MATH 307 — Winter 2020 — Final Exam

Instructions:

1. $(1+i)^{17}$ equals:

   A. $512(1+i)$
   B. $512(1-i)$
   C. $256(1+i)$
   D. $256(1-i)$
   E. None of these.

2. Let $\log z$ be the branch of the logarithm for which

$$\log re^{i\theta} = \log r + i\theta \quad \text{for} \quad \theta \in [\pi, 3\pi).$$

   If

$$z = \frac{2i}{\sqrt{3}+i},$$

   then $\log z$ equals:

   A. $4\pi i/3$
   B. $5\pi i/3$
   C. $7\pi i/3$
   D. $8\pi i/3$
   E. None of these.

3. $\dfrac{-2+i}{\sqrt{5}}$ equals:

   A. $e^{i\arctan(1/2)}$
   B. $e^{i(\pi-\arctan(1/2))}$
   C. $e^{i(\pi+\arctan(1/2))}$
   D. $e^{i(-\pi-\arctan(1/2))}$
   E. $e^{i(-\pi+\arctan(1/2))}$
   F. None of these.

4. The function e^z maps

    A. circles to vertical lines.
    B. horizontal lines to circles.
    C. vertical lines to rays through the origin.
    D. rays through the origin to horizontal lines.
    E. None of these.

5. $e^{2020\pi i/11}$ equals:

    A. $e^{-4i\pi/11}$
    B. $e^{-9i\pi/11}$
    C. $e^{-14i\pi/11}$
    D. $e^{-19i\pi/11}$
    E. None of these.

6. $\displaystyle\lim_{z\to e^{2i\pi/3}} \frac{z(z-e^{2i\pi/3})}{z^3-1}$ equals:

    A. $-\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$
    B. $\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$
    C. $-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$
    D. $\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$
    E. None of these.

7. Which of the following functions $u(x, y)$ is harmonic?

    A. $\operatorname{Im} e^z$
    B. $\operatorname{Re} e^{\bar{z}}$
    C. $\operatorname{Im} e^{\bar{z}}$
    D. $\operatorname{Re}(z^2)\operatorname{Im}(z^3) + \operatorname{Im}(z^2)\operatorname{Re}(z^3)$

E. All of these.

F. None of these.

8. For $z \in \mathbb{C} \setminus \{0\}$, let Γ_z be the set of all smooth paths from 1 to z that don't pass through 0. Which of the following is correct?

   A. $\displaystyle\int_\gamma \frac{dw}{w}$ is independent of $\gamma \in \Gamma_z$, for all $z \in \mathbb{C} \setminus \{0\}$.

   B. $\displaystyle\int_\gamma \frac{dw}{e^w - 1}$ is independent $\gamma \in \Gamma_z$, for all $z \in \mathbb{C} \setminus \{0\}$.

   C. $\displaystyle\int_\gamma \frac{dw}{e^{-w}}$ is independent of $\gamma \in \Gamma_z$, for all $z \in \mathbb{C} \setminus \{0\}$.

   D. $\displaystyle\int_\gamma \frac{dw}{\sinh w}$ is independent of $\gamma \in \Gamma_z$, for all $z \in \mathbb{C} \setminus \{0\}$.

   E. None of these.

9. Let $I = \displaystyle\int_{|z|=1} \frac{e^z}{z^2}\, dz$ and let $J = \displaystyle\int_{|z|=1} \frac{\cos z}{z^2}\, dz$.

   Which of the following is correct?

   A. $I = 0$ and $J = 0$.

   B. $I = 0$ and $J \neq 0$.

   C. $I \neq 0$ and $J = 0$

   D. $I \neq 0$ and $J \neq 0$.

   E. None of these.

10. Let γ be the positively oriented, circular path of radius 1/2 around -1. Let

$$I = \int_\gamma \frac{dz}{z^3 + 1}\, dz, \qquad J = \int_\gamma \frac{dz}{(z+1)^3}\, dz.$$

   Which of the following is correct?

   A. $I = 0$ and $J = 0$.

   B. $I = 0$ and $J \neq 0$.

C. $I \neq 0$ and $J = 0$

D. $I \neq 0$ and $J \neq 0$.

E. None of these.

11. Which of the following is a harmonic conjugate of $u(x, y) = \sinh x \sin y$?

A. $v(x, y) = \cos x \cosh y$

B. $v(x, y) = -\cos x \cosh y$

C. $v(x, y) = \cosh x \cos y$

D. $v(x, y) = -\cosh x \cos y$

E. None of these.

12. Let $f = u + iv$ be a continuous, complex-valued function on the connected open set U. Which of the following statements is equivalent to the analyticity of f on U?

A. $\dfrac{\partial f}{\partial z} = 0$

B. For every $z_0 \in U$, there are coefficients $a_0, a_1, \ldots$ such that

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges to $f(z)$ for all $z \in U$.

C. $u_x = v_y$ and $v_x = u_y$

D. $\displaystyle\int_\gamma f(z)\, dz = 0$ for all rectangular paths γ in U.

E. None of these.

13. $\displaystyle\int_{|z|=1} \frac{\sin z}{z^2}$ equals:

A. 0

B. 1

C. -1

D. $2\pi i$

E. $-2\pi i$

F. None of these.

14. Let γ be the positively oriented, triangular path with vertices 1, $e^{2\pi i/3}$, and $e^{4\pi i/3}$ Then

$$\int_\gamma \frac{dz}{z^2+1}$$

equals:

A. 0
B. 1
C. -1
D. $2\pi i$
E. $-2\pi i$
F. None of these.

15. The power series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$

A. converges pointwise in the disk $|z| \leq 1$.
B. converges uniformly in the disk $|z| < 1$.
C. converges pointwise in the annulus $|z| \geq 1$.
D. converges uniformly in the annulus $|z| > 1$.
E. None of these are correct.

16. The radii of convergence of the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \text{and} \qquad \sum_{n=0}^{\infty} n! z^n$$

are

A. 1 and 1, respectively.
B. 1 and ∞, respectively.
C. ∞ and 1, respectively.
D. ∞ and ∞, respectively.

E. None of these.

17. The sequence of functions $f_n(z) = z^n$

A. converges uniformly on $|z| < 1$.

B. does not converge uniformly on any disk centered at 0.

C. converges pointwise on $|z| \leq 1$.

D. diverges for infinitely many z with $|z| = 1$.

E. None of these.

18. Let $\operatorname{Log} z$ be the principal branch of the logarithm. Then the limit

$$\lim_{y \to 0^+} \left(\operatorname{Log}(iy - 1) - \operatorname{Log}(-iy - 1)\right)$$

equals:

A. -2π

B. $-2\pi i$

C. 2π

D. $2\pi i$

E. None of these.

19. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-ix}}{1 + x^2}\, dx$

A. $2\pi i \operatorname*{Res}_{z=i} \dfrac{e^{-iz}}{1 + z^2}$

B. $2\pi i \operatorname*{Res}_{z=-i} \dfrac{e^{-iz}}{1 + z^2}$

C. $2\pi i \left(\operatorname*{Res}_{z=i} \dfrac{e^{-iz}}{1 + z^2} + \operatorname*{Res}_{z=-i} \dfrac{e^{-iz}}{1 + z^2} \right)$

D. $2\pi i \left(\operatorname*{Res}_{z=i} \dfrac{e^{-iz}}{1 + z^2} - \operatorname*{Res}_{z=-i} \dfrac{e^{-iz}}{1 + z^2} \right)$

E. None of these.

20. The function

$$f(z) = \frac{\sin(iz)}{z(z^2+\pi^2)^2}$$

has

A. a removable singularity at $z = 0$ and simple poles at $z = i\pi$ and $z = -i\pi$.
B. simple poles at $z = 0$, $z = i\pi$, and $z = -i\pi$.
C. a simple poles at $z = 0$ and double poles at $z = i\pi$ and $z = -i\pi$.
D. a removable singularity at $z = 0$ and double poles at $z = i\pi$ and $z = -i\pi$.
E. None of the above.

21. The function

$$f(z) = \frac{z^2}{\sin\frac{1}{z}}$$

has

A. a removable singularity at $z = 0$.
B. a simple pole at $z = 0$.
C. a pole of order ≥ 2 at $z = 0$.
D. an essential singularity at $z = 0$.
E. None of these.

22. Let

$$f(z) = \frac{1}{z(z-1)(z^2-9)}.$$

Denote by L be the Laurent expansion of $f(z)$ around $z = 1$ that converges at $z = 5/2$. Then:

A. L diverges at both $z = -1/2$ and $z = 3/2$.
B. L diverges at $z = -1/2$ but converges at $z = 3/2$.
C. L converges at $z = -1/2$ but diverges at $z = 3/2$.
D. L converges at both $z = -1/2$ and $z = 3/2$.
E. None of these.

23. $\displaystyle\int_{|z-1|=3} \cot z \, dz$ equals

A. 0
B. $2\pi i$
C. $-2\pi i$
D. $4\pi i$
E. $-4\pi i$
F. None of these.

24. The residue of

$$f(z) = \frac{z^2 - 1}{\cos(\pi z) + 1}$$

at $z = 1$ is:

A. 0
B. $-1/\pi$
C. $2/\pi^2$
D. $-3/\pi$
E. $4/\pi^2$
F. None of these.

25. Suppose $f(z)$ is analytic on $\mathbb{C}$ except for

- removable singularities at $z = 0$ and $z = 1$,
- a double pole at $z = 1 + i$,
- and an essential singularity at $z = 3i$.

Then the radius of convergence of the Taylor expansion of $f(z)$ around $z = -1$ is:

A. 0
B. 1
C. 2
D. $\sqrt{5}$
E. $\sqrt{10}$
F. None of these.