1. Algebra

$*$ A *complex number*, z, is an expression of the form

$$z = a + bi, \quad \text{where} \quad a, b \in \mathbb{R}.$$

Set

$$\Re(z) = a \quad \text{(real part)}, \quad \Im(z) = b \quad \text{(imaginary part)}.$$

Write $\mathbb{C}$ for the set of all complex numbers:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}.$$

We view $\mathbb{R}$ as a subset of $\mathbb{C}$ by identifying $a + 0i \in \mathbb{C}$ and $a \in \mathbb{R}$. A complex number of the form $0 + bi$ is called *purely imaginary.*

Two complex numbers are equal when their real are equal and their imaginary parts are equal:

$$w = z \Longleftrightarrow \Re(w) = \Re(z) \quad \text{and} \quad \Im(w) = \Im(z)$$

$*$ Add and subtract complex numbers "as usual":

$$(a + bi) \pm (c + di) = (a + c) \pm (b + d)i$$

Equivalently,

$$\Re(z \pm w) = \Re(z) \pm \Re(w), \quad \Im(z \pm w) = \Im(z) \pm \Im(w).$$

$*$ Multiply complex numbers "as usual", subject to the extra rule $i^2 = -1$:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Equivalently,

$$\Re(zw) = \Re(z)\Re(w) - \Im(z)\Im(w), \quad \Im(zw) = \Re(z)\Im(w) + \Im(z)\Re(w).$$

$*$ Here's a formula for the z/w, $w \neq 0$. Do not memorize it! (See below.)

$$\frac{a + bi}{c + di} = \frac{(ac + bd)}{a^2 + b^2} + \frac{-ad + bc}{a^2 + b^2} i$$

We get reciprocals as a special case

$*$ Define the *complex conjugate*, $\bar{z}$, of z by

$$\bar{z} = a - bi.$$

Equivalently,

$$\Re(\bar{z}) = \Re(z), \quad \Im(\bar{z}) = -\Im(z).$$

Observe that

$$z \in \mathbb{R} \Longleftrightarrow z = \bar{z}, \quad z + \bar{z} = 2\Re(z) \in \mathbb{R}, \quad z - \bar{z} = 2i\Im(z) \in \mathbb{R}i$$

We have:

$$\overline{z \pm w} = \bar{z} \pm \bar{w}, \quad \overline{zw} = \bar{z}\bar{w}, \quad \overline{z/w} = \bar{z}/\bar{w}, \quad \overline{1/z} = 1/\bar{z}, \quad z\bar{z} = a^2 + b^2$$

Complex conjugates appear in the quadratic formula for negative discriminants. If $a, b, c \in \mathbb{R}$ and $D := b^2 - 4ac < 0$, then the roots of $az^2 + bz + c = 0$ are

$$z = \frac{-b + \sqrt{D}i}{2a} \quad \text{and} \quad \bar{z} = \frac{-b - \sqrt{D}i}{2a}.$$

Theorem: Every quadratic polynomial with real coefficients has two roots in $\mathbb{C}$, counted with multiplicity.

$*$ Extend the absolute value function from $\mathbb{R}$ to $\mathbb{C}$ by setting

$$|z| = \sqrt{a^2 + b^2}.$$

$|z|$ is also called the *modulus* of z. Notice that

$$z\bar{z} = |z|^2.$$

We have $|z| \geq 0$, with equality if and only if $z = 0$. Also,

$$|z + w| \leq |z| + |w| \quad \text{triangle inequality}, \quad |zw| = |z||w|.$$

The property $|zw| = |z||w|$ is equivalent to the identity

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$

Thus, a product of sums of squares is a sum of squares.

Observe that if $w \neq 0$, then

$$\frac{w}{z} = \frac{w\bar{z}}{z\bar{z}} = \frac{w\bar{z}}{|z|^2} = \frac{w\bar{z}}{a^2 + b^2}.$$

To divide z by w, just multiply numerator and denominator by $\bar{w}$ and then follow your nose. Reciprocals are easy:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{a^2 + b^2}.$$

It's useful to note that

$$|z| = 1 \Longleftrightarrow \frac{1}{z} = \bar{z}.$$

$3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, $33^2 + 56^2 = 65^2$

2. Geometry