MATH 307 — Worksheet #2

1. Let $\sqrt{\cdot}$ denote the branch of the square root defined by

$$\sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}, \quad \theta \in [0, 2\pi)$$

For which z does the identity $\sqrt{z^2} = z$ hold?

> **Solution:** Write $z = re^{i\theta}$, $\theta \in [0, 2\pi)$. Then
>
> $$z^2 = r^2 e^{i(2\theta)}$$
>
> If $0 \leq \theta < \pi$, then $0 \leq 2\theta < 2\pi$ and
>
> $$\sqrt{z^2} = \sqrt{r^2 e^{i(2\theta)}} = re^{i(2\theta)/2} = re^{i\theta} = z.$$
>
> If, however, $\pi \leq \theta < 2\pi$, then $2\pi \leq \theta < 4\pi$, in which case
>
> $$\sqrt{z^2} = \sqrt{r^2 e^{i(2\theta)}} = \sqrt{r^2 e^{i(2\theta - 2\pi)}} = re^{i(2\theta-2\pi)/2} = re^{i\theta}e^{-i\pi} = -z.$$
>
> Thus, $\sqrt{z^2} = \sqrt{z}$ if and only if $\theta \in [0, \pi)$.

2. Find all values.

   (a) $\log 1$

   > **Solution:**
   >
   > $$\log 1 = \log|1| + i \arg 1 = 2k\pi i$$

   (b) $\log(1 + i)$

   > **Solution:**
   >
   > $$\log(1+i) = \log|1+i| + i\arg(1+i) = \log\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right)$$

   (c) $(1 + i)^{1+i}$

Solution:

$$\begin{aligned}(1+i)^{1+i} &= e^{(1+i)\log(1+i)}\\ &= e^{(1+i)\left\{\log\sqrt{2}+i\left(\frac{\pi}{4}+2k\pi\right)\right\}}\\ &= e^{\log\sqrt{2}-\left(\frac{\pi}{4}+2k\pi\right)}e^{i(\log\sqrt{2}+\frac{\pi}{4}+2k\pi)}\\ &= \sqrt{2}e^{-\left(\frac{\pi}{4}+2k\pi\right)}\left\{\cos\left(\log\sqrt{2}+\frac{\pi}{4}+2k\pi\right)+i\sin\left(\log\sqrt{2}+\frac{\pi}{4}+2k\pi\right)\right\}\end{aligned}$$

3. Let $z = re^{i\theta}$. Express all values of z^z in the form $x + iy$.

Solution: Write $z = re^{i\theta}$.

$$\begin{aligned}z^z &= e^{z\log z}\\ &= e^{(r\cos\theta+ir\sin\theta)(\log r+i(\theta+2k\pi))}\\ &= e^{r\cos\theta\log r-(\theta+2k\pi)r\sin\theta}e^{i(r(\theta+2k\pi)\cos\theta+r\log r\sin\theta)}\\ &= r^{r\cos\theta}e^{-(\theta+2k\pi)r\sin\theta}\,\{\cos(r(\theta+2k\pi)\cos\theta+r\log r\sin\theta)\\ &\qquad +\, i\sin(r(\theta+2k\pi)\cos\theta+r\log r\sin\theta)\}\end{aligned}$$

4. Compute the limit or argue that it don't exist.

(a) $\lim\limits_{x\to\infty} e^{x+iy}$ (fixed y)

Solution:

$$\lim_{x\to\infty} e^{x+iy} = e^{iy}\lim_{x\to\infty} e^{x} = \infty$$

(b) $\lim\limits_{x\to-\infty} e^{x+iy}$ (fixed y)

Solution:

$$\lim_{x\to-\infty} e^{x+iy} = e^{iy}\lim_{x\to-\infty} e^{x} = 0$$

(c) $\lim\limits_{y\to\infty} e^{x+iy}$ (fixed x)

Solution:
$$\lim_{y\to\infty} e^{x+iy} = e^x \lim_{y\to\infty} e^{iy}$$
This limit does not exists; both real and imaginary parts of e^{iy} oscillate between -1 and 1.

(d) $\lim_{y\to-\infty} e^{x+iy}$ (fixed x)

Solution:
$$\lim_{y\to-\infty} e^{x+iy} = e^x \lim_{y\to-\infty} e^{iy}$$
This limit does not exists; both real and imaginary parts of e^{iy} oscillate between -1 and 1.

(e) $\lim_{|z|\to\infty} e^z$

Solution: This limit doesn't exist as $\lim_{|x|\to\infty} e^x$ doesn't.

(f) $\lim_{|z|\to\infty} |e^z|$

Solution: This limit doesn't exist as $\lim_{|x|\to\infty} |e^x| = \lim_{|x|\to\infty} e^x$ doesn't.

5. (a) Prove that $|a^b| = |a|^b$ for $a \in \mathbb{C}$ and $b \in \mathbb{R}$.

Solution:
$$\begin{aligned}
|a^b| &= |e^{b\log a}| \\
&= |e^{b(\log|a| + i\arg a)}| \\
&= e^{b\log|a|} && \text{because } b \text{ is real} \\
&= |a|^b.
\end{aligned}$$

(b) Prove that, for a fixed branch of log, $a^{b+c} = a^b a^c$.

(c) Prove that, for a fixed branch of log, $(ab)^c = a^c b^c$ valid for all complex a, b, c such that $\log(ab) = \log a + \log b$.

Solution:

$$\begin{aligned}
(ab)^c &= e^{c\log(ab)} \\
&= e^{b(\log a+\log b)} && \text{if } \log(ab)=\log a+\log b \\
&= e^{b\log a+c\log b} \\
&= e^{b\log a}e^{c\log b} \\
&= a^b a^c
\end{aligned}$$

6. Determine the set on which the function is analytic and compute its derivative.

(a) $\dfrac{1}{(z^3-1)(z^2+2)}$

Solution: The function is not differentiable on its domain,

$$z \neq 1, e^{2i\pi/3}, e^{4\pi/3}, \sqrt{2}i, -\sqrt{2}i.$$

For all other z,

$$\frac{d}{dz}\frac{1}{(z^3-1)(z^2+2)} = -(z^3-1)^{-2}3z^2(z^2+2)^{-1} - (z^3-1)(z^2+2)^{-2}2z.$$

(b) $\dfrac{1}{z+z^{-1}}$

Solution: The function is differentiable on its domain:

$$z \neq 0, i, -i.$$

For all other z,

$$\frac{d}{dz}\frac{1}{z+z^{-1}} = -(z+z^{-1})^{-2}(1-z^{-2}).$$

(c) $\dfrac{z}{z^n-2}$

Solution: The function is differentiable on its domain:

$$z \neq \sqrt{2}e^{2\pi ik/n}, \quad n = 0, \ldots, n-1.$$

For all other z,

$$\frac{d}{dz}\frac{z}{z^n-2} = (z^n-2)^{-1} + z(-2)(z^n-2)^{-2}nz^{n-1}.$$

7. Let

$$f(z) = \begin{cases} z^5/|z|^4 & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

(a) Show that

$$\lim_{z\to 0}\frac{f(z)}{z}$$

does not exist.

Solution:

$$\lim_{z\to 0}\frac{f(z)}{z} = \lim_{z\to 0}\frac{z^4}{|z|^4} = \lim_{z\to 0}\frac{z^4}{(z\bar{z})^2} = \lim_{z\to 0}\frac{z^2}{\bar{z}^2}$$

If $z \to 0$ along the real axis, $\bar{z} = z$ and the limit is 1.

Consider approaching 0 along the line $z = re^{i\pi/4}$. Then

$$\lim_{r\to 0}\frac{z^2}{\bar{z}^2} = \lim_{r\to 0}\frac{r^2e^{i\pi/2}}{r^2e^{-i\pi/2}} = \frac{i}{-i} = -1.$$

Thus the limit does not exist.

(b) Let $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. Show that

$$u(x,0) = x, \quad u(0,y) = 0, \quad v(x,0) = 0, \quad v(0,y) = y.$$

Solution: We have

$$f(x+0i) = \frac{x^5}{|x|^4} = \frac{x(x^4)}{x^4} = x + 0i$$

and

$$f(0+iy) = \frac{iy^5}{|iy|^4} = \frac{iy(y^4)}{y^4} = 0 + iy.$$

Therefore,

$$u(x,0) = \operatorname{Re} f(x+0i) = x, \quad v(x,0) = \operatorname{Im} f(x+0i) = 0$$

and

$$u(0,y) = \operatorname{Re} f(0+iy) = 0, \quad v(0,y) = \operatorname{Im} f(0+iy) = y.$$

(c) Conclude that the partial derivatives of u and v with respect to x and y exist, that the Cauchy-Riemann equations are satisfied, but $f'(0)$ does not exist. Why does this not contradict the Cauchy-Riemann theorem?

Solution: By definition, $f(0) = 0$, i.e., $u(0,0) = v(0,0) = 0$. Therefore,

$$\begin{aligned}
u_x(0,0) &= \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h} = 1 \\
u_y(0,0) &= \lim_{h\to 0} \frac{u(0,0+h) - u(0,0)}{h} = 0 \\
v_x(0,0) &= \lim_{h\to 0} \frac{v(0+h,0) - v(0,0)}{h} = 0 \\
v_y(0,0) &= \lim_{h\to 0} \frac{v(0,0+h)) - v(0,0)}{h} = 1
\end{aligned}$$

Thus, the Cauchy-Riemann equations are satisfied.

This doesn't contradict the Cauchy-Riemann theorem as aren't *continuous* at $z = 0$, a condition required by the theorem.

8. Find the real and imaginary parts of the function and verify that they satisfy the Cauchy-Riemann equations.

(a) $f(z) = z^3$

Solution:

$$z^3 = (x+iy)^3 = x^3 + 3x^2iy + 3x(iy)^2 + (iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$$\begin{aligned}
u_x &= 3x^2 - 3y^2 \\
u_y &= -6xy \\
v_x &= 6xy \\
v_y &= 3x^2y - 3y^2.
\end{aligned}$$

Thus, the Cauchy-Riemann equations are satisfied.

(b) ze^{-z}

Solution:

$ze^{-z} = (x+iy)e^{-x}(\cos y - i\sin y) = e^{-x}(x\cos y + y\sin y) + ie^{-x}(y\cos y - x\sin y)$

$$\begin{aligned}
u_x &= -e^{-x}(x\cos y + y\sin y) + e^{-x}\cos y \\
&= e^{-x}(\cos y - x\cos y - y\sin y) \\
u_y &= e^{-x}(-x\sin y + \sin y + y\cos y) \\
v_x &= -e^{-x}(y\cos y - x\sin y) + e^{-x}\sin y \\
&= e^{-x}(-y\cos y + x\sin y - \sin y) \\
v_y &= e^{-x}(\cos y - y\sin y - x\cos y).
\end{aligned}$$

Thus, the Cauchy-Riemann equations hold.

(c) $\cos 2z$

Solution: We have:

$$\cos(2z) = \cos(2x)\cosh(2y) - i\sin(2x)\sinh(2y)$$

$$\begin{aligned}
u_x &= -2\sin(2x)\cosh(2y) \\
u_y &= 2\cos(2x)\sinh(2y) \\
v_x &= -2\cos(2x)\sinh(2y) \\
v_y &= -2\sin(2x)\cosh(2y)
\end{aligned}$$

Thus, the Cauchy-Riemann equations hold.