1. Subspaces of $\mathbb{R}^n$

Definition 1.1. A *subspace of* $\mathbb{R}^n$ is a subset U of $\mathbb{R}^n$ that is *closed under addition*:

$$\text{if } u_1 \in U \text{ and } u_2 \in U \text{ then } u_1 + u_2 \in U,$$

and *closed under scalar multiplication*:

$$\text{if } t \in \mathbb{R} \text{ and } u \in U \text{ then } tu \in U.$$

Example 1.2. The set

$$U := \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^3 : u_1 - 2u_2 = 3u_3 \right\} \subseteq \mathbb{R}^3$$

is a subspace of $\mathbb{R}^3$.

Example 1.3. Lines, planes (graphically).

1.1. Nullspace.

Definition 1.4. Let $A \in \mathbb{R}^{m \times n}$. The *nullspace of* A, written $N(A)$, is the set of all vectors $x \in \mathbb{R}^n$ killed by A:

$$N(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$

Example 1.5. Let U be as in Example 1.2. Observe:

$$\begin{aligned} & & u_1 - 2u_2 &= 3u_3 \\ &\text{if and only if} & u_1 - 2u_2 - 3u_3 &= 0 \\ &\text{if and only if} & \begin{bmatrix} 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} &= [0]. \end{aligned}$$

Therefore,

$$U = N\left(\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}\right)$$

Theorem 1.6. *$N(A)$ is a subspace of $\mathbb{R}^n$.*

Proof. Closure under addition: Let $u_1, u_2 \in N(A)$. Then $Au_1 = 0$ and $Au_2 = 0$, by definition of $N(A)$. We compute:

$$A(u_1 + u_2) = Au_1 + Au_2 = 0 + 0 = 0.$$

Therefore, by the defintion of $N(A)$, $u_1 + u_2 \in N(A)$.

Closure under scalar multiplication: Let $t \in \mathbb{R}$ and let $u \in N(A)$. Then $Au = 0$, by definition of $N(A)$. We compute:

$$A(tu) = tAu = t0 = 0.$$

Therefore, by the defintion of $N(A)$, $tu \in N(A)$. $\square$

Exercise 1.7. Prove that $\{x \in \mathbb{R}^n : Ax = b\}$ is a subspace of $\mathbb{R}^n$ if and only if $b = 0$.

Definition 1.8. Let $\lambda \in \mathbb{R}$ and let $A \in \mathbb{R}^{m \times n}$. The λ-*eigenspace of* A, written $E_\lambda(A)$, is the set of all λ-eigenvectors of A:

$$E_\lambda(A) = \{x \in \mathbb{R}^n : Ax = \lambda x\}.$$

Corollary: $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.

Proof. $E_\lambda(A) = N(\lambda I - A)$, and nullspaces are subspaces. $\square$

Theorem 1.9. *Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then $N(A) = \{0\}$.*

Proof. Since $N(A)$ is a subspace of $\mathbb{R}^n$, $\{0\} \subseteq N(A)$. To prove the reverse inclusion, suppose that $x \in N(A)$, i.e., that $Ax = 0$. Since A is invertible,

$$x = A^{-1}Ax = A^{-1}0 = 0.$$

Therefore, $x \in \{0\}$. Since $x \in N(A)$ was arbitrary, $N(A) \subseteq \{0\}$. $\square$

1.2. **Orthogonal complement.**

Definition 1.10. Let S be a subset of $\mathbb{R}^n$. The *orthogonal complement of* S written $S^\perp$, is the set of vectors orthogonal to all elements of S:

$$S^\perp = \{v \in \mathbb{R}^n : u \cdot v = 0 \text{ for all } u \in S\}$$

Theorem 1.11. *$S^\perp$ is a subspace of $\mathbb{R}^n$.*

Proof. Closure under addition: Let $u_1, u_2 \in S^\perp$ and $v \in S$. We must show that $(u_1 + u_2) \cdot v = 0$. By definition of $S^\perp$, $u_1 \cdot v = 0$ and $u_2 \cdot v = 0$. Therefore,

$$(u_1 + u_2) \cdot v = u_1 \cdot v + u_2 \cdot v = 0 + 0 = 0,$$

as was to be shown.

Closure under scalar multiplication: Let $t \in \mathbb{R}$, let $u \in S^\perp$, and let $v \in S$. We must show that $(tu) \cdot v = 0$. By definition of $S^\perp$, $u \cdot v = 0$. Therefore,

$$(tu) \cdot v = t(u \cdot v) = t(0) = 0,$$

as was to be shown.

$\square$

The orthogonal complement of a finite set of vectors is a nullspce:

Theorem 1.12. *Suppose $S = \{a_1, \dots, a_k\}$ and let*

$$A = \begin{bmatrix} a_1 & \cdots a_k \end{bmatrix}.$$

Then

$$S^\perp = N(A^T).$$

Proof. Observe:

$$A^T x = \begin{bmatrix} a_1^T \\ \vdots \\ a_k^T \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ \vdots \\ a_k^T x \end{bmatrix} = \begin{bmatrix} a_1^T \cdot x \\ \vdots \\ a_k^T \cdot x \end{bmatrix}$$

Therefore, $A^T x = 0$ if and only if $a_j \cdot x = 0$ for all j, i.e., if and only if $x \in S^\perp$. Thus, $N(A^T) = S^\perp$. $\square$

We will see, later, that the orthogonal complement of *any* subset of $\mathbb{R}^n$ is a nullspace.

Example 1.13. Lines, planes

1.3. **Image and column space.**

Definition 1.14. Let $A \in \mathbb{R}^{m \times n}$ and let U be a subspace of $\mathbb{R}^n$. The *image of U under A*, written AU, is the set of all matrix-vector products Au for $u \in U$:

$$AU = \{Au : u \in U\} \subseteq \mathbb{R}^m.$$

Theorem 1.15. *AU is a subspace of $\mathbb{R}^m$.*

The image of $U = \mathbb{R}^n$ plays a special role and gets a special name.

Definition 1.16. Let $A \in \mathbb{R}^{m \times n}$. The image of $\mathbb{R}^n$ under A is called the *column space of* A and written $C(A)$:

$$C(A) = A\mathbb{R}^n = \{Ax : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

Theorem 1.17. *Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then $C(A) = \mathbb{R}^n$.*

Proof. Clearly, $C(A) \subset \mathbb{R}^n$. To prove the reverse inclusion, let $b \in \mathbb{R}^n$. Since A is invertible, we may set $x = A^{-1}b$. Then $b = Ax$, so $b \in C(A)$, by definition of $C(A)$. Since $b \in \mathbb{R}^n$ was arbitrary, $\mathbb{R}^n \subseteq C(A)$. ☐

Remark 1.18. We'll see, in ??, that A is invertible *if and only if* $C(A) = \mathbb{R}^n$.

1.4. **Linear combinations.**

Definition 1.19. Let $u_1, \dots, u_k \in \mathbb{R}^n$. A *linear combination of* $u_1, \dots, u_k$ is a vector of the form

$$t_1u_1 + \cdots + t_ku_k,$$

where $t_j \in \mathbb{R}$.

Example 1.20. The sum $u_1 + u_2$ is a linear combination of u_1 and u_2. The scalar multiple tu is a linear combination of u.

Example 1.21. Let $a_1, \dots, a_n \in \mathbb{R}^m$ and let $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$. Then the linear combinations of $a_1, \dots, a_n$ are precisely the vectors of the form Ax, for $x \in \mathbb{R}^n$:

$$x_1a_1 + \cdots + x_na_n = Ax.$$

Thus, the column space of A is the set of linear combinations of the column vectors of A:

$$C(A) = \{x_1a_1 + \cdots + x_na_n : x_1, \dots, x_n \in \mathbb{R}\}.$$

Exercise 1.22. Let S be a subset of $\mathbb{R}^n$. Prove the following statements.

(1) The sum of two linear combinations of elements of S is a linear combination of elements of S.
(2) A scalar multiple of a linear combination of elements of S is a linear combination of elements of S.
(3) A linear combination of linear combinations of elements of S is a linear combination of elements of S.

Theorem 1.23. *Let U be a subset of $\mathbb{R}^n$. Then U is a subspace of $\mathbb{R}^n$ if and only if U is* closed under linear combinations*, i.e., if and only if every linear combination of (finitely many) elements of U is, itself, an element of U.*

Corollary 1.24. *Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Then $C(A)$ is a subspace of $\mathbb{R}^m$.*

Example 1.25. Solution of a homogeneous system are linear combinations of *basic solutions*. (We'll give a more satisfying definition of basic solution later.)

1.5. **Span.**

Definition 1.26. Let S be a subset of $\mathbb{R}^n$. The *span of* S, written $\langle S \rangle$, is the set of all linear combinations of elements of S.

Theorem 1.27. *$\langle S \rangle$ is a subspace of $\mathbb{R}^n$.*

Proof. Do Exercise 1.22. ☐

Exercise 1.28. Prove that $\langle S \rangle$ is the smallest subspace of $\mathbb{R}^n$ containing S, i.e., that if U is a subspace of $\mathbb{R}^n$ and $S \subseteq U$ then $\langle S \rangle \subseteq U$. (Use Theorem 1.23.)

Exercise 1.29. Let S be a subset of $\mathbb{R}^n$. Prove that $S \subseteq \langle S \rangle$.

Theorem 1.30. *Let U be a subset of $\mathbb{R}^n$. Then U is a subspace of $\mathbb{R}^n$ if and only if $U = \langle U \rangle$.*

Proof. Suppose U is a subspace of $\mathbb{R}^n$. By Exercise 1.29, $U \subseteq \langle U \rangle$. By hypothesis, U is a subspace of $\mathbb{R}^n$; U obviously contains U. Therefore, by Exercise 1.29, $\langle U \rangle \subseteq U$. Having proved both inclusions, we conclude that $U = \langle U \rangle$.

Conversely, suppose $U = \langle U \rangle$. Then U is a subspace of $\mathbb{R}^n$ by Theorem 1.27 □

Exercise 1.31. Let S and T be subsets of $\mathbb{R}^n$. Prove that if $S \subseteq T$ then $\langle S \rangle \subseteq \langle T \rangle$. Is the converse true?

Theorem 1.32. *Let $u, v_1, \ldots, v_k \in \mathbb{R}^n$. Then $u \in \langle v_1, \ldots, v_k \rangle$ if and only if*

$$\langle v_1, \ldots, v_k \rangle = \langle u, v_1, \ldots, v_k \rangle. \tag{1}$$

Proof. Suppose $u \in \langle v_1, \ldots, v_k \rangle$. We must prove identity (1). By Exercise 1.31,

$$\langle v_1, \ldots, v_k \rangle \subseteq \langle u, v_1, \ldots, v_k \rangle.$$

To prove the reverse inclusion, let $x \in \langle u, v_1, \ldots, v_k \rangle$. Then there are scalars $r, s_1, \ldots, s_k$ such that

$$x = ru + s_1 v_1 + \cdots s_k v_k. \tag{2}$$

Since $u \in \langle v_1, \ldots, v_k \rangle$, by hypothesis, there are scalars $t_1, \ldots, t_k \in \mathbb{R}$ such that

$$u = t_1 v_1 + \cdots t_k v_k. \tag{3}$$

Substituting (3) into (2), we get

$$\begin{aligned} x &= r(t_1 v_1 + \cdots t_k v_k) + s_1 v_1 + \cdots s_k v_k \\ &= (rt_1 + s_1) v_1 + \cdots (rt_k + s_k) v_k. \end{aligned}$$

showing that x is a linear combination of $v_1, \ldots v_k$. Therefore, $x \in \langle v_1 \ldots, v_k \rangle$. Since $x \in \langle u, v_1, \ldots, v_k \rangle$ was chosen arbitrarily,

$$\langle u, v_1, \ldots, v_k \rangle \subseteq \langle v_1, \ldots, v_k \rangle,$$

completing the proof of (1).

Conversely, suppose that (1) holds. We must show that $u \in \langle v_1, \ldots, v_k \rangle$. But this is clear:

$$\begin{aligned} u &\in \langle u, v_1, \ldots, v_k \rangle && \text{by Exercise 1.29} \\ &= \langle v_1, \ldots, v_k \rangle && \text{by (1)} \end{aligned}$$ □

Definition 1.33. Let S be a subset of $\mathbb{R}^n$ and let U be a subspace of $\mathbb{R}^n$. We say that S *spans* U or that S *is a spanning set of* U if $U = \langle S \rangle$.

Exercise 1.34. Find vectors $v_1, \ldots, v_k$ such that $\{v_1, \ldots, v_k\}$ spans $N(A)$, where $A = ...$

Exercise 1.35. Find a finite spanning set for $E_\lambda(A)$, where $A = ...$ and $\lambda = ...$

Exercise 1.36. Find a vector v_1 such $\{v_1\}$ spans $\left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}^\perp$.

Exercise 1.37. Find a vector v_1 such $\{v_1\}$ spans $\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}^\perp$.

Exercise 1.38. Can you find two vectors v_1 and v_2 such $\{v_1, v_2\}$ spans $\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right\}^\perp$. Two *unit vectors*? Two *orthogonal* vectors? Two orthogonal unit vectors? A single vector?

Exercise 1.39. Suppose that $\{u_1, \ldots, u_k\}$ spans U. Prove that $\{Au_1, \ldots, Au_k\}$ spans AU.

1.6. **Column space, again.**

Example 1.40. By Example 1.21, the column space of a matrix is the span of its column vectors:

$$C\left(\begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}\right) = \langle a_1, \dots, a_k \rangle.$$

Theorem 1.41. *$C(A)$ is spanned by its pivot columns.*

We prove this theorem with the help of two lemmas ("helper theorems").

Lemma 1.42. *Let $v_1 \dots, v_k \in \mathbb{R}^n$ and let $X \in \mathbb{R}^{n\times n}$.*

$$\langle Xv_1, \dots, Xv_k \rangle = X\langle v_1, \dots, v_k \rangle.$$

Proof. Let

$$u \in \langle Xv_1, \dots, Xv_k \rangle,$$

with the goal of showing that

$$u \in X\langle v_1, \dots, v_k \rangle.$$

By definition of $\langle Xv_1, \dots, Xv_k \rangle$, there are scalars $t_1, \dots, t_k \in \mathbb{R}$ such that

$$u = t_1 X v_1 + \cdots t_k X v_k.$$

But $t_j X v_j = X(t_j v_j)$, so

$$u = X(t_1 v_1) + \cdots X(t_k v_k).$$

Therefore, setting

$$v := t_1 v_1 + \cdots t_k v_k,$$

we have

$$u = X(t_1 v_1 + \cdots t_k v_k) = Xv.$$

Evidently, $v \in \langle v_1, \dots, v_k \rangle$. Thus, $u = Xv$ with $v \in \langle v_1, \dots, v_k \rangle$. Therefore,

$$u \in X\langle v_1, \dots, v_k \rangle.$$

Since $u \in \langle Xv_1, \dots, Xv_k \rangle$ was arbitrary, we conclude that

$$\langle Xv_1, \dots, Xv_k \rangle \subseteq X\langle v_1, \dots, v_k \rangle.$$

Conversely, Let

$$u \in X\langle v_1, \dots, v_k \rangle,$$

with the goal of showing that

$$u \in \langle Xv_1, \dots, Xv_k \rangle.$$

Then there is an element $v \in \langle v_1, \dots, v_k \rangle$ such that $u = Xv$. As $v \in \langle v_1, \dots, v_k \rangle$, there are scalars $t_1, \dots, t_k$ such that

$$v = t_1 v_1 + \cdots t_k v_k.$$

Therefore,

$$\begin{aligned} u &= Xv \\ &= X(t_1 v_1 + \cdots + t_k v_k) \\ &= X(t_1 v_1) + \cdots X(t_k v_k) \\ &= t_1(Xv_1) + \cdots + t_k(Xv_k) \\ &\in \langle Xv_1, \dots, Xv_k \rangle. \end{aligned}$$

Since $u \in X\langle v_1, \dots, v_k \rangle$ was arbitrary, it follows that

$$X\langle v_1, \dots, v_k \rangle \subseteq \langle Xv_1, \dots, Xv_k \rangle.$$

Having proved the reverse inclusion above, statement (1) is proved.

$\square$

Lemma 1.43. *Theorem 1.41 holds when A is in reduced row echelon form.*

Proof. We must show that the nonpivot columns of A belong to the span of the pivot columns of A. Let $(1, j_i)$, …, (r, j_r) be the positions of the leading ones of A, so that $a_{j_1}, \ldots, a_{j_r}$ are the pivot columns of A. Since A is in reduced row echelon form, a leading one of A is the only nonzero element in its column. Therefore, $a_{j_i} = e_i \in \mathbb{R}^m$. Thus, it suffices to show that the nonpivot columns of A belong to $\langle e_1, \ldots, e_r \rangle \subseteq \mathbb{R}^m$.

Suppose a_j is a nonpivot column of A. Suppose that q pivot columns of A lie to the left of a_j. Then $1 \le q \le r$. (Why?) As A is in reduced row echelon form, $a_{i,j} = 0$ for $i > q$. Thus,

$$\begin{aligned}
a_j &= a_{1,j}e_1 + \cdots + a_{m,j}e_m && \text{(property of } \{e_1, \ldots, e_m\}) \\
&= a_{1,j}e_1 + \cdots + a_{q,j}e_q && \text{(as } a_{i,j} = 0 \text{ for } i > q) \\
&\in \langle e_1, \ldots, e_q \rangle && \text{(by definition of } \langle e_1, \ldots, e_q \rangle) \\
&\subseteq \langle e_1, \ldots, e_r \rangle && \text{(as } q \le r),
\end{aligned}$$

establishing the claim. □

Proof of Theorem 1.41. Let $\gamma \in \mathbb{R}^{m \times m}$ be an invertible matrix such that $B = \gamma A$ is in reduced row echelon form. Write

$$B = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}.$$

Let $(1, j_i)$, …, (r, j_r) be the positions of the leading ones of B, so that $a_{j_1}, \ldots, a_{j_r}$ are the pivot columns of B. Then, by Lemma ??,

$$\langle b_1, \ldots, b_n \rangle = \langle b_{j_1}, \ldots, b_{j_r} \rangle.$$

Write

$$A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}.$$

Since $B = \gamma A$, $b_j = \gamma a_j$ for all j. Substituting in to the above identity, we get

$$\langle \gamma a_1, \ldots, \gamma a_n \rangle = \langle \gamma a_{j_1}, \ldots, \gamma a_{j_r} \rangle.$$

By Lemma ??,

$$\langle \gamma a_1, \ldots, \gamma a_n \rangle = \gamma \langle a_1, \ldots, a_n \rangle$$

and

$$\langle \gamma a_{j_1}, \ldots, \gamma a_{j_r} \rangle = \gamma \langle a_{j_1}, \ldots, a_{j_r} \rangle.$$

Therefore,

$$\gamma \langle a_1, \ldots, a_n \rangle = \gamma \langle a_{j_1}, \ldots, a_{j_r} \rangle$$

Cancelling the γs (see Exercise 1.44), we get

$$\langle a_1, \ldots, a_n \rangle = \langle a_{j_1}, \ldots, a_{j_r} \rangle \qquad \square.$$

Exercise 1.44. Let U and V be subspaces of $\mathbb{R}^n$.

(1) Let $X \in \mathbb{R}^{m \times n}$. Prove: if $U \subseteq V$ then $XU \subseteq XV$.
(2) Show, by example, that $XU \subseteq XV$ need not imply $U \subseteq V$.
(3) Let $\gamma \in \mathbb{R}^{n \times n}$ be an invertible matrix. Prove: $U \subseteq V$ if and only if $\gamma U \subseteq \gamma V$.

1.7. **Sums of subspaces.**

Definition 1.45. Let U and V be subspaces of $\mathbb{R}^n$. The *sum of U and V*, written $U + V$ is the set of sums $u + v$ for $u \in U$ and $v \in V$:

$$U + V = \{u + v : u \in U, \, v \in V\} \subseteq \mathbb{R}^n.$$

Theorem 1.46. *$U + V$ is a subspace of $\mathbb{R}^n$.*

Theorem 1.47. $\langle S \rangle + \langle T \rangle = \langle S \cup T \rangle$.

Definition 1.48. Let $U_1, \ldots, U_k$ be subspaces of $\mathbb{R}^n$. The *sum of $U_1, \ldots, U_k$* is the set of sums $u_1 + \cdots + u_k$, where $u_j \in U_j$:

$$\sum_{j=1}^{k} U_j = U_1 + \cdots + U_k = \{u_1 + \cdots + u_k : u_1 \in U_1, \ldots, u_k \in U_k\}.$$

Theorem 1.49. *$U_1 + \cdots + U_k$ is a subspace of $\mathbb{R}^n$.*

Exercise 1.50. $\langle S_1 \rangle + \cdots + \langle S_k \rangle = \langle S_1 \cup \cdots \cup S_k \rangle$

Exercise 1.51. $\langle u_1, \ldots, u_k \rangle = \langle u_1 \rangle + \cdots + \langle u_k \rangle$

Exercise 1.52. Prove that $(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp$. Generalize to k subspaces.

Exercise 1.53. Prove that $A(U_1 + U_2) = AU_1 + AU_2$ and that $A^{-1}(V_1 + V_2) = A^{-1}V_1 + A^{-1}V_2$. Generalize to k summands.

2. Linear dependence and independence

Definition 2.1. Let $a_1, \ldots, a_n$ be a list of vectors in $\mathbb{R}^m$. A *linear dependence relation among* $a_1, \ldots, a_n$ is an identity of the form

$$x_1 a_1 + \cdots x_n a_n = 0,$$

where $x_1, \ldots, x_n \in \mathbb{R}$. Such a relation is *trivial* if $x_1 = 0, \ldots, x_n = 0$.

Remark 2.2. Writing

$$A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{m \times n}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m,$$

and noting that

$$x_1 a_1 + \cdots x_n a_n = Ax,$$

we see that a linear dependence relation among $v_1, \ldots, v_n$ is nothing more than an identity of the form

$$Ax = 0.$$

This linear relation is trivial if and only if $x = 0$.

Definition 2.3.

1. A finite list $a_1, \ldots, a_n$ of vectors in $\mathbb{R}^m$ is *linearly independent* if the only linear dependence relation among them is the trivial one. Otherwise, the list is said to be *linearly dependent*.
2. A set S (possibly infinite!) of vectors in $\mathbb{R}^n$ is *linearly independent* if, for every list finite $a_1, \ldots, a_n$ of pairwise distinct vectors drawn from S, the only linear dependence relation among $a_1, \ldots, a_n$ is the trivial one. Otherwise, is said to be *linearly dependent*.

(Lists are allowed to have repeated entries. Sets have no notion of repetition.)

Remark 2.4. Let $S = \{a_1, \ldots, a_n\}$ be a finite set of vectors in $\mathbb{R}^m$. Suppose $a_i \neq a_j$ if $i \neq j$. Then the set S is linearly independent if and only if the list $a_1, \ldots, a_n$ is linearly independent. (Why?)

Theorem 2.5.

1. *A list of vectors containing repeated entries is linearly dependent.*
2. *A list or set containing the zero vector is linearly dependent.*

Proof.

1. Let $a_1, \ldots, a_n$ be a list of vectors in $\mathbb{R}^m$. Suppose that $a_i = a_j$, where $1 \leq i < j \leq n$. Then the vectors $a_1, \ldots, a_n$ satisfy the nontrivial linear dependence relation

   $$x_1 a_1 + \cdots x_n a_n = 0,$$

   where $x_i = 1$, $x_j = -1$, and $x_k = 0$ if $k \neq i, j$. Therefore, the vectors $a_1, \ldots, a_n$ are linearly dependent.

(2) Let $a_1, \ldots, a_n$ be a list of vectors. Suppose it contains the zero vector; say $a_i = 0$, where $1 \leq i \leq n$. Then the vectors $a_1, \ldots, a_n$ satisfy the nontrivial linear dependence relation

$$x_1 a_1 + \cdots x_n a_n = 0,$$

where $x_i = 1$ and $x_j = 0$ if $j \neq i$. A set S containing the zero vector is linearly dependent as the list of elements drawn from S with the single entry 0 contains no repeated entries and is linearly dependent, by the above argument.

□

Example 2.6. The list of vectors $e_1, \ldots, e_m$ are linearly independent, where $e_j \in \mathbb{R}^m$ is the j-th standard basis vector.

To see this, observe that

$$x_1 e_1 + \cdots x_m e_m = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}.$$

Thus, $x_1 e_1 + \cdots + x_m e_m = 0$ if and only if the x_j are all zero. In other words, the only linear dependence relation satisfied by the vectors $e_1, \ldots, e_m$ is the trivial one.

Theorem 2.7. *Vectors $a_1, \ldots, a_n$ in $\mathbb{R}^m$ are linearly dependent if and only if the nullspace of the matrix*

$$A := \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$$

is nonzero.

Proof. By Remark 2.2, the vectors $a_1, \ldots, a_n$ satisfy a nontrivial linear dependence relation, i.e., are linearly dependent, if and only if $Ax = 0$ for some nonzero $x \in \mathbb{R}^n$, i.e., if and only if $N(A) \neq \{0\}$. □

Exercise 2.8.

(1) Prove that a superlist (resp., superset) of a linearly dependent list (resp., set) of vectors is linearly dependent.
(2) Prove that a sublist (resp., subset) of a linearly independent list (resp., set) of vectors is linearly independent.

Corollary 2.9. *Let A be an invertible matrix. Then the column vectors of A are linearly independent.*

Proof. Since A is invertible, $N(A) = \{0\}$ by Theorem 1.9. Therefore, by Theorem 2.7, the column vectors of A are linearly independent. □

Remark 2.10. We will see, in ??, that a square matrix A is invertible *if and only if* its column vectors are linearly independent.

Corollary 2.11. *Suppose $u_1, \ldots, u_{k+1}$ are distinct vectors in $\langle v_1, \ldots, v_k \rangle$. Then $\{u_1, \ldots, u_{k+1}\}$ is linearly dependent.*

Corollary 2.12. *A linearly independent set in $\mathbb{R}^n$ has at most n elements.*

Theorem 2.13. *Let $\gamma \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly independent if and only if the vectors $\gamma v_1, \ldots, \gamma v_k$ are.*

Proof. Suppose $v_1, \ldots, v_k$ are linearly independent. We want to show that $\gamma v_1, \ldots, \gamma v_k$ are, too Suppose $\gamma v_1, \ldots, \gamma v_k$ satisfy the linear dependence relation

$$0 = t_1(\gamma v_1) + \cdots t_k(\gamma v_k). \tag{4}$$

We'll show that this linear dependence relation is, necessarily, trivial. Since $t_j(\gamma v_j) = \gamma(t_j v_j)$,

$$\begin{aligned} 0 &= \gamma(t_1 v_1) + \cdots \gamma(t_k v_k) \\ &= \gamma(t_1 v_1 + \cdots + t_k v_k). \end{aligned}$$

Multiplying both sides of this identity by γ^{-1} yields

$$0 = t_1 v_1 + \cdots + t_k v_k.$$

Since $v_1, \ldots, v_k$ are linearly independent, the t_j must all be zero. Thus, the linear dependence relation (4) is trivial, as was to be shown.

Conversely, suppose the vectors $\gamma v_1, \ldots, \gamma v_k$ are linearly independent. By the above argument, applied with γv_j playing the role of v_j and γ^{-1} playing the role of γ, the vectors

$$\gamma^{-1}\gamma v_1, \ldots, \gamma^{-1}\gamma v_k$$

are linearly independent. But $\gamma^{-1}\gamma v_j = v_j$, so the vectors $v_1, \ldots, v_k$ are linearly independent, as was to be shown. □

Corollary 2.14. *The pivot columns of a matrix are linearly independent.*

Proof. Let γ be an invertible matrix such that $R := \gamma A$ is in reduced row echelon form. Let r be the number of nonzero rows in R. Then the pivot columns of R are $e_1, \ldots, e_r \in \mathbb{R}^m$, clearly linearly independent. The pivot columns of A are $\gamma^{-1}e_1, \ldots, \gamma^{-1}e_r$. These vectors are linearly independent by Theorem 2.13. □

Exercise 2.15. Let $A \in \mathbb{R}^{m\times n}$ and let $v_1 \ldots, v_k \in \mathbb{R}^n$.

(1) Prove that if $v_1, \ldots, v_k$ are linearly dependent, then so are $Av_1, \ldots, Av_k$.
(2) If $Av_1, \ldots, Av_k$ are linearly dependent (resp., independent), does it follow that $v_1, \ldots, v_k$ are?

Linearly independent sets are "minimal" spanning sets:

Theorem 2.16. *Let $S = \{v_1, \ldots, v_k\}$ be a set of vectors in $\mathbb{R}^n$. Then S is linearly dependent if and only if*

$$\langle S \setminus \{v_j\}\rangle = \langle S\rangle,$$

for some j with $1 \le j \le k$.

Proof. Suppose S is linearly dependent. Then there are scalars $t_1, \ldots, t_k$, not all zero, such that

$$0 = t_1 v_1 + \cdots t_k v_k.$$

We may assume, without loss of generality, that $t_k \neq 0$. Then

$$\begin{aligned} v_k &= -\frac{t_1}{t_k}v_1 - \cdots - \frac{t_{k-1}}{t_k}v_{k-1} \\ &\in \langle v_1, \ldots, v_{k-1}\rangle. \end{aligned}$$

By Theorem 1.32,

$$\langle v_1, \ldots, v_{k-1}\rangle = \langle v_1, \ldots, v_{k-1}, v_k\rangle.$$

Therefore,

$$\begin{aligned} \langle S \setminus \{v_k\}\rangle &= \langle v_1, \ldots, v_{k-1}\rangle \\ &= \langle v_1, \ldots, v_{k-1}, v_k\rangle \\ &= \langle S\rangle. \end{aligned}$$

as was to be shown.

Conversely, suppose that

$$\langle S \setminus \{v_j\}\rangle = \langle S\rangle.$$

for some j with $1 \le j \le k$. Then S is a k-element, spanning subset of $\langle S\rangle$, a subspace of $\mathbb{R}^n$ spanned by the $k-1$-element set $S \setminus \{v_j\}$. By Corollary 2.11, S is linearly dependent. □

Corollary 2.17. *Let U be a subspace of $\mathbb{R}^n$ and let S be a spanning subset of U. Then S is linearly independent if and only if, for every $v \in S$, the set $S \setminus \{v\}$ does not span U.*

Proof. Let S be a linearly independent, spanning subset of U and let T be a proper subset of U. We must show that T does not span U. Suppose, to the contrary, that T spans U. As $T \subsetneq S$, there is an element $u \in S$ with $u \notin T$.

Suppose the proper subset T of S spans U. We must show that S is linearly dependent. As $T \subsetneq S$, there is an element $u \in S$ with $u \notin T$. Since T spans U and

$$T \subseteq S \setminus \{u\} \subseteq U,$$

$S \setminus \{u\}$ must span U as well. As

$$u \in S \subset U = \langle S \setminus \{u\} \rangle,$$

the set

$$(S \setminus \{u\}) \cup \{u\}$$

is linearly dependent, by Theorem ??. But

$$(S \setminus \{u\}) \cup \{u\} = S,$$

so S is linearly dependent. $\square$

Theorem 2.18. *S is linearly independent if and only if every element of $\langle S \rangle$ can be written uniquely as a linear combination of elements of S.*

Definition 2.19. A set S of vectors in $\mathbb{R}^n$ is *orthogonal* if every pair of distinct vectors S are orthogonal, i.e.,

$$u \cdot v = 0 \text{ for all } u, v \in S \text{ with } u \neq v.$$

Theorem 2.20. *Let S be an orthogonal set in $\mathbb{R}^n$ with $0 \notin S$. Then S is linearly independent.*

2.1. **Linearly independent subspaces.**

2.1.1. *Two subspaces.*

Definition 2.21. Subspaces U_1 and U_2 of $\mathbb{R}^n$ are *linearly independent* if the pair (u_1, u_2) is linearly independent for all nonzero $u_1 \in U_1$ and all nonzero $u_2 \in U_2$.

Example 2.22. Let u_1 and u_2 be linearly independent vectors in $\mathbb{R}^n$. Then $\langle u_1 \rangle$ and $\langle u_2 \rangle$ are linearly independent. In other words, distinct lines are linearly independent.

To see this, let $x_1 \in \langle u_1 \rangle$ and let $x_2 \in \langle u_2 \rangle$. We need to show that x_1 and x_2 are linearly independent. So, suppose $s_1 x_1 + s_2 x_2 = 0$. We will show that $s_1 = 0$ and $s_2 = 0$. Since $x_1 \in \langle u_1 \rangle$, $x_1 = t_1 u_1$ for some nonero $t_1 \in \mathbb{R}$. Symmetrically, $x_2 = t_2 u_2$ for some nonzero $t_2 \in \mathbb{R}$. Substituting, we get

$$s_1 x_1 + s_2 x_2 = s_1 t_1 u_1 + s_2 t_2 u_2.$$

Since u_1 and u_2 are assumed linearly independent, we must have $s_1 t_1 = 0$ and $s_2 t_2 = 0$. Thus, $s_1 = 0$ and $s_2 = 0$ as t_1 and t_2 are nonzero.

Theorem 2.23. *U_1 and U_2 are linearly independent if and only if $U_1 \cap U_2 = \{0\}$.*

Example 2.24. Let

$$U = N\left(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\right), \quad V = N\left(\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\right).$$

Then

$$U \cap V = N\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\right) = \left\langle \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\rangle.$$

In particular $U \cap V \neq \{0\}$.

Exercise 2.25. Let u_1 and u_2 be linearly independent vectors in $\mathbb{R}^3$. Prove that $\{u_1\}^\perp$ and $\{u_2\}^\perp$ are linearly dependent. Give examples, to show that this result fails if $\mathbb{R}^3$ is replaced by $\mathbb{R}^n$ with $n > 3$.

Corollary 2.26. *Let λ_1 and λ_2 be distinct eigenvalues of A. Then the eigenspaces $E_{\lambda_1}(A)$ and $E_{\lambda_2}(A)$ are linearly independent.*

Proof. Do Exercise 2.27. $\square$

Exercise 2.27. Let λ_1 and λ_2 be distinct real numbers and let $A \in \mathbb{R}^{n\times n}$. Prove that

$$E_{\lambda_1}(A) \cap E_{\lambda_2}(A) = \{0\}.$$

Exercise 2.28. Let U_1 and U_2 be linearly independent subspaces of $\mathbb{R}^n$ and let S_1 and S_2 be subsets of U_1 and U_2, respectively. Then $S_1 \cup S_2$ is a linearly independent set if and only if both S_1 and S_2 are.

Definition 2.29. Subspaces U_1 and U_2 of $\mathbb{R}^n$ are orthogonal if every element of U_1 is orthogonal to every element of U_2, i.e.,

$$\text{if } u_1 \cdot u_2 = 0 \text{ for all } u_1 \in U_1 \text{ and all } u_2 \in U_2.$$

Example 2.30. Let U be a subspace of $\mathbb{R}^n$. Then U and $U^\perp$ are orthogonal.

Exercise 2.31. Let U_1 and U_2 are orthogonal subspaces of $\mathbb{R}^n$. Prove that U_1 and U_2 are linearly independent.

2.1.2. *k subspaces.*

Definition 2.32. Subspaces $U_1, \ldots, U_k$ of $\mathbb{R}^n$ are *linearly independent* if the sequence $(u_1, \ldots, u_k)$ is linearly independent for all $u_1 \in U_1, \ldots, u_k \in U_k$.

Theorem 2.33. *Let $A \in \mathbb{R}^{n\times n}$ and let Λ be the set of eigenvalues of A. Then the eigenspaces $E_\lambda(A)$, $\lambda \in \Lambda$, are linearly independent.*

Proof. Suppose not. Let Λ' be a subset of Λ, minimal with respect to the property that the eigenspaces $E_\lambda(A)$, $\lambda \in \Lambda'$, are linearly dependent. (This means: if Λ'' is a proper subset of Λ', then the $E_\lambda(A)$, $\lambda \in \Lambda''$, are linearly indepedent.) Note that $k \geq 2$. (Why?)

Suppose $\Lambda' = \{\lambda_1, \ldots, \lambda_k\}$ with the λ_j pairwise distinct. By the linear dependence of Λ', there are vectors $x_j \in E_{\lambda_j}(A)$ such that $\{x_1, \ldots, x_k\}$ is linearly dependent. Thus, there are scalars $t_1, \ldots, t_k$, not all zero, such that

$$0 = t_1 x_1 + \cdots t_k x_k. \tag{5}$$

In fact, by the minimality of Λ', $t_1, \ldots, t_k$ must *all* be nonzero. (Explain.) Multiply both sides of this identity by A:

$$0 = A(t_1 x_1 + \cdots t_k x_k) = t_1 A x_1 + \cdots + t_k A x_k.$$

Since x_j is a λ_j-eigenvector of A,

$$0 = t_1 \lambda_1 x_1 + \cdots t_k \lambda_k x_k. \tag{6}$$

Subtracing λ_k times (5) from (6) yields

$$\begin{aligned} 0 &= 0 - \lambda_k(0) \\ &= t_1\lambda_1 x_1 + \cdots t_k \lambda_k x_k - \lambda_k(t_1 x_1 + \cdots t_k x_k) \\ &= t_1(\lambda_1 - \lambda_k) + \cdots + t_{k-1}(\lambda_{k-1} - \lambda_k). \end{aligned}$$

(This the right hand side of this identity makes sense as $k \geq 2$.) Since the λ_j are pairwise distinct, $\lambda_j - \lambda_k \neq 0$ for $j \leq k-1$. The t_j being nonzero, $t_j(\lambda_j - \lambda_k) \neq 0$ for all $j \leq k-1$. Thus,

$$0 = t_1(\lambda_1 - \lambda_k) + \cdots + t_{k-1}(\lambda_{k-1} - \lambda_k)$$

is a nontrivial linear depedence relation among the eigenvectors $x_1, \ldots, x_{k-1}$ and, therefore, the eigenspaces $E_\lambda(A)$, $\lambda \in \Lambda'' := \{\lambda_1, \ldots, \lambda_{k-1}\}$, are linear dependent. But Λ'' has fewer elements than Λ', contradicting the minimality of the latter. $\square$

Theorem 2.34. *Let $U_1, \ldots, U_k$ be pairwise orthogonal subspaces of $\mathbb{R}^n$. Then $U_1, \ldots, U_k$ are linearly independent.*

Proof. Let $u_j \in U_j$ be a nonzero vector and suppose that

$$t_1 u_1 + \cdots t_k u_k = 0.$$

We must show that $t_j = 0$ for all j. Suppose $1 \leq j \leq k$. Take the dot product of each side of the above identity with u_j:

$$t_1(u_1 \cdot u_j) + \cdots + t_j(u_j \cdot u_j) + \cdots t_k(u_k \cdot u_j) = 0 \cdot u_j = 0.$$

Since the $U_1, \dots, U_k$ are pairwise orthogonal, $u_i \cdot u_j = 0$ if $i \neq j$. Therefore,

$$t_j(u_j \cdot u_j) = 0.$$

But $u_j \cdot u_j \neq 0$ as $u_j \neq 0$, so $t_j = 0$, as was to be shown. ☐

Exercise 2.35. Suppose $U_1, \dots, U_j$ are orthogonal. Prove that U_i and $\sum_{j \neq i} U_j$ are orthogonal for all i.

Theorem 2.36. *The following are equivalent for subspaces $U_1, \dots, U_k$ of $\mathbb{R}^n$:*

(1) *$U_1, \dots, U_k$ are linearly independent.*
(2) *U_i and $\sum_{j \neq i} U_i$ are linearly independent for all i, i.e.,*

$$U_i \cap \sum_{j \neq i} U_i = \{0\},$$

for all i.
(3) *Every element of $U_1 + \cdots U_k$ can be written uniquely in the form $u_1 + \cdots u_k$, where $u_1 \in U_1, \dots, u_k \in U_k$.*

Theorem 2.37. *Let $A \in \mathbb{R}^{n \times n}$ and let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of A. Then the eigenspaces $E_{\lambda_1}(A), \dots, E_{\lambda_k}(A)$ are linearly independent.*

3. Basis and dimension

Definition 3.1. Let U be a nonzero subspace of $\mathbb{R}^n$. A set B of vectors in U is a *basis of U* if B is linearly independent and B spans U.

Example 3.2. The set $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$.

Theorem 3.3. *Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then the set of column vectors of A is a basis of $\mathbb{R}^n$.*

Proof. By Corollary 2.9, the column vectors of A are linearly independent. To see that the column vectors of A span $\mathbb{R}^n$, observe:

$$\begin{aligned} \langle a_1, \dots, a_n \rangle &= C(A) && \text{(by definition of C(A))} \\ &= \mathbb{R}^n && \text{(by Theorem 1.17).} \end{aligned}$$ ☐

Theorem 3.4. *The set of pivot columns of A is a basis of $C(A)$.*

Proof. The pivot columns of A are linearly independent by Corollary 2.14. They span $C(A)$ by Theorem 1.41. ☐

Theorem 3.5. *Let U be a subspace of $\mathbb{R}^n$ and let S be a linearly independent subset of U. Then there is a basis B of U containing S.*

Proof. Let B be a linearly independent subset of U, continaing S, with the largest possible number of elements. Such a set exists as (i) a linearly independent subset of U containing S exists (S itself), and (ii) the sizes of linearly independent sets in $\mathbb{R}^n$ are bounded, by Corollary 2.12.

We will prove that B is a basis of U. Since B is linearly independent, by construction, we need only show that B spans U. Since $B \subset U$ and U is a subspace of $\mathbb{R}^n$, $\langle B \rangle \subseteq U$, by ??. To prove the reverse inclusion, let $u \in U$. If $u \notin \langle B \rangle$, when $B \cup \{u\}$ would be a linearly independent subset of U containing B, by Theorem ??, contradicting the maximality of B. Therefore, we must have $u \in \langle B \rangle$. Since $u \in U$ was arbitrary, we conclude that $U \subset \langle B \rangle$. Thus, B spans U, as was to be shown. ☐

Corollary 3.6. *Every nonzero subspace of* $\mathbb{R}^n$ *has a basis.*

Proof. Apply Theorem 3.5 with $S = \emptyset$. $\square$

Theorem 3.7. *Let* B_1 *and* B_2 *be bases of the subspace* U *of* $\mathbb{R}^n$*. Then* $|B_1| = |B_2|$.

Definition 3.8. Let U be a subspace of $\mathbb{R}^n$. The *dimension of* U, written $\dim U$, is the size of any basis of U.

Theorem 3.9. *Let* U *and* V *be subspaces of* $\mathbb{R}^n$ *with* $U \subseteq V$*. Then* $\dim U \leq \dim V$*, with equality holding if and only if* $U = V$*.*