Theorem 2.29. *Let $A \in \mathbb{R}^{n \times n}$ and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A. Then the eigenspaces $E_{\lambda_1}(A), \ldots, E_{\lambda_k}(A)$ are linearly independent.*

3. Basis and dimension

Definition 3.1. Let U be a nonzero subspace of $\mathbb{R}^n$. A set B of vectors in U is a *basis of U* if B is linearly independent and B spans U.

Theorem 3.2. *The set of pivot columns of A is a basis of $C(A)$.*