MATH 307 — Worksheet #5

1. Compute the integral. All curves are oriented counterclockwise.

   (a) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{z\cos z}{(z+2i)^2}\mathrm{d}z$, where C is the unit circle

   (b) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{z\cos z}{z-2i}\mathrm{d}z$, where C is the circle $|z-i|=2$

   (c) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{2e^{2z}}{z^2+1}\mathrm{d}z$, where C is the square with vertices at 1, $1+2i$, $-1+2i$, and -1.

   (d) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{2ze^{z}}{z^2+1}\mathrm{d}z$, where C is the circle $|z|=2$

   (e) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{e^{3z}}{z^3}\mathrm{d}z$, where C is the unit circle

   (f) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{e^{3z}}{z^3-2z^2}\mathrm{d}z$, where C is the unit circle $|z|=2$

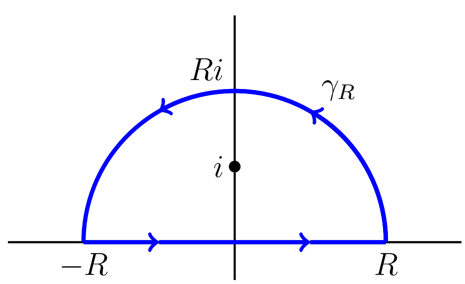
2. In this problem, we evaluate the real, improper integral

$$I := \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)^2}.$$

   (a) Let $R>0$. Use Cauchy's integral formula to compute

$$J_R := \int_{\gamma_R} \frac{\mathrm{d}z}{(z^2+1)^2},$$

   where γ_R is the simple, closed curve drawn in blue below.



(b) Let δ_R, be the semicircular portion of γ_R. Show that

$$|K_R| \leq \frac{\pi R}{(R^2-1)^2}, \qquad \text{where} \qquad K_R := \int_{\delta_R} \frac{\mathrm{d}z}{(z^2+1)^2}.$$

Hint: Show that $|z^2+1| \geq R^2 - 1$ for z on γ_R, then use the ML-bound.

(c) Briefly justify the identity

$$J_R = K_R + I_R, \qquad \text{where} \qquad I_R := \int_{-R}^{R} \frac{\mathrm{d}x}{(x^2+1)^2}.$$

Let $R \to \infty$ and evaluate I.