MATH 307 — Worksheet #2

1. Let $\sqrt{\cdot}$ denote the branch of the square root defined by

$$\sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}, \quad \theta \in [0, 2\pi)$$

   For which z does the identity $\sqrt{z^2} = z$ hold?

2. Find all values.

   (a) $\log 1$

   (b) $\log(1+i)$

   (c) $(1+i)^{1+i}$

3. Let $z = re^{i\theta}$. Express all values of z^z in the form $x + iy$.

4. Compute the limit or argue that it don't exist.

   (a) $\lim\limits_{x\to\infty} e^{x+iy}$ (fixed y)

   (b) $\lim\limits_{x\to-\infty} e^{x+iy}$ (fixed y)

   (c) $\lim\limits_{y\to\infty} e^{x+iy}$ (fixed x)

   (d) $\lim\limits_{y\to-\infty} e^{x+iy}$ (fixed x)

   (e) $\lim\limits_{|z|\to\infty} e^{z}$

   (f) $\lim\limits_{|z|\to\infty} |e^{z}|$

5. (a) Prove that $|a^b| = |a|^b$ for $a \in \mathbb{C}$ and $b \in \mathbb{R}$.

   (b) Prove that, for a fixed branch of log, $a^{b+c} = a^b a^c$.

   (c) Prove that, for a fixed branch of log, $(ab)^c = a^c b^c$ valid for all complex a, b, c such that $\log(ab) = \log a + \log b$.

6. Determine the set on which the function is analytic and compute its derivative.

   (a) $\dfrac{1}{(z^3-1)(z^2+2)}$

   (b) $\dfrac{1}{z+z^{-1}}$

   (c) $\dfrac{z}{z^n-2}$

7. Let

$$f(z) = \begin{cases} z^5/|z|^4 & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

   (a) Show that

$$\lim_{z\to 0} \frac{f(z)}{z}$$

   does not exist.

   (b) Let $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. Show that

$$u(x,0) = x, \quad u(0,y) = 0, \quad v(x,0) = 0, \quad v(0,y) = y.$$

   (c) Conclude that the partial derivatives of u and v with respect to x and y exist, that the Cauchy-Riemann equations are satisfied, but $f'(0)$ does not exist. Why does this not contradict the Cauchy-Riemann theorem?

8. Find the real and imaginary parts of the function and verify that they satisfy the Cauchy-Riemann equations.

   (a) $f(z) = z^3$

   (b) ze^{-z}

   (c) $\cos 2z$