MATH 307 — Worksheet #3

1. Compute the derivatives:

   (a) $\dfrac{\partial}{\partial \bar{z}}|z|^2$

   Solution:
   $$\frac{\partial}{\partial \bar{z}}|z|^2 = \frac{\partial}{\partial \bar{z}} z\bar{z} = z$$

   (b) $\dfrac{\partial}{\partial \bar{z}} y$

   Solution:
   $$\frac{\partial}{\partial \bar{z}} y = \frac{\partial}{\partial \bar{z}} \frac{1}{2i}(z - \bar{z}) = -\frac{1}{2i} = \frac{i}{2}$$

   (c) $\dfrac{\partial}{\partial \bar{z}} \dfrac{1-|z|}{1+|z|}$

   Solution:
   $$\begin{aligned}
   \frac{\partial}{\partial \bar{z}} \frac{1-|z|}{1+|z|} &= \frac{\partial}{\partial \bar{z}} \frac{1-(z\bar{z})^{1/2}}{1+(z\bar{z})^{1/2}} \\
   &= \frac{\frac{1}{2}(z\bar{z})^{-1/2} z(1+(z\bar{z})^{1/2}) - (1-(z\bar{z})^{1/2})\frac{1}{2}(z\bar{z})^{-1/2} z}{(1+(z\bar{z})^{1/2})^2} \\
   &= \frac{\frac{1}{2}|z|^{-1} z(1+|z| - (1-|z|))}{(1+|z|)^2} \\
   &= \frac{\frac{1}{2}|z|^{-1} z(2|z|)}{(1+(z\bar{z})^{1/2})^2} \\
   &= \frac{z}{(1+|z|)^2}
   \end{aligned}$$

2. Being uncomfortable with complex square roots, you're skeptical of the calculation
   $$\frac{\partial}{\partial \bar{z}}|z| = \frac{\partial}{\partial \bar{z}}\sqrt{z\bar{z}} = \frac{\partial}{\partial \bar{z}}\sqrt{z}\sqrt{\bar{z}} = \frac{\sqrt{z}}{2\sqrt{\bar{z}}} = \frac{\sqrt{z}}{2\sqrt{\bar{z}}} \cdot \frac{\sqrt{z}}{\sqrt{z}} = \frac{z}{2|z|}.$$
   Confirm its result by switching to Cartesian coordinates, i.e., evaluate
   $$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\sqrt{x^2+y^2}.$$

Solution:

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial}{\partial x}\sqrt{x^2+y^2}+i\frac{\partial}{\partial y}\sqrt{x^2+y^2}\right) &= \frac{1}{2}\left(\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}+i\frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}}\right)\\ &= \frac{1}{2}\frac{x+iy}{\sqrt{x^2+y^2}}\\ &= \frac{z}{2|z|}\end{aligned}$$

3. Let f be a complex-valued function on an open subset U of $\mathbb{C}$ and let $z \in U$. Explain the difference between the statements "f is differentiable at z" and "f is analytic at z".

Solution: The statement "f is differentiable at z" means that

$$\lim_{w\to z}\frac{f(w)-f(z)}{w-z}$$

exists, the derivative being the limiting value.

The statement "f is analytic at z" means that there is an $\varepsilon > 0$ such that f is differentiable at w for all w with $|w-x| < \varepsilon$. In other words, it means that that f is differentiable on a *neighborhood* of z.

4. At which z is $f'(z)$ differentiable? analytic?

   (a) $f(z) = x^3 + iy^3$

Solution: Write $u = x^3$, $v = y^3$. Then

$$u_x = 3x^2, \quad u_y = 0, \quad v_x = 0, \quad v_y = 3y^2.$$

The Cauchy-Riemann equations are satisfied only when $3x^2 = 3y^2$, i.e., when $x = \pm y$. In particular, f is not analytic at $z = x + iy$ when $x \neq \pm y$. On the other hand, since u and v have continuous partials on $\mathbb{C}$ and satisfy the Cauchy-Riemann equations along $y = \pm x$, if follows that f is differentiable along these lines. It's not analytic along these lines, however, because they have empty interior, i.e., they don't contain any disk.

(b) $f(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$

Solution: Write $u = x^3$, $v = y^3$. Then

$$u_x = 3x^2 + 3y^2 - 3, \quad u_y = 6xy, \quad v_x = 6xy, \quad v_y = 3y^2 + 3x^2 - 3.$$

The Cauchy-Riemann equations are satisfied only when $3x^2 = 3y^2$, i.e., when $x = 0$ or $y = \frac{1}{2}$. In particular, f is not analytic at $z = x + iy$ when $x \neq 0$ and $y \neq 1/2$. On the other hand, since u and v have continuous partials on $\mathbb{C}$ and satisfy the Cauchy-Riemann equations along $x = 0$ and $y = \frac{1}{2}$, if follows that f is differentiable these lines. It's not analytic along these lines, however, because they have empty interion, i.e., they don't contain any disk.

(c) $f(z) = \dfrac{z^2 + \bar{z}^2}{2} + iz\bar{z}$

Solution:
$$\frac{\partial f}{\partial \bar{z}} = \bar{z} + iz = (x - y) + i(x + y) = 0$$
if and only if $z = 0$. Thus, $f(z)$ is not analytic at $z \neq 0$. However, f is differentiable at $z = 0$ by the above equation together with the fact that the partials of $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are polynomials in x and y and, hence, continuous.

5. $(*)$ Find a $f = u + iv$ such that:

   1. u and v have continuous partials on $\mathbb{C}$,
   2. f is nowhere analytic,
   3. f is differentiable on the unit circle, $|z| = 1$.

6. Solve the equation $\dfrac{\partial u}{\partial \bar{z}} = 2x$.

Solution: Use the formula $x = \frac{1}{2}z + \bar{z}$ to express the equation as

$$\frac{\partial u}{\partial \bar{z}} = z + \bar{z}.$$

Now integrate with respect to $\bar{z}$:

$$u = z\bar{z} + \frac{1}{2}\bar{z}^2.$$

7. Show that $\dfrac{\partial^2 f}{\partial \bar{z}^2} = 0$ if and only if $f(z) = \bar{z}g(z) + h(z)$, where g and h are analytic.

Solution: Functions of the form $f(z) = g(z) + ih(z)$ are clearly solutions of the differential equation. Conversely, let f be a solution. We'll show it must have the required form. Since

$$\frac{\partial}{\partial \bar{z}} \frac{\partial f}{\partial \bar{z}} = 0,$$

$\partial f/\partial \bar{z}$ does not depend on $\bar{z}$, i.e.,

$$\frac{\partial f}{\partial \bar{z}} = g(z)$$

for some g. Moreover, g is analytic as

$$\frac{\partial g}{\partial \bar{z}} = \frac{\partial^2 f}{\partial \bar{z}^2} = 0.$$

Integrate this equation with respect to $\bar{z}$ to get

$$f(z) = \bar{z}g(z) + h(z)$$

for some h. (The function h is the "constant" of integration, which may depend on z as $\bar{z}$ is the variable of integration.) I claim that h is analytic. To see this, differentiate the equation $f(z) = \bar{z}g(z) + h(z)$ with respect to $\bar{z}$ and use the product rule:

$$\frac{\partial f}{\partial \bar{z}} = g(z) + \bar{z}\frac{\partial h}{\partial \bar{z}}$$

But $\partial f/\partial \bar{z} = g(z)$, so we get

$$\bar{z}\frac{\partial h}{\partial \bar{z}} = 0$$

Cancelling the $\bar{z}$, we conclude that $h(z)$ is analytic.

Remark: I've swept something under the rug, here. The above argument shows only that $\partial h/\partial \bar{z} = 0$ for $z \neq 0$. (You can't cancel zeros!) Can you figure out how to deduce the analyticity of $h(z)$ at $z = 0$?