MATH 307 — Quiz #2

Instructions: You have 50 minutes to solve all three problems. All answers should be exact – no decimal approximations. No calculators, notes, or other aids. Hand in only your solution booklet.

1. (3 points) Find an analytic function $g(z)$ such that

$$\frac{\partial}{\partial \bar{z}} e^x \cos(y) = g(\bar{z}).$$

Solution: Use the definition of $\partial/\partial\bar{z}$:

$$\begin{aligned}\frac{\partial}{\partial \bar{z}} e^x \cos(y) &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) e^x \cos(y) \\ &= \frac{1}{2} e^x(\cos y - i \sin y) \\ &= \frac{1}{2} e^{\bar{z}}.\end{aligned}$$

So take $g(z) = \frac{1}{2}e^z$.

2. (3 points) Find a harmonic conjugate of

$$u(x, y) = e^x \sin y + 2xy + y,$$

i.e., a harmonic function $v(x, y)$ such that $f = u + iv$ is analytic.

Solution: We differentiate:

$$u_x = e^x \sin y + 2y$$

We need $u_x = v_y$, so $v = \int u_x \, dy$:

$$v = -e^x \cos y + y^2 + C(x)$$

Differentiating, we get

$$v_x = -e^x \cos y + C'(x).$$

We need $v_x = -u_y$, i.e.,

$$v_x = -u_y \iff -e^x \cos y + C'(x) = -(e^x \cos y + 2x + 1) \iff C'(x) = -2x - 1$$

This identity holds with $C(x) = -x^2 - x$. Therefore,

$$v = -e^x \cos y + y^2 - x^2 - x$$

is a harmonic conjugate of u.

3. (3 points) For which z is the function

$$f(z) = \overline{e^{-\bar{z}^2}}$$

analytic? Justify your answer.

Solution:

Solution 1: Observe that $f(z) = e^{-z^2}$. Therefore, $f(z)$ is analytic on $\mathbb{C}$.

Solution 2: You showed on a worksheet that $g(z)$ is analytic on D if and only if $\overline{f(\bar{z})}$ is analytic on $\bar{D}$. Since $f(z) = \overline{g(\bar{z})}$ with $g(z) = e^{-z^2}$ and $g(z)$ is analytic on $\mathbb{C}$, it follows that $f(z)$ is analytic on $\bar{\mathbb{C}} = \mathbb{C}$.

Solution 3: (The "hard" way.) We have

$$f(z) = u(x, y) + iv(x, y),$$

where

$$u(x, y) = e^{y^2 - x^2} \cos(2xy) \quad \text{and} \quad v(x, y) = -e^{y^2 - x^2} \sin(2xy)$$

We compute the partials of u and v:

$$\begin{aligned}
u_x &= -2xe^{y^2-x^2} \cos(2xy) - 2ye^{y^2-x^2} \sin(2xy), \\
v_y &= -2ye^{y^2-x^2} \sin(2xy) - 2xe^{-y^2-x^2} \cos(2xy) = u_x, \\
u_y &= 2ye^{y^2-x^2} \cos(2xy) - 2xe^{y^2-x^2} \sin(2xy) \\
v_x &= 2xe^{y^2-x^2} \sin(2xy) - 2ye^{y^2-x^2} \cos(2xy) = -u_y
\end{aligned}$$

Since the partials of u and v are continuous and satisfy the Cauchy-Riemann equations on $\mathbb{C}$, $f(z)$ is analytic on $\mathbb{C}$.