MATH 307 — Worksheet #8

1. Evaluate the integral. All paths are positively oriented.

   (a) $\displaystyle\int_{|z|=1} \frac{z}{z^2+2z+5}\,dz$

   Solution: We have $z^2 + 2z + 5 = (z-\alpha)(z-\bar{\alpha})$, where $\alpha = -1+2i$. Neither α nor $\bar{\alpha}$ are lie inside the unit circle. Therefore,

   $$\int_{|z|=1} \frac{z}{z^2+2z+5}\,dz = 0,$$

   by Cauchy's Theorem.

   (b) $\displaystyle\int_{|z|=9} \frac{1}{e^z-1}\,dz$

   Solution: The function $f(z) = 1/(e^z-1)$ has a simple pole at $z=0$ with residue

   $$\lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{z}{e^z-1} = 1,$$

   by l'Hopital's rule. It has no other singularities. Therefore, by the residue theorem,

   $$\int_{|z|=9} \frac{1}{e^z-1}\,dz = 2\pi i \cdot 1 = 2\pi i$$

   (c) $\displaystyle\int_{|z|=8} \tan z\,dz$

   Solution: The function $\tan z$ has singularities at the zeros of $\cos z$:

   $$z_k = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}.$$

   These zeros are all simple. (Why?) Therefore, the z_k are all simple poles of $\tan z$. Since $\tan z$ is π-periodic, these poles all have the same residue:

   $$\lim_{z\to\pi/2} z\tan z = \lim_{z\to\pi/2} \frac{z\sin z}{\cos z} = \lim_{z\to\pi/2} \frac{\sin z + z\cos z}{\sin z} = 1,$$

   by l'Hopital's rule.
   There are six singularities of $\tan z$ in the circle $|z| = 8$:

   $$z_k = \pi/2 + k\pi, \quad k = -3, -2, -1, 0, 1, 2.$$

Therefore, by the Residue Theorem,

$$\int_{|z|=8} \tan z \, dz = 6 \cdot 2\pi i \cdot 1 = 12\pi i.$$

(d) $\displaystyle\int_{|z|=3} \frac{5z-2}{z(z-1)}\, dz$

Solution: The function $f(z) = (5z-2)/z(z-1)$ has a simple poles at $z = 0$ and at $z = 1$ with residues

$$\lim_{z\to 0} z f(z) = \frac{5(0)-2}{0-1} = 2$$

and

$$\lim_{z\to 0} z f(z) = \frac{5(1)-2}{1} = 3,$$

respectively. Both of these singularities lie inside the circle $|z| = 3$. Therefore, by the residue theorem,

$$\int_{|z|=3} \frac{5z-2}{z(z-1)}\, dz = 2\pi i(2+3) = 10\pi i.$$

(e) $\displaystyle\int_{\gamma} \frac{e^{-z^2}}{z^2}\, dz$, where γ is the square with vertices $\pm 1 \pm i$.

Solution: Since $e^{-z^2} \neq 0$ for all $z \in \mathbb{C}$, $z = 0$ is the only pole of $f(z) = e^{-z}/^2$; it has order 2. The Laurent expansion of $f(z)$ around $z = 0$ contains only even powers of z. Therefore,

$$\operatorname*{Res}_{z=0} f(z) = 0$$

and

$$\int_{\gamma} \frac{e^{-z^2}}{z^2}\, dz = 0$$

by the Residue Theorem.

(f) $\displaystyle\int_{|z|=1/2} \frac{1}{(1-z)^3}\, dz$

Solution: The function $f(z) = 1/(1-z)^3$ is analytic on and inside $|z| = 1/2$. Therefore,

$$\int_{|z|=1/2} \frac{1}{(1-z)^3}\,dz = 0,$$

by Cauchy's Theorem.

(g) $\displaystyle\int_{|z-1|=1/2} \frac{1}{(1-z)^3}\,dz$

Solution: The function $f(z) = 1/(1-z)^3$ has a single triple pole inside $|z-1| = 1/2$, at $z = 1$. Since the Laurent expansion of $f(z)$ around $z = 1$ is

$$f(z) = \frac{-1}{(z-1)^3},$$

the residue at $z = 1$ is 0. Therefore,

$$\int_{|z-1|=1/2} \frac{1}{(1-z)^3}\,dz = 0$$

by the Residue Theorem.

(h) $\displaystyle\int_{|z-1|=1/2} \frac{e^z}{(1-z)^3}\,dz$

Solution: The function $f(z) = e^z/(1-z)^3$ has a single triple pole inside $|z-1| = 1/2$, at $z = 1$. The Laurent expansion of $f(z)$ around $z = 1$ has the form

$$\begin{aligned} f(z) &= \frac{-1}{(z-1)^3}\left(e + e(z-1) + \frac{e}{2}(z-1)^2 + \cdots\right) \\ &= \frac{-e}{(z-1)^3} + \frac{-e}{(z-1)^2} + \frac{-e/2}{z-1} + \cdots . \end{aligned}$$

Therefore, by the Residue Theorem,

$$\int_{|z-1|=1/2} \frac{e^z}{(1-z)^3}\,dz = 2\pi i(-e/2) = -e\pi i.$$

(i) $\displaystyle\int_{|z|=3} \frac{\cos(z+2)}{z(z+2)^3}\,dz$

Solution: The function $f(z) = \cos(z+2)/z(z+2)^3$ has a simple pole at $z = 0$, a triple pole at $z = -2$, and no other singularities.

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} z f(z) = \frac{\cos 2}{(0+2)^3} = \frac{\cos 2}{8}.$$

To compute the residue at $z = -2$, let $g(z) = \cos(z+2)/z$. Since $g(z)$ is analytic at $z = -2$, it's Laurent expansion around $z = -2$ is a Taylor expansion:

$$g(z) = a_0 + a_1(z+2) + a_2(z+2)^2 + \cdots .$$

Therefore, the Laurent expansion of $f(z)$ around $z = -2$ has the form

$$f(z) = \frac{a_0}{(z+2)^3} + \frac{a_1}{(z+2)^2} + \frac{a_2}{(z+2)^2} + \cdots .$$

It follows that $\operatorname{Res}_{z=-2} f(z) = a_2$. We can compute a_2 by using the usual formula for the Taylor coefficient.

$$\begin{aligned}
g'(z) &= \frac{-z\sin(z+2) + \cos(z+2)}{z^2}, \\
g''(z) &= \frac{(-(\sin(z+2) + z\cos(z+2)) - \sin(z+2))z^2 - 2z(-z\sin(z+2) + \cos(z+2))}{z^4} \\
g''(-2) &= \frac{(-(-2)(1))(-2)^2 - 2(-2)(1)}{(-2)^4} \\
&= \frac{3}{4} \\
a_2 &= \frac{g''(-2)}{2!} \\
&= \frac{3}{8}
\end{aligned}$$

Therefore,

$$\int_{|z|=3} \frac{\cos(z+2)}{z(z+2)^3}\,dz = 2\pi i\left(\frac{\cos 2}{8} + \frac{3}{8}\right) = \frac{\pi i}{4}(\cos 2 + 3).$$