MATH 307 — Worksheet #9

Evaluating definite integrals using the residue theorem

Theorem I. Let $R(x, y)$ be a rational function of x and y whose denominator doesn't vanish on $|z| = 1$. Then

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = 2\pi i \sum\{\text{residues of } f(z) \text{ inside } |z| = 1\},$$

where

$$f(z) = \frac{1}{iz} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)$$

Theorem II.

1. Let $f(z)$ be analytic on the closed upper half-plane

   $$\mathcal{H}^* = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$$

   except for finitely many singularities in the open upper half-plane

   $$\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

   Suppose there are positive constants M, p, and R_0 with $p > 1$ such that

   $$|f(z)| \leq \frac{M}{z^p} \quad \text{for all } z \in \mathcal{H} \text{ with } |z| \geq R_0.$$

   Then

   $$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum\{\text{residues of } f(z) \text{ in } \mathcal{H}\}.$$

2. Let $f(z)$ be analytic on the closed lower half-plane

   $$\mathcal{L}^* = \{z \in \mathbb{C} : \operatorname{Im} z \leq 0\}$$

   except for finitely many singularities in the open lower half-plane

   $$\mathcal{L} = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}.$$

   Suppose there are positive constants M, p, and R_0 with $p > 1$ such that

   $$|f(z)| \leq \frac{M}{z^p} \quad \text{for all } z \in \mathcal{L} \text{ with } |z| \geq R_0.$$

   Then

   $$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum\{\text{residues of } f(z) \text{ in } \mathcal{L}\}.$$

3. Both 1. and 2. hold when $f = P/Q$ is a rational function such that

   (a) $\deg Q \geq \deg P + 2$, and

   (b) Q has no real roots.

Problems

1. Evaluate the definite integral.

(a) $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 - 2x + 4}$

Solution: We have

$$\deg(x^2 - 2x + 4) = 2 \geq 2 + 0 = 2 + \deg 1.$$

Moreover $x^2 - 2x + 4$ has roots α and $\bar{\alpha}$,

$$\alpha = \frac{2 + \sqrt{(-2)^2 - 4(1)(4)}}{2} = 1 + \sqrt{3}i,$$

neither of which lie on the real line. Therefore, by Theorem II(3),

$$I = 2\pi i \operatorname*{Res}_{z=\alpha} \frac{1}{(z-\alpha)(z-\bar{\alpha})} = \frac{2\pi i}{\alpha - \bar{\alpha}} = \frac{2\pi i}{2\sqrt{3}i} = \frac{\pi}{\sqrt{3}}.$$

(b) $I = \displaystyle\int_{0}^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2}$

Solution: Let $z = e^{i\theta}$, so that

$$\sin\theta = \frac{z^2 - 1}{2iz} \quad \text{and} \quad d\theta = \frac{dz}{iz}.$$

It follows that

$$\begin{aligned}
I &= \int_{|z|=1} \frac{1}{\left(5 - 3\left(\frac{z^2-1}{2iz}\right)\right)^2} \frac{dz}{iz} \\
&= \int_{|z|=1} \frac{(2iz)^2}{\left(5(2iz) - 3\left(z^2 - 1\right)\right)^2} \frac{dz}{iz} \\
&= \int_{|z|=1} \frac{4iz\,dz}{\left(3z^2 - 10iz - 3\right)^2} \\
&= \frac{4i}{9} \int_{|z|=1} \frac{z\,dz}{(z-3i)^2(z-i/3)^2}
\end{aligned}$$

The integrand has no singularities on $|z| = 1$ and its only singularity inside $|z| = 1$ is a double pole at $z = i/3$.

We have:

$$\begin{aligned}
\operatorname*{Res}_{z=i/3} \frac{z}{(z-3i)^2(z-i/3)^2} &= \frac{d}{dz}\bigg|_{z=i/3} (z-i/3)^2 \frac{z}{(z-3i)^2(z-i/3)^2} \\
&= \frac{d}{dz}\bigg|_{z=i/3} \frac{z}{(z-3i)^2} \\
&= \frac{-z-3i}{(z-3i)^3}\bigg|_{z=i/3} \\
&= -\frac{45}{256}
\end{aligned}$$

Therefore,

$$I = 2\pi i \cdot \frac{4i}{9} \cdot \frac{-45}{256} = \frac{5\pi}{32}.$$

(c) $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}$ where $a > |b|$

Solution: Let $z = e^{i\theta}$, so that

$$\sin\theta = \frac{z^2-1}{2iz} \quad \text{and} \quad d\theta = \frac{dz}{iz}.$$

It follows that

$$\begin{aligned}
I &= \int_{|z|=1} \frac{1}{a + b\left(\frac{z^2-1}{2iz}\right)} \frac{dz}{iz} \\
&= 2\int_{|z|=1} \frac{dz}{bz^2 + 2iaz - b} \\
&= \frac{2}{b}\int_{|z|=1} \frac{dz}{(z-i\alpha)(z-i\beta)},
\end{aligned}$$

$$\alpha = \frac{-a + \sqrt{a^2-b^2}}{b}, \quad \beta = \frac{-a - \sqrt{a^2-b^2}}{b}$$

Since $a > |b|$, $\beta < -1$ and $i\beta$ lies outside the closed unit disk. Noting that

$\alpha = 1/\beta$, it follows that α lies inside the open unit disk. Therefore,

$$\begin{aligned}
I &= 2\pi i \cdot \frac{2}{b} \cdot \operatorname*{Res}_{z=i\alpha} \frac{1}{(z-i\alpha)(z-i\beta)} \\
&= 2\pi i \cdot \frac{2}{b} \cdot \frac{1}{i(\alpha-\beta)} \\
&= 2\pi i \cdot \frac{2}{b} \cdot \frac{b}{2i\sqrt{a^2-b^2}} \\
&= \frac{2\pi}{\sqrt{a^2-b^2}}.
\end{aligned}$$

(d) $I = \displaystyle\int_{-\infty}^{\infty} \frac{1}{x^6+1} dx$

Solution: Let $f(z) = 1/(z^6+1)$. Then by Theorem II(3),

$$I = 2\pi i \sum \{\text{residues of } f(z) \text{ in } \mathcal{H}\}.$$

The roots of the polynomial z^6+1 are

$$z_k = e^{(\pi i + 2k\pi)/6}, \quad k = 0, 1, 2, 3, 4, 5.$$

The roots in the upper half-plane are

$$z_0 = e^{\pi i/6}, \quad z_1 = e^{3\pi i/6}, \quad \text{and} \quad z_2 = e^{5\pi i/6}.$$

Being simple zeros of z^6+1, the points z_k are simple poles of $f(z)$. Therefore,

$$\begin{aligned}
\operatorname*{Res}_{z=z_k} f(z) &= \lim_{z\to z_k} \frac{z-z_k}{z^6+1} \\
&= \lim_{z\to z_k} \frac{1}{6z^5} \\
&= \frac{1}{6z_k^5},
\end{aligned}$$

by l'Hopital's rule.

Thus,

$$\begin{aligned}
\operatorname*{Res}_{z=z_0} f(z) &= \frac{1}{6}e^{-5\pi i} \\
&= \frac{1}{6}\left(\frac{-\sqrt{3}-i}{2}\right) \\
&= -\frac{\sqrt{3}}{12} - \frac{i}{12} \\
\operatorname*{Res}_{z=z_1} f(z) &= \frac{1}{6}e^{-15\pi i/6} \\
&= \frac{1}{6}e^{-\pi i/2} \\
&= -\frac{i}{6} \\
\operatorname*{Res}_{z=z_0} f(z) &= \frac{1}{6}e^{-25\pi i} \\
&= \frac{1}{6}e^{-\pi i/6} \\
&= \frac{\sqrt{3}}{12} - \frac{i}{12}
\end{aligned}$$

Therefore,

$$I = 2\pi i\left(-\frac{\sqrt{3}}{6} - \frac{i}{12} - \frac{i}{6} + \frac{\sqrt{3}}{12} - \frac{i}{12}\right) = \frac{2\pi}{3}.$$