

# MATH 307 — Worksheet #1

1. Express the following in the form  $x + iy$ :

(a)  $\frac{2i - 1}{5 + 6i}$

**Solution:**  $\frac{2i - 1}{5 + 6i} = \frac{(2i - 1)(5 - 6i)}{5^2 + 6^2} = \frac{1}{61}(7 + 16i)$

(b)  $(3 - 2i)^3$

**Solution:**  $(3 - 2i)^3 = 3^3 - 3^2(2i) + 3(2i)^2 - (2i)^3 = -9 - 46i$

2. Let  $z = x + iy$ . Express the following in the form  $u(x, y) + iv(x, y)$ .

(a)  $1 - z^2$

**Solution:**

$$1 - z^2 = 1 - (x + iy)^2 = 1 - (x^2 - y^2 + 2ixy) = (1 - x^2 + y^2) - 2ixy$$

(b)  $\frac{1}{z^2}$

**Solution:**

$$\frac{1}{z^2} = \frac{\overline{z^2}}{|z^2|^2} = \frac{\bar{z}^2}{|z|^4} = \frac{(x - iy)^2}{(x^2 + y^2)^2} = \frac{\bar{z}^2}{|z|^4} = \frac{x^2 - y^2}{(x^2 + y^2)^2} + i \frac{-2xy}{(x^2 + y^2)^2}$$

(c)  $\frac{1}{z^3}$

**Solution:**

$$\frac{1}{z^3} = \frac{\overline{z^3}}{|z^3|^2} = \frac{\bar{z}^3}{|z|^6} = \frac{(x - iy)^3}{(x^2 + y^2)^3} = \frac{x^3 + 3x(-iy)^2 + 3x^2(-iy) + (-iy)^3}{(x^2 + y^2)^3} = \frac{x^3 - 3xy^2}{(x^2 + y^2)^3} + i \frac{-3x^2y + y^3}{(x^2 + y^2)^3}$$

(d)  $z^3$

**Solution:**

$$z^3 = (x + iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$$

3. Verify the identities  $\operatorname{Re}(iz) = -\operatorname{Im}(z)$  and  $\operatorname{Im}(iz) = \operatorname{Re}(z)$ .

**Solution:**

$$\operatorname{Re}(iz) = \operatorname{Re}(i(x + iy)) = \operatorname{Re}(-y + ix) = -y = -\operatorname{Im} z$$

$$\operatorname{Im}(iz) = \operatorname{Im}(i(x + iy)) = \operatorname{Im}(-y + ix) = x = \operatorname{Re} z$$

4. For which  $z$  does the identity  $\operatorname{Re}(z^2) = \operatorname{Re}(z)^2$  hold?

**Solution:** With  $z = x + iy$ ,  $\operatorname{Re}(z^2) = x^2 - y^2$  and  $\operatorname{Re}(z)^2 = x^2$ . Thus,  $\operatorname{Re}(z^2) = \operatorname{Re}(z)^2$  holds if and only if  $y = 0$ , i.e., if and only if  $z \in \mathbb{R}$ .

5. Express  $\frac{i^3(1-i)}{2(1+i\sqrt{3})}$  in the form  $re^{i\theta}$  with  $r > 0$  and  $\theta \in [5\pi, 7\pi)$ .

**Solution:** We have

$$r = \left| \frac{i^3(1-i)}{2(1+i\sqrt{3})} \right| = \frac{|i^3||1-i|}{|2||1+i\sqrt{3}|} = \frac{1\sqrt{2}}{2\sqrt{4}} = \frac{1}{2\sqrt{2}}$$

and

$$\begin{aligned} \arg \frac{i^3(1-i)}{2(1+i\sqrt{3})} &= 3\arg i + \arg(1-i) - \arg 2 - \arg(1+i\sqrt{3}) + 2k\pi \\ &= 3\frac{\pi}{2} - \frac{\pi}{4} - 0 - \frac{\pi}{3} + 2k\pi \\ &= \frac{11\pi}{12} + 2k\pi. \end{aligned}$$

As

$$11\pi/12 + 2k\pi \in [5\pi, 7\pi) \iff k = 3,$$

we set

$$\theta := \frac{11\pi}{12} + 6\pi = \frac{83\pi}{12} \in [5\pi, 7\pi).$$

Thus,

$$\frac{i^3(1-i)}{2(1+i\sqrt{3})} = \frac{1}{2\sqrt{2}}e^{83\pi i/12}.$$

6. Let  $a, b, c, d \in \mathbb{R}$  be such that  $cd \neq 0$  and let  $z \in \mathbb{C} \setminus \mathbb{R}$ .

(a) Express  $\operatorname{Im} \frac{az+b}{cz+d}$  in terms of  $\operatorname{Im} z$ .

**Solution:**

$$\frac{az+b}{cz+d} = \frac{(az+b)\overline{(cz+d)}}{|cz+d|^2} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2}$$

Note that  $ac|z|^2$ ,  $bd$ , and  $|cz+d|^2$  are real with  $|cz+d|^2 > 0$ . Therefore,

$$\begin{aligned} \operatorname{Im} \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2} &= \operatorname{Im} \frac{adz + bc\bar{z}}{|cz+d|^2} \\ &= \frac{ad}{|cz+d|^2} \operatorname{Im} z + \frac{bc}{|cz+d|^2} \operatorname{Im} \bar{z} = \frac{ad-bc}{|cz+d|^2} \operatorname{Im} z. \end{aligned}$$

(b) When is  $\operatorname{Im} \frac{az+b}{cz+d}$  equal to 0?

**Solution:** As  $z \in \mathbb{C} \setminus \mathbb{R}$ ,  $\operatorname{Im} z \neq 0$ . Therefore,  $\operatorname{Im} \frac{az+b}{cz+d} = 0$  if and only if  $ad-bc = 0$ .

7. Describe and sketch the set solution set.

(a)  $|z-i| = 2$

**Solution:** The set of points at distance 2 from  $i$ , i.e., the circle centered at  $i$  with radius 2.

(b)  $|z + i| = |z - 1|$

**Solution:** The set of points equidistant from  $-i$  and 1. These lie on the line  $y = -x$ .

(c)  $|z + 2i| + |z - 2i| = 6$

**Solution:** The set of points the sum of whose distances from  $-2i$  and  $2i$  is equal to 6 is an ellipse with foci at  $-2i$  and  $2i$ . Let's find its radii. By symmetry, the axes of the ellipse lie along the real and imaginary axes. Suppose  $\pm ai$ ,  $a > 0$ , are the points of the ellipse along the imaginary axis. Then

$$6 = |-2i - ai| + |2i - ai| = (a + 2) + (a - 2) = 2a \implies a = 3$$

Suppose  $\pm bi$ ,  $b > 0$ , are the points of the ellipse along the real axis. By the Pythagorean theorem,

$$\left(\frac{6}{2}\right)^2 = 2^2 + b^2 \implies b = \sqrt{5}$$

Thus, the Cartesian equation of the ellipse is

$$\left(\frac{x}{\sqrt{5}}\right)^2 + \left(\frac{y}{3}\right)^2 = 1.$$

(d)  $|z + 3| - |z - 3| = 4$

**Solution:** The set of points the difference of whose distances from  $(-3, 0)$  and  $(3, 0)$  is equal to  $\pm 4$  is a hyperbola with foci at  $-3$  and  $3$ . By the symmetries of the equation  $|z + 3| - |z - 3| = 4$ , the hyperbola is in standard position and orientation. Thus, it has an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

for some  $a, b > 0$ . Since  $(a, 0)$  is on the curve,

$$4 = |a + 3| - |a - 3| = a + 3 - (3 - a) = 2a \implies a = 2$$

To determine  $b$ , first find another point on the hyperbola. Let's find the points  $z = 3 \pm iy$  that lie on the hyperbola. By the defining equation,

$$|3 + iy + 3| - |3 + iy - 3| = 4.$$

The solutions of this equation are  $y = \pm \frac{5}{2}$ . Plugging  $x = 3$ ,  $y = \frac{5}{2}$  into

$$\frac{x^2}{2^2} + \frac{y^2}{b^2} = 1,$$

and solving for  $b$ , we get  $b = \sqrt{5}$ . Thus, the hyperbola has cartesian equation

$$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{\sqrt{5}}\right)^2 = 1.$$

(e)  $\text{Im } z^2 = 4$

**Solution:** As  $\text{Im } z^2 = 2xy$ ,

$$\text{Im } z^2 = 4 \iff 2xy = 4 \iff xy = 2.$$

The graph of  $xy = 2$  is a rectangular hyperbola.

8. Solve the equation.

(a)  $z^2 + 2z + (1 - i) = 0$

**Solution:** By the quadratic formula and a bit of algebra,

$$z = \frac{-2 \pm \sqrt{2^2 - 4(1)(1 - i)}}{2} = -1 \pm \sqrt{i}.$$

The square roots of  $i$  are  $\pm \frac{1+i}{\sqrt{2}}$ . Therefore,

$$z = -1 \pm \frac{1+i}{\sqrt{2}}.$$

(b)  $z^2 + (2i - 3)z + 5 - i = 0$

**Solution:** By the quadratic formula a bit of algebra,

$$z = \frac{(3 - 2i) \pm \sqrt{-15 - 8i}}{2}.$$

To find the square roots of  $-15 - 8i$ , we convert to polar coordinates:

$$r := |-15 - 8i| = 17, \quad \tan \theta = \frac{8}{15}$$

(We *cannot* write  $\theta = \arctan(8/15)$ . Why not?) Since  $\pi/2 < \theta < \pi$ ,

$$\cos \frac{\theta}{2} = -\sqrt{\frac{\cos \theta + 1}{2}} = -\sqrt{\frac{\frac{-15}{17} + 1}{2}} = -\frac{1}{\sqrt{17}}.$$

(It's easy to miss this minus sign!)

$$\sin \frac{\theta}{2} = \sqrt{1 - \cos^2 \frac{\theta}{2}} = \sqrt{1 - \frac{1}{17}} = \frac{4}{\sqrt{17}}$$

Therefore, the square roots of  $-15 - 8i$  are

$$\pm \sqrt{r} \left( \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) = \pm \sqrt{17} \left( -\frac{1}{\sqrt{17}} + \frac{4i}{\sqrt{17}} \right) = \pm(-1 + 4i)$$

Therefore,

$$z = \frac{(3 - 2i) \pm (-1 + 4i)}{2} = 1 + i, 2 - 3i.$$

9. Given  $x, y \in \mathbb{R}$ , show that

$$a = \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} \quad b = \text{sign}(y) \sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}}.$$

is the unique solution to  $(a + ib)^2 = x + iy$  with  $a \geq 0$ . Here,

$$\text{sign}(y) = \begin{cases} 1 & \text{if } y > 0, \\ 0 & \text{if } y = 0, \\ -1 & \text{if } y < 0. \end{cases}$$

**Solution:** Suppose  $(a + bi)^2 = x + iy$ . Equating real and imaginary parts,

$$(a + bi)^2 = x + iy \iff (a^2 - b^2) + 2iab = x + iy. \iff a^2 - b^2 = x \text{ and } 2ab = y$$

We consider the case  $y \neq 0$ , the case  $y = 0$  being simpler. If  $y \neq 0$ , then  $a \neq 0$  and  $b = y/2a$ . Substituting this into  $a^2 - b^2 = x$ , we get

$$a^2 - \frac{y^2}{4a^2} = x \iff 4(a^2)^2 - 4x(a^2) - y^2 = 0$$

$$a^2 = \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} = \frac{x \pm \sqrt{x^2 + y^2}}{2}$$

Noting that  $x - \sqrt{x^2 + y^2} < 0$ ,

$$a^2 = \frac{x - \sqrt{x^2 + y^2}}{2}$$

has no solution  $a \in \mathbb{R}$ . Thus,

$$a = \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}}.$$

We solve for  $b$ :

$$\begin{aligned} \frac{1}{\sqrt{2}a} &= \sqrt{\frac{1}{x + \sqrt{x^2 + y^2}}} = \sqrt{\frac{x - \sqrt{x^2 + y^2}}{x^2 - (x^2 + y^2)}} \\ &= \sqrt{\frac{-x + \sqrt{x^2 + y^2}}{y^2}} = \text{sign}(y) \frac{\sqrt{-x + \sqrt{x^2 + y^2}}}{y} \end{aligned}$$

Thus,

$$b = \frac{y}{2a} = \text{sign}(y) \sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}}.$$

10. Prove the identities:

$$\cos z = \cosh(iz), \quad \cos(iz) = \cosh z, \quad \sin z = -i \sinh(iz), \quad \sin(iz) = i \sinh z$$

**Solution:** The identity  $\cos z = \cosh(iz)$  follows directly from the definitions and

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{i} \frac{e^{iz} - e^{-iz}}{2} = -i \sinh(iz)$$

Replacing  $z$  with  $iz$  in these identities, we get

$$\begin{aligned}\cos(iz) &= \cosh(i(iz)) = \cosh(-z) = \cosh z, \\ \sin(iz) &= -i \sinh(i(iz)) = -i \sinh(-z) = i \sinh z.\end{aligned}$$

11. Prove the identities:

$$\begin{aligned}\cos(x + iy) &= \cos x \cosh y - i \sin x \sinh y, \\ \sin(x + iy) &= \sin x \cosh y + i \cos x \sinh y\end{aligned}$$

**Solution:** Using the identities proved above,

$$\begin{aligned}\cos(x + iy) &= \cos x \cos(iy) - \sin x \sin(iy) \\ &= \cos x \cosh y - \sin x (i \sinh y) \\ &= \cos x \cosh y - i \sin x \sinh y,\end{aligned}$$

$$\begin{aligned}\sin(x + iy) &= \sin x \cos(iy) + \cos x \sin(iy) \\ &= \sin x \cosh y + \cos x (i \sinh y) \\ &= \sin x \cosh y + i \cos x \sinh y.\end{aligned}$$

12. Prove the identity:

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

Deduce that

$$\lim_{y \rightarrow \infty} |\cos z| = \infty \quad \text{and} \quad \lim_{y \rightarrow \infty} |\sin z| = \infty$$



**Solution:**

$$\begin{aligned}
 |\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\
 &= \cos^2 x \cosh^2 y + (1 - \cos^2 x) \sinh^2 y \\
 &= \cos^2 x (\cosh^2 y - \sinh^2 y) + \sinh^2 y \\
 &= \cos^2 x + \sinh^2 y
 \end{aligned}$$

$$\begin{aligned}
 |\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
 &= \sin^2 x \cosh^2 y + (1 - \sin^2 x) \sinh^2 y \\
 &= \sin^2 x (\cosh^2 y - \sinh^2 y) + \sinh^2 y \\
 &= \sin^2 x + \sinh^2 y
 \end{aligned}$$

Since  $\sinh y \rightarrow \infty$  as  $y \rightarrow \infty$  and  $|\sinh y| \leq |\cos z|$ ,  $|\sin z|$  by the above, the statement about the limits follows.

13. Solve the equation  $|\cot z| = 1$ .

**Solution:** We have:

$$\begin{aligned}
 &|\cot z| = 1 \\
 \iff &|\cos z|^2 = |\sin z|^2 \\
 \iff &\cos^2 x + \sinh^2 y = \sin^2 x + \sinh^2 y \quad (\text{see above}) \\
 \iff &\cos^2 x = \sin^2 x \\
 \iff &\cos x = \pm \sin x \\
 \iff &x = \pi \pm \frac{\pi}{4}
 \end{aligned}$$

Thus,

$$|\cot z| = 1 \iff \operatorname{Re} z = k\pi \pm \frac{\pi}{4}.$$

14. Solve the equations:

(a)  $e^{\bar{z}} = \overline{e^z}$

**Solution:** We have:

$$\begin{aligned} e^{\bar{z}} &= e^{x-iy} = e^x(\cos(-y) + i\sin(-y)) \\ &= e^x(\cos y - i\sin y) = \overline{e^x(\cos y + i\sin y)} = \overline{e^z}. \end{aligned}$$

Thus, this identity holds for all  $z$ .

(b)  $\cos(i\bar{z}) = \overline{\cos(iz)}$

**Solution:** We showed above that  $e^{\bar{z}} = \overline{e^z}$ . Therefore,

$$\cos(i\bar{z}) = \cosh \bar{z} = \frac{e^{\bar{z}} + e^{-\bar{z}}}{2} = \frac{\overline{e^z + e^{-z}}}{2} = \overline{\cosh z} = \overline{\cos(iz)}$$

for all  $z$ .