MATH 307 — Worksheet #1

1. Express the following in the form $x + iy$:

   (a) $\dfrac{2i-1}{5+6i}$

   Solution: $\dfrac{2i-1}{5+6i} = \dfrac{(2i-1)(5-6i)}{5^2+6^2} = \dfrac{1}{61}(7+16i)$

   (b) $(3-2i)^3$

   Solution: $(3-2i)^3 = 3^3 - 3^2(2i) + 3(2i)^2 - (2i)^3 = -9 - 46i$

2. Let $z = x + iy$. Express the following in the form $u(x,y) + iv(x,y)$.

   (a) $1 - z^2$

   Solution:

   $$1 - z^2 = 1 - (x+iy)^2 = 1 - (x^2 - y^2 + 2ixy) = (1 - x^2 + y^2) - 2ixy$$

   (b) $\dfrac{1}{z^2}$

   Solution:

   $$\frac{1}{z^2} = \frac{\overline{z^2}}{|z^2|^2} = \frac{\bar{z}^2}{|z|^4} = \frac{(x-iy)^2}{(x^2+y^2)^2} = \frac{\bar{z}^2}{|z|^4} = \frac{x^2-y^2}{(x^2+y^2)^2} + i\frac{-2xy}{(x^2+y^2)^2}$$

   (c) $\dfrac{1}{z^3}$

   Solution:

   $$\frac{1}{z^3} = \frac{\overline{z^3}}{|z^3|^2} = \frac{\bar{z}^3}{|z|^6} = \frac{(x-iy)^3}{(x^2+y^2)^3} =$$

   $$\frac{x^3 + 3x(-iy)^2 + 3x^2(-iy) + (-iy)^3}{(x^2+y^2)^3} = \frac{x^3 - 3xy^2}{(x^2+y^2)^3} + i\frac{-3xy^2 + y^3}{(x^2+y^2)^3}$$

(d) z^3

> **Solution:**
> $$z^3 = (x+iy)^3 = x^3 - ixy^2 + i(x^2y - y^3)$$

3. Verify the identities $\operatorname{Re}(iz) = -\operatorname{Im}(z)$ and $\operatorname{Im}(iz) = \operatorname{Re}(z)$.

> **Solution:**
> $$\operatorname{Re}(iz) = \operatorname{Re}(i(x+iy)) = \operatorname{Re}(-y+ix) = -y = -\operatorname{Im} z$$
> $$\operatorname{Im}(iz) = \operatorname{Im}(i(x+iy)) = \operatorname{Im}(-y+ix) = x = \operatorname{Re} z$$

4. For which z does the identity $\operatorname{Re}(z^2) = \operatorname{Re}(z)^2$ hold?

> **Solution:** With $z = x + iy$, $\operatorname{Re}(z^2) = x^2 - y^2$ and $\operatorname{Re}(z)^2 = x^2$. Thus, $\operatorname{Re}(z^2) = \operatorname{Re}(z)^2$ holds if and only if $y = 0$, i.e., if and only if $z \in \mathbb{R}$.

5. Express $\dfrac{i^3(1-i)}{2(1+i\sqrt{3})}$ in the form $re^{i\theta}$ with $r > 0$ and $\theta \in [5\pi, 7\pi)$.

> **Solution:** We have
> $$r = \left|\frac{i^3(1-i)}{2(1+i\sqrt{3})}\right| = \frac{|i^3||1-i|}{|2||1+i\sqrt{3}|} = \frac{1\sqrt{2}}{2\sqrt{4}} = \frac{1}{2\sqrt{2}}$$
> and
> $$\begin{aligned} \arg\frac{i^3(1-i)}{2(1+i\sqrt{3})} &= 3\arg i + \arg(1-i) - \arg 2 - \arg(1+i\sqrt{3}) + 2k\pi \\ &= 3\frac{\pi}{2} - \frac{\pi}{4} - 0 - \frac{\pi}{3} + 2k\pi \\ &= \frac{11\pi}{12} + 2k\pi. \end{aligned}$$
> As
> $$11\pi/12 + 2k\pi \in [5\pi, 7\pi) \iff k = 3,$$

we set

$$\theta := \frac{11\pi}{12} + 6\pi = \frac{83\pi}{12} \in [5\pi, 7\pi).$$

Thus,

$$\frac{i^3(1-i)}{2(1+i\sqrt{3})} = \frac{1}{2\sqrt{2}} e^{83\pi i/12}.$$

6. Let $a, b, c, d \in \mathbb{R}$ be such that $cd \neq 0$ and let $z \in \mathbb{C} \setminus \mathbb{R}$.

   (a) Express $\operatorname{Im} \dfrac{az+b}{cz+d}$ in terms of $\operatorname{Im} z$.

   Solution:

   $$\frac{az+b}{cz+d} = \frac{(az+b)\overline{(cz+d)}}{|cz+d|^2} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2}$$

   Note that $ac|z|^2$, bd, and $|cz+d|^2$ are real with $|cz+d|^2 > 0$. Therefore,

   $$\begin{aligned} \operatorname{Im} \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2} &= \operatorname{Im} \frac{adz + bc\bar{z}}{|cz+d|^2} \\ &= \frac{ad}{|cz+d|^2} \operatorname{Im} z + \frac{bc}{|cz+d|^2} \operatorname{Im} \bar{z} = \frac{ad-bc}{|cz+d|^2} \operatorname{Im} z. \end{aligned}$$

   (b) When is $\operatorname{Im} \dfrac{az+b}{cz+d}$ equal to 0?

   Solution: As $z \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Im} z \neq 0$. Therefore, $\operatorname{Im} \dfrac{az+b}{cz+d} = 0$ if and only if $ad - bc = 0$.

7. Describe and sketch the set solution set.

   (a) $|z - i| = 2$

Solution: The set of points at distance 2 from i, i.e., the circle centered at i with radius 2.

(b) $|z+i| = |z-1|$

Solution: The set of points equidistant from $-i$ and 1. These lie on the line $y = -x$.

(c) $|z+2i| + |z-2i| = 6$

Solution: The set of points the sum of whose distances from $-2i$ and $2i$ is equal to 6 is an ellipse with foci at $-2i$ and $2i$. Let's find its radii. By symmetry, the axes of the ellipse lie along the real and imaginary axes. Suppose $\pm ai$, $a > 0$, are the points of the ellipse along the imaginary axis. Then

$$6 = |-2i - ai| + |2i - ai| = (a+2) + (a-2) = 2a \Longrightarrow a = 3$$

Suppose $\pm bi$, $b > 0$, are the points of the ellipse along the real axis. By the Pythagorean theorem,

$$\left(\frac{6}{2}\right)^2 = 2^2 + b^2 \Longrightarrow b = \sqrt{5}$$

Thus, the Cartesian equation of the ellipse is

$$\left(\frac{x}{\sqrt{5}}\right)^2 + \left(\frac{y}{3}\right)^2 = 1.$$

(d) $|z+3| - |z-3| = 4$

Solution: The set of points the difference of whose distances from $(-3,0)$ and $(3,0)$ is equal to ± 4 is a hyperbola with foci at -3 and 3. By the symmetries of the equation $|z+3| - |z-3| = 4$, the hyperbola is in standard position and orientation. Thus, it has an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

for some $a, b > 0$. Since $(a,0)$ is on the curve,

$$4 = |a+3| - |a-3| = a + 3 - (3-a) = 2a \Longrightarrow a = 2$$

To determine b, first find another point on the hyperbola. Let's find the points $z = 3 \pm iy$ that lie on the hyperbola. By the defining equation,

$$|3 + iy + 3| - |3 + iy - 3| = 4.$$

The solutions of this equation are $y = \pm\frac{5}{2}$. Plugging $x = 3$, $y = \frac{5}{2}$ into

$$\frac{x^2}{2^2} + \frac{y^2}{b^2} = 1,$$

and solving for b, we get $b = \sqrt{5}$. Thus, the hyperbola has cartesian equation

$$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{\sqrt{5}}\right)^2 = 1.$$

(e) $\operatorname{Im} z^2 = 4$

Solution: As $\operatorname{Im} z^2 = 2xy$,

$$\operatorname{Im} z^2 = 4 \iff 2xy = 4 \iff xy = 2.$$

The graph of $xy = 2$ is a rectangular hyperbola.

8. Solve the equation.

(a) $z^2 + 2z + (1 - i) = 0$

Solution: By the quadratic formula and a bit of algebra,

$$z = \frac{-2 \pm \sqrt{2^2 - 4(1)(1 - i)}}{2} = -1 \pm \sqrt{i}.$$

The square roots of i are $\pm\dfrac{1+i}{\sqrt{2}}$. Therefore,

$$z = -1 \pm \frac{1+i}{\sqrt{2}}.$$

(b) $z^2 + (2i - 3)z + 5 - i = 0$

Solution: By the quadratic formula a bit of algebra,

$$z = \frac{(3-2i) \pm \sqrt{-15-8i}}{2}.$$

To find the square roots of $-15-8i$, we convert to polar coordinates:

$$r := |-15-8i| = 17, \qquad \tan\theta = \frac{8}{15}$$

(We *cannot* write $\theta = \arctan(8/15)$. Why not?) Since $\pi/2 < \theta/2 < \pi$,

$$\cos\frac{\theta}{2} = -\sqrt{\frac{\cos\theta+1}{2}} = -\sqrt{\frac{\frac{-15}{17}+1}{2}} = -\frac{1}{\sqrt{17}}.$$

(It's easy to miss this minus sign!)

$$\sin\frac{\theta}{2} = \sqrt{1-\cos^2\frac{\theta}{2}} = \sqrt{1-\frac{1}{17}} = \frac{4}{\sqrt{17}}$$

Therefore, the square roots of $-15-8i$ are

$$\pm\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \pm\sqrt{17}\left(-\frac{1}{\sqrt{17}} + \frac{4i}{\sqrt{17}}\right) = \pm(-1+4i)$$

Therefore,

$$z = \frac{(3-2i)\pm(-1+4i)}{2} = 1+i,\ 2-3i.$$

9. Given $x, y \in \mathbb{R}$, show that

$$a = \sqrt{\frac{x+\sqrt{x^2+y^2}}{2}} \quad b = \operatorname{sign}(y)\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}.$$

is the unique solution to $(a+ib)^2 = x+iy$ with $a \geq 0$. Here,

$$\operatorname{sign}(y) = \begin{cases} 1 & \text{if } y > 0, \\ 0 & \text{if } y = 0, \\ -1 & \text{if } y < 0. \end{cases}$$

Solution: Suppose $(a+bi)^2 = x+iy$. Equating real and imaginary parts,

$$(a+bi)^2 = x+iy \iff (a^2-b^2)+2iab = x+iy. \iff a^2-b^2 = x \text{ and } 2ab = y$$

We consider the case $y \neq 0$, the case $y = 0$ being simpler. If $y \neq 0$, then $a \neq 0$ and $b = y/2a$. Substituting this into $a^2 - b^2 = x$, we get

$$a^2 - \frac{y^2}{4a^2} = x \iff 4(a^2)^2 - 4x(a^2) - y^2 = 0$$

$$a^2 = \frac{4x \pm \sqrt{16x^2+16y^2}}{8} = \frac{x \pm \sqrt{x^2+y^2}}{2}$$

Noting that $x - \sqrt{x^2+y^2} < 0$,

$$a^2 = \frac{x - \sqrt{x^2+y^2}}{2}$$

has no solution $a \in \mathbb{R}$. Thus,

$$a = \sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}.$$

We solve for b:

$$\begin{aligned}\frac{1}{\sqrt{2}a} = \sqrt{\frac{1}{x+\sqrt{x^2+y^2}}} &= \sqrt{\frac{x-\sqrt{x^2+y^2}}{x^2-(x^2+y^2)}} \\ &= \sqrt{\frac{-x+\sqrt{x^2+y^2}}{y^2}} = \operatorname{sign}(y)\frac{\sqrt{-x+\sqrt{x^2+y^2}}}{y}\end{aligned}$$

Thus,

$$b = \frac{y}{2a} = \operatorname{sign}(y)\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}.$$

10. Prove the identities:

$$\cos z = \cosh(iz), \quad \cos(iz) = \cosh z, \quad \sin z = -i\sinh(iz), \quad \sin(iz) = i\sinh z$$

Solution: The identity $\cos z = \cosh(iz)$ follows directly from the definitions and

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{2} = -i\sinh(iz)$$

Replacing z with iz in these identities, we get

$$\cos(iz) = \cosh(i(iz)) = \cosh(-z) = \cosh z,$$

$$\sin(iz) = -i\sinh(i(iz)) = -i\sinh(-z) = i\sinh z.$$

11. Prove the identities:

$$\cos(x+iy) = \cos x\cosh y - i\sin x\sinh y,$$
$$\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$$

Solution: Using the identities proved above,

$$\begin{aligned}\cos(x+iy) &= \cos x\cos(iy) - \sin x\sin(iy) \\ &= \cos x\cosh y - \sin x(i\sinh y) \\ &= \cos x\cosh y - i\sin x\sinh y,\end{aligned}$$

$$\begin{aligned}\sin(x+iy) &= \sin x\cos(iy) + \cos x\sin(iy) \\ &= \sin x\cosh y + \cos x(i\sinh y) \\ &= \sin x\cosh y + i\cos x\sinh y.\end{aligned}$$

12. Prove the identity:

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

Deduce that

$$\lim_{y\to\infty} |\cos z| = \infty \quad \text{and} \quad \lim_{y\to\infty} |\sin z| = \infty$$

Solution:

$$\begin{aligned}|\cos z|^2 &= \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y\\ &= \cos^2 x\cosh^2 y + (1-\cos^2 x)\sinh^2 y\\ &= \cos^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y\\ &= \cos^2 x + \sinh^2 y\end{aligned}$$

$$\begin{aligned}|\sin z|^2 &= \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y\\ &= \sin^2 x\cosh^2 y + (1-\sin^2 x)\sinh^2 y\\ &= \sin^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y\\ &= \sin^2 x + \sinh^2 y\end{aligned}$$

Since $\sinh y \to \infty$ as $y \to \infty$ and $|\sinh y| \le |\cos z|,\ |\sin z|$ by the above , the statement about the limits follows.

13. Solve the equation $|\cot z| = 1$.

Solution: We have:

$$\begin{aligned} & |\cot z| = 1\\ \Longleftrightarrow\quad & |\cos z|^2 = |\sin z|^2\\ \Longleftrightarrow\quad & \cos^2 x + \sinh^2 y = \sin^2 x + \sinh^2 y \quad \text{(see above)}\\ \Longleftrightarrow\quad & \cos^2 x = \sin^2 x\\ \Longleftrightarrow\quad & \cos x = \pm\sin x\\ \Longleftrightarrow\quad & x = \pi \pm \frac{\pi}{4}\end{aligned}$$

Thus,

$$|\cot z| = 1 \Longleftrightarrow \operatorname{Re} z = k\pi \pm \frac{\pi}{4}.$$

14. Solve the equations:

(a) $e^{\bar{z}} = \overline{e^z}$

Solution: We have:

$$\begin{aligned} e^{\bar{z}} = e^{x-iy} = e^x(\cos(-y)) + i\sin(-y)) \\ = e^x(\cos y - i\sin y) = \overline{e^x(\cos y + i\sin y)} = \overline{e^z}. \end{aligned}$$

Thus, this identity holds for all z.

(b) $\cos(i\bar{z}) = \overline{\cos(iz)}$

Solution: We showed above that $e^{\bar{z}} = \overline{e^z}$. Therefore,

$$\cos(i\bar{z}) = \cosh\bar{z} = \frac{e^{\bar{z}} + e^{-\bar{z}}}{2} = \overline{\frac{e^z + e^{-z}}{2}} = \overline{\cosh z} = \overline{\cos(iz)}$$

for all z.