MATH 307 — Worksheet #6

1. For what z does
$$\sum_{n=0}^{\infty} \frac{1}{(z^2+1)^n}$$
converge? What is the sum of the series?

Solution: The series converges whenever $|z^2+1| < 1$. The sum is
$$\frac{1}{1-(z^2+1)} = \frac{1}{z^2}.$$

2. Find the limit of the sequence of functions
$$f_n(x) = \arctan(nx), \qquad x \in \mathbb{R}.$$
Is the convergence uniform?

Solution: We have:
$$\lim_{n\to\infty} \arctan(n0) = \lim_{n\to\infty} \arctan(0) = \lim_{n\to\infty} 0 = 0$$
If $x > 0$, then
$$\lim_{n\to\infty} \arctan(nx) = \lim_{y\to\infty} \arctan(y) = \frac{\pi}{2}$$
. If $x < 0$, then
$$\lim_{n\to\infty} \arctan(nx) = \lim_{y\to-\infty} \arctan(y) = -\frac{\pi}{2}$$
. Therefore,
$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} \frac{\pi}{2} & \text{if } x > 0, \\ 0 & \text{if } x - 0, \\ -\frac{\pi}{2} & \text{if } x < 0. \end{cases}$$
Note that f is discontinuous at $x = 0$. Since f_n is continuous and the limit of a uniformly convergent sequence of continuous functions is continuous, the convergence of f_n to f cannot be uniform.

3. Find the limit of the sequence of functions

$$f_n(x) = \begin{cases} 0 & \text{if } x < n \\ 1 & \text{if } x \geq n. \end{cases}$$

Is the convergence uniform?

Solution: Since $f_n(x) = 0$ for all $n \geq x$, if follows that $f_n(x) \to 0$ for all x, i.e., $f_n \to 0 =: f$ pointwise. The convergence is not uniform, however, since for any n,

$$|f_n(n) - f(n)| = |1 - 0| = 1.$$

4. Show that

$$\sum_{n \to \infty} e^{-n} \sin(nz)$$

is analytic on the region $A = \{z : -1 < \operatorname{Im} z < 1\}$.

Hint: Show, using the Weierstrass M-test, that $\sum \operatorname{Re} f_n$ and $\sum \operatorname{Im} f_n$ both converge uniformly on $A_c = \{z : -c < \operatorname{Im} z < c\}$ for all c with $0 \leq c < 1$. The identity $\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$ is helpful.

Solution: Let

$$f_n(z) = e^{-n} \sin(nz).$$

Since

$$\sin(nz) = \sin(x)\cosh(ny) + i\cos(x)\sinh(ny),$$

we can set

$$g_n(z) := \operatorname{Re} f_n(z) = e^{-n} \sin(x)\cosh(ny)$$

and

$$h_n(z) := \operatorname{Im} f_n(z) = e^{-n} \cos(x)\sinh(ny).$$

Suppose $z \in A_c$.

$$\begin{aligned}
|g_n(z)| &= e^{-n}|\sin(x)||\cosh(ny)| \\
&= e^{-n}|\cosh(ny)| \\
&= \frac{1}{2}e^{-n}(e^{\frac{ny}{2}} + e^{-\frac{ny}{2}}) \\
&= \frac{1}{2}(e^{n\frac{y-1}{2}} + e^{n\frac{-y-1}{2}}) \\
&\leq \frac{1}{2}(e^{n\frac{c-1}{2}} + e^{n\frac{c-1}{2}}) && \text{(as } z \in A_c\text{)} \\
&= e^{n\frac{c-1}{2}} \\
&=: M_n
\end{aligned}$$

Since $0 \leq c < 1$, $\frac{c-1}{2} < 0$ and $|e^{\frac{c-1}{2}}| < 1$. Therefore,

$$\sum M_n = \sum \left(e^{\frac{c-1}{2}}\right)^n$$

converges. Therefore, by the Weierstrass M-test, $\sum g_n$ converges uniformly on A_c and, therefore, is analytic threre by the analytic convergence theorem. Since $\bigcup A_c = A$, $\sum g_n$ is analytic on A.

The argument for h_n is similar.

5. Show that the series

$$\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$$

is analytic on $|z| < 1$ and on $|z| > 1$.

Hint: To prove analyticity for $|z| < 1$, first show that $|1 + z^{2n}| < \rho$ for all z with $|z| < \rho < 1$. For $|z| > 1$, consider consider the change of variable $z \leftrightarrow \frac{1}{z}$.

Solution: Let

$$f_n(z) = \frac{z^n}{1+z^{2n}}.$$

Suppose $|z| < \rho < 1$. Then $|1 + z^{2n}| > 1 - \rho$. To see this, interpret $1 + z^{2n}$ as the distance from z^{2n} to -1 and note that z^{2n} is inside the circle of radius ρ centered at

$z = 0$. It follows that

$$\left|\frac{z^n}{1+z^{2n}}\right| < \frac{\rho^n}{1-\rho} =: M_n.$$

By the Weierstrass M-test and the analytic convergence theorem, $\sum f_n(z)$ is analytic on $|z| < \rho$. Since this holds for all ρ with $0 < \rho < 1$, $\sum f_n$ is analytic on $|z| < 1$.

To prove analyticity for $|z| > 1$, note that $f_n(\frac{1}{z}) = f_n(z)$. Let $\sigma > 1$, by the above analysis, $\sum f_n(\frac{1}{z})$ converges uniformly on $|\frac{1}{z}| < \rho := \frac{1}{\sigma}$, i.e., for $|z| > \sigma$. But $\sum f_n(z) = \sum_n f_n(\frac{1}{z})$.

6. Determine the radius of convergence R of the power series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$

Solution: $R = 1$

(b) $\displaystyle\sum_{n=0}^{\infty} nz^n$

Solution: $R = 1$

(c) $\displaystyle\sum_{n=0}^{\infty} n^2 z^n$

Solution: $R = 1$

(d) $\displaystyle\sum_{n=1}^{\infty} n!\frac{z^n}{n^n}$

Solution: $R = e$

(e) $\displaystyle\sum_{n=0}^{\infty} n^n z^n$

Solution: $R = 0$

7. Determine the radius of convergence R of the Taylor expansion of $f(z)$ around $z = z_0$.

(a) $f(z) = \dfrac{\sin z}{z^2 + 4}, \;\; z_0 = 0$

Solution: $R = 2$

(b) $f(z) = \dfrac{(z+1)}{(z-1)(z-4)}, \;\; z_0 = 2$

Solution: $R = 1$

(c) $f(z) = \dfrac{e^z}{z^2 - z}, \;\; z_0 = 4i$

Solution: $R = 4$

(d) $f(z) = \dfrac{z}{e^z + 1}, \;\; z_0 = 0$

Solution: $R = \pi$

(e) $f(z) = e^{-z^2/2} \sinh(z+2), \;\; z_0 = 0$

Solution: $R = \infty$

(f) $f(z) = \tanh(2z), \;\; z_0 = 0$

Solution: $R = \frac{\pi}{2}$