Differential operators

- $(z, \bar{z})$ and (x, y) are related by in invertible, linear change of variable:

$$\begin{aligned} z &= x + iy \\ \bar{z} &= x - iy \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} x &= \frac{1}{2}(z + \bar{z}) \\ y &= \frac{1}{2i}(z - \bar{z}) \end{aligned}$$

- View $(z, \bar{z})$ as coordinates on $\mathbb{C} \cong \mathbb{R}^2$.

$$\frac{\partial x}{\partial z} = \qquad , \qquad \frac{\partial y}{\partial z} = \qquad , \qquad \frac{\partial x}{\partial \bar{z}} = \qquad , \qquad \frac{\partial y}{\partial \bar{z}} =$$

$$\frac{\partial x}{\partial z} = \frac{1}{2}, \qquad \frac{\partial y}{\partial z} = \frac{1}{2i}, \qquad \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \qquad \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i}$$

$$\begin{aligned}\frac{\partial f}{\partial z} &= \\ &= \\ &= \end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \\ &= \\ &= \end{aligned}$$

Wirtinger's differential operators:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Example: Compute $\dfrac{\partial z}{\partial z}$ and $\dfrac{\partial z}{\partial \bar{z}}$.

Exercise: Show that $\dfrac{\partial \bar{f}}{\partial z} = \overline{\dfrac{\partial f}{\partial \bar{z}}}$ and that $\dfrac{\partial \bar{f}}{\partial \bar{z}} = \overline{\dfrac{\partial f}{\partial z}}$.

The Cauchy-Riemann equations and the $\dfrac{\partial}{\partial \bar{z}}$ operator

Theorem: The Cauchy-Riemann equations hold for f if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Proof:

Corollary: Let f be a continuously differentiable function on the open set G. Then

$$f \text{ is analytic on } G \iff \frac{\partial f}{\partial \bar{z}} = 0 \text{ on } G,$$

in which case

$$\frac{df}{dz} = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = i\frac{\partial f}{\partial y} \quad \text{on } G.$$

Proof: The first statement follows from the above theorem and the Cauchy-Riemann theorem. You should check the second statement as an exercise.

Example: Show that $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$ is analytic on $\mathbb{C}$.

§2.5 Harmonic functions

A function u is *harmonic* on an open subset G of $\mathbb{C}$ if

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Theorem: If f is analytic on G and has continuous second-order partial derivatives[1], then $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic on G.

Proof:

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[1] Redundant hypothesis. We'll see later that analytic functions have continuous partials of all orders.

Let u be harmonic on G. A *harmonic conjugate* v of u is a (necessarily harmonic) function v on G such that $u + iv$ is analytic on G.

Example: Show that $u(x, y) = x^3 - 3xy^2 + 2xy$ is harmonic on $\mathbb{C}$. Then find a harmonic conjugate of u.

Example: Show that $u(x, y) = \frac{1}{2}\log(x^2 + y^2)$ is harmonic on $\mathbb{C} - \{0\}$. Then find a harmonic conjugate of u.

Example: Write $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ in the coordinates $(z, \bar{z})$.

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

Theorem: If f and g is analytic, then $f + \bar{g}$ is harmonic.

Proof:

Corollary: If f are analytic, then $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic.

Proof: