MATH 307 – Supplementary problems

Matthew Greenberg

January 19, 2020

1 MATH 307 — Tutorial — 01.24.2020

1. Express the following in the form $x + iy$:

   (a) $\dfrac{2i-1}{5+6i}$

   Solution: $\dfrac{2i-1}{5+6i} = \dfrac{(2i-1)(5-6i)}{5^2+6^2} = \dfrac{1}{61}(7+16i)$

   (b) $(3-2i)^3$

   Solution: $(3-2i)^3 = 3^3 - 3^2(2i) + 3(2i)^2 - (2i)^3 = -9 - 46i$

2. Let $z = x + iy$. Express the following in the form $u(x,y) + iv(x,y)$.

   (a) $1 - z^2$

   Solution:

   $$1 - z^2 = 1 - (x+iy)^2 = 1 - (x^2 - y^2 + 2ixy) = (1 - x^2 + y^2) - 2ixy$$

   (b) $\dfrac{1}{z^2}$

   Solution:

   $$\frac{1}{z^2} = \frac{\overline{z^2}}{|z^2|^2} = \frac{\bar{z}^2}{|z|^4} = \frac{(x-iy)^2}{(x^2+y^2)^2} = \frac{\bar{z}^2}{|z|^4} = \frac{x^2-y^2}{(x^2+y^2)^2} + i\frac{-2xy}{(x^2+y^2)^2}$$

   (c) z^3

Solution:

$$\frac{1}{z^3} = \frac{\overline{z^3}}{|z^3|^2} = \frac{\bar{z}^3}{|z|^6} = \frac{(x-iy)^3}{(x^2+y^2)^3} =$$
$$\frac{x^3+3x(-iy)^2+3x^2(-iy)+(-iy)^3}{(x^2+y^2)^3} = \frac{x^3-3xy^2}{(x^2+y^2)^3} + i\frac{-3xy^2+y^3}{(x^2+y^2)^3}$$

3. Verify the identities $\operatorname{Re}(iz) = -\operatorname{Im}(z)$ and $\operatorname{Im}(iz) = \operatorname{Re}(z)$.

Solution:

$$\operatorname{Re}(iz) = \operatorname{Re}(i(x+iy)) = \operatorname{Re}(-y+ix) = -y = -\operatorname{Im} z$$
$$\operatorname{Im}(iz) = \operatorname{Im}(i(x+iy)) = \operatorname{Im}(-y+ix) = x = \operatorname{Re} z$$

4. For which z does the identity $\operatorname{Re}(z^2) = \operatorname{Re}(z)^2$ hold?

Solution: With $z = x+iy$, $\operatorname{Re}(z^2) = x^2 - y^2$ and $\operatorname{Re}(z)^2 = x^2$. Thus, $\operatorname{Re}(z^2) = \operatorname{Re}(z)^2$ holds if and only if $y = 0$, i.e., if and only if $z \in \mathbb{R}$.

5. Express $\dfrac{i^3(1-i)}{2(1+i\sqrt{3})}$ in the form $re^{i\theta}$ with $r > 0$ and $\theta \in [5\pi, 7\pi)$.

Solution: We have

$$r = \left|\frac{i^3(1-i)}{2(1+i\sqrt{3})}\right| = \frac{|i^3||1-i|}{|2||1+i\sqrt{3}|} = \frac{1\sqrt{2}}{2\sqrt{4}} = \frac{1}{2\sqrt{2}}$$

and

$$\begin{aligned}
\arg\frac{i^3(1-i)}{2(1+i\sqrt{3})} &= 3\arg i + \arg(1-i) - \arg 2 - \arg(1+i\sqrt{3}) + 2k\pi \\
&= 3\frac{\pi}{2} - \frac{\pi}{4} - 0 - \frac{\pi}{3} + 2k\pi \\
&= \frac{11\pi}{12} + 2k\pi.
\end{aligned}$$

As

$$11\pi/12 + 2k\pi \in [5\pi, 7\pi) \iff k = 3,$$

we set

$$\theta := \frac{11\pi}{12} + 6\pi = \frac{83\pi}{12} \in [5\pi, 7\pi).$$

Thus,

$$\frac{i^3(1-i)}{2(1+i\sqrt{3})} = \frac{1}{2\sqrt{2}} e^{83\pi i/12}.$$

6. Let $a, b, c, d \in \mathbb{R}$ be such that $cd \neq 0$ and let $z \in \mathbb{C} \setminus \mathbb{R}$.

   (a) Express $\operatorname{Im} \dfrac{az+b}{cz+d}$ in terms of $\operatorname{Im} z$.

   Solution:

   $$\frac{az+b}{cz+d} = \frac{(az+b)\overline{(cz+d)}}{|cz+d|^2} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2}$$

   Note that $ac|z|^2$, bd, and $|cz+d|^2$ are real with $|cz+d|^2 > 0$. Therefore,

   $$\begin{aligned} \operatorname{Im} \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2} &= \operatorname{Im} \frac{adz + bc\bar{z}}{|cz+d|^2} \\ &= \frac{ad}{|cz+d|^2} \operatorname{Im} z + \frac{bc}{|cz+d|^2} \operatorname{Im} \bar{z} = \frac{ad-bc}{|cz+d|^2} \operatorname{Im} z. \end{aligned}$$

   (b) When is $\operatorname{Im} \dfrac{az+b}{cz+d}$ equal to 0?

   Solution: As $z \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Im} z \neq 0$. Therefore, $\operatorname{Im} \dfrac{az+b}{cz+d} = 0$ if and only if $ad - bc = 0$.

7. Describe and sketch the set solution set.

   (a) $|z - i| = 2$

   Solution: The set of points at distance 2 from i, i.e., the circle centered at i with radius 2.

   (b) $|z + i| = |z - 1|$

   Solution: The set of points equidistant from $-i$ and 1. These lie on the line $y = -x$.

(c) $|z+2i|+|z-2i|=6$

Solution: The set of points the sum of whose distances from $-2i$ and $2i$ is equal to 6 is an ellipse with foci at $-2i$ and $2i$. Let's find its radii. By symmetry, the axes of the ellipse lie along the real and imaginary axes. Suppose $\pm ai$, $a>0$, are the points of the ellipse along the imaginary axis. Then

$$6=|-2i-ai|+|2i-ai|=(a+2)+(a-2)=2a \Longrightarrow a=3$$

Suppose $\pm bi$, $b>0$, are the points of the ellipse along the real axis. By the Pythagorean theorem,

$$\left(\frac{6}{2}\right)^2=2^2+b^2 \Longrightarrow b=\sqrt{5}$$

Thus, the Cartesian equation of the ellipse is

$$\left(\frac{x}{\sqrt{5}}\right)^2+\left(\frac{y}{3}\right)^2=1.$$

(d) $|z+3|-|z-3|=4$

Solution: The set of points the difference of whose distances from $(-3,0)$ and $(3,0)$ is equal to 4 is an ellipse with foci at $-3i$ and $3i$. Let $(\pm a,0)$, $a>0$, be the intersection points of this hyperbola with the x-axis. Since $(a,0)$ is on the curve,

$$4=|a+3|-|a-3|=a+3-(3-a)=2a \Longrightarrow a=2$$

Let $b=\sqrt{3^2-a^2}=\sqrt{5}$. Then the hyperbola has cartesian equation

$$\left(\frac{x}{2}\right)^2-\left(\frac{y}{\sqrt{5}}\right)^2=1.$$

(e) $\operatorname{Im} z^2=4$

Solution: As $\operatorname{Im} z^2=2xy$,

$$\operatorname{Im} z^2=0 \Longleftrightarrow 2xy=0 \Longleftrightarrow x=0 \text{ or } y=0.$$

Thus, the solution set of $\operatorname{Im} z^2=0$ is the union of the real and imaginary axes.

8. Solve the equation.

(a) $z^2 + 2z + (1 - i) = 0$

Solution: By the quadratic formula and a bit of algebra,

$$z = \frac{-2 \pm \sqrt{2^2 - 4(1)(1-i)}}{2} = -1 \pm \sqrt{i}.$$

The square roots of i are $\pm\dfrac{1+i}{\sqrt{2}}$. Therefore,

$$z = -1 \pm \frac{1+i}{\sqrt{2}}.$$

(b) $z^2 + (2i - 3)z + 5 - i = 0$

Solution: By the quadratic formula a bit of algebra,

$$z = \frac{(3-2i) \pm \sqrt{-15-8i}}{2}.$$

To find the square roots of $-15 - 8i$, we convert to polar coordinates:

$$r := |-15 - 8i| = 17, \qquad \tan\theta = \frac{8}{15}$$

(We *cannot* write $\theta = \arctan(8/15)$. Why not?) Since $\pi/2 < \theta/2 < \pi$,

$$\cos\frac{\theta}{2} = -\sqrt{\frac{\cos\theta + 1}{2}} = -\sqrt{\frac{\frac{-15}{17} + 1}{2}} = -\frac{1}{\sqrt{17}}.$$

(It's easy to miss this minus sign!)

$$\sin\frac{\theta}{2} = \sqrt{1 - \cos^2\frac{\theta}{2}} = \sqrt{1 - \frac{1}{17}} = \frac{4}{\sqrt{17}}$$

Therefore, the square roots of $-15 - 8i$ are

$$\pm\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \pm\sqrt{17}\left(-\frac{1}{\sqrt{17}} + \frac{4i}{\sqrt{17}}\right) = \pm(-1+4i)$$

Therefore,

$$z = \frac{(3-2i) \pm (-1+4i)}{2} = 1 + i,\ 2 - 3i.$$

9. Given $x, y \in \mathbb{R}$, show that

$$a = \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} \quad b = \operatorname{sign}(y)\sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}}. \tag{1}$$

is the unique solution to $(a + ib)^2 = x + iy$ with $a \geq 0$.

Solution: Suppose $(a + bi)^2 = x + iy$. Equating real and imaginary parts,

$$(a + bi)^2 = x + iy \iff (a^2 - b^2) + 2iab = x + iy. \iff a^2 - b^2 = x \text{ and } 2ab = y$$

We consider the case $y \neq 0$, the case $y = 0$ being simpler. If $y \neq 0$, then $a \neq 0$ and $b = y/2a$. Substituting this into $a^2 - b^2 = x$, we get

$$a^2 - \frac{y^2}{4a^2} = x \iff 4(a^2)^2 - 4x(a^2) - y^2 = 0$$

$$a^2 = \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} = \frac{x \pm \sqrt{x^2 + y^2}}{2}$$

Noting that $x - \sqrt{x^2 + y^2} < 0$,

$$a^2 = \frac{x - \sqrt{x^2 + y^2}}{2}$$

has no solution $a \in \mathbb{R}$. Thus,

$$a = \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}}.$$

We solve for b:

$$\frac{1}{\sqrt{2}a} = \sqrt{\frac{1}{x + \sqrt{x^2 + y^2}}} = \sqrt{\frac{x - \sqrt{x^2 + y^2}}{x^2 - (x^2 + y^2)}}$$

$$= \sqrt{\frac{-x + \sqrt{x^2 + y^2}}{y^2}} = \operatorname{sign}(y)\frac{\sqrt{-x + \sqrt{x^2 + y^2}}}{y}$$

Thus,

$$b = \frac{y}{2a} = \operatorname{sign}(y)\sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}}.$$

10. Prove the identities:

$$\cos z = \cosh(iz), \quad \cos(iz) = \cosh z, \quad \sin z = -i\sinh(iz), \quad \sin(iz) = i\sinh z$$

Solution: The identity $\cos z = \cosh(iz)$ follows directly from the definitions and

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{2} = -i\sinh(iz)$$

Replacing z with iz in these identities, we get

$$\cos(iz) = \cosh(i(iz)) = \cosh(-z) = \cosh z,$$

$$\sin(iz) = -i\sinh(i(iz)) = -i\sinh(-z) = i\sinh z.$$

11. Prove the identities:

$$\begin{aligned}\cos(x+iy) &= \cos x\cosh y - i\sin x\sinh y,\\ \sin(x+iy) &= \sin x\cosh y + i\cos x\sinh y\end{aligned}$$

Solution: Using the identities proved above,

$$\begin{aligned}\cos(x+iy) &= \cos x\cos(iy) - \sin x\sin(iy)\\ &= \cos x\cosh y - \sin x(i\sinh y)\\ &= \cos x\cosh y - i\sin x\sinh y,\end{aligned}$$

$$\begin{aligned}\sin(x+iy) &= \sin x\cos(iy) + \cos x\sin(iy)\\ &= \sin x\cosh y + \cos x(i\sinh y)\\ &= \sin x\cosh y + i\cos x\sinh y.\end{aligned}$$

12. Prove the identity:

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

Deduce that

$$\lim_{y\to\infty} |\cos z| = \infty \quad \text{and} \quad \lim_{y\to\infty} |\sin z| = \infty$$

Solution:

$$\begin{aligned}
|\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\
&= \cos^2 x \cosh^2 y + (1 - \cos^2 x) \sinh^2 y \\
&= \cos^2 x (\cosh^2 y - \sinh^2 y) + \sinh^2 y \\
&= \cos^2 x + \sinh^2 y
\end{aligned}$$

$$\begin{aligned}
|\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
&= \sin^2 x \cosh^2 y + (1 - \sin^2 x) \sinh^2 y \\
&= \sin^2 x (\cosh^2 y - \sinh^2 y) + \sinh^2 y \\
&= \sin^2 x + \sinh^2 y
\end{aligned}$$

Since $\sinh y \to \infty$ as $y \to \infty$ and $|\sinh y| \leq |\cos z|, \ |\sin z|$ by the above , the statement about the limits follows.

13. Solve the equation $|\cot z| = 1$

Solution: We have:

$$\begin{aligned}
& & |\cot z| &= 1 \\
&\Longleftrightarrow & |\cos z|^2 &= |\sin z|^2 \\
&\Longleftrightarrow & \cos^2 x + \sinh^2 y &= \sin^2 x + \sinh^2 y \quad \text{(see above)} \\
&\Longleftrightarrow & \cos^2 x &= \sin^2 x \\
&\Longleftrightarrow & \cos x &= \pm \sin x \\
&\Longleftrightarrow & x &= \pi \pm \frac{\pi}{4}
\end{aligned}$$

Thus,

$$|\cot z| = 1 \Longleftrightarrow \operatorname{Re} z = k\pi \pm \frac{\pi}{4}.$$

14. Find all solutions:

(a) $e^{\bar{z}} = \overline{e^z}$

Solution: We have:

$$e^{\bar{z}} = e^{x-iy} = e^x(\cos(-y)) + i\sin(-y))$$
$$= e^x(\cos y - i\sin y) = \overline{e^x(\cos y + i\sin y)} = \overline{e^z}.$$

Thus, this identity holds for all z.

(b) $\cos(i\bar{z}) = \overline{\cos(iz)}$

Solution: We showed above that $e^{\bar{z}} = \overline{e^z}$. Therefore,

$$\cos(i\bar{z}) = \cosh\bar{z} = \frac{e^{\bar{z}} + e^{-\bar{z}}}{2} = \frac{\overline{e^z + e^{-z}}}{2} = \overline{\cosh z} = \overline{\cos(iz)}$$

for all z.

2 The Complex Plane and Elementary Functions

1. Simplify:
   (a) $(2-3i)(i+8)$

   Solution: $(2-3i)(i+8) = 19 - 22i$

   (b) $(3-2i)^2$

   Solution: $(3-2i)^2 = 5 - 12i$

   (c) $(1+i)^4$

   Solution: $(1+i)^4 = 1^4 + 4i + 6i^2 + 4i^3 + i^4 = -4$

2. Find real and imaginary parts in terms of $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.
   (a) $\dfrac{z}{z+1}$

Solution:

$$\begin{aligned}\frac{z}{z+1} = \frac{x+iy}{(x+1)+iy} &= \frac{(x+iy)((x+1)-iy)}{(x+1)^2+y^2} \\ &= \frac{x(x+1)+y^2-ixy+i(x+1)y}{(x+1)^2+y^2} = \frac{x(x+1)+y^2+iy}{(x+1)^2+y^2}\end{aligned}$$

Therefore, $\operatorname{Re}\dfrac{z}{z+1} = \dfrac{x(x+1)+y^2}{(x+1)^2+y^2}$ and $\operatorname{Im}\dfrac{z}{z+1} = \dfrac{y}{(x+1)^2+y^2}$.

3. Is the identity $\operatorname{Re}(wz) = \operatorname{Re}(w)\operatorname{Re}(z)$ valid? Prove or give a counterexample.

Solution: No, it isn't. For example,

$$\operatorname{Re}(i \cdot i) = \operatorname{Re}(-1) = 1 \neq 1 = 1 \cdot 1 = \operatorname{Re}(i)\operatorname{Re}(i).$$

4. Evaluate $\left|\dfrac{i(i+1)^3(4i+5)}{(2+3i)^2}\right|$.

Solution:

$$\left|\frac{i(i+1)^3(4i+5)}{(2+3i)^2}\right| = \frac{|i||i+1|^3|4i+5|}{|2+3i|^2} = \frac{1 \cdot \sqrt{2}^3\sqrt{41}}{\sqrt{13}^2} = \frac{2\sqrt{82}}{13}$$

5. $z(\bar{z}+2) = 3$

Solution: The solutions of this equation are $z = -1$ and $z = 3$:

$$\begin{aligned} z(\bar{z}+2) = 3 &\iff |z|^2 + 2z - 3 = 0 \\ &\iff (x^2+y^2+2x-3) + 2iy = 0 \\ &\iff x^2+y^2+2x-3 = 0 \text{ and } y = 0 \\ &\iff x^2 - 2x - 3 = 0 \text{ and } y = 0 \\ &\iff (x = -1 \text{ or } x = 3) \text{ and } y = 0. \end{aligned}$$

6. Find the square roots of:

(a) i

Solution: By (1) with $x = 0$ and $y = 1$, the square roots of i are $\pm(a + ib)$, where

$$a = \frac{1}{\sqrt{2}}, \quad b = \frac{1}{\sqrt{2}}.$$

Alternatively, since $i = e^{i\pi/2}$, the square roots of i are

$$\pm e^{i\pi/4} = \pm\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right).$$

(b) $-i$

Solution: By (1) with $x = 0$ and $y = -1$, the square roots of $-i$ are $\pm(a + ib)$, where

$$a = \frac{1}{\sqrt{2}}, \quad b = -\frac{1}{\sqrt{2}}.$$

Alternatively, since $-i = e^{-i\pi/2}$, the square roots of i are

$$\pm e^{-i\pi/4} = \pm\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \pm\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right).$$

$$\cos\frac{\pi}{8}$$

(c) $1 + i$

Solution: By (1) with $x = 1$ and $y = 1$, the square roots of $1 + i$ are $\pm(a + ib)$, where

$$a = \sqrt{\frac{1 + \sqrt{2}}{2}}, \quad b = \sqrt{\frac{-1 + \sqrt{2}}{2}}. \qquad (*)$$

Alternatively, the square roots of $1 + i = \sqrt{2}e^{i\pi/4}$ are

$$\pm\sqrt[4]{2}e^{i\pi/8} = \pm\sqrt[4]{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

We express $\cos(\pi/8)$ and $\sin(\pi/8)$ in terms of radicals. Using the identity $\cos 2x = 2\cos^2 x - 1$,

$$\cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8} - 1$$

Solving for $\cos(\pi/8)$, we get

$$\cos\frac{\pi}{8} = \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\frac{\sqrt{2}+1}{\sqrt{2}}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} = \frac{\sqrt{\sqrt{2}+2}}{2}$$

By the Pythagorean theorem,

$$\sin^2\frac{\pi}{8} = 1-\cos^2\frac{\pi}{8} = 1-\frac{2+\sqrt{2}}{4} = \frac{2-\sqrt{2}}{4}.$$

Therefore,

$$\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

and

$$\pm\sqrt[4]{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right) = \pm 2^{-3/4}\left(\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}\right). \qquad (**)$$

I leave it to you to reconcile $(*)$ and $(**)$.

7. Find all solutions:
   (a) $z^2 + 4z - (4-6i) = 0$

Solution: By the quadratic formula and a bit of algebra,

$$z = -2 \pm \sqrt{8-6i}.$$

To find the square roots of $8-6i$, we convert to polar coordinates:

$$r := |8-6i| = \sqrt{8^2+6^2} = 10, \quad \theta := \arctan\left(-\frac{6}{8}\right)$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{\cos\theta+1}{2}} = \sqrt{\frac{\frac{8}{10}+1}{2}} = \frac{3}{\sqrt{10}}$$

$$\sin\frac{\theta}{2} = \sqrt{1-\cos^2\frac{\theta}{2}} = \sqrt{1-\frac{9}{10}} = \frac{1}{\sqrt{10}}$$

Therefore, the square roots of $8-6i$ are

$$\pm\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \pm\sqrt{10}\left(\frac{3}{\sqrt{10}} + \frac{i}{\sqrt{10}}\right) = \pm(3-i)$$

Therefore,

$$z = -4 \pm (3-i) = 1-i,\ -5+i.$$

(b) $z^4 - 1 = 0$

Solution:

$$z^4 - 1 = (z^2-1)(z^2+1) = (z-1)(z+1)(z-i)(z+i)$$

Therefore,

$$z = -1,\ 1,\ -i,\ i.$$

(c) $z^4 + 1 = 0$

Solution:

$$z^4 + 1 = z^4 - i^2 = (z^2 - i)(z^2 + i)$$

By 6(a), the square roots of i are $\pm(1+i)/\sqrt{2}$. By 6(b), the square roots of $-i$ are $\pm(1-i)/\sqrt{2}$ Thus, the solutions of $z^4 + 1 = 0$ are

$$z = \frac{\pm 1 \pm i}{\sqrt{2}},$$

with all four possible combinations of signs.

(d) $z^8 - 1 = 0$

Solution:

$$z^8 - 1 = (z^4 - 1)(z^4 + 1)$$

Therefore, by (b) and (c), the solutions of $z^8 - 1 = 0$ are

$$z = \pm 1,\ \pm i,\ \frac{\pm 1 \pm i}{\sqrt{2}} \text{ (all four sign combinations).}$$

(e) $x^4 - i = 0$

Solution:

$$z^4 - i = \left(z^2 - \frac{1}{\sqrt{2}}(1+i)\right)\left(z^2 + \frac{1}{\sqrt{2}}(1+i)\right)$$

By 6(c), the square roots of $1+i$ are $\pm(a+ib)$, where

$$a = \sqrt{\frac{1+\sqrt{2}}{2}}, \quad b = \sqrt{\frac{-1+\sqrt{2}}{2}}.$$

Therefore, the square roots of $(1+i)/\sqrt{2}$ are $\pm 2^{-1/4}(a+ib)$. The square roots of $-(1+i)/\sqrt{2}$ are i times the square roots of $(1+i)/\sqrt{2}$: $\pm 2^{-1/4}(-b+ia)$.

Thus, the solutions of $z^4 - i = 0$ are

$$z = \pm 2^{-1/4}(a+ib),\ \pm 2^{-1/4}(-b+ia).$$

8. Suppose $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$. Show that

$$\operatorname{Log}(wz) = \operatorname{Log} w + \operatorname{Log} z.$$

Solution: Since $\operatorname{Re} z > 0$, $-\pi/2 < \operatorname{Arg} z < \pi/2$. Thus, $-\pi < \operatorname{Arg} z + \operatorname{Arg} w < \pi$. It follows that $\operatorname{Arg}(wz) = \operatorname{Arg} w + \operatorname{Arg} z$. Therefore,

$$\begin{aligned}\operatorname{Log}(wz) = \log|wz| + i\operatorname{Arg}(wz) &= \log|w| + \log|z| + i(\operatorname{Arg} w + \operatorname{Arg} z)\\ &= (\log|w| + i\operatorname{Arg} w) + (\log|z| + i\operatorname{Arg} z) = \operatorname{Log} w + \operatorname{Log} z.\end{aligned}$$

9. Write all values of the following expressions in the form $x + iy$.

(a) $\operatorname{Log}(\operatorname{Log} i)$

Solution:

$$\operatorname{Log} i = \operatorname{Log}|i| + i\operatorname{Arg} i = \log 1 + \frac{i\pi}{2} = \frac{i\pi}{2}$$

$$\log(\operatorname{Log} i) = \operatorname{Log}\frac{i\pi}{2} = \log\left|\frac{i\pi}{2}\right| + i\arg\frac{i\pi}{2} = \log\frac{\pi}{2} + i\left(\frac{\pi}{2} + 2k\pi\right)$$

(b) $\sin(e^i)$

Solution:

$$\begin{aligned}\sin(e^i) &= \sin(\cos 1 + i\sin 1)\\ &= \sin(\cos 1)\cosh(\sin 1) + i\cos(\cos 1)\sinh(\sin 1)\end{aligned}$$

(c) $(-3)^{\sqrt{2}}$

Solution:

$$\begin{aligned}
(-3)^{\sqrt{2}} &= e^{\sqrt{2}\log(-3)} = e^{\sqrt{2}(\log|-3|+i\arg(-3))} \\
&= e^{\sqrt{2}(\log 3+i(\pi+2k\pi))} \\
&= e^{\sqrt{2}\log 3} e^{i(\pi+2k\pi)} \\
&= 3^{\sqrt{2}}\left(\cos\sqrt{2}(\pi+2k\pi) + i\sin\sqrt{2}(\pi+2k\pi)\right)
\end{aligned}$$

10. Find all solutions:

   (a) $e^{\bar{z}} = \overline{e^z}$

Solution: We have:

$$\begin{aligned}
e^{\bar{z}} = e^{x-iy} = e^x(\cos(-y)) + i\sin(-y)) \\
= e^x(\cos y - i\sin y) = \overline{e^x(\cos y + i\sin y)} = \overline{e^z}.
\end{aligned}$$

Thus, this identity holds for all z.

   (b) $\sinh z + \cosh z = i$

Solution: We have:

$$\sinh z + \cosh z = \frac{e^z - e^{-z}}{2} + \frac{e^z + e^{-z}}{2} = e^z$$

$e^z = i$ if and only if $z = i(\pi/2 + 2k\pi)$.

11. Find the argument of $\dfrac{3+3i}{\sqrt{3}+i}$ in the interval $[5\pi, 7\pi)$.

12. Using the principal branch of the logarithm, compute:

   (a) $\mathrm{Log}(1+i\sqrt{3})$

Solution:

$$\mathrm{Log}(1+i\sqrt{3}) = \log|1+i\sqrt{3}| + i\,\mathrm{Arg}(1+i\sqrt{3}) = \log 2 + i\frac{\pi}{3}$$

   (b) $(1+i)^{1+i}$

Solution:

$$\begin{aligned} i^{1+i} &= e^{i\,\mathrm{Log}(1+i)} \\ &= e^{i(\log|1+i|+i\,\mathrm{Arg}(1+i))} \\ &= e^{i(\log 2+i\pi/4)} \\ &= e^{-\pi/4+i\log 2} \\ &= e^{-\pi/4}(\cos(\log 2)+i\sin(\log 2)) \end{aligned}$$

13. Compute $|e^{i\pi^2}|$.

1. $e^{\mathrm{Log}\,z} = z$ for all z
2. $\mathrm{Log}\,e^z = z$ for all z.

Which of the following statements is true?

1. Only (1) is correct.
2. Only (2) is correct.
3. Both (1) and (2) are correct.
4. Neither (1) nor (2) are correct

Choose a branch of $\log z$ such that $\log z$ is continuous on the positive real axis and

$$e^{\frac{1}{2}\log x} = -\sqrt{x}$$

for all $x > 0$. Justify your answer.

Choose a branch of the square root function such that $\sqrt{1} = 1$ and $\sqrt{i} = -\dfrac{1+i}{\sqrt{2}}$. Justify your answer.

Solve $e^{2z} = -1+\sqrt{3}$

Solve: $|e^{iz}| = 2$. What's wrong with the following argument?

$$|e^{iz}|^2 = |\cos z + i\sin z|^2 = \cos^2 z + \sin^2 z = 1 \quad \text{for all } z \in \mathbb{C}.$$

14. (a) Does the identity $\overline{\mathrm{Log}\,z} = \mathrm{Log}\,\bar{z}$ hold for all z in the domain of continuity of $\mathrm{Log}\,z$? Prove or give a counterexample.

Solution: Noting that $\mathrm{Arg}\,\bar{z} = -\mathrm{Arg}\,z$ for all z in the domain of continuity of $\mathrm{Log}\,z$,

$$\overline{\mathrm{Log}\,z} = \overline{\log|z| + i\,\mathrm{Arg}\,z} = \log|z| - i\,\mathrm{Arg}\,z = \log|\bar{z}| + i\,\mathrm{Arg}\,\bar{z} = \mathrm{Log}\,\bar{z}$$

(b) Let $\log z$ denote the branch of the logarithm for which $\arg z \in [0, 2\pi)$. Does the identity $\overline{\operatorname{Log} z} = \operatorname{Log} \bar{z}$ hold for all z in the domain of continuity of $\operatorname{Log} z$? Prove or give a counterexample.

Solution: The identity does not hold for $z = i$:

$$\log \bar{i} = \log(-i) = \log|-i| + i\arg(-i) = i\frac{3\pi}{2},$$
$$\overline{\log i} = \overline{\log|i| + i\arg i} = \overline{i\frac{\pi}{2}} = -i\frac{\pi}{2}$$