MATH 307 — Worksheet #1

1. Express the following in the form $x + iy$:

   (a) $\dfrac{2i - 1}{5 + 6i}$

   > **Solution:** $\dfrac{2i - 1}{5 + 6i} = \dfrac{(2i - 1)(5 - 6i)}{5^2 + 6^2} = \dfrac{1}{61}(7 + 16i)$

   (b) $(3 - 2i)^3$

   > **Solution:** $(3 - 2i)^3 = 3^3 - 3^2(2i) + 3(2i)^2 - (2i)^3 = -9 - 46i$

2. Let $z = x + iy$. Express the following in the form $u(x, y) + iv(x, y)$.

   (a) $1 - z^2$

   > **Solution:**
   > $$1 - z^2 = 1 - (x + iy)^2 = 1 - (x^2 - y^2 + 2ixy) = (1 - x^2 + y^2) - 2ixy$$

   (b) $\dfrac{1}{z^2}$

   > **Solution:**
   > $$\frac{1}{z^2} = \frac{\bar{z}^2}{|z^2|^2} = \frac{\bar{z}^2}{|z|^4} = \frac{(x - iy)^2}{(x^2 + y^2)^2} = \frac{\bar{z}^2}{|z|^4} = \frac{x^2 - y^2}{(x^2 + y^2)^2} + i\frac{-2xy}{(x^2 + y^2)^2}$$

   (c) z^3

   > **Solution:**
   > $$\frac{1}{z^3} = \frac{\bar{z}^3}{|z^3|^2} = \frac{\bar{z}^3}{|z|^6} = \frac{(x - iy)^3}{(x^2 + y^2)^3} =$$
   > $$\frac{x^3 + 3x(-iy)^2 + 3x^2(-iy) + (-iy)^3}{(x^2 + y^2)^3} = \frac{x^3 - 3xy^2}{(x^2 + y^2)^3} + i\frac{-3xy^2 + y^3}{(x^2 + y^2)^3}$$

3. Verify the identities $\operatorname{Re}(iz) = -\operatorname{Im}(z)$ and $\operatorname{Im}(iz) = \operatorname{Re}(z)$.

Solution:
$$\operatorname{Re}(iz) = \operatorname{Re}(i(x+iy)) = \operatorname{Re}(-y+ix) = -y = -\operatorname{Im} z$$
$$\operatorname{Im}(iz) = \operatorname{Im}(i(x+iy)) = \operatorname{Im}(-y+ix) = x = \operatorname{Re} z$$

4. For which z does the identity $\operatorname{Re}(z^2) = \operatorname{Re}(z)^2$ hold?

Solution: With $z = x + iy$, $\operatorname{Re}(z^2) = x^2 - y^2$ and $\operatorname{Re}(z)^2 = x^2$. Thus, $\operatorname{Re}(z^2) = \operatorname{Re}(z)^2$ holds if and only if $y = 0$, i.e., if and only if $z \in \mathbb{R}$.

5. Express $\dfrac{i^3(1-i)}{2(1+i\sqrt{3})}$ in the form $re^{i\theta}$ with $r > 0$ and $\theta \in [5\pi, 7\pi)$.

Solution: We have
$$r = \left| \frac{i^3(1-i)}{2(1+i\sqrt{3})} \right| = \frac{|i^3||1-i|}{|2||1+i\sqrt{3}|} = \frac{1\sqrt{2}}{2\sqrt{4}} = \frac{1}{2\sqrt{2}}$$
and
$$\arg \frac{i^3(1-i)}{2(1+i\sqrt{3})} = 3\arg i + \arg(1-i) - \arg 2 - \arg(1+i\sqrt{3}) + 2k\pi$$
$$= 3\frac{\pi}{2} - \frac{\pi}{4} - 0 - \frac{\pi}{3} + 2k\pi$$
$$= \frac{11\pi}{12} + 2k\pi.$$

As
$$11\pi/12 + 2k\pi \in [5\pi, 7\pi) \iff k = 3,$$
we set
$$\theta := \frac{11\pi}{12} + 6\pi = \frac{83\pi}{12} \in [5\pi, 7\pi).$$
Thus,
$$\frac{i^3(1-i)}{2(1+i\sqrt{3})} = \frac{1}{2\sqrt{2}}e^{83\pi i/12}.$$

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6. Let $a, b, c, d \in \mathbb{R}$ be such that $cd \neq 0$ and let $z \in \mathbb{C} \setminus \mathbb{R}$.

(a) Express $\operatorname{Im} \dfrac{az + b}{cz + d}$ in terms of $\operatorname{Im} z$.

> **Solution:**
>
> $$\frac{az + b}{cz + d} = \frac{(az + b)\overline{(cz + d)}}{|cz + d|^2} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz + d|^2}$$
>
> Note that $ac|z|^2$, bd, and $|cz + d|^2$ are real with $|cz + d|^2 > 0$. Therefore,
>
> $$\operatorname{Im} \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz + d|^2} = \operatorname{Im} \frac{adz + bc\bar{z}}{|cz + d|^2}$$
>
> $$= \frac{ad}{|cz + d|^2} \operatorname{Im} z + \frac{bc}{|cz + d|^2} \operatorname{Im} \bar{z} = \frac{ad - bc}{|cz + d|^2} \operatorname{Im} z.$$

(b) When is $\operatorname{Im} \dfrac{az + b}{cz + d}$ equal to 0?

> **Solution:** As $z \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Im} z \neq 0$. Therefore, $\operatorname{Im} \dfrac{az + b}{cz + d} = 0$ if and only if $ad - bc = 0$.

7. Describe and sketch the set solution set.

(a) $|z - i| = 2$

> **Solution:** The set of points at distance 2 from i, i.e., the circle centered at i with radius 2.

(b) $|z + i| = |z - 1|$

> **Solution:** The set of points equidistant from $-i$ and 1. These lie on the line $y = -x$.

(c) $|z + 2i| + |z - 2i| = 6$

> **Solution:** The set of points the sum of whose distances from $-2i$ and $2i$ is equal to 6 is an ellipse with foci at $-2i$ and $2i$. Let's find its radii. By symmetry, the

axes of the ellipse lie along the real and imaginary axes. Suppose $\pm ai$, $a > 0$, are the points of the ellipse along the imaginary axis. Then

$$6 = |-2i - ai| + |2i - ai| = (a+2) + (a-2) = 2a \implies a = 3$$

Suppose $\pm bi$, $b > 0$, are the points of the ellipse along the real axis. By the Pythagorean theorem,

$$\left(\frac{6}{2}\right)^2 = 2^2 + b^2 \implies b = \sqrt{5}$$

Thus, the Cartesian equation of the ellipse is

$$\left(\frac{x}{\sqrt{5}}\right)^2 + \left(\frac{y}{3}\right)^2 = 1.$$

(d) $|z + 3| - |z - 3| = 4$

Solution: The set of points the difference of whose distances from $(-3, 0)$ and $(3, 0)$ is equal to 4 is an ellipse with foci at $-3i$ and $3i$. Let $(\pm a, 0)$, $a > 0$, be the intersection points of this hyperbola with the x-axis. Since $(a, 0)$ is on the curve,

$$4 = |a + 3| - |a - 3| = a + 3 - (3 - a) = 2a \implies a = 2$$

Let $b = \sqrt{3^2 - a^2} = \sqrt{5}$. Then the hyperbola has cartesian equation

$$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{\sqrt{5}}\right)^2 = 1.$$

(e) $\operatorname{Im} z^2 = 4$

Solution: As $\operatorname{Im} z^2 = 2xy$,

$$\operatorname{Im} z^2 = 0 \iff 2xy = 0 \iff x = 0 \text{ or } y = 0.$$

Thus, the solution set of $\operatorname{Im} z^2 = 0$ is the union of the real and imaginary axes.

8. Solve the equation.

(a) $z^2 + 2z + (1 - i) = 0$

Solution: By the quadratic formula and a bit of algebra,

$$z = \frac{-2 \pm \sqrt{2^2 - 4(1)(1 - i)}}{2} = -1 \pm \sqrt{i}.$$

The square roots of i are $\pm \dfrac{1 + i}{\sqrt{2}}$. Therefore,

$$z = -1 \pm \frac{1 + i}{\sqrt{2}}.$$

(b) $z^2 + (2i - 3)z + 5 - i = 0$

Solution: By the quadratic formula a bit of algebra,

$$z = \frac{(3 - 2i) \pm \sqrt{-15 - 8i}}{2}.$$

To find the square roots of $-15 - 8i$, we convert to polar coordinates:

$$r := |-15 - 8i| = 17, \qquad \tan \theta = \frac{8}{15}$$

(We *cannot* write $\theta = \arctan(8/15)$. Why not?) Since $\pi/2 < \theta/2 < \pi$,

$$\cos \frac{\theta}{2} = -\sqrt{\frac{\cos \theta + 1}{2}} = -\sqrt{\frac{\frac{-15}{17} + 1}{2}} = -\frac{1}{\sqrt{17}}.$$

(It's easy to miss this minus sign!)

$$\sin \frac{\theta}{2} = \sqrt{1 - \cos^2 \frac{\theta}{2}} = \sqrt{1 - \frac{1}{17}} = \frac{4}{\sqrt{17}}$$

Therefore, the square roots of $-15 - 8i$ are

$$\pm \sqrt{r} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) = \pm \sqrt{17} \left(-\frac{1}{\sqrt{17}} + \frac{4i}{\sqrt{17}} \right) = \pm(-1 + 4i)$$

Therefore,

$$z = \frac{(3 - 2i) \pm (-1 + 4i)}{2} = 1 + i, \ 2 - 3i.$$

9. Given $x, y \in \mathbb{R}$, show that

$$a = \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} \quad b = \text{sign}(y)\sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}}.$$

is the unique solution to $(a + ib)^2 = x + iy$ with $a \geq 0$.

Solution: Suppose $(a + bi)^2 = x + iy$. Equating real and imaginary parts,

$$(a + bi)^2 = x + iy \iff (a^2 - b^2) + 2iab = x + iy. \iff a^2 - b^2 = x \text{ and } 2ab = y$$

We consider the case $y \neq 0$, the case $y = 0$ being simpler. If $y \neq 0$, then $a \neq 0$ and $b = y/2a$. Substituting this into $a^2 - b^2 = x$, we get

$$a^2 - \frac{y^2}{4a^2} = x \iff 4(a^2)^2 - 4x(a^2) - y^2 = 0$$

$$a^2 = \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} = \frac{x \pm \sqrt{x^2 + y^2}}{2}$$

Noting that $x - \sqrt{x^2 + y^2} < 0$,

$$a^2 = \frac{x - \sqrt{x^2 + y^2}}{2}$$

has no solution $a \in \mathbb{R}$. Thus,

$$a = \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}}.$$

We solve for b:

$$\frac{1}{\sqrt{2}a} = \sqrt{\frac{1}{x + \sqrt{x^2 + y^2}}} = \sqrt{\frac{x - \sqrt{x^2 + y^2}}{x^2 - (x^2 + y^2)}}$$

$$= \sqrt{\frac{-x + \sqrt{x^2 + y^2}}{y^2}} = \text{sign}(y)\frac{\sqrt{-x + \sqrt{x^2 + y^2}}}{y}$$

Thus,

$$b = \frac{y}{2a} = \text{sign}(y)\sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}}.$$

10. Prove the identities:

$$\cos z = \cosh(iz), \quad \cos(iz) = \cosh z, \quad \sin z = -i\sinh(iz), \quad \sin(iz) = i\sinh z$$

Solution: The identity $\cos z = \cosh(iz)$ follows directly from the definitions and

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{2} = -i\sinh(iz)$$

Replacing z with iz in these identities, we get

$$\cos(iz) = \cosh(i(iz)) = \cosh(-z) = \cosh z,$$

$$\sin(iz) = -i\sinh(i(iz)) = -i\sinh(-z) = i\sinh z.$$

11. Prove the identities:

$$\cos(x + iy) = \cos x \cosh y - i\sin x \sinh y,$$
$$\sin(x + iy) = \sin x \cosh y + i\cos x \sinh y$$

Solution: Using the identities proved above,

$$\begin{aligned}
\cos(x + iy) &= \cos x \cos(iy) - \sin x \sin(iy) \\
&= \cos x \cosh y - \sin x(i\sinh y) \\
&= \cos x \cosh y - i\sin x \sinh y,
\end{aligned}$$

$$\begin{aligned}
\sin(x + iy) &= \sin x \cos(iy) + \cos x \sin(iy) \\
&= \sin x \cosh y + \cos x(i\sinh y) \\
&= \sin x \cosh y + i\cos x \sinh y.
\end{aligned}$$

12. Prove the identity:
$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

Deduce that
$$\lim_{y\to\infty} |\cos z| = \infty \quad \text{and} \quad \lim_{y\to\infty} |\sin z| = \infty$$

Solution:

$$
\begin{aligned}
|\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\
&= \cos^2 x \cosh^2 y + (1 - \cos^2 x) \sinh^2 y \\
&= \cos^2 x (\cosh^2 y - \sinh^2 y) + \sinh^2 y \\
&= \cos^2 x + \sinh^2 y
\end{aligned}
$$

$$
\begin{aligned}
|\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
&= \sin^2 x \cosh^2 y + (1 - \sin^2 x) \sinh^2 y \\
&= \sin^2 x (\cosh^2 y - \sinh^2 y) + \sinh^2 y \\
&= \sin^2 x + \sinh^2 y
\end{aligned}
$$

Since $\sinh y \to \infty$ as $y \to \infty$ and $|\sinh y| \le |\cos z|, |\sin z|$ by the above , the statement about the limits follows.

13. Solve the equation $|\cot z| = 1$.

Solution: We have:
$$
\begin{aligned}
& |\cot z| = 1 \\
\Longleftrightarrow \quad & |\cos z|^2 = |\sin z|^2 \\
\Longleftrightarrow \quad & \cos^2 x + \sinh^2 y = \sin^2 x + \sinh^2 y \quad \text{(see above)} \\
\Longleftrightarrow \quad & \cos^2 x = \sin^2 x \\
\Longleftrightarrow \quad & \cos x = \pm \sin x \\
\Longleftrightarrow \quad & x = \pi \pm \frac{\pi}{4}
\end{aligned}
$$

Thus,
$$|\cot z| = 1 \Longleftrightarrow \operatorname{Re} z = k\pi \pm \frac{\pi}{4}.$$

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14. Solve the equations:

(a) $e^{\bar{z}} = \overline{e^z}$

Solution: We have:

$$e^{\bar{z}} = e^{x - iy} = e^x(\cos(-y)) + i\sin(-y))$$
$$= e^x(\cos y - i\sin y) = \overline{e^x(\cos y + i\sin y)} = \overline{e^z}.$$

Thus, this identity holds for all z.

(b) $\cos(i\bar{z}) = \overline{\cos(iz)}$

Solution: We showed above that $e^{\bar{z}} = \overline{e^z}$. Therefore,

$$\cos(i\bar{z}) = \cosh \bar{z} = \frac{e^{\bar{z}} + e^{-\bar{z}}}{2} = \frac{\overline{e^z + e^{-z}}}{2} = \overline{\cosh z} = \overline{\cos(iz)}$$

for all z.