MATH 307 — Worksheet #7

1. Find the Laurent expansion of
$$f(z) = \frac{1}{z(z^2+1)}$$
valid in the given region.

   (a) $0 < |z| < 1$

Solution:

$$\begin{aligned} f(z) &= \frac{1}{z}\frac{1}{1-(-z^2)} \\ &= \frac{1}{z}\sum_{n=0}^{\infty}(-z^2)^n \\ &= \sum_{n=0}^{\infty}(-1)^n z^{2n-1} \end{aligned}$$

   (b) $|z| > 1$

Solution:

$$\begin{aligned} f(z) &= \frac{1}{z^3}\frac{1}{1-(-z^{-2})} \\ &= \frac{1}{z^3}\sum_{n=0}^{\infty}(-z^{-2})^n \\ &= \sum_{n=0}^{\infty}(-1)^n z^{-2n-3} \end{aligned}$$

2. Find the Laurent expansions of
$$f(z) = \frac{1}{1+z^2} + \frac{1}{3-z}$$
valid in the given region.

   (a) $|z| < 1$

Solution:

$$\begin{aligned} f(z) &= \frac{1}{1-(-z^2)} + \frac{1}{3}\frac{1}{1-z/3} \\ &= \sum_{n=0}^{\infty}(-z^2)^n + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n z^{2n} + \sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}} \end{aligned}$$

(b) $1 < |z| < 3$

Solution:

$$\begin{aligned} f(z) &= \frac{1}{z^2}\frac{1}{1-(-z^{-2})} + \frac{1}{3}\frac{1}{1-z/3} \\ &= \sum_{n=0}^{\infty}(-z^{-2})^n + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n z^{-2n} + \sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}} \end{aligned}$$

(c) $|z| > 3$

Solution:

$$\begin{aligned} f(z) &= \frac{1}{z^2}\frac{1}{1-(-z^{-2})} - \frac{1}{z}\frac{1}{1-3/z} \\ &= \sum_{n=0}^{\infty}(-z^{-2})^n + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n z^{-2n} + \sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}} \end{aligned}$$

3. Find the poles of the $f(z)$. For each such pole, a, determine:

- $\operatorname{ord}_a f(z)$,
- $\operatorname{res}_a f(z)$,
- the annuli of convergence of the Laurent expansions of $f(z)$ around a.

(a) $f(z) = \dfrac{e^z(z-3)}{(z-1)(z-5)}$

Solution: The poles of $f(z)$, both simple, are at 1 and 5.

$$\operatorname{res}_1 f(z) = \lim_{z\to 1}(z-1)\frac{e^z(z-3)}{(z-1)(z-5)} = \frac{e^1(1-3)}{1-5} = \frac{e}{2}$$
$$\operatorname{res}_5 f(z) = \lim_{z\to 5}(z-5)\frac{e^z(z-3)}{(z-1)(z-5)} = \frac{e^5(5-3)}{5-1} = \frac{e^5}{2}$$

The annuli of convergence of the Laurent expansions of $f(z)$ around $z=1$ are $0 < |z-1| < 4$ and $|z-1| > 4$.

The annuli of convergence of the Laurent expansions of $f(z)$ around $z=5$ are $0 < |z-5| < 4$ and $|z-5| > 4$.

(b) $f(z) = \dfrac{e^z - 1}{z}$

Solution: Since $\operatorname{ord}_0(e^z-1) = 1$ and $\operatorname{ord}_0 z = 1$, $f(z)$ has a pole of order $1-1=0$ at $z=0$. In other words, $z=0$ is a removable singularity of $f(z)$; $f(z)$ has no other singularities.

Since 0 is a removable singularity of $f(z)$, $\operatorname{res}_0 f(z) = 0$.

The annulus of convergence of the Laurent expansion of $f(z)$ around $z=0$ is $0 < |z| < \infty$.

(c) $f(z) = \dfrac{e^z - 2}{z}$

Solution: Since $e^z - 2$ and z vanish to orders 0 and 1, respectively, at $z=0$, $f(z)$ has a simple pole there. It has no other singularities.

$$\operatorname{res}_0 f(z) = \lim_{z\to 0} z\frac{e^z-2}{z} = e^0 - 2 = -1.$$

The annulus of convergence of the Laurent expansion of $f(z)$ around $z = 0$ is $0 < |z| < \infty$.

(d) $f(z) = \dfrac{\cos z}{1 - z}$

Solution: Since $\cos z$ and $1 - z$ vanish to orders 0 and 1, respectively, at $z = 1$, $f(z)$ has a simple pole there. It has no other singularities.

$$\operatorname{res}_1 f(z) = \lim_{z \to 1} (z - 1) \frac{\cos z}{1 - z} = -\cos 1$$

The annulus of convergence of the Laurent expansion of $f(z)$ around $z = 1$ is $0 < |z - 1| < \infty$.

(e) $f(z) = \dfrac{z^2 - 1}{\cos(\pi z) + 1}$

Solution: Write $g(z)$ and $h(z)$ for the numerator and denominator of $f(z)$, respectively. $g(z)$ has simple zeros at $z = \pm 1$ while $h(z)$ has a zero when $\cos(\pi z) = -1$, i.e., when $z = k$, k an odd integer. Since

$$h'(z) = -\pi \sin(\pi z), \quad h'(2k+1) = 0,$$
$$h''(z) = -\pi^2 \cos(\pi z), \quad h''(2k+1) = -\pi^2 \neq 0,$$

$h(z)$ has a zero of order 2 at each odd integer. It follows that

$$\operatorname{ord}_1 f(z) = \operatorname{ord}_{-1} f(z) = 1, \quad \operatorname{ord}_k = -2, \quad k \text{ an odd integer} \neq \pm 1.$$

Since $f(z)$ has simple poles at $z = \pm 1$, We have:

$$\operatorname{res}_{\pm 1} f(z) = 2 \frac{g'(\pm 1)}{h''(\pm 1)} = 2 \frac{\pm 2}{-\pi^2} = \mp \frac{4}{\pi^2}$$

Let k be an odd integer, $k \neq \pm 1$. Then

$$\operatorname{res}_k f(z) = 2 \frac{g'(k)}{h''(k)} - \frac{2}{3} \frac{g(k) h'''(k)}{h''(k)^2} = 2 \frac{2k}{-\pi^2} - \frac{2}{3} \frac{0}{\pi^4} = -\frac{4k}{\pi^2}.$$

The annuli of convergence of $f(z)$ around k are

$$0 < |z - k| < 2, \quad 2 < |z - k| < 4, \quad 4 < |z - k| < 6, \quad \ldots.$$

4. Find and classify the singularities of $f(z)$ (removable, pole of order n, essential singularity).

(a) $f(z) = \sin \dfrac{1}{z}$

Solution: The only singularity of $f(z)$ is at $z = 0$. It's an essential singularity as the Laurent expansion of $f(z)$ around $z = 0$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{2n+1}},$$

has infinitely many terms with negative powers of z.

(b) $f(z) = \csc \dfrac{1}{z}$

Solution: $f(z)$ has singularities at $z = 0$ and $z = 1/k\pi$, $k \in \mathbb{Z}$, $k \neq 0$.
Since $1/k\pi \to 0$ as $k \to \infty$, 0 is not an isolated singularity of $f(z)$. Therefore, it must be an essential singularity.
Suppose $k \neq 0$. Then $f(z)$ has a simple pole at $1/k\pi$:

$$\operatorname{ord}_{k\pi} \sin z = 1 \Longrightarrow \operatorname{ord}_{1/k\pi} \sin \frac{1}{z} = 1 \Longrightarrow \operatorname{ord}_{1/k\pi} \csc \frac{1}{z} = -1.$$