MATH 307 — Worksheet #9

Evaluating definite integrals using the residue theorem

Theorem I. Let $R(x, y)$ be a rational function of x and y whose denominator doesn't vanish on $|z| = 1$. Then

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = 2\pi i \sum \{\text{residues of } f(z) \text{ inside } |z| = 1\},$$

where

$$f(z) = \frac{1}{iz} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)$$

Theorem II.

1. Let $f(z)$ be analytic on the closed upper half-plane

$$\mathcal{H}^* = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$$

except for finitely many singularities in the open upper half-plane

$$\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

Suppose there are positive constants M, p, and R_0 with $p > 1$ such that

$$|f(z)| \leq \frac{M}{z^p} \quad \text{for all } z \in \mathcal{H} \text{ with } |z| \geq R_0.$$

Then

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \{\text{residues of } f(z) \text{ in } \mathcal{H}\}.$$

2. Let $f(z)$ be analytic on the closed lower half-plane

$$\mathcal{L}^* = \{z \in \mathbb{C} : \operatorname{Im} z \leq 0\}$$

except for finitely many singularities in the open lower half-plane

$$\mathcal{L} = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}.$$

Suppose there are positive constants M, p, and R_0 with $p > 1$ such that

$$|f(z)| \leq \frac{M}{z^p} \quad \text{for all } z \in \mathcal{L} \text{ with } |z| \geq R_0.$$

Then

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \{\text{residues of } f(z) \text{ in } \mathcal{L}\}.$$

3. Both 1. and 2. hold when $f = P/Q$ is a rational function such that

   (a) $\deg Q \geq \deg P + 2$, and

   (b) Q has no real roots.

Problems

1. Evaluate the definite integral.

   (a) $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 - 2x + 4}$

   (b) $I = \displaystyle\int_{0}^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2}$

   (c) $I = \displaystyle\int_{0}^{2\pi} \frac{d\theta}{a + b\sin\theta}$ where $a > |b|$

   (d) $I = \displaystyle\int_{-\infty}^{\infty} \frac{1}{x^6 + 1} dx$