BASIC COMPLEX ANALYSIS

Third Edition

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Preface

This text is intended for undergraduates in mathematics, the physical sciences, and engineering who are taking complex analysis for the first time. Two years of calculus, up through calculus of several variables and Green's theorem, are adequate preparation for the course. The text contains some references to linear algebra and basic facts about ϵ-δ analysis, but the extent to which they are emphasized can be adjusted by the instructor depending on the background and needs of the class.

The book has a generous number of examples, exercises, and applications. We have made a special effort to motivate students by making the book readable for self-study and have provided plenty of material to help students gain an intuitive understanding of the subject. Our arrangement enables application-oriented students to skip the more technical parts without sacrificing an understanding of the main theoretical points. Applications include electric potentials, heat conduction, hydrodynamics (studied with the aid of harmonic functions and conformal mappings), Laplace transforms, asymptotic expansions, the Gamma function, and Bessel functions.

The core of Chapters 1 to 6 can be taught in a one-semester course for mathematics majors. In applied mathematics courses, if some of the technical parts of Chapter 2 and parts of Chapter 6 are omitted, then parts of Chapters 7 and 8 can be covered in one semester. It is healthy for mathematics majors to see as many of the applications as possible, for they are an integral part of the cultural and historical heritage of mathematics.

Symbols The symbols used in this text are, for the most part, standard. The set of real numbers is denoted $\mathbf{R}$, while $\mathbf{C}$ denotes the set of complex numbers. "Iff" stands for "if and only if" (except in definitions, where we write only "if"). The end of a proof is marked $\blacksquare$, the end of the proof of a lemma in the middle of a proof of a theorem is marked $\blacktriangledown$ and occasionally, the end of an example in the text is marked $\blacklozenge$. The notation $]a, b[$ represents the open interval consisting of all real numbers x satisfying $a < x < b$. This is to avoid confusion with the ordered pair notation (a, b). The notation $f : A \subset \mathbf{C} \to \mathbf{C}$ means that the mapping f maps the domain A, which is a subset of $\mathbf{C}$, into $\mathbf{C}$, and we write $z \mapsto f(z)$ for the effect of f on the point $z \in A$. Occasionally, $\Rightarrow$ is used to mean "implies". The set theoretic difference of the sets A and B is denoted by $A \backslash B$, while their union and intersection are denoted by $A \cup B$ and $A \cap B$. The definitions, theorems, propositions, lemmas, and examples are numbered consecutively for easy cross reference; for example, Definition 6.2.3 refers to the third item in Section 6.2.

Classic texts Despite the large numbers of texts written in recent years, some of the older classics remain the best. A few that are worth looking at are A. Hurwitz and R. Courant, *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen* (Berlin: Julius Springer, 1925); E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Fourth Edition (London: Cambridge University Press, 1927); E. T. Titchmarsh, *The Theory of Functions*, 2d ed. (NewYork: Oxford University Press, 1939, reprinted 1985); and K. Knopp, *Theory of Functions* (New York: Dover, 1947). The reader who wishes further information on various of the more advanced topics can profitably consult E. Hille, *Analytic Function Theory*, 2 volumes, (Boston: Ginn, 1959); L. V. Ahlfors, *Complex Analysis* (New York: McGraw-Hill, 1966); W. Rudin, *Real and Complex Analysis* (New York: McGraw-Hill, 1969); and P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (New York: McGraw-Hill, 1953). Some additional references are given throughout the text.

The modern treatment of complex analysis did not evolve rapidly or smoothly. The numerous creators of this area of mathematics traveled over many rough roads and encountered many blind alleys before the superior routes were found. An appreciation of the history of mathematics and its intimate connection to the physical sciences is important to every student's education. We recommend looking at M. Klein's *Mathematical Thought from Ancient to Modern Times* (London: Oxford University Press, 1972).

Third edition The third edition features an *Instructor's Supplement* as well as a *Student Guide*. Answers to the odd numbered problems are in the back of the book and exercises with solutions in the Student Guide are marked with a bullet (•) in the text.

We have streamlined a number of features in the text, such as the treatment of Cauchy's Theorem. We have substantially rewritten Chapter 4 on the evaluation of integrals, making the treatment less encyclopedic. An *Internet Supplement* is available free from http://www.whfreeman.com/ (look in the mathematics section) or from http://cds.caltech.edu/~marsden/ (look under "books"). It contains additional information for those who want to delve into some topics in a little more depth.

Acknowledgments We are grateful to the many readers who supplied corrections and comments for this edition. There are too many to be thanked individually, but we would like especially to mention (more or less chronologically) M. Buchner (who helped significantly with the First Edition), C. Risk, P. Roeder, W. Barker, G. Hill, J. Seitz, J. Brudowski, H. O. Cordes, M. Choi, W. T. Stallings, E. Green, R. Iltis, N. Starr, D. Fowler, L. L. Campbell, D. Goldschmidt, T. Kato, J. Mesirov, P. Kenshaft, K. L. Teo, G. Bergmann, J. Harrison and C. Daniels. Finally, we thank Barbara Marsden for her accurate typsetting of this new edition.

Chapter 1

Analytic Functions

In this chapter the basic ideas about complex numbers and analytic functions are introduced. The organization of the text is analogous to that of an elementary calculus textbook, which begins by introducing $\mathbf{R}$, the set of real numbers, and functions $f(x)$ of a real variable x. One then studies the theory and practice of differentiation and integration of functions of a real variable. Similarly, in complex analysis we begin by introducing $\mathbf{C}$, the set of complex numbers z. We then study functions $f(z)$ of a complex variable z, which are differentiable in a complex sense; these are called analytic functions.

The analogy between real and complex variables is, however, a little deceptive, because complex analysis is a surprisingly richer theory; a lot more can be said about an analytic function than about a differentiable function of a real variable, as will be fully developed in subsequent chapters.

In addition to becoming familiar with the theory, the student should strive to gain facility with the standard (or "elementary") functions such as polynomials, e^z, $\log z$, $\sin z$—as in calculus. These functions are studied in §1.3 and appear frequently throughout the text.

1.1 Introduction to Complex Numbers

The following discussion will assume some familiarity with the main properties of real numbers. The real number system resulted from the search for a system (an abstract set together with certain rules) that included the rationals but that also provided solutions to such polynomial equations as $x^2 - 2 = 0$.

Historical Perspective Historically, a similar consideration gave rise to an extension of the real numbers. As early as the sixteenth century, Geronimo Cardano considered quadratic (and cubic) equations such as $x^2 + 2x + 2 = 0$, which is satisfied by no real number x. The quadratic formula $\left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$ yields "formal" expressions for the two solutions of the equation $ax^2 + bx + c = 0$. But this formula may involve square roots of negative numbers; for example, $-1 \pm \sqrt{-1}$ for the equation $x^2 + 2x + 2 = 0$. Cardano noticed that if these "complex numbers" were treated as ordinary numbers with the added rule that $\sqrt{-1} \cdot \sqrt{-1} = -1$, they did indeed solve the equations.

The important expression $\sqrt{-1}$ is now given the widely accepted designation $i = \sqrt{-1}$. (An alternative convention is followed by many electrical engineers, who prefer the symbol $j = \sqrt{-1}$ since they wish to reserve the symbol i for electric current.) However, in the past it was felt that no meaning could actually be assigned to such expressions, which were therefore termed "imaginary." Gradually, especially as a result of the work of Leonhard Euler in the eighteenth century, these imaginary quantities came to play an important role. For example, Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ revealed the existence of a profound relationship between complex numbers and the trigonometric functions. The rule $e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2}$ was found to summarize the rules for expanding sine and cosine of a sum of two angles in a neat way, and this result alone indicated that some meaning should be attached to these "imaginary" numbers.

However, it took nearly three hundred years until the work of Casper Wessel (ca. 1797), Jean Robert Argand (1806), Karl Friedrich Gauss (1831), Sir William R. Hamilton (1837), and others, when "imaginary" numbers were recognized as legitimate mathematical objects, and it was realized that there is nothing "imaginary" about them at all (although this term is still used).

The complex analysis that is the subject of this book was developed in the nineteenth century, mainly by Augustin Cauchy (1789–1857). Later his theory was made more rigorous and extended by such mathematicians as Peter Dirichlet (1805–1859), Karl Weierstrass (1815–1897), and Georg Friedrich Bernhard Riemann (1826–1866).

The search for a method to describe heat conduction influenced the development of the theory, which has found many uses outside mathematics. Subsequent chapters will discuss some of these applications to problems in physics and engineering, such as hydrodynamics and electrostatics. The theory also has mathematical applications to problems that at first do not seem to involve complex numbers. For example, the proof that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2},$$

or that

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin(\alpha\pi)},$$

(where $0 < \alpha < 1$), or that

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = \frac{2\pi}{\sqrt{a^2-1}},$$

may be difficult or, in some cases, impossible using elementary calculus, but these identities can be readily proved using the techniques of complex variables.

The Complex Number System Complex analysis has become an indispensable and standard tool of the working mathematician, physicist, and engineer. Neglect of it can prove to be a severe handicap in most areas of research and application involving mathematical ideas and techniques. The first objective of this section will be to define complex numbers and to show that the usual algebraic manipulations hold. To begin, recall that the xy plane, denoted by $\mathbb{R}^2$, consists of all ordered pairs (x, y) of real numbers.

Definition 1.1.1 *The system of complex numbers, denoted* $\mathbb{C}$, *is the set* $\mathbb{R}^2$ *together with the usual rules of vector addition and scalar multiplication by a real number* a, *namely,*

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\ a(x, y) &= (ax, ay)\end{aligned}$$

and with the operation of complex multiplication, defined by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

We will need to explain where this strange rule of multiplication comes from! Rather than using (x, y) to represent a complex number, we will find it more convenient to return to more standard notation as follows. Let us identify *real* numbers x with points on the x axis; thus x and $(x, 0)$ stand for the same point $(x, 0)$ in $\mathbb{R}^2$. The y axis will be called the ***imaginary axis***, and the unit point $(0, 1)$ will be denoted i. Thus, by definition, $i = (0, 1)$. Then

$$(x, y) = x + yi$$

because the right side of the equation stands for

$$(x, 0) + y(0, 1) = (x, 0) + (0, y) = (x, y).$$

Using $y = (y, 0)$ and Definition 1.1.1 of complex multiplication, we get

$$iy = (0, 1)(y, 0) = (0 \cdot y - 1 \cdot 0, y \cdot 1 + 0 \cdot 0) = (0, y) = y(0, 1) = yi,$$

so we can also write $(x, y) = x + iy$. A single symbol such as $z = a + ib$ is generally used to indicate a complex number. The notation $z \in \mathbb{C}$ means that z belongs to the set of complex numbers.

Note that

$$i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, (1 \cdot 0 + 0 \cdot 1)) = (-1, 0) = -1,$$

so we do have the property we want:

$$i^2 = -1.$$

If we remember this equation, then the rule for multiplication of complex numbers is also easy to remember and motivate:

$$\begin{aligned}(a+ib)(c+id) &= ac+iad+ibc+i^2bd\\ &= (ac-bd)+i(ad+bc).\end{aligned}$$

For example, $2+3i$ is the complex number $(2,3)$, and

$$(2+3i)(1-4i) = 2-12i^2+3i-8i = 14-5i$$

is another way of saying that

$$(2,3)(1,-4) = (2\cdot 1-3(-4), 3\cdot 1+2(-4)) = (14,-5).$$

The reason for using the expression $a+bi$ is twofold. First, it is conventional. Second, the rule $i^2=-1$ is easier to use than the rule $(a,b)(c,d)=(ac-bd, bc+ad)$, although both rules produce the same result.

Because multiplication of real numbers is associative, commutative, and distributive, it is reasonable to expect that multiplication of complex numbers is also; that is, for all complex numbers z, w, and s we have

$$(zw)s = z(ws), \quad zw = wz, \quad \text{and} \quad z(w+s) = zw+zs.$$

Let us verify the first of these properties; the others can be similarly verified.

Let $z=a+ib$, $w=c+id$, and $s=e+if$. Then $zw=(ac-bd)+i(bc+ad)$, so

$$(zw)s = e(ac-bd)-f(bc+ad)+i[e(bc+ad)+f(ac-bd)].$$

Similarly,

$$\begin{aligned}z(ws) &= (a+bi)[(ce-df)+i(cf+de)]\\ &= a(ce-df)-b(cf+de)+i[a(cf+de)+b(ce-df)].\end{aligned}$$

Comparing these expressions and accepting the usual properties of real numbers, we conclude that $(zw)s=z(ws)$. Thus we can write, without ambiguity, an expression like $z^n = z\cdot\ldots\cdot z$ (n times).

Note that $a+ib=c+id$ means $a=c$ and $b=d$ (since this is what equality means in $\mathbb{R}^2$) and that 0 stands for $0+i0=(0,0)$. Thus $a+ib=0$ means that *both* $a=0$ and $b=0$.

In what sense are these complex numbers an extension of the reals? We have already said that if a is real we also write a to stand for $a+0i=(a,0)$. In other words, the reals $\mathbb{R}$ are identified with the x axis in $\mathbb{C}=\mathbb{R}^2$; we are thus regarding the real numbers as those complex numbers $a+bi$ for which $b=0$. If, in the expression $a+bi$, the term $a=0$, we call $bi=0+bi$ a ***pure imaginary number***. In the expression $a+bi$ we say that a is the ***real part*** and b is the ***imaginary part***. This is sometimes written $\operatorname{Re} z = a$, $\operatorname{Im} z = b$, where $z=a+bi$. Note that $\operatorname{Re} z$ and $\operatorname{Im} z$ are always real numbers (see Figure 1.1.1).

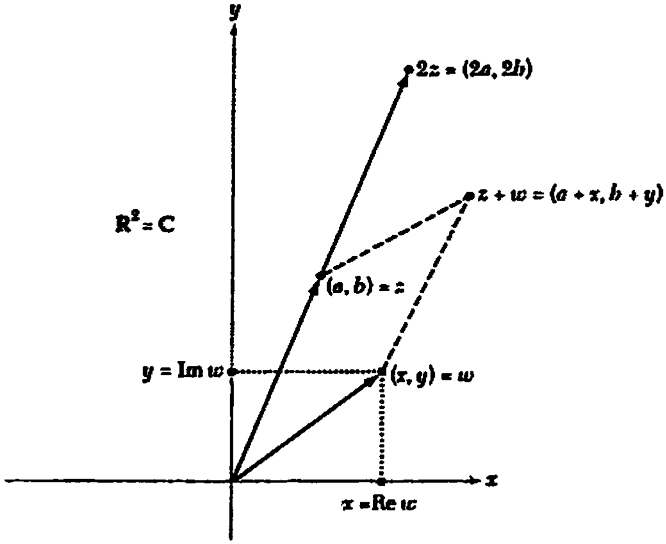


Figure 1.1.1: The geometry of complex numbers.

Algebraic Properties The complex numbers obey all the algebraic rules that ordinary real numbers do. For example, it will be shown in the following discussion that multiplicative inverses exist for nonzero elements. This means that if $z \neq 0$, then there is a (complex) number z' such that $zz' = 1$, and we write $z' = z^{-1}$. We can write this expression unambiguously (in other words, z' is uniquely determined), because if $zz'' = 1$ as well, then $z' = z' \cdot 1 = z'(zz'') = (z'z)z'' = 1 \cdot z'' = z''$, and so $z'' = z'$. To show that z' exists, suppose that $z = a + ib \neq 0$. Then at least one of $a \neq 0, b \neq 0$ holds, and so $a^2 + b^2 \neq 0$. To find z', we set $z' = a' + b'i$. The condition $zz' = 1$ imposes conditions that will enable us to compute a' and b'. Computing the product gives $zz' = (aa' - bb') + (ab' + a'b)i$. The linear equations $aa' - bb' = 1$ and $ab' + a'b = 0$ can be solved for a' and b' giving $a' = a/(a^2 + b^2)$ and $b' = -b/(a^2 + b^2)$, since $a^2 + b^2 \neq 0$. Thus for $z = a + ib \neq 0$, we may write

$$z^{-1} = \frac{a}{a^2 + b^2} - \frac{ib}{a^2 + b^2}.$$

Having found this candidate for z^{-1} it is now a straightforward, albeit tedious, computation to check that it works.

If z and w are complex numbers with $w \neq 0$, then the symbol z/w means zw^{-1}; we call z/w the ***quotient*** of z by w. Thus $z^{-1} = 1/z$. To compute z^{-1}, the following series of equations is common and is a useful way to remember the preceding formula for z^{-1}:

$$\frac{1}{a + ib} = \frac{a - ib}{(a + ib)(a - ib)} = \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

In short, all the usual algebraic rules for manipulating real numbers, fractions, polynomials, and so on, hold for complex numbers.

Formally, the system of complex numbers is an example of a ***field***. The crucial rules for a field, stated here for reference, are

Addition rules

(i) $z + w = w + z$

(ii) $z + (w + s) = (z + w) + s$

(iii) $z + 0 = z$

(iv) $z + (-z) = 0$

Multiplication rules

(i) $zw = wz$

(ii) $(zw)s = z(ws)$

(iii) $1z = z$

(iv) $z(z^{-1}) = 1$ for $z \neq 0$

Distributive law $z(w + s) = zw + zs$

In summary, we have

Theorem 1.1.2 *The complex numbers* $\mathbf{C}$ *form a field.*

The student is cautioned that we generally do not define inequalities like $z \leq w$, for complex z and w. If one requires the usual ordering properties for reals to hold, then *such an ordering is impossible* for complex numbers.[1] Thus in this text the notation $z \leq w$ will be avoided unless z and w happen to be real.

Roots of Quadratic Equations As mentioned previously, one of the reasons for using complex numbers is to enable us to take square roots of negative real numbers. That this can, in fact, be done for all complex numbers is verified in the next proposition.

Proposition 1.1.3 *Let $z \in \mathbf{C}$. Then there exists a complex number $w \in \mathbf{C}$ such that $w^2 = z$. (Notice that $-w$ also satisfies this equation.)*

[1] This statement can be proved as follows. Suppose that such an ordering exists. Then either $i \geq 0$ or $i \leq 0$. Suppose that $i \geq 0$. Then $i \cdot i \geq 0$, so $-1 \geq 0$, which is absurd. Alternatively, suppose that $i \leq 0$. Then $-i \geq 0$, so $(-i)(-i) \geq 0$, that is, $-1 \geq 0$, again absurd. If $z = a + ib$ and $w = c + id$, we could say that $z \leq w$ iff $a \leq c$ and $b \leq d$. This is an ordering of sorts, but it does not satisfy all the rules that might be required, such as those obeyed by real numbers.

Proof (We shall give a purely algebraic proof here; another proof, based on polar coordinates, is given in §1.2.) Let $z = a + bi$. We want to find $w = x + iy$ such that $z = w^2$; *i.e.*, $a + bi = (x + iy)^2 = (x^2 - y^2) + (2xy)i$, and so we must simultaneously solve $x^2 - y^2 = a$ and $2xy = b$. The existence of such solutions is geometrically clear from examination of the graphs of the two equations. These graphs are shown in Figure 1.1.2 for the case in which both a and b are positive. From the graphs it is clear that there should be two solutions which are negatives of each other. In the following paragraph, these will be obtained algebraically.

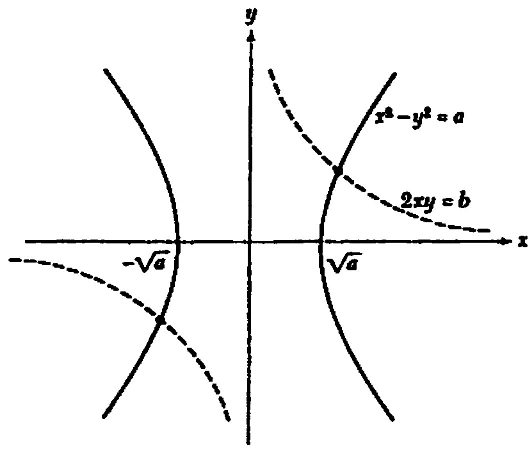


Figure 1.1.2: Graphs of the curves $x^2 - y^2 = a$ and $2xy = b$.

We know that $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = a^2 + b^2$. Hence $x^2 + y^2 = \sqrt{a^2 + b^2}$, so $x^2 = \left(a + \sqrt{a^2 + b^2}\right)/2$ and $y^2 = \left(-a + \sqrt{a^2 + b^2}\right)/2$. If we let

$$\alpha = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \quad \text{and} \quad \beta = \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}},$$

where $\sqrt{\ }$ denotes the positive square root of positive real numbers, then, in the event that b is positive, we have either $x = \alpha, y = \beta$ or $x = -\alpha, y = -\beta$; in the event that b is negative, we have either $x = \alpha, y = -\beta$ or $x = -\alpha, y = \beta$. We conclude that the equation $w^2 = z$ has solutions $\pm(\alpha + \mu\beta i)$, where $\mu = 1$ if $b \geq 0$ and $\mu = -1$ if $b < 0$. ■

The formula for square roots developed in this proof is worth summarizing explicitly. Namely, *the two (complex) square roots of $a + ib$ are given by*

$$\sqrt{a + ib} = \pm(\alpha + \mu\beta i),$$

where α and β are given by the displayed formula preceding this one and where $\mu = 1$ if $b \geq 0$ and $\mu = -1$ if $b < 0$. From the expressions for α and β we can conclude three things:

1. The square roots of a complex number are real if and only if the complex number is real and positive.
2. The square roots of a complex number are purely imaginary if and only if the complex number is real and negative.
3. The two square roots of a number coincide if and only if the complex number is zero.

(The student should check these conclusions.)

We can easily check that the quadratic equation $az^2 + bz + c = 0$ for complex numbers a, b, c has solutions $z = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$, where now the square root denotes the two square roots just constructed.

Worked Examples

Example 1.1.4 *Prove that* $1/i = -i$ *and that* $1/(i+1) = (1-i)/2$.

Solution First,

$$\frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = -i$$

because $i \cdot -i = -(i^2) = -(-1) = 1$. Also,

$$\frac{1}{i+1} = \frac{1}{i+1}\frac{1-i}{1-i} = \frac{1-i}{2}$$

since $(1+i)(1-i) = 1+1 = 2$.

Example 1.1.5 *Find the real and imaginary parts of* $(z+2)/(z-1)$ *where* $z = x + iy$.

Solution We start by writing the fraction in terms of the real and imaginary parts of z and "rationalizing the denominator". Namely,

$$\begin{aligned}\frac{z+2}{z-1} = \frac{(x+2)+iy}{(x-1)+iy} &= \frac{(x+2)+iy}{(x-1)+iy} \cdot \frac{(x-1)-iy}{(x-1)-iy} \\ &= \frac{(x+2)(x-1)+y^2+i[y(x-1)-y(x+2)]}{(x-1)^2+y^2}.\end{aligned}$$

Hence,

$$\operatorname{Re}\frac{z+2}{z-1} = \frac{x^2+x-2+y^2}{(x-1)^2+y^2}$$

and

$$\operatorname{Im}\frac{z+2}{z-1} = \frac{-3y}{(x-1)^2+y^2}.$$

Example 1.1.6 *Solve the equation* $z^4 + i = 0$ *for* z.

Solution Let $z^2 = w$. Then the equation becomes $w^2 + i = 0$. Now we use the formula $\sqrt{a+ib} = \pm(\alpha + \mu\beta i)$ we developed for taking square roots. Letting $a = 0$ and $b = -1$, we get

$$w = \pm\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right).$$

Consider the equation $z^2 = (1-i)/\sqrt{2}$. Using the same formula for square roots, but now letting $a = 1/\sqrt{2}$ and $b = -1/\sqrt{2}$, we obtain the two solutions

$$z = \pm\left(\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2}i\right).$$

From the second possible value for w we obtain two further solutions:

$$z = \pm\left(\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2+\sqrt{2}}}{2}i\right).$$

In the next section, de Moivre's formula will be developed, which will enable us to find the nth root of any complex number rather simply.

Example 1.1.7 *Prove that, for complex numbers z and w,*

$$\operatorname{Re}(z+w) = \operatorname{Re} z + \operatorname{Re} w$$

and

$$\operatorname{Im}(z+w) = \operatorname{Im} z + \operatorname{Im} w.$$

Solution Let $z = x + iy$ and $w = a + ib$. Then $z + w = (x+a) + i(y+b)$, so $\operatorname{Re}(z+w) = x + a = \operatorname{Re} z + \operatorname{Re} w$. Similarly, $\operatorname{Im}(z+w) = y + b = \operatorname{Im} z + \operatorname{Im} w$.

Exercises

1. Express the following complex numbers in the form $a + ib$:
   (a) $(2+3i) + (4+i)$
   (b) $\dfrac{2+3i}{4+i}$
   (c) $\dfrac{1}{i} + \dfrac{3}{1+i}$
2. Express the following complex numbers in the form $a + bi$:
   (a) $(2+3i)(4+i)$
   (b) $(8+6i)^2$

(c) $\left(1+\frac{3}{1+i}\right)^2$

3. Find the solutions to $z^2 = 3 - 4i$.

4. Find the solutions to the following equations:

   (a) $(z+1)^2 = 3 + 4i$
   (b) $z^4 - i = 0$

5. Find the real and imaginary parts of the following, where $z = x + iy$:

   (a) $\frac{1}{z^2}$
   (b) $\frac{1}{3z+2}$

6. Find the real and imaginary parts of the following, where $z = x + iy$:

   (a) $\frac{z+1}{2z-5}$
   (b) z^3

7. Is it true that $\operatorname{Re}(zw) = (\operatorname{Re} z)(\operatorname{Re} w)$?

8. If a is real and z is complex, prove that $\operatorname{Re}(az) = a \operatorname{Re} z$ and $\operatorname{Im}(az) = a \operatorname{Im} z$. Generally, show that $\operatorname{Re} : \mathbf{C} \to \mathbf{R}$ is a real linear map; that is, $\operatorname{Re}(az + bw) = a \operatorname{Re} z + b \operatorname{Re} w$ for a, b real and z, w complex.

9.* Show that $\operatorname{Re}(iz) = -\operatorname{Im}(z)$ and that $\operatorname{Im}(iz) = \operatorname{Re}(z)$ for any complex number z.

10. (a) Fix a complex number $z = x + iy$ and consider the linear mapping $\phi_z : \mathbf{R}^2 \to \mathbf{R}^2$ (that is, of $\mathbf{C} \to \mathbf{C}$) defined by $\phi_z(w) = z \cdot w$ (that is, multiplication by z). Prove that the matrix of ϕ_z in the standard basis $(1,0), (0,1)$ of $\mathbf{R}^2$ is given by

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

   (b) Show that $\phi_{z_1 z_2} = \phi_{z_1} \circ \phi_{z_2}$.

11. Assuming that they work for real numbers, show that the nine rules given for a field also work for complex numbers.

12. Using only the axioms for a field, give a formal proof (including all details) for the following:

   (a) $\frac{1}{z_1 z_2} = \frac{1}{z_1} \cdot \frac{1}{z_2}$

(b) $\dfrac{1}{z_1}+\dfrac{1}{z_2}=\dfrac{z_1+z_2}{z_1 z_2}$

13.* Let $(x-iy)/(x+iy)=a+ib$. Prove that $a^2+b^2=1$.

14. Prove the binomial theorem for complex numbers; that is, letting z, w be complex numbers and n be a positive integer,

$$(z+w)^n = z^n + \binom{n}{1} z^{n-1}w + \binom{n}{2} z^{n-2}w^2 + \ldots + \binom{n}{n} w^n,$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Use induction on n.

15. Show that z is real if and only if $\operatorname{Re} z = z$.

16. Prove that, for each integer k,

$$i^{4k}=1, i^{4k+1}=i, i^{4k+2}=-1, i^{4k+3}=-i.$$

Show how this result gives a formula for i^n for all n by writing $n=4k+j, 0\le j\le 3$.

17. Simplify the following:

(a) $(1+i)^4$
(b) $(-i)^{-1}$

18. Simplify the following:

(a) $(1-i)^{-1}$
(b) $\dfrac{1+i}{1-i}$

19. Simplify the following:

(a) $\sqrt{1+\sqrt{i}}$
(b) $\sqrt{1+i}$
(c)* $\sqrt{\sqrt{-i}}$

20. Show that the following rules uniquely determine complex multiplication on $\mathbf{C}=\mathbf{R}^2$:

(a) $(z_1+z_2)w = z_1 w + z_2 w$
(b) $z_1 z_2 = z_2 z_1$
(c) $i\cdot i=-1$
(d) $z_1(z_2 z_3) = (z_1 z_2) z_3$
(e) If z_1 and z_2 are real, $z_1 \cdot z_2$ is the usual product of real numbers.

1.2 Properties of Complex Numbers

It is important to be able to visualize mathematical concepts and to develop geometric intuition—an ability especially valuable in complex analysis. In this section we define and give a geometric interpretation for several concepts: the *absolute value*, *argument*, *polar representation*, and *complex conjugate* of a complex number.

Addition of Complex Numbers In the preceding section a complex number was defined to be a point in the plane $\mathbb{R}^2$. Thus, a complex number may be thought of geometrically as a (two-dimensional) vector and pictured as an arrow from the origin to the point in $\mathbb{R}^2$ given by the complex number (see Figure 1.2.1).

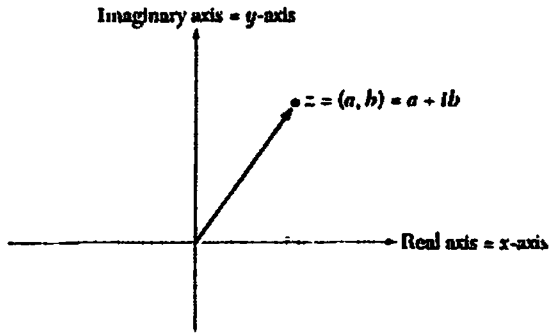


Figure 1.2.1: Vector representation of complex numbers.

Because the points $(x, 0) \in \mathbb{R}^2$ correspond to real numbers, the horizontal or x axis is called the ***real axis***. Similarly, the vertical axis (the y axis) is called the ***imaginary axis***, because points on it have the form $iy = (0, y)$ for y real.

As we already saw in Figure 1.1.1, the addition of complex numbers can be pictured as addition of vectors (an explicit example is given in Figure 1.2.2).

Polar Representation of Complex Numbers To understand the geometric meaning of multiplying two complex numbers, we will write them in what is called polar coordinate form. Recall that the ***length*** of the vector $(a, b) = a + ib$ is defined to be $\sqrt{a^2 + b^2}$. Suppose the vector makes an angle θ with the positive direction of the real axis, where $0 \le \theta < 2\pi$ (see Figure 1.2.3).

Thus, $\tan\theta = b/a$. Since $a = r\cos\theta$ and $b = r\sin\theta$, we have

$$a + bi = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta).$$

This way of writing the complex number is called the ***polar coordinate representation***. The length of the vector $z = (a, b) = a + ib$ is denoted $|z|$ and is called the ***norm***, or ***modulus***, or ***absolute value*** of z. The angle θ is called the ***argument*** of the complex number and is denoted $\theta = \arg z$.

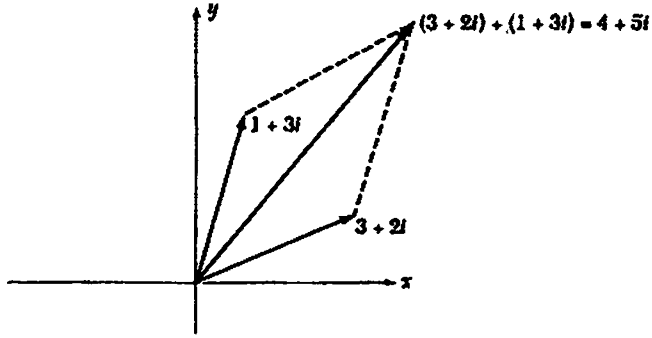


Figure 1.2.2: Addition of complex numbers.

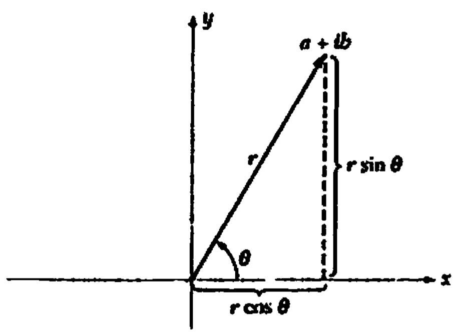


Figure 1.2.3: Polar coordinate representation of complex numbers.

If we restrict θ to the interval $0 \leq \theta < 2\pi$, then each nonzero complex number has an unambiguously defined argument. (We learn this in trigonometry.) However, it is clear that we can add integral multiples of 2π to θ and still obtain the same complex number. In fact, we shall find it convenient to be flexible in our requirements for the values that θ is to assume. For example, we could equally well allow the range of θ to be $-\pi < \theta \leq \pi$. Such an interval must always be specified or be clearly understood.

Once an interval of length 2π is specified, then for each $z \neq 0$, a unique θ is determined that lies within that specified interval. It is clear that any $\theta \in \mathbb{R}$ can be brought into our specified interval by the addition of some (positive or negative) integral multiple of 2π. For these reasons it is sometimes best to think of $\arg z$ as the set of possible values of the angle. If θ is one possible value, then so is $\theta + 2\pi n$ for any integer n, and we can sometimes think of $\arg z$ as $\{\theta + 2\pi n \mid n \text{ is an integer}\}$. Specification of a particular range for the angle is known as choosing a ***branch of the argument***.

Multiplication of Complex Numbers The polar representation of complex numbers helps us understand the geometric meaning of the product of two complex numbers. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2[(\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2)] + i[(\cos\theta_1 \cdot \sin\theta_2 + \cos\theta_2 \cdot \sin\theta_1)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)], \end{aligned}$$

by the addition formula for the sine and cosine functions used in trigonometry. Thus, we have proven

Proposition 1.2.1 *For any complex numbers z_1 and z_2,*

$$|z_1 z_2| = |z_1| \cdot |z_2| \quad \textit{and} \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}.$$

In other words, the product of two complex numbers is the complex number that has a length equal to the product of the lengths of the two complex numbers and an argument equal to the sum of the arguments of those numbers. This is the basic geometric representation of complex multiplication (see Figure 1.2.4).

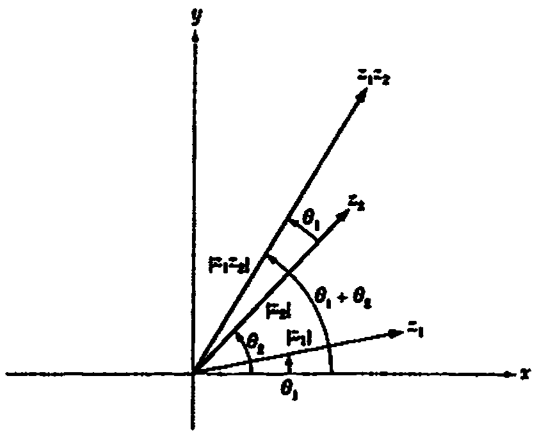


Figure 1.2.4: Multiplication of complex numbers.

The second equality in Proposition 1.2.1 means that the sets of possible values for the left and right sides are the same, that is, that the two sides can be made to agree by the addition of the appropriate multiple of 2π to one side. If a particular branch is desired and $\arg z_1 + \arg z_2$ lies outside the interval that we specify, we should adjust it by a multiple of 2π to bring it within that interval. For example, if our interval is $[0, 2\pi[$ and $z_1 = -1$ and $z_2 = -i$, then $\arg z_1 = \pi$ and $\arg z_2 = 3\pi/2$ (see Figure 1.2.5), but $z_1 z_2 = i$, so $\arg(z_1 z_2) = \pi/2$, and $\arg z_1 + \arg z_2 = \pi + 3\pi/2 = 2\pi + \pi/2$. We can obtain the correct answer by subtracting 2π to bring it within the interval $[0, 2\pi[$.

Multiplication of complex numbers can be analyzed in another useful way. Let $z \in \mathbf{C}$ and define $\psi_z : \mathbf{C} \to \mathbf{C}$ by $\psi_z(w) = wz$; that is, ψ_z is the map "multiplication

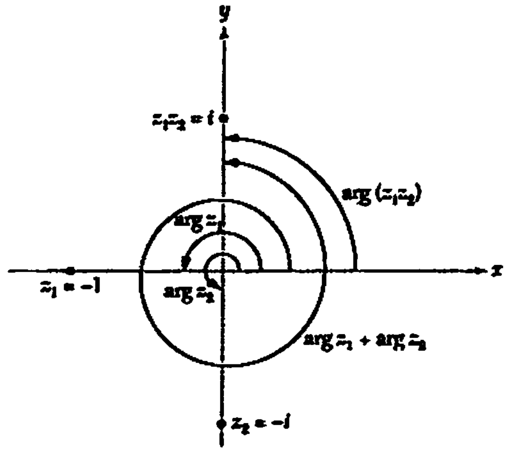


Figure 1.2.5: Multiplication of the complex numbers -1 and $-i$.

by z^n. By Proposition 1.2.1, *the effect of this map is to rotate a complex number through an angle equal to* $\arg z$ *in the counterclockwise direction and to stretch its length by the factor* $|z|$. For example, ψ_i (multiplication by i) rotates complex numbers by $\pi/2$ in the counterclockwise direction (see Figure 1.2.6).

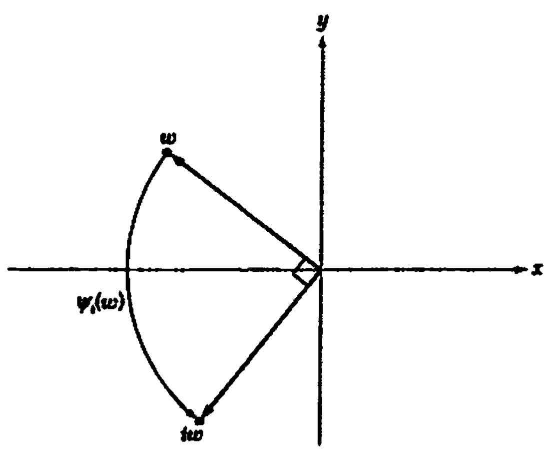


Figure 1.2.6: Multiplication by i.

The map ψ_z is a ***linear transformation*** on the plane, in the sense that

$$\psi_z(\lambda w_1 + \mu w_2) = \lambda\psi_z(w_1) + \mu\psi_z(w_2),$$

where λ, μ are real numbers and w_1, w_2 are complex numbers. Any linear transformation of the plane to itself can be represented by a matrix, as we learn in linear algebra. If $z = a + ib = (a, b)$, then the matrix of ψ_z is

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

since

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix}$$

(see Exercise 10, §1.1).

De Moivre's Formula The formula we derived for multiplication, using the polar coordinate representation, provides more than geometric intuition. We can use it to obtain a formula for the nth power of a complex number. This formula can then be used to find the nth roots of any complex number.

Proposition 1.2.2 (De Moivre's Formula) *If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then*

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Proof By Proposition 1.2.1,

$$z^2 = r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)] = r^2(\cos 2\theta + i\sin 2\theta).$$

Multiplying again by z gives

$$z^3 = z \cdot z^2 = r \cdot r^2[\cos(2\theta + \theta) + i\sin(2\theta + \theta)] = r^3(\cos 3\theta + i\sin 3\theta).$$

This procedure may be continued by induction to obtain the desired result for any integer n. ■

Let w be a complex number; that is, let $w \in \mathbf{C}$. Using de Moivre's formula will help us solve the equation $z^n = w$ for z when w is given. Suppose that $w = r(\cos\theta + i\sin\theta)$ and $z = \rho(\cos\psi + i\sin\psi)$. Then de Moivre's formula gives $z^n = \rho^n(\cos n\psi + i\sin n\psi)$. It follows that $\rho^n = r = |w|$ by uniqueness of the polar representation and $n\psi = \theta + k(2\pi)$, where k is some integer. Thus

$$z = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{k}{n}2\pi\right) + i\sin\left(\frac{\theta}{n} + \frac{k}{n}2\pi\right)\right].$$

Each value of $k = 0, 1, \ldots, n-1$ gives a different value of z. Any other value of k merely repeats one of the values of z corresponding to $k = 0, 1, 2, \ldots, n-1$. Thus there are exactly n nth roots of a (nonzero) complex number.

An example will help illustrate how to use this theory. Consider the problem of finding the three solutions to the equation $z^3 = 1 = 1(\cos 0 + i \sin 0)$. The preceding formula gives them as follows:

$$z = \cos \frac{k2\pi}{3} + i \sin \frac{k2\pi}{3},$$

where $k = 0, 1, 2$. In other words, the solutions are

$$z = 1, \quad -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad -\frac{1}{2} - \frac{i\sqrt{3}}{2}.$$

This procedure for finding roots is summarized as follows.

Corollary 1.2.3 *Let w be a nonzero complex number with polar representation $w = r(\cos\theta + i\sin\theta)$. Then the nth roots of w are given by the n complex numbers*

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i \sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right] \quad k = 0, 1, \ldots, n-1.$$

As a special case of this formula we note that the n roots of 1 (that is, the nth roots of unity) are 1 and $n - 1$ other points equally spaced around the unit circle, as illustrated in Figure 1.2.7 for the case $n = 8$.

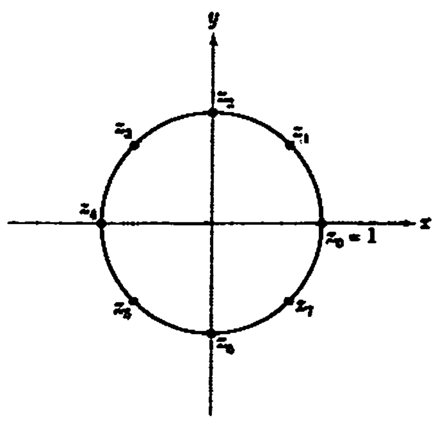


Figure 1.2.7: The eighth roots of unity.

Complex Conjugation Subsequent chapters will include many references to the simple idea of conjugation, which is defined as follows: If $z = a + ib$, then $\bar{z}$, the ***complex conjugate*** of z, is defined by $\bar{z} = a - ib$. Complex conjugation can be pictured geometrically as reflection in the real axis (see Figure 1.2.8).

The next proposition summarizes the main properties of complex conjugation.

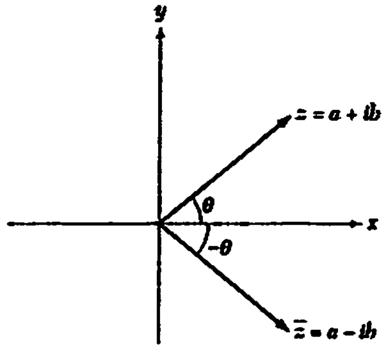


Figure 1.2.8: Complex conjugation.

Proposition 1.2.4 *The following properties hold for complex numbers:*

(i) $\overline{z+z'} = \bar{z} + \overline{z'}$.

(ii) $\overline{zz'} = \bar{z}\overline{z'}$.

(iii) $\overline{z/z'} = \bar{z}/\overline{z'}$ *for* $z' \neq 0$.

(iv) $z\bar{z} = |z|^2$ *and hence if* $z \neq 0$, *we have* $z^{-1} = \bar{z}/|z|^2$.

(v) $z = \bar{z}$ *if and only if* z *is real.*

(vi) $\operatorname{Re} z = (z+\bar{z})/2$ *and* $\operatorname{Im} z = (z-\bar{z})/2i$.

(vii) $\bar{\bar{z}} = z$.

Proof

(i) Let $z = a + ib$ and let $z' = a' + ib'$. Then $z + z' = a + a' + i(b + b')$, and so $\overline{z+z'} = (a+a') - i(b+b') = a - ib + a' - ib' = \bar{z} + \overline{z'}$.

(ii) Let $z = a + ib$ and let $z' = a' + ib'$. Then

$$\overline{zz'} = \overline{(aa' - bb') + i(ab' + a'b)} = (aa' - bb') - i(ab' + a'b).$$

On the other hand, $\bar{z}\overline{z'} = (a - ib)(a' - ib') = (aa' - bb') - i(ab' + a'b)$.

(iii) By (ii) we have $\overline{z'z/z'} = \overline{z'}\,\overline{z/z'} = \bar{z}$. Hence, $\overline{z/z'} = \bar{z}/\overline{z'}$.

(iv) $z\bar{z} = (a+ib)(a-ib) = a^2 + b^2 = |z|^2$.

(v) If $a + ib = a - ib$, then $ib = -ib$, and so $b = 0$.

(vi) This assertion is clear by the definition of $\bar{z}$.

(vii) This assertion is also clear by the definition of complex conjugation. ■

The absolute value of a complex number $|z| = |a + ib| = \sqrt{a^2 + b^2}$, which is the usual Euclidean length of the vector representing the complex number, has already been defined. From Proposition 1.2.4(iv), note that $|z|$ is also given by $|z|^2 = z\bar{z}$. The absolute value of a complex number is encountered throughout complex analysis; the following properties of the absolute value are quite basic.

Proposition 1.2.5 (i) $|zz'| = |z| \cdot |z'|$.

(ii) *If* $z' \neq 0$, *then* $|z/z'| = |z|/|z'|$.

(iii) $-|z| \leq \operatorname{Re} z \leq |z|$ *and* $-|z| \leq \operatorname{Im} z \leq |z|$; *that is,* $|\operatorname{Re} z| \leq |z|$ *and* $|\operatorname{Im} z| \leq |z|$.

(iv) $|\bar{z}| = |z|$.

(v) $|z + z'| \leq |z| + |z'|$.

(vi) $|z - z'| \geq ||z| - |z'||$.

(vii) $|z_1 w_1 + \ldots + z_n w_n| \leq \sqrt{|z_1|^2 + \ldots + |z_n|^2}\sqrt{|w_1|^2 + \ldots + |w_n|^2}$.

Statement (iv) is clear geometrically from Figure 1.2.8, (v) is called the ***triangle inequality*** for vectors in $\mathbb{R}^2$ (see Figure 1.2.9) and (vii) is referred to as the ***Cauchy-Schwarz inequality***. By repeated application of (v) we get the general statement $|z_1 + \ldots + z_n| \leq |z_1| + \ldots + |z_n|$.

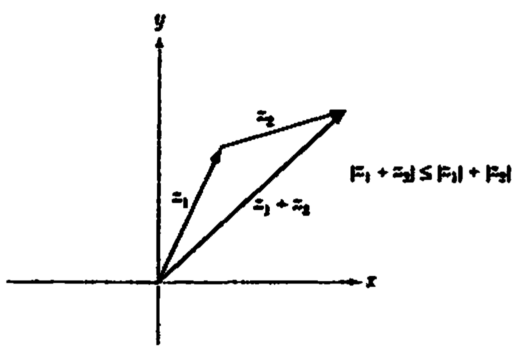


Figure 1.2.9: Triangle inequality.

Proof

(i) This equality was shown in Proposition 1.2.1.

(ii) By (i), $|z'||z/z'| = |z' \cdot (z/z')| = |z|$, so $|z/z'| = |z|/|z'|$.

(iii) If $z = a + ib$, then $-\sqrt{a^2+b^2} \le a \le \sqrt{a^2+b^2}$ since $b^2 \ge 0$. The other inequality asserted in (iii) is similarly proved.

(iv) If $z = a + ib$, then $\bar{z} = a - ib$, and we clearly have $|z| = \sqrt{a^2+b^2} = \sqrt{a^2+(-b)^2} = |\bar{z}|$.

(v) By Proposition 1.2.4(iv),

$$\begin{aligned} |z+z'|^2 &= (z+z')\overline{(z+z')} \\ &= (z+z')(\bar{z}+\overline{z'}) \\ &= z\bar{z} + z'\overline{z'} + z'\bar{z} + z\overline{z'}. \end{aligned}$$

But $z\overline{z'}$ is the conjugate of $z'\bar{z}$ (Why?), so by Proposition 1.2.4(vi) and (iii) in this proof,

$$|z|^2 + |z'|^2 + 2\operatorname{Re} z'\bar{z} \le |z|^2 + |z'|^2 + 2|z'\bar{z}| = |z|^2 + |z'|^2 + 2|z||z'|.$$

But this equals $(|z| + |z'|)^2$, so we get our result.

(vi) By applying (v) to z' and $z - z'$ we get

$$|z| = |z' + (z - z')| \le |z'| + |z - z'|,$$

so $|z - z'| \ge |z| - |z'|$. By interchanging the roles of z and z', we similarly get $|z - z'| \ge |z'| - |z| = -(|z| - |z'|)$, which is what we originally claimed.

(vii) This inequality is less evident and the proof of it requires a slight mathematical trick (see Exercise 22 for a different proof). Let us suppose that not all the $w_k = 0$ (or else the result is clear). Let

$$v = \sum_{k=1}^{n} |z_k|^2, \quad t = \sum_{k=1}^{n} |w_k|^2, \quad s = \sum_{k=1}^{n} z_k w_k, \quad \text{and} \quad c = s/t.$$

Now consider the sum

$$\sum_{k=1}^{n} |z_k - c\bar{w}_k|^2$$

which is ≥ 0 and equals

$$\begin{aligned} v + |c|^2 t - c\sum_{k=1}^{n} \bar{z}_k \bar{w}_k - \bar{c}\sum_{k=1}^{n} z_k w_k &= v + |c|^2 t - 2\operatorname{Re} \bar{c}s \\ &= v + \frac{|s|^2}{t} - 2\operatorname{Re}\frac{\bar{s}s}{t}. \end{aligned}$$

Since t is real and $s\bar{s} = |s|^2$ is real, $v + (|s|^2/t) - 2(|s|^2/t) = v - |s|^2/t \ge 0$. Hence $|s|^2 \le vt$, which is the desired result. ■

Worked Examples

Example 1.2.6 *Solve $z^8 = 1$ for z.*

Solution Since $1 = \cos k2\pi + i \sin k2\pi$ when k equals any integer, Corollary 1.2.3 gives

$$\begin{aligned} z &= \cos\frac{k2\pi}{8} + i\sin\frac{k2\pi}{8} \quad k = 0, 1, 2, \dots, 7 \\ &= 1, \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, i, \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -1, \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, -i, \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}. \end{aligned}$$

These may be pictured as points evenly spaced on the circle in the complex plane (see Figure 1.2.10).

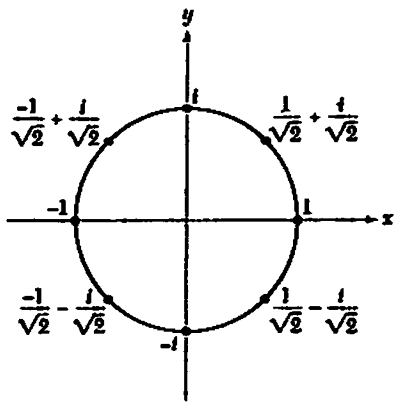


Figure 1.2.10: The eight 8th roots of unity.

Example 1.2.7 *Show that*

$$\overline{\left[\frac{(3+7i)^2}{(8+6i)}\right]} = \frac{(3-7i)^2}{(8-6i)}.$$

Solution The point here is that it is not necessary first to work out $(3+7i)^2/(8+6i)$ if we simply use the properties of complex conjugation, namely, $\overline{z^2} = (\bar{z})^2$ and $\overline{z/z'} = \bar{z}/\bar{z}'$. Thus we obtain

$$\overline{\left[\frac{(3+7i)^2}{(8+6i)}\right]} = \frac{\overline{(3+7i)^2}}{\overline{(8+6i)}} = \frac{\left(\overline{3+7i}\right)^2}{\overline{(8+6i)}} = \frac{(3-7i)^2}{(8-6i)}.$$

Example 1.2.8 *Show that the maximum absolute value of $z^2 + 1$ on the unit disk $|z| \le 1$ is 2.*

Solution By the triangle inequality, $|z^2+1| \le |z^2|+1 = |z|^2+1 \le 1^2+1 = 2$, since $|z| \le 1$ thus $|z^2+1|$ does not exceed 2 on the disk. Since the value 2 is achieved at $z=1$, the maximum is 2.

Example 1.2.9 *Express* $\cos 3\theta$ *in terms of* $\cos\theta$ *and* $\sin\theta$ *using de Moivre's formula.*

Solution De Moivre's formula for $r=1$ and $n=3$ gives the identity

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

The left side of this equation, when expanded (see Exercise 14 of §1.1), becomes

$$\cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$

By equating real and imaginary parts, we get

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

and the additional formula

$$\sin 3\theta = -\sin^3\theta + 3\cos^2\theta\sin\theta.$$

Example 1.2.10 *Write the equation of a straight line, of a circle, and of an ellipse using complex notation.*

Solution The straight line is most conveniently expressed in parametric form: $z = a + bt, a, b \in \mathbb{C}, t \in \mathbb{R}$, which represents a line in the direction of b and passing through the point a.

The circle can be expressed as $|z-a| = r$ (radius r, center a).

The ellipse can be expressed as $|z-d|+|z+d| = 2a$; the foci are located at $\pm d$ and the semimajor axis equals a.

These equations, in which $|\cdot|$ is interpreted as length, coincide with the geometric definitions of these loci.

Exercises

1. Solve the following equations:

   (a) $z^5 - 2 = 0$

   (b) $z^4 + i = 0$

2. Solve the following equations:

   (a) $z^6 + 8 = 0$

   (b) $z^3 - 4 = 0$

3. What is the complex conjugate of $(3+8i)^4/(1+i)^{10}$?

4. What is the complex conjugate of $(8-2i)^{10}/(4+6i)^5$?

5. Express $\cos 5x$ and $\sin 5x$ in terms of $\cos x$ and $\sin x$.

6. Express $\cos 6x$ and $\sin 6x$ in terms of $\cos x$ and $\sin x$.

7. Find the absolute value of $[i(2+3i)(5-2i)]/(-2-i)$.

8. Find the absolute value of $(2-3i)^2/(8+6i)^2$.

9.• Let w be an nth root of unity, $w \neq 1$. Show that $1+w+w^2+\ldots+w^{n-1}=0$.

10. Show that the roots of a polynomial with real coefficients occur in conjugate pairs.

11. If $a, b \in \mathbf{C}$, prove the ***parallelogram identity***: $|a-b|^2+|a+b|^2 = 2(|a|^2+|b|^2)$.

12. Interpret the identity in Exercise 11 geometrically.

13. When does equality hold in the triangle inequality $|z_1+z_2+\ldots+z_n| \leq |z_1|+|z_2|+\ldots+|z_n|$? Interpret your result geometrically.

14. Assuming either $|z|=1$ or $|w|=1$ and $\bar{z}w \neq 1$, prove that

$$\left|\frac{z-w}{1-\bar{z}w}\right| = 1.$$

15. Does $z^2 = |z|^2$? If so, prove this equality. If not, for what z is it true?

16.• Letting $z = x+iy$, prove that $|x|+|y| \leq \sqrt{2}|z|$.

17.• Let $z = a+ib$ and $z' = a'+ib'$. Prove that $|zz'| = |z||z'|$ by evaluating each side.

18. Prove the following:

    (a) $\arg \bar{z} = -\arg z (\operatorname{mod} 2\pi)$

    (b) $\arg(z/w) = \arg z - \arg w (\operatorname{mod} 2\pi)$

    (c) $|z| = 0$ if and only if $z = 0$

19. What is the equation of the circle with radius 3 and center $8+5i$ in complex notation?

20. Using the formula $z^{-1} = \bar{z}/|z|^2$, show how to construct z^{-1} geometrically.

21. Describe the set of all z such that $\operatorname{Im}(z+5) = 0$.

22.* Prove ***Lagrange's identity***:

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \left(\sum_{k=1}^{n}|z_k|^2\right)\left(\sum_{k=1}^{n}|w_k|^2\right) - \sum_{k<j}|z_k\bar{w}_j - z_j\bar{w}_k|^2.$$

Deduce the Cauchy-Schwarz inequality from your proof.

23.* Given $a \in \mathbb{C}$, find the maximum of $|z^n + a|$ for those z with $|z| \le 1$.

24. Compute the least upper bound (that is, supremum) of the set of all real numbers of the form Re $(iz^3 + 1)$ such that $|z| < 2$.

25.* Prove ***Lagrange's trigonometric identity***:

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{1}{2} + \frac{\sin\left(n + \frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}.$$

(Assume that $\sin(\theta/2) \neq 0$.)

26. Suppose that the complex numbers z_1, z_2, z_3 satisfy the equation

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{z_1 - z_3}{z_2 - z_3}.$$

Prove that $|z_2 - z_1| = |z_3 - z_1| = |z_2 - z_3|$. *Hint:* Argue geometrically, interpreting the meaning of each statement.

27. Give a necessary and sufficient condition for

(a) z_1, z_2, z_3 to lie on a straight line.

(b) z_1, z_2, z_3, z_4 to lie on a straight line or a circle.

28. Prove the identity

$$\left(\sin\frac{\pi}{n}\right)\left(\sin\frac{2\pi}{n}\right)\ldots\left(\sin\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

Hint: The given product can be written as $1/2^{n-1}$ times the product of the nonzero roots of the polynomial $(1 - z)^n - 1$.

29. Let w be an nth root of unity, $w \neq 1$. Evaluate $1 + 2w + 3w^2 + \ldots + nw^{n-1}$.

30. Show that the correspondence of the complex number $z = a + bi$ with the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \psi_z$ noted in the text preceding Proposition 1.2.2 has the following properties:

(a) $\psi_{zw} = \psi_z\psi_w$.

(b) $\psi_{z+w} = \psi_z + \psi_w$.

(c) $\psi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(d) $\lambda\psi_z = \psi_{\lambda z}$ if λ is real.

(e) $\psi_{\bar{z}} = (\psi_z)^t$ (the transposed matrix).

(f) $\psi_{1/z} = (\psi_z)^{-1}$.

(g) z is real if and only if $\psi_z = (\psi_z)^t$.

(h) $|z| = 1$ if and only if ψ_z is an orthogonal matrix.

1.3 Some Elementary Functions

We learn about the trigonometric functions sine and cosine, as well as the exponential function and the logarithmic function in elementary calculus. Recall that the trigonometric functions may be defined in terms of the ratios of sides of a right-angled triangle. The definition of "angle" may be extended to include any real value, and thus $\cos\theta$ and $\sin\theta$ become real-valued functions of the real variable θ. It is a basic fact that $\cos\theta$ and $\sin\theta$ are differentiable, with derivatives given by $d(\cos\theta)/d\theta = -\sin\theta$ and $d(\sin\theta)/d\theta = \cos\theta$. Alternatively, $\cos\theta$ and $\sin\theta$ can be defined by their power series:

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots . \end{aligned}$$

The proof of convergence of these series can be found in Chapter 3 and in many calculus texts.[2] Alternatively, $\sin x$ can be defined as the unique solution $f(x)$ to the differential equation $f''(x) + f(x) = 0$ satisfying $f(0) = 0, f'(0) = 1$; and $\cos x$ can be defined as the unique solution to $f''(x) + f(x) = 0, f(0) = 1, f'(0) = 0$ (again, see a calculus text for proofs).

Exponential Function The exponential function, denoted e^x, may be defined as the unique solution to the differential equation $f'(x) = f(x)$, subject to the initial condition that $f(0) = 1$; one has to show that a unique solution exists. The exponential function can also be defined by its power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

We accept from calculus the fact that e^x is a positive, strictly increasing function of x. Therefore, for $y > 0, \log y$ can be defined as the inverse function of e^x; that

[2] An example is J. Marsden and A. Weinstein, *Calculus*, Second Edition (New York: Springer-Verlag, 1985), Chapter 12.

is, $e^{\log y} = y$. Another approach that is often used in calculus books is to *begin* by defining

$$\log y = \int_1^y \frac{1}{t} dt$$

for $y > 0$ and then to define e^x as the inverse function of $\log y$. (Many calculus books write $\ln y$ for the logarithm to the base e. As in most advanced mathematics, throughout this book we will write $\log y$ for $\ln y$.)

In this section these functions will be extended to the complex plane. In other words, the functions $\sin z, \cos z, e^z$, and $\log z$ will be defined for complex z, and their restrictions to the real line will be the usual $\sin x, \cos x, e^x$, and $\log x$. The extension to complex numbers should be natural in the sense that many of the familiar properties of sin, cos, exp, and log are retained.

We first extend the exponential function. We know from calculus that for real x, e^x can be represented by its Maclaurin series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

Thus, it would be reasonable to define e^{iy} by

$$1 + \frac{(iy)}{1!} + \frac{(iy)^2}{2!} + \dots$$

for $y \in \mathbb{R}$. Of course, this definition is not quite legitimate, as convergence of series in $\mathbb{C}$ has not yet been discussed. Chapter 3 will show that this series does indeed represent a well-defined complex number for each y, but for the moment the series is used *informally* as the basis for the definition that follows, which will be precise. A slight rearrangement of the series (using Exercise 16, §1.1) shows that

$$e^{iy} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right),$$

which we recognize as being $\cos y + i \sin y$. Thus we *define*

$$e^{iy} = \cos y + i \sin y.$$

So far, we have defined e^z for z along both the real and imaginary axes. How do we define $e^z = e^{x+iy}$? We desire our extension of the exponential to retain the familiar properties, and among these is the law of exponents: $e^{a+b} = e^a \cdot e^b$. This requirement forces us to define $e^{x+iy} = e^x \cdot e^{iy}$. This can be stated in a formal definition.

Definition 1.3.1 *If $z = x + iy$, then e^z is defined by $e^x(\cos y + i \sin y)$.*

Note that if z is real (that is, if $y = 0$), this definition agrees with the usual exponential function e^x. The student is cautioned that we are not, at this stage, fully justified in thinking of e^z as "e raised to the "power" of z," since we have not yet, for example, established laws of exponents for complex numbers.

There is another, again purely formal, reason for defining $e^{iy} = \cos y + i \sin y$. If we write $e^{iy} = f(y) + ig(y)$, we note that since we want $e^0 = 1$, we should have $f(0) = 1$, and $g(0) = 0$. If the exponential function is to have the familiar differentiation properties, we will need

$$ie^{iy} = f'(y) + ig'(y),$$

so when $y = 0$ we get $f'(0) = 0, g'(0) = 1$. Differentiating again gives us

$$-e^{iy} = f''(y) + ig''(y).$$

Comparing this equation with $e^{iy} = f(y) + ig(y)$, we conclude that $f''(y) + f(y) = 0, f(0) = 1$, and $f'(0) = 0$. Therefore, $f(y) = \cos y$ by the definition of $\cos y$ in terms of differential equations. Similarly, we find that $g''(y)+g(y) = 0, g(0) = 0, g'(0) = 1$, and hence $g(y) = \sin y$. Thus, we would obtain $e^{iy} = \cos y + i \sin y$ as in Definition 1.3.1.

Some of the important properties of e^z are summarized in the following proposition. To state it, we recall the definition of a periodic function. A function $f : \mathbb{C} \to \mathbb{C}$ is called ***periodic*** if there exists a $w \in \mathbb{C}$ (called a ***period***) such that $f(z + w) = f(z)$ for all $z \in \mathbb{C}$.

Proposition 1.3.2

(i) $e^{z+w} = e^z e^w$ *for all* $z, w \in \mathbb{C}$.

(ii) e^z *is never zero.*

(iii) *If* x *is real, then* $e^x > 1$ *when* $x > 0$ *and* $0 < e^x < 1$ *when* $x < 0$.

(iv) $|e^{x+iy}| = e^x$.

(v) $e^{\pi i/2} = i, e^{\pi i} = -1, e^{3\pi i/2} = -i, e^{2\pi i} = 1$.

(vi) e^z *is periodic; each period for* e^z *has the form* $2\pi n i$, *for some integer* n.

(vii) $e^z = 1$ *iff* $z = 2n\pi i$ *for some integer* n *(positive, negative, or zero).*

Proof

(i) Let $z = x + iy$, and let $w = s + it$. By our definition of e^z,

$$\begin{aligned} e^{z+w} &= e^{(x+s)+i(y+t)} \\ &= e^{x+s}[\cos(y+t) + i\sin(y+t)] \\ &= e^x e^s[(\cos y \cos t - \sin y \sin t) + i(\sin y \cos t + \cos y \sin t)] \\ &= [e^x(\cos y + i \sin y)][e^s(\cos t + i \sin t)] \end{aligned}$$

using the addition formulas for sine and cosine and the property $e^{x+s} = e^x \cdot e^s$ for real numbers x and s. Thus $e^{z+w} = e^z \cdot e^w$ for all complex numbers z and w.

(ii) For any z, we have $e^z \cdot e^{-z} = e^0 = 1$ since we know that the usual exponential satisfies $e^0 = 1$. Thus e^z can never be zero, because if it were, then $e^z \cdot e^{-z}$ would be zero, which is not true.

(iii) We may accept this from calculus. For example,[3] obviously

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots > 1 \quad \text{when} \quad x > 0.$$

(iv) Using $|zz'| = |z||z'|$ (see Proposition 1.2.5) and the facts that $e^x > 0$ and $\cos^2 y + \sin^2 y = 1$, we get

$$\begin{aligned} |e^{x+iy}| &= |e^x e^{iy}| = |e^x||e^{iy}| \\ &= e^x|\cos y + i\sin y| = e^x. \end{aligned}$$

(v) By definition, $e^{\pi i/2} = \cos(\pi/2) + i\sin(\pi/2) = i$. The proofs of the other formulas are similar.

(vi) Suppose that $e^{z+w} = e^z$ for all $z \in \mathbb{C}$. Setting $z = 0$, we get $e^w = 1$. If $w = s + ti$, then, using (iv), $e^w = 1$ implies that $e^s = 1$, so $s = 0$. Hence any period is of the form ti, for some $t \in \mathbb{R}$. Suppose that $e^{ti} = 1$, that is, that $\cos t + i\sin t = 1$. Then $\cos t = 1, \sin t = 0$; thus, $t = 2\pi n$ for some integer n.

(vii) $e^0 = 1$, as we have seen, and $e^{2\pi n i} = 1$ because e^z is periodic, by (vi). Conversely, $e^z = 1$ implies that $e^{z+z'} = e^{z'}$ for all z'; so by (vi), $z = 2\pi n i$ for some integer n. ■

How can we picture e^{iy}? Since $e^{iy} = (\cos y, \sin y)$, it moves along the unit circle in a counterclockwise direction as y goes from 0 to 2π. It reaches i at $y = \pi/2, -1$ at $\pi, -i$ at $3\pi/2$, and 1 again at 2π. Thus, e^{iy} is the point on the unit circle with argument y (see Figure 1.3.1).

Note that in exponential form, the polar representation of a complex number becomes

$$z = |z|e^{i(\arg z)}$$

which is sometimes abbreviated to $z = re^{i\theta}$.

Trigonometric Functions Next we wish to extend the definitions of cosine and sine to the complex plane. The extension of the exponential to the complex plane suggests a way to extend the definitions of sine and cosine. We have $e^{iy} = \cos y + i\sin y$, and $e^{-iy} = \cos y - i\sin y$, which implies that

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} \quad \text{and} \quad \cos y = \frac{e^{iy} + e^{-iy}}{2}.$$

[3] Another proof utilizing the definition of e^x in terms of differential equations is as follows. Recall that e^x is the unique solution to $f'(x) = f(x)$ with $e^0 = 1$ (x real). Since e^x is continuous and is never zero, it must be strictly positive. Hence $(e^x)' = e^x$ is always positive and consequently e^x is strictly increasing. Thus for $x > 0$, $e^x > 1$. Similarly, for $x < 0$, we have $e^x < 1$.

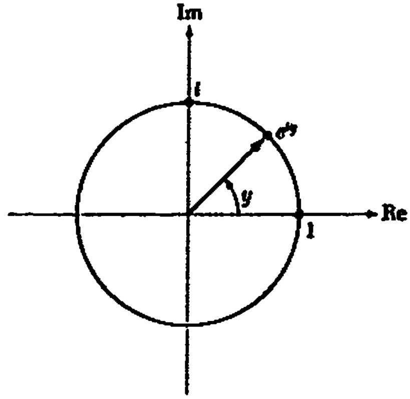


Figure 1.3.1: Points on the unit circle.

But since e^{iz} is now defined for any $z \in \mathbb{C}$, we are led to formulate the following definition.

Definition 1.3.3 *The complex sine and cosine functions are defined by*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \textit{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

for any complex number z.

Again, if z is real, these definitions agree with the usual definitions of sine and cosine learned in elementary calculus.

The next proposition lists some of the properties of the sine and cosine functions that have now been defined over the whole of $\mathbb{C}$ and not merely on $\mathbb{R}$.

Proposition 1.3.4

(i) $\sin^2 z + \cos^2 z = 1$.

(ii) $\sin(z + w) = \sin z \cdot \cos w + \cos z \cdot \sin w$ *and*
$\cos(z + w) = \cos z \cdot \cos w - \sin z \cdot \sin w$.

Again the student is cautioned that these formulas, although plausible, must be proved, since at this stage we know their validity only when w and z are real.

Proof Using the definitions, we have

$$\begin{aligned}\sin^2 z + \cos^2 z &= \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 \\ &= \frac{e^{2iz} - 2 + e^{-2iz}}{-4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4} = 1,\end{aligned}$$

which proves (i). To prove (ii), write

$$\begin{aligned}&\sin z\cdot\cos w+\cos z\cdot\sin w\\&\qquad=\frac{e^{iz}-e^{-iz}}{2i}\cdot\frac{e^{iw}+e^{-iw}}{2}+\frac{e^{iz}+e^{-iz}}{2}\cdot\frac{e^{iw}-e^{-iw}}{2i},\end{aligned}$$

which, using $e^{iz}e^{iw}=e^{i(z+w)}$ and noting cancellations between the two terms, simplifies to

$$\frac{e^{i(z+w)}-e^{-i(z+w)}}{4i}+\frac{e^{i(z+w)}-e^{-i(z+w)}}{4i}=\frac{e^{i(z+w)}-e^{-i(z+w)}}{2i}=\sin(z+w).$$

The student can similarly check the addition formula for $\cos(z+w)$. ■

In addition to $\cos z$ and $\sin z$, we can define $\tan z=(\sin z)/(\cos z)$ when $\cos z\neq 0$, and similarly obtain the other trigonometric functions.

Logarithm Function We now define the logarithm in a way that agrees with the usual definition of $\log x$ when x is real and positive. In the real case we can view the logarithm as the inverse of the exponential (that is, $\log x=y$ is the solution of $e^y=x$). When we allow z to range over $\mathbf{C}$, we must be more careful, because the exponential is periodic and thus cannot have a unique inverse. Furthermore, the exponential is never zero, so we cannot expect to be able to define the logarithm at zero. Thus, we must be careful in our choice of the domain in $\mathbf{C}$ on which we can define the logarithm. The next proposition indicates how this may be done.

Proposition 1.3.5 *Let A_{y_0} denote the set of complex numbers $x+iy$ such that $y_0\le y<y_0+2\pi$; symbolically,*

$$A_{y_0}=\{x+iy\mid x\in\mathbf{R}\quad\text{and}\quad y_0\le y<y_0+2\pi\}.$$

Then e^z maps A_{y_0} in a one-to-one manner onto the set $\mathbf{C}\backslash\{0\}$.

Recall that a map is ***one-to-one*** when the map takes every two distinct points to two distinct points; in other words, two distinct points never get mapped to the same point. A map is ***onto*** a set B when every point of B is the image of some point under the mapping. The notation $\mathbf{C}\backslash\{0\}$ means the whole plane $\mathbf{C}$ minus the point 0; that is, the plane with the origin removed.

Proof If $e^{z_1}=e^{z_2}$, then $e^{z_1-z_2}=1$, so $z_1-z_2=2\pi in$ for some integer n, by Proposition 1.3.2. But because z_1 and z_2 both lie in A_{y_0}, where the difference between the imaginary parts of any points is less than 2π, we must have $z_1=z_2$. This argument shows that e^z is one-to-one. Let $w\in\mathbf{C}$ with $w\neq 0$. We claim the equation $e^z=w$ has a solution z in A_{y_0}. The equation $e^{x+iy}=w$ is equivalent to the two equations $e^x=|w|$ and $e^{iy}=w/|w|$. (Why?) The solution of the first equation is $x=\log|w|$, where "log" is the ordinary logarithm (with base e) defined on the positive part of the real axis. The second equation has infinitely many solutions y, each differing by integral multiples of 2π, but exactly one of these is the interval $[y_0, y_0 + 2\pi[$. This y is merely $\arg w$, where the specified range for the arg function is $[y_0, y_0 + 2\pi[$. Thus e^z is onto $\mathbf{C}\backslash\{0\}$. ■

The sets defined in this proposition are shown in Figure 1.3.2. Here e^z maps the horizontal strip between $y_0 i$ and $(y_0 + 2\pi)i$ one-to-one onto $\mathbf{C}\backslash\{0\}$. (The notation $z \mapsto f(z)$ is used to indicate that z is sent to $f(z)$ under the mapping f.)

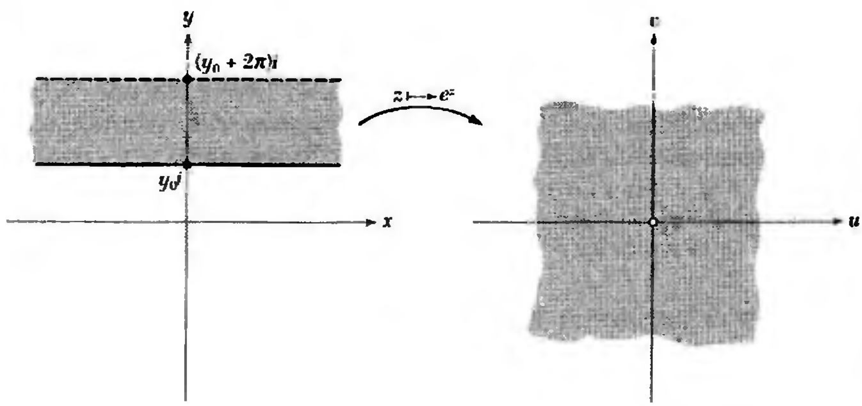


Figure 1.3.2: e^z as a one-to-one function onto $\mathbf{C}\backslash\{0\}$.

In the proof of Proposition 1.3.5 an explicit expression was derived for the inverse of e^z restricted to the strip $y_0 \le \operatorname{Im} z < y_0 + 2\pi$, and this expression is stated formally in the following definition.

Definition 1.3.6 *The function* $\log : \mathbf{C}\backslash\{0\} \to \mathbf{C}$, *with range* $y_0 \le \operatorname{Im} \log z < y_0 + 2\pi$, *is defined by*

$$\log z = \log |z| + i \arg z,$$

where $\arg z$ *takes values in the interval* $[y_0, y_0 + 2\pi[$ *and* $\log |z|$ *is the usual logarithm of the positive real number* $|z|$.

This function is sometimes referred to as the "*branch* of the logarithm function lying in $\{x + iy \mid y_0 \le y < y_0 + 2\pi\}$." But we must remember that the function $\log z$ is well defined only when we specify an interval of length 2π in which $\arg z$ takes its values, that is, when a specific branch is chosen.

For example, suppose that the specified interval for the argument is $[0, 2\pi[$. Then $\log(1 + i) = \log \sqrt{2} + i\pi/4$. However, if the specified interval is $[\pi, 3\pi[$, then $\log(1 + i) = \log \sqrt{2} + i9\pi/4$. Any particular branch of the logarithm defined in this way undergoes a sudden jump as z moves across the ray $\arg z = y_0$. To avoid this jumping, one can restrict the domain to $y_0 < y < y_0 + 2\pi$. This idea will be important in §1.6.

Proposition 1.3.7 *The logarithm $\log z$ is the inverse of e^z in the following sense: For any branch of $\log z$, we have $e^{\log z} = z$, and if we choose the branch lying in $y_0 \le y < y_0 + 2\pi$, then $\log(e^z) = z$ for $z = x + iy$ and $y_0 \le y < y_0 + 2\pi$.*

Proof Since $\log z = \log|z| + i\arg z$, we have

$$e^{\log z} = e^{\log|z|}e^{i\arg z} = |z|e^{i\arg z} = z.$$

Conversely, suppose that $z = x + iy$ and $y_0 \le y < y_0 + 2\pi$. By definition, $\log e^z = \log|e^z| + i\arg e^z$. But $|e^z| = e^x$ and $\arg e^z = y$ by our choice of branch. Thus, $\log e^z = \log e^x + iy = x + iy = z$. ■

The logarithm defined on $\mathbb{C}\backslash\{0\}$ behaves the same way with respect to products as the logarithm restricted to the positive part of the real axis.

Proposition 1.3.8 *If $z_1, z_2 \in \mathbb{C}\backslash\{0\}$, then $\log(z_1z_2) = \log z_1 + \log z_2$ (up to the addition of integral multiples of $2\pi i$).*

Proof By definition, $\log z_1z_2 = \log|z_1z_2| + i\arg(z_1z_2)$, where an interval $[y_0, y_0 + 2\pi[$ has been chosen for the values of the arg function. We know that $\log|z_1z_2| = \log|z_1||z_2| = \log|z_1| + \log|z_2|$ and $\arg(z_1z_2) = \arg z_1 + \arg z_2$ (up to integral multiples of 2π). Thus $\log z_1z_2 = (\log|z_1| + i\arg z_1) + (\log|z_2| + i\arg z_2) = \log z_1 + \log z_2$ (up to integral multiples of $2\pi i$). ■

To illustrate this proposition, let us find $\log[(-1 - i)(1 - i)]$, where the range for the arg function is chosen as, for instance, $[0, 2\pi[$. Thus,

$$\log[(-1 - i)(1 - i)] = \log(-2) = \log 2 + \pi i.$$

On the other hand, $\log(-1 - i) = \log\sqrt{2} + i5\pi/4$ and $\log(1 - i) = \log\sqrt{2} + i7\pi/4$. Thus,

$$\log(-1 - i) + \log(1 - i) = \log 2 + i3\pi = (\log 2 + \pi i) + 2\pi i,$$

so in this case, when $z_1 = -1 - i$ and $z_2 = 1 - i$, $\log z_1z_2$ differs from $\log z_1 + \log z_2$ by $2\pi i$.

The basic property in Proposition 1.3.8 can help one remember the definition of $\log z$ by writing $\log z = \log(re^{i\theta}) = \log r + \log e^{i\theta} = \log|z| + i\arg z$.

Complex Powers We are now in a position to define the expression a^b where $a, b \in \mathbb{C}$ and $a \ne 0$ (read "a raised to the power of b"). Of course, however we define a^b, the definition should reduce to the usual one in which a and b are real numbers. Notice that a can also be written $e^{\log a}$ by Proposition 1.3.7. Thus, if b is an integer, we have $a^b = (e^{\log a})^b = e^{b\log a}$. This last equality holds since if n is an integer and z is any complex number, $(e^z)^n = e^z \dots e^z = e^{nz}$ by Proposition 1.3.2(i). Thus we are led to formulate the following definition.

Definition 1.3.9 *Let $a, b \in \mathbb{C}$ with $a \neq 0$. Then a^b is defined to be $e^{b \log a}$; it is understood that some interval $[y_0, y_0 + 2\pi[$ (that is, some branch of* log*) has been chosen within which the* arg *function takes its values.*

It is important to understand precisely what this definition involves. Note especially that in general $\log z$ is "multiple-valued"; that is, $\log z$ can be assigned many different values because different intervals $[y_0 + 2\pi[$ can be chosen. This is not surprising, for if $b = 1/q$, where q is an integer, then our previous work with de Moivre's formula would lead us to expect that a^b is one of the qth roots of a and thus should have q distinct values. The following theorem elucidates this point.

Proposition 1.3.10 *Let $a, b \in \mathbb{C}, a \neq 0$. Then a^b is single-valued (that is, the value of a^b does not depend on the choice of branch for* log*) if and only if b is an integer. If b is a real, rational number, and if $b = p/q$ is in its lowest terms (in other words, if p and q have no common factor), then a^b has exactly q distinct values, namely, the q roots of a^p. If b is real and irrational or if b has a nonzero imaginary part, then a^b has infinitely many values. When a^b has distinct values, these values differ by factors of the form $e^{2\pi nbi}$.*

Proof Choose some interval, for example, $[0, 2\pi[$, for the values of the arg function. Let $\log z$ be the corresponding branch of the logarithm. If we were to choose any other branch of the log function, we would obtain $\log a + 2\pi ni$ rather than $\log a$, for some integer n. Thus $a^b = e^{b \log a + 2\pi nbi} = e^{b \log a} \cdot e^{2\pi nbi}$, where the value of n depends on the branch of logarithm (that is, on the interval chosen for the values of the arg function). By Proposition 1.3.2, $e^{2\pi nbi}$ remains the same for different values of n if and only if b is an integer. Similarly, $e^{2\pi nip/q}$ has q distinct values if p and q have no common factor. If b is irrational, and if $e^{2\pi nbi} = e^{2\pi mbi}$, it follows that $e^{(2\pi bi)(n-m)} = 1$ and hence $b(n - m)$ is an integer; since b is irrational, this implies that $n - m = 0$. Thus if b is irrational, $e^{2\pi nbi}$ has infinitely many distinct values. If b is of the form $x + iy, y \neq 0$, then $e^{2\pi nbi} = e^{-2\pi ny} \cdot e^{2\pi nix}$, which also has infinitely many distinct values. ■

To repeat: When we write $e^{b \log a}$, it is understood that some branch of log has been chosen, and accordingly $e^{b \log a}$ has a single well-defined value. But as we change the branch of log, we get values for $e^{b \log a}$ that differ by factors of $e^{2\pi inb}$. This is what we mean when we say that $a^b = e^{b \log a}$ is "multiple-valued".

An example should make this clear. Let $a = 1+i$ and let b be some real irrational number. Then the infinitely many different possible values of a^b are given by

$$(1 + i)^b = e^{b[\log(1+i)+2\pi ni]} = e^{b(\log \sqrt{2}+i\pi/4+2\pi ni)} = (e^{b \log \sqrt{2}+ib\pi/4})e^{b2\pi ni}$$

as n takes on all integral values (corresponding to different choices of the branch). For instance, if we used the branch corresponding to $[-\pi, \pi[$ or $[0, 2\pi[$, we would set $n = 0$.

Some general properties of a^b are found in the exercises at the end of this section, but we are now interested in the special case when b is of the form $1/n$, because this gives the nth root.

The nth Root Function We know that $\sqrt[n]{z}$ has exactly n values for $z \neq 0$. To make it a specific function we single out a branch of log as described in the preceding paragraphs.

Definition 1.3.11 *The nth root function is defined by*

$$\sqrt[n]{z} = z^{1/n} = e^{(\log z)/n}$$

for a specific choice of branch of $\log z$*; with this choice,* $\sqrt[n]{z} = e^{(\log z)/n}$ *is called a branch of the nth root function.*

The next proposition verifies a familiar property of root functions.

Proposition 1.3.12 *The function* $\sqrt[n]{z}$ *so defined is an nth root of z; that is,* $(\sqrt[n]{z})^n = z$*. It is obtained as follows. If* $z = re^{i\theta}$*, then*

$$\sqrt[n]{z} = \sqrt[n]{r}e^{i\theta/n},$$

where θ *is chosen so that it lies within a particular interval corresponding to the branch choice. As we add multiples of* 2π *to* θ*, we run through the n nth roots of z. On the right-hand side,* $\sqrt[n]{r}$ *is the usual positive real nth root of the positive real number r.*

Proof By definition, $\sqrt[n]{z} = e^{(\log z)/n}$. But $\log z = \log r + i\theta$, so

$$e^{(\log z)/n} = e^{(\log r)/n} \cdot e^{i\theta/n} = \sqrt[n]{r}e^{i\theta/n}.$$

The assertion is then clear. ■

The reader should now take the time to become convinced that this way of describing the n nth roots of z is the same as that described in Corollary 1.2.3.

Geometry of the Elementary Functions To further understand the functions z^n, $\sqrt[n]{z}, e^z$, and $\log z$, we shall consider the geometric interpretation of each in the remainder of this section. Let us begin with the power function z^n and let $n = 2$. We know that z^2 has length $|z|^2$ and argument $2\arg z$. Thus the map $z \mapsto z^2$ *squares lengths* and *doubles arguments* (see Figure 1.3.3).

From this doubling of angles it follows that the power function z^2 maps the first quadrant to the whole upper half plane (see Figure 1.3.4). Similarly, the upper half plane is mapped to the whole plane.

Now consider the square root function $\sqrt{z} = \sqrt{r}e^{i\theta/2}$. Suppose that we choose a branch by using the interval $0 \le \theta < 2\pi$. Then $0 \le \theta/2 < \pi$, so $\sqrt{z}$ will always lie in the upper half plane, and the angles thus are cut in half. The situation is similar to that involving the exponential function in that $z \mapsto \sqrt{z}$ is the inverse for $z \mapsto z^2$ when the latter is restricted to a region on which it is one-to-one. In like manner, if we choose the branch $-\pi \le \theta < \pi$, we have $-\pi/2 \le \theta/2 < \pi/2$, so $\sqrt{z}$ takes its values in the right half plane instead of the upper half plane. (Generally, any "half

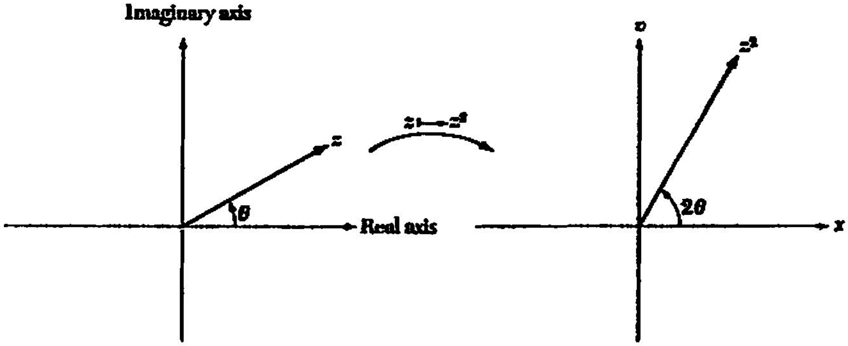


Figure 1.3.3: Squaring function.

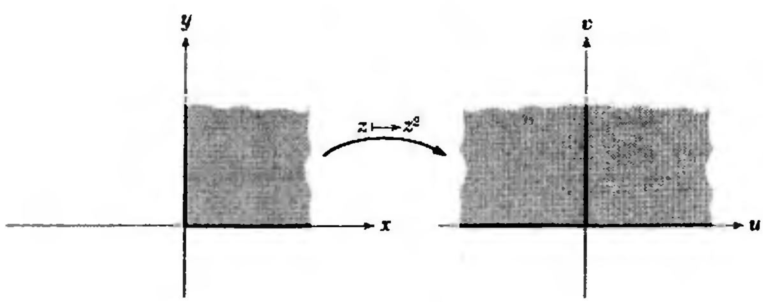


Figure 1.3.4: Effect of the squaring function on the first quadrant.

plane" could be used—see Figure 1.3.5.) If we choose a specific branch of $\sqrt{z}$, we also choose which of the two possible square roots we shall obtain.

Various geometric statements can be made concerning the map $z \mapsto z^2$ that also give information about the inverse, $z \mapsto \sqrt{z}$. For example, a circle of radius r described by the set of points $re^{i\theta}, 0 \leq \theta < 2\pi$, is mapped to $r^2e^{i2\theta}$, a circle of radius r^2; as $re^{i\theta}$ moves once around the first circle, the image point moves twice around (see Figure 1.3.6). The inverse map does the opposite: as z moves along the circle $re^{i\theta}$ of radius r, $\sqrt{z}$ moves half as fast along the circle $\sqrt{r}e^{i\theta/2}$ of radius $\sqrt{r}$.

Domains on which $z \mapsto e^z$ and $z \mapsto \log z$ are inverses have already been discussed (see Figure 1.3.2). Note that the lines $y =$ constant, described by the points $x + iy$ as x varies, are mapped by the function $z \mapsto e^z$ to points $e^x e^{iy}$, which is a ray with argument y. As x ranges from $-\infty$ to $+\infty$, the image point on the ray goes from 0 out to infinity (see Figure 1.3.7).

Similarly, the vertical line $x =$ constant is mapped to a circle of radius e^x. If we restrict y to an interval of length 2π, the image circle is described once, but if y is unrestricted, the image circle is described infinitely many times as y ranges from

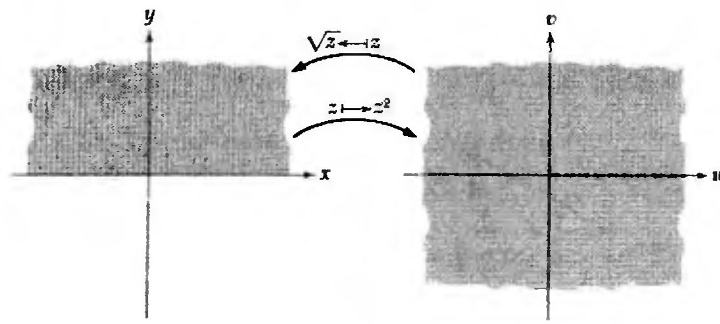


Figure 1.3.5: Squaring function and its inverse.

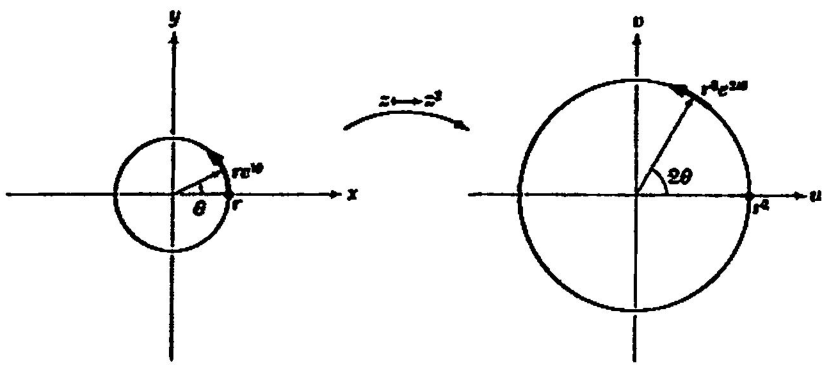


Figure 1.3.6: Effect of the squaring function on a circle of radius r.

$-\infty$ to $+\infty$. The logarithm, being the inverse of e^z, maps points in the opposite direction to e^z, as shown in Figure 1.3.7. Because of the special nature of the striplike regions in Figures 1.3.2 and 1.3.7 (on them e^z is one-to-one) and because of the periodicity of e^z, these regions deserve a name. They are usually called ***period strips*** of e^z.

Worked Examples

Example 1.3.13 *Find the real and imaginary parts of* $\exp(e^z)$. *(It is common to use* $\exp w$ *as another way of writing* e^w*.)*

Solution Let $z = x + iy$; then $e^z = e^x \cos y + ie^x \sin y$. Thus,

$$\exp e^z = e^{e^x \cos y}[\cos(e^x \sin y) + i \sin(e^x \sin y)].$$

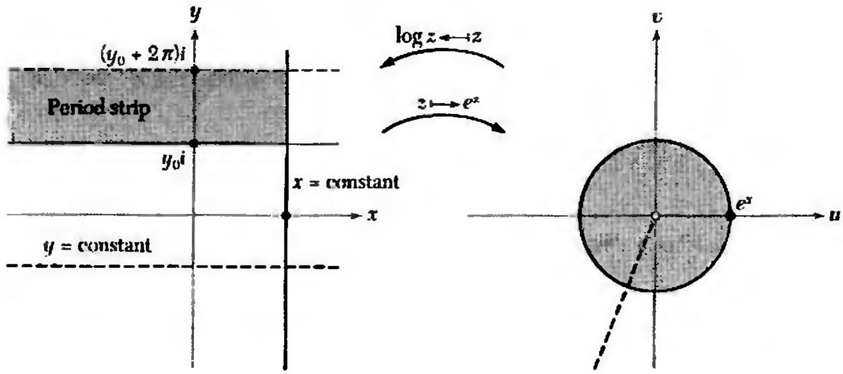


Figure 1.3.7: Geometry of e^z and $\log z$.

Therefore,

$$\operatorname{Re}(\exp e^z) = (e^{e^x \cos y})\cos(e^x \sin y) \quad \text{and} \quad \operatorname{Im}(\exp e^z) = (e^{e^x \cos y})\sin(e^x \sin y).$$

Example 1.3.14 *Find all the values of* i^i.

Solution

$$i^i = e^{i \log i} = e^{i[\log 1 + (i\pi/2) + (2\pi n)i]} = (e^{-2\pi n})e^{-\pi/2} = e^{-2\pi(n+1/4)}.$$

All the values of i^i are given by the last expression as n takes integral values, $n = 0, \pm 1, \pm 2, \ldots$.

Example 1.3.15 *Solve* $\cos z = \frac{1}{2}$ *for* z.

Solution We know that $z_n = \pm(\pi/3 + 2\pi n)$, where n is an integer, solves the equation $\cos z = \frac{1}{2}$; we shall show that $z_n, n = 0, \pm 1, \ldots$, are the *only* solutions; that is, there are no solutions off the real axis. We are given

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2}.$$

Therefore, $e^{2iz} - e^{iz} + 1 = 0$, and so by the quadratic formula, $e^{iz} = \frac{1}{2} \pm \sqrt{3}i/2$. Hence $iz = \log(\frac{1}{2} \pm \sqrt{3}i/2) = \pm\log(\frac{1}{2} + \sqrt{3}i/2)$, since $\frac{1}{2} + \sqrt{3}i/2$ and $\frac{1}{2} - \sqrt{3}i/2$ are checked to be reciprocals of one another. We thus obtain

$$z = \pm i \log\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \pm i\left(\log 1 + \frac{\pi}{3}i + 2\pi n i\right) = \pm\left(\frac{\pi}{3} + 2\pi n\right).$$

Example 1.3.16 *Consider the mapping* $z \mapsto \sin z$. *Show that lines parallel to the real axis are mapped to ellipses and that lines parallel to the imaginary axis are mapped to hyperbolas.*

Solution Using Proposition 1.3.4 (also see Example 1.3.14), we get

$$\begin{aligned}\sin z &= \sin(x+iy) = \sin x \cos(iy) + \sin(iy)\cos x \\ &= \sin x \cosh y + i \sinh y \cos x\end{aligned}$$

where

$$\cosh y = \frac{e^y + e^{-y}}{2} \quad \text{and} \quad \sinh y = \frac{e^y - e^{-y}}{2}.$$

Suppose that $y = y_0$ is constant; if we write $\sin z = u + iv$, then we have

$$\frac{u^2}{\cosh^2 y_0} + \frac{v^2}{\sinh^2 y_0} = 1$$

since $\sin^2 x + \cos^2 x = 1$. This is an ellipse.

Similarly, if $x = x_0$ is constant, from $\cosh^2 y - \sinh^2 y = 1$ we obtain

$$\frac{u^2}{\sin^2 x_0} - \frac{v^2}{\cos^2 x_0} = 1,$$

which is a hyperbola.

Exercises

1. Express in the form $a + bi$:
   (a) e^{2+i}
   (b) $\sin(1+i)$
2. Express in the form $a + bi$:
   (a) e^{3-i}
   (b) $\cos(2+3i)$
3. Solve
   (a) $\cos z = \frac{3}{4} + \frac{i}{4}$
   (b) $\cos z = 4$
4. Solve
   (a) $\sin z = \frac{3}{4} + \frac{i}{4}$
   (b) $\sin z = 4$
5. Find all the values of
   (a) $\log 1$
   (b) $\log i$

6. Find all the values of

   (a) $\log(-i)$

   (b) $\log(1+i)$

7. Find all the values of

   (a) $(-i)^i$

   (b) $(1+i)^{1+i}$

8. Find all the values of

   (a) $(-1)^i$

   (b) 2^i

9. For what values of z is $\overline{(e^{iz})} = e^{i\bar{z}}$?

10. Let $\sqrt{\cdot}$ denote the particular square root defined by

$$\sqrt{r(\cos\theta + i\sin\theta)} = r^{1/2}[\cos(\theta/2) + i\sin(\theta/2)], 0 \le \theta < 2\pi;$$

   the other square root is

$$r^{1/2}\{\cos[(\theta+2\pi)/2] + i\sin[(\theta+2\pi)/2]\}.$$

   For what values of z does the equation $\sqrt{z^2} = z$ hold?

11.* Along which rays through the origin (a ray is determined by $\arg z =$ constant) does $\lim_{z\to\infty} |e^z|$ exist?

12. Prove the identity

$$z = \tan\left[\frac{1}{i}\log\left(\frac{1+iz}{1-iz}\right)^{1/2}\right].$$

13. Simplify e^{z^2}, e^{iz}, and $e^{1/z}$, where $z = x + iy$. For $e^{1/z}$ we specify that $z \neq 0$.

14. Examine the behavior of e^{x+iy} as $x \to \pm\infty$ and the behavior of e^{x+iy} as $y \to \pm\infty$.

15.* Prove that $\sin(-z) = -\sin z$; $\cos(-z) = \cos z$; $\sin(\pi/2 - z) = \cos z$.

16. Define sinh and cosh on all of $\mathbf{C}$ by $\sinh z = (e^z - e^{-z})/2$ and $\cosh z = (e^z + e^{-z})/2$. Prove that

   (a) $\cosh^2 z - \sinh^2 z = 1$

   (b) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

   (c) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

(d) $\sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y$

(e) $\cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y$

17.• Use the equation $\sin z = \sin x \cosh y + i \sinh y \cos x$ where $z = x+iy$ to prove that $|\sinh y| \le |\sin z| \le |\cosh y|$.

18. If b is real, prove that $|a^b| = |a|^b$.

19. Is it true that $|a^b| = |a|^{|b|}$ for all $a, b \in \mathbb{C}$?

20. (a) For complex numbers a, b, c, prove that $a^b a^c = a^{b+c}$, using a fixed branch of log.

    (b) Show that $(ab)^c = a^c b^c$ if we choose branches so that $\log(ab) = \log a + \log b$ (with no extra $2\pi n i$).

21.• Using polar coordinates, show that $z \mapsto z + 1/z$ maps the circle $|z| = 1$ to the interval $[-2, 2]$ on the x axis.

22. (a) The map $z \mapsto z^3$ maps the first quadrant onto what?

    (b) Discuss the geometry of $z \mapsto \sqrt[3]{z}$ as was done in the text for $\sqrt{z}$.

23.• The map $z \mapsto 1/z$ takes the exterior of the unit circle to the interior (excluding zero) and vice versa. To what are lines $\arg z =$ constant mapped?

24. What are the images of vertical and horizontal lines under $z \mapsto \cos z$?

25. Under what conditions does $\log a^b = b \log a$ for complex numbers a, b? (Use the branch of log with $-\pi \le \theta < \pi$.)

26. (a) Show that under the map $z \mapsto z^2$, lines parallel to the real axis are mapped to parabolas.

    (b) Show that under (a branch of) $z \mapsto \sqrt{z}$, lines parallel to the real axis are mapped to hyperbolas.

27. Show that the n nth roots of unity are $1, w, w^2, w^3, \ldots, w^{n-1}$, where $w = e^{2\pi i/n}$.

28. Show that the trigonometric identities can be deduced if $e^{i(x_1+x_2)} = e^{ix_1} \cdot e^{ix_2}$ is assumed.

29.• Show that $\sin z = 0$ iff $z = k\pi.\ k = 0, \pm 1, \pm 2, \ldots.$

30. Show that the sine and cosine are periodic with minimum period 2π; that is, that

    (a) $\sin(z + 2\pi) = \sin z$ for all z.

    (b) $\cos(z + 2\pi) = \cos z$ for all z.

    (c) $\sin(z + \omega) = \sin z$ for all z implies $\omega = 2\pi n$ for some integer n.

(d) $\cos(z+\omega) = \cos z$ for all z implies $\omega = 2\pi n$ for some integer n.

31. Find the maximum of $|\cos z|$ on the square

$$0 \le \operatorname{Re} z \le 2\pi, 0 \le \operatorname{Im} z \le 2\pi.$$

32. Show that $\log z = 0$ iff $z = 1$, using the branch with $-\pi < \arg z \le \pi$.

33. Compute the following quantities numerically to two significant figures:
(a) $e^{3.2+6.1i}$ (b) $\log(1.2 - 3.0i)$ (c) $\sin(8.1i - 3.2)$

34.• Show that the function $\sin z$ maps the strip $-\pi/2 < \operatorname{Re} z < \pi/2$ onto the set $\mathbb{C}\backslash\{z \mid \operatorname{Im} z = 0 \text{ and } |\operatorname{Re} z| \ge 1\}$.

35.• Discuss the inverse functions $\sin^{-1} z$ and $\cos^{-1} z$. For example, is $\sin z$ one-to-one on the set defined by $0 \le \operatorname{Re} z < 2\pi$?

1.4 Continuous Functions

In this section and the next, the fundamental notions of continuity and differentiability for complex-valued functions of a complex variable will be analyzed. The results are similar to those learned in the calculus of functions of real variables. These sections will be concerned mostly with the underlying theory, which is applied to the elementary functions in §1.6.

Since $\mathbb{C}$ is $\mathbb{R}^2$ with the extra structure of complex multiplication, many geometric concepts can be translated from $\mathbb{R}^2$ into complex notation. This has already been done for the absolute value, $|z|$, which is the same as the norm, or length, of z regarded as a vector in $\mathbb{R}^2$. Furthermore, we will use calculus for functions of two variables in the study of functions of a complex variable.

Open Sets We will need the notion of an open set. A set $A \subset \mathbb{C} = \mathbb{R}^2$ is called ***open*** when, for each point z_0 in A, there is a real number $\epsilon > 0$ such that $z \in A$ whenever $|z - z_0| < \epsilon$. See Figure 1.4.1. The value of ϵ may depend on z_0; as z_0 gets close to the "edge" of A, ϵ gets smaller. Intuitively, a set is open if it does not contain any of its "boundary" or "edge" points.

For a number $r > 0$, the ***r neighborhood*** or ***r disk*** around a point z_0 in $\mathbb{C}$ is defined to be the set $D(z_0;r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$. For practice, the student should prove that for each $w_0 \in \mathbb{C}$ and $r > 0$, the disk $A = \{z \in \mathbb{C} \mid |z - w_0| < r\}$ is itself open. A ***deleted r neighborhood*** is an r neighborhood whose center point has been removed. Thus a deleted r-neighborhood has the form $D(z_0;r)\backslash\{z_0\}$, which stands for the set $D(z_0;r)$ minus the singleton set $\{z_0\}$. See Figure 1.4.2.

A ***neighborhood*** of a point z_0 is, by definition, a set containing some r disk around z_0. Notice that *a set A is open iff for each z_0 in A, there is an r neighborhood of z_0 wholly contained in A.*

The basic properties of open sets are collected in the next proposition.

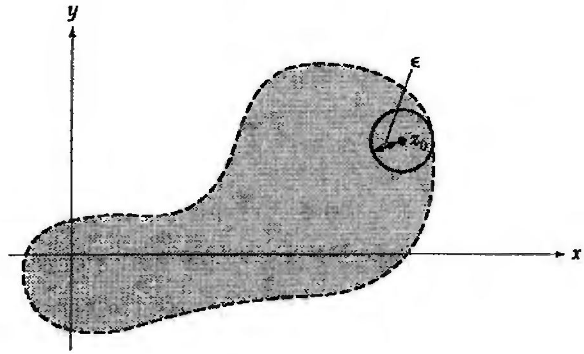
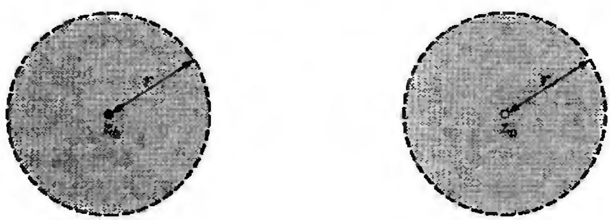


Figure 1.4.1: Open set.



(a) (b)

Figure 1.4.2: (a) r-Neighborhood. (b) Deleted r-neighborhood.

Proposition 1.4.1 (i) $\mathbb{C}$ *is open.*

(ii) *The empty set* $\varnothing$ *is open.*

(iii) *The union of any collection of open subsets of* $\mathbb{C}$ *is open.*

(iv) *The intersection of any finite collection of open subsets of* $\mathbb{C}$ *is open.*

Proof The first two assertions hold almost by definition; the first because any ϵ will work for any point z_0, and the second because there are no points for which we are required to find such an ϵ. The reader is asked to supply proofs of the last two in Exercises 19 and 20 at the end of this section. ■

Mappings, Limits, and Continuity Let A be a subset of $\mathbb{C}$. Recall that a mapping $f : A \to \mathbb{C}$ is an assignment of a specific point $f(z)$ in $\mathbb{C}$ to each point z in A. The set A is called the ***domain*** of f, and we say f is ***defined on*** A. When the domain and the range (the set of values f assumes) are both subsets of $\mathbb{C}$, as here, we speak of f as a ***complex function of a complex variable.*** Alternatively, we can think of f as a map $f : A \subset \mathbb{R}^2 \to \mathbb{R}^2$; then f is called a vector-valued function of two real variables. For $f : A \subset \mathbb{C} \to \mathbb{C}$, we can let $z = x + iy = (x, y)$ and define $u(x, y) = \operatorname{Re} f(z)$ and $v(x, y) = \operatorname{Im} f(z)$. Then u and v are the components of f thought of as a vector function. Hence we may uniquely write $f(x + iy) = u(x, y) + iv(x, y)$, where u and v are real-valued functions defined on A.

Next we consider the idea of limit in the setting of complex numbers.

Definition 1.4.2 *Let f be defined on a set containing some deleted r neighborhood of z_0. We say that f has the limit a as $z \to z_0$ and write*

$$\lim_{z \to z_0} f(z) = a.$$

when, for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $z \in D(z_0; r)$ satisfying $z \neq z_0$ and $|z - z_0| < \delta$, we have $|f(z) - a| < \epsilon$.

The expression in this definition has the same intuitive meaning as it has in calculus; namely, $f(z)$ is close to a whenever z is close to z_0. It is not necessary to define f on a *whole* deleted neighborhood to have a valid theory of limits, but deleted neighborhoods are used here for the sake of simplicity and also because such usage will be appropriate later in the text.

Just as with real numbers and real-valued functions, a function can have no more than one limit at a point, and limits behave well with respect to algebraic operations. This is the content of the next two propositions.

Proposition 1.4.3 *Limits are unique if they exist.*

Proof Suppose that $\lim_{z \to z_0} f(z) = a$ and $\lim_{z \to z_0} f(z) = b$ with $a \neq b$. Let $2\epsilon = |a - b|$, so that $\epsilon > 0$. There is a $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies that $|f(z) - a| < \epsilon$ and $|f(z) - b| < \epsilon$. Choose such a point $z \neq z_0$ (because f is defined in a deleted neighborhood of z_0). Then, by the triangle inequality, $|a - b| \leq |a - f(z)| + |f(z) - b| < 2\epsilon$, a contradiction. Thus $a = b$. ■

Proposition 1.4.4 *If $\lim_{z \to z_0} f(z) = a$ and $\lim_{z \to z_0} g(z) = b$, then*

(i) $\lim_{z \to z_0} [f(z) + g(z)] = a + b$.

(ii) $\lim_{z \to z_0} [f(z) g(z)] = ab$.

(iii) $\lim_{z \to z_0} [f(z)/g(z)] = a/b$ *if* $b \neq 0$.

Proof Only assertion (ii) will be proved here. The proof of assertion (i) is easy, and proof of assertion (iii) is slightly more challenging, but the reader can get the necessary clues from the corresponding real-variable case. To prove assertion (ii), we write

$$\begin{aligned} |f(z)g(z) - ab| &\leq |f(z)g(z) - f(z)b| + |f(z)b - ab| \quad \text{(triangle inequality)} \\ &= |f(z)||g(z) - b| + |f(z) - a||b| \quad \text{(factoring).} \end{aligned}$$

To estimate each term, we choose $\delta_1 > 0$ so that $0 < |z - z_0| < \delta_1$ implies that $|f(z) - a| < 1$, and thus $|f(z)| < |a| + 1$, since $|f(z) - a| \geq |f(z)| - |a|$, by Proposition 1.2.5(vi). Given $\epsilon > 0$, select positive numbers δ_2 and δ_3 so that $0 < |z - z_o| < \delta_2$ implies $|f(z) - a| < \epsilon/2(|b| + 1)$ and $0 < |z - z_o| < \delta_3$ implies $|g(z) - b| < \epsilon/2(|a| + 1)$. Let δ be the smallest of $\delta_1, \delta_2, \delta_3$. If $0 < |z - z_o| < \delta$, we have

$$\begin{aligned} |f(z)g(z) - ab| &\leq |f(z)|\,|g(z) - b| + |f(z) - a|\,|b| \\ &< \frac{\epsilon}{2(|a| + 1)}\,|f(z)| + \frac{\epsilon}{2(|b| + 1)}\,|b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $\lim_{z \to z_o} f(z)g(z) = ab$ as claimed. ■

Definition 1.4.5 *Let $A \subset \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$ be a function. We say f* ***is continuous at*** *$z_0 \in A$ if and only if*

$$\lim_{z \to z_0} f(z) = f(z_0)$$

and that f ***is continuous on*** *A if f is continuous at each point z_0 in A.*

This definition has the same intuitive meaning as it has in elementary calculus: If z is close to z_0, then $f(z)$ is close to $f(z_0)$. From Proposition 1.4.4 we deduce that if f and g are continuous on A, then so are the sum $f + g$ and the product fg, and so is f/g if $g(z_0) \neq 0$ for all points z_0 in A. It is also true that a composition of continuous functions is continuous.

Proposition 1.4.6 (i) *If $\lim_{z \to z_0} f(z) = a$ and h is a function defined on a neighborhood of a and is continuous at a, then $\lim_{z \to z_0} h(f(z)) = h(a)$.*

(ii) *If f is a continuous function on an open set A in $\mathbb{C}$ and h is continuous on $f(A)$, then the composite function $(h \circ f)(z) = h(f(z))$ is continuous on A.*

Proof Given $\epsilon > 0$, there is a $\delta_1 > 0$ such that $|h(w) - h(a)| < \epsilon$ whenever $|w - a| < \delta_1$ and a $\delta > 0$ such that $|f(z) - a| < \delta_1$ whenever $0 < |z - z_0| < \delta$. Therefore we get $|h(f(z)) - h(a)| < \epsilon$ whenever $0 < |z - z_0| < \delta$, which establishes (i). A proof of (ii) follows from (i) and is requested in Exercise 22 at the end of this section. ■

Sequences The concept of convergent sequences of complex numbers is analogous to that for sequences of real numbers studied in calculus. A sequence z_n, $n = 1, 2, 3, \ldots$ of points of $\mathbb{C}$ ***converges to*** z_0 if and only if for every $\epsilon > 0$, there is an integer N such that $n \geq N$ implies $|z_n - z_0| < \epsilon$. The limit of a sequence is expressed as

Limits of sequences have the same properties, obtained by the same proofs, as limits of functions. For example, the limit is unique if it exists; and if $z_n \to z_0$ and $w_n \to w_0$, then

(i) $z_n + w_n \to z_0 + w_0$.

(ii) $z_n w_n \to z_0 w_0$.

(iii) $z_n/w_n \to z_0/w_0$ (if w_0 and w_n are not 0).

Also, $z_n \to z_0$ iff $\operatorname{Re} z_n \to \operatorname{Re} z_0$ and $\operatorname{Im} z_n \to \operatorname{Im} z_0$. A proof of this for functions is requested in Exercise 2 at the end of this section.

A sequence z_n is called a ***Cauchy sequence*** if for every $\epsilon > 0$ there is an integer N such that $|z_n - z_m| < \epsilon$ whenever both $n \geq N$ and $m \geq N$. A basic property of real numbers, which we will accept without proof, is that every Cauchy sequence in $\mathbb{R}$ converges. More precisely, if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers, then there is a real number x_0 such that $\lim_{n\to\infty} x_n = x_0$. This is equivalent to the ***completeness*** of the real number system.[4] From the fact that $z_n \to z_0$ iff $\operatorname{Re} z_n \to \operatorname{Re} z_0$ and $\operatorname{Im} z_n \to \operatorname{Im} z_0$, we can conclude that *every Cauchy sequence in* $\mathbb{C}$ *converges*. This is a technical point, but is useful in convergence proofs, as we shall see in Chapter 3.

It should be noted that a link exists between sequences and continuity; namely, $f : A \subset \mathbb{C} \to \mathbb{C}$ is continuous iff for every convergent sequence $z_n \to z_0$ of points in A (that is, $z_n \in A$ and $z_0 \in A$), we have $f(z_n) \to f(z_0)$. The student is requested to prove this in Exercise 18 at the end of this section.

Closed Sets A subset F of $\mathbb{C}$ is said to be ***closed*** if its complement, $\mathbb{C}\backslash F = \{z \in \mathbb{C} \mid z \notin F\}$, is open. By taking complements and using Proposition 1.4.1, one discovers the following properties of closed sets.

Proposition 1.4.7

(i) *The empty set is closed.*

(ii) $\mathbb{C}$ *is closed.*

(iii) *The intersection of any collection of closed subsets of* $\mathbb{C}$ *is closed.*

(iv) *The union of any finite collection of closed subsets of* $\mathbb{C}$ *is closed.*

Closed and open sets are important for their relationships to continuous functions and to sequences and for other constructions we will see later.

Proposition 1.4.8 *A set* $F \subset \mathbb{C}$ *is closed iff whenever* $z_1, z_2, z_3, \ldots$ *is a sequence of points in* F *such that* $w = \lim_{n\to\infty} z_n$ *exists, then* $w \in F$.

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[4]See, for example, J. Marsden and M. Hoffman, *Elementary Classical Analysis*, Second Edition (New York: W. H. Freeman and Company, 1993).

Proof Suppose F is closed and z_n is a sequence of points in F. If $D(w;r)$ is any disk around w, then by the definition of convergence, z_n is in $D(w;r)$ for large enough n. Thus, $D(w;r)$ cannot be contained in the complement of F. Since that complement is open, w must not be in the complement of F. Therefore, it must be in F.

If F is not closed, then the complement is not open. In other words, there is a point w in $\mathbb{C}\backslash F$ such that no neighborhood of w is contained in $\mathbb{C}\backslash F$. In particular, we may pick points z_n in $F \cap D(w;1/n)$; this yields a convergent sequence of points of F whose limit is not in F. ■

Proposition 1.4.9 *If $f : \mathbb{C} \to \mathbb{C}$, the following are equivalent:*

(i) *f is continuous.*

(ii) *The inverse image of every closed set is closed.*

(iii) *The inverse image of every open set is open.*

Proof To show that (i) implies (ii), suppose f is continuous and F is closed. Let $z_1, z_2, z_3, \ldots$ be a sequence of points in $f^{-1}(F)$ and suppose that $z_n \to w$.[5] Since f is continuous, $f(z_n) \to f(w)$. But the points $f(z_n)$ are in the closed set F, and so $f(w)$ is also in F. That is, w is in $f^{-1}(F)$. Proposition 1.4.8 shows that $f^{-1}(F)$ is closed.

To show that (ii) implies (iii), let U be open. Then $F = \mathbb{C}\backslash U$ is closed. If (ii) holds, then $f^{-1}(F)$ is closed. Therefore, $\mathbb{C}\backslash f^{-1}(F) = f^{-1}(\mathbb{C}\backslash F) = f^{-1}(U)$ is open.

To show that (iii) implies (i), fix z_0 and let $\epsilon > 0$. Then z_0 is a member of the open set $f^{-1}(D(f(z_0);\epsilon))$. Hence there is a $\delta > 0$ with

$$D(z_0;\delta) \subset f^{-1}(D(f(z_0);\epsilon)).$$

This says precisely that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$. We thus get exactly the inequality needed to establish continuity. ■

To handle continuity on a subset of $\mathbb{C}$, it is convenient to introduce the notion of relatively open and closed sets. If $A \subset \mathbb{C}$, a subset B of A is called ***open relative to*** A if $B = A \cap U$ for some open set U. It is said to be ***closed relative to*** A if $B = A \cap F$ for some closed set F. This leads to the following proposition, whose proof is left to the reader.

Proposition 1.4.10 *If $f : A \to \mathbb{C}$, the following are equivalent:*

(i) *f is continuous.*

(ii) *The inverse image of every closed set is closed relative to A.*

(iii) *The inverse image of every open set is open relative to A.*

[illegible]

Connected Sets This subsection and the next study two important classes of sets which occupy to some extent the place in the theory of complex variables held by intervals and by closed bounded intervals in the theory of functions of a real variable. These are the connected sets and the compact sets.

A connected set should be one that "consists of one piece". This may be approached from a positive point of view—"Any point can be connected to any other"—or from a negative point of view—"The set cannot be split into two parts". This leads to two possible definitions.

Definition 1.4.11 *A set $C \subset \mathbb{C}$ is path-connected if for every pair of points a, b in C there is a continuous map $\gamma : [0, 1] \to C$ with $\gamma(0) = a$ and $\gamma(1) = b$. We call γ a path joining a and b.*

One can often easily tell if a set is path-connected, as is shown in Figure 1.4.3. The negative point of view suggests a slightly different definition.

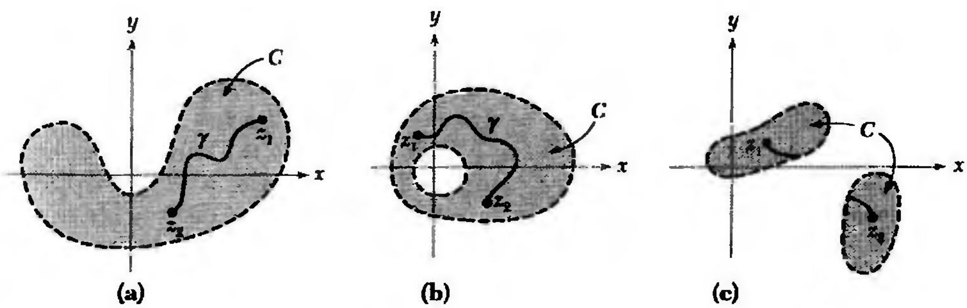

Figure 1.4.3: Regions in (a) and (b) are connected while the region in (c) is not.

Definition 1.4.12 *A set $C \subset \mathbb{C}$ is not connected (see Figure 1.4.4) if there are open sets U and V such that*

(i) $C \subset U \cup V$

(ii) $C \cap U \neq \varnothing$ *and* $C \cap V \neq \varnothing$

(iii) $(C \cap U) \cap (C \cap V) = \varnothing$

If a set fails to be "not connected", it is called connected.

The notions of relatively open and closed sets allow this to be rephrased in terms of subsets of C. Since the intersection of C with U is the same as its intersection with the complement of V, the set $C \cap U$ is both open and closed relative to C, as is $C \cap V$. This proves the next result.

Proposition 1.4.13 *A set C is connected if and only if the only subsets of C that are both open and closed relative to C are the empty set and C itself.*

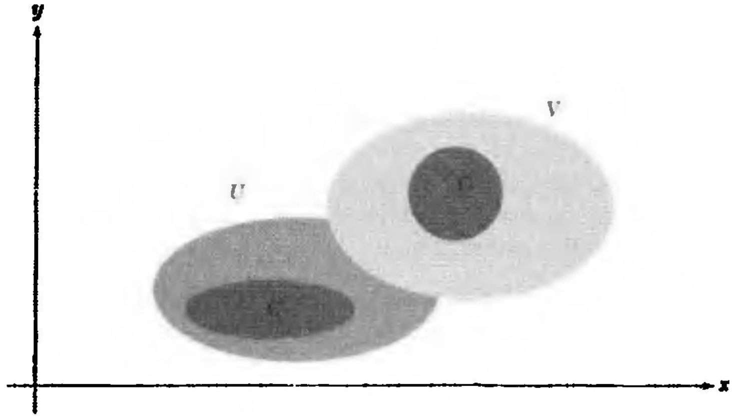

Figure 1.4.4: The set C is not connected.

The next two propositions give the relationship between the two definitions. The two notions are not in general equivalent, but they are for open sets. The proof of this last assertion (given below in Proposition 1.4.15) illustrates a fairly typical way of using the notion of connectivity. One shows that a certain property holds everywhere in C by showing that the set of places where it holds is not empty and is both relatively open and relatively closed.

Proposition 1.4.14 *A path-connected set is connected.*

Proof Suppose C is a path-connected set and D is a nonempty subset of C that is both open and closed relative to C. If $C \neq D$, there is a point z_1 in D and a point z_2 in $C \setminus D$. Let $\gamma : [a, b] \to C$ be a continuous path joining z_1 to z_2. Let $B = \gamma^{-1}(D)$. Then B is a subset of the interval $[a, b]$, since γ is continuous. (See Proposition 1.4.10.) Since a is in B, B is not empty, and $[a, b] \setminus B$ is not empty since it contains b.

This argument shows that it is sufficient to prove the theorem for the case of an interval $[a, b]$. We thus need to establish that intervals on the real line are connected. A proof uses the least upper bound property (or some other characterization of the fact that the system of real numbers is complete). Let $x = \sup B$ (that is, the least upper bound of B). We find that x is in B since B is closed. Since B is open there is a neighborhood of x contained in B (note that $x \neq b$, since b is in $[a, b] \setminus B$). Thus, for some $\epsilon > 0$, the point $x + \epsilon$ is in B. Thus x cannot be the least upper bound. This contradiction shows that such a set B cannot exist. ■

A connected set need not be path-connected,[6] but if it is open it must be. In fact, more is true.

[6] A standard example is given by letting C be the union of the graph of $y = \sin 1/x$, where $x > 0$, and the line segment $-1 \leq y \leq 1, x = 0$. This set is connected but not path-connected.

Proposition 1.4.15 *If C is an open connected set and a and b are in C, then there is a differentiable path $\gamma : [0,1] \to C$ with $\gamma(0) = a$ and $\gamma(1) = b$.*[7]

Proof Let a be in C. If z_0 is in C, then since C is open, there is an $\epsilon > 0$ such that the disk $D(z_0;\epsilon)$ is contained in C. By combining a path from a to z_0 with one from z_0 to z that stays in this disk, we see that z_0 can be connected to a by a differentiable path if and only if the same is true for every point z in $D(z_0;\epsilon)$. This shows that both the sets

$$A = \{z \in \mathbf{C} \mid z \text{ can be connected to } a \text{ by a differentiable path}\}$$

and

$$B = \{z \in \mathbf{C} \mid z \text{ cannot be so connected to } a\}$$

are open. Since C is connected, either A or B must be empty. Obviously it must be B. See Figure 1.4.5. ■

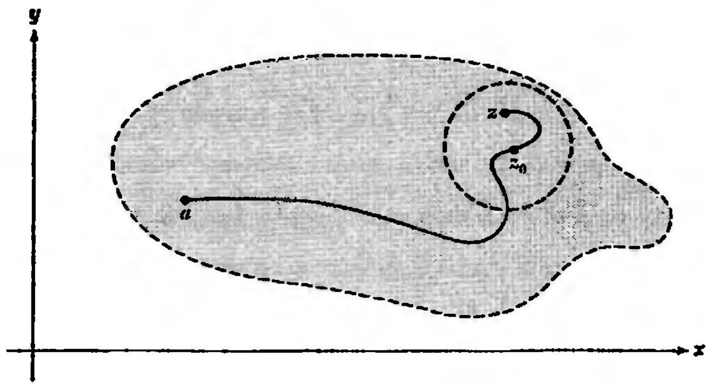

Figure 1.4.5: An open connected set is path-connected.

Because of the importance of open connected sets, they are often designated by a special term. Although the usage is not completely standard in the literature, the words ***region*** and ***domain*** are often used. In this text these terms will be used synonymously to mean an open connected subset of $\mathbf{C}$. The reader should be careful to check the meanings when these words are encountered in other texts.

The notion of connected sets will be of use to us several times. One observation is that a continuous function cannot break apart a connected set.

Proposition 1.4.16 *If f is a continuous function defined on a connected set C, then the image set $f(C)$ is also connected.*

[7] Differentiability of γ means that each component of γ is differentiable in the usual sense of one-variable calculus.

Proof If U and V are open sets that disconnect $f(C)$, then $f^{-1}(U)$ and $f^{-1}(V)$ are open sets disconnecting C. ■

Be careful. This proposition works in the opposite direction from the one about open and closed sets. For continuous functions, the inverse images of open sets are open and the inverse images of closed sets are closed. But it is the direct images that are guaranteed to be connected and *not* the inverse images of connected sets. (Can you think of an example?) The same sort of thing will happen with the class of sets studied in the next subsection, the compact sets.

Compact Sets The next special class of sets we introduce is that of the compact sets. These will turn out to be those subsets K of $\mathbb{C}$ that are bounded in the sense that there is a number M such that $|z| \le M$ for every z in K and that are closed. One of the nice properties of such sets is that every sequence of points in the set must have a subsequence which converges to some point in the set. For example, the sequence $1, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{5}{3}, \frac{1}{4}, \frac{7}{4}, \frac{1}{5}, \frac{9}{5}, \ldots$ of points in $]0, 2[$ has the subsequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$, which converges to the point 0, which is not in the open interval $]0, 2[$ but is in the closed interval $[0, 2]$. Note that in the claimed property, the sequence itself is not asserted to converge. All that is claimed is that some subsequence does; the example shows that this is necessary.

As often happens in mathematics, the study consists of three parts:

(i) An easily recognized characterization: closed and bounded

(ii) A property we want: the existence of convergent subsequences

(iii) A technical definition useful in proofs and problems

In the case at hand, the technical definition involves the relationship between compactness and open sets. A collection of open sets U_α for α in some index set $\mathcal{A}$ is called a ***cover*** (or an ***open cover***) of a set K if K is contained in their union: $K \subset \cup_{\alpha\in\mathcal{A}} U_\alpha$. For example, the collection of all open disks of radius 2 is an open cover of $\mathbb{C}$:

$$U_z = D(z; 2) \quad \mathbb{C} \subset \cup_{z\in\mathbb{C}} D(z; 2).$$

It may be, as here, that the covering process has been wasteful, using more sets than needed. In that case we may use only some of the sets and talk of a ***subcover***, for example, $\mathbb{C} \subset \cup_{n,m\in\mathbb{Z}} D(n + mi; 2)$, where $\mathbb{Z}$ denotes the set of integers.

Definition 1.4.17 *A set K is compact if every open cover of K has a finite subcover.*

That is, if U_α is any collection of open sets whose union contains K, then there is a finite subcollection $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_k}$ such that $K \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_k}$.

Proposition 1.4.18 *The following conditions are equivalent for a subset K of $\mathbb{C}$ (or of $\mathbb{R}$):*

(i) *K is closed and bounded.*

(ii) *Every sequence of points in K has a subsequence which converges to some point in K.*

(iii) *K is compact.*

This proposition requires a deeper study of the completeness properties of the real numbers than it is necessary for us to go into here, so the proof is omitted. It may be found in most advanced calculus or analysis texts.[5] It is easy to see why (i) is necessary for (ii) and (iii). If K is not bounded we can select z_1 in K and then successively choose z_2 with $|z_2| > |z_1| + 1$ and, in general, z_n with $|z_n| > |z_{n-1}| + 1$. This gives a sequence with no convergent subsequence. The open disks $D(0; n), n = 1, 2, 3, \ldots$, would be an open cover with no finite subcover.

If K is a set in $\mathbf{C}$ that is not closed, then there is a point w in $\mathbf{C}\backslash K$ and a sequence $z_1, z_2, \ldots$ of points in K that converges to w. Since the sequence converges, w is the only possible limit of a subsequence, so no subsequence can converge to a point of K. The sets $\{z \text{ such that } |z - w| > 1/n\}$ for $n = 1, 2, 3, \ldots$ form an open cover of K with no finite subcover.

The utility of the technical Definition 1.4.17 is illustrated in the following results.

Proposition 1.4.19 *If K is a compact set and f is a continuous function defined on K, then the image set $f(K)$ is also compact.*

Proof If U_α is an open cover of $f(K)$, then the sets $f^{-1}(U_\alpha)$ form an open cover of K. Selection of a finite subcover gives

$$K \subset f^{-1}(U_{\alpha_1}) \cup \ldots \cup f^{-1}(U_{\alpha_k})$$

so that $f(K) \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_k}$. ■

Theorem 1.4.20 (Extreme Value Theorem) *If K is a compact set and $f : K \to \mathbb{R}$ is continuous, then f attains finite maximum and minimum values.*

Proof The image $f(K)$ is compact, hence closed and bounded. Since it is bounded, the numbers $M = \sup\{f(z) \mid z \in K\}$ and $m = \inf\{f(z) \mid z \in K\}$ are finite. Since $f(K)$ is closed, m and M are included in $f(K)$. ■

Another illustration of the use of compactness is given by the following lemma, which asserts that the distance from a compact set to a closed set is positive. That is, there must be a definite gap between the two sets.

Lemma 1.4.21 (Distance Lemma) *Suppose K is compact, C is closed, and $K \cap C = \varnothing$. Then the distance $d(K, C)$ from K to C is greater than 0. That is, there is a number $\rho > 0$ such that $|z - w| > \rho$ whenever z is in K and w is in C.*

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[5] See, for example, J. Marsden and M. Hoffman, *Elementary Classical Analysis*, Second Edition (New York: W. H. Freeman and Company, 1993).

Proof The complement of C, namely the set $U = \mathbb{C}\backslash C$, is an open set and $K \subset U$, so that each point z in K is the center of some disk $D(z; \rho(z)) \subset U$. The collection of smaller disks $D(z; \rho(z)/2)$ also covers K, and by compactness there is a finite number of disks, which we denote by $D_k = D(z_k; \rho(z_k)/2), k = 1, 2, 3, \ldots, N$ that cover K. (See Figure 1.4.6.) Let $\rho_k = \rho(z_k)/2$ and $\rho = \min(\rho_1, \rho_2, \ldots, \rho_N)$. If z is in K and w is in C, then z is in D_k for some k, and so $|z - z_k| < \rho_k$. But $|w - z_k| > \rho(z_k) = 2\rho_k$. Thus, $|z - w| > \rho_k \geq \rho$. ■

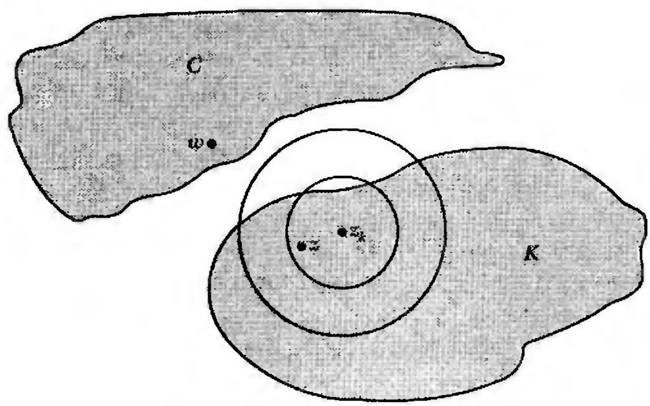

Figure 1.4.6: The distance between a closed set C and a compact set K is greater than zero.

Uniform Continuity Remember that a function is said to be continuous on a set K if it is continuous at each point of K. This is called a ***local property*** since it is defined in terms of the behavior of the function at or near each point and can be determined for each point by looking only near the point and not at the whole set at once. This is in contrast to ***global properties*** of a function, which depend on its behavior on the whole set.

An example of a global property is boundedness. Saying that a function f is bounded by some number M on a set K is an assertion that depends on the whole set at once. If the function is continuous it is certainly bounded near each point, but that would not automatically say that it is bounded on the whole set. For example, the function $f(x) = 1/x$ is continuous on the open interval $]0, 1[$ but is certainly not bounded there. We have seen that if a function f is continuous on a compact set K, then it is bounded on K and in fact the bounds are attained. Thus, compactness of K allowed us to carry the local boundedness near each point given by continuity over to the whole set. Compactness often can be used to make such a shift from a local property to a global one. The following is a global version of the notion of continuity.

Definition 1.4.22 *A function $f : A \to \mathbb{C}$ (or $\mathbb{R}$) is **uniformly continuous** on A*

if for every choice of $\epsilon > 0$ *there is a* $\delta > 0$ *such that* $|f(s) - f(t)| < \epsilon$ *whenever* s *and* t *are in* A *and* $|s - t| < \delta$.

Notice that the difference between this and the definition of ordinary continuity is that now the choice of δ can be made so that the same δ will work everywhere in the set A. Obviously, uniformly continuous functions are continuous. On a compact set the opposite is true as well.

Proposition 1.4.23 *A continuous function on a compact set is uniformly continuous.*

Proof Suppose f is a continuous function on a compact set K, and let $\epsilon > 0$. For each point t in K, there is a number $\delta(t)$ such that $|f(s) - f(t)| < \epsilon/2$ whenever $|s - t| < \delta(t)$. The open sets $D(t; \delta(t)/2)$ cover K, so by compactness there are a finite number of points $t_1, t_2, \ldots, t_N$ such that the sets $D_k = D(t_k; \delta(t_k)/2)$ cover K. Let $\delta_k = \delta(t_k)/2$ and set δ equal to the minimum of $\delta_1, \delta_2, \ldots, \delta_N$. If $|s - t| < \delta$, then t is in D_k for some k, and so $|t - t_k| < \delta_k$. Thus $|f(t) - f(t_k)| < \epsilon/2$. But also,

$$|s - t_k| = |s - t + t - t_k| \leq |s - t| + |t - t_k| \leq \delta + \delta_k \leq \delta(t_k)$$

and so $|f(s) - f(t_k)| \leq \epsilon/2$. Thus

$$\begin{aligned} |f(s) - f(t)| &= |f(s) - f(t_k) + f(t_k) - f(t)| \\ &\leq |f(s) - f(t_k)| + |f(t_k) - f(t)| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

We have produced a single δ that works everywhere in K, and so f is uniformly continuous. ■

Path-Covering Lemma The notion of uniform continuity is a very powerful one that will be useful to us several times. We use it first in conjunction with the Distance Lemma and some of the properties of compact sets to establish a useful geometric lemma about curves in open subsets of the complex plane. This lemma will be useful later in the text, particularly for studying integrals along such curves. It says that the curve can be covered by a finite number of disks centered along the curve in such a way that each disk is contained in the open set and each contains the centers of both the preceding and the succeeding disks along the curve. (See Figure 1.4.7.)

Lemma 1.4.24 (Path-Covering Lemma) *Suppose* $\gamma : [a, b] \to G$ *is a continuous path from the interval* $[a, b]$ *into an open subset* G *of* $\mathbb{C}$. *Then there are a number* $\rho > 0$ *and a subdivision of the interval* $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ *such that*

(i) $D(\gamma(t_k); \rho) \subset G$ *for all* k

(ii) $\gamma(t) \in D(\gamma(t_0); \rho)$ *for* $t_0 \leq t \leq t_1$

(iii) $\gamma(t) \in D(\gamma(t_k); \rho)$ *for* $t_{k-1} \leq t \leq t_{k+1}$

(iv) $\gamma(t) \in D(\gamma(t_n); \rho)$ *for* $t_{n-1} \leq t \leq t_n$

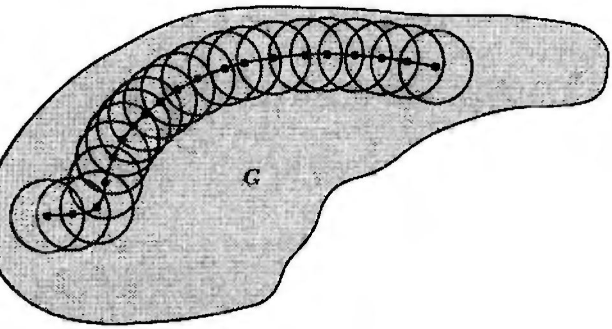

Figure 1.4.7: A continuous path in an open set can be covered by a finite number of well-overlapping disks.

Proof Since γ is continuous and the closed interval $[a, b]$ is compact, the image curve $K = \gamma([a, b])$ is compact. By the Distance Lemma 1.4.21 there is a number ρ such that each point on the curve is a distance at least ρ from the complement of G. Therefore, $D(\gamma(t); \rho) \subset G$ for every t in $[a, b]$. Also, since γ is continuous on the compact set $[a, b]$, it is uniformly continuous, and there is a number $\delta > 0$ such that $|\gamma(t) - \gamma(s)| < \rho$ whenever $|s - t| < \delta$. Thus if the subdivision is chosen fine enough so that $t_k - t_{k-1} < \delta$ for all $k = 1, 2, 3, \ldots, N$, then the conclusions of the theorem hold. ■

Riemann Sphere and Point at Infinity For some purposes it is convenient to introduce a point ∞ in addition to the points $z \in \mathbb{C}$. One must be careful in doing so, since it can lead to confusion and abuse of the symbol ∞. But with care it can be useful, and we certainly want to be able to talk intelligently about infinite limits and limits at infinity.

In contrast to the real line, to which $+\infty$ and $-\infty$ can be added, we have only one ∞ for $\mathbb{C}$. The reason is that $\mathbb{C}$ has no natural ordering as $\mathbb{R}$ does. Formally we add a symbol ∞ to $\mathbb{C}$ to obtain the ***extended complex plane***, $\mathbb{C}$, and define operations with ∞ by the rules

$$\begin{aligned} z + \infty &= \infty \\ z \cdot \infty &= \infty \qquad \text{provided } z \neq 0 \\ \infty + \infty &= \infty \\ \infty \cdot \infty &= \infty \\ \frac{z}{\infty} &= 0 \end{aligned}$$

for $z \in \mathbb{C}$. Notice that some things are not defined: $\infty/\infty, 0 \cdot \infty, \infty - \infty$, and so forth are ***indeterminate forms*** for essentially the same reasons that they are in the calculus of real numbers. We also define appropriate limit concepts:

$\lim_{z\to\infty} f(z) = z_0$ *means: For any* $\epsilon > 0$, *there is an* $R > 0$ *such that* $|f(z) - z_0| < \epsilon$ *whenever* $|z| \geq R$.

$\lim_{z\to z_0} f(z) = \infty$ *means: For any* $R > 0$, *there is a* $\delta > 0$ *such that* $|f(z)| > R$ *whenever* $|z - z_0| < \delta$.

For sequences:

$\lim_{z\to\infty} z_n = \infty$ *means: For any* $R > 0$, *there is an* $N > 0$ *such that* $|z_n| > R$ *whenever* $n > N$.

Thus a point $z \in \mathbb{C}$ is "close to ∞" when it lies outside a large circle. This type of closeness can be pictured geometrically by means of the *Riemann sphere* shown in Figure 1.4.8. By the method of *stereographic projection* illustrated in this figure, a point z' on the sphere is associated with each point z in $\mathbb{C}$. Exactly one point on the sphere $\mathbf{S}$ has been omitted—the "north" pole. We assign ∞ in $\bar{\mathbb{C}}$ to the north pole of $\mathbf{S}$. We see geometrically that z is close to ∞ if and only if the corresponding points are close on the Riemann sphere in the usual sense of closeness in $\mathbb{R}^3$. Proof of this is requested in Exercise 24.

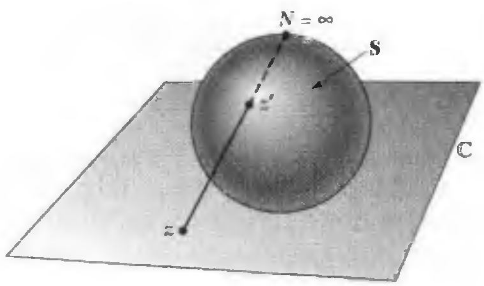

Figure 1.4.8: Riemann sphere.

The Riemann sphere $\mathbf{S}$ represents a convenient geometric picture of the extended plane $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The sphere does point up one fact about the extended plane that is sometimes useful in further theory. Since $\mathbf{S}$ is a closed bounded subset of $\mathbb{R}^3$, it is compact. Therefore every sequence in it has a convergent subsequence. Since stereographic projection makes convergence on $\mathbf{S}$ coincide with convergence of the sequence of corresponding points in $\bar{\mathbb{C}}$, the same is true there. That is, $\bar{\mathbb{C}}$ is compact. Every sequence of points in $\bar{\mathbb{C}}$ must have a subsequence convergent in $\bar{\mathbb{C}}$. *Caution:* Since the convergence is in the extended plane, the limit might be ∞, in that case we would normally say that the limit does not exist. Basically we have thrown in the point at infinity as another available limit so that sequences that did not formerly have a limit now have one. The sphere can be used both to help visualize and to make precise some notions about the behavior of functions "at infinity" that we will meet in future chapters.

Worked Examples

Example 1.4.25 *Where is the function*

$$f(z) = \frac{z^3 + 2z + 1}{z^3 + 1}$$

continuous?

Solution Since sums, products, and quotients of continuous functions are continuous except where the denominator is 0, this function is continuous on the whole plane except at the cube roots of -1. In other words, this function is continuous on the set $\mathbb{C}\backslash\{e^{\pi i/3}, e^{5\pi i/3}, -1\}$.

Example 1.4.26 *Show that the set* $\{z \mid \operatorname{Re} z > 0\}$ *is open.*

Solution A proof can be based on the following properties of complex numbers (see Exercise 1): *If* $w \in \mathbb{C}$, *then*

(i) $|\operatorname{Re} w| \leq |w|$

(ii) $|\operatorname{Im} w| \leq |w|$

(iii) $|w| \leq |\operatorname{Re} w| + |\operatorname{Im} w|$

Let $U = \{z \mid \operatorname{Re} z > 0\}$ and let z_0 be in U. We claim that the disk $D(z_0; \operatorname{Re} z_0)$ lies in U. To see this, let z be in this disk. Then $|\operatorname{Re} z - \operatorname{Re} z_0| = |\operatorname{Re}(z - z_0)| \leq |z - z_0| < \operatorname{Re} z_0$, and so $\operatorname{Re} z > 0$ and z is in U. Thus, $D(z_0; \operatorname{Re} z_0)$ is a neighborhood of z_0 that is contained in U. Since this can be done for any point z_0 which is in U, the set U is open. See Figure 1.4.9.

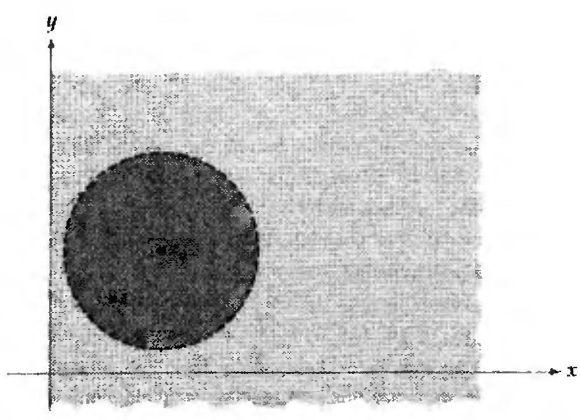

Figure 1.4.9: Open right half-plane.

Example 1.4.27 *Prove the following statement: Let $A \subset \mathbb{C}$ be an open set and $z_0 \in A$, and suppose that $D_r = \{z$ such that $|z - z_0| \leq r\} \subset A$. Then there is a number $\rho > r$ such that $D(z_0; \rho) \subset A$.*

Solution We know from the Extreme Value Theorem 1.4.20 that a continuous real-valued function on a closed bounded set in $\mathbb{C}$ attains its maximum and minimum at some point of the set. For z in D_r let $f(z) = \inf\{|z - w|$ such that $w \in \mathbb{C}\backslash A\}$. (Here "inf" means the greatest lower bound.) In other words, $f(z)$ is the distance from z to the complement of A. Since A is open, $f(z) > 0$ for each z in D_r. We can also verify that f is continuous. Thus f assumes its minimum at some point z_1 in D_r. Let $\rho = f(z_1) + r$, and check that this ρ has the desired properties. See Figure 1.4.10.

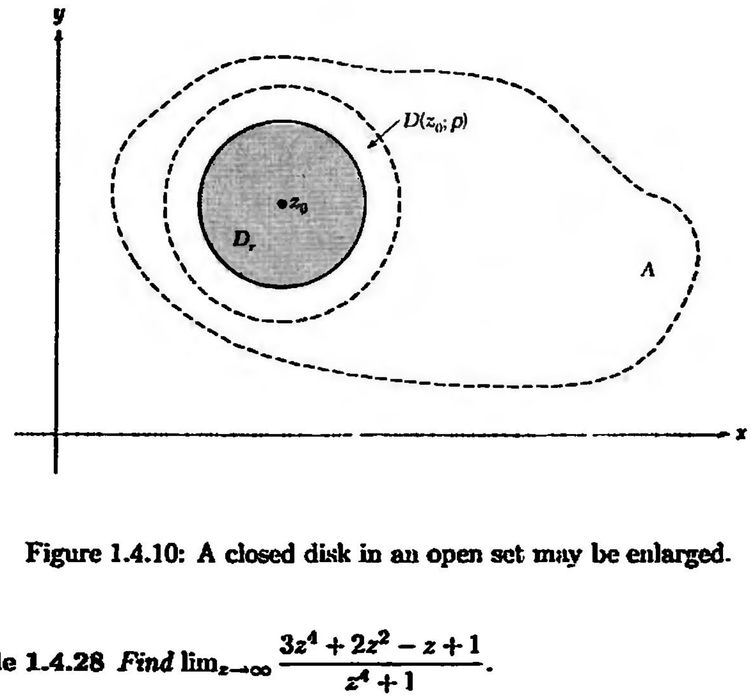

Figure 1.4.10: A closed disk in an open set may be enlarged.

Example 1.4.28 *Find* $\lim_{z\to\infty} \dfrac{3z^4 + 2z^2 - z + 1}{z^4 + 1}$.

Solution

$$\lim_{z\to\infty} \frac{3z^4 + 2z^2 - z + 1}{z^4 + 1} = \lim_{z\to\infty} \frac{3 + 2z^{-2} - z^{-3} + z^{-4}}{1 + z^{-4}} = 3$$

using $\lim_{z\to\infty} z^{-1} = 0$ and the basic properties of limits.

Exercises

1. Show that if $w \in \mathbb{C}$, then

(a) $|\operatorname{Re} w| \leq |w|$

(b) $|\operatorname{Im} w| \leq |w|$

(c) $|w| \leq |\operatorname{Re} w| + |\operatorname{Im} w|$

2.* (a) Show that

$$|\operatorname{Re} z_1 - \operatorname{Re} z_2| \leq |z_1 - z_2| \leq |\operatorname{Re} z_1 - \operatorname{Re} z_2| + |\operatorname{Im} z_1 - \operatorname{Im} z_2|$$

for any two complex numbers z_1 and z_2.

(b) If $f(z) = u(x,y) + iv(x,y)$, show that

$$\lim_{z \to z_0} f(z) = \lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) + i \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y)$$

exists if both limits on the right of the equation exist. Conversely, if the limit on the left exists, show that both limits on the right exist as well and equality holds. Show that $f(z)$ is continuous iff u and v are.

3. Prove: If f is continuous and $f(z_0) \neq 0$, there is a neighborhood of z_0 on which f is $\neq 0$.

4. If $z_0 \in \mathbf{C}$, show that the set $\{z_0\}$ is closed.

5. Prove: The complement of a finite number of points is an open set.

6. Use the fact that a function is continuous if and only if the inverse image of every open set is open to show that a composition of two continuous functions is continuous.

7. Show that $f(z) = \bar{z}$ is continuous.

8. Show that $f(z) = |z|$ is continuous.

9. What is the largest set on which the function $f(z) = 1/(1-e^z)$ is continuous?

10. Prove or find a counterexample if false: If $\lim_{z \to z_0} f(z) = a$, h is defined at the points $f(z)$, and $\lim_{w \to a} h(w) = c$, then $\lim_{z \to z_0} h(f(z)) = c$. [*Hint*: Could we have $h(a) \neq c$?]

11. For what z does the sequence $z_n = nz^n$ converge?

12.* Define $f : \mathbf{C} \to \mathbf{C}$ by setting $f(0) = 0$ and by setting $f(r[\cos\theta + i\sin\theta]) = \sin\theta$ if $r > 0$. Show that f is discontinuous at 0 but is continuous everywhere else.

13. For each of the following sets, state (i) whether or not it is open and (ii) whether or not it is closed.

(a) $\{z \text{ such that } |z| < 1\}$

(b) $\{z \mid 0 < |z| \leq 1\}$

(c) $\{z \mid 1 \leq \operatorname{Re} z \leq 2\}$

14. For each of the following sets, state (i) whether or not it is open and (ii) whether or not it is closed.

    (a) $\{z \mid \operatorname{Im} z > 2\}$
    (b) $\{z \mid 1 \leq |z| \leq 2\}$
    (c) $\{z \mid -1 < \operatorname{Re} z \leq 2\}$

15. For each of the following sets, state (i) whether or not it is connected and (ii) whether or not it is compact.

    (a) $\{z \mid 1 \leq |z| \leq 2\}$
    (b) $\{z \text{ such that } |z| \leq 3 \text{ and } |\operatorname{Re} z| \geq 1\}$
    (c) $\{z \text{ such that } |\operatorname{Re} z| \leq 1\}$
    (d) $\{z \text{ such that } |\operatorname{Re} z| \geq 1\}$

16. For each of the following sets, state (i) whether or not it is connected and (ii) whether or not it is compact.

    (a) $\{z \mid 1 < \operatorname{Re} z \leq 2\}$
    (b) $\{z \mid 2 \leq |z| \leq 3\}$
    (c) $\{z \text{ such that } |z| \leq 5 \text{ and } |\operatorname{Im} z| \geq 1\}$

17. If $A \subset \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$, show that $\mathbb{C}\backslash f^{-1}(A) = f^{-1}(\mathbb{C}\backslash A)$.

18. Show that $f : A \subset \mathbb{C} \to \mathbb{C}$ is continuous if and only if $z_n \to z_0$ in A implies that $f(z_n) \to f(z_0)$.

19. Show that the union of any collection of open subsets of $\mathbb{C}$ is open.

20. Show that the intersection of any finite collection of open subsets of $\mathbb{C}$ is open.

21. Give an example to show that the statement in Exercise 20 is false if the word "finite" is omitted.

22. Prove part (ii) of Proposition 1.4.6 by using part (i).

23. Show that if $|z| > 1$, then $\lim_{n\to\infty}(z^n/n) = \infty$.

24. Introduce the ***chordal metric*** ρ on $\bar{\mathbb{C}}$ by setting $\rho(z_1, z_2) = d(z_1', z_2')$ where z_1' and z_2' are the corresponding points on the Riemann sphere and d is the usual distance between points in $\mathbb{R}^3$.

    (a) Show that $z_n \to z$ in $\mathbb{C}$ if and only if $\rho(z_n, z) \to 0$.
    (b) Show that $z_n \to \infty$ if and only if $\rho(z_n, \infty) \to 0$.
    (c) If $f(z) = (az + b)/(cz + d)$ and $ad - bc \neq 0$, show that f is continuous at ∞.

1.5 Basic Properties of Analytic Functions

Although continuity is an important concept, its importance in complex analysis is overshadowed by that of the complex derivative. There are several approaches to the theory of complex differentiation. We shall begin by defining the derivative as the limit of difference quotients in the same spirit as in calculus. Many properties of the derivative including useful computation rules follow from the properties of limits just as they do in calculus. However, there are some surprising and beautiful results special to the complex theory.

Several different words are used to describe functions that are differentiable in the complex sense, for example, "regular", "holomorphic", and "analytic". We will use the term "analytic" since it is used in calculus to describe functions for which the Taylor series converges to the value of the function. An elegant result of complex analysis justifies this choice of language. Indeed, we will see in Chapter 3 that, in sharp distinction from the case of a single real variable, the assumption that a function is differentiable in the sense of complex variables guarantees the validity of the Taylor expansion of that function.

Definition 1.5.1 *Let $f : A \to \mathbf{C}$ where $A \subset \mathbf{C}$ is an open set. The function f is said to be* ***differentiable (in the complex sense)*** *at $z_0 \in A$ if*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. This limit is denoted by $f'(z_0)$, or sometimes by $(df/dz)(z_0)$. Thus, $f'(z_0)$ is a complex number. The function f is said to be ***analytic on A*** *if f is complex-differentiable at each $z_0 \in A$. The word* ***"holomorphic"****, which is sometimes used, is synonymous with the word "analytic." The phrase* ***"analytic at z_0"*** *means f is analytic on a neighborhood of z_0.*

Note that the quotient

$$\frac{f(z) - f(z_0)}{z - z_0}$$

is undefined at $z = z_0$, and this is the primary reason why deleted neighborhoods were used in the definition of limit.

Although the definition of the derivative $f'(z_0)$ is similar to that of the usual derivative of a function of a real variable and they share many similar properties, the complex case is much richer. Note also that in the definition of $f'(z_0)$, we are dividing by the *complex* number $z - z_0$ and the special nature of division by complex numbers is a key consideration. The limit as $z \to z_0$ is taken for an arbitrary z approaching z_0 but not along any particular direction.

The existence of f' implies a great deal about f. It will be proven in §2.4 that if f' exists, then all the derivatives of f exist (that is, f'', the (complex) derivative of f', exists, and so on). This is in contrast to the case of a function $g(x)$ of the real variable x, in which $g'(x)$ can exist without the existence of $g''(x)$.

The analysis of what are called the Cauchy-Riemann equations in Theorem 1.5.8 will show how the complex derivative of f is related to the usual partial derivatives of f as a function of the real variables (x, y) and will supply a useful criterion for determining the existence of $f'(z_0)$. As in elementary calculus, continuity of f does not imply differentiability; for example, $f(z) = |z|$ is continuous but is not differentiable (see Exercise 10 at the end of this section). However, as in one-variable calculus, a differentiable function must be continuous.

Proposition 1.5.2 *If $f'(z_0)$ exists, then f is continuous at z_0.*

Proof By the sum rule for limits, we need only show that

$$\lim_{z\to z_0} [f(z) - f(z_0)] = 0.$$

But

$$\lim_{z\to z_0} [f(z) - f(z_0)] = \lim_{z\to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0}(z - z_0)\right],$$

which, by the product rule for limits, equals $f'(z_0) \cdot 0 = 0$. ■

The usual rules of calculus—the product rule, the quotient rule, the chain rule, and the inverse function rule—can be used when differentiating analytic functions. We now explore these rules in detail.

Proposition 1.5.3 *Suppose that f and g are analytic on A, where $A \subset \mathbb{C}$ is an open set. Then*

(i) *$af + bg$ is analytic on A and $(af + bg)'(z) = af'(z) + bg'(z)$ for any complex numbers a and b.*

(ii) *fg is analytic on A and $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.*

(iii) *If $g(z) \neq 0$ for all $z \in A$, then f/g is analytic on A and*

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - g'(z)f(z)}{[g(z)]^2}.$$

(iv) *Any polynomial $a_0 + a_1 z + \ldots + a_n z^n$ is analytic on all of $\mathbb{C}$ with derivative $a_1 + 2a_2 z + \ldots + na_n z^{n-1}$.*

(v) *Any rational function*

$$\frac{a_0 + a_1 z + \ldots + a_n z^n}{b_0 + b_1 z + \ldots + b_m z^m}$$

is analytic on the open set consisting of all z except those (at most, m) points where the denominator is zero. (See Review Exercise 24 for Chapter 1.)

Proof The proofs of (i), (ii), and (iii) are all similar to the proofs of the corresponding results found in calculus. The procedure can be illustrated with a proof of (ii). Applying the limit theorems and the fact that $\lim_{z\to z_0} f(z) = f(z_0)$ (Proposition 1.5.2), we get

$$\begin{aligned}
&\lim_{z\to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} \\
&\quad = \lim_{z\to z_0} \left[\frac{f(z)g(z) - f(z)g(z_0)}{z - z_0} + \frac{f(z)g(z_0) - f(z_0)g(z_0)}{z - z_0}\right] \\
&\quad = \lim_{z\to z_0} \left[f(z)\frac{g(z) - g(z_0)}{z - z_0}\right] + \lim_{z\to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} g(z_0)\right] \\
&\quad = f(z_0)g'(z_0) + f'(z_0)g(z_0).
\end{aligned}$$

To prove (iv) we must first show that $f' = 0$ if f is constant. This is immediate from the definition of derivative because $f(z) - f(z_0) = 0$. It is equally easy to prove that $dz/dz = 1$. Then, using (ii), we can prove that

$$\frac{d}{dz}z^2 = 1 \cdot z + z \cdot 1 = 2z$$

and

$$\frac{d}{dz}z^3 = \frac{d}{dz}(z \cdot z^2) = 1 \cdot z^2 + z \cdot 2z = 3z^2.$$

In general, we see by induction that $dz^n/dz = nz^{n-1}$. Then (iv) follows from this and (i), and (v) follows from (iv) and (iii). ∎

For example,

$$\frac{d}{dz}(z^2 + 8z - 2) = 2z + 8$$

and

$$\frac{d}{dz}\left(\frac{1}{z+1}\right) = -\frac{1}{(z+1)^2}.$$

The student will also recall that one of the most important rules for differentiation is the chain rule, or "function of a function" rule. To illustrate,

$$\frac{d}{dz}\left[(z^3 + 1)^{10}\right] = 10(z^3 + 1)^9 \cdot 3z^2 = 30z^2(z^3 + 1)^9.$$

This procedure for differentiating should be familiar; it is justified by the next result.

Theorem 1.5.4 (Chain Rule) *Let $f : A \to \mathbf{C}$ and $g : B \to \mathbf{C}$ be analytic (A, B are open sets) and let $f(A) \subset B$. Then $g \circ f : A \to \mathbf{C}$ defined by $(g \circ f)(z) = g(f(z))$ is analytic and*

$$\frac{d}{dz}(g \circ f)(z) = g'(f(z)) \cdot f'(z).$$

The basic idea of the proof of this theorem is that if $w = f(z)$ and $w_0 = f(z_0)$, then

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = \frac{g(w) - g(w_0)}{w - w_0} \cdot \frac{f(z) - f(z_0)}{z - z_0},$$

and if we let $z \to z_0$, we also have $w \to w_0$, and the right side of the preceding equation thus becomes $g'(w_0)f'(z_0)$. The trouble is that even if $z \neq z_0$, we could have $w = w_0$. Because of this possibility, we give a more careful proof. (Although the chain rule here can be deduced from the chain rule for the usual derivative for functions of several variables—see the proof of 1.5.8—a separate proof is instructive.)

Proof Let $w_0 = f(z_0)$, and define, for $w \in B$,

$$h(w) = \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0)$$

if $w \neq w_0$ and $h(w_0) = 0$. Since $g'(w_0)$ exists, h is continuous. Since the composite of continuous functions is continuous,

$$\lim_{z \to z_0} h(f(z)) = h(w_0) = 0.$$

From the definition of h and letting $w = f(z)$, we get $(g \circ f)(z) - g(w_0) = [h(f(z)) + g'(w_0)][f(z) - w_0]$. Note that this still holds if $f(z) = w_0$.For $z \neq z_0$, we get

$$\frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} = [h(f(z)) + g'(w_0)] \frac{f(z) - f(z_0)}{z - z_0}.$$

As $z \to z_0$, the right side of the equation converges to $[0 + g'(w_0)] \cdot [f'(z_0)]$, so the theorem is proved. ■

An argument similar to the one just given proves a slightly different version of the chain rule. Namely, if $\gamma :]a, b[\to \mathbb{C}$ is differentiable, we can differentiate the curve $\sigma(t) = f(\gamma(t))$ and obtain $\sigma'(t) = f'(\gamma(t)) \cdot \gamma'(t)$. Here $\gamma'(t)$ is the derivative of γ as a function $]a, b[\to \mathbb{R}^2$; that is, if $\gamma(t) = (x(t), y(t))$, then $\gamma'(t) = (x'(t), y'(t)) = x'(t) + iy'(t)$.

We now use the chain rule to prove a complex version of the following theorem from calculus: A function whose derivative is identically 0 must be constant. The result illustrates the importance of *regions*, or open connected sets, in which we may, by Proposition 1.4.15, connect any two points by a differentiable path.

Proposition 1.5.5 *Let $A \subset \mathbb{C}$ be open and connected and let $f : A \to \mathbb{C}$ be analytic. If $f'(z) = 0$ on A, then f is constant on A.*

Proof Let $z_1, z_2 \in A$. We want to show that $f(z_1) = f(z_2)$. Let $\gamma(t)$ be a path joining z_1 to z_2. By the chain rule, $df(\gamma(t))/dt = f'(\gamma(t)) \cdot \gamma'(t) = 0$, since $f' = 0$. Thus if $f = u + iv$, we have $du(\gamma(t))/dt = 0$ and $dv(\gamma(t))/dt = 0$. From calculus, we know this implies that $u(\gamma(t))$ and $v(\gamma(t))$ are constant functions of t. Comparing the values at $t = a$ and $t = b$ gives us $f(z_1) = f(z_2)$. ■

Clearly, connectedness is needed because if A consisted of two disjoint pieces, we could let $f = 1$ on one piece and $f = 0$ on the other. Then $f'(z)$ would equal 0 but f would not be constant on A.

Conformal Maps The existence of the complex derivative f' places severe but very useful restrictions on f. The first of these restrictions will be briefly discussed here. Another restriction will be mentioned when the Cauchy-Riemann equations are analyzed in Theorem 1.5.8.

It will be shown that "infinitesimally" near a point z_0 at which $f'(z_0) \neq 0$, f is a rotation through the angle $\arg f'(z_0)$ and a magnification by the factor $|f'(z_0)|$. The term "infinitesimally" is defined more precisely below, but intuitively it means that locally f is approximately a rotation together with a magnification (see Figure 1.5.1). If $f'(z_0) = 0$, the structure of f is more complicated. (This point will be studied further in Chapter 6.)

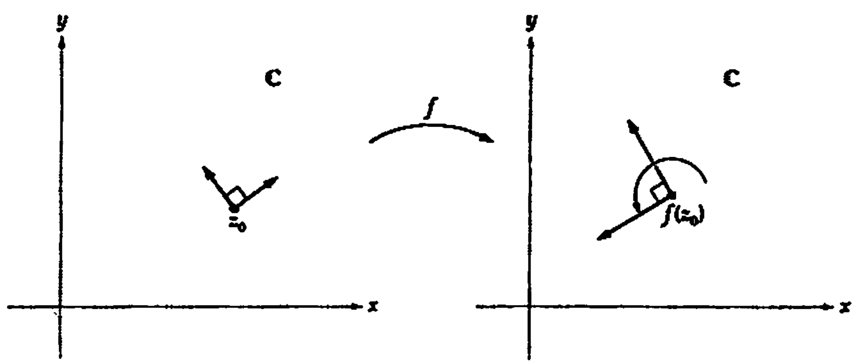

Figure 1.5.1: Conformal at z_0.

Definition 1.5.6 *A map $f : A \to \mathbf{C}$ is called **conformal** at z_0 if there exist a $\theta \in [0, 2\pi[$ and an $r > 0$ such that for any curve $\gamma(t)$ that is differentiable at $t = 0$, for which $\gamma(t) \in A$ and $\gamma(0) = z_0$, and that satisfies $\gamma'(0) \neq 0$, the curve $\sigma(t) = f(\gamma(t))$ is differentiable at $t = 0$ and, setting $u = \sigma'(0)$ and $v = \gamma'(0)$, we have $|u| = r|v|$ and $\arg u = \arg v + \theta (\operatorname{mod} 2\pi)$. A map is called **conformal** when it is conformal at every point.*

Thus a conformal map merely rotates and stretches tangent vectors to curves. This is the precise meaning of "infinitesimal" as previously used. It should be noted that a conformal map *preserves angles* between intersecting curves. (By definition, the angle between two curves is the angle between their tangent vectors— see Figure 1.5.2).

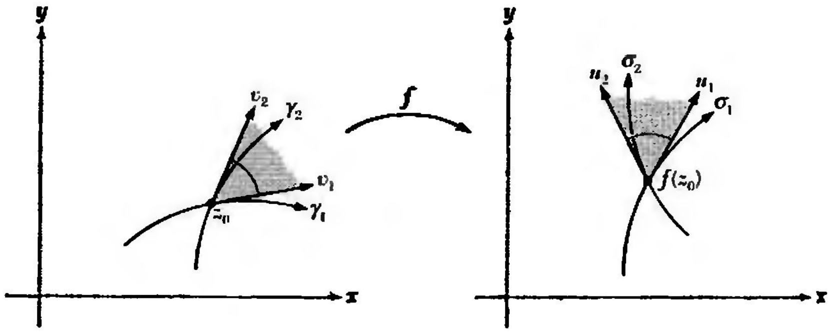

Figure 1.5.2: Preservation of angles by a conformal map.

Theorem 1.5.7 (Conformal Mapping Theorem) *If $f : A \to \mathbb{C}$ is analytic and if $f'(z_0) \neq 0$, then f is conformal at z_0 with $\theta = \arg f'(z_0)$ and $r = |f'(z_0)|$, fulfilling Definition 1.5.6.*

The proof of this theorem is remarkably simple.

Proof Using the preceding notation and the chain rule, we get $u = \sigma'(0) = f'(z_0) \cdot \gamma'(0) = f'(z_0) \cdot v$. Thus $\arg u = \arg f'(z_0) + \arg v (\text{mod } 2\pi)$ and $|u| = |f'(z_0)| \cdot |v|$, as required. ∎

The point of this proof is that the tangent vector v to any curve is multiplied by a fixed complex number, namely, $f'(z_0)$, no matter in which direction v is pointing. This is because, in the definition of $f'(z_0)$, $\lim_{z \to z_0}$ is "taken through all possible directions" as $z \to z_0$.

Cauchy-Riemann Equations Recall that if $f : A \subset \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}^2$ and if $f(x, y) = (u(x, y), v(x, y)) = u(x, y) + iv(x, y)$, then the ***Jacobian matrix*** of f is defined as the matrix of partial derivatives given by

$$Df(x, y) = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

at each point (x, y). We shall relate these partial derivatives to the complex derivative. From the point of view of real variables, f is called ***differentiable*** with derivative the matrix $Df(x_0, y_0)$ at (x_0, y_0) iff for any $\epsilon > 0$, there is a $\delta > 0$ such that $|(x, y) - (x_0, y_0)| < \delta$ implies

$$|f(x, y) - f(x_0, y_0) - Df(x_0, y_0)[(x, y) - (x_0, y_0)]| \leq \epsilon |(x, y) - (x_0, y_0)|$$

where $Df(x_0, y_0) \cdot [(x, y) - (x_0, y_0)]$ means the matrix $Df(x_0, y_0)$ applied to the (column) vector

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

and $|w|$ stands for the length of a vector w.

Here are some facts from the calculus of several variables that we shall use.[9] If f is differentiable, then the usual partials $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x$, and $\partial v/\partial y$ exist and $Df(x_0, y_0)$ is given by the Jacobian matrix. The expression $Df(x_0, y_0)[w]$ represents the derivative of f in the direction w. If the partials exist and are continuous, then f is differentiable. Generally, then, differentiability is a bit stronger than existence of the individual partials. The main result connecting the partial derivatives and analyticity is stated in the next theorem.

Theorem 1.5.8 (Cauchy-Riemann Theorem) *Suppose A is an open set in $\mathbb{C}$ and $f : A \subset \mathbb{C} \to \mathbb{C}$ is a given function. Then $f'(z_0)$ exists if and only if f is differentiable in the sense of real variables and at $(x_0, y_0) = z_0$, the functions u, v satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(called the Cauchy-Riemann equations).

Thus, if $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x$, and $\partial v/\partial y$ exist, are continuous on A, and satisfy the Cauchy-Riemann equations, then f is analytic on A.

If $f'(z_0)$ does exist, then

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

Proof Let us first show that if $f'(z_0)$ exists, then u and v satisfy the Cauchy-Riemann equations. In the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

let us take the special case that $z = x + iy_0$. Then

$$\begin{aligned} \frac{f(z) - f(z_0)}{z - z_0} &= \frac{u(x, y_0) + iv(x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{x - x_0} \\ &= \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i\frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}. \end{aligned}$$

[9]Proofs of the following statements are not included here but can be found in any advanced calculus text, such as J. Marsden and M. Hoffman, *Elementary Classical Analysis*, Second Edition (New York: W. H. Freeman and Company, 1993), Ch. 6.

As $x \to x_0$, the left side of the equation converges to the limit $f'(z_0)$. Thus both the real and imaginary parts of the right side must converge to a limit (see Exercise 2 of §1.4). From the definition of partial derivatives, this limit is $(\partial u/\partial x)(x_0, y_0) + i(\partial v/\partial x)(x_0, y_0)$. Thus $f'(z_0) = \partial u/\partial x + i\partial v/\partial x$ evaluated at (x_0, y_0).

Next let $z = x_0 + iy$. Then we similarly have

$$\begin{aligned}\frac{f(z) - f(z_0)}{z - z_0} &= \frac{u(x_0, y) + iv(x_0, y) - u(x_0, y_0) - iv(x_0, y_0)}{i(y - y_0)} \\ &= \frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} + \frac{v(x_0, y) - v(x_0, y_0)}{y - y_0}.\end{aligned}$$

As $y \to y_0$, we get

$$\frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Thus, since $f'(z_0)$ exists and has the same value regardless of how z approaches z_0, we get

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

By comparing real and imaginary parts of these equations, we derive the Cauchy-Riemann equations as well as the two formulas for $f'(z_0)$.

Another argument for this direction of the proof and one for the opposite implication may be based on the matrix representation for complex multiplication developed in Exercise 10 of §1.1.

Lemma 1.5.9 *A matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

represents, under matrix multiplication, multiplication by a complex number iff $a = d$ and $b = -c$. The complex number in question is $a + ic = d - ib$.

Proof First, let us consider multiplication by the complex number $a+ic$. It sends $x + iy$ to $(a + ic)(x + iy) = ax - cy + i(ay + cx)$, which is the same as

$$\begin{pmatrix} a & -c \\ c & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - cy \\ cx + ay \end{pmatrix}.$$

Conversely, let us suppose that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = z \cdot (x + iy)$$

for a complex number $z = \alpha + i\beta$. Then we get

$$ax + by = \alpha x - \beta y \quad \text{and} \quad cx + dy = \alpha y + \beta x$$

for all x, y. This implies (setting $x = 1, y = 0$, then $x = 0, y = 1$) that $a = \alpha, b = -\beta, c = \beta$, and $d = \alpha$, and so the proof is complete. ■

We can now complete the proof of Theorem 1.5.8.

From the definition of f', the statement that $f'(z_0)$ exists is equivalent to the following statement: For any $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \epsilon|z - z_0|.$$

First let us suppose that $f'(z_0)$ exists. By definition, $Df(x_0, y_0)$ is the unique matrix with the property that for any $\epsilon > 0$ there is a $\delta > 0$ such that, setting $z = (x, y)$ and $z_0 = (x_0, y_0)$, $0 < |z - z_0| < \delta$ implies

$$|f(z) - f(z_0) - Df(z_0)(z - z_0)| < \epsilon|z - z_0|.$$

If we compare this equation with the preceding one and recall that multiplication by a complex number is a linear map, we conclude that f is differentiable in the sense of real variables and that the matrix $Df(z_0)$ represents multiplication by the complex number $f'(z_0)$. Thus, applying the lemma to the matrix

$$Df(z_0) = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

with $a = \partial u/\partial x, b = \partial u/\partial y, c = \partial v/\partial x, d = \partial v/\partial y$, we have $a = d, b = -c$, which are the Cauchy-Riemann equations.

Conversely, if the Cauchy-Riemann equations hold, $Df(z_0)$ represents multiplication by a complex number (by the lemma) and then, as above, the definition of differentiability in the sense of real variables reduces to that for the complex derivative.

The formula for $f'(z_0)$ follows from the last statement of the lemma. ■

We can also express the Cauchy-Riemann equations in terms of polar coordinates, but care must be exercised because the change of coordinates defined by $r = \sqrt{x^2 + y^2}$ and $\theta = \arg(x + iy)$ is a differentiable change only if θ is restricted to the *open* interval $]0, 2\pi[$ or any other open interval of length 2π and if the origin $(r = 0)$ is omitted. Without such a restriction θ is discontinuous, because it jumps by 2π on crossing the x axis. Using $\partial x/\partial r = \cos\theta, \partial y/\partial r = \sin\theta$, we see that the Cauchy-Riemann equations are equivalent to saying that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

on a region contained in a region such as those shown in Figure 1.5.3. Here we are employing standard abuse of notation by writing $u(r, \theta) = u(r\cos\theta, r\sin\theta)$. (For a more precise statement, see Exercise 12 at the end of this section.)

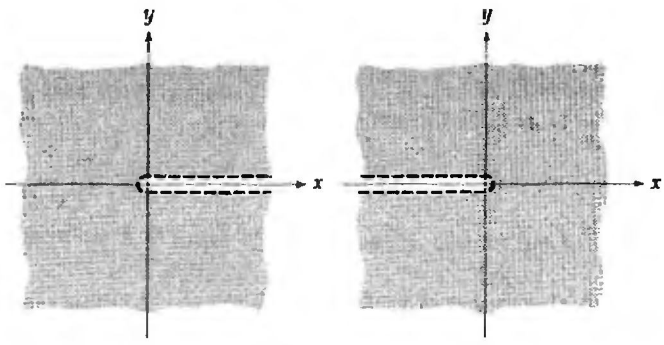

Figure 1.5.3: Two regions of validity of polar coordinates.

Inverse Functions A basic result of real analysis is the Inverse Function Theorem: *A continuously differentiable function is one-to-one and onto an open set and has a differentiable inverse in some neighborhood of a point where the Jacobian determinant of the derivative matrix is not* 0. We will give proof here of the complex counterpart of this result, which assumes that the derivative f' is continuous and depends on the corresponding theorem for functions of real variables. After we have proved Cauchy's Theorem in Chapter 2, we will see that the continuity of f' is automatic, and in Chapter 6 we will prove the theorem in another way that does not depend on the real-variable theorem. The proof given here, however, illustrates the relationship between real and complex variables and the relevance of the Cauchy-Riemann equations.

Theorem 1.5.10 (Inverse Function Theorem) *Let $f : A \to \mathbb{C}$ be analytic (with f' continuous) and assume that $f'(z_0) \neq 0$. Then there exists a neighborhood U of z_0 and a neighborhood V of $f(z_0)$ such that $f : U \to V$ is a bijection (that is, is one-to-one and onto) and its inverse function f^{-1} is analytic with derivative given by*

$$\frac{d}{dw}f^{-1}(w) = \frac{1}{f'(z)} \quad \textit{where} \quad w = f(z).$$

The student is cautioned that application of the Inverse Function Theorem allows one only to conclude the existence of a *local* inverse for f. For example, let us consider $f(z) = z^2$ defined on $A = \mathbb{C}\backslash\{0\}$. Then $f'(z) = 2z \neq 0$ at each point of A. The Inverse Function Theorem says that f has a unique local analytic inverse, which is, in fact, merely some branch of the square root function. But f is not one-to-one on all of A, since, for example, $f(1) = f(-1)$. Thus f will be one-to-one only within sufficiently small neighborhoods surrounding each point.

To prove this theorem, let us recall the statement for real variables in two dimensions.

Theorem 1.5.11 (Real-Variable Inverse Function Theorem) *If $f : A \subset \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable and $Df(x_0, y_0)$ has a nonzero determinant, then there are neighborhoods U of (x_0, y_0) and V of $f(x_0, y_0)$ such that $f : U \to V$ is a bijection, $f^{-1} : V \to U$ is differentiable, and*

$$Df^{-1}(f(x,y)) = [Df(x,y)]^{-1}$$

(this is the inverse of the matrix of partials).

The proof of this theorem may be found in advanced calculus texts. See, for instance, J. Marsden and M. Hoffman, *Elementary Classical Analysis*, Second Edition (New York: W. H. Freeman and Company, 1993), Ch. 7.

Accepting this statement and assuming that f' in Theorem 1.5.10 is continuous, we can complete the proof.

Proof of Theorem 1.5.10 For analytic functions such as $f(z)$, we have seen that the matrix of partial derivatives is

$$Df = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{-\partial v}{\partial x} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial x} \end{pmatrix},$$

which has determinant

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(z)|^2$$

since $f'(z) = \partial u/\partial x + i\partial v/\partial x$. All these functions are to be evaluated at the point $(x_0, y_0) = z_0$. Now $f'(z_0) \neq 0$, so $\operatorname{Det} Df(x_0, y_0) = |f'(z_0)|^2 \neq 0$. Thus the real-variable Inverse Function Theorem applies. By the Cauchy-Riemann Theorem 1.5.8 we need only verify that the entries of $[Df(x,y)]^{-1}$ satisfy the Cauchy-Riemann equations and give $(f^{-1})'$ as stated.

As we have just seen,

$$Df = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

and the inverse of this matrix is

$$\frac{1}{\operatorname{Det} Df}\begin{pmatrix} \dfrac{\partial v}{\partial y} & \dfrac{-\partial u}{\partial y} \\ \dfrac{-\partial v}{\partial x} & \dfrac{\partial u}{\partial x} \end{pmatrix}.$$

Thus if we write $f^{-1}(x,y) = t(x,y) + is(x,y)$, then, comparing

$$D(f^{-1}) = \begin{pmatrix} \dfrac{\partial t}{\partial x} & \dfrac{\partial t}{\partial y} \\ \dfrac{\partial s}{\partial x} & \dfrac{\partial s}{\partial y} \end{pmatrix}$$

with the inverse matrix for Df, we get

$$\frac{\partial t}{\partial x} = \frac{1}{\operatorname{Det} Df}\frac{\partial v}{\partial y} = \frac{1}{\operatorname{Det} Df}\frac{\partial u}{\partial x}$$

and

$$\frac{\partial s}{\partial x} = \frac{1}{\operatorname{Det} Df}\frac{-\partial v}{\partial x} = \frac{1}{\operatorname{Det} Df}\frac{\partial u}{\partial y},$$

which are evaluated at $f(x_0, y_0)$. Similarly,

$$\frac{\partial t}{\partial y} = \frac{1}{\operatorname{Det} Df}\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial s}{\partial y} = \frac{1}{\operatorname{Det} Df}\frac{\partial v}{\partial y}.$$

Thus the Cauchy-Riemann equations hold for t and s since they hold for u and v. Therefore, f^{-1} is complex-differentiable. From the Cauchy-Riemann Theorem we see that at the point $f(z_0)$,

$$(f^{-1})' = \frac{\partial t}{\partial x} + i\frac{\partial s}{\partial x} = \frac{1}{\operatorname{Det} Df}\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right) = \frac{\overline{f'(z_0)}}{|f'(z_0)|^2} = \frac{1}{f'(z_0)}. \quad \blacksquare$$

The real and imaginary parts of an analytic function must satisfy the Cauchy-Riemann equations. Manipulation of these equations leads directly to another very important property, which we now isolate. A twice continuously differentiable function $u : A \to \mathbb{R}$ defined on an open set A is called ***harmonic*** if

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The expression $\nabla^2 u$ is called the ***Laplacian*** of u and is one of the most basic operations in mathematics and physics. Harmonic functions play a fundamental role in the physical examples discussed later in Chapters 5 and 8. For the moment let us study them from the mathematical point of view. For $\nabla^2 u = 0$ to make sense, the function u must be twice differentiable. In Chapter 3 an analytic function will be shown to be *infinitely* differentiable. Thus its real and imaginary parts are infinitely differentiable. Let us accept (or assume) these properties here. In particular, the second partial derivatives are continuous, and so a standard result of calculus says that the mixed partials are equal. The Cauchy-Riemann equations may then be used to show that the functions are harmonic.

Proposition 1.5.12 *If f is analytic on an open set A and $f = u + iv$ (that is, if $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$), then u and v are harmonic on A.*

Proof We use the Cauchy-Riemann equations, $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$. Differentiating the first equation with respect to x and the second equation with respect to y, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

As above, the second partials are symmetric because they are continuous:

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}.$$

Therefore, adding the equations in the preceding display gives us

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0.$$

The equation for v is proved in the same way. ■

If u and v are real-valued functions defined on an open subset A of $\mathbf{C}$ such that the complex-valued function $f = u + iv$ is analytic on A, we sat that u and v are ***harmonic conjugates*** on A. For example,

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy$$

are harmonic conjugates of each other since they are the real and imaginary parts of $f(z) = z^2$. These functions are linked geometrically as well as algebraically. The level curves of u and v passing through any nonzero point intersect at right angles at that point, which is illustrated in Figure 1.5.4. The next Proposition asserts that this occurs generally for the real and imaginary parts of an analytic function at points where its derivative is not 0. The proof uses the Cauchy-Riemann equations to show that the dot product of the tangent vectors (or the normal vectors) is 0.

Proposition 1.5.13 *Let u and v be harmonic conjugates on a region A. Suppose that the equations*

$$u(x, y) = \text{constant} = c_1 \quad \textit{and} \quad v(x, y) = \text{constant} = c_2$$

define smooth curves. Then these curves intersect orthogonally (see Figure 1.5.4).

We shall accept from calculus the fact that $u(x, y) = c_1$ defines a smooth curve if the gradient grad $u(x, y) = (\partial u/\partial x, \partial u/\partial y) = (\partial u/\partial x) + i(\partial u/\partial y)$ is nonzero for x and y satisfying $u(x, y) = c_1$. (The student should be aware of this fact even though it is a technical point that does not play a major role in concrete examples.) It is also true that the vector grad u is perpendicular to that curve (see Figure 1.5.5).

This perpendicularity property can be explained as follows. If $(x(t), y(t))$ is the curve, then $u(x(t), y(t)) = c_1$, a constant, so

$$\frac{d}{dt}[u(x(t), y(t))] = 0,$$

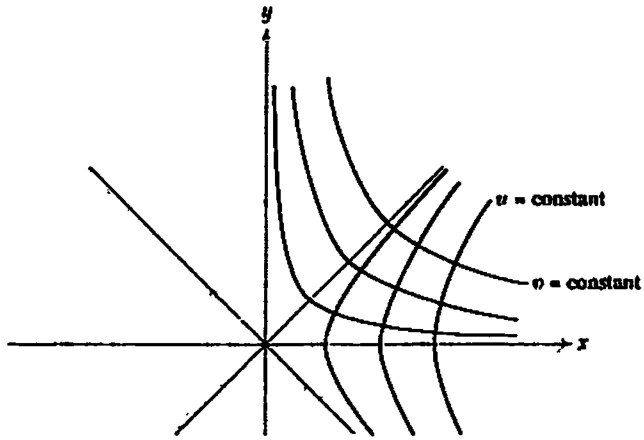

Figure 1.5.4: Harmonic conjugates: $u = x^2 - y^2, v = 2xy, f(z) = z^2$.

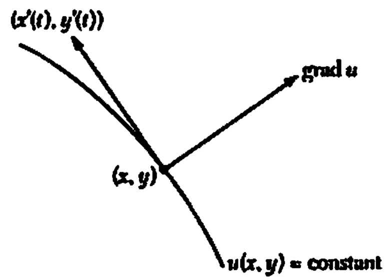

Figure 1.5.5: Gradients are orthogonal to level sets.

and thus by the chain rule,

$$\frac{\partial u}{\partial x} \cdot x'(t) + \frac{\partial u}{\partial y} \cdot y'(t) = 0.$$

That is,

$$\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot (x'(t), y'(t)) = 0.$$

Proof of Proposition 1.5.13 By the above remarks, it suffices to show that grad u and grad v are perpendicular. Their inner product is

$$\operatorname{grad} u \cdot \operatorname{grad} v = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y},$$

which is zero by the Cauchy-Riemann equations. ■

This orthogonality property of harmonic conjugates has an important physical interpretation, which will be used in Chapter 5. Another way to see why this property should hold is to consider conformal maps and the Inverse Function Theorem. This is illustrated in Figure 1.5.6.

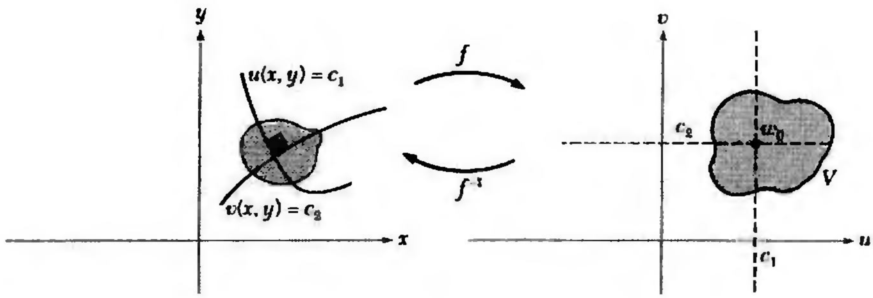

Figure 1.5.6: Since f and f^{-1} are analytic, they are conformal and so preserve orthogonality.

If $f = u+iv$ is analytic and $f'(z_0) \neq 0$, then f^{-1} is analytic on a neighborhood V of $w_0 = f(z_0)$ and $(f^{-1})(w_0) \neq 0$ by the Inverse Function Theorem. If $w_0 = c_1+ic_2$, then the curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are the images of the vertical and horizontal lines through w_0 under the mapping f^{-1}. They should cross at right angles since f^{-1} is conformal by Theorem 1.5.7.

Proposition 1.5.12 says that the real part of an analytic function is harmonic. A natural question is the opposite one: Is every harmonic function the real part of an analytic function? More precisely: Given a harmonic function u on a set A, need there be a harmonic conjugate v such that $f = u + iv$ is analytic on A? The full answer is a little tricky and depends on the nature of the set A. However, the answer is simpler if we confine ourselves to small neighborhoods. Indeed, the property of being harmonic is what is called a *local property*. The function u is harmonic on a set if $\nabla^2 u = 0$ holds at each point of that set. Therefore, it makes sense to study this property in a neighborhood of any point.

Proposition 1.5.14 *If u is a twice continuously differentiable harmonic function on an open set A and $z_0 \in A$, then there is some neighborhood of z_0 on which u is the real part of an analytic function.*

In other words, there exist an $r > 0$ and a function v defined on the open disk $D(z_0;r)$ such that u and v are harmonic conjugates on $D(z_0;r)$. In fact, $D(z_0;r)$ may be taken to be the largest disk centered at z_0 and contained in A. A direct proof of this is outlined in Exercise 32. A different proof of a slightly stronger result will be given in Chapter 2. Since the Cauchy-Riemann equations must hold, v is uniquely determined up to the addition of a constant. These equations may be used as a method for finding v when u is given.

Worked Examples

Example 1.5.15 *Where is the function* $f(z) = (z^3 + 2z + 1)/(z^3 + 1)$ *analytic? Compute the derivative.*

Solution By Proposition 1.5.3(iii) f is analytic on the set $A = \{z \in \mathbb{C} | z^3 + 1 \neq 0\}$; that is, f is analytic on the whole plane except the cube roots of $-1 = e^{\pi i}$, namely, the points $e^{\pi i/3}, e^{\pi i}$, and $e^{5\pi i/3}$. By the formula for differentiating a quotient, the derivative is

$$f'(z) = \frac{(z^3+1)(3z^2+2) - (z^3+2z+1)(3z^2)}{(z^3+1)^2} = \frac{(2-4z^3)}{(z^3+1)^2}.$$

Example 1.5.16 *Consider* $f(z) = z^3 + 1$. *Study the infinitesimal behavior of* f *at* $z_0 = i$.

Solution We use the Conformal Mapping Theorem 1.5.7. In this case $f'(z_0) = 3i^2 = -3$. Thus f rotates locally by $\pi = \arg(-3)$ and multiplies lengths by $3 = |f'(z_0)|$. More precisely, if c is any curve through $z_0 = i$, the image curve will, at $f(z_0)$, have its tangent vector rotated by π and stretched by a factor 3.

Example 1.5.17 *Show that* $f(z) = \bar{z}$ *is not analytic.*

Solution Let $f(z) = u(x,y) + iv(x,y) = x - iy$ where $z = (x,y) = x + iy$. Thus, $u(x,y) = x, v(x,y) = -y$. But $\partial u/\partial x = 1$ and $\partial v/\partial y = -1$ and hence $\partial u/\partial x \neq \partial v/\partial y$, so the Cauchy-Riemann equations do not hold. Therefore, $f(z) = \bar{z}$ cannot be analytic, by the Cauchy-Riemann Theorem 1.5.8.

Example 1.5.18 *We know by Proposition 1.5.3 that* $f(z) = z^3 + 1$ *is analytic. Verify the Cauchy-Riemann equations for this function.*

Solution If $f(z) = u(x,y) + iv(x,y)$ when $z = (x,y) = x + iy$, then in this case $u(x,y) = x^3 - 3xy^2 + 1$ and $v(x,y) = 3x^2y - y^3$. Therefore, $\partial u/\partial x = 3x^2 - 3y^2, \partial u/\partial y = -6xy, \partial v/\partial x = 6xy$, and $\partial v/\partial y = 3x^2 - 3y^2$, from which we see that $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$.

Example 1.5.19 *Let* A *be an open subset of* $\mathbb{C}$ *and* $A^* = \{z \mid \bar{z} \in A\}$. *Suppose* f *is analytic on* A, *and define a function* g *on* A^* *by* $g(z) = \overline{f(\bar{z})}$. *Show that* g *is analytic on* A^*.

Solution If $f(z) = u(x,y) + iv(x,y)$, then $g(z) = \overline{f(\bar{z})} = u(x,-y) - iv(x,-y)$. We check the Cauchy-Riemann equations for g as follows:

$$\begin{aligned}
\frac{\partial}{\partial x}(\operatorname{Re} g) &= \frac{\partial}{\partial x}u(x,-y) = \left.\frac{\partial u}{\partial x}\right|_{(x,-y)} = \left.\frac{\partial v}{\partial y}\right|_{(x,-y)} \\
&= \frac{\partial}{\partial y}[-v(x,-y)] = \frac{\partial}{\partial y}(\operatorname{Im} g)
\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial y}(\operatorname{Re} g) &= \frac{\partial}{\partial y}u(x,-y) = -\frac{\partial u}{\partial y}\bigg|_{(x,-y)} = \frac{\partial v}{\partial x}\bigg|_{(x,-y)} \\ &= -\frac{\partial}{\partial x}[-v(x,-y)] = -\frac{\partial}{\partial x}(\operatorname{Im} g).\end{aligned}$$

Since the Cauchy-Riemann equations hold and g is differentiable in the sense of real variables (why?), it is analytic on A^* by the Cauchy-Riemann Theorem 1.5.8. (One could also solve this exercise by direct appeal to the definition of complex differentiability.)

Example 1.5.20 *Find the harmonic conjugates of the following harmonic functions on* $\mathbb{C}$*:*
(a) $u(x,y) = x^2 - y^2$
(b) $u(x,y) = \sin x \cosh y$

The reader might recognize $x^2 - y^2$ and $\sin x \cosh y$ as the real parts of z^2 and $\sin z, z = x+iy$. From this observation it follows that the conjugates, up to addition of constants, are $2xy$ and $\sinh y \cos x$. (We shall see in the next section that $\sin z$ is analytic.) It is instructive, however, to solve the problem directly using the Cauchy-Riemann equations, because the student might not always recognize an appropriate analytic $f(z)$ by inspection.

Solution To solve (a), suppose that v is a harmonic conjugate of u. By the Cauchy-Riemann equations,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x.$$

Therefore, $v = 2yx + g_1(y)$ and $v = 2xy + g_2(x)$. Hence $g_1(y) = g_2(x) =$ constant, and so $v(x,y) = 2yx+$ constant. To find the harmonic conjugate v for part (b), we use the Cauchy-Riemann equations again and write

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\sin x \sinh y \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \cos x \cosh y.$$

The first equation implies that $v = \cos x \sinh y + g_1(y)$ and the second equation implies that $v = \cos x \sinh y + g_2(x)$. Hence $g_1(y) = g_2(x) =$ constant. Therefore $v(x,y) = \cos x \sinh y +$ constant.

Example 1.5.21 *Suppose f is an analytic function on a region (an open connected set) A and that $|f(z)|$ is constant on A. Show that f is constant on A.*

Solution We use the Cauchy-Riemann equations to show that $f'(z) = 0$ everywhere in A. Let $f = u + iv$. Then $|f|^2 = u^2 + v^2 = c$ is constant. If $c = 0$, then $|f(z)| = 0$ and so $f(z) = 0$ for all z in A. If $c \neq 0$, we take derivatives of $u^2 + v^2 = c$ with respect to x and y to obtain

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

By the Cauchy-Riemann equations these become

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \quad \text{and} \quad v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0.$$

As a system of equations for the two unknowns $\partial u/\partial x$ and $\partial u/\partial y$, the matrix of coefficients has determinant $u^2 + v^2 = c$, which is not 0. Thus the only solution is $\partial u/\partial x = \partial u/\partial y = 0$ at all points of A. Therefore $f'(z) = \partial u/\partial x + i(\partial v/\partial x) = 0$ everywhere in A. Since A is connected, f is constant (by Proposition 1.5.5).

Exercises

1. Determine the sets on which the following functions are analytic, and compute their derivatives:

   (a) $(z + 1)^3$

   (b) $z + \frac{1}{z}$

   (c) $\left(\dfrac{1}{z-1}\right)^{10}$

   (d) $\dfrac{1}{(z^3 - 1)(z^2 + 2)}$

2. Determine the sets on which the following functions are analytic, and compute their derivatives:

   (a) $3z^2 + 7z + 5$

   (b) $(2z + 3)^4$

   (c) $\dfrac{3z - 1}{3 - z}$

3. On what sets are the following functions analytic? Compute the derivative for each.

   (a) z^n, n being an integer (positive or negative)

   (b) $\dfrac{1}{(z + 1/z)^2}$

   (c) $z/(z^n - 2), n$ being a positive integer

4. For $\gamma :]a,b[\to \mathbb{C}$ differentiable and $f : A \to \mathbb{C}$ analytic with $\gamma(]a,b[) \subset A$, prove that $\sigma = f \circ \gamma$ is differentiable with $\sigma'(t) = f'(\gamma(t)) \cdot \gamma'(t)$ by imitating the proof of the chain rule (1.5.4).

5. Study the infinitesimal behavior of the following functions at the indicated points:

   (a) $f(z) = z + 3, z_0 = 3 + 4i$

   (b) $f(z) = z^6 + 3z, z_0 = 0$

   (c) $f(z) = \dfrac{z^2 + z + 1}{z - 1}, z_0 = 0$

6. Study the infinitesimal behavior of the following functions at the indicated points:

   (a) $f(z) = 2z + 5, z_0 = 5 + 6i$

   (b) $f(z) = z^4 + 4z, z_0 = i$

   (c) $f(z) = 1/(z - 1), z_0 = i$

7. Prove that $df^{-1}/dw = 1/f'(z)$ where $w = f(z)$ by differentiating $f^{-1}(f(z)) = z$, using the chain rule. Assume that f^{-1} is defined and is analytic.

8.• Use the Inverse Function Theorem to show that if $f : A \to \mathbb{C}$ is analytic and $f'(z) \neq 0$ for all $z \in A$, then f maps open sets in A to open sets.

9. Verify the Cauchy-Riemann equations for the function $f(z) = z^2 + 3z + 2$.

10. Prove that $f(z) = |z|$ is not analytic.

11.• Show, by changing variables, that the Cauchy-Riemann equations in terms of polar coordinates become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

12. Perform the computation in Exercise 11 by the following procedure. Let f be defined on the open set $A \subset \mathbb{C}$ (that is, $f : A \subset \mathbb{C} \to \mathbb{C}$) and suppose that $f(z) = u(z) + iv(z)$. Let $T :]0, 2\pi[\times \mathbb{R}^+ \to \mathbb{R}^2$, where $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$, be given by $T(\theta, r) = (r\cos\theta, r\sin\theta)$. Thus T is one-to-one and onto the set $\mathbb{R}^2 \backslash \{(x, 0) \mid x \geq 0\}$. Define

$$\tilde{u}(\theta, r) = u(r\cos\theta, r\sin\theta) \quad \text{and} \quad \tilde{v}(\theta, r) = v(r\cos\theta, r\sin\theta).$$

Show that

   (a) T is continuously differentiable and has a continuously differentiable inverse.

(b) f is analytic on $A \backslash \{x+iy \mid y=0, x \geq 0\}$ if and only if $(\tilde{u}, \tilde{v}) : T^{-1}(A) \to \mathbb{R}^2$ is differentiable and we have

$$\frac{\partial \tilde{u}}{\partial r} = \frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta} \quad \text{and} \quad \frac{\partial \tilde{v}}{\partial r} = -\frac{1}{r}\frac{\partial \tilde{u}}{\partial \theta}$$

on $T^{-1}(A)$.

13. Define the symbol $\partial f/\partial \bar{z}$ by

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right).$$

Show that the Cauchy-Riemann equations are equivalent to $\partial f/\partial \bar{z} = 0$. *Note:* It is sometimes said, because of this result, that analytic functions are not functions of $\bar{z}$ but of z alone. This statement can be made more precise as follows. Given $f(x,y)$, write $x = \frac{1}{2}(z+\bar{z})$ and $y = (1/2i)(z-\bar{z})$. Then f becomes a function of z and $\bar{z}$ and the chain rule gives

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right).$$

14. Define the symbol $\partial f/\partial z$ by

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right).$$

(a) Show that if f is analytic, then $f' = \partial f/\partial z$.

(b) If $f(z) = z$, show that $\partial f/\partial z = 1$ and $\partial f/\partial \bar{z} = 0$.

(c) If $f(z) = \bar{z}$, show that $\partial f/\partial z = 0$ and $\partial f/\partial \bar{z} = 1$.

(d) Show that the symbols $\partial/\partial z$ and $\partial/\partial \bar{z}$ obey the sum, product, and scalar multiple rules for derivatives.

(e) Show that the expression $\sum_{n=0}^{N}\sum_{m=0}^{M} a_{nm} z^n \bar{z}^m$ is an analytic function of z if and only if $a_{nm} = 0$ whenever $m \neq 0$.

15. Suppose that f is an analytic function on the disk $D = \{z \text{ such that } |z| < 1\}$ and that $\operatorname{Re} f(z) = 3$ for all z in D. Show that f is constant on D.

16.* (a) Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function defined on a connected open set A. If $au(x,y) + bv(x,y) = c$ in A, where a, b, c are real constants not all 0, prove that $f(z)$ is constant in A.

(b) Is the result obtained in (a) still valid if a, b, c are complex constants?

17.* Suppose f is analytic on the set $A = \{z \mid \operatorname{Re} z > 1\}$ and that $\partial u/\partial x + \partial v/\partial y = 0$ on A. Show that there are a real constant c and a complex constant d such that $f(z) = -icz + d$ on A.

18. Let $f(z) = z^5/|z|^4$ if $z \neq 0$ and $f(0) = 0$.

    (a) Show that $f(z)/z$ does not have a limit as $z \to 0$.

    (b) If $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, show that $u(x, 0) = x, v(0, y) = y, u(0, y) = v(x, 0) = 0$.

    (c) Conclude that the partials of u, v exist and that the Cauchy-Riemann equations hold but that $f'(0)$ does not exist. Does this conclusion contradict the Cauchy-Riemann Theorem?

    (d) Repeat exercise (c), letting $f = 1$ on the x and y axes and 0 elsewhere.

    (e) Repeat exercise (c), letting $f(z) = \sqrt{|xy|}$.

19. Let $f(z) = (z + 1)/(z - 1)$.

    (a) Where is f analytic?

    (b) Is f conformal at $z = 0$?

    (c) What are the images of the x and y axes under f?

    (d) At what angle do these images intersect?

20. Let f be an analytic function on an open connected set A and suppose that $f^{(n+1)}(z)$ (the $n + 1$st derivative) exists and is zero on A. Show that f is a polynomial of degree $\leq n$.

21. On what set is $u(x, y) = \operatorname{Re}(z/(z^3 - 1))$ harmonic?

22.* Verify directly that the real and imaginary parts of $f(z) = z^4$ are harmonic.

23. On what sets are each of the following functions harmonic?

    (a) $u(x, y) = \operatorname{Im}(z^2 + 3z + 1)$

    (b) $u(x, y) = \dfrac{x - 1}{x^2 + y^2 - 2x + 1}$

24. On what sets are each of the following functions harmonic?

    (a) $u(x, y) = \operatorname{Im}(z + 1/z)$

    (b) $u(x, y) = \dfrac{y}{(x - 1)^2 + y^2}$

25. Let $f : A \to \mathbf{C}$ be analytic and let $w : B \to \mathbf{R}$ be harmonic with $f(A) \subset B$. Show that $w \circ f : A \to \mathbf{R}$ is harmonic.

26. If u is harmonic, show that, in terms of polar coordinates,

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

Hint: Use the Cauchy-Riemann equations in polar form (Exercise 11).

27. (a) Show that $u(x,y) = e^x \cos y$ is harmonic on $\mathbf{C}$.

    (b) Find a harmonic conjugate $v(x,y)$ for u on $\mathbf{C}$ such that $v(0,0) = 0$.

    (c) Show that $f(z) = e^z$ is analytic on $\mathbf{C}$.

28. Show that $u(x,y) = x^3 - 3xy^2$ is harmonic on $\mathbf{C}$ and find a harmonic conjugate v such that $v(0,0) = 2$.

29. If $u(z)$ is harmonic and $v(z)$ is harmonic, then are the following harmonic?

    (a) $u(v(z), 0)$

    (b) $u(z) \cdot v(z)$

    (c) $u(z) + v(z)$

30. Consider the function $f(z) = 1/z$. Draw the contours $u = \operatorname{Re} f =$ constant and $v = \operatorname{Im} f =$ constant. How do they intersect? Is it *always* true that grad u is parallel to the curve $v =$ constant?

31. Let u have continuous second partials on an open set A and let $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$. Let $f = \partial u/\partial x - i\partial u/\partial y$. Show that f is analytic.

32. Suppose u is a twice continuously differentiable real-valued harmonic function on a disk $D(z_0; r)$ centered at $z_0 = x_0 + iy_0$. For $(x_1, y_1) \in D(z_0; r)$, show that the equation

$$v(x_1, y_1) = c + \int_{y_0}^{y_1} \frac{\partial u}{\partial x}(x_1, y)dy - \int_{x_0}^{x_1} \frac{\partial u}{\partial y}(x, y_0)dx$$

defines a harmonic conjugate for u on $D(z_0; r)$ with $v(x_0, y_0) = c$.

1.6 Differentiation of the Elementary Functions

Exponential Function and Logarithm This section discusses differentiability properties of the elementary functions introduced in §1.3 and we will begin with the exponential function and its inverse, the logarithm.

Proposition 1.6.1 *The map $f : \mathbf{C} \to \mathbf{C}, z \mapsto e^z$, is analytic on $\mathbf{C}$ and*

$$\frac{de^z}{dz} = e^z.$$

Proof By definition, $f(z) = e^x(\cos y + i \sin y)$, so the real and imaginary parts are $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. These are C^∞ (infinitely often differentiable) functions, so f is differentiable in the sense of real variables. To show that f is analytic, we must verify the Cauchy-Riemann equations. To do so, we first compute the partial derivatives

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = -e^x \sin y$$
$$\frac{\partial v}{\partial x} = e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y.$$

Thus, $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$, so by the Cauchy-Riemann Theorem 1.5.8, f is analytic. Finally,

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x(\cos y + i \sin y) = e^z. \quad \blacksquare$$

A function that is defined and analytic on the whole plane $\mathbb{C}$ is called ***entire***. Thus, $f(z) = e^z$ is an entire function.

Using the differentiation rules (Proposition 1.5.3) as in elementary calculus, we can differentiate e^z in combination with various other functions. For instance, $e^{z^2} + 1$ is entire because $z \mapsto z^2$ and $w \mapsto e^w$ are analytic, so by the chain rule, $z \mapsto e^{z^2}$ is analytic. By the chain rule and the sum rule, $(d/dz)(e^{z^2} + 1) = 2ze^{z^2}$.

We recall that $\log z : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ is an inverse for e^z when e^z is restricted to a period strip $\{x + iy \mid y_0 \le y < y_0 + 2\pi\}$. However, for differentiability of $\log z$ we must restrict $\log z$ to a set that is smaller than $\mathbb{C}\backslash\{0\}$. The reason is simple: $\log z = \log|z| + i \arg z$ for, say, $0 \le \arg z < 2\pi$. But the arg function is discontinuous; it jumps by 2π as we cross the positive real axis. If we remove these points, then we are excluding the usual positive reals on which we want $\log z$ defined. Therefore, it is convenient to use the branch $-\pi < \arg z < \pi$. Then an appropriate set on which $\log z$ is analytic is given as follows.

Proposition 1.6.2 *Let A be the open set that is $\mathbb{C}$ minus the negative real axis including zero (that is, $\mathbb{C}\backslash\{x + iy \mid x \le 0 \text{ and } y = 0\}$). Define a branch of* log *on A by*

$$\log z = \log|z| + i \arg z \qquad -\pi < \arg z < \pi,$$

which is called the principal branch of the logarithm. Then $\log z$ *is analytic on A with*

$$\frac{d}{dz}\log z = \frac{1}{z}.$$

Analogous statements hold for other branches.

First Proof (using the Inverse Function Theorem) From §1.3 we know that $\log z$ is the unique inverse of the function $f(z) = e^z$ restricted to the set $\{z \mid z = x+iy, -\pi < y < \pi\}$. Since $de^z/dz = e^z \neq 0$, the Inverse Function Theorem implies that locally, e^z has an analytic inverse. Since the inverse is unique, it must be $\log z$. The derivative of $f^{-1}(w)$ is $1/f'(z)$. In this case $f'(z) = f(z) = w$, and so $df^{-1}/dw = 1/w$ at each point $w \in A$. $\blacksquare$

Second Proof (using the Cauchy-Riemann equations in polar form) In polar form, $\log z = \log r + i\theta$, and so $u(r,\theta) = \log r, v(r,\theta) = \theta$, which are C^∞ functions of r, θ. Also, the Cauchy-Riemann equations may be expressed in polar coordinates, as explained in §1.5 and Exercises 11 and 12 at the end of that section. But

$$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad -\frac{1}{r}\frac{\partial u}{\partial \theta} = 0 = \frac{\partial v}{\partial r}$$

and so the Cauchy-Riemann equations hold. We also have

$$\frac{d}{dz}\log z = \frac{\partial}{\partial x}\log r + i\frac{\partial \theta}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} + i\frac{\partial \theta}{\partial x}.$$

It is obvious that on $A, \partial r/\partial x = x/r$ and $\partial\theta/\partial x = -y/r^2$, using, for example, $\theta = \tan^{-1}(y/x)$, so

$$\frac{d}{dz}\log z = \frac{x}{r^2} - \frac{iy}{r^2} = \frac{\bar{z}}{|z|^2} = \frac{1}{z} \quad \blacksquare$$

The domain on which the principal branch of $\log z$ is analytic is given in Figure 1.6.1. Here is an example of another branch: $\log z = \log|z| + i\arg z, 0 < \arg z < 2\pi$, is analytic on $\mathbb{C}\backslash\{x + iy \mid x \geq 0, y = 0\}$. We will use the principal branch unless otherwise stated.

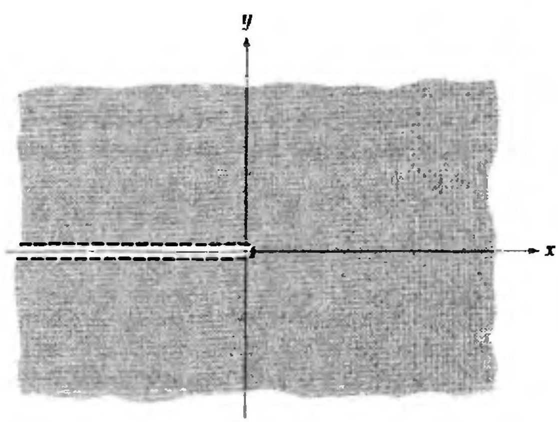

Figure 1.6.1: Domain of $\log z$.

When using $\log z$ in compositions, we must be careful to stay in the domain of log. For example, consider $f(z) = \log z^2$ using the principal branch of log. This function is analytic on the set $A = \{z \mid z \neq 0 \text{ and } \arg z \neq \pm\pi/2\}$ by the following reasoning. Proposition 1.5.3 shows that z^2 is analytic on all of $\mathbb{C}$. The image of A under the map $z \mapsto z^2$ is precisely $\mathbb{C}\backslash\{x + iy \mid x \leq 0, y = 0\}$ (Why?), which is the

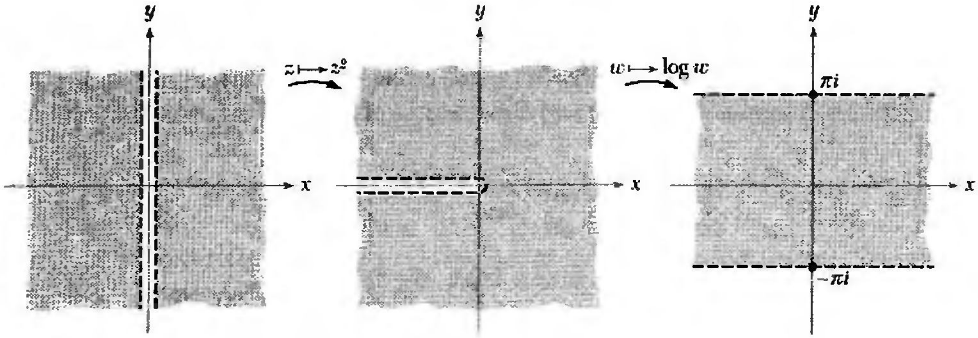

Figure 1.6.2: Composing the squaring and the log functions.

set on which the principal branch of log is defined and analytic. By the chain rule, $z \mapsto \log z^2$ is then analytic on A (see Figure 1.6.2).

The function $f(z) = \log z^2$ also illustrates that caution must be exercised in manipulating logarithms. Consider the two functions $\log z^2$ and $2\log z$. If we consider all possible values of the logarithm, then the two collections are the same. If, however, we pick a particular branch, for example, the principal branch, and use it for both, then the two are *not* always the same. For example, if $z = -1 + i$, then $\arg z = 3\pi/4$ and $\log z^2 = \log 2 - \pi i/2$, while $2 \log z = \log 2 + 3\pi i/2$. The function $\log z^2$ is analytic on the plane with the imaginary axis deleted, while $2 \log z$ is analytic on the plane with the negative real axis deleted.

The Trigonometric Functions Now that we have established properties of e^z, the differentiation of the sine and cosine functions follows readily.

Proposition 1.6.3 *The sine and cosine functions* $\sin z$ *and* $\cos z$ *are entire functions with derivatives given by*

$$\frac{d}{dz}\sin z = \cos z \quad \textit{and} \quad \frac{d}{dz}\cos z = -\sin z.$$

Proof By definition, $\sin z = (e^{iz} - e^{-iz})/2i$; using the sum rule and the chain rule and the fact that the exponential function is entire, we conclude that $\sin z$ is entire and that $d(\sin z)/dz = \{ie^{iz} - (-ie^{-iz})\}/2i = (e^{iz} + e^{-iz})/2 = \cos z$. Similarly,

$$\begin{aligned} \frac{d}{dz}\cos z &= \frac{d}{dz}\frac{1}{2}(e^{iz} + e^{-iz}) = \frac{i}{2}(e^{iz} - e^{-iz}) \\ &= -\frac{1}{2i}(e^{iz} - e^{-iz}) = -\sin z. \quad \blacksquare \end{aligned}$$

We can also discuss $\sin^{-1} z$ and $\cos^{-1} z$ in somewhat the same way that we discussed $\log z$ (which is $\exp^{-1} z$ with appropriate domains and ranges). These functions are analyzed in Exercise 6 at the end of this section.

The Power Function Let a and b be complex numbers. Recall that $a^b = e^{b \log a}$, which is, in general, multivalued according to the different branches for log that we choose. We now consider the functions $z \mapsto z^b$ and $z \mapsto a^z$. Although these functions appear to be similar, their properties of analyticity are quite different. The case in which b is an integer was covered in Proposition 1.5.3. The situation for general b is slightly more complicated.

Proposition 1.6.4 (i) *For any choice of branch for the* log *function, the function $z \mapsto a^z$ is entire and has derivative $z \mapsto (\log a)a^z$.*

(ii) *Choose a branch of the* log *function—for example, the principal branch. Then the function $z \mapsto z^b$ is analytic on the domain of the branch of* log *chosen and the derivative is $z \mapsto bz^{b-1}$.*

Proof

(i) $a^z = e^{z \log a}$. By the chain rule this function is analytic on $\mathbb{C}$ with derivative $(\log a)e^{z \log a} = (\log a)a^z$($\log a$ is merely a constant).

(ii) $z^b = e^{b \log z}$. This function is analytic on the domain of $\log z$, since $w \mapsto e^{bw}$ is entire. By the chain rule,

$$\frac{d}{dz} z^b = \frac{b}{z} e^{b \log z} = \frac{b}{z} z^b.$$

(That this equals bz^{b-1} follows from Exercise 20 at the end of §1.3.) ∎

If b is a non-negative integer we know that z^b is entire (with derivative bz^{b-1}). In general, however, z^b is analytic only on the domain of $\log z$.

Let us emphasize what is stated in (ii). If we choose the principal branch of $\log z$, which has domain $\mathbb{C}\backslash\{x + iy \mid y = 0 \text{ and } x \leq 0\}$ and range $\mathbb{R} \times]-\pi, \pi[$ (Why?), then $z \mapsto z^a$ is analytic on $\mathbb{C}\backslash\{x + iy \mid y = 0, x \leq 0\}$ (see Figure 1.6.3). We could also choose the branch of log that has domain $\mathbb{C}\backslash\{x + ix \mid x \geq 0\}$ and range $\mathbb{C} \times]-7\pi/4, \pi/4[$; then $z \mapsto z^a$ would be analytic on $\mathbb{C}\backslash\{x + ix \mid x \geq 0\}$.

The nth Root Function One of the nth roots of z is given by $z^{1/n}$, for a choice of branch of $\log z$. The other roots are given by the other choices of branches, as in §1.3. The principal branch is the one that is usually used. Thus, from Proposition 1.6.4(ii), we get the following as a special case.

Proposition 1.6.5 *The function $z \mapsto z^{1/n} = \sqrt[n]{z}$ is analytic on the domain of* $\log z$ *(for example, the principal branch) and has derivative*

$$\left(\frac{1}{n}\right) z^{(1/n)-1}.$$

As with $\log z$, we must exercise care with the functions $z \mapsto z^b, z \mapsto \sqrt[n]{z}$ when composing with other functions to be sure we stay in the domain of analyticity. The procedure is illustrated for the square root function in Worked Example 1.6.8.

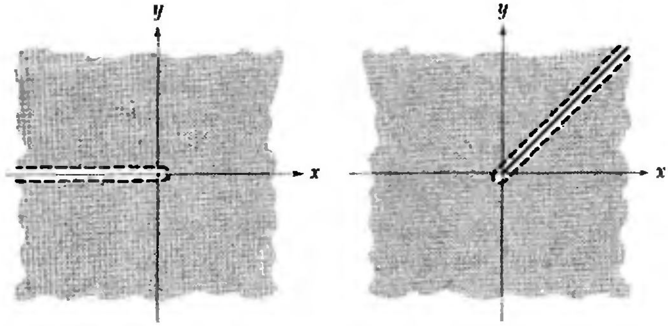

Figure 1.6.3: Regions of analyticity for $z \mapsto z^a$.

Worked Examples

Example 1.6.6 *Differentiate the following functions, giving the appropriate region on which the functions are analytic:*

(a) e^{e^z}

(b) $\sin(e^z)$

(c) $e^z/(z^2+3)$

(d) $\sqrt{e^z+1}$

(e) $\cos \bar{z}$

(f) $1/(e^z-1)$

(g) $\log(e^z+1)$

Solution

(a) The function e^z is entire, so by the chain rule, $z \mapsto e^{e^z}$ is entire. The chain rule also tells us that the derivative at z is $e^z e^{e^z}$.

(b) Both $z \mapsto e^z$ and $w \mapsto \sin w$ are entire, so by the chain rule $z \mapsto \sin e^z$ is entire and the derivative at z is $(\cos e^z)e^z$.

(c) The map $z \mapsto e^z$ is entire and the map $z \mapsto 1/(z^2+3)$ is analytic on $\mathbb{C}\backslash\{\pm\sqrt{3}i\}$. Hence $z \mapsto e^z/(z^2+3)$ is analytic on $\mathbb{C}\backslash\{\pm\sqrt{3}i\}$ and has derivative at $z \neq \pm\sqrt{3}i$ given by

$$\frac{e^z}{z^2+3} - \frac{e^z \cdot 2z}{(z^2+3)^2} = \frac{e^z(z^2-2z+3)}{(z^2+3)^2}.$$

(d) Choose the branch of the function $w \mapsto \sqrt{w}$ that is analytic on $\mathbb{C}\backslash\{x+iy \mid y=0, x\leq 0\}$. Then we must choose the region A such that if $z\in A$, then e^z+1 is not both real and ≤ 0. Notice that e^z is real iff $y=\operatorname{Im} z = n\pi$ for some integer n (Why?). When n is even, e^z is positive; when n is odd, e^z is negative. Here $|e^z| = e^x$, where $x=\operatorname{Re} z$ and $e^x\geq 1$ iff $x\geq 0$. Therefore if we define $A=\mathbb{C}\backslash\{x+iy \mid x\geq 0, y=(2n+1)\pi, n \text{ an integer }\}$ (as in Figure 1.6.4), then e^z+1 is real and ≤ 0 iff $z\notin A$. Since e^z+1 is entire, it is certainly analytic on A. By the chain rule, $\sqrt{e^z+1}$ is analytic on A with derivative at z given by $(e^z+1)^{-1/2}(e^z)/2$.

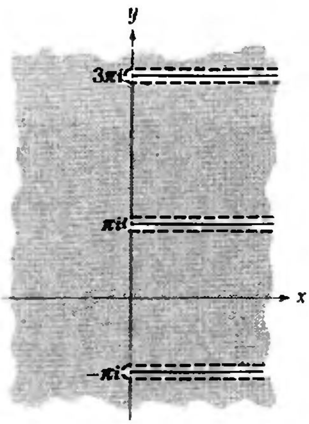

Figure 1.6.4: Region of analyticity of $\sqrt{e^z+1}$.

(e) Since $z=x+iy$, by Proposition 1.3.4, $\cos\bar{z}=\cos(x-iy)=\cos x\cos(-iy)-\sin x\sin(-iy) = \cos x\cosh y + i\sin x\sinh y$, so $u(x,y)=\cos x\cosh y$ and $v(x,y)=\sin x\sinh y$. Thus $\partial u/\partial x = -\sin x\cosh y, \partial v/\partial y = \sin x\cosh y$. If $\cos\bar{z}$ were analytic, $\partial u/\partial x$ would equal $\partial v/\partial y$, which would occur iff $\sin x = 0$ (that is, if $x=0$, or if $x=\pi n, n=\pm1,\pm2,\ldots$). Thus, there is not an open (nonempty) set A on which $z\mapsto\cos\bar{z}$ is analytic.

(f) By Proposition 1.5.3(iii) and the fact that $z\mapsto e^z-1$ is entire, we conclude that $z\mapsto 1/(e^z-1)$ is analytic on the set on which $e^z-1\neq 0$; namely, the set $A=\mathbb{C}\backslash\{z=2\pi ni \mid n=0,\pm1,\pm2,\ldots\}$. The derivative at z is $-e^z/(e^z-1)^2$.

(g) Since (the principal branch of) the log is defined and is analytic on the same region as the square root, namely, $A=\mathbb{C}\backslash\{x+iy \mid y=0, x\leq 0\}$, we can use the results of (d). By the chain rule and the results of (d), $z\mapsto\log(e^z+1)$ is analytic on the region depicted in Figure 1.6.4.

Example 1.6.7 *Verify directly that after mapping by the function e^z, the angles between lines parallel to the coordinate axes are preserved.*

Solution The line determined by $y = y_0$ is mapped to the ray $\{x + ix \tan y_0 \mid x > 0\}$, and the line determined by $x = x_0$ is mapped to the circle $|z| = e^{x_0}$; see Figure 1.3.7. The angle between any such ray and the tangent vector to the circle at the point of contact of the ray and the circle is $\pi/2$. Thus angles are preserved. This is consistent with the Conformal Mapping Theorem 1.5.7.

Example 1.6.8 *Show that a branch of the function $w \mapsto \sqrt{w}$ can be defined in such a way that $z \mapsto \sqrt{z^2 - 1}$ is analytic in the region shaded in Figure 1.6.5, and using the notations of that figure, show that $\sqrt{z^2 - 1} = \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2}$, where $0 < \theta_1 < 2\pi, -\pi < \theta_2 < \pi$.*

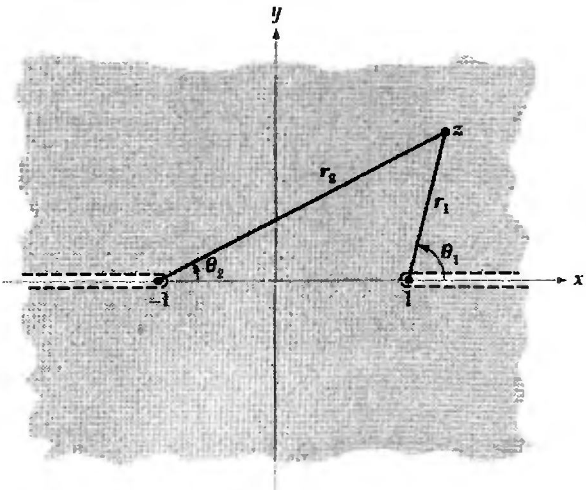

Figure 1.6.5: Domain of analyticity of $\sqrt{z^2 - 1}$.

Solution If $\sqrt{z-1}$ is a square root of $z - 1$ and $\sqrt{z+1}$ is a square root of $z + 1$, then $\sqrt{z-1} \cdot \sqrt{z+1}$ is a square root of $z^2 - 1$ (Why?). For $z \mapsto \sqrt{z-1}$ we may choose $\sqrt{\ }$ defined and analytic on $\mathbb{C}\backslash\{x + iy \mid y = 0 \text{ and } x \geq 0\}$; thus $z \mapsto \sqrt{z-1}$ is analytic on the region $\mathbb{C}\backslash\{x + iy \mid y = 0, x \geq 1\}$. For the map $z \mapsto \sqrt{z+1}$, we may choose $\sqrt{\ }$ defined and analytic on the region $\mathbb{C}\backslash\{x+iy \mid y = 0 \text{ and } x \leq 0\}$; therefore, $z \mapsto \sqrt{z+1}$ is analytic on $\mathbb{C}\backslash\{x + iy \mid y = 0, x \leq -1\}$. Thus $z \mapsto \sqrt{z-1}\sqrt{z+1}$ is analytic on $\mathbb{C}\backslash\{x + iy \mid y = 0, |x| \geq 1\}$ with the appropriate branches of $\sqrt{\ }$ as indicated. With these branches we have

$$\sqrt{z-1} = \sqrt{r_1} e^{i\theta_1/2} \quad \text{and} \quad \sqrt{z+1} = \sqrt{r_2} e^{i\theta_2/2},$$

so $\sqrt{z-1}\sqrt{z+1} = \sqrt{r_1 r_2} e^{i(\theta_1+\theta_2)/2}$. Since $z \mapsto \sqrt{z-1}\sqrt{z+1}$ is analytic on $\mathbb{C}\backslash\{x + iy \mid y = 0, |x| \geq 1\}$, it should correspond to some branch of the square root function in the map $z \mapsto \sqrt{z^2-1}$. To see which one, note that $z \in \mathbf{C}\backslash\{x+iy \mid y=0, |x| \geq 1\}$ iff $z^2-1 \in \mathbf{C}\backslash\{x+iy \mid y=0, x \geq 0\}$ (Why?), so if we take the square root defined and analytic on $\mathbf{C}\backslash\{x+iy \mid y=0, x \geq 0\}$, then $z \mapsto \sqrt{z^2-1}$ is analytic on $\mathbf{C}\backslash\{x+iy \mid y=0, |x| \geq 1\}$.

The symbol $\sqrt{\cdot}$ has been used here to mean different branches of the square root function, the particular branch being clear from the context. Thus, one normally thinks of $\sqrt{\cdot}$ together with a choice of branch.

Exercises

1. Differentiate and give the appropriate region of analyticity for each of the following:

   (a) $z^2 + z$

   (b) $1/z$

   (c) $\sin z / \cos z$

   (d) $\exp\left(\dfrac{z^3+1}{z-1}\right)$

2. Differentiate and give the appropriate region of analyticity for each of the following:

   (a)• 3^z

   (b) $\log(z+1)$

   (c)• $z^{(1+i)}$

   (d) $\sqrt{z}$

   (e)• $\sqrt[3]{z}$

3. Determine whether the following complex limits exist and find their value if they do:

   (a) $\lim_{z\to 0} \dfrac{e^z-1}{z}$

   (b) $\lim_{z\to 0} \dfrac{\sin |z|}{z}$

4. Determine whether the following complex limits exist and find their value if they do:

   (a) $\lim_{z\to 1} \dfrac{\log z}{z-1}$

   (b) $\lim_{z\to 1} \dfrac{\bar{z}-1}{z-1}$

5. Is it true that $|\sin z| \leq 1$ for all $z \in \mathbf{C}$?

6.• Solve $\sin z = w$ and show how to choose a domain and thus how to pick a particular branch of $\sin^{-1} z$ so that it is analytic on the domain. Give the derivative of this branch of $\sin^{-1} z$; see Exercise 35 at the end of §1.3.

7. Let $f(z) = 1/(1-z)$; is it continuous on the interior of the unit circle?

8. Let $u(x,y)$ and $v(x,y)$ be real-valued functions on an open set $A \subset \mathbb{R}^2 = \mathbb{C}$ and suppose that they satisfy the Cauchy-Riemann equations on A. Show that

$$u_1(x,y) = [u(x,y)]^2 - [v(x,y)]^2 \quad \text{and} \quad v_1(x,y) = 2u(x,y)v(x,y)$$

satisfy the Cauchy-Riemann equations on A and that the functions

$$u_2(x,y) = e^{u(x,y)} \cos v(x,y) \quad \text{and} \quad v_2(x,y) = e^{u(x,y)} \sin v(x,y)$$

also satisfy the Cauchy-Riemann equations on A. Can you do this without performing any computations?

9. Find the region of analyticity and the derivative of each of the following functions:

   (a) $\dfrac{z}{z^2-1}$

   (b) $e^{z+(1/z)}$

10.• Find the region of analyticity and the derivative of each of the following functions:

   (a) $\sqrt{z^3-1}$

   (b) $\sin\sqrt{z}$

11. Find the minimum of $|e^{z^2}|$ for those z with $|z| \le 1$.

12. Prove Proposition 1.6.5 using the method of the first proof of Proposition 1.6.2.

13.• Where is $z \mapsto 2^{z^2}$ analytic? $z \mapsto z^{2z}$?

14. Define a branch of $\sqrt{1+\sqrt{z}}$ and show that it is analytic.

Review Exercises for Chapter 1

1. Compute the following quantities: (a) e^i (b) $\log(1+i)$ (c) $\sin i$ (d) 3^i (e) $e^{2\log(-1)}$

2. For what values of z is $\log z^2 = 2\log z$ if the principal branch of the logarithm is used on both sides of the equation?

3. Find the eighth roots of i.

4. Find all numbers z such that $z^2 = 1 + i$.

5. Solve $\cos z = \sqrt{3}$ for z.

6. Solve $\sin z = \sqrt{3}$ for z.

7. Describe geometrically the set of points $z \in \mathbb{C}$ satisfying

   (a) $|z+i| = |z-i|$
   (b) $|z-1| = 3|z-2|$

8. Describe geometrically the set of points $z \in \mathbb{C}$ satisfying

   (a) $|z-1| = |z+1|$
   (b) $|z-1| = 2|z|$

9. Differentiate the following expressions on appropriate regions:

   (a) $z^3 + 8$
   (b) $\dfrac{1}{z^3+1}$
   (c) $\exp(z^4 - 1)$
   (d) $\sin(\log z^2)$

10. On what set is $\sqrt{z^2-2}$ analytic? Compute the derivative.

11. Describe the sets on which the following functions are analytic and compute their derivatives:

    (a) $e^{1/z}$
    (b) $\dfrac{1}{(1-\sin z)^2}$
    (c) $\dfrac{e^{az}}{a^2+z^2}$ for a real

12.• Repeat Review Exercise 11 for the following functions:

    (a) $\exp\left(\dfrac{1}{1-az}\right)$ for $a \in \mathbb{C}$

(b) $\dfrac{\sin z}{z}$

13. Can a single-valued (analytic) branch of $\log z$ be defined on the following sets?

    (a) $\{z \mid 1 < |z| < 2\}$

    (b) $\{z \mid \operatorname{Re} z > 0\}$

    (c) $\{z \mid \operatorname{Re} z > \operatorname{Im} z\}$

14.* Show that the map $z \mapsto z + 1/z$ maps the circle $\{z \text{ such that } |z| = c\}$ onto the ellipse described by

$$\{w = u + iv \mid u = (c + 1/c)\cos\theta, v = (c - 1/c)\sin\theta, 0 \le \theta \le 2\pi\}.$$

Can we allow $c = 1$?

15. Let f be analytic on A. Define $g : A \to \mathbf{C}$ by $g(z) = \overline{f(z)}$. When is g analytic?

16. Find the real and imaginary parts of $f(z) = z^3$ and verify directly that they satisfy the Cauchy-Riemann equations.

17. Let $f(x + iy) = (x^2 + 2y) + i(x^2 + y^2)$. At what points does $f'(z_0)$ exist?

18. Let $f : A \subset \mathbf{C} \to \mathbf{C}$ be analytic on an open set A. Let $A^* = \{\bar{z} \mid z \in A\}$.

    (a) Describe A^* geometrically.

    (b) Define $g : A^* \to \mathbf{C}$ by $g(z) = \overline{[f(\bar{z})]^2}$. Show that g is analytic.

19.* Suppose that $f : A \subset \mathbf{C} \to \mathbf{C}$ is analytic on the open connected set A and that $f(z)$ is real for all $z \in A$. Show that f is constant.

20. Prove the Cauchy-Riemann equations as follows. Let $f : A \subset \mathbf{C} \to \mathbf{C}$ be differentiable at $z_0 = x_0 + iy_0$. Let $g_1(t) = t + iy_0$ and $g_2(t) = x_0 + it$. Apply the chain rule to $f \circ g_1$ and $f \circ g_2$ to prove the result.

21. Let $f(z)$ be analytic in the disk $|z - 1| < 1$. Suppose that $f'(z) = 1/z, f(1) = 0$. Prove that $f(z) = \log z$.

22.* Use the Inverse Function Theorem to prove the following result. Let $f : A \subset \mathbf{C} \to \mathbf{C}$ be analytic (where A is open and connected) and suppose that $f(A) \subset \{z \text{ such that } |z| = 3\}$. Then f is constant.

23. Prove that

$$\lim_{h \to 0} \frac{(z_0 + h)^n - z_0^n}{h} = nz_0^{n-1}$$

for any $z_0 \in \mathbf{C}$.

24. (a) If a polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ has a root c, then show that we can write $p(z) = (z - c)h(z)$, where $h(z)$ is a polynomial of degree $n - 1$. (Use division of polynomials to show that $z - c$ divides $p(z)$.)

(b) Use part (a) to show that p can have no more than n roots.

(c) When is c a root of both $p(z)$ and $p'(z)$?

25.• On what set is the function $z \mapsto z^2$ analytic? Compute its derivative.

26. Let g be analytic on the open set A. Let $B = \{z \in A \mid g(z) \neq 0\}$. Show that B is open and that $1/g$ is analytic on B.

27. Find and plot all solutions of $z^3 = -8i$.

28. Let $f : A \subset \mathbf{C} \to \mathbf{C}$ be analytic on the open set A and let $f'(z_0) \neq 0$ for every $z_0 \in A$. Show that $\{\operatorname{Re} f(z) \mid z \in A\} \subset \mathbf{R}$ is open.

29.• Show that $u(x, y) = x^3 - 3xy^2, v(x, y) = 3x^2y - y^3$ satisfy the Cauchy-Riemann equations. Comment on the result.

30. Prove that the following functions are continuous at $z = 0$:

(a) $f(z) = \begin{cases} (\operatorname{Re} z^2)^2/|z|^2 & z \neq 0 \\ 0 & z = 0 \end{cases}$

(b) $f(z) = |z|$

31. At what points z are the following functions differentiable?

(a) $f(z) = |z|^2$

(b) $f(z) = y - ix$

32.• Use de Moivre's theorem to find the sum $\sin x + \sin 2x + \ldots + \sin nx$.

33. For the function $u(x, y) = y^3 - 3x^2y$,

(a) Show that u is harmonic (see Proposition 1.5.12).

(b) Determine a conjugate function $v(x, y)$ such that $u + iv$ is analytic.

34. Consider the function $w(z) = 1/z$. Draw the level curves $u = \operatorname{Re}(w(z)) =$ constant. Discuss.

35. Determine the four different values of z that are mapped to unity by the function $w(z) = z^4$.

36. Suppose that $f(z)$ is analytic and satisfies the condition $|f(z)^2 - 1| < 1$ in a region Ω. Show that either $\operatorname{Re} f(z) > 0$ or $\operatorname{Re} f(z) < 0$ throughout Ω.

37.• Suppose that $f : \mathbf{C} \to \mathbf{C}$ is continuous and that $f(z) = f(2z)$ for all $z \in \mathbf{C}$. Show that f is constant on $\mathbf{C}$.

38. Suppose that $f : \mathbf{C} \to \mathbf{C}$ is entire and that $f(2z) = 2f(z)$ for all $z \in \mathbf{C}$. Show that there is a constant c such that $f(z) = cz$ for all z. (You might want to use Exercise 37.)

Chapter 2

Cauchy's Theorem

An attractive feature of complex analysis is that it is based on a few simple, yet powerful theorems from which most of the results of the subject follow. Foremost among these theorems is a remarkable result called Cauchy's Theorem, which is one of the keys to the development of the rest of the subject and its applications.

2.1 Contour Integrals

Definitions and Basic Properties To state Cauchy's Theorem, we first define contour integrals and study their basic properties.

Let $h : [a, b] \subset \mathbf{R} \to \mathbf{C}$ be a complex-valued function of one real variable and let u and v be its real and imaginary parts; that is, set $h(t) = u(t) + iv(t)$. Suppose, for the sake of simplicity, that u and v are continuous. Define the ***integral*** of h to be the complex number

$$\int_a^b h(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt,$$

where the integrals of u and v have their usual meaning from single-variable calculus. We want to extend this definition to integrals of functions along curves in $\mathbf{C}$.

To accomplish this, we will need a few definitions. A continuous ***curve*** or ***contour*** in $\mathbf{C}$ is, by definition, a continuous map $\gamma : [a, b] \to \mathbf{C}$. The curve is called ***piecewise*** C^1 if we can divide up the interval $[a, b]$ into finitely many subintervals $a = a_0 < a_1 < \ldots < a_n = b$ such that the derivative $\gamma'(t)$ exists on each open subinterval $]a_i, a_{i+1}[$ and is continuous on $[a_i, a_{i+1}]$; continuity on $[a_i, a_{i+1}]$ means that the limits $\lim_{t \to a_i+} \gamma'(t)$ and $\lim_{t \to a_{i+1}-} \gamma'(t)$ exist (see, for example, Figure 2.1.1). Unless otherwise specified, curves will always be assumed to be continuous and piecewise C^1.

Definition 2.1.1 *Suppose that f is continuous and defined on an open set $A \subset \mathbf{C}$ and that $\gamma : [a, b] \to \mathbf{C}$ is a piecewise smooth curve satisfying $\gamma([a, b]) \subset A$. The*

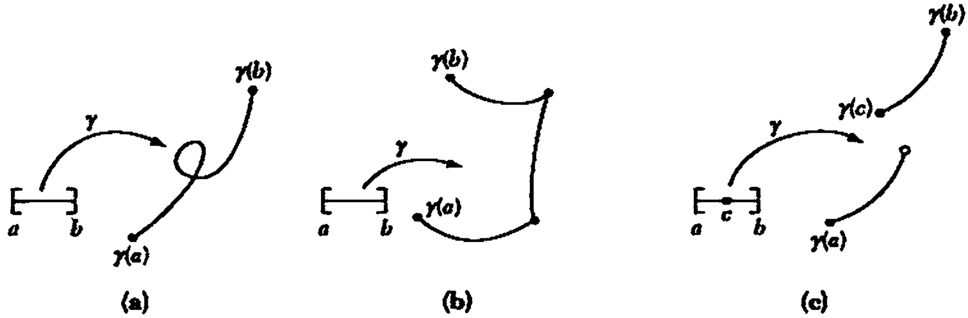

Figure 2.1.1: Curves in the complex plane $\mathbb{C}$. (a) Smooth curve. (b) Piecewise C^1 curve. (c) Discontinuous curve.

expression

$$\int_\gamma f = \int_\gamma f(z)dz = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)dt$$

is called the integral of f along γ.

The definition is analogous to the following definition of a line integral from vector calculus: Let $P(x,y)$ and $Q(x,y)$ be real-valued functions of x and y and let γ be a curve. Define

$$\int_\gamma P(x,y)dx + Q(x,y)dy = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} \left[P(x(t),y(t))\frac{dx}{dt} + Q(x(t),y(t))\frac{dy}{dt}\right] dt,$$

where $\gamma(t) = (x(t), y(t))$. The two definitions are related as follows.

Proposition 2.1.2 *If $f(z) = u(x,y) + iv(x,y)$, then*

$$\int_\gamma f = \int_\gamma [u(x,y)dx - v(x,y)dy] + i\int_\gamma [u(x,y)dy + v(x,y)dx].$$

Proof According to the definition we must work out $f(\gamma(t))\gamma'(t)$. We do this as follows:

$$\begin{aligned} f(\gamma(t))\gamma'(t) &= [u(x(t),y(t)) + iv(x(t),y(t))] \cdot [x'(t) + iy'(t)] \\ &= [u(x(t),y(t))x'(t) - v(x(t),y(t))y'(t)] \\ &\quad + i[v(x(t),y(t))x'(t) + u(x(t),y(t))y'(t)]. \end{aligned}$$

Integrating both sides over $[a_i, a_{i+1}]$ with respect to t and using definition 2.1.1 then gives the desired result. ■

The formula in this proposition can easily be remembered by formally writing

$$f(z)dz = (u + iv)(dx + i\,dy) = u\,dx - v\,dy + i(v\,dx + u\,dy).$$

For a curve $\gamma : [a, b] \to \mathbf{C}$, we define the ***opposite curve***, $-\gamma : [a, b] \to \mathbf{C}$, by setting $(-\gamma)(t) = \gamma(a + b - t)$. The curve $-\gamma$ is thus γ traversed in the opposite sense (see Figure 2.1.2).

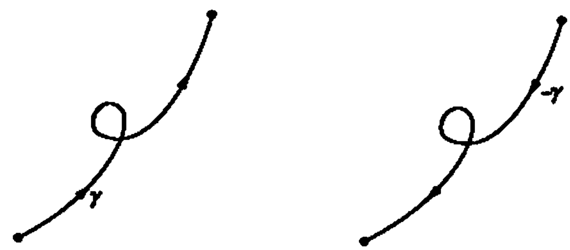

Figure 2.1.2: Opposite curve.

We also want to define the ***join*** or ***sum*** or ***union*** $\gamma_1 + \gamma_2$ of two curves γ_1 and γ_2. Intuitively, we want to join them at their endpoints to make a single curve (see Figure 2.1.3). Precisely, suppose that $\gamma_1 : [a, b] \to \mathbf{C}$ and that $\gamma_2 : [b, c] \to \mathbf{C}$, with $\gamma_1(b) = \gamma_2(b)$. Define $\gamma_1 + \gamma_2 : [a, c] \to \mathbf{C}$ by

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{if} \quad t \in [a, b] \\ \gamma_2(t) & \text{if} \quad t \in [b, c]. \end{cases}$$

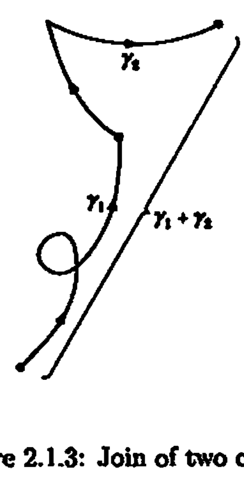

Figure 2.1.3: Join of two curves.

Clearly, if γ_1 and γ_2 are piecewise smooth, then so is $\gamma_1 + \gamma_2$. If the intervals $[a, b]$ and $[b, c]$ for γ_1 and γ_2 are not of this special form (the first interval ends where the second begins), then the formula is a little more complicated, but the special form will suffice for this text. The general sum $\gamma_1 + \ldots + \gamma_n$ is defined similarly.

The next proposition gives some properties of the integral that follow from the definitions given in this section. The student is asked to prove them in Exercise 6 at the end of this section.

Proposition 2.1.3 *For (continuous) functions f, g, complex constants c_1, c_2, and piecewise C^1 curves $\gamma, \gamma_1, \gamma_2$, the following hold:*

(a) $\displaystyle\int_\gamma (c_1 f + c_2 g) = c_1 \int_\gamma f + c_2 \int_\gamma g$

(b) $\displaystyle\int_{-\gamma} f = -\int_\gamma f$

(c) $\displaystyle\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$

Of course, more general statements (that follow from the preceding) could be made, namely,

$$\int_\gamma \sum_{i=1}^n c_i f_i = \sum_{i=1}^n \left(c_i \int_\gamma f_i \right)$$

and

$$\int_{\gamma_1+\ldots+\gamma_n} f = \sum_{i=1}^n \int_{\gamma_i} f.$$

To compute specific examples it is sometimes convenient to use the formula in Proposition 2.1.2. However, it may be that we are not given γ as a map but are told only that it is, for example, "the straight-line joining 0 to $i + 1$" or "the unit circle traversed counterclockwise". To use the definition, we need to choose some explicit map $\gamma(t)$ that describes this geometrically given curve. Obviously, the same geometric curve can be described in different ways, so the question arises whether the integral $\int_\gamma f$ is independent of that description.

To answer this question, we use the following definition.

Definition 2.1.4 *Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve. A piecewise smooth curve $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ is called a* ***reparametrization*** *of γ if there is a C^1 function $\alpha : [a, b] \to [\tilde{a}, \tilde{b}]$ with $\alpha'(t) > 0, \alpha(a) = \tilde{a}$, and $\alpha(b) = \tilde{b}$ such that $\gamma(t) = \tilde{\gamma}(\alpha(t))$ (see Figure 2.1.4).*

The conditions $\alpha'(t) > 0$ (hence α is increasing), $\alpha(a) = \tilde{a}$, and $\alpha(b) = \tilde{b}$ imply that $\tilde{\gamma}$ traverses the curve in the same sense as γ does. This is the precise meaning of the statement that γ and $\tilde{\gamma}$ represent the same (oriented) geometric curve. Also, the points in $[\tilde{a}, \tilde{b}]$ at which $\tilde{\gamma}'$ does not exist correspond under α to the points of $[a, b]$ at which γ' does not exist. (This is because α has a strictly increasing C^1 inverse.)

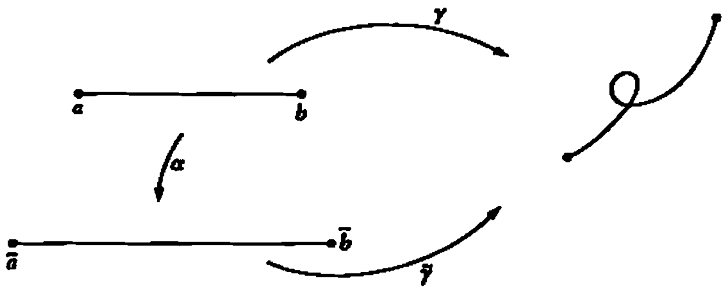

Figure 2.1.4: Reparametrization.

Proposition 2.1.5 *If $\tilde{\gamma}$ is a reparametrization of γ, then*

$$\int_\gamma f = \int_{\tilde{\gamma}} f$$

for any continuous f defined on an open set containing the image of γ = image of $\tilde{\gamma}$.

Proof We can, by breaking up $[a,b]$ into subintervals, assume that γ is C^1. By definition

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

By the chain rule, $\gamma'(t) = d\gamma(t)/dt = d\tilde{\gamma}(\alpha(t))/dt = \tilde{\gamma}'(\alpha(t))\alpha'(t)$. Let $s = \alpha(t)$ be a new variable, so that $s = \tilde{a}$ when $t = a$ and $s = \tilde{b}$ when $t = b$. Then

$$\begin{aligned}\int_a^b f(\gamma(t))\gamma'(t)dt &= \int_a^b f(\tilde{\gamma}(\alpha(t)))\tilde{\gamma}'(\alpha(t))\frac{d\alpha}{dt}dt \\ &= \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(s))\tilde{\gamma}'(s)ds.\end{aligned}$$

Changing variables in a complex integral (here, from t to s) is justified by applying the usual real-variables rule to its real and imaginary parts. ■

This proposition "justifies" the use of any parametrization γ that describes a given oriented geometric curve to evaluate an integral.[1] As an example, let us evaluate $\int_\gamma x\,dz$, where γ is the straight-line from $z = 0$ to $z = 1 + i$ (see Figure 2.1.5).

[1]Strictly speaking, this statement is not quite correct, since two maps with the same image need not be reparametrizations of one another. However, they are reparametrizations if we ignore points where $\gamma'(t) = 0$. The proposition can be generalized to cover this situation as well, but the complications that result from generalizing it to cover this case have been omitted to simplify the exposition.

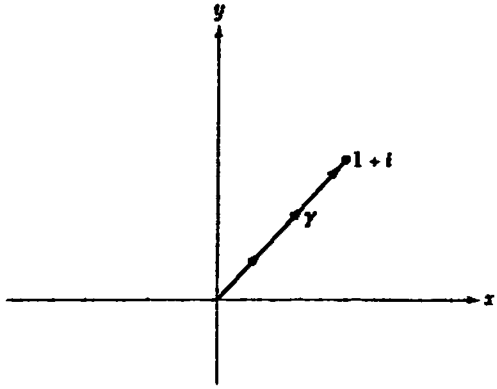

Figure 2.1.5: Curve from 0 to $1+i$.

We choose the curve $\gamma : [0,1] \to \mathbb{C}$, defined by $\gamma(t) = t + it$. Of course, when we write x in $\int_\gamma x\,dz$, we mean the function that gives the real part of any complex number (that is, $f(z) = x = \operatorname{Re} z$). Thus,

$$\int_\gamma x\,dz = \int_0^1 [\operatorname{Re}\gamma(t)]\gamma'(t)dt.$$

Hence,

$$\int_\gamma x\,dz = \int_0^1 t(1+i)dt = \frac{1+i}{2}.$$

An orientation is often described by saying that "γ goes from z_1 to z_2." However, if γ is a closed curve, with $z_1 = z_2$, we need a different prescription. When solving examples, where the curves are always easy to visualize, the student should assume that a closed curve γ is traversed in the *counterclockwise direction* unless advised to the contrary.

From calculus, recall that the ***arc length*** of a curve $\gamma : [a,b] \to \mathbb{C}$ is defined by

$$l(\gamma) = \int_a^b |\gamma'(t)|dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt.$$

Arc length, too, is independent of the parametrization, by a similar proof to that of Proposition 2.1.5. The reader should be familiar with the fact that the arc length of the unit circle is 2π, the perimeter of the unit square is 4, so on.

The next result gives an important way to estimate integrals.

Proposition 2.1.6 *Let f be continuous on an open set A and let γ be a piecewise C^1 curve in A. If there is a constant $M \geq 0$ such that $|f(z)| \leq M$ for all points z*

on γ (that is, for all z of the form $\gamma(t)$ for some t), then

$$\left|\int_\gamma f\right| \leq Ml(\gamma).$$

More generally, we have

$$\left|\int_\gamma f\right| \leq \int_\gamma |f||dz|$$

where the latter integral is defined by

$$\int_\gamma |f||dz| = \int_a^b |f(\gamma(t))||\gamma'(t)|dt.$$

Proof For a complex-valued function $g(t)$ on $[a, b]$, we have

$$\operatorname{Re}\int_a^b g(t)dt = \int_a^b \operatorname{Re} g(t)dt$$

since $\int_a^b g(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$ if $g(t) = u(t) + iv(t)$. Let us use this fact to prove that

$$\left|\int_a^b g(t)dt\right| \leq \int_a^b |g(t)|dt.$$

(We learn in calculus how to prove this for g that are real-valued, but here g is *complex-valued.*) For our proof, we let $\int_a^b g(t)dt = re^{i\theta}$ for fixed r and θ, where $r \geq 0$, so that $r = e^{-i\theta}\int_a^b g(t)dt = \int_a^b e^{-i\theta}g(t)dt$. Thus,

$$r = \operatorname{Re} r = \operatorname{Re}\int_a^b e^{-i\theta}g(t)dt = \int_a^b \operatorname{Re}(e^{-i\theta}g(t))dt.$$

By Proposition 1.2.3(iii), $\operatorname{Re}(e^{-i\theta}g(t)) \leq |e^{-i\theta}g(t)| = |g(t)|$, since $|e^{-i\theta}| = 1$. Thus, $\int_a^b \operatorname{Re}(e^{-i\theta}g(t))dt \leq \int_a^b |g(t)|dt$, so

$$\left|\int_a^b g(t)dt\right| = r \leq \int_a^b |g(t)|dt.$$

Using this and $|zz'| = |z||z'|$, we get

$$\left|\int_\gamma f\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \leq \int_a^b |f(\gamma(t))\gamma'(t)|dt = \int_a^b |f(\gamma(t))||\gamma'(t)|dt.$$

Since $|f(\gamma(t))| \leq M$, the preceding expression is bounded by $M\int_a^b |\gamma'(t)|dt = Ml(\gamma)$. ■

This proposition provides a basic tool that we shall use in subsequent proofs to estimate the size of integrals. The student might try to prove this result directly in terms of the expression

$$\int_\gamma f = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx)$$

to be convinced that the result is not altogether trivial.

Fundamental Theorem of Calculus for Contour Integrals The Fundamental Theorem of Calculus is a basic fact in the calculus of real-valued functions. It says that the integral of the derivative of a function is the difference of the values of the function at the endpoints of the interval of integration and that the indefinite integral of a function is an antiderivative for the function. Both of these assertions have important analogues for complex path integrals.

Theorem 2.1.7 (Fundamental Theorem of Calculus—Contour Integrals) *Suppose that $\gamma : [0,1] \to \mathbb{C}$ is a piecewise smooth curve and that F is a function defined and analytic on an open set G containing γ. Assume F' is continuous (later we will see that this is redundant). Then*

$$\int_\gamma F'(z)dz = F(\gamma(1)) - F(\gamma(0)).$$

In particular, if $\gamma(0) = \gamma(1)$, then

$$\int_\gamma F'(z)dz = 0.$$

Proof The chain rule and the definition of the path integral will be used to reduce the problem to the standard Fundamental Theorem of Calculus for real-valued functions of a real variable. Let g, u, and v be defined by

$$F(\gamma(t)) = g(t) = u(t) + iv(t).$$

Break the parameter interval into pieces on which γ is smooth, so, by the chain rule,

$$F'(\gamma(t))\gamma'(t) = g'(t) = u'(t) + iv'(t).$$

We apply the Fundamental Theorem of Calculus on each subinterval and get a telescoping sum:

$$\begin{aligned}
\int_\gamma F'(z)dz &= \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} F'(\gamma(t))\gamma'(t)dt \\
&= \sum_{i=0}^{n-1} \left[\int_{a_i}^{a_{i-1}} u'(t)dt + i \int_{a_i}^{a_{i+1}} v'(t)dt \right] \\
&= \sum_{i=0}^{n-1} [u(a_{i+1}) - u(a_i)] + i[v(a_{i+1}) - v(a_i)] \\
&= [u(a_n) + iv(a_n)] - [u(a_0) + iv(a_0)] \\
&= F(\gamma(1)) - F(\gamma(0)). \quad \blacksquare
\end{aligned}$$

Using this result can save a lot of effort in working examples. For instance, consider $\int_\gamma z^3 dz$ where γ is the portion of the ellipse $x^2 + 4y^2 = 1$ that joins $z = 1$ to $z = i/2$. To evaluate the integral we note that $z^3 = \frac{1}{4}(dz^4/dz)$, so

$$\int_\gamma z^3 dz = \frac{z^4}{4}\bigg|_1^{i/2} = \left(\frac{1}{4}\right)\left(\frac{i}{2}\right)^4 - \left(\frac{1}{4}\right)(1)^4 = -\frac{15}{64}.$$

Notice that we did not even need to parametrize the curve! By applying the Fundamental Theorem of Calculus, we would have obtained the same answer for any curve joining these two points. We will investigate the independence of the value of an integral from the particular path used in the next subsection.

The Fundamental Theorem of Calculus has many applications and ramifications, one of which is the following proof of a property of open connected sets, which first appeared as Proposition 1.5.5. It is the analogue in the complex domain of the following principle so useful in calculus: A function whose derivative is identically 0 is constant.

Corollary 2.1.8 *If f is a function defined and analytic on an open connected set $G \subset \mathbf{C}$, and if $f'(z) = 0$ for every point z in G, then f is constant on G.*

Proof Fix a point z_0 in G and suppose that z is any other point in G. By Proposition 1.4.15 there is a smooth path γ from z_0 to z in G. By Theorem 2.1.7, $f(z) - f(z_0) = \int_\gamma f'(\zeta)d\zeta = 0$. Therefore $f(z) = f(z_0)$. The value of f at any point of G is thus the same as its value at z_0. That is, f is constant on G. $\blacksquare$

Path Independence of Integrals The idea that an indefinite integral is an antiderivative does not carry over directly to the complex domain. What should we mean by the integral between two points? There are many possible paths. The connection comes up in the study of one of the central questions we will study in this chapter: Under what conditions is the value of an integral independent of which path is selected between the two points? Consider the following two examples.

Example 1 Let $z_0 = 1$ and $z_1 = -1$ and let $f(z) = 3z^2$. Then $F(z) = z^3$ is an antiderivative for f everywhere in the complex plane. Therefore, by the Fundamental Theorem of Calculus, no matter what path γ we take from z_0 to z_1 we will have $\int_\gamma f(z)dz = F(z_1) - F(z_0) = 1^3 - (-1)^3 = 2$. The value of the integral does not depend on the particular path selected, but only on the function and the two endpoints. ♦

Example 2 Again let $z_0 = 1$ and $z_1 = -1$, but now take $f(z) = 1/z$. Let γ_1 be the upper half of the unit circle from 1 to -1. Then γ_1 is parametrized by $\gamma(t) = e^{it}$ for $0 \le t \le \pi$. Thus,

$$\int_{\gamma_1} f(z)dz = \int_0^\pi f(\gamma_1(t))\gamma_1'(t)dt = \int_0^\pi e^{-it}ie^{it}dt = \pi i.$$

Now let γ_2 be the lower half of the unit circle from 1 to -1. Then γ_2 is parametrized by $\gamma_2(t) = e^{-it}$ for $0 \le t \le \pi$ and

$$\int_{\gamma_2} f(z)dz = \int_0^\pi f(\gamma_2(t))\gamma_2'(t)dt = \int_0^\pi e^{it}(-ie^{-it})dt = -\pi i.$$

The values of the integral between z_0 and z_1 now are different for the two different paths. ♦

The dependence on the path in the second example is related to the problem of antiderivatives. "The" antiderivative of $f(z)$ ought to be $\log z$. As we saw in Chapter 1, it is possible to define a branch of the logarithm function that is analytic along either one of the two curves, but it is *not* possible to define consistently a single branch of the logarithm on an open set containing both these curves at once. This way of looking at the difficulty is made precise in the next theorem.

Theorem 2.1.9 (Path Independence Theorem) *Suppose f is a continuous function on an open connected set $G \subset \mathbb{C}$. Then the following are equivalent:*

(i) *Integrals are path-independent: If z_0 and z_1 are any two points in G and γ_0 and γ_1 are paths in G from z_0 to z_1, then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

(ii) *Integrals around closed curves are 0: If Γ is a closed curve (loop) lying in G, then $\int_\Gamma f(z)dz = 0$.*

(iii) *There is a (global) antiderivative for f on G: There is a function F defined and analytic on all of G such that $F'(z) = f(z)$ for all z in G.*

Proof The equivalence of (i) and (ii) is obtained in the direction (ii) $\Rightarrow$ (i) by joining the curves γ_0 and $-\gamma_1$ to form a closed curve Γ and in the direction (i) $\Rightarrow$ (ii) by picking two points z_0 and z_1 along a closed curve and thinking of it as made up of two curves from one point to the other and then back to the first. The construction is illustrated in Figure 2.1.6, and the computation runs as follows:

$$\int_{\Gamma} f(z)dz = \int_{\gamma_0} f(z)dz + \int_{-\gamma_1} f(z)dz = \int_{\gamma_0} f(z)dz - \int_{\gamma_1} f(z)dz.$$

Thus, the integral along the closed loop Γ is 0 if and only if the integrals along the paths γ_0 and γ_1 are equal.

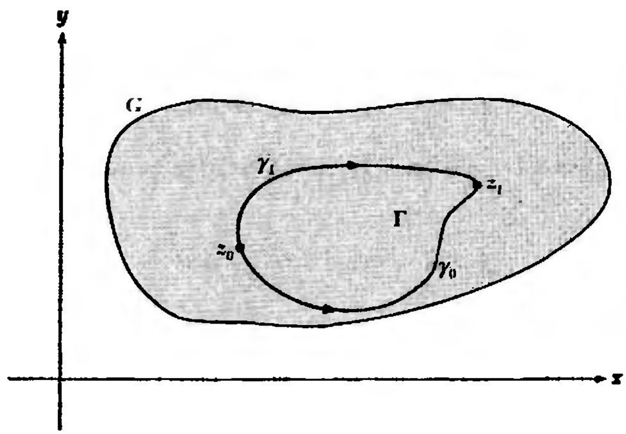

Figure 2.1.6: $\gamma_0 - \gamma_1 = \Gamma$.

The implication (iii) $\Rightarrow$ (i) follows from the Fundamental Theorem. The value of the integral is $F(z_1) - F(z_0)$ regardless of which path is selected.

To show that (i) $\Rightarrow$ (iii), we will attempt to use an integral ending at z to define the value of the antiderivative at z. Let z_0 be any point fixed in G, and let z be any other point in G. Since G is open and connected, it is path-connected, and by Proposition 1.4.15 there is at least one smooth path in G from z_0 to z. Let γ be any such path and set $F(z) = \int_\gamma f(\zeta)d\zeta$. This defines a function F on G in a nonambiguous way since (i) says that the value $F(z)$ depends only on z and *not* on the particular path selected so long as it stays in G. (Of course, it also depends on z_0, but that is fixed for the entire discussion.) We say that F is ***well defined.*** Our remaining task is to check that F is differentiable and that $F' = f$. This computation is illustrated in Figure 2.1.7.

Let $\epsilon > 0$. Since G is open and f is continuous at z, there is a number $\delta > 0$ such that the disk $D(z;\delta) \subset G$ and $|f(\zeta) - f(z)| < \epsilon$ whenever $|\zeta - z| < \delta$. Suppose $|w - z| < \delta$. Connect z to w by a straight-line segment ρ. Then all of ρ lies in

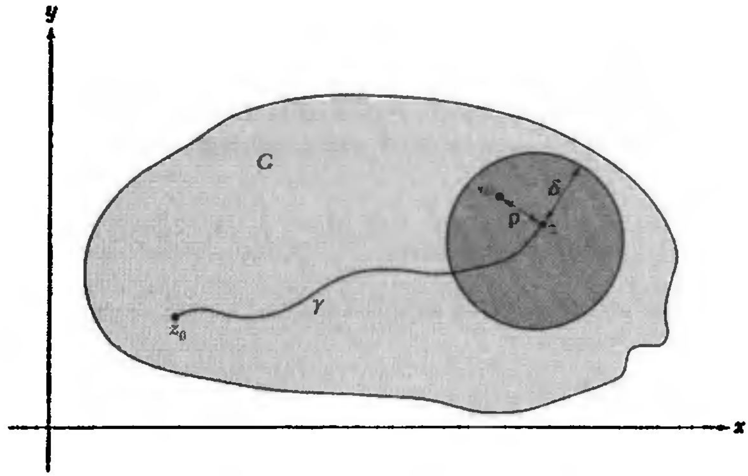

Figure 2.1.7: Defining an antiderivative by an integral.

$D(z;\delta)$ and

$$F(w) - F(z) = \int_{\gamma+\rho} f(\zeta)d\zeta - \int_{\gamma} f(\zeta)d\zeta = \int_{\rho} f(\zeta)d\zeta.$$

Thus,

$$\begin{aligned}
\left|\frac{F(w) - F(z)}{w - z} - f(z)\right| &= \frac{|F(w) - F(z) - (w - z)f(z)|}{|w - z|} \\
&= \frac{|\int_\rho f(\zeta)d\zeta - f(z)\int_\rho 1d\zeta|}{|w - z|} \\
&= \frac{|\int_\rho [f(\zeta) - f(z)]d\zeta|}{|w - z|} \\
&\le \frac{\epsilon \operatorname{length}(\rho)}{|w - z|} = \frac{\epsilon|w - z|}{|w - z|} = \epsilon.
\end{aligned}$$

Thus the limit of the difference quotient is $f(z)$, so F is differentiable and $F' = f$. as desired. ■

The reader who is familiar with conservative force fields from vector calculus may recognize the constructions in the last proof. The integral of a force field along a path defines the work done by it (or in moving against it) along that path. The field is called ***conservative*** if the net work done along a closed path is always 0 or equivalently if the work done between two points is independent of the path taken between those points. If it is, then such an integral defines a quantity, called the ***potential energy***, whose gradient is the original force field.

Worked Examples

Example 2.1.10 *Evaluate each of the following integrals:*

a) $\int_\gamma x\,dz$ *(γ is the circumference of the unit square)*

b) $\int_\gamma e^z dz$ *(γ is the part of the unit circle joining* 1 *to* i *in a counterclockwise direction)*

Solution To solve (a), define $\gamma : [0,4] \to \mathbf{C}$ as follows: $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ where the four sides of the unit square are

$$\gamma_1(t) = t + 0i; 0 \le t \le 1 \qquad \gamma_3(t) = (3-t) + i; 2 \le t \le 3$$
$$\gamma_2(t) = 1 + (t-1)i; 1 \le t \le 2 \quad \gamma_4(t) = 0 + (4-t)i; 3 \le t \le 4 .$$

We compute as follows:

$$\begin{aligned}
\int_{\gamma_1} x\,dz &= \int_0^1 [\mathrm{Re}(\gamma_1(t))]\gamma_1'(t)dt = \int_0^1 t\,dt = \frac{1}{2} \\
\int_{\gamma_2} x\,dz &= \int_1^2 [\mathrm{Re}(\gamma_2(t))]\gamma_2'(t)dt = \int_1^2 i\,dt = i \\
\int_{\gamma_3} x\,dz &= \int_2^3 [\mathrm{Re}(\gamma_3(t))]\gamma_3'(t)dt = \int_2^3 -(3-t)dt = -\frac{1}{2} \\
\int_{\gamma_4} x\,dz &= \int_3^4 [\mathrm{Re}(\gamma_4(t))]\gamma_4'(t)dt = \int_3^4 0\,dt = 0.
\end{aligned}$$

Hence,

$$\int_\gamma x\,dz = \int_{\gamma_1} x\,dz + \int_{\gamma_2} x\,dz + \int_{\gamma_3} x\,dz + \int_{\gamma_4} x\,dz = \frac{1}{2} + i - \frac{1}{2} + 0 = i.$$

To solve (b), note that e^z is the derivative of e^z, and e^z is analytic on all of $\mathbf{C}$. Thus, whatever parametrization we use for the part of the unit circle joining 1 to i in a counterclockwise direction, we will have $\int_\gamma e^z dz = e^i - e^1$ by the Fundamental Theorem. A second, less elegant solution is to use the original definition to evaluate the integral directly. Define $\gamma(t) = \cos t + i \sin t, 0 \le t \le \pi/2$. Hence,

$$\begin{aligned}
\int_\gamma e^z dz &= \int_0^{\pi/2} e^{\cos t + i \sin t}(-\sin t + i\cos t)dt \\
&= \int_0^{\pi/2} [-e^{\cos t}\cos(\sin t)\cdot \sin t - e^{\cos t}\sin(\sin t)\cdot \cos t]dt \\
&\quad + i\int_0^{\pi/2} [-e^{\cos t}\sin(\sin t)\cdot \sin t + e^{\cos t}\cos(\sin t)\cdot \cos t]dt \\
&= e^{\cos t}\cos(\sin t)\Big|_0^{\pi/2} + ie^{\cos t}\sin(\sin t)\Big|_0^{\pi/2} \\
&= e^{\cos t + i \sin t}\Big|_0^{\pi/2} = e^i - e^1 .
\end{aligned}$$

Example 2.1.11 *Let γ be the upper half of the unit circle described counterclockwise. Show that*

$$\left|\int_\gamma \frac{e^z}{z} dz\right| \le \pi e.$$

Solution We use Proposition 2.1.6. The arc length of γ is

$$l(\gamma) = \int_0^\pi |\gamma'(t)| dt = \pi$$

since we can take $\gamma(t) = e^{it}, 0 \le t \le \pi$, and $\gamma'(t) = ie^{it}$ and thus $|\gamma'(t)| = 1$. Of course, this is what we would expect. The absolute value of e^z/z, with $z = e^{it} = \cos t + i \sin t$, is estimated by

$$\left|\frac{e^z}{z}\right| = \frac{e^{\cos t}}{1} \le e$$

since $\cos t \le 1$. Thus $e = M$ is a bound for $|e^z/z|$ along γ, and therefore,

$$\left|\int_\gamma \frac{e^z}{z} dz\right| \le Ml(\gamma) = e\pi.$$

Example 2.1.12 *Let γ be the circle of radius r around $a \in \mathbb{C}$. Evaluate the integral*

$$\int_\gamma (z-a)^n dz$$

for all integers $n = 0, \pm 1, \pm 2, \ldots$.

Solution First, let $n \ge 0$. Then

$$(z-a)^n = \frac{d}{dz}\frac{1}{n+1}(z-a)^{n+1}$$

is the derivative of an analytic function, so by the Fundamental Theorem 2.1.7,

$$\int_\gamma (z-a)^n \cdot dz = 0.$$

Second, let $n \le -2$. Then again

$$(z-a)^n = \frac{d}{dz}\frac{1}{n+1}(z-a)^{n+1},$$

which is analytic on the set $A = \mathbb{C}\backslash\{a\}$. (Note that this formula fails if $n = -1$.) Since γ lies in A, the Fundamental Theorem again shows that $\int_\gamma (z-a)^n dz = 0$.

Finally, let $n = -1$. We proceed directly and parametrize γ by $\gamma(\theta) = re^{i\theta} + a, 0 \le \theta \le 2\pi$ (see Figure 2.1.8).

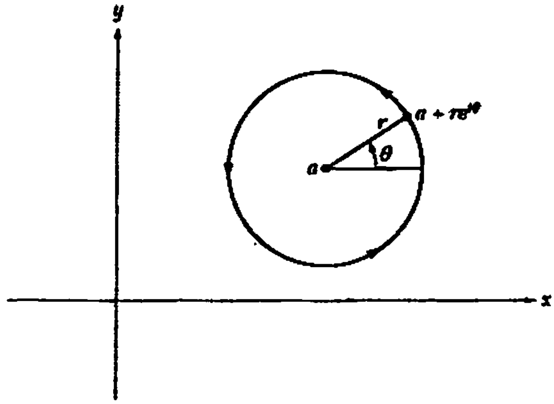

Figure 2.1.8: Parametrization of the circle of radius r and center a.

By the chain rule, $\gamma'(\theta) = rie^{i\theta}$, so

$$\int_\gamma \frac{1}{z-a}dz = \int_0^{2\pi} \frac{1}{(re^{i\theta}+a)-a} ire^{i\theta}d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i.$$

In summary,

$$\int_\gamma (z-a)^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}.$$

This is a useful formula and we shall have occasion to use it later.

Exercises

1. Evaluate the following:

   (a) $\int_\gamma y\,dz$, where γ is the union of the line segments joining 0 to i and then to $i+2$

   (b) $\int_\gamma \sin 2z\,dz$, where γ is the line segment joining $i+1$ to $-i$

   (c) $\int_\gamma ze^{z^2}dz$, where γ is the unit circle

2. Evaluate the following:

   (a) $\int_\gamma x\,dz$, where γ is the union of the line segments joining 0 to i and then to $i+2$

(b) $\int_\gamma (z^2 + 2z + 3)dz$, where γ is the straight-line segment joining 1 to $2+i$

(c) $\int_\gamma \frac{1}{z-1}dz$, where γ is the circle of radius 2 centered at 1 traveled once counterclockwise

3. Evaluate $\int_\gamma (1/z)dz$, where γ is the circle of radius 1 centered at 2 traveled once counterclockwise.

4. Evaluate $\int_\gamma \frac{1}{z^2 - 2z}dz$, where γ is the curve in Exercise 3.

5. Does $\text{Re}\left\{\int_\gamma f\,dz\right\} = \int_\gamma \text{Re}\, f\,dz$?

6. Prove Proposition 2.1.3.

7. Evaluate the following integrals:

   (a) $\int_\gamma \bar{z}\,dz$, where γ is the unit circle traversed once in a counterclockwise direction

   (b) $\int_\gamma (x^2 - y^2)dz$, where γ is the straight-line from 0 to i

8.* Evaluate $\int_\gamma \bar{z}^2 dz$ along two paths joining $(0,0)$ to $(1,1)$ as follows:

   (a) γ is the straight-line joining $(0,0)$ to $(1,1)$.

   (b) γ is the broken line joining $(0,0)$ to $(1,0)$, then joining $(1,0)$ to $(1,1)$.

   In view of your answers to (a) and (b) and the Fundamental Theorem, could $\bar{z}^2$ be the derivative of any analytic function $F(z)$?

9. Find a number M such that

$$\left|\int_\gamma \frac{dz}{2+z^2}\right| \leq M,$$

   where γ is the upper half of the unit circle.

10.* Let C be the arc of the circle $|z| = 2$ that lies in the first quadrant. Show that

$$\left|\int_C \frac{dz}{z^2+1}\right| \leq \frac{\pi}{3}.$$

11. Evaluate the following:

(a) $\displaystyle\int_{|z|=1} \frac{dz}{z}$; $\displaystyle\int_{|z|=1} \frac{dz}{|z|}$; $\displaystyle\int_{|z|=1} \frac{|dz|}{z}$; $\displaystyle\int_{|z|=1} \left|\frac{dz}{z}\right|$

(b) $\displaystyle\int_\gamma z^2 dz$, where γ is the curve given by $\gamma(t) = e^{it}\sin^3 t, 0 \leq t \leq \pi/2$.

12. Let γ be a closed curve lying entirely in the set $\mathbb{C}\backslash\{z \mid \operatorname{Re} z \leq 0\}$. Show that $\displaystyle\int_\gamma (1/z)dz = 0$.

13. Evaluate $\displaystyle\int_\gamma z \sin z^2 dz$, where γ is the unit circle.

14. Give some conditions on a closed curve γ that will guarantee that $\displaystyle\int_\gamma (1/z)dz = 0$.

15. Let γ be the unit circle. Prove that

$$\left|\int_\gamma \frac{\sin z}{z^2} dz\right| \leq 2\pi e.$$

16.* Show that the arc length $l(\gamma)$ of a curve γ is unchanged if γ is reparametrized.

2.2 Cauchy's Theorem—A First Look

One form of Cauchy's Theorem states that *if γ is a simple closed curve* (the word "simple" meaning that γ intersects itself only at its endpoints) *and if f is analytic on and inside γ, then*

$$\int_\gamma f = 0.$$

This remarkable theorem lies at the heart of complex analysis, and this section is devoted to its proof (see Figure 2.2.1).

If the function f is not analytic on the whole region inside γ, then the integral may or may not be 0. For example, let γ be the unit circle and $f(z) = 1/z$. Then f is analytic at all points except $z = 0$, and indeed the integral is *not zero*. In fact,

$$\int_\gamma f = 2\pi i$$

by Worked Example 2.1.12. On the other hand, if $f(z) = 1/z^2$, then f is still analytic at all points except $z = 0$, but now the integral *is* 0. This value of 0 results *not* from Cauchy's Theorem f is not analytic everywhere inside γ—but rather from the fact that f has an antiderivative on $\mathbb{C}\backslash\{0\}$ namely, f is the derivative of $-1/z$. More generally, the Path Independence Theorem 2.1.9, shows that Cauchy's Theorem is valid if there is an antiderivative of f. This is made explicit in Theorem 2.2.5.

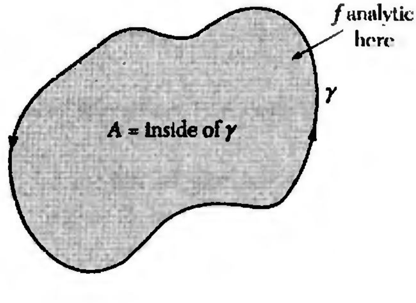

Figure 2.2.1: Cauchy's Theorem: $\int_\gamma f = 0$.

Green's Theorem Our proof of Cauchy's Theorem uses Green's Theorem from vector calculus, which states that, for continuously differentiable functions $P(x,y)$ and $Q(x,y)$,

$$\int_\gamma P(x,y)dx + Q(x,y)dy = \iint_A \left[\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\right] dx\,dy.$$

Recall that if $\gamma : [a,b] \to \mathbb{C}$, $\gamma(t) = (x(t), y(t))$, then we define the line integrals by

$$\int_\gamma P(x,y)dx = \int_a^b P(x(t),y(t))\, x'(t)dt$$

and

$$\int_\gamma Q(x,y)dy = \int_a^b Q(x(t),y(t))\, y'(t)dt.$$

In Green's Theorem, A represents the "inside" of γ, γ is traversed in a counterclockwise direction, and P and Q are sufficiently smooth— class C^1 is sufficient, but we shall take a closer look at this later.

Green's Theorem is a fundamental result from multivariable calculus that every student should know. Recall that the basic idea of the proof is really quite simple: We use the technique of evaluation of multiple integrals by iterated integration (which in turn is related to equality of mixed partial derivatives) and the Fundamental Theorem of Calculus.[2]

Part of the job of developing a context for Green's Theorem is to make precise notions like the *inside* of γ. Intuitively, the meaning of "inside" should be clear. We shall come back to issues like this as we proceed.

---

[2]See a calculus text such as J. Marsden and A. Weinstein, *Calculus III* (New York: Springer-Verlag, 1985), 908–911, or J. Marsden and A. Tromba, *Vector Calculus*, Fourth Edition (New York: W. H. Freeman and Company, 1996), §8.1, for a proof of Green's Theorem.

Preliminary Version of Cauchy's Theorem The following statement is preliminary in the sense that it assumes that a satisfactory context for Green's Theorem has been developed. We will come back to this point and make things more precise in due course.

Theorem 2.2.1 *Suppose that f is analytic, with the derivative f' continuous on and inside a simple closed curve γ. Then*

$$\int_\gamma f = 0.$$

Proof Setting $f = u + iv$, we have

$$\begin{aligned}\int_\gamma f &= \int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx + i\,dy)\\ &= \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx).\end{aligned}$$

By applying Green's Theorem to each integral, we get

$$\int_\gamma f = \iint_A \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dx\,dy + i\iint_A \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dx\,dy.$$

Both terms are zero by the Cauchy-Riemann equations. ∎

A closer look at the proof of Cauchy's Theorem given in §2.3 shows that one need not assume that f' is continuous. Amazingly, the continuity of f' follows automatically, but this is not obvious.

We also need to do some additional chores, such as eliminating the assumption that the curve is simple. In many cases, the assumption of simplicity of the curve can be avoided by viewing the path as being made up of two or more simple pieces. In Figure 2.2.2, the "figure eight" can be treated as two simple loops. We will discuss this in §2.3 as well.

Here is a simple example of Cauchy's Theorem. Let γ be the unit square and $f(z) = \sin(e^{z^2})$. Then f is analytic on and inside γ (in fact, f is entire), so $\int_\gamma f = 0$.

Deformation Theorem It is important to be able to study functions that are not analytic on the entire inside of γ and whose integral therefore might not be zero. For example, $f(z) = 1/z$ fails to be analytic at $z = 0$, and $\int_\gamma f = 2\pi i$ where γ is the unit circle. (The point $z = 0$ is called a ***singularity*** of f.) To study such functions it is important to be able to replace $\int_\gamma f$ by $\int_{\tilde\gamma} f$, where $\tilde\gamma$ is a less complicated curve (say, a circle). The strategy is that $\int_{\tilde\gamma} f$ might be easier to evaluate. The procedure that allows us to pass from γ to $\tilde\gamma$ is based on Cauchy's Theorem and is as follows.

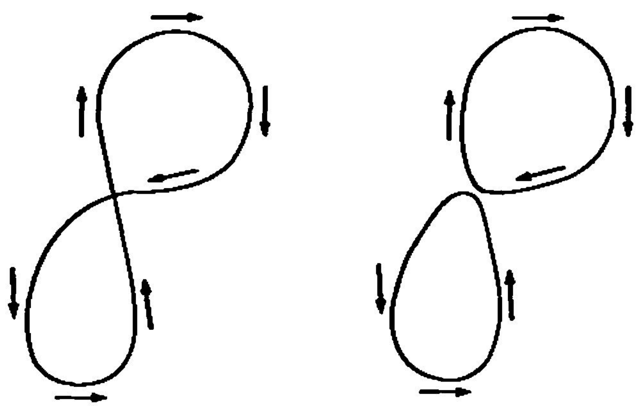

Figure 2.2.2: Treating a nonsimple curve as made up of several simple loops.

Theorem 2.2.2 (Preliminary Version of the Deformation Theorem) *Let f be analytic on a region A and let γ be a simple closed curve in A. Suppose that γ can be continuously deformed to another simple closed curve $\tilde{\gamma}$ without passing outside the region A. (We say that γ is homotopic to $\tilde{\gamma}$ in A.) (The precise definition of "homotopic" is given in §2.3, and the assumption the curves are simple will be eliminated.) Then*

$$\int_\gamma f = \int_{\tilde{\gamma}} f.$$

The Deformation Theorem is illustrated in Figure 2.2.3.

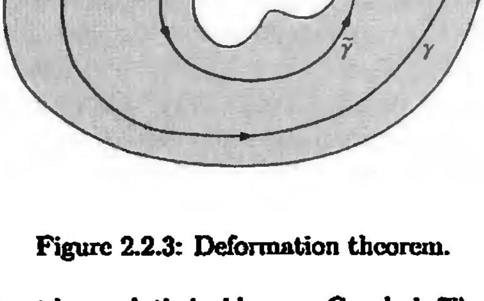

Figure 2.2.3: Deformation theorem.

Note that f need not be analytic inside γ, so Cauchy's Theorem does *not* imply that the integrals in the preceding equation are zero. It is implicit in the statement of the Deformation Theorem that both γ and $\tilde{\gamma}$ are traversed in a counterclockwise direction. As with Cauchy's Theorem itself, we will come back to this theorem in the next section and have a closer, more careful look at the notion of "deformation" as well as a more careful look at the following proof.

Proof Consider Figure 2.2.4, in which a curve γ_0 is drawn joining γ to $\tilde{\gamma}$; we assume such a curve can in fact be drawn (it can in all "practical" examples). We get a new curve consisting of γ, then γ_0, then $-\tilde{\gamma}$, and then $-\gamma_0$, in that order.

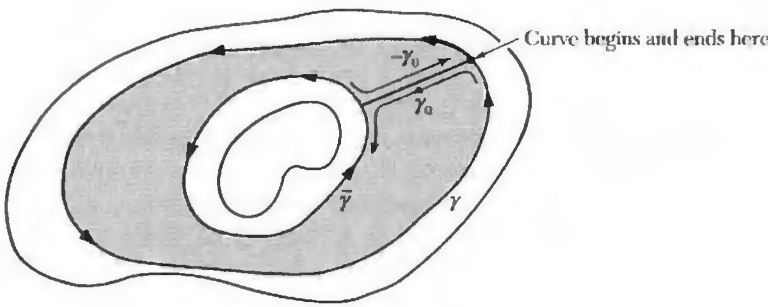

Figure 2.2.4: Curve used to prove the Deformation Theorem.

The inside of this curve is the shaded region in Figure 2.2.4. The function f is, by assumption, analytic on this region, so Cauchy's Theorem gives

$$0 = \int_{\gamma+\gamma_0-\tilde{\gamma}-\gamma_0} f = \int_\gamma f + \int_{\gamma_0} f - \int_{\tilde{\gamma}} f - \int_{\gamma_0} f = \int_\gamma f - \int_{\tilde{\gamma}} f.$$

Thus, $\int_\gamma f = \int_{\tilde{\gamma}} f$ as required. Strictly speaking, this new curve is not a simple closed curve, but such an objection can be taken care of by drawing two parallel copies of γ_0 and taking the limit as these copies converge together. ■

Simply Connected Regions A region $A \subset \mathbb{C}$ is called ***simply connected*** if A is connected and every closed curve γ in A can be deformed in A to some constant curve $\tilde{\gamma}(t) = z_0 \in A$: we also say that γ is ***homotopic to a point*** or is ***contractible to a point***. Intuitively, a region is simply connected when it has no holes; this is because a curve that loops around a hole cannot be shrunk down to a point in A without leaving A (see Figure 2.2.5). Therefore, the domain on which a function like $f(z) = 1/z$, which has a singularity, is analytic is *not* simply connected.

We can rewrite Cauchy's Theorem in terms of simply connected regions as follows.

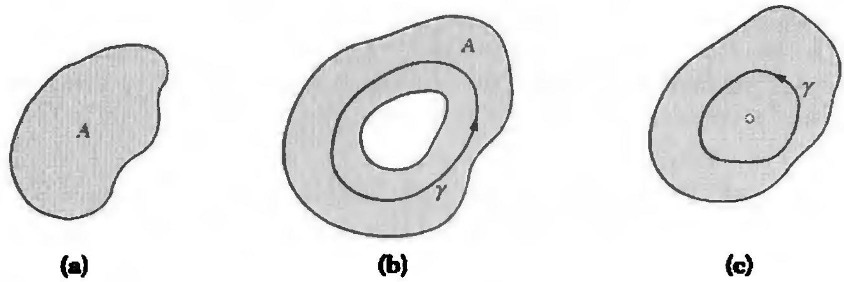

Figure 2.2.5: (a) Simply connected region. (b). (c) Non-simply connected regions.

Theorem 2.2.3 (Cauchy's Theorem for a Simply Connected Region) *If f is analytic on a simply connected region G and γ is a closed curve in G, then*

$$\int_\gamma f = 0.$$

Independence of Path and Antiderivatives In the Path Independence Theorem 2.1.9 we saw how to relate the vanishing of integrals along closed curves to path independence of integrals between points and to the existence of antiderivatives on regions. We can exploit these ideas in the present context.

Proposition 2.2.4 *Suppose that f is analytic on a simply connected region A. Then for any two curves γ_1 and γ_2 joining two points z_0 and z_1 in A (as in Figure 2.2.6), we have*

$$\int_{\gamma_1} f = \int_{\gamma_2} f.$$

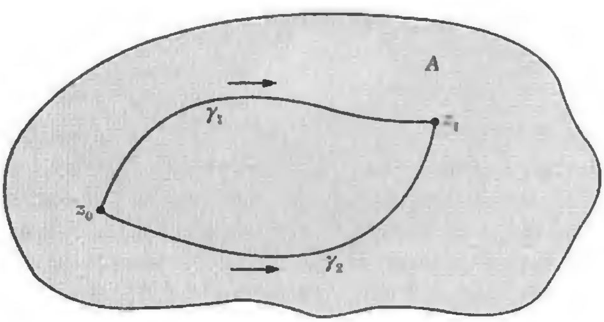

Figure 2.2.6: Independence of path.

Proof[3] Consider the closed curve $\gamma = \gamma_2 - \gamma_1$. By Cauchy's Theorem,

$$0 = \int_\gamma f = \int_{\gamma_2} f - \int_{\gamma_1} f,$$

and therefore,

$$\int_{\gamma_2} f = \int_{\gamma_1} f,$$

as required. ■

Just as in Theorem 2.1.9, we also get the existence of an antiderivative for f on the region.

Theorem 2.2.5 (Antiderivative Theorem) *Let f be defined and analytic on a simply connected region A. Then there is an analytic function F defined on A that is unique up to an additive constant, such that $F'(z) = f(z)$ for all z in A. We call F the* ***antiderivative*** *of f on A.*

Proof The existence of the antiderivative follows from the Path Independence Theorem 2.1.9. (Strictly speaking, we should get rid of the assumption of simple curves first.) The uniqueness assertion means that if F_0 is any other such function, then $F_0(z) = F(z) + C$ for some constant C. This follows because the region A is connected and

$$(F_0 - F)'(z) = F_0'(z) - F'(z) = f(z) - f(z) = 0$$

for all z in A. Thus, $F_0 - F$ is constant on A by Corollary 2.1.8. ■

More on the Logarithm If A is not simply connected, the conclusions of the preceding proposition need not hold. For example, if $A = \mathbb{C}\backslash\{0\}$ and $f(z) = 1/z$, there is no F defined on all of A with $F' = f$ since the integral of f around the unit circle is not zero (see Example 2.1.12). In some sense, F ought to be the logarithm, but we cannot define this in a consistent way on all of A. However, on any simply connected region not containing 0 we can find such an F as the following proposition shows.

Proposition 2.2.6 (Existence of Logarithms) *Let A be a simply connected region and assume that $0 \notin A$. Then there is an analytic function $F(z)$, unique up to the addition of multiples of $2\pi i$, such that $e^{F(z)} = z$.*

---

[3] In this proof we assume that γ is a *simple* closed curve. We will show in the next section that this is not necessary.

Proof By the Antiderivative Theorem 2.2.5, there is an analytic function F with $F'(z) = 1/z$ on A. Fix a point $z_0 \in A$. Then z_0 lies in the domain of some branch of the log function defined in §1.6. If we adjust F by adding a constant so that $F(z_0) = \log z_0$, then at $z_0, e^{F(z_0)} = z_0$. We now want to show that $e^{F(z)} = z$ is true on all of A. To do this, we let $g(z) = e^{F(z)}/z$. Then, since $0 \notin A, g$ is analytic on A, and since $F'(z) = 1/z$,

$$g'(z) = \frac{z \cdot \frac{1}{z} \cdot e^{F(z)} - 1 \cdot e^{F(z)}}{z^2} = 0.$$

Thus g is constant on A. But $g = 1$ at z_0, so g is 1 on all of A. Therefore, $e^{F(z)} = z$ on all of A.

For uniqueness, let F and G be functions analytic on A and suppose that $e^{F(z)} = z$ and $e^{G(z)} = z$. Then $e^{F(z)-G(z)} = 1$, so at a fixed $z_0, F(z_0) - G(z_0) = 2\pi ni$ for some integer n. But $F'(z) = 1/z = G'(z)$, so we have $d(F - G)/dz = 0$, from which we conclude (from the fact that a function with zero derivative on a connected region is a constant) that $F - G = 2\pi ni$ on all of A. ■

We write $F(z) = \log z$ and call such a choice of F a ***branch of the logarithm function*** on A. Clearly, this procedure generalizes that in §1.6 and we get the usual log as defined in that section if A is $\mathbb{C}$ minus 0 and the negative real axis. Note that this A is simply connected. However, the A in this proposition can be more complicated, as depicted in Figure 2.2.7.

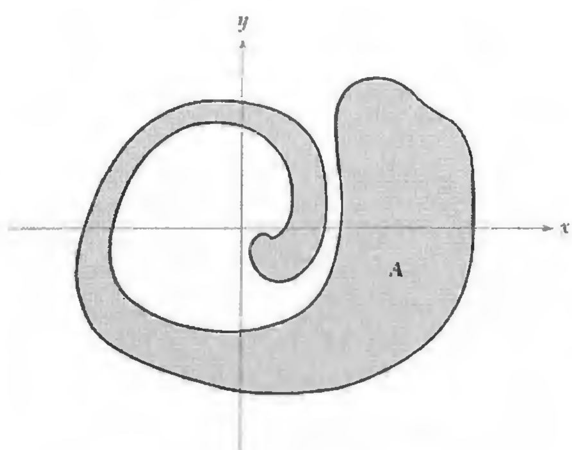

Figure 2.2.7: A possible domain for the log function.

Worked Examples

Example 2.2.7 *Evaluate the following integrals:*

(a) $\int_\gamma e^z dz$, *where γ is the perimeter of the unit square*

(b) $\int_\gamma 1/z^2 dz$, *where γ is the unit circle*

(c) $\int_\gamma 1/z\, dz$, *where γ is the circle $3 + e^{i\theta}, 0 \le \theta \le 2\pi$*

(d) $\int_\gamma z^2 dz$, *where γ is the segment joining $1 + i$ to 2*

Solution To solve (a), notice that e^z is entire; thus, by Cauchy's Theorem, $\int_\gamma e^z dz = 0$, since γ is a simple closed curve. Alternatively, e^z is the derivative of e^z, and since γ is closed we can apply the Path Independence Theorem 2.1.9.

To evaluate the integral in (b), note that $1/z^2$ is defined and analytic on $\mathbb{C}\backslash\{0\}$ and is the derivative of $-1/z$, which is defined and analytic on $\mathbb{C}\backslash\{0\}$. By the Path Independence Theorem 2.1.9 and the fact that the unit circle lies in $\mathbb{C}\backslash\{0\}$, we have $\int_\gamma (1/z^2)dz = 0$. Alternatively, we can use Worked Example 2.1.12 for our solution.

Next, we solve (c). The circle $\gamma = 3 + e^{i\theta}, 0 \le \theta \le 2\pi$, does not pass through or include 0 in its interior. Hence $1/z$ is analytic on γ and the interior of γ, so by Cauchy's Theorem, $\int_\gamma (1/z)dz = 0$. An alternative but less direct solution is the following. The region $\{x+iy \mid x > 0\}$ is simply connected and $1/z$ is analytic on it. Therefore, by Proposition 2.2.6, $1/z$ is the derivative of some analytic function $F(z)$ (one of the branches of $\log z$) and thus, since γ is closed, the Path Independence Theorem gives $\int_\gamma (1/z)dz = 0$.

Finally, to evaluate the integral in (d), note that z^2 is entire and is the derivative of $z^3/3$, which is also entire. By Theorem 2.1.7,

$$\int_\gamma z^2 dz = \frac{1}{3}z^3\bigg|_{1+i}^{2} = \frac{(2)^3}{3} - \frac{(1+i)^3}{3} = \frac{10}{3} - \frac{2i}{3}.$$

Example 2.2.8 *Use the Deformation Theorem to argue informally that if γ is a simple closed curve (not necessarily a circle) containing 0, then*

$$\int_\gamma \frac{1}{z} dz = 2\pi i.$$

Solution The inside of γ contains 0, so we can find an $r > 0$ such that the circle $\tilde{\gamma}$ of radius r and centered at 0 lies entirely inside γ. Our intuition tells us that we can deform γ to $\tilde{\gamma}$ without passing through 0 (that is, by staying in the region $A = \mathbb{C}\backslash\{0\}$ of analyticity of $1/z$; see Figure 2.2.8). Therefore, the Deformation Theorem and the calculation in Worked Example 2.1.12 give the required answer:

$$\int_\gamma \frac{1}{z} dz = \int_{\tilde{\gamma}} \frac{1}{z} dz = 2\pi i.$$

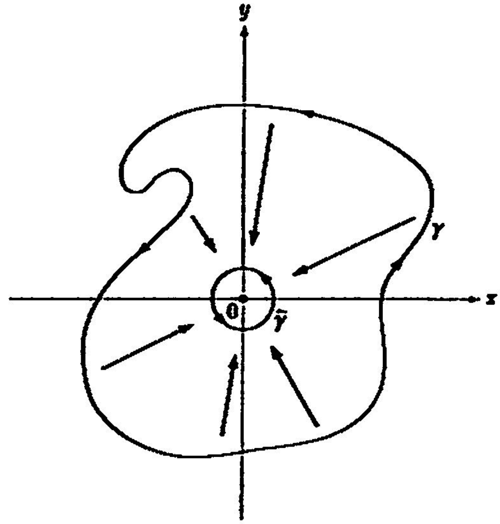

Figure 2.2.8: Deformation of γ to the circle $\tilde{\gamma}$.

Example 2.2.9 *Outline a proof of the following extension of the Deformation Theorem: Suppose that $\gamma_1, \ldots, \gamma_n$ are nonoverlapping simple closed curves and that γ is a simple closed curve with f analytic on the region between γ and $\gamma_1, \ldots, \gamma_n$ (see Figure 2.2.9). Then*

$$\int_\gamma f = \sum_{k=1}^{n} \int_{\gamma_k} f.$$

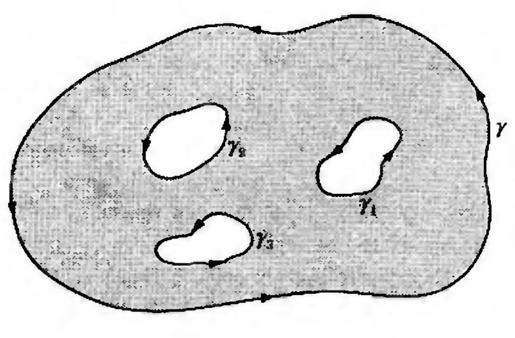

Figure 2.2.9: Generalized Deformation Theorem.

Solution Draw curves $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n$ joining γ to $\gamma_1, \ldots, \gamma_n$, respectively, as shown in Figure 2.2.10(left). Let ρ denote the curve drawn in Figure 2.2.10(right). The inside of ρ is a region of analyticity of f, so $\int_\rho f = 0$. But ρ consists of $\gamma, -\gamma_1, -\gamma_2, \ldots, -\gamma_n$, and each $\tilde{\gamma}_i$ traversed twice in opposite directions, so the contributions from these last portions cancel. Thus,

$$0 = \int_\gamma f + \int_{-\gamma_1} f + \cdots + \int_{-\gamma_n} f = \int_\gamma f - \sum_{i=1}^{n} \int_{\gamma_i} f$$

as required.

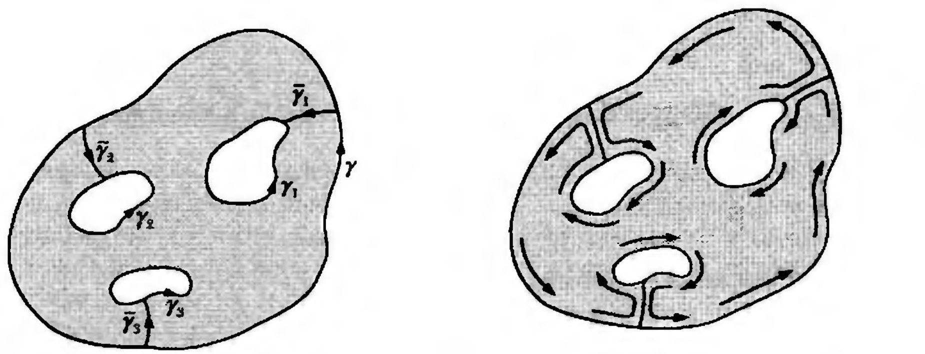

Figure 2.2.10: Path used to prove the Generalized Deformation Theorem.

Example 2.2.10 *Let $f(z)$ be analytic on a simply connected region A, except possibly not analytic at $z_0 \in A$. Suppose, however, that f is bounded in absolute value near z_0. Show that, for any simple closed curve γ containing z_0, $\int_\gamma f = 0$.*

Solution Let $\epsilon > 0$ and small enough so that the circle γ_ϵ of radius ϵ and center z_0 lies inside γ. By the Deformation Theorem,

$$\int_\gamma f = \int_{\gamma_\epsilon} f.$$

By assumption there is a constant M with $|f(z)| < M$ near z_0. Thus, if ϵ is small enough to make this estimate valid, then

$$\left|\int_\gamma f\right| = \left|\int_{\gamma_\epsilon} f\right| \le 2\pi\epsilon M.$$

This holds for all small enough positive ϵ. Letting ϵ approach 0, we conclude that $\int_\gamma f = 0$.

Exercises

1. Evaluate the following integrals:

   (a) $\int_\gamma (z^3+3)dz$, where γ is the upper half of the unit circle

   (b) $\int_\gamma (z^3+3)dz$, where γ is the unit circle

   (c) $\int_\gamma e^{1/z}dz$, where γ is a circle of radius 3 centered at $5i+1$

   (d) $\int_\gamma \cos[3+1/(z-3)]dz$, where γ is a unit square with corners at $0, 1, 1+i$, and i

2. Let γ be a simple closed curve containing 0. Argue informally that

$$\int_\gamma \frac{1}{z^2}dz = 0.$$

3. Let f be entire. Evaluate

$$\int_0^{2\pi} f(z_0 + re^{i\theta})e^{ki\theta}d\theta$$

   for k an integer, $k \geq 1$.

4.* Discuss the validity of the formula $\log z = \log r + i\theta$ for log on the region A shown in Figure 2.2.7.

5. For what simple closed curves γ does the equation

$$\int_\gamma \frac{dz}{z^2+z+1} = 0$$

   hold?

6. Evaluate $\int_\gamma (z - (1/z))dz$, where γ is the straight-line path from 1 to i.

7. Does Cauchy's Theorem hold separately for the real and imaginary parts of f? If so, prove that it does; if not, give a counterexample.

8.* Let γ_1 be the circle of radius 1 and let γ_2 be the circle of radius 2 (traversed counterclockwise and centered at the origin). Show that

$$\int_{\gamma_1} \frac{dz}{z^3(z^2+10)} = \int_{\gamma_2} \frac{dz}{z^3(z^2+10)}.$$

9. Evaluate $\int_\gamma \sqrt{z}\,dz$, where γ is the upper half of the unit circle: first, directly, and second, using the Fundamental Theorem 2.1.7.

10.* Evaluate $\int_\gamma \sqrt{z^2-1}\,dz$, where γ is a circle of radius $\frac{1}{2}$ centered at 0.

11. Evaluate

$$\int_\gamma \frac{2z^2-15z+30}{z^3-10z^2+32z-32}dz,$$

where γ is the circle $|z|=3$. *Hint*: Use partial fractions; one root of the denominator is $z=2$.

2.3 A Closer Look at Cauchy's Theorem

In this section we take another look at some of the issues that were treated informally in the previous section. The strategy is to start by carefully examining Cauchy's Theorem for a rectangle and then to use the theorem in this special case, together with subdivision arguments, to build up to more general regions in a systematic way.

Recall that the basic theme of Cauchy's Theorem is that *if a function is analytic everywhere inside a closed contour, then its integral around that contour must be* 0. The principal goal of this section is to give a proof of a form of the theorem known as a ***homotopy version of Cauchy's Theorem***. This approach extends and sharpens the idea presented in the preceding section of the continuous deformation of a curve. The primary objective will be the precise formulation and proof of deformation theorems which say, roughly, that if a curve is continuously deformed through a region in which a function is analytic, then the integral along the curve does not change. The reader will also notice that in this section references are made not to "simple closed curves" but only to "closed curves."

Cauchy's Theorem for a Rectangle We begin with a careful statement of Cauchy's Theorem in this case.

Theorem 2.3.1 (Cauchy's Theorem for a Rectangle) *Suppose R is a rectangular path with sides parallel to the axes and that f is a function defined and analytic on an open set G containing R and its interior. Then $\int_R f=0$.*

There are several methods to prove Cauchy's Theorem for a rectangle. One way, which fits the spirit of the previous section, is to prove a strong version of Green's Theorem for rectangles[4]. Another technique, the one that we follow, is a bisection technique due to Édouard Goursat in 1884. It was Goursat[5] who first noticed that

[4] F. Acker, The missing link, *Mathematical Intelligencer*, 18 (1996), 4–9.

[5] *Acta Mathematica*, 4 (1884), 197–200 and *Transactions of the American Mathematical Society*, 1 (1900), 14–16.

one does not need to assume that the derivative of f is continuous. Surprisingly, this follows automatically, which is a rather different situation than that for real functions of several variables.

Besides these techniques, there have been many other proofs of Cauchy's Theorem. For example, Pringsheim[6] uses triangles rather than rectangles, which has some advantages. Cauchy's original proof (for which the assumptions of continuity of the derivative were not made clear), had the content of Green's Theorem implicit in the argument—in fact Green did not formulate Green's Theorem as such until about 1830, whereas Cauchy presented his theorem in 1825.[7] There are also interesting proofs based on "homology" given by Ahlfors.[8]

Local Version of Cauchy's Theorem Before proving Cauchy's Theorem for a rectangle, we indicate how it can already be used to prove a limited but still important and more general case of Cauchy's Theorem.

Theorem 2.3.2 (Cauchy's Theorem for a Disk) *Suppose that $f : D \to \mathbf{C}$ is analytic on a disk $D = D(z_0; \rho) \subset \mathbf{C}$. Then*

(i) *f has an antiderivative on D: that is, there is a function $F : D \to \mathbf{C}$ that is analytic on D and that satisfies $F'(z) = f(z)$ for all z in D.*

(ii) *If Γ is any closed curve in D, then $\int_\Gamma f = 0$.*

From the discussion in §2.1 on the path independence of integrals (see Theorem 2.1.9), we know that (i) and (ii) are equivalent in the sense that whichever we establish first, the other will follow readily from it. Our problem is how to obtain either one of them. In the proof of the Path Independence Theorem 2.1.9, it was shown that (ii) follows easily from (i), and the construction of an antiderivative to get (i) was facilitated by the path independence of integrals. The strategy for proceeding is quite interesting.

1. Prove (ii) directly for the very special case in which Γ is the boundary of a rectangle.

2. Show that this limited version of path independence is enough to carry out a construction of an antiderivative similar to that in the proof of the Path Independence Theorem.

3. With (i) thus established, part (ii) in its full generality follows as in the Path Independence Theorem.

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[6] *Transactions of the American Mathematical Society*, 2 (1902)

[7] In his *Mémoire sur les intégrales définies prises entre des limites imaginaires*.

[8] L. Ahlfors, *Complex Analysis*. Second Edition (New York: McGraw-Hill, 1966).

'roof of Cauchy's Theorem for a Rectangle A subtle technical point worth ·peating: Care must be taken because we do not know in advance that the deriva- ·ve of f is continuous. In fact, we will use Cauchy's Theorem itself to eventually rove that f' is *automatically continuous*. Now let's get down to the proof.

Let P be the perimeter of R and Δ the length of its diagonal. Divide the ·ctangle R into four congruent smaller rectangles $R^{(1)}, R^{(2)}, R^{(3)}$, and $R^{(4)}$. If ıch is oriented in the counterclockwise direction, then cancellation along common lges leaves

$$\int_R f = \int_{R^{(1)}} f + \int_{R^{(2)}} f + \int_{R^{(3)}} f + \int_{R^{(4)}} f.$$

ince

$$\left|\int_R f\right| \leq \left|\int_{R^{(1)}} f\right| + \left|\int_{R^{(2)}} f\right| + \left|\int_{R^{(3)}} f\right| + \left|\int_{R^{(4)}} f\right|,$$

ıere must be at least one of the rectangles for which $\left|\int_{R^{(i)}} f\right| \geq \frac{1}{4}\left|\int_R f\right|$. Call this ıbrectangle R_1. Notice that the perimeter and diagonal of R_1 are half those of R Figure 2.3.1).

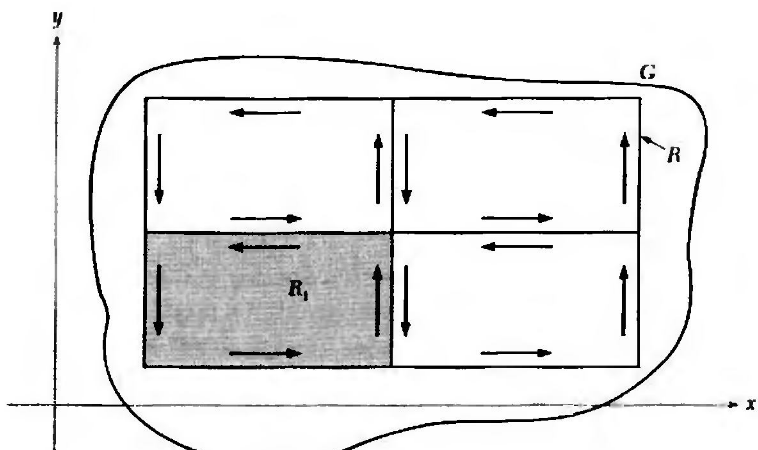


Figure 2.3.1: Bisection procedure.

Now repeat this bisection process, obtaining a sequence $R_1, R_2, R_3, \ldots$ of smaller ınd smaller rectangles that have the following properties:

(i) $\left|\int_{R_n} f\right| \geq \frac{1}{4}\left|\int_{R_{n-1}} f\right| \geq \cdots \geq \frac{1}{4^n}\left|\int_R f\right|$

(ii) Perimeter$(R_n) = \frac{1}{2^n}$ perimeter$(R) = \frac{P}{2^n}$

(iii) $\text{Diagonal}(R_n) = \frac{1}{2^n}\,\text{diagonal}(R) = \frac{\Delta}{2^n}$ (see Figure 2.3.2)

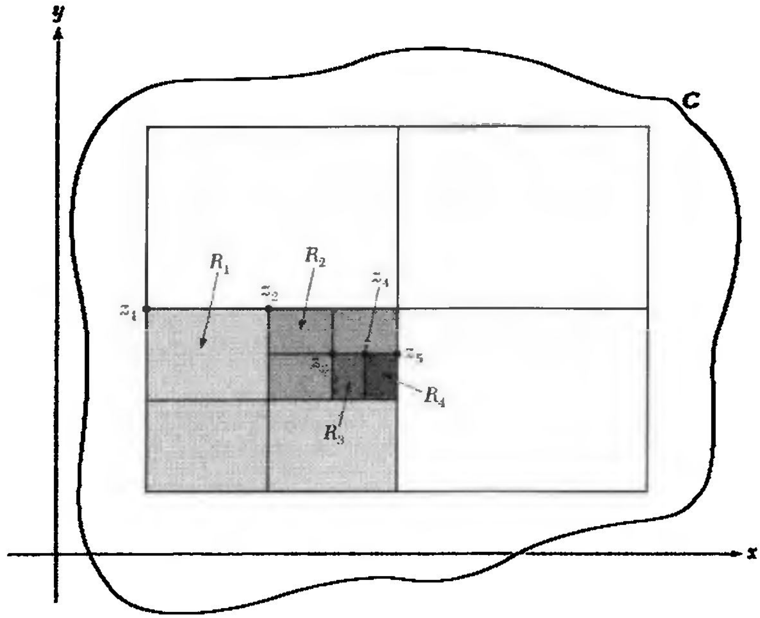


Figure 2.3.2: Goursat's repeated bisection process for the proof of Cauchy's Theorem for a rectangle.

Since these rectangles are nested one within another and have diagonals tending to 0, they must shrink down to a single point w_0. To be precise, let z_n be the upper left-hand corner of R_n. If $m > n$, then $|z_n - z_m| \le$ diagonal $(R_n) = \Delta/2^n$, and thus $\{z_n\}$ forms a Cauchy sequence that must converge to some point w_0. If z is any point on the rectangle R_n, then since all z_k with $k \ge n$ are within R_n, z can be no farther from w_0 than the length of the diagonal of R_n. That is, $|z - w_0| \le \Delta/2^n$ for z in R_n.

From (i) we see that $|\int_R f| \le 4^n |\int_{R_n} f|$. To obtain a sufficiently good estimate on the right side of this inequality, we use the differentiability of f at the point w_0.

For $\epsilon > 0$, there is a number $\delta > 0$ such that

$$\left|\frac{f(z) - f(w_0)}{z - w_0} - f'(w_0)\right| < \epsilon$$

whenever $|z - w_0| < \delta$. If we choose n large enough that $\Delta/2^n$ is less than δ, then

$$|f(z) - f(w_0) - (z - w_0)f'(w_0)| < \epsilon|z - w_0| \le \epsilon\frac{\Delta}{2^n}$$

for all points z on the rectangle R_n. Furthermore, by the Path Independence Theorem 2.1.9,

$$\int_{R_n} 1dz = 0 \quad \text{and} \quad \int_{R_n} (z-w_0)dz = 0.$$

Since z is an antiderivative for $1, (z-w_0)^2/2$ is an antiderivative for $(z-w_0)$, and the path R_n is closed. Thus,

$$\begin{aligned}
\left|\int_R f\right| &\le 4^n\left|\int_{R_n} f\right| \\
&= 4^n\left|\int_{R_n} f(z)dz - f(w_0)\int_{R_n} 1dz - f'(w_0)\int_{R_n}(z-w_0)dz\right| \\
&\le 4^n\left|\int_{R_n}[f(z)-f(w_0)-(z-w_0)f'(w_0)]dz\right| \\
&\le 4^n\int_{R_n}|f(z)-f(w_0)-(z-w_0)f'(w_0)|\,|dz| \\
&\le 4^n\left(\frac{\epsilon\Delta}{2^n}\right)\cdot \text{perimeter}\,(R_n) \\
&\le \epsilon\Delta P.
\end{aligned}$$

Since this is true for every $\epsilon > 0$, we must have $|\int_R f| = 0$ and so $\int_R f = 0$, as desired. ■

Back to Cauchy's Theorem on a Disk For most of the rest of this section, "curve" means "piecewise C^1 curve." However, at one point in the technical development it will become important to drop this piecewise C^1 restriction and consider continuous curves.[9]

We can now carry out the second step of the proof of Cauchy's Theorem for a disk (Theorem 2.3.2). Since the function f is analytic on the disk $D = D(z_0;\rho)$, the result for a rectangle just proved shows that the integral of f is 0 around any rectangle in D. This is enough to carry out a construction of an antiderivative for f very much like that done in the proof of the Path Independence Theorem 2.1.9 and thus to establish part (i) of the theorem.

We will again define the antiderivative $F(z)$ as an integral from z to z_0. However, we do not yet know that such an integral is path independent. Instead we will specify a particular choice of path and use the new information available— the analyticity of f and the geometry of the situation together with the rectangular case of Cauchy's Theorem —to show that we get an antiderivative. For the duration of this proof we will use the notation $\langle\langle a,b\rangle\rangle$ to denote the polygonal path proceeding from a point a to a point b in two segments, first parallel to the x axis, then parallel to the y axis, as in Figure 2.3.3.

[9]The technical treatment of integration over continuous curves is given in the Internet Supplement.

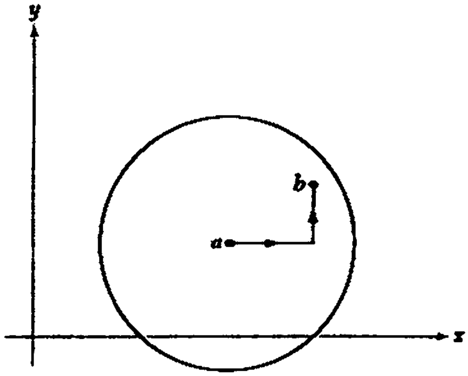


Figure 2.3.3: The path $\langle\!\langle a, b\rangle\!\rangle$.

If the point b is in a disk $D(a;\delta)$ centered at a, then the path $\langle\!\langle a, b\rangle\!\rangle$ is contained in that disk. Thus, for $z \in D$, we may define a function $F(z)$ by

$$F(z) = \int_{\langle\!\langle z_0, z\rangle\!\rangle} f(\xi)d\xi.$$

We want to show that $F'(z) = f(z)$. To do this we need to show that

$$\lim_{w \to z} = \frac{F(w) - F(z)}{w - z} = f(z).$$

Fixing $z \in D$ and $\epsilon > 0$, we use the fact that D is open and f is continuous on D to choose $\delta > 0$ small enough that $D(z;\delta) \subset D$ and $|f(z) - f(\xi)| < \epsilon$ for $\xi \in D(z;\delta)$. If $w \in D(z;\delta)$, then the path $\langle\!\langle z, w\rangle\!\rangle$ is contained in $D(z;\delta)$ and hence in D. The paths $\langle\!\langle z_0, z\rangle\!\rangle$ and $\langle\!\langle z_0, w\rangle\!\rangle$ are also contained in D, and these three paths fit together in a nice way with a rectangular path R also contained in D and having one corner at z; see Figure 2.3.4. We can write, for the two cases in Figure 2.3.4,

$$\int_{\langle\!\langle z_0, z\rangle\!\rangle} f(\xi)d\xi \pm \int_R f(\xi)d\xi + \int_{\langle\!\langle z, w\rangle\!\rangle} f(\xi)d\xi = \int_{\langle\!\langle z_0, w\rangle\!\rangle} f(\xi)d\xi.$$

By the Cauchy theorem for a rectangle, $\int_R f(\xi)d\xi = 0$, so the preceding equation becomes

$$F(z) + \int_{\langle\!\langle z, w\rangle\!\rangle} f(\xi)d\xi = F(w).$$

Neither side of the right triangle defined by $\langle\!\langle z, w\rangle\!\rangle$ can be any longer than its

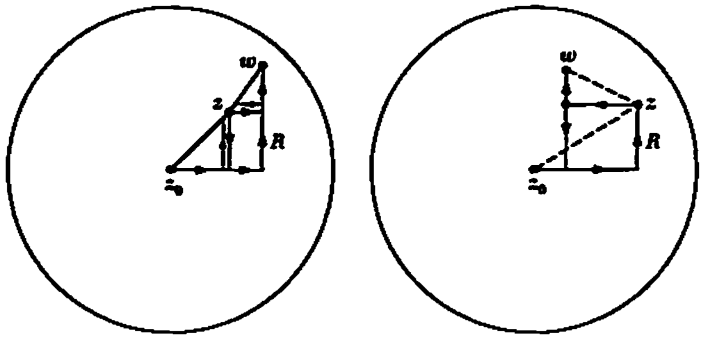


Figure 2.3.4: Two possible configurations for R, z_0, z, and w.

hypotenuse, which has length $|z-w|$, so length$(\langle\langle z, w\rangle\rangle) \leq 2|z-w|$, and thus

$$\begin{aligned}
\left|\frac{F(w)-F(z)}{w-z}-f(z)\right| &= \frac{1}{|w-z|}\left|\int_{\langle\langle z,w\rangle\rangle} f(\xi)d\xi - f(z)(w-z)\right| \\
&= \frac{1}{|w-z|}\left|\int_{\langle\langle z,w\rangle\rangle} f(\xi)d\xi - f(z)\int_{\langle\langle z,w\rangle\rangle} 1d\xi\right| \\
&= \frac{1}{|w-z|}\left|\int_{\langle\langle z,w\rangle\rangle} [f(\xi)-f(z)]d\xi\right| \\
&\leq \frac{1}{|w-z|}\int_{\langle\langle z,w\rangle\rangle} |f(\xi)-f(z)||d\xi| \\
&\leq \frac{1}{|w-z|}\epsilon \text{ length}(\langle\langle z,w\rangle\rangle) \leq \frac{1}{|w-z|}\epsilon\cdot 2|w-z| = 2\epsilon.
\end{aligned}$$

Thus,

$$\lim_{w\to z} = \frac{F(w)-F(z)}{w-z} = f(z)$$

and therefore $F'(z) = f(z)$, as desired. This establishes part (i) of the theorem. Since f has an antiderivative defined everywhere on D and γ is a closed curve in D, we have $\int_\gamma f = 0$ by the Path Independence Theorem 2.1.9. This establishes part (ii) of the theorem and so the proof of Cauchy's Theorem in a disk (Theorem 2.3.2) is now complete. ■

Deleted Neighborhoods For technical reasons that will be apparent in §2.4, it will be useful to have the following variant of Cauchy's Theorem for a rectangle.

Lemma 2.3.3 *Suppose that R is a rectangular path with sides parallel to the axes, that f is a function defined on an open set G containing R and its interior, and* *that f is analytic on G except at some fixed point z_1 in G which is not on the path R. Suppose that at z_1, the function f satisfies $\lim_{z \to z_1}(z - z_1)f(z) = 0$. Then $\int_R f = 0$.*

Notice that the limit condition in this lemma holds under any of the following three situations:

(i) If f is bounded in a deleted neighborhood of z_1

(ii) If f is continuous on G

(iii) If $\lim_{z \to z_1} f(z)$ exists

Proof If z_1 is outside R, then the situation is really just that of the Cauchy theorem for a rectangle (Theorem 2.3.1), so we may assume that z_1 is in the interior of R. For $\epsilon > 0$, there is a number $\delta > 0$ such that $|z - z_1||f(z)| < \epsilon$ whenever $|z - z_1| < \delta$. Choose δ small enough to do this and so that the square S of side length δ centered at z_1 lies entirely within R. Then everywhere along S we have $|f(z)| < \epsilon/|z - z_1|$. Now divide R into nine subrectangles by extending the sides of S, as shown in Figure 2.3.5.

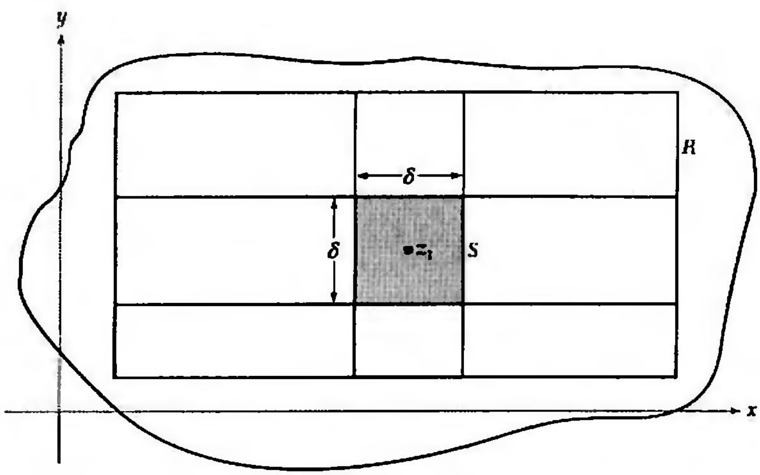


Figure 2.3.5: Construction of S and subdivision of R for the proof of Lemma 2.3.3.

By Cauchy's Theorem for a rectangle, the integrals of f around all eight of the subrectangles other than S are 0, so $\int_S f = \int_R f$. But along S we have

$$|f(z)| < \frac{\epsilon}{|z - z_1|} \leq \frac{\epsilon}{\delta/2} = \frac{2\epsilon}{\delta}$$

since $|z - z_1| \geq \delta/2$ along S. Thus,

$$\left|\int_S f\right| \leq \text{length}\,(S)\frac{2\epsilon}{\delta} = 4\delta\frac{2\epsilon}{\delta} = 8\epsilon.$$

Therefore, $|\int_R f| = |\int_S f| \leq 8\epsilon$ for every $\epsilon > 0$. Thus we must have $|\int_R f| = 0$ and so $\int_R f = 0$, as desired. ■

If we strengthen the assumption on f and assume that it is continuous at z_1, then we can drop the stipulation that z_1 not be on the path R.

Lemma 2.3.4 *Suppose that R is a rectangular path with sides parallel to the axes, that f is a function defined and continuous on an open set G containing R and its interior, and that f is analytic on G except at some fixed point z_1 in G. Then $\int_R f = 0$.*

The only real problem is to make sure that the integral is well behaved if the rectangle R happens to pass through z_1. In that case, the subdivision is a little different, but the estimates are simpler.

Proof Again let $\epsilon > 0$. We may choose δ so that $|f(z) - f(z_1)| < \epsilon$ whenever $|z - z_1| < \delta$. Adding a constant to f, we may assume $f(z_1) = 0$. If z_1 is not on R, then Lemma 2.3.3 applies. If it is on R, let S be half a square of side δ, and subdivide R as shown in Figure 2.3.6.

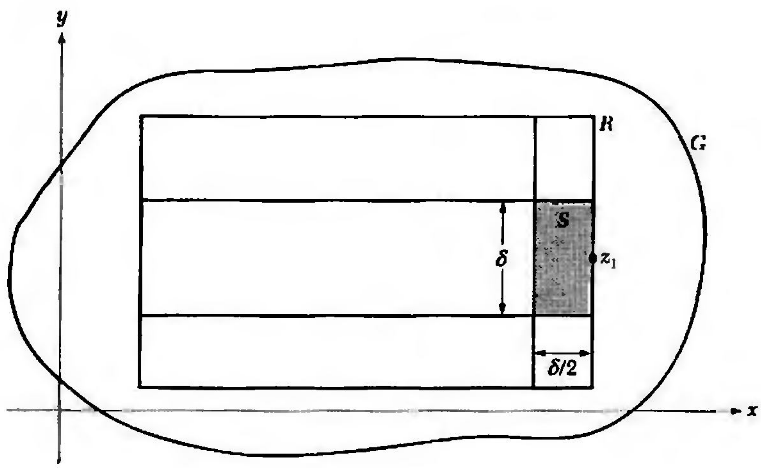


Figure 2.3.6: What happens if z_1 lies on R.

By Cauchy's Theorem for rectangles, the integrals of f around all five of the subrectangles other than S are 0, so $\int_S f = \int_R f$. Along S we have $|f(z)| < \epsilon$. Thus, if we also require $\delta < 1/3$, we get

$$\left|\int_S f\right| \leq \text{length}\,(S)\epsilon = 3\delta\epsilon < \epsilon.$$

Therefore, $|\int_R f| = |\int_S f| \leq \epsilon$ for every $\epsilon > 0$, so we must have $|\int_R f| = 0$. Thus, $\int_R f = 0$, as desired. ■

Strengthened Cauchy's Theorem for a Disk If we use Lemma 2.3.4 instead of Cauchy's Theorem for a rectangle in the proof of Cauchy's Theorem for a disk, we obtain the corresponding conclusions.

Theorem 2.3.5 (Strengthened Cauchy's Theorem for a Disk) *The conclusions of Cauchy's Theorem for a disk (Theorem 2.3.2) hold if we assume only that the function f is continuous on D and analytic on $D\backslash\{z_1\}$ for some fixed z_1 in D.*

Notice that continuity at z_1 is assumed. Again this is needed to apply Lemma 2.3.4 and to make sure that the integral $\int_\gamma f$ is defined even if γ passes through z_1. Notice also that a more complicated but parallel version of the same argument will produce the same conclusion if the number of "bad" points in G is finite instead of just one.

Homotopy and Simply Connected Regions To extend Cauchy's Theorem to more general regions than disks or rectangles and to prove the deformation theorems, we first clarify the concept of deforming curves or homotopy that was discussed informally in §2.2. There are two situations to be treated: two different curves between the same two endpoints and two closed curves that might not cross at all. For convenience we will assume that all curves are parametrized by the interval $[0, 1]$ unless specified otherwise. (This can always be done by reparametrizing if necessary.)

Definition 2.3.6 *Suppose $\gamma_0 : [0, 1] \to G$ and $\gamma_1 : [0, 1] \to G$ are two continuous curves from z_0 to z_1 in a set G. We say that γ_0 is* ***homotopic with fixed endpoints*** *to γ_1 in G if there is a continuous function $H : [0, 1] \times [0, 1] \to G$ from the unit square $[0, 1] \times [0, 1]$ into G such that*

(i) $H(0, t) = \gamma_0(t)$ *for* $0 \le t \le 1$.

(ii) $H(1, t) = \gamma_1(t)$ *for* $0 \le t \le 1$.

(iii) $H(s, 0) = z_0$ *for* $0 \le s \le 1$.

(iv) $H(s, 1) = z_1$ *for* $0 \le s \le 1$.

The idea behind this definition is simple. As s ranges from 0 to 1, we have a family of curves that continuously change, or deform, from γ_0 to γ_1, as in Figure 2.3.7. The reader should be aware that the picture need not be as simple in appearance as this illustration. The curves may twist and turn and cross over themselves or each other. No assumption is made that the curves are simple, but usually this does not matter. A little more notation may make the matter clearer. If we put $\gamma_s(t) = H(s, t)$, then each γ_s is a continuous curve from z_0 to z_1 in G. The initial curve is γ_0, and it corresponds to the left edge of the unit square. The final curve is γ_1, and it corresponds to the right edge of the square. The entire bottom edge goes to z_0 and the entire top edge to z_1. The curves γ_s are a continuously changing family of intermediate curves.

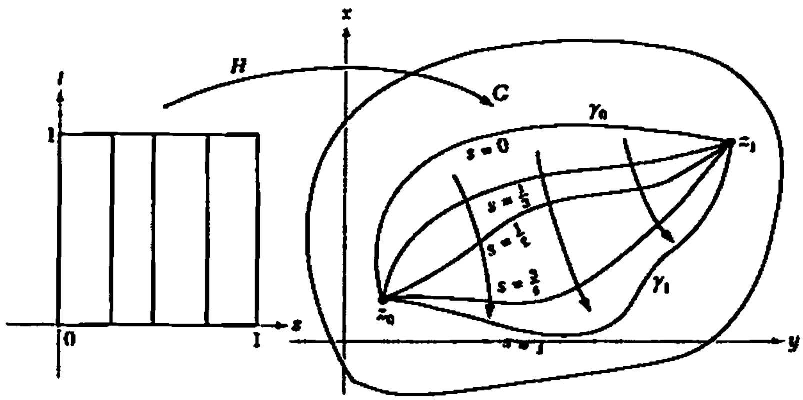


Figure 2.3.7: Fixed-endpoint homotopy.

For example, the straight-line segment from 0 to $1+i$, which is parametrized by $\gamma_0(t) = t + ti$, is homotopic with fixed endpoints to the parabolic path from 0 to $1+i$ parametrized by $\gamma_1(t) = t + t^2 i$; see Figure 2.3.8.

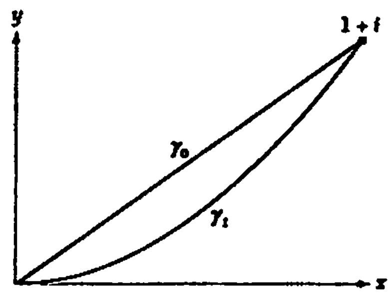


Figure 2.3.8: A straight-line path and a parabolic path from 0 to $1+i$.

One possible homotopy from one curve to the other is

$$H(s,t) = t + t^{1+s} i.$$

There is more than one way to get a homotopy between these curves. Another way makes $H(s,t)$ follow the straight-line between $t+ti$ and $t+t^2i$:

$$H(s,t) = s(t+t^2 i) + (1-s)(t+ti) = t + [st^2 + (1-s)t]i.$$

A slightly different definition is called for in the deformation of one *closed* curve to another.

Definition 2.3.7 *Suppose $\gamma_0 : [0,1] \to G$ and $\gamma_1 : [0,1] \to G$ are two continuous closed curves in a set G. We say that γ_0 and γ_1 are* **homotopic as closed curves**

in G if there is a continuous function $H : [0,1] \times [0,1] \to G$ from the unit square $[0,1] \times [0,1]$ into G such that

(i) $H(0,t) = \gamma_0(t)$ *for* $0 \le t \le 1$.

(ii) $H(1,t) = \gamma_1(t)$ *for* $0 \le t \le 1$.

(iii) $H(s,0) = H(s,1)$ *for* $0 \le s \le 1$.

Again, if we put $\gamma_s(t) = H(s,t)$, then each γ_s is a continuous curve in G. The third condition says that each of them is a closed curve; see Figure 2.3.9.

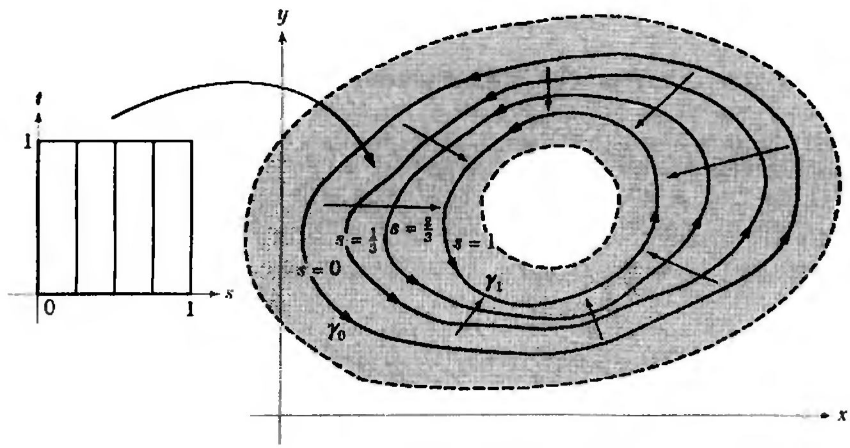


Figure 2.3.9: Closed-curve homotopy.

For example, the unit circle can be parametrized by $\gamma_0(t) = \cos t + i \sin t$ and the ellipse $x^2/4 + y^2 = 1$ by $\gamma_1(t) = 2\cos t + i \sin t$. These curves are homotopic as closed curves in the annulus $G = \{z \mid \frac{1}{2} < |z| < 3\}$. One possibility for the homotopy is $H(s,t) = (1+s)\cos t + i \sin t$. (See Figure 2.3.10.)

If the hole were not in the middle of G in Figure 2.3.10 but we had instead the solid disk $D = \{z \text{ such that } |z| < 3\}$, then either of the two curves could be continuously deformed down to a point. For example, $H(s,t) = (1-s)\gamma_0(t)$ is a homotopy that shrinks the circle γ_0 down to a constant curve at the point 0. The intermediate curves γ_s are circles of radius $(1-s)$ centered at 0. If γ_0 were any other curve in D, then the same definition on H would give a homotopy that continuously changes the scale of the curve until it shrinks down to a point. Thus any curve in D is homotopic to a point in D. If there were a hole in the set as there is in the annulus in Figure 2.3.10, then this shrinking procedure could not be done if the curve surrounded the hole. This leads us to a more precise definition of the notion of simply connected regions that was introduced informally in §2.2.

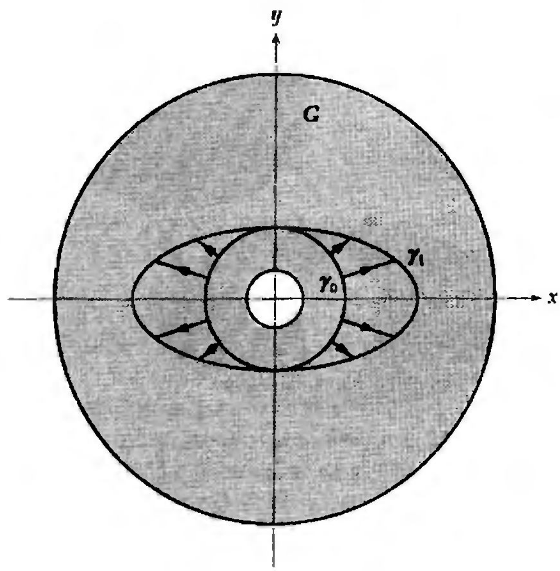


Figure 2.3.10: A circle homotopic to an ellipse.

Definition 2.3.8 *A connected set G is called **simply connected** if every closed curve γ in G is homotopic (as a closed curve) to a point in G, that is, to some constant curve.*

The second homotopy between the straight line and the parabola in Figure 2.3.8, which followed along straight-line segments, and the homotopy of a circle down to a point in the disk are suggestive and lead to the definition of two important classes of simply connected sets. Recall that if z_0 and z_1 are any two points and $0 \le s \le 1$, then the point $sz_1 + (1-s)z_0$ lies on the straight-line segment between the two.

Definition 2.3.9 *A set A is called **convex** if it contains the straight-line segment between every pair of its points. That is, if z_0 and z_1 are in A, then so is $sz_1 + (1-s)z_0$ for every number s between* 0 *and* 1 (Figure 2.3.11).

Proposition 2.3.10 *If A is a convex region, then any two closed curves in A are homotopic as closed curves in A, and any two curves with the same endpoints are homotopic with fixed endpoints.*

Proof Let $\gamma_0 : [0,1] \to G$ and $\gamma_1 : [0,1] \to G$ denote the two curves mentioned in the proposition and define $H(s,t)$ by $H(s,t) = s\gamma_1(t) + (1-s)\gamma_0(t)$. Then $H(s,t)$ lies on the straight-line segment between $\gamma_0(t)$ and $\gamma_1(t)$ and so is in the set A. It is a continuous function, since γ_0 and γ_1 are continuous. At $s = 0$ we get $\gamma_0(t)$,

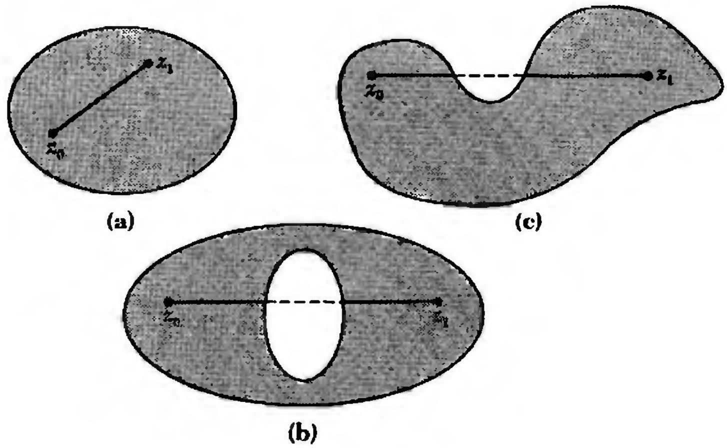


Figure 2.3.11: (a) Convex set. (b) and (c) Sets that are not convex.

and at $s = 1$ we get $\gamma_1(t)$. If they are closed curves, then

$$H(s,0) = s\gamma_1(0) + (1-s)\gamma_0(0) = s\gamma_1(1) + (1-s)\gamma_0(1) = H(s,1),$$

so it is a closed curve homotopy between the two. If they both go from z_0 to z_1, then $H(s,0) = s\gamma_1(0) + (1-s)\gamma_0(0) = sz_0 + (1-s)z_0 = z_0$ and $H(s,1) = s\gamma_1(1) + (1-s)\gamma_0(1) = sz_1 + (1-s)z_1 = z_1$, so H is a fixed endpoint homotopy between the two. ■

Corollary 2.3.11 *A convex region is simply connected.*

Proof Let z_0 be any point in the convex region A, and let γ be any closed curve in A. The constant curve at z_0, $\gamma_1(t) = z_0$ for all t, is certainly closed, and the two are homotopic by Proposition 2.3.10. ■

A slightly more general type of simply connected region called a ***starlike*** (or ***star-shaped***) ***region*** will be considered in the exercises. For more complicated regions, we often rely on our geometric intuition to determine when two curves are homotopic. In other words, we try to decide whether we can continuously deform one curve to the other without leaving our region. One reason is that we rarely use the homotopies H explicitly in practice; they are usually theoretical tools whose existence allows us to claim something else, such as the equality of two integrals. Also in many situations homotopies might be quite complicated to write down. However, we must be prepared to justify our geometric intuition either with an explicit H or a proof of its existence in any particular situation.

Theorem 2.3.12 (Deformation Theorem) *Suppose that f is an analytic function on an open set G and that γ_0 and γ_1 are piecewise C^1 curves in G.*

(i) *If γ_0 and γ_1 are paths from z_0 to z_1 and are homotopic in G with fixed endpoints, then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

(ii) *If γ_0 and γ_1 are closed curves which are homotopic as closed curves in G, then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

Proof The homotopy assumption means that there is a continuous function $H : [0,1] \times [0,1] \to G$ from the unit square into G which implements a continuous deformation from γ_0 to γ_1 in G. For each value of s, the function $\gamma_s(t) = H(s,t)$ is an intermediate curve taken on during the deformation. Similarly, for each fixed value of t, the function $\lambda_t(s) = H(s,t)$ traces out a curve crossing from $H(0,t) = \gamma_0(t)$ to $H(1,t) = \gamma_1(t)$. Thus a grid of horizontal and vertical lines in the square defines a corresponding grid of curves in G with the left edge of the square corresponding to γ_0 and the right edge to γ_1. In the fixed-endpoint case, $\lambda_0(s)$ is a constant curve at z_0 and $\lambda_1(s)$ is a constant curve at z_1. In the closed-curve case, they are the same curve, from $\gamma_0(0)(= \gamma_0(1))$ to $\gamma_1(0)(= \gamma_1(1))$. See Figures 2.3.12 and 2.3.13. The reader is cautioned that the grid of curves in G need not look as nice as this illustration, since it may twist and cross over itself, becoming somewhat entangled in appearance like a fishnet thrown on the beach. Fortunately, this does not matter for the proof.

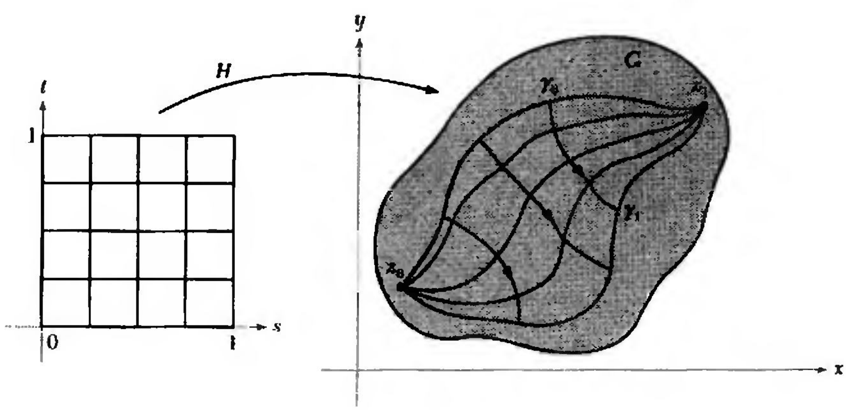


Figure 2.3.12: Fixed-endpoint homotopy.

The idea of the proof is to use uniform continuity to restrict the problem to small disks, use Cauchy's Theorem for a disk, and then put the pieces back together to

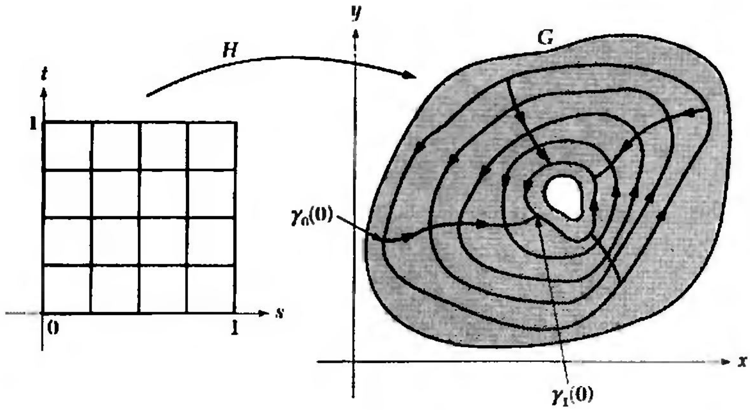


Figure 2.3.13: Closed-curve homotopy.

obtain the desired result. We wish to partition the square $[0,1] \times [0,1]$ into smaller squares by choosing intermediate points $0 = s_0 < s_1 < s_2 < \ldots < s_n = 1$ and $0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$ close enough together that each small square of the resulting grid is mapped into a disk that is wholly contained in G, as shown in Figure 2.3.14.

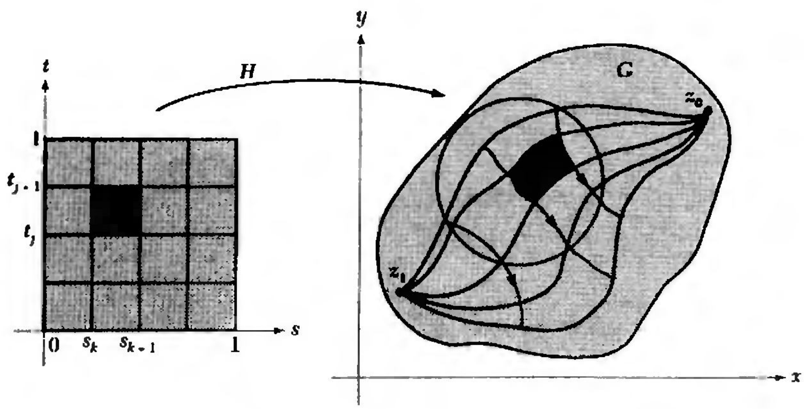


Figure 2.3.14: Subdivision for proof of the Deformation Theorem.

We will then be able to apply Cauchy's Theorem for a disk to the integral around each of these small paths. Making the subdivision is no problem. The function H is continuous on the compact set $[0,1] \times [0,1]$, so its image is a compact subset of G by Proposition 1.4.19. By the Distance Lemma 1.4.21, it stays a positive distance ρ away from the closed set $\mathbb{C} \backslash G$. That is, $|H(s,t) - z| < \rho$ implies that $z \in G$. But we know (by Proposition 1.4.23) that H is actually uniformly continuous on the

;uare. Therefore there is a number δ such that $|H(s,t) - H(s',t')| < \rho$ whenever

$$\text{distance}((s,t),(s',t')) = \sqrt{(s-s')^2 + (t-t')^2} < \delta.$$

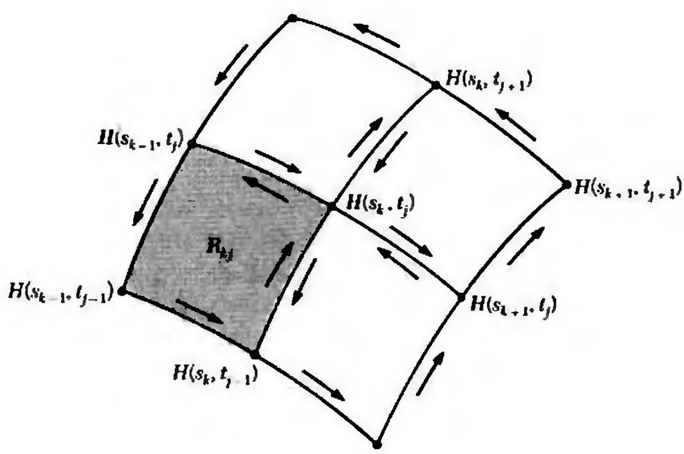
we choose the intermediate points equally spaced to break $[0,1] \times [0,1]$ into small ;uares with edge length $1/n$, the diagonal of each subsquare will have length less :at δ if $n > \sqrt{2}/\delta$. If R_{kj} is the rectangle with corners at

$$(s_{k-1}, t_{j-1}), (s_k, t_{j-1}), (s_k, t_j), (s_{k-1}, t_j),$$

ıen the whole rectangle is mapped into the disk $D_{kj} = D(H(s_{k-1}, t_{j-1}); \rho)$ that contained in G. Let Γ_{kj} be the closed curve described by $H(R_{kj})$ oriented by ıking R_{kj} to be oriented in the counterclockwise direction.The image of each edge ! each of the subsquares R_{kj} enters as part of two of the closed curves Γ_{kj} and :th opposite orientation, except those subsquares along the outer edge where t or is 0 or 1. Notice that these edges actually piece together to make up the curves ,ıt) and $\lambda_t(s)$. If we sum the integrals around all the loops Γ_{kj}, all the edges used ʌice will cancel out and leave only

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\Gamma_{kj}} f = \int_{\lambda_0} f + \int_{\gamma_1} f - \int_{\lambda_1} f - \int_{\gamma_0} f$$

ee Figure 2.3.15).



'igure 2.3.15: Cancellation of edges of the subsquares in the proof of the deforma-ion theorems.

Since Γ_{kj} is a closed curve lying entirely within the disk D_{kj} on which the unction f is analytic, Cauchy's Theorem for a disk implies that each integral in he sum on the left is 0, so the right side is also 0. Thus,

$$0 = \int_{\lambda_0} f + \int_{\gamma_1} f - \int_{\lambda_1} f - \int_{\gamma_0} f.$$

That is,

$$\int_{\lambda_1} f + \int_{\gamma_0} f = \int_{\lambda_0} f + \int_{\gamma_1} f.$$

Up to this point the proofs for the fixed-endpoint case and the closed-curve case have been the same. Now they diverge a bit.

For the fixed-endpoint case, $\lambda_0(s) = H(s, 0) = z_0$ for all s, and $\lambda_1(s) = H(s, 1) = z_1$ for all s. Both are constant curves, that is, single points, so $\int_{\lambda_0} f = \int_{\lambda_1} f = 0$.

For the closed-curve case, λ_0 and λ_1 are the same curve:

$$\lambda_0(s) = H(s, 0) = H(s, 1) = \lambda_1(s)$$

for all s, so $\int_{\lambda_0} f = \int_{\lambda_1} f$. In either case, the equation $\int_{\lambda_1} f + \int_{\gamma_0} f = \int_{\lambda_0} f + \int_{\gamma_1} f$ becomes $\int_{\gamma_0} f = \int_{\gamma_1} f$, which is exactly what we want.

The proof just given is actually not quite valid. The difficulty may appear to be an uninteresting technical subtlety, but it is crucial nevertheless. The function H has been assumed to be continuous, but no assumption was made about differentiability. Thus the curves $\gamma_s(t)$ and $\lambda_t(s)$ are continuous but need not be piecewise C^1. Unfortunately, all our theory about contour integrals is based on piecewise C^1 curves. Thus, the integrals appearing previously do not necessarily make sense. They would, and everything would be all right, if all the curves in question were piecewise C^1. Therefore we make one more provisional definition and assumption.

Definition 2.3.13 *A homotopy $H : [0,1] \times [0,1] \to G$ is called **smooth** if the intermediate curves $\gamma_s(t)$ are piecewise C^1 functions of t for each s and the cross curves $\lambda_t(s)$ are piecewise C^1 functions of s for each t.*

Assuming that the homotopies in the Deformation Theorem are smooth, all the curves in the preceding proof are piecewise C^1, the integrals all make sense; and the proof is valid. With this additional assumption in place, we will refer to the theorem as the *Smooth Deformation Theorem*. The technical discussion of how to relax the smoothness assumption is given in the Internet Supplement.

Theorem 2.3.14 (Homotopy Form of Cauchy's Theorem) *Let f be analytic on a region G. Let γ be a closed curve in G which is homotopic to a point in G. Then*

$$\int_\gamma f = 0.$$

Proof The curve γ is homotopic in G to a constant curve $\lambda(t) = z_0$ for all t. Therefore, $\int_\gamma f = \int_\lambda f = 0$. ■

We can also prove Cauchy's Theorem for a simply connected region in this context (see Theorem 2.2.3). Indeed, every closed curve γ in G is homotopic to a point in G, so the result follows from the homotopy form of Cauchy's Theorem.

Old Results in the New Setting In the Path Independence Theorem 2.1.9 we saw that the existence of antiderivatives is closely tied to path independence of integrals and to the vanishing of integrals around closed curves. Notice that the Antiderivative Theorem 2.2.5, and Theorem 2.2.6 on the existence of logarithms, are valid with the sharper meaning of simply connected, as discussed in this section.

The Deformation Theorem, Cauchy's Theorem, and all these consequences were proved from the conclusions of Cauchy's Theorem for a rectangle. From Theorem 2.2.5, we see that all these conclusions remain valid if we assume merely that f is continuous on G and analytic on $G\backslash\{z_1\}$ for some fixed z_1 in G. In §2.4 it will be shown that this assumption implies that f is analytic on G so that such a weakening of the hypotheses of the theorems is only apparent. But it is necessary for the logical development of the theory.

Worked Examples

Example 2.3.15 *Let A be the region bounded by the x axis and the curve $\sigma(\theta) = Re^{i\theta}, 0 \le \theta \le \pi$, where $R > 0$ is fixed. Let $f(z) = e^{z^2}/(2R - z)^2$. Show that for each closed curve γ in A, $\int_\gamma f = 0$.*

Solution First observe that f fails to be analytic only when $z = 2R$ and hence f is analytic on A, since $2R$ lies outside A (see Figure 2.3.16).

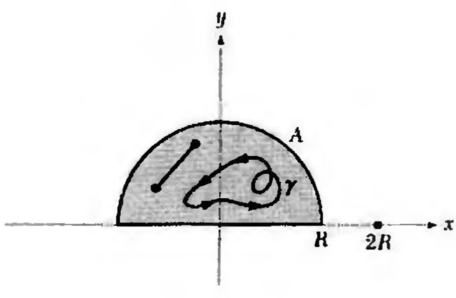


Figure 2.3.16: Convex region.

We claim that A is simply connected. That any two points in A can be joined by a straight-line lying in A (that is, that A is convex) is obvious geometrically and also is a simple matter to check (which the student should do). Hence A is simply connected, by Corollary 2.3.11. By Cauchy's Theorem, $\int_\gamma f = 0$ for any closed curve in A.

Example 2.3.16 *Let $A = \{z \in \mathbb{C} \mid 1 < |z| < 4\}$. First intuitively, then precisely, show that A is not simply connected. Also show precisely that the circles $|z| = 2$ and $|z| = 3$ are homotopic in A.*

Solution Intuitively, the circle $|z| = 2$ cannot be contracted continuously to a point without passing over the hole in A; that is, the set $\{z \in \mathbb{C}$ such that $|z| \leq 1\}$.

Precisely, the function $1/z$ is analytic on A, and if A were simply connected, then we would have $\int_\gamma (1/z)dz = 0$ for any closed curve in A. But if we let $\gamma(t) = 2e^{it}, 0 \leq t \leq 2\pi$, then we obtain

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{1}{2e^{it}} \cdot 2ie^{it} dt = 2\pi i.$$

Hence A is not simply connected.

Let the circles $|z| = 2$ and $|z| = 3$ be parametrized by $\gamma_1(t) = 2e^{it}$ and $\gamma_2(t) = 3e^{it}$, for $0 \leq t \leq 2\pi$ respectively. Then $H(t, s) = (2 + s)e^{it}$ defines a homotopy between γ_1 and γ_2 in A. The effect of H is illustrated in Figure 2.3.17.

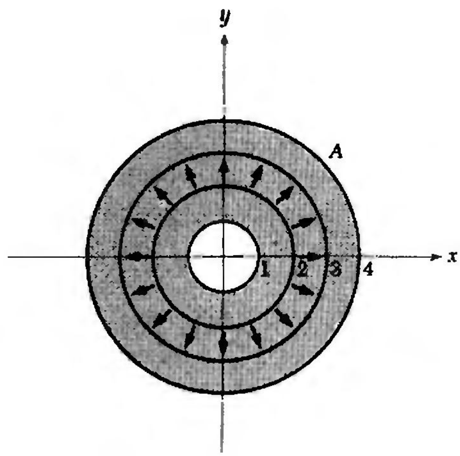


Figure 2.3.17: Region that is not simply connected.

Exercises

1. Prove that $\mathbb{C} \backslash \{0\}$ is not simply connected.

2. Show that every disk is convex.

3. A region A is called ***star-shaped with respect to*** z_0 if it contains the line segment between each of its points and z_0, that is, if $z \in A$ and $0 \leq s \leq 1$ imply that $sz_0 + (1 - s)z \in A$. The region is called ***star-shaped*** if there is at least one such point in A. Show that a star-shaped set is simply connected.

4. Show that a set A is convex if and only if it is star shaped with respect to each of its points (see Exercise 3).

5. Let G be the region built as a union of two rectangular regions $G = \{z$ such that $|\operatorname{Re} z| < 1$ and $|\operatorname{Im} z| < 3\} \cup \{z$ such that $|\operatorname{Re} z| < 3$ and $|\operatorname{Im} z| < 1\}$. (This set is illustrated in Figure 2.3.18.) Show that G is star shaped. (See Exercise 3.)

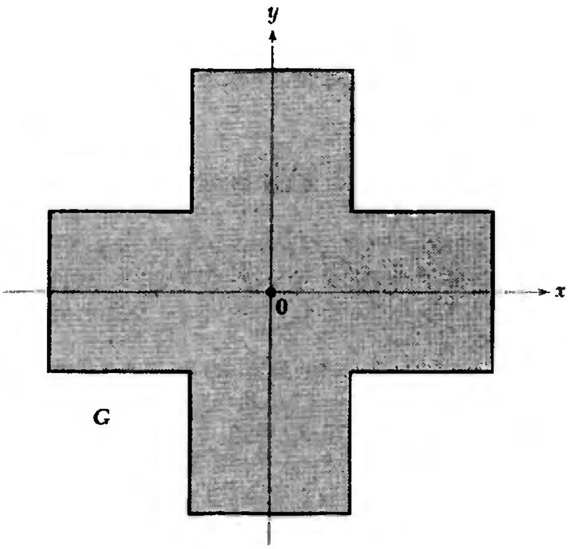


Figure 2.3.18: A star-shaped nonconvex region.

6. Complete the proof of Proposition 2.2.4.

7. Evaluate the following integrals without performing an explicit computation:

(a) $\displaystyle\int_\gamma \frac{dz}{z}$, where $\gamma(t) = \cos t + 2i \sin t, 0 \le t \le 2\pi$

(b) $\displaystyle\int_\gamma \frac{dz}{z^2}$, where γ is defined as in (a)

(c) $\displaystyle\int_\gamma \frac{e^z dz}{z}$, where $\gamma(t) = 2 + e^{it}, 0 \le t \le 2\pi$

(d) $\displaystyle\int_\gamma \frac{dz}{z^2 - 1}$, where γ is a circle of radius 1 centered at 1

8. Evaluate $\displaystyle\int_\gamma dz/z$, where γ is the line segment joining 1 to i.

9. (a) Let γ be a curve homotopic to the unit circle in $\mathbb{C}\setminus\{0\}$. Evaluate $\int_\gamma dz/z$.

(b) Evaluate $\displaystyle\int_\gamma dz/z$, where γ is the curve $\gamma(t) = 3\cos t + i4 \sin t, 0 \le t \le 2\pi$.

10.* Evaluate the following:

(a) $\displaystyle\int_{|z|=\frac{1}{2}} \frac{dz}{(1-z)^3}$

(b) $\displaystyle\int_{|z-1|=\frac{1}{2}} \frac{dz}{(1-z)^3}$

(c) $\displaystyle\int_{|z+1|=\frac{1}{2}} \frac{dz}{(1-z)^3}$

2.4 Cauchy's Integral Formula

One of the attractions of the theory of functions of one complex variable is that many powerful results can be derived from theoretically attractive results such as Cauchy's Theorem. We are now in a position to begin to draw some of these important consequences.

Among these many consequences, we shall see that a differentiable function must be infinitely differentiable and, in fact, analytic in the sense that the Taylor series converges to the function in some disk. The Fundamental Theorem of Algebra, that every polynomial has a complex root, will be a side benefit.

A stepping stone to these results is Cauchy's Integral Formula, a consequence of Cauchy's Theorem. It says that the values of an analytic function are completely determined everywhere inside a closed curve by its values along the curve and it gives an explicit formula for these values.

Index of a Closed Path There is a useful formula that expresses how many times a curve γ winds around a given point z_0 (see Figure 2.4.1). This number of times is called the *index* of γ with respect to z_0. The term "index" will be formally defined in Definition 2.4.1.

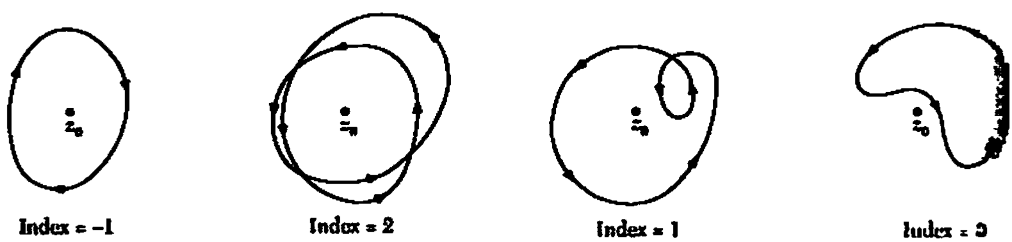


Figure 2.4.1: Index of a curve around a point.

The formula we shall use to compute the index is based on the computation done in Worked Example 2.1.12: If γ is the unit circle $\gamma(t) = e^{it}, 0 \le t \le 2\pi$, then

$$2\pi i = \int_\gamma \frac{dz}{z}.$$

If $\gamma(t) = e^{it}, 0 \le t \le 2\pi n$, then γ encircles the origin n times, and we find in the same way that

$$n = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}.$$

Now let us suppose that another closed curve $\tilde{\gamma}$ can be deformed to γ without passing through zero (that is, that $\tilde{\gamma}$ and γ are homotopic in the region $A = \mathbb{C}\backslash\{0\}$). Then again,

$$n = \frac{1}{2\pi i}\int_{\tilde{\gamma}} \frac{dz}{z} = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$$

by the Deformation Theorem (see §2.2 or §2.3). Since $\tilde{\gamma}$ and γ are homotopic in $\mathbb{C}\backslash\{0\}$, it is reasonable that they wind around 0 the same number of times. Generally, for any point $z_0 \in \mathbb{C}$, the number of times a curve $\tilde{\gamma}$ winds around z_0 is seen to be

$$n = \frac{1}{2\pi i}\int_{\tilde{\gamma}} \frac{dz}{z - z_0}$$

by a similar argument. As a consequence,

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0} = \left\{ \begin{array}{rl} \pm 1 & \text{if } z_0 \text{ is inside } \gamma \\ 0 & \text{if } z_0 \text{ is outside } \gamma \end{array} \right\}$$

for a *simple* closed curve γ. If one prefers, this establishes the definition of what we mean by the inside and outside of the curve. Classically, the notion of "inside" of a curve is often defined using the difficult ***Jordan curve theorem***, which states, roughly speaking, that a simple closed curve divides the plane uniquely into two connected pieces, exactly one of which is bounded, the inside.

These ideas lead to the formulation in the following definition.

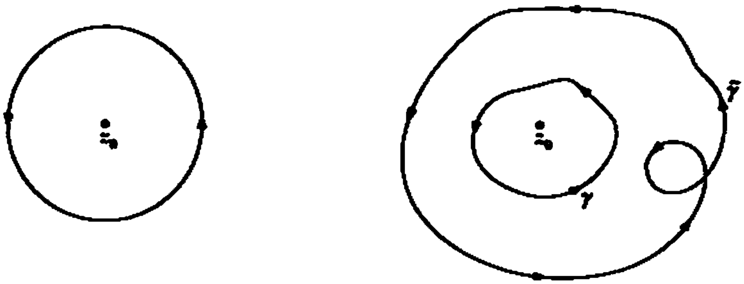
Definition 2.4.1 *Let γ be a closed curve in $\mathbb{C}$ and $z_0 \in \mathbb{C}$ be a point not on γ. Then the **index** of γ with respect to z_0 (also called the **winding number** of γ with respect to z_0) is defined by*

$$I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0}.$$

*We say that γ **winds around** z_0, $I(\gamma; z_0)$ times.*

The discussion that preceded this definition proves the following proposition, which is illustrated in Figure 2.4.2.

Proposition 2.4.2 (i) *The circle $\gamma(t) = z_0 + re^{it}$, where $r > 0$ is the radius and the parameter range is $0 \le t \le 2\pi n$, has index n with respect to z_0, while the circle $-\gamma(t) = z_0 + re^{-it}$, where again $0 \le t \le 2\pi n$, has index $-n$.*



Index of circle $\quad$ $\tilde{\gamma}$ and γ have the same index with respect to z_0

Figure 2.4.2: Index = the number of times that z_0 is encircled.

(ii) *If z_0 does not lie on either $\tilde{\gamma}$ or γ and if $\tilde{\gamma}$ and γ are homotopic in $\mathbf{C}\backslash\{z_0\}$, then*

$$I(\tilde{\gamma}; z_0) = I(\gamma; z_0).$$

Since homotopies can sometimes be awkward to deal with directly, it is customary merely to give an intuitive geometric argument that $I(\gamma; z_0)$ has a certain value, but again the student should be prepared to give a complete proof when called for (see Worked Example 2.4.12 at the end of this section).

The next result provides a check that the index $I(\gamma; z_0)$ is always an integer. This should be the case if Definition 2.4.1 actually represents the ideas illustrated in the figures.

Theorem 2.4.3 *Let $\gamma[a, b] \to \mathbf{C}$ be a (piecewise C^1) closed curve and z_0 a point not on γ; then $I(\gamma; z_0)$ is an integer.*

Proof Let

$$g(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0} ds.$$

At points where the integrand is continuous, the Fundamental Theorem of Calculus gives

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0}.$$

The right-hand side is the "logarithmic derivative" of $\gamma(t) - z_0$. Motivated by this observation and using the product rule for derivatives, we can write the preceding display as

$$\frac{d}{dt} e^{-g(t)}[\gamma(t) - z_0] = 0$$

at points where $g'(t)$ exists. Thus, $e^{-g(t)}[\gamma(t) - z_0]$ is piecewise constant on $[a, b]$. But $e^{-g(t)}[\gamma(t) - z_0]$ is continuous and therefore must be constant on $[a, b]$. This constant value is

$$e^{-g(a)}[\gamma(a) - z_0],$$

so we get $e^{-g(b)}[\gamma(b) - z_0] = e^{-g(a)}[\gamma(a) - z_0]$. But $\gamma(b) = \gamma(a)$, so $e^{-g(b)} = e^{-g(a)}$. On the other hand, $g(a) = 0$; hence $e^{-g(b)} = 1$. Thus $g(b) = 2\pi n i$ for an integer n, and the theorem follows. ■

The index, $I(\gamma; z) = (1/2\pi i)\int_\gamma d\zeta/(\zeta - z)$, is a continuous function of z as long as z does not cross γ. (Why?) But we have just seen that if γ is a closed curve, then its value must be an integer. Thus it must stay constant except when z crosses the curve. *Caution*: This need not be true if γ is not a closed curve.

The ***inside*** of a closed curve γ is defined by $\{z \mid I(\gamma; z) \neq 0\}$; this definition agrees with the intuitive ideas illustrated in Figure 2.4.1.

Derivation of Cauchy's Integral Formula Cauchy's Theorem will now be used to derive a useful formula relating the value of an analytic function at z_0 to a certain integral.

Theorem 2.4.4 (Cauchy's Integral Formula)[10] *Let f be analytic on a region A, let γ be a closed curve in A that is homotopic to a point, and let $z_0 \in A$ be a point not on γ. Then*

$$f(z_0) \cdot I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz.$$

This formula is often applied when γ is a simple closed curve and z_0 is inside γ. Then $I(\gamma; z_0) = 1$, so the formula becomes

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz.$$

The preceding formula is remarkable, for it says that the values of f on γ completely determine the values of f inside γ. In other words, the value of f is determined by its "boundary values."

Proof The proof makes a clever use of the analyticity of f and the technical strengthening of Cauchy's Theorem for which we laid the groundwork in the strengthened Cauchy's Theorem 2.3.5 for a disk, in which the function was allowed

[10] Cauchy's Integral Formula can be strengthened by requiring only that f be continuous on γ and analytic inside γ. This change makes little difference in solving most examples. For the proof of the strengthened theorem, the methods given in the Internet Supplement for Chapter 2 and an approximation argument may be used. See also E. Hille, *Analytic Function Theory*, Volume I (Boston: Ginn and Company, 1959.)

to be merely continuous and not necessarily analytic at one point. (See also Worked Example 2.2.10 and the remarks following Proposition 2.3.14.) Let

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0 \\ f'(z_0) & \text{if } z = z_0 \end{cases}.$$

Then g is analytic except perhaps at z_0, and it is continuous at z_0 since f is differentiable there. Thus $\int_\gamma g = 0$, so

$$0 = \int_\gamma g(z)dz = \int_\gamma \frac{f(z)}{z - z_0}dz - \int_\gamma \frac{f(z_0)}{z - z_0}dz$$

and

$$\int_\gamma \frac{f(z_0)}{z - z_0}dz = f(z_0)\int_\gamma \frac{1}{z - z_0}dz = 2\pi i f(z_0)I(\gamma; z_0),$$

and therefore the theorem follows. ■

Cauchy's Integral Formula 2.4.4 is extremely useful for computations. For example, if γ is the unit circle, we can immediately calculate

$$\int_\gamma \frac{e^z}{z}dz = 2\pi i \cdot e^0 = 2\pi i.$$

In Cauchy's Integral Formula, we simply choose $f(z) = e^z$ and $z_0 = 0$.

Note that in Cauchy's Integral Formula, it is f and not the integrand $f(z)/(z - z_0)$ that is analytic on A; the integrand is analytic only on $A\backslash\{z_0\}$, so we cannot use Cauchy's Theorem to conclude that the integral is zero—in fact, the integral is usually nonzero.

Integrals of Cauchy Type Cauchy's Integral Formula is a special and powerful formula for the value of f at z_0. We will now use it to show that all the higher derivatives of f also exist. The central trick in the proof is an idea that is often useful. If we start assuming only that we know the values of a function along a curve, then we can consider integrals along the curve as defining a new function called an ***integral of Cauchy type***.

Theorem 2.4.5 (Differentiability of Cauchy-Type Integrals) *Suppose γ is a curve in $\mathbb{C}$ and g is a continuous function defined along the image $\gamma([a,b])$. Set*

$$G(z) = \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)}{\zeta - z}d\zeta.$$

Then G is analytic on $\mathbb{C}\backslash\gamma([a,b])$; in fact, G is infinitely differentiable, with the kth derivative given by

$$G^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta - z)^{k+1}}d\zeta, \quad k = 1, 2, 3, \ldots.$$

The formula for the derivative can be remembered by "differentiating with respect to z under the integral sign":

$$\frac{d}{dz}G(z) = \frac{1}{2\pi i}\frac{d}{dz}\int_\gamma \frac{g(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{\partial}{\partial z}\left(\frac{g(\zeta)}{\zeta - z}\right)d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta - z)^2}d\zeta.$$

The formal proof justifies this procedure and appears at the end of this section.

Existence of Higher Derivatives Using integrals of Cauchy type, we can show that a differentiable function of one complex variable is actually infinitely differentiable and at the same time give a formula for all the derivatives.

Theorem 2.4.6 (Cauchy Integral Formula for Derivatives) *Let f be analytic on a region A. Then all the derivatives of f exist on A. Furthermore, for z_0 in A and γ any closed curve homotopic to a point in A with z_0 not on γ, we have*

$$f^{(k)}(z_0) \cdot I(\gamma; z_0) = \frac{k!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}d\zeta \quad k = 1, 2, 3, \ldots,$$

where $f^{(k)}$ denotes the kth derivative of f.

Proof Since A is open and z_0 is not on γ, we can find a small circle γ_0 centered at z_0 with interior in A and such that γ does not cut across γ_0. (See Figure 2.4.3.) For z in A and not on γ, define

$$G(z) = f(z) \cdot I(\gamma; z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta.$$

This is an integral of Cauchy type, so it is infinitely differentiable on $A\backslash\gamma$, and

$$G^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}d\zeta.$$

But

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)}{\zeta - z}d\zeta \quad \text{and} \quad I(\gamma; z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}d\zeta$$

are also integrals of Cauchy type, so they are infinitely differentiable near z_0. As mentioned earlier, the index is constant except when z crosses the curve. In particular, it is constant inside γ_0. Thus $G^{(k)}(z_0) = f^{(k)}(z_0)I(\gamma; z_0)$. Combining this with the preceding formula for $G^{(k)}(z)$ gives the desired result. ■

Cauchy's Inequalities and Liouville's Theorem We continue developing the consequences of the Cauchy Theorem with an important set of inequalities for the derivatives of an analytic function and its consequences.

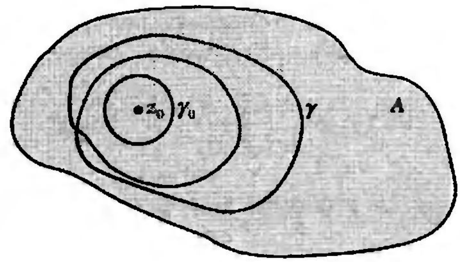


Figure 2.4.3: Let γ_0 be a circle centered at z_0 and small enough that it does not meet γ.

Theorem 2.4.7 (Cauchy's Inequalities) *Let f be analytic on a region A and let γ be a circle with radius R and center z_0 that lies in A. Assume that the disk $\{z$ such that $|z - z_0| < R\}$ also lies in A. Suppose that $|f(z)| \le M$ for all z on γ. Then, for any $k = 0, 1, 2, \ldots$,*

$$|f^{(k)}(z_0)| \le \frac{k!}{R^k} M.$$

Proof Since $I(\gamma; z_0) = 1$, from Cauchy's Integral Formula 2.4.4 we obtain

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta,$$

and hence

$$|f^{(k)}(z_0)| = \frac{k!}{2\pi} \left| \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta \right|.$$

Now

$$\left| \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \right| \le \frac{M}{R^{k+1}},$$

since $|\zeta - z_0| = R$ for ζ on γ, so

$$|f^{(k)}(z_0)| \le \frac{k!}{2\pi} \cdot \frac{M}{R^{k+1}} \cdot l(\gamma).$$

But $l(\gamma) = 2\pi R$, so we get our result. ■

This result states that although the kth derivatives of f can go to infinity as $k \to \infty$, *they cannot grow too fast* as $k \to \infty$; specifically, they can grow no faster than a constant times $k!/R^k$. We can use Cauchy's Inequalities to derive the following surprising result: *The only bounded entire functions are constants.*

Theorem 2.4.8 (Liouville's Theorem)[11] *If f is entire and there is a constant M such that $|f(z)| \le M$ for all $z \in \mathbb{C}$, then f is constant.*

Proof The Cauchy Inequalities 2.4.7 with $k = 1$ show that for any $z_0 \in \mathbb{C}$, the inequality $|f'(z_0)| \le M/R$ holds. Holding z_0 fixed and letting $R \to \infty$, we conclude that $|f'(z_0)| = 0$ and therefore that $f'(z_0) = 0$. This is true for every z_0 in $\mathbb{C}$, so f is constant. ■

This is again a quite different property than any that could possibly hold for functions of a real variable. Certainly, there are many nonconstant bounded smooth functions of a real variable, such as $f(x) = \sin x$.

Fundamental Theorem of Algebra Next we shall prove a result that *appears* to be elementary and that the student has, in the past, probably taken for granted. Algebraically, the theorem is quite difficult.[12] However, there is a simple proof that uses Liouville's theorem.

Theorem 2.4.9 (Fundamental Theorem of Algebra) *Let $a_0, a_1, \ldots, a_n$ be a collection of $n+1$ complex numbers and suppose that $n \ge 1$ and $a_n \ne 0$. Let $p(z) = a_0 + a_1 z + \ldots + a_n z^n$. Then there exists a point $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Proof Suppose that $p(z_0) \ne 0$ for all $z_0 \in \mathbb{C}$. Then $f(z) = 1/p(z)$ is entire. Now $p(z)$ and hence $f(z)$ is not constant (because $a_n \ne 0$), so it suffices, by Liouville's theorem, to show that $f(z)$ is bounded.

To do so, we first show that $p(z) \to \infty$ as $z \to \infty$, or equivalently, that $f(z) \to 0$ as $z \to \infty$. In other words, we prove that, given $M > 0$, there is a number $K > 0$ such that $|z| > K$ implies $|p(z)| > M$. From $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ we have $|p(z)| \ge |a_n||z|^n - |a_0| - |a_1||z| - \ldots - |a_{n-1}||z|^{n-1}$. (We set $a_n z^n = p(z) - z_0 - z_1 z - \ldots - z_{n-1} z^{n-1}$ and apply the triangle inequality.) Let $a = |a_0| + |a_1| + \ldots + |a_{n-1}|$. If $|z| > 1$, then

$$\begin{aligned} |p(z)| &\ge |z|^{n-1}\left(|a_n||z| - \frac{|a_0|}{|z|^{n-1}} - \frac{|a_1|}{|z|^{n-2}} - \ldots - \frac{|a_{n-1}|}{1}\right) \\ &\ge |z|^{n-1}(|a_n||z| - a). \end{aligned}$$

Let $K = \max\{1, (M+a)/|a_n|\}$; then, if $|z| > K$, we have $|p(z)| \ge M$.

Thus if $|z| > K$, we have $1/|p(z)| < 1/M$. But on the set of z for which $|z| \le K$, the function $1/p(z)$ is bounded in absolute value because it is continuous. If this

[11] According to E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Fourth Edition (London: Cambridge University Press, 1927), p. 105. Liouville's theorem is incorrectly attributed to Liouville by Borchardt (whom others copied), who heard it in Liouville's lectures in 1847. It is due to Cauchy, in *Comptes Rendus*, 19 (1844), 1377–1378, although it may have been known to Gauss earlier (see the next footnote).

[12] It was first proved by Karl Friedrich Gauss in his doctoral thesis in 1799. The present proof appears to be essentially due to Gauss as well (*Comm. Soc. Gott.*, 3 (1816), 59–64).

bound for $1/p(z)$ is denoted by L, then on $\mathbf{C}$ we have $1/|p(z)| < \max(1/M, L)$, so $|f(z)|$ is bounded on $\mathbf{C}$. ■

By Review Exercise 24 at the end of Chapter 1, the polynomial p can have no more than n roots. It follows by repeated factoring that p has *exactly* n roots if they are counted according to their multiplicity.

Another argument for showing that $f(z) \to 0$ as $z \to \infty$ that is a little simpler but accepts the validity of various limit theorems is as follows:

$$\begin{aligned} f(z) &= \frac{1}{a_n z^n + a_{n-1} z^{n-1} + \ldots + z_0} \\ &= \frac{1/z^n}{a_n + a_{n-1}(1/z) + a_{n-2}(1/z^2) + \ldots + a_0(1/z^n)}. \end{aligned}$$

Letting $z \to \infty$, we get

$$\lim_{z \to \infty} f(z) = \frac{0}{a_n + 0 + \ldots + 0} = 0$$

since $a_n \neq 0$.

The existence of higher derivatives, when combined with that of antiderivatives (Theorem 2.2.5), has some important consequences. Two of these are a partial converse to Cauchy's Theorem and an extension of the existence of logarithms.

Morera's Theorem Cauchy's Theorem says roughly that the integral of an analytic function around a closed curve is 0. Morera's Theorem says that if we know the function is continuous we can get the converse implication.

Theorem 2.4.10 (Morera's Theorem) *Let f be continuous on a region A, and suppose that $\int_\gamma f = 0$ for every closed curve in A. Then f is analytic on A, and $f = F'$ for some analytic function F on A.*

Proof The existence of the antiderivative follows from the vanishing of integrals around closed curves and the Path Independence Theorem 2.1.9. The antiderivative F is certainly analytic (its derivative is f). Therefore, by the Cauchy Integral Formula for Derivatives 2.4.6, it is infinitely differentiable. In particular, $F'' = f'$ exists. ■

In applying Morera's Theorem, one often wishes only to show that f is analytic on a region. If the region is not simply connected, f might not have an antiderivative on the whole region. But to show differentiability near a point one may restrict attention to a small neighborhood of the point and to special curves if convenient. This idea is illustrated in the following corollary and Worked Examples 2.4.16 and 2.4.17.

Corollary 2.4.11 *Let f be continuous on a region A and analytic on $A \backslash \{z_0\}$ for a point $z_0 \in A$. Then f is analytic on A.*

Proof To show analyticity at z_0, we may restrict attention to a small disk $D(z_0; \epsilon) \subset A$. If γ is any closed curve in this disk, then $\int_\gamma f = 0$, by the strengthened Cauchy theorem for a disk (2.3.5). Thus Morera's Theorem 2.4.10 implies that f is analytic on this disk. We already know it is analytic on the rest of A. ■

More on Logarithms In Theorem 2.2.6, we used the existence of antiderivatives to obtain logarithms on simply connected regions not containing the origin. With the existence of higher derivatives we can get a more general version of this result.

Proposition 2.4.12 (Logarithms of Functions) *Let $f(z)$ be a function that is analytic and never 0 on a simply connected region A. Then there is a function $g(z)$ analytic on A and unique up to the addition of a constant multiple of $2\pi i$ such that $e^{g(z)} = f(z)$ for all z in A.*

Proof In effect, we are looking for a logarithm for $f(z)$. If $g(z)$ is to be such a function, we must have $f'(z) = e^{g(z)}g'(z) = f(z)g'(z)$. Since $f(z)$ is never 0, this says that $g'(z) = f'(z)/f(z)$. Thus, g must be an antiderivative for f'/f on the connected open set A. The difference between two such antiderivatives would have derivative 0 and so be constant on A. Since the new function would still have to be a logarithm for $f(z)$, that constant would be an integer multiple of $2\pi i$. It remains to show that there is such a function.

Since f'' exists, f' is analytic on A. Since $f(z)$ is never 0, the quotient $f'(z)/f(z)$ is analytic on the simply connected region A. From the Antiderivative Theorem (see Theorem 2.2.5), there is a function $g : A \to \mathbb{C}$ such that $g'(z) = f'(z)/f(z)$ for all z in A. Fix z_0 in A. By adjusting our antiderivative by adding a constant, we may assume that $g(z_0)$ is any convenient choice of a value of $\log(f(z_0))$. Let $h(z) = e^{g(z)}/f(z)$. Then

$$h'(z) = \frac{f(z)e^{g(z)}g'(z) - e^{g(z)}f'(z)}{(f(z))^2} = \frac{f(z)e^{g(z)}f'(z)/f(z) - e^{g(z)}f'(z)}{(f(z))^2} = 0.$$

Since $h'(z)$ is identically 0 on the connected open set A, the function h must be constant on A. But $h(z_0) = e^{\log(f(z_0))}/f(z_0) = 1$. Therefore, $e^{g(z)}/f(z) = h(z) = h(z_0) = 1$ for all z in A. Thus, $e^{g(z)} = f(z)$ on A, as required. ■

If we apply this result to the function $f(z) = z$ on a simply connected region not containing 0, we recover our earlier results on logarithms (see Theorem 2.2.6).

Technical Proof of Theorem 2.4.5 We will prove Theorem 2.4.5 with a somewhat weaker assumption on g than continuity. All we assume is that the function is bounded and integrable along γ. We call such functions ***admissible***. First we use several facts from advanced calculus that were developed in §1.4. The image curve γ is a compact set since it is a continuous image of a closed bounded interval. If z_0 is not on γ, then by the Distance Lemma 1.4.21, it lies at a positive distance

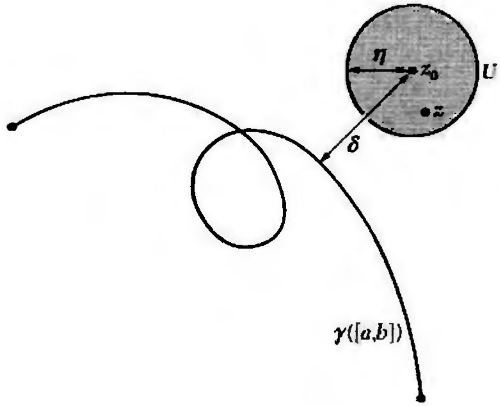


Figure 2.4.4: A point z_0 not on a curve γ is a positive distance from γ.

δ from it. If we let $\eta = \delta/2$ and U be the η disk around z_0, then $z \in U$ and ζ on γ implies that $|z-\zeta| \geq \eta$ since $|z-\zeta| \geq |\zeta - z_0| - |z_0 - z| \geq 2\eta - \eta$ (see Figure 2.4.4).

We begin with the case $k = 1$. We want to show that

$$\lim_{z \to z_0} \left[\frac{G(z) - G(z_0)}{z - z_0} - \frac{1}{2\pi i} \int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^2} d\zeta \right] = 0.$$

The expression in brackets may be written

$$\frac{G(z) - G(z_0)}{z - z_0} - \frac{1}{2\pi i} \int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^2} d\zeta = \frac{(z - z_0)}{2\pi i} \int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^2(\zeta - z)} d\zeta,$$

where we have used the identity

$$\frac{1}{z - z_0} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \right) - \frac{1}{(\zeta - z_0)^2} = \frac{z - z_0}{(\zeta - z_0)^2(\zeta - z)}.$$

Let the η neighborhood U of z_0 be constructed as previously described and let M be the maximum of g on γ. Then $|(\zeta - z_0)^2(\zeta - z)| \geq \eta^2 \cdot \eta = \eta^3$, so we have the estimate $|g(\zeta)/[(\zeta - z_0)^2(\zeta - z)]| \leq M\eta^{-3}$ (a fixed constant independent of ζ on γ and $z, z_0 \in U$). Thus,

$$\left| \frac{z - z_0}{2\pi i} \int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^2(\zeta - z)} d\zeta \right| \leq |z - z_0| \frac{M\eta^{-3}}{2\pi} l(\gamma).$$

This expression approaches 0 as $z \to z_0$, so the limit is 0, as we wanted.

To prove the general case, we proceed by induction on k. Suppose the theorem is known to hold for all admissible functions and all values of k from 1 to $n-1$. We want to prove that it works for $k = n$. We phrase the induction hypothesis this way since we will apply it not only to g, but also to $g(\zeta)/(\zeta - z_0)$, which is also bounded and integrable along γ. We know that G can be differentiated $n-1$ times on $\mathbb{C}\backslash\gamma$ and that

$$G^{(n-1)}(z) = \frac{(n-1)!}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta - z)^n}d\zeta.$$

Let $z_0 \in \mathbb{C}\backslash\gamma([a,b])$. Using the identity

$$\frac{1}{(\zeta - z)^n} = \frac{1}{(\zeta - z)^{n-1}(\zeta - z_0)} + \frac{z - z_0}{(\zeta - z)^n(\zeta - z_0)},$$

we obtain

$$\begin{aligned} &G^{(n-1)}(z) - G^{(n-1)}(z_0) \\ &= \frac{(n-1)!}{2\pi i}\left[\int_\gamma \frac{g(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)}d\zeta - \int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^n}d\zeta\right] \\ &\quad + \frac{(n-1)!}{2\pi i}(z - z_0)\int_\gamma \frac{g(\zeta)}{(\zeta - z)^n(\zeta - z_0)}d\zeta. \end{aligned}$$

We can conclude from this equation that $G^{(n-1)}$ is continuous at z_0, for the following reason. By applying the induction hypothesis to $g(\zeta)/(\zeta - z_0)$, we see that

$$\int_\gamma \frac{g(\zeta)}{(\zeta - z)^{(n-1)}(\zeta - z_0)}d\zeta$$

is analytic as a function of z on the set $\mathbb{C}\backslash\gamma([a,b])$ and thus is continuous in z. Therefore,

$$\int_\gamma \frac{g(\zeta)}{(\zeta - z)^{(n-1)}(\zeta - z_0)}d\zeta \to \int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^n}d\zeta$$

as $z \to z_0$. If the distance from z_0 to γ is 2η, if $|g(z)| < M$ on γ, and if $|z - z_0| < \eta$, we have

$$\left|\int_\gamma \frac{g(\zeta)}{(\zeta - z)^n(\zeta - z_0)}d\zeta\right| < \frac{M}{\eta^{n+1}}\cdot l(\gamma),$$

where $l(\gamma)$ is the length of γ. Hence

$$|z - z_0|\left|\int_\gamma \frac{g(\zeta)}{(\zeta - z)^n(\zeta - z_0)}d\zeta\right| \to 0$$

as $z \to z_0$, and therefore $G^{(n-1)}$ is continuous on $\mathbb{C}\backslash\gamma([a,b])$.

From the equation for $G^{(n-1)}(z) - G^{(n-1)}(z_0)$, we obtain

$$\begin{aligned} &\frac{G^{(n-1)}(z) - G^{(n-1)}(z_0)}{z - z_0} \\ &= \frac{(n-1)!}{2\pi i}\frac{1}{(z - z_0)}\left[\int_\gamma \frac{g(\zeta)}{(\zeta - z)^{(n-1)}(\zeta - z_0)}d\zeta - \int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^n}d\zeta\right] \\ &\quad + \frac{(n-1)!}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta - z)^n(\zeta - z_0)}d\zeta. \end{aligned}$$

By applying the induction hypothesis to $g(\zeta)/(\zeta - z_0)$, we see that the first term on the right side of the preceding equation converges to

$$\frac{(n-1)(n-1)!}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta$$

as $z \to z_0$. We have already shown that $G^{(n-1)}$ is continuous on $\mathbb{C}\setminus\gamma([a,b])$, and this fact applied to $g(\zeta)/(\zeta - z_0)$, instead of to $g(\zeta)$, implies that

$$\int_\gamma \frac{g(\zeta)}{(\zeta - z)^n(\zeta - z_0)}d\zeta \to \int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta$$

as $z \to z_0$. Thus we have shown that as $z \to z_0$,

$$\frac{G^{(n-1)}(z) - G^{(n-1)}(z_0)}{z - z_0}$$

converges to

$$\begin{aligned}(n-1)\frac{(n-1)!}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta + \frac{(n-1)!}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta \\ = \frac{n!}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta.\end{aligned}$$

This concludes the induction and thus proves the theorem. ■

Worked Examples

Example 2.4.13 *Consider the curve γ defined by $\gamma(t) = (\cos t, 3\sin t), 0 \le t \le 4\pi$ Show that $I(\gamma;0) = 2$.*

Solution The strategy is to show that γ is homotopic in $\mathbb{C}\setminus\{0\}$ to a circle $\tilde{\gamma}$ that is centered around the origin and that is traversed twice in the counterclockwise direction (that is, $\tilde{\gamma}(t) = e^{it}, 0 \le t \le 4\pi$). Once this is done, by Proposition 2.4.2. $I(\gamma;0) = I(\tilde{\gamma};0) = 2$.

A suitable homotopy is $H(t,s) = \cos t + i(3-2s)\sin t$; note that H is continuous. $H(t,0) = \gamma(t)$ and $H(t,1) = \tilde{\gamma}(t)$, and H is never zero (see Figure 2.4.5).

Example 2.4.14 *Evaluate*

$$\int_\gamma \frac{\cos z}{z}dz \quad \textit{and} \quad \int_\gamma \frac{\sin z}{z^2}dz,$$

where γ is the unit circle.

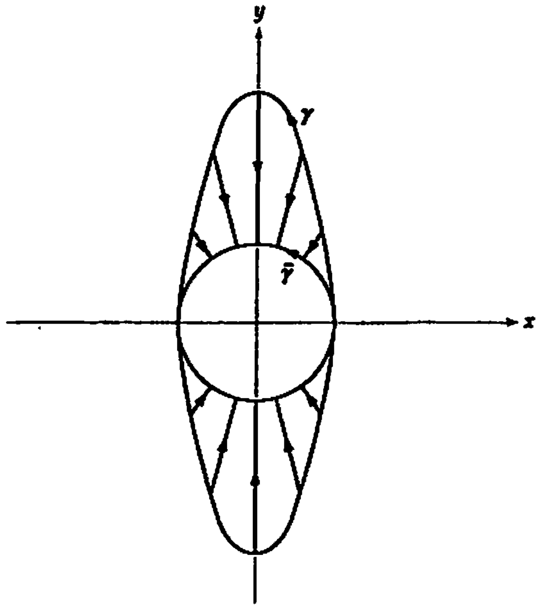


Figure 2.4.5: Homotopy of $\gamma(t) = (\cos t, 3\sin t)$ to $\tilde{\gamma}(t) = (\cos t, \sin t)$.

Solution The circle γ is contractible to a point in the region in which $\cos z$ is analytic, since in fact $\cos z$ is entire. Therefore, we can apply Cauchy's Integral Formula 2.4.4, observing that $I(\gamma; 0) = 1$, to obtain

$$1 = \cos 0 = \frac{1}{2\pi i}\int_\gamma \frac{\cos z}{z}dz,$$

so

$$\int_\gamma \frac{\cos z}{z}dz = 2\pi i.$$

By the Cauchy Integral Formula for Derivatives 2.4.6, we have

$$\sin'(0) = \frac{1}{2\pi i}\int_\gamma \frac{\sin z}{z^2}dz,$$

that is,

$$\int_\gamma \frac{\sin z}{z^2}dz = 2\pi i \cos 0 = 2\pi i.$$

Example 2.4.15 *This example, which deals with analytic functions defined by integrals, generalizes Theorem 2.4.5. Let $f(z, w)$ be a continuous function of z, w for*

z in a region A and w on a curve γ. For each w on γ assume that f is analytic in z. Let

$$F(z) = \int_\gamma f(z, w)\,dw.$$

Show that F is analytic and

$$F'(z) = \int_\gamma \frac{\partial f}{\partial z}(z, w)\,dw,$$

where $\partial f/\partial z$ denotes the derivative of f with respect to z with w held fixed.

Solution Let $z_0 \in A$. Let γ_0 be a circle in A around z_0 whose interior also lies in A. For z inside γ_0,

$$f(z, w) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(\zeta, w)}{\zeta - z}\,d\zeta$$

by Cauchy's Integral Formula 2.4.4. Thus

$$F(z) = \frac{1}{2\pi i}\int_\gamma \left[\int_{\gamma_0} \frac{f(\zeta, w)}{\zeta - z}\,d\zeta\right] dw.$$

Next we claim that we may invert the order of integration, thus obtaining

$$F(z) = \frac{1}{2\pi i}\int_{\gamma_0} \left[\int_\gamma \frac{f(\zeta, w)}{\zeta - z}\,dw\right] d\zeta = \frac{1}{2\pi i}\int_{\gamma_0} \frac{F(\zeta)}{\zeta - z}\,d\zeta.$$

This procedure is justifiable because the integrand is continuous and when written out in terms of real integrals has the form

$$\int_a^b\int_\alpha^\beta h(s, t)\,ds\,dt + i\int_a^b\int_\alpha^\beta k(s, t)\,ds\,dt.$$

We know from advanced calculus that this order can be interchanged (Fubini's theorem)[13].

Thus,

$$F(z) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{F(\zeta)}{\zeta - z}\,d\zeta,$$

and so by Theorem 2.4.5, F is analytic inside γ_0 and

$$\begin{aligned} F'(z) &= \frac{1}{2\pi i}\int_{\gamma_0} \frac{F(\zeta)}{(\zeta - z)^2}\,d\zeta = \frac{1}{2\pi i}\int_{\gamma_0}\int_\gamma \frac{f(\zeta, w)}{(\zeta - z)^2}\,dw\,d\zeta \\ &= \frac{1}{2\pi i}\int_\gamma\int_{\gamma_0} \frac{f(\zeta, w)}{(\zeta - z)^2}\,d\zeta\,dw = \int_\gamma \frac{\partial f}{\partial z}(\zeta, w)\,dw, \end{aligned}$$

[13] See, for instance, J. Marsden and M. Hoffman, *Elementary Classical Analysis*, Second Edition (New York: W. H. Freeman and Company, 1993), Chapter 9.

again by Cauchy's Integral Formula. Since z_0 is arbitrary we obtain the desired result. *Remark:* The function f should be analytic in z but it needs to only be integrable in the w variable, as is evident from the preceding proof; we merely need an adequate hypothesis to justify interchanging the order of integration.

Example 2.4.16 *Prove the following assertion: Suppose that f is continuous on a region A and that for each z_0 in A there is a disk $D = D(z_0;\rho)$ such that $\int_R f = 0$ for every rectangular path R in D with sides parallel to the axes. Show that f is analytic on A* (see Figure 2.4.6).

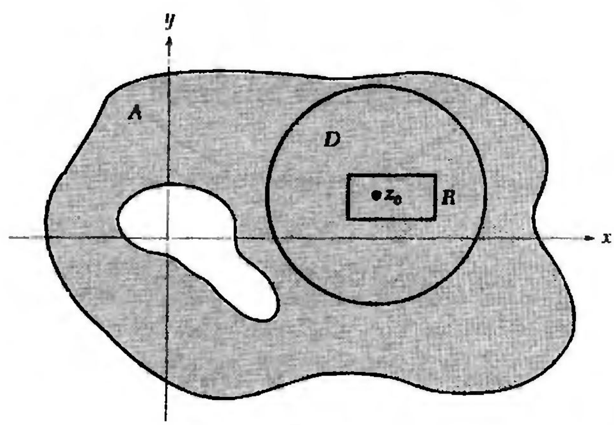


Figure 2.4.6: If $\int_R f = 0$, then f is analytic.

Solution Let z_0 be in A. The vanishing of $\int_R f$ for rectangles in D was the conclusion of Cauchy's Theorem 2.3.2 for a rectangle and the tool used in the construction of the antiderivative for f in the proof of Cauchy's Theorem for a disk. Thus the antiderivative exists on D (not necessarily on all of A at once). Analyticity on D follows as the proof of Morera's Theorem 2.4.10, so f is analytic near z_0. Since z_0 was an arbitrary point in A, f is analytic on A.

Example 2.4.17 *Prove the following: Suppose A is a region that intersects the real axis and that f is a function continuous on A and analytic on $A\backslash\mathbb{R}$. Then f is analytic on A.*

Solution We know f is analytic everywhere in A except on the real axis, so suppose $z_0 \in \mathbb{R}$. Since A is open there is a disk $D = D(z_0;\rho) \subset A$. Let R be a rectangular path in this disk with sides parallel to the axes. If R does not touch or cross the real axis, then $\int_R f = 0$ by Cauchy's Theorem. If it does cross, as in Figure 2.4.7, then $\int_R f = \int_{R_1} f + \int_{R_2} f$, where R_1 and R_2 are rectangles with one

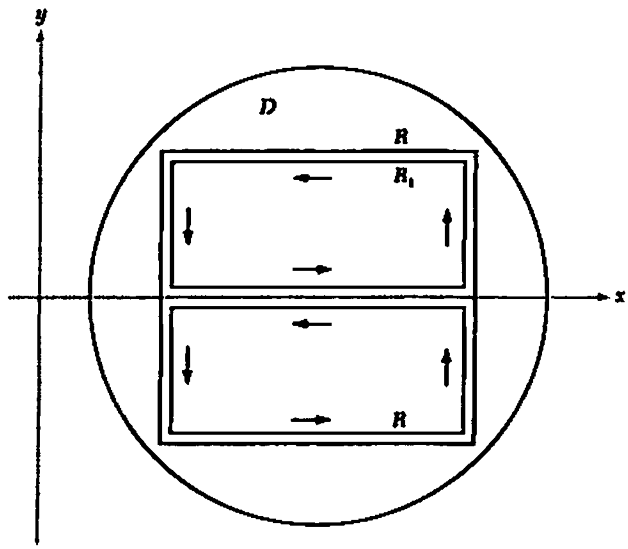


Figure 2.4.7: Construction used to show $\int_R f = 0$.

edge on the axis. (The edges on the axis are traversed in opposite directions and so cancel out.)

Thus it is enough to show that $\int_R f = 0$ when R is a rectangle with one side on the real axis, as in Figure 2.4.8. Let a and b be the ends of the edge on the axis and note that $b - a < \rho$.

Let $\epsilon > 0$. Since f is continuous, it is uniformly continuous on the compact set composed of R and its interior, so there is a $\delta > 0$ such that $|f(z_1) - f(z_2)| < \epsilon$ whenever $|z_1 - z_2| \leq \delta$ and z_1 and z_2 are in this set. We may also choose δ to be less than ϵ. Let M be the maximum of $|f(z)|$ on R and its interior, and let S be another rectangle the same as R except that the edge on the axis is moved a distance δ from the axis. Then with the notation of Figure 2.4.8,

$$\begin{aligned}
\left|\int_R f - \int_S f\right| &= \left|\int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f - \int_{\gamma_4} f\right| \\
&\leq \left|\int_{\gamma_1} f\right| + \left|\int_{\gamma_2} f - \int_{\gamma_4} f\right| + \left|\int_{\gamma_3} f\right| \\
&\leq \delta M + \left|\int_a^b [f(x) - f(x + \delta i)]dx\right| + \delta M \\
&\leq 2\delta M + \int_a^b |f(x) - f(x + \delta i)|dx \\
&\leq 2\delta M + \epsilon(b - a) \leq \epsilon(2M + \rho).
\end{aligned}$$

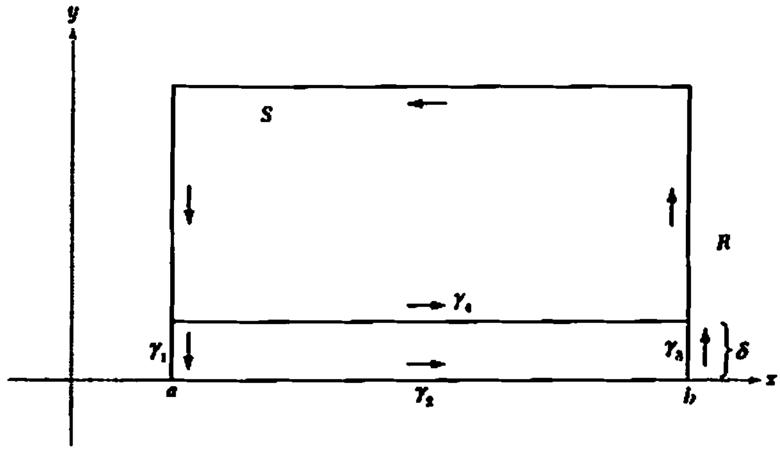


Figure 2.4.8: The rectangle pulled away slightly from the real axis.

Since this holds for every $\epsilon > 0$, we must have $\int_R f - \int_S f = 0$. But $\int_S f = 0$ by Cauchy's Theorem since S does not cross the axis and lies entirely within a region in which F is known to be analytic. Thus $\int_R f = 0$. We have shown that the conditions of the preceding Worked Example apply, so f is analytic on A.

Exercises

1. Evaluate the following integrals:

   (a) $\displaystyle\int_\gamma \frac{z^2}{z-1}dz$, where γ is a circle of radius 2, centered at 0

   (b) $\displaystyle\int_\gamma \frac{e^z}{z^2}dz$, where γ is the unit circle

2. Evaluate the following integrals:

   (a) $\displaystyle\int_\gamma \frac{z^2-1}{z^2+1}dz$, where γ is a circle of radius 2, centered at 0

   (b) $\displaystyle\int_\gamma \frac{\sin e^z}{z}dz$, where γ is the unit circle

3. Let f be entire and assume that $|f(z)| \le M|z|^n$ for large $|z|$, for a constant M, for some integer n. Show that f is a polynomial of degree $\le n$.

4.* Let f be analytic "inside and on" a simple closed curve γ. Suppose that $f = 0$ on γ. Show that $f = 0$ inside γ.

5. Evaluate the following integrals:

(a) $\displaystyle\int_\gamma \frac{dz}{z^3}$, where γ is the square with vertices $-1-i, 1-i, 1+i, -1+i$

(b) $\displaystyle\int_\gamma \frac{\sin z}{z^4}dz$, where γ is the unit circle

6. Let f be analytic on a region A and let γ be a closed curve in A. For any $z_0 \in A$ not on γ, show that

$$\int_\gamma \frac{f'(\zeta)}{\zeta - z_0}d\zeta = \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2}d\zeta.$$

Can you think of a way to generalize this result?

7. Suppose that $f(z)$ is analytic on the set $|z| < 1$ and that it satisfies the inequality $|f(z)| \leq 1$. What estimate can be made about $|f'(0)|$?

8.* Suppose that f is entire and that $\lim_{z\to\infty} f(z)/z = 0$. Prove that f is constant.

9. Prove that if γ is a circle, $\gamma(t) = z_0 + re^{it}, 0 \leq t \leq 2\pi$, then for every z inside γ (that is, $|z - z_0| < r$), $I(\gamma; z) = 1$.

10.* Use Worked Example 2.4.15 to show that

$$F(z) = \int_0^1 e^{-z^2x^2}dx$$

is analytic in z. What is $F'(z)$?

11. Show that if F is analytic on A, then so is f where

$$f(z) = \frac{F(z) - F(z_0)}{z - z_0}$$

if $z \neq z_0$ and $f(z_0) = F'(z_0)$ where z_0 is some point in A.

12. Prove that if the image of γ lies in a simply connected region A and if $z_0 \notin A$, then $I(\gamma; z_0) = 0$.

13. Use Worked Example 2.1.12 (where appropriate) and Cauchy's Integral Formula 2.4.4 to evaluate the following integrals; γ is the circle $|z| = 2$ in each case.

(a) $\displaystyle\int_\gamma \frac{dz}{z^2 - 1}$

(b) $\displaystyle\int_\gamma \frac{dz}{z^2 + z + 1}$

(c) $\displaystyle\int_\gamma \frac{dz}{z^2 - 8}$

(d) $\displaystyle\int_\gamma \frac{dz}{z^2+2z-3}$

14.* Prove that $\int_0^\pi e^{\cos\theta}\cos(\sin\theta)d\theta = \pi$ by considering $\int_\gamma (e^z/z)dz$, where γ is the unit circle.

15. Evaluate

$$\int_C \frac{|z|e^z}{z^2}dz,$$

where C is the circumference of the circle of radius 2 around the origin.

16. Consider the function $f(z) = 1/z^2$.

(a) It satisfies $\int_\gamma f(z)dz = 0$ for all closed contours γ (not passing through the origin) but is not analytic at $z = 0$. Does this statement contradict Morera's Theorem?

(b) It is bounded as $z \to \infty$ but is not a constant. Does this statement contradict Liouville's theorem?

17. Let $f(z)$ be entire and let $|f(z)| \geq 1$ on the whole complex plane. Prove that f is constant.

18.* Does $\displaystyle\int_{|z|=1} \frac{e^z}{z^2}dz = 0$? Does $\displaystyle\int_{|z|=1} \frac{\cos z}{z^2}dz = 0$?

19. Evaluate

(a) $\displaystyle\int_{|z-1|=2} \frac{dz}{z^2-2i}$

(b) $\displaystyle\int_{|z|=2} \frac{dz}{z^2(z^2+16)}$

20. Prove that for closed curves γ_1, γ_2,

$$I(-\gamma_1; z_0) = -I(\gamma_1; z_0)$$

and

$$I(\gamma_1 + \gamma_2; z_0) = I(\gamma_1; z_0) + I(\gamma_2; z_0).$$

Interpret these results geometrically.

21.* Let f be analytic inside and on the circle $\gamma : |z - z_0| = R$. Prove that

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} - f'(z_0) = \frac{1}{2\pi i}\int_\gamma \left[\frac{1}{(z-z_1)(z-z_2)} - \frac{1}{(z-z_0)^2}\right] f(z)dz$$

for z_1, z_2 inside γ.

2.5 Maximum Modulus Theorem and Harmonic Functions

One of the most powerful consequences of the Cauchy Integral Formula is the *Maximum Modulus Theorem*, also called the *Maximum Modulus Principle*. It states that if f is a nonconstant analytic function on a region A, then $|f|$ cannot have a local maximum anywhere *inside* A—it can attain a maximum only on the *boundary* of A. This theorem and the Cauchy Integral Formula will be used to develop some of the important properties of harmonic functions.

Maximum Modulus Theorem The central idea of the Maximum Modulus Principle can perhaps best be stated as follows: If an analytic function has a local maximum (of its absolute value) at a point, then it must be constant near that point. A preliminary version of the theorem follows.

Theorem 2.5.1 (Maximum Modulus Principle—Local Version)
Let f be analytic on a region A and suppose that $|f|$ has a relative maximum at $z_0 \in A$. (That is, $|f(z)| \le |f(z_0)|$ for all z in some neighborhood of z_0.) Then f is constant in some neighborhood of z_0.

The proof rests on a striking consequence of the Cauchy Integral Formula: *The value of an analytic function at the center of a circle is the average of its values around the circle.* All this will be made precise shortly, but the local version of the principle follows essentially because the average of a function cannot be greater than or equal to the values of the function unless they are all equal. We develop this idea in preparation for proving the Maximum Modulus Theorem.

Theorem 2.5.2 (Mean Value Property) *Let f be analytic inside and on a circle of radius r and center z_0 (that is, analytic on a region containing the circle and its interior). Then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})d\theta. \tag{2.5.1}$$

Proof By Cauchy's Integral Formula 2.4.4,

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz,$$

where $\gamma(\theta) = z_0 + re^{i\theta}, 0 \le \theta \le 2\pi$. However, by definition of the integral,

$$\begin{aligned}\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})d\theta. \quad \blacksquare\end{aligned}$$

It is worth noting that as long as we are integrating all the way around the circle, it does not matter through what range of 2π the angle goes. A change of variable shows that, for example, $\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta = \int_{-\pi}^{\pi} f(z_0 + re^{i\theta})d\theta$.

The Mean Value Property will now be used to establish the local version of the Maximum Modulus Principle. The idea is that if $f(z_0)$ is at least as great as all the other values of f near z_0 and also equal to the average of those values around small circles centered at z_0, then $|f(z)|$ must be constant near z_0. Once we know that $|f|$ is constant, it follows from the Cauchy-Riemann equations that f is itself constant.

Proof of Theorem 2.5.1 Suppose that f is analytic and has a relative maximum at z_0 so that $|f(z)| \leq |f(z_0)|$ on some disk $D_0 = D(z_0; r_0)$. We want to show that $|f(z)| = |f(z_0)|$ on D_0, so suppose instead that there is a point z_1 in D_0 where strict inequality holds: $|f(z_1)| < |f(z_0)|$. Let $z_1 = z_0 + re^{ia}$ with $r < r_0$. Since f is continuous, there are positive numbers ϵ and δ such that

$$|f(z_0 + re^{i\theta})| < |f(z_0)| - \delta$$

whenever $|\theta - a| < \epsilon$. Equivalently,

$$|f(z_0 + re^{i(a+\phi)})| < |f(z_0)| - \delta$$

whenever $|\phi| < \epsilon$. We now obtain a contradiction by using the Mean Value Property and considering separately that part of the integral over the circle where we know the function is smaller (see Figure 2.5.1)

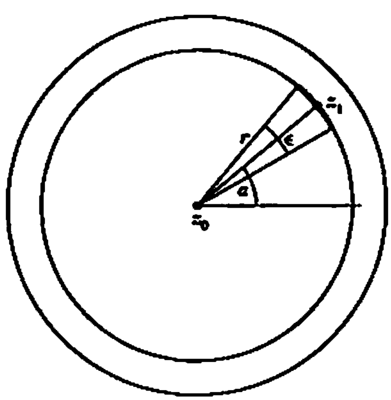


Figure 2.5.1: Construction for the proof of the Maximum Modulus Principle—local version. $|\phi| < \epsilon$ gives a part of the circle where $|f|$ is known to be smaller.

To carry this out, write

$$\begin{aligned}
|f(z_0)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(z_0 + re^{i(\alpha+\phi)})d\phi\right| \\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{-\epsilon} f(z_0 + re^{i(\alpha+\phi)})d\phi + \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} f(z_0 + re^{i(\alpha+\phi)})d\phi \right. \\
&\quad \left. + \frac{1}{2\pi}\int_{\epsilon}^{\pi} f(z_0 + re^{i(\alpha+\phi)})d\phi\right| \\
&\le \frac{1}{2\pi}\int_{-\pi}^{-\epsilon} \left|f(z_0 + re^{i(\alpha+\phi)})\right| d\phi + \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} \left|f(z_0 + re^{i(\alpha+\phi)})\right| d\phi \\
&\quad + \frac{1}{2\pi}\int_{\epsilon}^{\pi} \left|f(z_0 + re^{i(\alpha+\phi)})\right| d\phi.
\end{aligned}$$

In the first and third integrals the integrand is no greater than $|f(z_0)|$ and the interval length is $\pi - \epsilon$. Thus each of these integrals is no more than $|f(z_0)|(\pi - \epsilon)$. In the middle integral, the interval length is 2ϵ and the integrand is less than $|f(z_0)| - \delta$. Hence this integral is no more than $(|f(z_0)| - \delta)2\epsilon$. Putting these together gives

$$|f(z_0)| < \frac{1}{2\pi}[|f(z_0)|(\pi - \epsilon) + (|f(z_0)| - \delta)2\epsilon + |f(z_0)|(\pi - \epsilon)]$$

i.e.,

$$|f(z_0)| < |f(z_0)| - \frac{\epsilon\delta}{\pi}.$$

This impossibility shows that there can be no such point z in D_0 with $|f(z)| < |f(z_0)|$. The only remaining possibility is that $|f(z)| = |f(z_0)|$ for all z in D_0.

Thus $|f|$ is constant on D_0. Use of the Cauchy-Riemann equations as in Worked Example 1.5.21 shows that the function f itself must be constant. This is exactly what we wanted. ■

The local version of the Maximum Modulus Principle says that an analytic function cannot have a local maximum point unless it is constant near that point We will see in Chapter 6 that more is true. A function analytic on an open *connected* set cannot have a local maximum anywhere in that set unless it is constant on the whole set.

We now turn our attention here to a somewhat different global version of the principle. We investigate absolute maxima, that is, the largest value $|f(z)|$ takes anywhere in the set. We shall show that this can be found only on the edge or boundary of the set. In §1.4, we saw that a real-valued function continuous on a closed bounded set actually attains a finite maximum but that it might not if the set fails to be closed or bounded.

Closure and Boundary The intuition in §1.4 was that a set is closed if it contains all its boundary points and open if it contains none of them. Thus if we start with a set A and add to it any of its boundary points which happen to be missing we should obtain a closed set containing A. This is true, but there are some technical problems. One is that we really have no definition yet for "boundary."

Definition 2.5.3 *The closure of a set $A \subset \mathbf{C}$, denoted by $\bar{A}$ or by $\mathrm{cl}(A)$, consists of A together with the limit points of all convergent sequences of points of A.*

This produces the desired result, the smallest closed set containing A.

Proposition 2.5.4 *If $A \subset \mathbf{C}$, then*

(i) $A \subset \mathrm{cl}(A)$.

(ii) *A is closed if and only if $A = \mathrm{cl}(A)$.*

(iii) *If $A \subset C$ and C is closed, then $\mathrm{cl}(A) \subset C$.*

(iv) *$\mathrm{cl}(A)$ is closed.*

Proof The first assertion is immediate from the definition. The basic tool for the remainder is Proposition 1.4.8, which states that a set is closed if and only if it contains the limits of all convergent sequences of its points. If we let

$$\mathrm{limit}(A) = \{w \mid \text{there is a sequence of points in } A \text{ convergent to } w\},$$

then $A \subset \mathrm{limit}(A)$, since constant sequences certainly converge. The closure was defined by $\mathrm{cl}(A) = A \cup \mathrm{limit}(A)$, so we actually have $\mathrm{cl}(A) = \mathrm{limit}(A)$. But Proposition 1.4.8 says exactly that A is closed if and only if $\mathrm{limit}(A) \subset A$, so (ii) is established. It also shows that if C is closed and $A \subset C$, then $\mathrm{limit}(A) \subset C$, so we have (iii). The only remaining gap is to show that $\mathrm{cl}(A)$ is actually closed. To do this we need only show that $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$ that is, $\mathrm{limit}(A) = \mathrm{limit}(\mathrm{limit}(A))$. Since $\mathrm{limit}(A) \subset \mathrm{limit}(\mathrm{limit}(A))$ automatically, it remains to show that $\mathrm{limit}(\mathrm{limit}(A)) \subset \mathrm{limit}(A)$. Suppose $z_1, z_2, z_3, \ldots$ is a sequence of points in $\mathrm{limit}(A)$ such that $\lim_{n\to\infty} z_n = w$. We want to show that w is in $\mathrm{limit}(A)$. Each z_n is in $\mathrm{limit}(A)$, so there are points w_n in A with $|w_n - z_n| < 1/n$. This forces $\lim_{n\to\infty} w_n = w$, so $w \in \mathrm{limit}(A)$, as desired. ■

The boundary of a set A is the set of points on the "edge" of A. If w is in the boundary, we should be able to approach it through A and through the complement of A. This leads to the following definition.

Definition 2.5.5 *The boundary of a set $A \subset \mathbf{C}$ is defined by*

$$\mathrm{bd}(A) = \mathrm{cl}(A) \cap \mathrm{cl}(\mathbf{C}\backslash A).$$

It is not hard to see that $\mathrm{cl}(A) = A \cup \mathrm{bd}(A)$. (See Worked Example 2.5.16.)

Global Maximum Modulus Principle Now we are ready for the promised global version of the Maximum Modulus Principle.

Theorem 2.5.6 (Maximum Modulus Principle) *Let A be an open, connected, bounded set in $\mathbb{C}$ and suppose $f : \text{cl}(A) \to \mathbb{C}$ is analytic on A and continuous on* $\text{cl}(A)$. *Then $|f|$ has a finite maximum value on* $\text{cl}(A)$ *which is attained at some point on the boundary of A. If it is also attained in the interior of A, then f must be constant on* $\text{cl}(A)$.

This theorem states that the maximum of f occurs on the boundary of A and that if that maximum is attained on A itself, then f must be constant. This is a very striking result and is certainly a very special property of analytic functions. The values of $|f|$ inside a region A must be smaller than the largest value of $|f|$ on the boundary of A. One must exercise some care. For example, the Maximum Modulus Principle in this form need *not* be true if A is not bounded. In such a case the function need not be bounded on A even if it is on $\text{bd}(A)$. (See Exercise 3.) In applications of this theorem, A will often be the inside of a simple closed curve γ, so $\text{cl}(A)$ will be $A \cup \gamma$ and $\text{bd}(A)$ will be γ.

It is reasonably clear that if A is bounded, so is its closure. If $|z| \le B$ for all z in A and $z_1, z_2, z_3, \ldots$ is a sequence in A converging to w, then $|z_n|$ converges to $|w|$, so $|w| \le B$. Thus $\text{cl}(A) = \text{limit } (A)$ is also bounded by B.

From the Extreme Value Theorem 1.4.20, we know that a continuous real-valued function on a closed bounded set attains a maximum on that set. It follows that if $M' = \sup\{|f(z)| \text{ such that } z \in \text{cl}(A)\}$, then $M' = |f(a)|$ for some $a \in \text{cl}(A)$.

Proof of the Maximum Modulus Principle Since the real-valued function $|f|$ is continuous on the closed bounded set $\text{cl}(A)$, the Extreme Value Theorem says that it attains a finite maximum value M at some point in $\text{cl}(A)$. If it is not attained in the interior, then it must be attained somewhere on the boundary since it is attained somewhere. We next show that if it is attained in the interior, then f must be constant on A. By continuity f will be constant on all of $\text{cl}(A)$, so M is attained on the boundary as well in this case also.

Suppose there is a point a in A at which $|f(a)| = M$. Define subsets of A by

$$A_1 = \{z \in A \mid f(z) = f(a)\} \quad \text{and} \quad A_2 = A \setminus \text{cl}(A_1).$$

If z is in A but not in A_2, then it must be in $\text{cl}(A_1)$. Choose a sequence in A_1 converging to z. Since f is continuous on $\text{cl}(A)$ and has value $f(a)$ at each point in the sequence, we also have $f(z) = a$ and $z \in A_1$. Thus $A \subset A_1 \cup A_2$. Since $A_1 \subset \text{cl}(A_1)$, we certainly have $A_1 \cap A_2 = \varnothing$. The set A_2 is the intersection of the open sets A and $\mathbb{C} \setminus \text{cl}(A_1)$, so it is open. Finally, if $z_0 \in A_1$, then $|f(z_0)| = |f(a)| = M$ Thus, $|f|$ has a local maximum at z_0 and must be constantly equal to $f(a)$ on a disk centered at z_0. Since A is open, we can take the radius small enough so that this disk is contained in A and hence in A_1. This shows that A_1 is open. If the sets A_1 and A_2 were both nonempty, they would disconnect the connected set A. Since $a \in A_1$, we must have $A_2 = \varnothing$ and $A = A_1$. Thus $f(z) = f(a)$ for all z in A and hence also in $\text{cl}(A)$ as claimed. ■

Schwarz Lemma The next theorem is an example of an application of the Maximum Modulus Theorem. This result is not one of the most basic results of the theory, but it further indicates the type of severe restrictions that analyticity imposes. This result will be quite useful in Chapter 5.

Lemma 2.5.7 (Schwarz Lemma) *Let f be analytic on the open unit disk $A = \{z \in \mathbb{C} \mid |z| < 1\}$ with $f(0) = 0$ and $|f(z)| \le 1$ for each z in A. Then $|f'(0)| \le 1$ and $|f(z)| \le |z|$ for each z in A. If $|f'(0)| = 1$ or if there is a point z_0 other than 0 in A with $|f(z_0)| = |z_0|$, then there is a constant c with $|c| = 1$ and $f(z) = cz$ for all z in A.*

Proof Let $g(z) = f(z)/z$ if $z \ne 0$ and $g(0) = f'(0)$. The function g is analytic on A because it is continuous on A and analytic on $A\backslash\{0\}$ (see Corollary 2.4.11 to Morera's Theorem). Let

$$A_r = \{z \text{ such that } |z| \le r\}$$

for $0 < r < 1$ (see Figure 2.5.2). Then g is analytic on A_r, and on $|z| = r$, $|g(z)| = |f(z)/z| \le 1/r$. By the Maximum Modulus Principle 2.5.6, $|g(z)| \le 1/r$ on all of A_r; that is, $|f(z)| \le |z|/r$ on A_r. Holding $z \in A$ fixed, we can let $r \to 1$ to obtain $|f(z)| \le |z|$. Clearly, $|g(0)| \le 1$; that is, $|f'(0)| \le 1$.

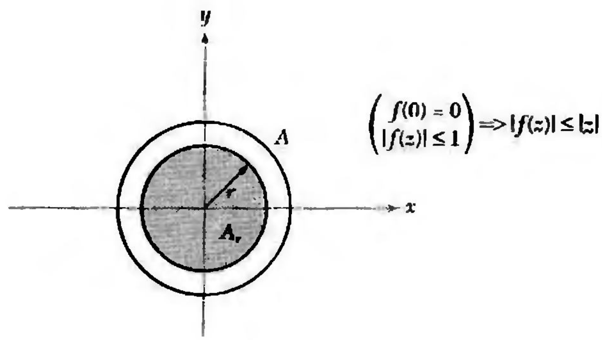


Figure 2.5.2: Schwarz Lemma

If $|f(z_0)| = |z_0|$, $z_0 \ne 0$, then $|g(z_0)| = 1$ is maximized in A_r, where $|z_0| < r < 1$, so g is constant on A_r. The constant is independent of r. (Why?) Similarly, if $|f'(0)| = 1$, then $|g|$ has a local maximum at 0, so g is constant on A. ∎

The Schwarz Lemma is a tool for many useful geometric results of complex analysis. A generalization that is useful for obtaining accurate estimates of bounds for functions is known as the ***Lindelöf Principle***, which is as follows: *Suppose that f and g are analytic on $|z| < 1$, that g maps $|z| < 1$ one to one onto a set G, that $f(0) = g(0)$, and that the range of f is contained in G. Then $|f'(0)| \le |g'(0)|$*

and the image of $|z| < r$, *for* $r < 1$, *under* f *is contained in its image under* g. This principle is often used with g a linear fractional transformation, i.e., $g(z) = (az+b)/(cz+d)$. As will be proved in Chapter 5, they take circles into circles, so the g image of the disk $|z| < r$ is usually easy to find (see Exercise 4 for further details).[14]

Harmonic Functions and Harmonic Conjugates If f is analytic on A and $f = u + iv$, we know that u and v are infinitely differentiable and are harmonic (by Theorem 2.4.6 and Proposition 1.5.12). Let us now show that the converse is also true.

Proposition 2.5.8 *Let A be a region in $\mathbb{C}$ and let u be a twice continuously differentiable harmonic function on A. Then u is C^∞, and in a neighborhood of each point $z_0 \in A$, u is the real part of some analytic function. If A is simply connected, there is an analytic function f on A such that $u = \operatorname{Re} f$.*

Thus, a harmonic function is always the real part of an analytic function f (or the imaginary part of the analytic function if), at least locally, and on all of the domain of that function if the domain is simply connected.

Proof We prove the last statement of the theorem first. Consider the function $g = (\partial u/\partial x) - i(\partial u/\partial y)$. We claim that g is analytic. Setting $g = U + iV$ where $U = \partial u/\partial x$ and $V = -\partial u/\partial y$, we must check that U and V have continuous first partials and that they satisfy the Cauchy-Riemann equations. Indeed, the functions $\partial U/\partial x = \partial^2 u/\partial x^2$ and $\partial V/\partial y = -\partial^2 u/\partial y^2$ are continuous by assumption and are equal since $\nabla^2 u = 0$. Also, by the equality of mixed partials,

$$\frac{\partial U}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y} = -\frac{\partial V}{\partial x}.$$

Thus, we conclude that g is analytic. Furthermore, if A is simply connected there is an analytic function f on A such that $f' = g$ (by the Antiderivative Theorem 2.2.5). Let $f = \tilde{u} + i\tilde{v}$. Then $f' = (\partial\tilde{u}/\partial x) - i(\partial\tilde{u}/\partial y)$, and thus $\partial\tilde{u}/\partial x = \partial u/\partial x$ and $\partial\tilde{u}/\partial y = \partial u/\partial y$. Thus $\tilde{u}$ differs from u by a constant. Adjusting f by subtracting this constant, we get $u = \operatorname{Re} f$.

Now we prove the first statement. If D is a disk around z_0 in A, it is simply connected. Therefore, as a result of what we have just proven, we can write $u = \operatorname{Re} f$ for some analytic f on D. Thus since f is C^∞, u is also C^∞ on a neighborhood of each point in A, so is C^∞ on A. ■

Recall that when there is an analytic function f such that u and v are related by $f = u + iv$, we say that u and v are ***harmonic conjugates***. Since if is analytic, $-v$ and u are also harmonic conjugates. Be careful! The order matters

[14] For a useful survey of some of the more geometric results and a bibliography, see T. H. MacGregor, Geometric Problems in Complex Analysis, *Am. Math. Monthly*, May (1972), 447

If v is a harmonic conjugate of u, then u is probably not a harmonic conjugate of v. Instead, $-u$ is! The preceding proposition says that *on any simply connected region A, any harmonic function has a harmonic conjugate $v = \operatorname{Im} f$.* Since the Cauchy-Riemann equations ($\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$) must hold, v is uniquely determined up to the addition of a constant. These equations may be used as a practical method of finding v when u is given (see Worked Example 1.5.20). Another way of obtaining the harmonic conjugate of u on a disk, by defining it directly in terms of an integral, was indicated in Exercise 32 of §1.5.

Mean Value Property One reason why Proposition 2.5.8 is important is that it enables us to deduce properties of harmonic functions from corresponding properties of analytic functions. This is done in the next theorem.

Theorem 2.5.9 (Mean Value Property for Harmonic Functions) *Let u be harmonic on a region containing a circle of radius r around $z_0 = x_0 + iy_0$ and its interior. Then*

$$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta. \tag{2.5.2}$$

Proof By Proposition 2.5.8, there is an analytic function f defined on a region containing this circle and its interior such that $u = \operatorname{Re} f$. This containing region may be chosen to be a slightly larger disk. The existence of a slightly larger circle in A is intuitively clear; the precise proof is given in Worked Example 1.4.27. By the Mean Value Property for f,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta.$$

Taking the real part of both sides of this equation gives the desired result. ■

Maximum Principle for Harmonic Functions From the Mean Value Property we can deduce, in a way similar to the way we deduced Theorem 2.5.1, the following fact.

Theorem 2.5.10 (Local Maximum Principle for Harmonic Functions) *Let u be harmonic on a region A. Suppose that u has a relative maximum at $z_0 \in A$ (that is, $u(z) \le u(z_0)$ for z near z_0). Then u is constant in a neighborhood of z_0.*

In this theorem "maximum" can be replaced by "minimum" (see Exercise 6).

Instead of actually going through a proof for $u(z)$ similar to the proof of Theorem 2.5.1, we can use that result to give a quick proof.

Proof On a disk around $z_0, u = \text{Re} f$ for some analytic f. Then $e^{f(z)}$ is analytic and $|e^{f(z)}| = e^{u(z)}$. Thus, since e^x is strictly increasing in x for all real x, the maxima of u are the same as those of $|e^f|$. By Theorem 2.5.1, e^f is constant in a neighborhood of z_0; therefore, e^u and hence u are also (again because e^x is strictly increasing for x real). ■

From this result we deduce, exactly as the Maximum Modulus Principle was deduced from its local version, the following "global" version.

Theorem 2.5.11 (Global Maximum Principle for Harmonic Functions) *Suppose that $A \subset \mathbb{C}$ is an open, connected, and bounded set. Let $u : \text{cl}(A) \to \mathbb{R}$ be continuous and harmonic on A and let M be the maximum of u on* bd(A). *Then*

(i) $u(x, y) \leq M$ *for all* $(x, y) \in A$.

(ii) *If* $u(x, y) = M$ *for some* $(x, y) \in A$, *then u is constant on A.*

There is a corresponding result for the minimum. Let m denote the minimum of u on bd(A). Then

(i) $u(x, y) \geq m$ for $(x, y) \in A$.

(ii) If $u(x, y) = m$ for some $(x, y) \in A$, then u is constant.

The Minimum Principle for Harmonic Functions may be deduced by applying the Maximum Principle to $-u$.

Dirichlet Problem for the Disk and Poisson's Formula There is a very important problem common to mathematics, physics, and engineering called the ***Dirichlet Problem.*** It is this: Let A be an open bounded region and let u_0 be a given continuous function on bd(A). Find a real-valued function u on cl(A) that is continuous on cl(A) and harmonic on A and that equals u_0 on bd(A).

There are (reasonably difficult) theorems stating that if the boundary bd(A) is "sufficiently smooth," then there always is a solution u. However, we can easily show that the solution is always unique.

Theorem 2.5.12 (Uniqueness for the Dirichlet Problem) *The solution to the Dirichlet Problem is unique (assuming that there is a solution).*

Proof Let u and $\tilde{u}$ be two solutions. Let $\phi = u - \tilde{u}$. Then ϕ is harmonic and $\phi = 0$ on bd(A). We must show that $\phi = 0$.

By the maximum principle for harmonic functions, $\phi(x, y) \leq 0$ inside A. Similarly, from the minimum principle, $\phi(x, y) \geq 0$ on A. Thus $\phi = 0$. ■

We want to find the solution to the Dirichlet Problem for the case where the region is an open disk. To do so we derive a formula that explicitly expresses the values of the solution in terms of its values on the boundary of the disk.

Theorem 2.5.13 (Poisson's Formula) *Assume that u is defined and continuous on the closed disk $\{z$ such that $|z| \le r\}$ and is harmonic on the open disk $D(0;r) = \{z$ such that $|z| < r\}$. Then for $\rho < r$, we have the real form of Poisson's Formula*

$$u(\rho e^{i\phi}) = \frac{r^2-\rho^2}{2\pi}\int_0^{2\pi}\frac{u(re^{i\theta})}{r^2-2r\rho\cos(\phi-\theta)+\rho^2}d\theta,$$

which is equivalent to the complex form of Poisson's Formula

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}d\theta.$$

The technical parts of the following proof require an acquaintance with the idea of *uniform convergence*. The student who has not studied uniform convergence in advanced calculus may wish to reread this proof after studying §3.1, where the relevant ideas are discussed. Notice that if we set $z = 0$ in the complex form of Poisson's Formula, we recover the Mean Value Property of harmonic functions.

Proof First note that since u is harmonic on $D(0;r)$ and $D(0;r)$ is simply connected, there is an analytic function f defined on $D(0,r)$ such that $u = \operatorname{Re} f$. Next, let $0 < s < r$ and let γ_s be the circle $|z| = s$. Then, by Cauchy's Integral Formula 2.4.4, we have

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_s}\frac{f(\zeta)}{\zeta - z}d\zeta$$

for all z such that $|z| < s$. We next manipulate this expression into a form suitable for taking real parts. To do so, let $\tilde{z} = s^2/\bar{z}$, which is called the ***reflection*** of z in the circle $|\zeta| = s$. Reflection is pictured geometrically in Figure 2.5.3.

Thus if z lies inside the circle, then $\tilde{z}$ lies outside the circle, and therefore

$$\frac{1}{2\pi i}\int_{\gamma_s}\frac{f(\zeta)}{\zeta-\tilde{z}}d\zeta = 0$$

for $|z| < s$. Subtracting the preceding integral from

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_s}\frac{f(\zeta)}{\zeta - z}d\zeta,$$

we obtain

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_s} f(\zeta)\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\tilde{z}}\right)d\zeta.$$

Observing that $|\zeta| = s$, we can simplify as follows:

$$\begin{aligned}\frac{1}{\zeta-z}-\frac{1}{\zeta-\tilde{z}} &= \frac{1}{\zeta-z}-\frac{1}{\zeta-|\zeta|^2/\bar{z}} = \frac{1}{\zeta-z}-\frac{\bar{z}}{\zeta(\bar{z}-\bar{\zeta})}\\ &= \frac{-\zeta\bar{z}+|\zeta|^2+\zeta\bar{z}-|z|^2}{\zeta|\zeta-z|^2} = \frac{|\zeta|^2-|z|^2}{\zeta|\zeta-z|^2}.\end{aligned}$$

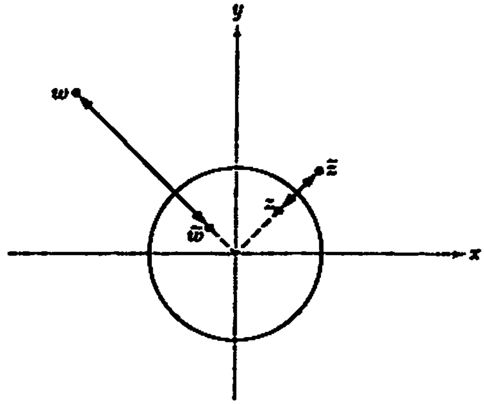


Figure 2.5.3: Reflection of a complex number in a circle.

Hence, we have

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_s} \frac{f(\zeta)(|\zeta|^2 - |z|^2)}{\zeta|\zeta - z|^2}d\zeta;$$

that is,

$$f(\rho e^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(se^{i\theta})(s^2 - \rho^2)}{|se^{i\theta} - \rho e^{i\phi}|^2}\,d\theta,$$

where $\rho < s$. Noting that $|se^{i\theta} - \rho e^{i\phi}|^2 = s^2 + \rho^2 - 2s\rho\cos(\phi - \theta)$ and taking the real parts on both sides of the equations, we obtain

$$u(\rho e^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{u(se^{i\theta})(s^2 - \rho^2)d\theta}{s^2 + \rho^2 - 2s\rho\cos(\phi - \theta)}.$$

Keeping ρ and ϕ fixed, note that this formula is valid for any s such that $\rho < s < r$. Since u is continuous on the closure of $D(0;r)$ and since the function $s^2 + \rho^2 - 2s\rho\cos(\phi - \theta)$ is never zero whenever $s > \rho$, we conclude that for $s > \rho$.

$$[u(se^{i\theta})(s^2 - \rho^2)]/[s^2 + \rho^2 - 2s\rho\cos(\phi - \theta)]$$

is a continuous function of s and θ and hence (with ρ, ϕ fixed) is uniformly continuous on the compact set, $0 \le \theta \le 2\pi, (r + \rho)/2 \le s \le r$. Consequently, as $s \to r$,

$$\frac{u(se^{i\theta})(s^2 - \rho^2)}{s^2 + \rho^2 - 2s\rho\cos(\phi - \theta)} \to \frac{u(re^{i\theta})(r^2 - \rho^2)}{r^2 + \rho^2 - 2r\rho\cos(\phi - \theta)}$$

uniformly in θ, which implies that as $s \to r$,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{u(se^{i\theta})(s^2 - r^2)}{s^2 + \rho^2 - 2s\rho\cos(\phi - \theta)}d\theta \to \frac{1}{2\pi}\int_0^{2\pi} \frac{u(re^{i\theta})(r^2 - \rho^2)}{r^2 + \rho^2 - 2r\rho\cos(\phi - \theta)}d\theta.$$

Solution $A \subset \mathrm{cl}(A)$ and $\mathrm{bd}(A) \subset \mathrm{cl}(A)$, so $A \cup \mathrm{bd}(A) \subset \mathrm{cl}(A)$. In the other direction, if z is in $\mathrm{cl}(A)$ and not in A, then

$$z \in (\lim(A)) \cap (\mathbb{C}\backslash A) \subset \mathrm{cl}(A) \cap \mathrm{cl}(\mathbb{C}\backslash A) = \mathrm{bd}(A).$$

Thus, $\mathrm{cl}(A) \subset A \cup \mathrm{bd}(A)$.

Example 2.5.17 *Suppose $u : \mathbb{C} \to \mathbb{R}$ is a bounded harmonic function. Show that u must be constant.*

Solution Since u is harmonic on the simply connected set $\mathbb{C}$, Proposition 2.5.8 says that there is a harmonic function $v : \mathbb{C} \to \mathbb{R}$ such that $f = u + iv : \mathbb{C} \to \mathbb{C}$ is analytic. We have $|e^{f(z)}| = |e^{u(z)+iv(z)}| = e^{u(z)}$. Since $u(z)$ remains bounded, so does the entire function $e^{f(z)}$. By Liouville's theorem, $e^{f(z)}$ must be a constant, say, K. For each z, $f(z)$ must be a choice of logarithm for K. But, since f is continuous, it cannot switch from one value of $\log K$ to another, so f must be constant. Thus $u = \mathrm{Re}\, f$ is also constant.

Example 2.5.18 *Suppose f and g are one to one analytic functions from the unit disk D onto D that satisfy $f(0) = g(0)$ and $g'(0) = f'(0) \neq 0$. Show that $f(z) = g(z)$ for all z in D.*

Solution The function $h(z) = g^{-1}(f(z))$ is analytic from D to D and $h(0) = g^{-1}(f(0)) = g^{-1}(g(0)) = 0$. Since $g(h(z)) = f(z)$, we have $g'(h(0)).h'(0) = f'(0)$, so $h'(0) = f'(0)/g'(0) = 1$. The Schwarz Lemma shows that $h(z) = cz$ for a constant c; since $h'(0) = 1, c = 1$. Thus $f(z) = g(h(z)) = g(z)$. We will see in Chapter 5 that the assumption that f and g are one to one *forces* the derivative to be nonzero, so the assumption of nonzero derivatives is really superfluous.

Exercises

1. Find the maximum of $|e^z|$ on $|z| \leq 1$.

2. Find the maximum of $|\cos z|$ on $[0, 2\pi] \times [0, 2\pi]$.

3. Give an example to show that the interpretation of the Maximum Modulus Principle that reads "The absolute value of an analytic function on a region is always smaller than its maximum on the boundary of the region" is false if the region is not bounded. The region in your example should be something other than all of $\mathbb{C}$ so that the boundary is not empty.

4.* (a) Let the mapping T be defined by $T(z) = R(z - z_0)/(R^2 - \bar{z}_0 z)$. Show that for $|z_0| < R$, T takes the open disk of radius R one to one onto the disk of radius 1 and takes z_0 to the origin. *Hint:* Use the Maximum Modulus Theorem and verify that $z_0 \mapsto 0$ and $|z| = R$ implies that $|Tz| = 1$.

13. Let f be analytic and let $f'(z) \neq 0$ on a region A. Let $z_0 \in A$ and assume that $f(z_0) \neq 0$. Given $\epsilon > 0$, show that there exist a $z \in A$ and a $\zeta \in A$ such that $|z - z_0| < \epsilon, |\zeta - z_0| < \epsilon$, and

$$|f(z)| > |f(z_0)| \qquad |f(\zeta)| < |f(z_0)|.$$

Hint: Use the Maximum Modulus Theorem.

14. Prove ***Hadamard's Three-circle Theorem***: Let f be analytic on a region containing the set R in Figure 2.5.4. Let $R = \{z \mid r_1 \leq |z| \leq r_3\}$ and assume $0 < r_1 < r_2 < r_3$. Let M_1, M_2, M_3 be the maxima of $|f|$ on the circles $|z| = r_1, r_2, r_3$, respectively. Then we have the inequality

$$M_2^{\log(r_3/r_1)} \leq M_1^{\log(r_3/r_2)} M_3^{\log(r_2/r_1)}.$$

Hint: Let $\lambda = -\log(M_3/M_1)/\log(r_3/r_1)$ and consider $g(z) = z^\lambda f(z)$. Apply the maximum principle to g; be careful about the domain of analyticity of g

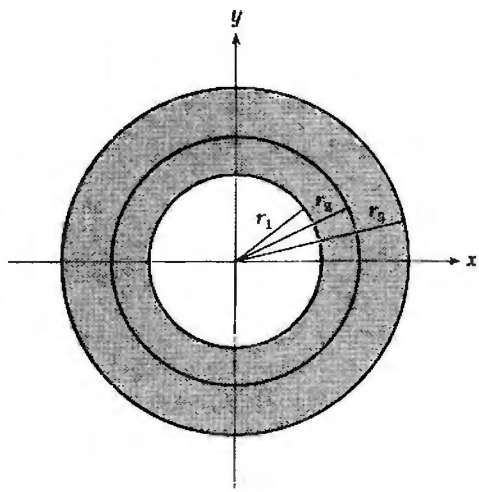


Figure 2.5.4: Hadamard's Three-circle Theorem.

15. Let g be analytic on $\{z \text{ such that } |z| < 1\}$ and assume that $|g(z)| = |z|$ for all $|z| < 1$. Show that $g(z) = e^{i\theta} z$ for some constant $\theta \in [0, 2\pi]$. *Hint:* Use the Schwarz Lemma.

16.* Prove: If u is continuous and satisfies the Mean Value Property, then u is C^∞ and is harmonic. *Hint:* Use Poisson's Formula.

17. Evaluate $\displaystyle\int_\gamma \frac{dz}{z^2 - 1}$ where γ is the circle $|z| = 2$.

18. The function $f(z)$ is analytic over the whole complex plane and $\operatorname{Im} f \leq 1$. Prove that f is a constant.

8.* Let f be entire and let $|f(z)| \leq M$ for z on the circle $|z| = R$; let R be fixed. Prove that

$$|f^{(k)}(re^{i\theta})| \leq \frac{k!M}{(R-r)^k} \quad k = 0, 1, 2, \ldots$$

for all $0 \leq r < R$.

9. Find a harmonic conjugate for $u(x, y) = \dfrac{x^2 + y^2 - x}{(x-1)^2 + y^2}$ on a suitable domain.

10. Let f be analytic on A and let $f'(z_0) \neq 0$. Show that if γ is a sufficiently small circle centered at z_0, then

$$\frac{2\pi i}{f'(z_0)} = \int_\gamma \frac{dz}{f(z) - f(z_0)}.$$

Hint: Use the Inverse Function Theorem.

11. Evaluate $\displaystyle\int_0^{2\pi} e^{-i\theta} e^{e^{i\theta}}\, d\theta$.

12.* Let f and g be analytic in a region A and let $g'(z) \neq 0$ for all $z \in A$; let g be one to one and let γ be a closed curve in A. Then for z not on γ, prove that

$$f(z)I(\gamma; z) = \frac{g'(z)}{2\pi i} \int_\gamma \frac{f(\zeta)}{g(\zeta) - g(z)} d\zeta.$$

Hint: Apply the Cauchy Integral Formula to $h(\zeta) = f(\zeta)(\zeta - z)/(g(\zeta) - g(z))$ for $z \neq \zeta$ and $h(\zeta) = f(\zeta)/g'(\zeta)$. Apply this result to the case in which $g(z) = e^z$.

13. Simplify: $e^{\log i}$; $\log i$; $\log(-i)$; $i^{\log(-1)}$.

14. Let $A = \mathbf{C}$ minus the negative real axis and zero. Show that $\log z = \int_{\gamma_z} d\zeta/\zeta$, where γ_z is any curve in A joining 1 to z. Is A simply connected?

15. Let f be analytic on a region A and let f be nonzero. Let γ be a closed curve homotopic to a point in A. Show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0.$$

16.* Let f be analytic on and inside the unit circle. Suppose that the image of the unit circle $|z| = 1$ lies in the disk $D = \{z \text{ such that } |z - z_0| < r\}$. Show that the image of the whole inside of the unit circle lies in D. Illustrate with e^z.

17. Is $\displaystyle\int_\gamma x\,dx + x\,dy$ always zero if γ is a closed curve?

Chapter 3

Series Representation of Analytic Functions

In Chapter 2 we defined an analytic function to be one that has a derivative in the sense of complex functions of a complex variable. There is an important alternative way to view an analytic function. In some developments of complex function theory, a function f is called *analytic* if, near each point z_0 in its domain, it is locally representable as a convergent power series centered at z_0.[1] As we shall see, this series must be the *Taylor series* of f centered at z_0, namely

$$f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2}f''(z_0)(z - z_0)^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

In other words, one could alternatively define an analytic function to be one which is infinitely differentiable and whose Taylor series converges to the function. We will reconcile this alternative approach with ours in this chapter.

With real variables, both the question of infinite differentiability as well as the convergence of the Taylor series of a given function can present a problem. For example, the function

$$f(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ -x^2 & \text{for } x < 0 \end{cases}$$

is differentiable, but $f'(x) = 2|x|$. Thus, the second derivative does not exist at 0. Even if all the derivatives exist, the Taylor series might not converge to the function. The function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

[1] See, for example, H. Cartan, *Elementary Theory of Analytic Functions of One or Several Complex Variables* (Reading, Mass.: Addison-Wesley, 1963 and New York: Dover Publications, 1995).

is an example. Using induction one can check that $f^{(k)}(0)$ exists for all k (in the real-variable sense) and $f^{(k)}(0) = 0$. Here all coefficients of the Taylor series at 0 are 0, so the resulting series is zero, which does not equal $f(x)$ in any nontrivial interval around 0.

A nice thing about complex analysis is that neither of these difficulties arises. This reinforces the fact that assuming the existence of a complex derivative is much stronger than assuming the existence of a real derivative. We discovered in Chapter 2 that as soon as the first derivative exists on a region, all the higher ones must also. We will find in this chapter that the second difficulty also disappears. If f is analytic on a region A and z_0 is in A, then the Taylor series of f centered at z_0 automatically converges to f on the largest open disk centered at z_0 and contained in A.

The reader is probably familiar with the geometric series

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=0}^{\infty} t^n.$$

which is valid provided $|t| < 1$. We will show in the first section that this works just as well for complex as for real numbers. In §3.2 we will use it to expand the integrand in the Cauchy Integral Formula as an infinite series, integrate the series term by term, and use the Cauchy Integral Formula for Derivatives 2.4.6 to recognize the resulting coefficients as the correct ones for a Taylor series.

Building on the preparation in §3.1, in §3.3 we investigate the series representation of a function analytic on a deleted neighborhood, that is, a function with an isolated singularity. The resulting series, called the *Laurent series*, yields valuable information about the behavior of functions near singularities, and this behavior is the key to the subject of residues and its subsequent applications.

3.1 Convergent Series of Analytic Functions

We shall use the Cauchy Integral Formula 2.4.4 to determine when the limit of a convergent sequence or series of analytic functions is an analytic function and when the derivative (or integral) of the limit is the limit of the derivative (or integral) of the terms in the sequence or series. The basic type of convergence studied in this chapter is uniform convergence; the Weierstrass M Test is a basic tool used to determine such convergence. In §3.2 we shall be especially interested in the special case of power series, but we should be aware that some important functions are convergent series that are not power series, such as the Riemann zeta function (see Worked Example 3.1.15).

The proofs of the first few results are slightly technical, and since they are analogous to the case of real series, they appear at the end of the section.

Convergence of Sequences and Series We begin with some basic definitions and terminology.

Definition 3.1.1 *A sequence $z_n, n = 1, 2, 3, \ldots$ of complex numbers is said to converge to a complex number z_0 if, for each $\epsilon > 0$, there is an integer N such that*

$$n \geq N \quad \textit{implies that} \quad |z_n - z_0| < \epsilon.$$

Convergence of z_n to z_0 is denoted by $z_n \to z_0$.

An infinite series $\sum_{k=1}^{\infty} a_k$ of complex numbers is said to converge to S, and we write

$$\sum_{k=1}^{\infty} a_k = S$$

if the sequence of partial sums defined by $s_n = \sum_{k=1}^{n} a_k$ converges to S.

The limit of a convergent sequence is unique; that is, a sequence can converge to only one point z_0. (This and other properties of limits were discussed in §1.4.) A sequence z_n converges iff it is a ***Cauchy sequence***, in other words, if, for each $\epsilon > 0$, there is an N such that $n, m \geq N$ implies that $|z_n - z_m| < \epsilon$. (Equivalently, the definition of Cauchy sequence can read: For each $\epsilon > 0$ there is an N such that $n \geq N$ implies that $|z_n - z_{n+p}| < \epsilon$ for every integer $p = 0, 1, 2, \ldots$.) This property of $\mathbb{C} = \mathbb{R}^2$ follows from the corresponding property of $\mathbb{R}$, and we shall accept it from advanced calculus.

Corresponding statements for the series $\sum_{k=1}^{\infty} a_k$ can be made if we consider the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$. Since $s_{n+p} - s_n = \sum_{k=n+1}^{n+p} a_k$, the Cauchy criterion for sequences becomes the ***Cauchy criterion for series***:

> *$\sum_{k=1}^{\infty} a_k$ converges iff, for each $\epsilon > 0$, there is an N such that $n \geq N$ implies that*
>
> $$\left| \sum_{k=n+1}^{n+p} a_k \right| < \epsilon \quad \textit{for all} \quad p = 1, 2, 3, \ldots.$$

As a particular case of the Cauchy criterion, with $p = 1$ we see that

$$\textit{if} \quad \sum_{k=1}^{\infty} a_k \quad \textit{converges, then} \ a_k \to 0.$$

The converse is not necessarily true, as the harmonic series $\sum_{k=1}^{\infty} 1/k$ from calculus demonstrates.

As with real series, a complex series $\sum_{k=1}^{\infty} a_k$ is said to ***converge absolutely*** if $\sum_{k=1}^{\infty} |a_k|$ converges. Using the Cauchy criterion for series, we get the following proposition.

Proposition 3.1.2 *If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then it converges.*

The proof of this proposition is found at the end of the section. The example $\sum_{k=1}^{\infty}(-1)^k/k$ from calculus shows that the converse need not be true; that is, this is an example of a series that converges, but not absolutely.

This proposition is important because $\sum_{k=1}^{\infty}|a_k|$ is a *real* series, and the usual tests for real series that we know from calculus can be applied. Some of those tests are included in the next proposition (again the proof appears at the end of the section).

Proposition 3.1.3 *The following tests for the convergence of series hold.*

(i) ***Geometric series:*** *If $|r| < 1$, then $\sum_{n=0}^{\infty} r^n$ converges to $1/(1-r)$ and diverges (does not converge) if $|r| \geq 1$.*

(ii) ***Comparison test:*** *If $\sum_{k=1}^{\infty} b_k$ converges and $0 \leq a_k \leq b_k$, then $\sum_{k=1}^{\infty} a_k$ converges; if $\sum_{k=1}^{\infty} c_k$ diverges and $0 \leq c_k \leq d_k$, then $\sum_{k=1}^{\infty} d_k$ diverges.*

(iii) ***p-series test:*** *$\sum_{n=1}^{\infty} n^{-p}$ converges if $p > 1$ and diverges to ∞ (that is, the partial sums increase without bound) if $p \leq 1$.*

(iv) ***Ratio test:*** *Suppose that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists and is strictly less than 1. Then $\sum_{n=1}^{\infty} a_n$ converges absolutely. If the limit is strictly greater than 1, the series diverges. If the limit equals 1, the test is inconclusive.*

(v) ***Root test:*** *Suppose that $\lim_{n\to\infty}(|a_n|)^{1/n}$ exists and is strictly less than 1. Then $\sum_{n=1}^{\infty} a_n$ converges absolutely. If the limit is strictly greater than 1, the series diverges; if the limit equals 1, the test is inconclusive.*

There are a few other tests that we shall occasionally call upon from calculus such as the alternating series test and the integral test. We assume the reader will review them as the need arises.

Uniform Convergence Suppose that $f_n : A \to \mathbf{C}$ is a sequence of functions all defined on the set A. The sequence is said to ***converge pointwise*** iff, for each $z \in A$, the sequence $f_n(z)$ converges. The limit defines a new function $f(z)$ on A. A more important kind of convergence is called uniform convergence and is defined as follows.

Definition 3.1.4 *A sequence $f_n : A \to \mathbf{C}$ of functions defined on a set A is said to* ***converge uniformly*** *to a function f if, for each $\epsilon > 0$, there is an N such that $n \geq N$ implies that $|f_n(z) - f(z)| < \epsilon$ for all $z \in A$. This is written "$f_n \to f$ uniformly on A."*

A series $\sum_{k=1}^{\infty} g_k(z)$ is said to ***converge pointwise*** *if the corresponding partial sums $s_n(z) = \sum_{k=1}^{n} g_k(z)$ converge pointwise. A series $\sum_{k=1}^{\infty} g_k(z)$ is said to* ***converge uniformly*** *iff $s_n(z)$ converges uniformly.*

Evidently, *uniform convergence implies pointwise convergence*. The difference between uniform and pointwise convergence is as follows. For pointwise convergence, given $\epsilon > 0$, the N required is allowed to vary from point to point, whereas for uniform convergence we must be able to find a *single N that works for all z*.

It is difficult to draw the graph of a complex valued function of a complex variable, since it would require four real dimensions, but the corresponding notions for real-valued functions are instructive to illustrate. The geometric meaning of uniform convergence is shown in Figure 3.1.1.

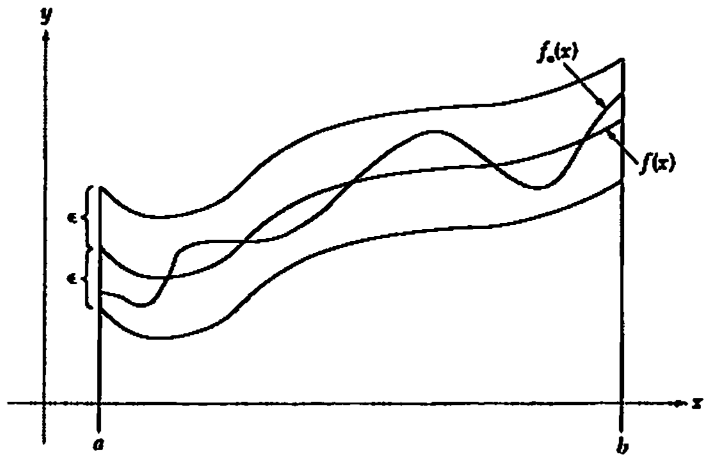


Figure 3.1.1: Uniform convergence on an interval $[a, b]$.

If $\epsilon > 0$, then for large enough n, the graph $y = f_n(x)$ must stay inside the "ϵ-tube" around the graph of f (the tube's width is measured in the vertical direction).

The concept of uniformity depends not only on the functions involved but also on the set on which we are working. Convergence might be uniform on one set but not on a larger set. The following example illustrates this point. The sequence of functions $f_n(x) = x^n$ converges pointwise to the zero function $f(x) = 0$ for x in the half-open interval $[0, 1[$, but the convergence is not uniform. The function value x^n takes much longer to get close to 0 for x close to 1 than for x close to 0; by taking x close enough to 1, we need arbitrarily large values of n. The convergence is uniform on any closed subinterval $[0, r]$ with $r < 1$. Since the worst case is at $x = r$, whatever n works there also works for all smaller x. See Figure 3.1.2.

Theorem 3.1.5 (Cauchy Criterion)

(i) *A sequence $f_n(z)$ converges uniformly on A iff, for each $\epsilon > 0$, there is an N such that $n \geq N$ implies that $|f_n(z) - f_{n+p}(z)| < \epsilon$ for all $z \in A$ and all $p = 1, 2, 3, \ldots$.*

(ii) *A series $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on A iff, for each $\epsilon > 0$, there is an*

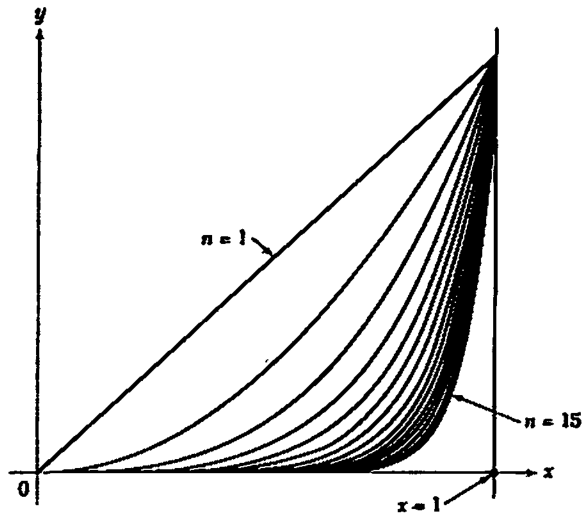


Figure 3.1.2: Convergence of x^n to 0 is not uniform on $\{x \mid 0 \leq x < 1\}$.

N such that $n \geq N$ implies that

$$\left|\sum_{k=n+1}^{n+p} g_k(z)\right| < \epsilon$$

for all $z \in A$ and all $p = 1, 2, \ldots$.

The next result states a basic property of uniform convergence.

Proposition 3.1.6 *If the sequence f_n consists of continuous functions defined on A and if $f_n \to f$ uniformly, then f is continuous on A. Similarly, if the functions $g_k(z)$ are continuous and $g(z) = \sum_{k=1}^{\infty} g_k(z)$ converges uniformly on A, then g is continuous on A.*

Propositions 3.1.5 and 3.1.6 are proved at the end of this section.

Thus, *a uniform limit of continuous functions is continuous*. If the convergence is not uniform, then the limit might be discontinuous. For example, let

$$f_n(x) = \begin{cases} -1 & \text{for } x \leq -1/n \\ nx & \text{for } -1/n < x < 1/n \\ 1 & \text{for } 1/n \leq x \end{cases}$$

and

$$f(x) = \begin{cases} -1 & \text{for } -\infty < x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } 0 < x < \infty, \end{cases}$$

as illustrated in Figure 3.1.3. The functions f_n converge pointwise to f on the whole line, but the convergence is not uniform on any interval that contains 0, since for very small nonzero values of x, n may have to be quite large to bring $f_n(x)$ within a specified distance of $f(x)$. Each of the functions f_n is continuous, but the limit function is not.

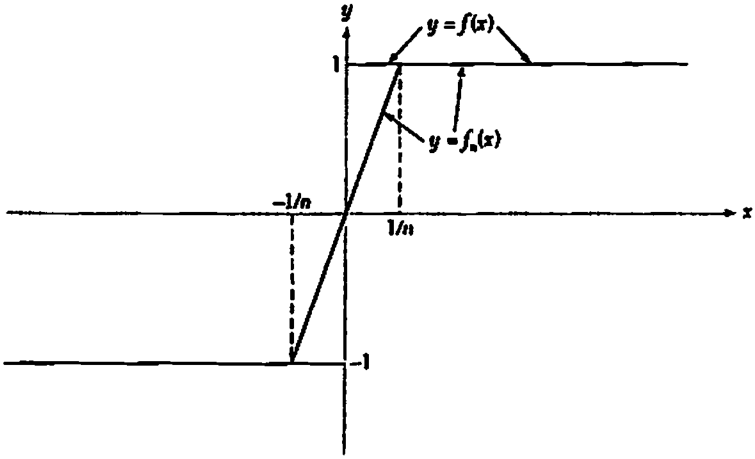


Figure 3.1.3: A nonuniform limit of continuous functions need not be continuous.

Weierstrass M Test The Weierstrass M Test is one of the most useful theoretical and practical tools for showing that a series converges uniformly. It does not always apply, but it is effective in many cases.

Theorem 3.1.7 (Weierstrass M Test) *Let g_n be a sequence of functions defined on a set $A \subset \mathbb{C}$. Suppose that there is a sequence of real constants $M_n \geq 0$ such that both of the following conditions hold.*

(i) $|g_n(z)| \leq M_n$ *for all* $z \in A$

(ii) $\sum_{n=1}^{\infty} M_n$ *converges*

Then $\sum_{n=1}^{\infty} g_n$ *converges absolutely and uniformly on* A.

Proof Since $\sum M_n$ converges, for any $\epsilon > 0$ there is an N such that $n \geq N$ implies $\sum_{k=n+1}^{n+p} M_k < \epsilon$ for all $p = 1, 2, 3, \ldots$. (Absolute value bars are not needed because $M_n \geq 0$.) Thus, $n \geq N$ implies

$$\left| \sum_{k=n+1}^{n+p} g_k(z) \right| \leq \sum_{k=n+1}^{n+p} |g_k(z)| \leq \sum_{k=n+1}^{n+p} M_k < \epsilon,$$

so by the Cauchy Criterion 3.1.5, we have convergence, both absolute and uniform, as desired. ■

Example Consider the series $g(z) = \sum_{n=1}^{\infty} z^n/n$. We claim that *this series converges uniformly on each of the sets* $A_r = \{z \text{ such that } |z| \le r\}$, where $0 \le r < 1$. (We cannot let $r = 1$.) Here $g_n(z) = z^n/n$ and $|g_n(z)| = |z|^n/n \le r^n/n$ since $|z| \le r$.

To show this, we let $M_n = r^n/n$. Since $0 \le M_n \le r^n$, the series $\sum M_n$ converges by comparison to the convergent geometric series $\sum_{n=0}^{\infty} r^n$. Therefore, our series converges uniformly on A_r by the Weierstrass M Test. It converges pointwise on the set $A = \{z \in \mathbb{C} \mid |z| < 1\}$ since each z in A is in A_r for r close enough to 1. (See Figure 3.1.4).

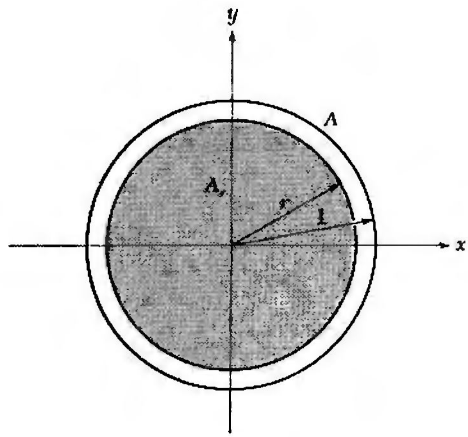


Figure 3.1.4: Region of convergence of $\sum(z^n/n)$: uniformly on A_r, pointwise on A.

This series does not, however, converge uniformly on A. Indeed, if it did, $\sum x^n/n$ would converge uniformly on $[0, 1[$. Suppose that this were true. Then for any $\epsilon > 0$ there would be an N such that $n \ge N$ would imply that

$$\frac{x^n}{n} + \frac{x^{n+1}}{n+1} + \ldots + \frac{x^{n+p}}{n+p} < \epsilon$$

for all $x \in [0, 1[$ and $p = 0, 1, 2, \ldots$. But the harmonic-type series

$$\frac{1}{N} + \frac{1}{N+1} + \ldots$$

diverges to infinity (that is, the partial sums $\to \infty$), so we can choose p such that

$$\frac{1}{N} + \ldots + \frac{1}{N+p} > 2\epsilon.$$

Next, we choose x so close to 1 that $x^{N+p} > 1/2$. Then

$$\frac{x^N}{N} + \ldots + \frac{x^{N+p}}{N+p} > x^{N+p}\left(\frac{1}{N} + \ldots + \frac{1}{N+p}\right) > \epsilon,$$

which is a contradiction. However, note that $g(z)$ is still continuous on A because it is continuous at each z, since each z lies in some A_r on which we do have uniform convergence. ♦

Series of Analytic Functions The next result, formulated by Karl Weierstrass in approximately 1860, is one of the main theorems concerning the convergence of analytic functions.

Theorem 3.1.8 (Analytic Convergence Theorem) (i) *Let A be an open set in $\mathbb{C}$ and let f_n be a sequence of analytic functions defined on A. If $f_n \to f$ uniformly on every closed disk contained in A, then f is analytic. Furthermore, $f_n' \to f'$ pointwise on A and uniformly on every closed disk in A (see Figure 3.1.5).*

(ii) *If g_k is a sequence of analytic functions defined on an open set A in $\mathbb{C}$ and $g(z) = \sum_{k=1}^{\infty} g_k(z)$ converges uniformly on every closed disk in A, then g is analytic on A and $g'(z) = \sum_{k=1}^{\infty} g_k'(z)$ pointwise on A and also uniformly on every closed disk contained in A.*

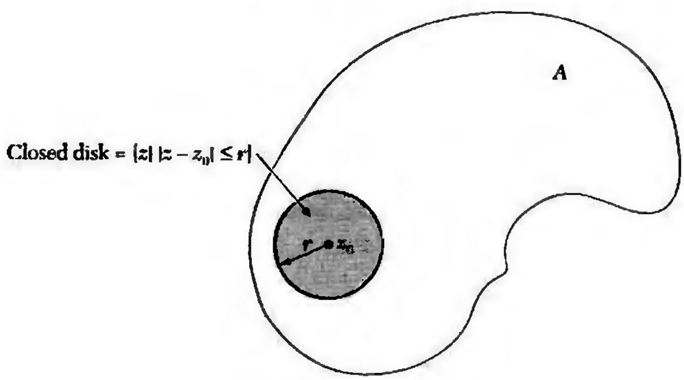


Figure 3.1.5: Uniform convergence on closed disks in the set A.

This theorem reveals yet another remarkable property of analytic functions that is not shared by functions of a real variable (compare §2.4). Uniform convergence usually is not sufficient to justify differentiation of a series term by term, but for analytic functions it is sufficient.

The proof of the Analytic Convergence Theorem 3.1.8 depends on Morera's Theorem and Cauchy's Integral Formula, which were studied in §2.4. To prepare for this proof let us first analyze a result concerning integration of sequences and series.

Proposition 3.1.9 *Let $\gamma : [a,b] \to A$ be a curve in a region A and let f_n be a sequence of continuous functions defined on $\gamma([a,b])$ which converges uniformly to*

f on $\gamma([a,b])$. Then

$$\int_\gamma f_n \to \int_\gamma f.$$

Similarly, if $\sum_{n=1}^\infty g_n(z)$ converges uniformly on γ, then we can interchange infinite sums and integrals:

$$\int_\gamma \left(\sum_{n=1}^\infty g_n(z)\right) dz = \sum_{n=1}^\infty \int_\gamma g_n(z)dz.$$

Proof The function f is continuous by Proposition 3.1.6 and so is integrable. Given $\epsilon > 0$, we can choose A such that $n \geq N$ implies that $|f_n(z) - f(z)| < \epsilon$ for all z on γ. Then, by Proposition 2.1.6,

$$\left|\int_\gamma f_n - \int_\gamma f\right| = \left|\int_\gamma f_n - f\right| \leq \int_\gamma |f_n(z) - f(z)||dz| < \epsilon l(\gamma)$$

from which the first assertion follows. The second assertion is obtained by applying the first to the partial sums. (The student should write out the details.) ∎

Proof of the Analytic Convergence Theorem 3.1.8 As usual, it suffices to prove (i). Let $z_0 \in A$ and let $\{z \text{ such that } |z - z_0| \leq r\}$ be a closed disk around z_0 entirely contained in A. (Why does such a disk exist?) Consider $D(z_0; r) = \{z \text{ such that } |z - z_0| < r\}$, which is a simply connected region because it is convex. Since $f_n \to f$ uniformly on the set $\{z \text{ such that } |z - z_0| \leq r\}$, it is clear that $f_n \to f$ uniformly on $D(z_0; r)$. We wish to show that f is analytic on $D(z_0; r)$. To do this we use Morera's Theorem 2.4.10. By Proposition 3.1.6, f is continuous on $D(z_0; r)$. Let γ be any closed curve in $D(z_0; r)$. Since f_n is analytic, $\int_\gamma f_n = 0$ by Cauchy's Theorem and by the fact that $D(z_0; r)$ is simply connected. But by Proposition 3.1.9, $\int_\gamma f_n \to \int_\gamma f$, so $\int_\gamma f = 0$. Thus by Morera's Theorem f is analytic on $D(z_0; r)$.

We must still show that $f'_n \to f'$ uniformly on closed disks. To do this we use Cauchy's Integral Formula for Derivatives 2.4.6. Let $B = \{z \text{ such that } |z - z_0| \leq r\}$ be a closed disk in A. We can draw a circle γ of radius $\rho > r$ centered at z_0 that contains B entirely in its interior (see Worked Example 1.4.27 and Figure 3.1.6).

For any $z \in B$,

$$f'_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(\zeta)}{(\zeta - z)^2}d\zeta \quad \text{and} \quad f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}d\zeta$$

by the Cauchy Integral Formula. By hypothesis, $f_n \to f$ uniformly on the closed disk $\{z \text{ such that } |z - z_0| \leq \rho\}$, which lies entirely in A. Then, given $\epsilon > 0$, we pick N such that $n \geq N$ implies that $|f_n(z) - f(z)| < \epsilon$ for all z in this disk (which we

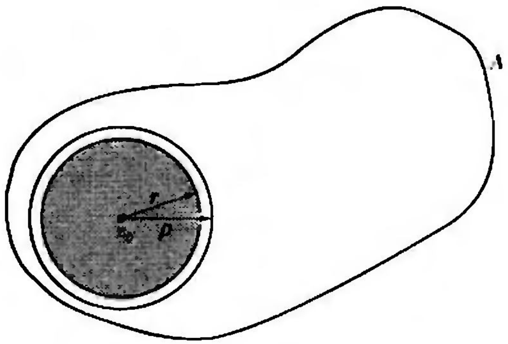


Figure 3.1.6: A closed disk in an open set can be slightly enlarged.

can do by hypothesis). Since γ is the boundary of this disk, $n \geq N$ implies that $|f_n(\zeta) - f(\zeta)| < \epsilon$ on γ. Note that

$$|f_n'(z) - f'(z)| = \left|\frac{1}{2\pi i}\int_\gamma \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2}d\zeta\right|$$

and observe that for ζ and γ and $z \in B$, $|\zeta - z| \geq \rho - r$. Hence $n \geq N$ implies that

$$|f_n'(z) - f'(z)| \leq \frac{1}{2\pi} \cdot \frac{\epsilon}{(\rho - r)^2} \cdot l(\gamma) = \frac{\epsilon\rho}{(\rho - r)^2}.$$

Since ρ and r are fixed constants that are independent of $z \in B$, we get the desired result. ■

Applying the Analytic Convergence Theorem 3.1.8 repeatedly we see that the kth derivatives $f_n^{(k)}$ converge to $f^{(k)}$ uniformly on closed disks in A. Notice also that this theorem does not assume uniform convergence on all of A. For example, $\sum_{n=1}^{\infty} z^n/n$ on $A = \{z$ such that $|z| < 1\}$ converges uniformly on the sets $A_r = \{z$ such that $|z| \leq r\}$ for $0 \leq r < 1$ (as we saw in the preceding example) and hence converges uniformly on all closed disks in A. Thus we can conclude that $\sum z^n/n$ is analytic on A and that the derivative is $\sum z^{n-1}$, which also converges on A. However, as that example demonstrated, we do have pointwise but not uniform convergence on A; convergence is uniform only on each closed subdisk in A.

Technical Proofs Now we provide the missing proofs.

Proof of Proposition 3.1.2 By the Cauchy Criterion 3.1.5, given $\epsilon > 0$ there is an N such that $n \geq N$ implies

$$\sum_{k=n+1}^{n+p} |a_k| < \epsilon \qquad p = 1, 2, \ldots .$$

But

$$\left|\sum_{k=n+1}^{n+p} a_k\right| \le \sum_{k=n+1}^{n+p} |a_k| < \epsilon$$

by the triangle inequality (see §1.2). Thus by the Cauchy Criterion 3.1.5, $\sum_{k=1}^{\infty} a_k$ converges. ■

Proof of Proposition 3.1.3

(i) By basic algebra,

$$1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

if $r \neq 1$. Since $r^{n+1} \to 0$ as $n \to \infty$ if $|r| < 1$, and since $|r|^{n+1} \to \infty$ if $|r| > 1$. we have convergence if $|r| < 1$ and divergence if $|r| > 1$. Obviously, $\sum_{n=0}^{\infty} r^n$ diverges if $|r| = 1$, since r^n does not converge to zero.

(ii) The partial sums of the series $\sum_{k=1}^{\infty} b_k$ form a Cauchy sequence and thus the partial sums of the series $\sum_{k=1}^{\infty} a_k$ also form a Cauchy sequence, since for any k and p we have $a_k + a_{k+1} + \ldots + a_{k+p} \le b_k + b_{k+1} + \ldots + b_{k+p}$. Hence $\sum_{k=1}^{\infty} a_k$ converges. A positive series can diverge only to $+\infty$, so given $M > 0$, we can find k_0 such that $k \ge k_0$ implies $c_1 + c_2 + \ldots + c_k \ge M$ Therefore, for $k \ge k_0, d_1 + d_2 + \ldots + d_k \ge M$, so $\sum_{k=1}^{\infty} d_k$ also diverges to ∞.

(iii) First suppose that $p \le 1$; in this case $1/n^p \ge 1/n$ for all $n = 1, 2, \ldots$. Therefore, by (ii), $\sum_{n=1}^{\infty} 1/n^p$ will diverge if $\sum_{n=1}^{\infty} 1/n$ diverges. We now recall the proof of this from calculus:[2] If $s_k = 1/1 + 1/2 + \ldots + 1/k$, then s_k is a strictly increasing sequence of positive real numbers. Write s_{2^k} as follows:

$$\begin{aligned} s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &\quad + \ldots + \left(\frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k}\right) \\ &\ge 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \ldots + \left(\frac{1}{2}\right) = 1 + \frac{k}{2}. \end{aligned}$$

Hence s_k can be made arbitrarily large if k is made sufficiently large; thus $\sum_{n=1}^{\infty} 1/n$ diverges.

Now suppose that $p > 1$. If we let

$$s_k = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{k^p}$$

[2] We can also prove (iii) by using the integral test for positive series (see any calculus text The demonstration given here also proves the *Cauchy condensation test*: Let $\sum a_n$ be a series of positive terms with $a_{n+1} \le a_n$. Then $\sum a_n$ converges iff $\sum_{j=1}^{\infty} 2^j a_{2^j}$ converges (see G. J. Porter An alternative to the integral test for infinite series, *Am. Math. Monthly*, **79** (1972), 634).

then s_k is an increasing sequence of positive real numbers. On the other hand,

$$\begin{aligned} s_{2^k-1} &= \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \dots \\ &\quad + \left(\frac{1}{(2^{k-1})^p} + \dots + \frac{1}{(2^k-1)^p}\right) \le \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + \frac{2^{k-1}}{(2^{k-1})^p} \\ &= \frac{1}{1^{p-1}} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \dots + \frac{1}{(2^{k-1})^{p-1}} < \frac{1}{1-1/2^{p-1}}. \end{aligned}$$

(Why?) Thus the sequence $\{s_k\}$ is bounded from above by $1/(1-1/2^{p-1})$; hence $\sum_{n=1}^{\infty} 1/n^p$ converges.

(iv) Suppose that $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = r < 1$. Choose r' such that $r < r' < 1$ and let N be such that $n \ge N$ implies

$$\left|\frac{a_{n+1}}{a_n}\right| < r'.$$

For $k \ge N$, we have

$$|a_k| \le r'|a_{k-1}| \le (r')^2|a_{k-2}| \le \dots \le (r')^{k-N}|a_N|.$$

The series $\sum_{k=N}^{\infty}|a_k|$ converges by comparison to the convergent geometric series with ratio r'. If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r > 1,$$

choose r' such that $1 < r' < r$ and let N be such that $n \ge N$ implies that $\left|\dfrac{a_{n+1}}{a_n}\right| > r'$. Hence $|a_{N+p}| > (r')^p|a_N|$, and so $\lim_{n\to\infty}|a_N| = \infty$, whereas the limit would have to be zero if the sum converged (see Exercise 10). Thus, $\sum_{k=1}^{\infty} a_k$ diverges. To see that the test fails if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, consider the two series $1 + 1 + 1 + \dots$ and $\sum_{n=1}^{\infty} 1/n^p$ for $p > 1$. In both cases $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, but the first series diverges and the second converges.

(v) Suppose that $\lim_{n\to\infty}|a_n|^{1/n} = r < 1$. Choose r' such that $r < r' < 1$ and N such that $n \ge N$ implies that $|a_n|^{1/n} < r'$, in other words, that $|a_n| < (r')^n$. The series

$$|a_1| + |a_2| + \dots + |a_{N-1}| + (r')^N + (r')^{N+1} + \dots$$

converges to

$$|a_1| + |a_2| + \dots + |a_{N-1}| + \frac{(r')^N}{1-r'},$$

so by (ii), $\sum_{k=1}^{\infty} |a_k|$ converges. If $\lim_{n\to\infty} |a_n|^{1/n} = r > 1$, choose $1 < r' < r$ and N such that $n \geq N$ implies that $|a_n|^{1/n} > r'$ or, in other words, that $|a_n| > (r')^n$. Hence $\lim_{n\to\infty} |a_n| = \infty$. Therefore, $\sum_{k=1}^{\infty} a_k$ diverges.

To show that the test fails when $\lim_{n\to\infty} |a_n|^{1/n} = 1$, we use these limits from calculus:

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = 1$$

(take logarithms and use L'Hôpital's rule to show that $(\log x)/x \to 0$ as $x \to \infty$). But $\sum_{n=1}^{\infty} 1/n$ diverges and $\sum_{n=1}^{\infty} 1/n^2$ converges. ■

Proof of 3.1.5 (Cauchy Criterion)

(i) First we prove the "if" part. Let $f(z) = \lim_{n\to\infty} f_n(z)$, which exists because for each z, $f_n(z)$ is a Cauchy sequence. We wish to show that $f_n \to f$ uniformly on A. Given $\epsilon > 0$, choose N such that $|f_n(z) - f_{n+p}(z)| < \epsilon/2$ for $n \geq N$ and all $p \geq 1$. The first step is to show that for any z and any $n \geq N, |f_n(z) - f(z)| < \epsilon$. For $z \in A$, choose p large enough so that $|f_{n+p}(z) - f(z)| < \epsilon/2$, which is possible by pointwise convergence. Then, by the triangle inequality,

$$|f_n(z) - f(z)| \leq |f_n(z) - f_{n+p}(z)| + |f_{n+p}(z) - f(z)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

(Notice that although p depends on z, N does not.)

Conversely, if $f_n \to f$ uniformly, given $\epsilon > 0$ choose N such that $n \geq N$ implies $|f_n(z) - f(z)| < \epsilon/2$ for all z. Since $n + p \geq N$,

$$|f_n(z) - f_{n+p}(z)| \leq |f_n(z) - f(z)| + |f(z) - f_{n+p}(z)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

(ii) By applying (i) to the partial sums, we deduce (ii). ■

Proof of Proposition 3.1.6 It suffices to prove the assertion for sequences (Why?). We wish to show that for $z_0 \in A$, given $\epsilon > 0$, there is a $\delta > 0$ such that $|z - z_0| < \delta$ implies that $|f(z) - f(z_0)| < \epsilon$. Choose N such that $|f_N(z) - f(z)| < \epsilon/3$ for all $z \in A$. Since f_N is continuous, there is a $\delta > 0$ such that $|f_N(z) - f_N(z_0)| < \epsilon/3$ if $|z - z_0| < \delta$. Thus,

$$\begin{aligned} |f(z) - f(z_0)| &\leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \quad ■ \end{aligned}$$

Note that in the last step we need an N that is independent of z to conclude that both $|f_N(z) - f(z)| < \epsilon/3$ and $|f_N(z_0) - f(z_0)| < \epsilon/3$.

Worked Examples

Example 3.1.10 *Show that the sequence of functions $f_n(x) = \sin(x/n)$ converges uniformly to the constant function $f(x) = 0$ for x in the interval $[0, \pi]$.*

Solution From calculus, $\sin\theta$ is increasing and $\sin\theta \le \theta$ for $0 \le \theta \le \pi/2$. Thus, if $x \in [0,\pi]$ and $n \ge 2$, then $|f_n(x) - f(x)| = |\sin(x/n)| \le \sin(\pi/n) \le \pi/n$. (See Figure 3.1.7.) Therefore $|f_n(x) - f(x)| < \epsilon$ provided $n > \max(2, \pi/\epsilon)$. The same n works for all x in the interval, and so the convergence is uniform on $[0,\pi]$.

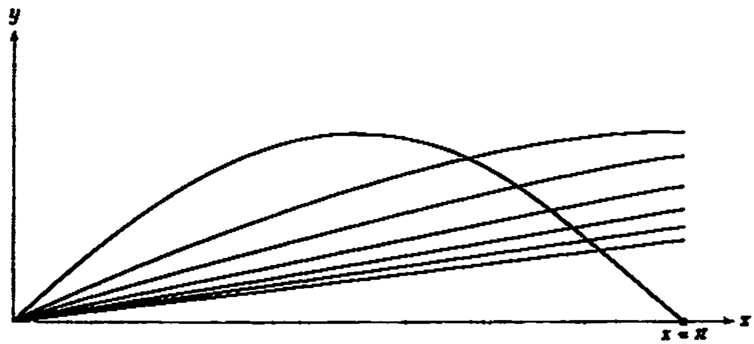


Figure 3.1.7: $y = \sin(x/n)$ for $n = 1$ through 7.

Example 3.1.11 *Show that the sequence of functions* $f_n(x) = \arctan(nx)$ *converges for* x *in the interval* $[-5,5]$ *to the function*

$$f(x) = \begin{cases} -\pi/2 & \text{for} \quad x < 0 \\ 0 & \text{for} \quad x = 0 \\ \pi/2 & \text{for} \quad x > 0 \end{cases}$$

but that the convergence is not uniform. (See Figure 3.1.8.)

Solution If $x > 0$, then $|f_n(x) - f(x)| = |\arctan(nx) - \pi/2|$. We know that $\arctan(y)$ is an increasing function of y with limit $\pi/2$ as $y \to \infty$. Therefore $|\arctan(nx) - \pi/2| < \epsilon$ if and only if $nx > \tan(\pi/2 - \epsilon)$. For any particular value of x, large enough values of n will work, but by taking x close to 0 we can force the required n to be quite large. Thus we have convergence but not uniform convergence. (Similar discussions apply for $x \le 0$.) One can see indirectly that the convergence must not be uniform. If it were, then the limit function would be continuous, by Proposition 3.1.6. But it is not.

The next three examples develop the important special case of the geometric series and show how the tools of this section can be applied to it to obtain some interesting results. The behavior of these examples is typical of the more general power series studied in the next section.

Example 3.1.12 *Show that the series* $\sum_{n=0}^{\infty} z^n$ *converges on the open unit disk* $D = D(0;1)$ *to the analytic function* $f(z) = 1/(1-z)$. *Prove that the convergence is uniform and absolute on every closed disk* $D_r = \{z \text{ such that } |z| \le r\}$ *with* $r < 1$.

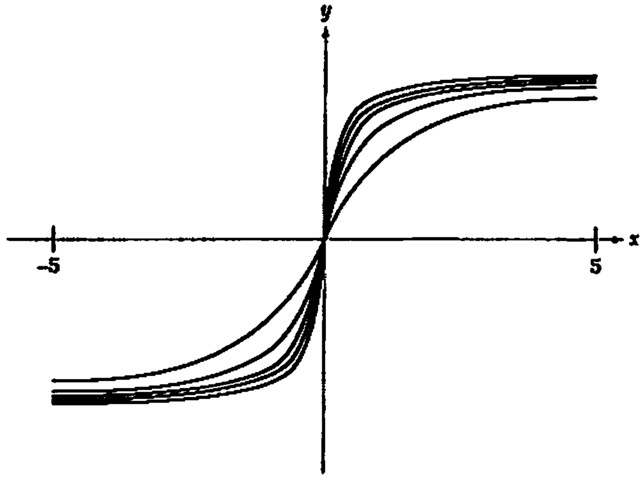


Figure 3.1.8: $y = \arctan(nx)$ for $n = 1$ through 5.

Solution If $z \in D$, then $z \in D_r$ whenever $|z| \leq r < 1$. Hence convergence at z follows from the second assertion. To prove this, suppose z is in D_r. Then $|z^n| < r^n$. Since $\sum r^n$ converges (Proposition 3.1.3 (i)), the Weierstrass M Test applies with $M_n = r^n$, and our series converges uniformly and absolutely on D_r. We have run into one of the shortcomings of a tool like the Weierstrass M Test. We have shown that the series converges but have not identified the limit. To do this notice that

$$1 - z^{n+1} = (1 - z)(1 + z + z^2 + \ldots + z^n),$$

so

$$\left|\frac{1}{1-z} - \sum_{k=0}^{n} z^k\right| = \frac{|z|^{n+1}}{|1-z|} \leq \frac{r^{n+1}}{1-r}.$$

Since $r < 1$, this goes to 0 as $n \to \infty$ and we have our result.

Example 3.1.13 *Show that the series $\sum_{n=1}^{\infty} nz^{n-1} = \sum_{n=0}^{\infty}(n+1)z^n$ converges on the open unit disk D to $g(z) = 1/(1-z)^2$. The convergence is uniform and absolute on every closed disk contained in D.*

Solution If B is any closed disk contained in D, then $B \subset D_r$ for some closed disk D_r, which is proved as in the last example. The series $\sum z^n$ converges uniformly and absolutely to $f(z) = 1/(1-z)$ on D_r and so on B. By the Analytic Convergence Theorem 3.1.8 (ii), the series of derivatives converges uniformly on every closed disk in D to $f'(z)$. That is, $\sum_{n=1}^{\infty} nz^{n-1} = f'(z) = 1/(1-z)^2$, as desired. The convergence is absolute by comparison. If $|z| \leq r < 1$, then $|nz^{n-1}| < nr^{n-1}$, but $\sum nr^{n-1}$ converges by the argument just given.

Example 3.1.14 *Show that the series* $\sum_{n=1}^{\infty}(-1)^{n-1}z^n/n$ *converges uniformly and absolutely to* $\log(1+z)$ *on the open unit disk, where* $\log(\rho e^{i\theta}) = \log\rho + i\theta$ *with* $-\pi < \theta < \pi$.

Solution We know that the given formula for log defines a branch of logarithm on the disk $D(1;1)$. In fact, it is the same as that described by the construction $\log w = \int_\gamma (1/\zeta)d\zeta$, where γ is a straight-line path from 1 to w. By the path independence guaranteed by Cauchy's Theorem we can integrate first along a circular arc (constant $r = 1$) and then along a ray from the origin (constant θ) to get from 1 to $w = \rho e^{i\theta}$ (see Figure 3.1.9).

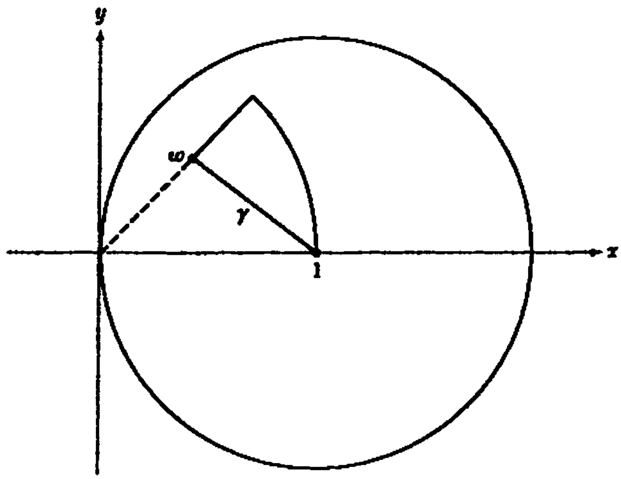


Figure 3.1.9: Path for computing $\log w$ on $D(1;1)$.

This construction shows that

$$\int_\gamma \frac{1}{\zeta}d\zeta = \int_0^\theta e^{-i\phi}ie^{i\phi}d\phi + \int_1^\rho \frac{1}{re^{i\theta}}e^{i\theta}dr = i\theta + \log\rho.$$

Changing variables to $\xi = \zeta - 1$ gives

$$\log w = \int_\mu \frac{1}{\xi+1}d\xi = \int_\mu \frac{1}{1-(-\xi)}d\xi,$$

the path μ being a straight line from 1 to $z = w - 1$ in the open unit disk $D = D(0;1)$. By Worked Example 3.1.12, the integrand may be expanded in an infinite series $\sum_{n=0}^{\infty}(-\xi)^n$, which converges uniformly on μ. The Analytic Convergence Theorem allows us to integrate term by term to obtain

$$\begin{aligned}\log w &= \int_\mu \left[\sum (-\xi)^n\right] d\xi = \sum \left[\int_\mu (-\xi)^n d\xi\right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (w-1)^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(w-1)^n}{n}.\end{aligned}$$

This works for every w in $D(1;1)$. Setting $z = w - 1$ gives

$$\log(z+1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

for all z in $D(0;1)$. Again, the convergence is uniform and absolute on any D_r with $r < 1$. Indeed, since $|z| \le r$ implies

$$\left|\frac{(-1)^n z^n}{n}\right| \le \frac{r^n}{n} \le r^n$$

and $\sum r^n$ converges, the Weierstrass M Test applies with $M_n = r^n$.

Example 3.1.15 *Show that the Riemann ζ function, defined by*

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

is analytic on the region $A = \{z \mid \operatorname{Re} z > 1\}$. Write a convergent series for $\zeta'(z)$ on that set.

Solution We use the Analytic Convergence Theorem 3.1.8. We must be careful to try to prove uniform convergence only on closed disks in A and not on all of A. In this example we do not in fact have uniform convergence on all of A (see Exercise 8).

Let B be a closed disk in A and let δ be its distance from the line $\operatorname{Re} z = 1$ (see Figure 3.1.10). We shall show that $\sum_{n=1}^{\infty} n^{-z}$ converges uniformly on B. Here $n^{-z} = e^{-z\log n}$, where $\log n$ means the usual log of real numbers. Now

$$|n^{-z}| = |e^{-z\log n}| = e^{-x\log n} = n^{-x}.$$

But $x \ge 1 + \delta$ if $z \in B$, and so $|n^{-z}| \le n^{-(1+\delta)}$ for all $z \in B$. Let us, therefore choose $M_n = n^{-(1+\delta)}$.

By Proposition 3.1.3(iii), $\sum_{n=1}^{\infty} M_n$ converges. Thus, by the Weierstrass M Test, our series $\sum_{n=1}^{\infty} n^{-z}$ converges uniformly on B. Hence ζ is analytic on A. Also by the Analytic Convergence Theorem 3.1.8, we can differentiate term by term to give

$$\zeta'(z) = -\sum_{n=1}^{\infty} (\log n) n^{-z},$$

which we know must also converge on A (and uniformly on closed disks in A).

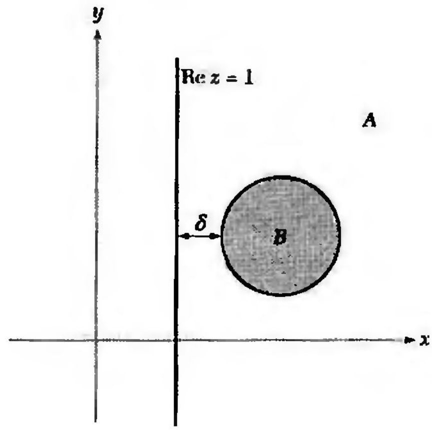


Figure 3.1.10: Domain of analyticity of the Riemann zeta function.

Exercises

1. Do the following sequences converge, and if so, what is their limit?

   (a) $z_n = (-1)^n + \dfrac{i}{n+1}$

   (b) $z_n = \dfrac{n!}{n^n} i^n$

2. Let c be a complex constant. Let $z_0 = 0$ and $z_1 = c$, and define a sequence by putting $z_{n+1} = z_n^2 + c$.

   (a) Show that if $|c| > 2$, then $\lim_{n\to\infty} z_n = \infty$. *Hint:* Let $r = |c| - 1$ and use induction to show that $|z_n| \geq |c| r^{n-1}$ for all n.

   (b) Show that if $|c| \leq 2$ and there is a value of k with $|z_k| > 2$, then $\lim_{n\to\infty} z_n = \infty$. *Hint:* Let $r = |z_k| - 1$, and show that $|z_{k+p}| \geq |z_k| r^p$ for all $p \geq 0$.[3]

3. What is the limit of the sequence $f_n(x) = (1+x)^{1/n}$ defined for $x \geq 0$? Does it converge uniformly?

4. (a) Show that the series $\sum_{n=0}^{\infty} 1/(n^2+z^2)$ converges on the set $\mathbb{C}\backslash\{z = ni \mid n$ is an integer $\}$.

   (b) Show that the convergence is uniform and absolute on each closed disk contained in this region.

[3]Those values of c for which the sequence z_n defined in this problem stays bounded form a very interesting set with many pretty patterns called the *Mandelbrot set*. See A. K. Dewdney, Computer Recreations, *Scientific American*, August 1985.

5. (a) Show that the sequence of functions $f_n(z) = z^n$ converges uniformly to the zero function $f(z) = 0$ on every closed disk $D_r = \{z$ such that $|z| \le r\}$ with $r < 1$.

   (b) Is the convergence uniform on the open unit disk $D(0;1)$?

6. (a) Show that the sequence of functions $f_n(x) = \cos(x/n)$ converges uniformly to the constant function $f(x) = 1$ for $x \in [0, \pi]$.

   (b) Show that it converges pointwise to 1 on all of $\mathbb{R}$.

   (c) Is the convergence uniform on all of $\mathbb{R}$?

7. Test the following series for absolute convergence and convergence:

   (a) $\displaystyle\sum_{2}^{\infty} \frac{i^n}{\log n}$

   (b) $\displaystyle\sum_{1}^{\infty} \frac{i^n}{n}$

8.* Prove that $\zeta(z) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ does not converge uniformly on $A = \{z \mid \operatorname{Re} z > 1\}$.

9. If $\sum_{k=1}^{\infty} g_k(z)$ is a uniformly convergent series of continuous functions and if $z_n \to z$, show that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} g_k(z_n) = \sum_{k=1}^{\infty} g_k(z).$$

10. If $\sum_{k=1}^{\infty} a_k$ converges, prove that $a_k \to 0$. If $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly, show that $g_k \to 0$ uniformly.

11. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{z^n}$ is analytic on $A = \{z$ such that $|z| > 1\}$.

12.* Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!z^n}$ is analytic on $\mathbb{C}\backslash\{0\}$. Compute its integral around the unit circle.

13. Show that $\sum_{n=1}^{\infty} e^{-n} \sin nz$ is analytic in the region $A = \{z \mid -1 < \operatorname{Im} z < 1\}$

14. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$ converges in both the interior and exterior of the unit circle and represents an analytic function in each region.

15. Show that $\sum_{n=1}^{\infty} (\log n)^k n^{-z}$ is analytic on $\{z \mid \operatorname{Re} z > 1\}$. *Hint:* Use the result of Worked Example 3.1.15.

16.* Let f be an analytic function on the disk $D(0;2)$ such that $|f(z)| \leq 7$ for all $z \in D(0;2)$. Prove that there exists a $\delta > 0$ such that if $z_1, z_2 \in D(0;1)$, and if $|z_1 - z_2| < \delta$, then $|f(z_1) - f(z_2)| < 1/10$. Find a numerical value of δ independent of f that has this property. *Hint:* Use the Cauchy Integral Formula.

17. If $f_n(z) \to f(z)$ uniformly on a region A, and if f_n is analytic on A, is it true that $f_n'(z) \to f'(z)$ uniformly on A?

18. Prove that $f_n \to f$ uniformly on every closed disk in a region A iff $f_n \to f$ uniformly on every compact (closed and bounded) subset of A.

19. Find a suitable region in which $\displaystyle\sum_{n=1}^{\infty} \frac{(2z-1)^n}{n}$ is analytic.

20.* Let f_n be analytic on a bounded region A and continuous on $\mathrm{cl}(A)$, $n = 1, 2, 3, \ldots$. Suppose that the functions f_n converge uniformly on $\mathrm{bd}(A)$. Prove that the functions f_n converge uniformly to an analytic function on A. *Hint:* Use the Maximum Modulus Theorem.

3.2 Power Series and Taylor's Theorem

This section will consider special kinds of series called power series, which have the form $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. We shall examine their convergence properties and show that a function is analytic iff it is locally representable as a convergent power series. To obtain this representation, we first need to establish Taylor's Theorem, which asserts that if f is analytic on an open disk centered at z_0, then the ***Taylor series*** of f,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

converges on the disk and equals $f(z)$ everywhere on that disk.

In proving the results of this section we shall use the techniques developed in §3.1 and Cauchy's Integral Formula 2.4.4.

Convergence of Power Series A ***power series*** is a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

where a_n and z_0 are fixed complex numbers. Each term $a_n(z - z_0)^n$ is entire, so in proving that the sum is analytic on a region, we can use the Analytic Convergence Theorem 3.1.8. The basic fact to remember about power series is that the appropriate domain of analyticity is the interior of a circle centered at z_0. This is established in the following theorem.

Theorem 3.2.1 (Power Series Convergence) *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series. There is a unique number $R \geq 0$, possibly $+\infty$, called the* ***radius of convergence****, such that if $|z-z_0| < R$, the series converges, and if $|z-z_0| > R$, the series diverges. Furthermore, the convergence is uniform and absolute on every closed disk in $A = \{z \in \mathbb{C}$ such that $|z-z_0| < R\}$. No general statement about convergence can be made if $|z-z_0| = R$. (See Figure 3.2.1.)*

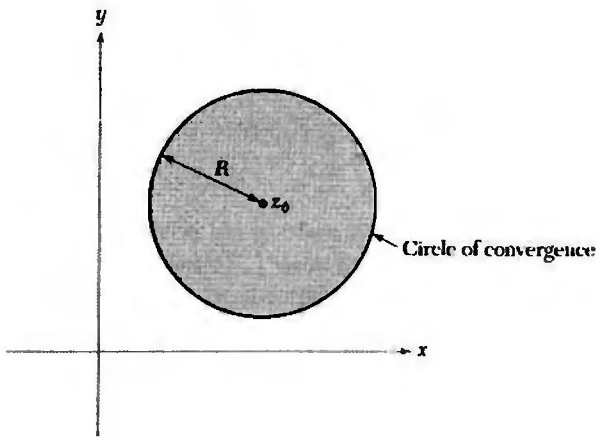


Figure 3.2.1: Convergence of power series. Series converges within circle; series diverges outside circle.

Thus, the series converges on the region $A = \{z \in \mathbb{C}$ such that $|z-z_0| < R\}$, and it diverges at z if $|z-z_0| > R$. The circle $|z-z_0| = R$ is called the ***circle of convergence*** of the given power series. Practical methods of calculating R use the ratio and root tests and will be given shortly.

The overall strategy is to let $R = \sup\{r \geq 0 \mid \sum_{n=0}^{\infty} |a_n| r^n \text{ converges }\}$, where sup means the least upper bound of that set of real numbers, and then to show that R has the desired properties. The following lemma will be useful.

Lemma 3.2.2 (Abel-Weierstrass Lemma) *Suppose that $r_0 \geq 0$ and that the inequality $|a_n| r_0^n \leq M$ holds for all integers $n \geq 0$, where M is some constant. Then for $r < r_0$, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly and absolutely on the closed disk $A_r = \{z$ such that $|z-z_0| \leq r\}$.*

Proof For $z \in A_r$ we have

$$|a_n(z-z_0)^n| \leq |a_n| r^n = |a_n| r_0^n \left(\frac{r}{r_0}\right)^n \leq M \left(\frac{r}{r_0}\right)^n.$$

Let

$$M_n = M\left(\frac{r}{r_0}\right)^n.$$

Since $r/r_0 < 1$, the series $\sum M_n$ converges. Thus, by the Weierstrass M Test 3.1.7, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly and absolutely on A_r. ■

Proof of Theorem 3.2.1 Let $r_0 < R$. By the definition of R there is an $r_1, r_0 < r_1 \leq R$ such that $\sum |a_n| r_1^n$ converges. Therefore, $\sum_{n=0}^{\infty} |a_n| r_0^n$ converges by the comparison test. The terms $|a_n| r_0^n$ are bounded (in fact $\to 0$), and so by the Abel-Weierstrass Lemma, the series converges uniformly and absolutely on A_r for any $r < r_0$. Since any z with $|z - z_0| < R$ lies in some A_r and since we can always choose r_0 such that $r < r_0 < R$, we have convergence at z.

Now suppose that $|z_1 - z_0| > R$ and $\sum a_n(z_1 - z_0)^n$ converges. We shall derive a contradiction. The terms $a_n(z_1 - z_0)^n$ are bounded in absolute value because they approach zero. Thus, by the Abel-Weierstrass Lemma, if $R < r < |z_1 - z_0|$, then $\sum a_n(z_1 - z_0)^n$ converges absolutely if $z_1 \in A_r$. Therefore, $\sum |a_n| r^n$ converges. But this would mean, by definition of R, that $R < R$.

We have proved that the convergence is uniform and absolute on every *strictly smaller* closed disk A_r and hence on any closed disk in A. ■

Combining the Analytic Convergence and Power Series Convergence Theorems, we may deduce the following theorem.

Theorem 3.2.3 (Analyticity of Power Series) *A power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is an analytic function on the inside of its circle of convergence.*

We also know that we can differentiate convergent series of analytic functions term by term. This leads to another interesting theorem.

Theorem 3.2.4 (Differentiation of Power Series) *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

be the analytic function defined on the inside of the circle of convergence of the given power series. Then $f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$, and this series has the same circle of convergence as $\sum a_n(z - z_0)^n$. Furthermore, the coefficients a_n are given by $a_n = f^{(n)}(z_0)/n!$.

Proof We know from the Analytic Convergence Theorem 3.1.8 that the derivative $f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ converges on $A = D(z_0; R) = \{z \in \mathbb{C}$ such that $|z - z_0| < R\}$. To show that the derived series has the same circle of convergence as the original series, we need only show that it diverges for $|z - z_0| > R$. If it did converge at some point z_1 with $|z_1 - z_0| = r_0 > R$, then $n a_n r_0^{n-1}$ would be bounded. Thus $a_n r_0^n = (n a_n r_0^{n-1})(r_0/n)$ would also be bounded, so $\sum a_n(z - z_0)^n$ would converge for $R \leq |z - z_0| < r_0$ by the Abel-Weierstrass Lemma. But this contradicts the maximal property of R from the Power Series Convergence Theorem 3.2.1. This establishes the assertion about the radius of convergence.

To identify the coefficients, set $z = z_0$ in the formula defining $f(z)$ to find $f(z_0) = a_0$. Proceeding inductively, we find

$$f^{(n)}(z) = n!a_n + \sum_{k=n+1}^{\infty} k(k-1)(k-2)\dots(k-n+1)(z-z_0)^{k-n},$$

and setting $z = z_0$, we get $f^{(n)}(z_0) = n!a_n$. ■

It is important to notice just what has been done in the last assertion of this theorem. The coefficients of a power series around a particular center are completely determined by the function that series represents. Thus if two apparently different series have been obtained for the same function about the same center, they must in fact be the same.

Theorem 3.2.5 (Uniqueness of Power Series) *Power series expansions around the same center are unique. If*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = f(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

for all z in some nontrivial disk $D(z_0;r)$ with $r > 0$, then $a_n = b_n$ for $n = 0,1,2,3,\dots$.

Proof The last assertion of the differentiation of power series theorem says $a_n = f^{(n)}(z_0)/n! = b_n$. ■

Uniqueness of power series may be used in a number of ways. In particular, it says that whatever tricks we can use to find a convergent power series representing a function, it must be the Taylor series. It can also help us use power series in the solution of differential equations and other problems. Several of these tricks and ideas for the manipulation and application of power series are demonstrated in the worked examples.

We will now obtain some practical methods of computing the radius of convergence R. (The method of defining R given in the proof of Theorem 3.2.1 is not terribly useful for computing R in specific examples.)

Proposition 3.2.6 *Consider a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.*

(i) ***Ratio test:*** *If*

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

exists, then it equals R, the radius of convergence of the series.

(ii) ***Root test:*** *If $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists, then $R = 1/\rho$ is the radius of convergence. (Set $R = \infty$ if $\rho = 0$.)*

Proof To prove both cases we show that

$$R = \sup\left\{ r \geq 0 \;\middle|\; \sum_{n=0}^{\infty} |a_n| r^n < \infty \right\}.$$

(i) By the ratio test (Proposition 3.1.3) we know that $\sum_{n=0}^{\infty} |a_n| r^n$ converges or diverges if

$$\lim_{n\to\infty} \frac{|a_{n+1}r^{n+1}|}{|a_n r^n|} < 1 \quad \text{or} \quad > 1,$$

that is, according to whether

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} > r \quad \text{or} \quad \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} < r.$$

Thus, by the characterization of R in the Power Series Convergence Theorem 3.2.1, the limit equals R.

(ii) By the root test (Proposition 3.1.3) we know that $\sum_{n=0}^{\infty} |a_n| r^n$ converges or diverges if $\lim_{n\to\infty}(|a_n| r^n)^{1/n} < 1$ or > 1, that is, according to whether

$$r < 1/\lim_{n\to\infty} |a_n|^{1/n} \quad \text{or} \quad r > 1/\lim_{n\to\infty} |a_n|^{1/n}.$$

The result follows as in (i). ■

For example:

- The series $\sum_{n=0}^{\infty} z^n$ has radius of convergence 1 since $a_n = 1$, and thus we have $\lim_{n\to\infty} |a_n/a_{n+1}| = 1$.
- The series $\sum_{n=0}^{\infty} z^n/n!$ has radius of convergence $R = +\infty$ (that is, the function is entire), since $a_n = 1/n!$, and so $|a_n/a_{n+1}| = n + 1 \to \infty$.
- The series $\sum_{n=0}^{\infty} n! z^n$ has radius of convergence $R = 0$ because $|a_n/a_{n+1}| = 1/(n+1) \to 0$. (This function thus does not have a nontrivial region of analyticity.)

By refining the root test, it is possible to show that $R = 1/\rho$ where $\rho = \operatorname{limsup}_{n\to\infty} \sqrt[n]{|a_n|}$, which always exists. In this statement, the "limsup" means, by definition,

$$\operatorname{limsup}_{n\to\infty} c_n = \lim_{n\to\infty} (\sup\{c_n, c_{n+1}, \ldots\}).$$

This is known as ***Hadamard's formula*** for the radius of convergence. There is no analogous refinement for the ratio test (known to us).

Taylor's Theorem It is obvious from the preceding computations that if $f : A \to \mathbb{C}$ equals, in a small disk around each $z_0 \in A$, a convergent power series, then f is analytic. The converse is also true: If f is analytic it equals, on every disk in its domain, a convergent power series. This is made explicit in the following theorem.

Theorem 3.2.7 (Taylor's Theorem) *Let f be analytic on an open set A in $\mathbb{C}$. Let $z_0 \in A$ and let $A_r = \{z \text{ such that } |z - z_0| < r\}$ be contained in A (usually the largest open disk possible is used: if $r = \infty, A_r = A = \mathbb{C}$) (see Figure 3.2.2). Then for every $z \in A_r$, the series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

converges on A_r (that is, has a radius of convergence $\geq r$), and we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

*(We use the convention $0! = 1$.) The series of this equation is called the **Taylor series** of f around the point z_0.*

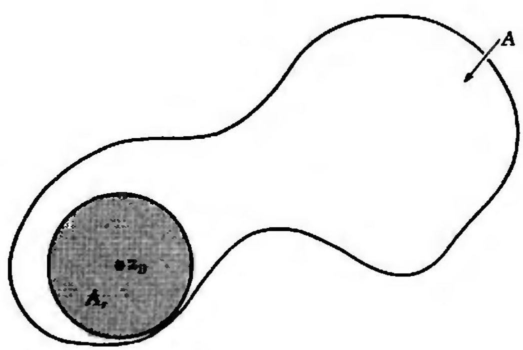


Figure 3.2.2: Taylor's Theorem.

Before proving this result let us study an example that illustrates its usefulness. Consider $f(z) = e^z$. Here f is analytic, and $f^{(n)}(z) = e^z$ for all n, so $f^{(n)}(0) = 1$ and thus

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

which is valid for all $z \in \mathbb{C}$, since e^z is entire.

Table 3.2.1 lists the Taylor series of some common elementary functions. The Taylor series around the point $z_0 = 0$ is sometimes called the ***Maclaurin series***.

Table 3.2.1 **Some Common Expansions**

Function	Taylor series around 0	Where valid
$\dfrac{1}{1-z}$	$\sum_{n=0}^{\infty} z^n$ (geometric series)	$\lvert z\rvert < 1$
e^z	$\sum_{n=0}^{\infty} \dfrac{z^n}{n!}$	all z
$\sin z$	$z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{z^{2n-1}}{(2n-1)!}$	all z
$\cos z$	$1 - \dfrac{z^2}{2} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \dfrac{z^{2n}}{(2n)!}$	all z
$\log(1+z)$ (principal branch)	$\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} z^n$	$\lvert z\rvert < 1$
$(1+z)^\alpha$ (principal branch)	$\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$ (binomial series)	$\lvert z\rvert < 1$ but all z if α is a positive integer

In the binomial series, $\alpha \in \mathbf{C}$ is fixed,

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!},$$

and we let $\binom{\alpha}{n}$ be zero if α is an integer $< n$ and let $\binom{\alpha}{0} = 1$.

All the series in Table 3.2.1 are important and useful. They may be established by taking successive derivatives and using Taylor's Theorem. The binomial series for α a positive integer should be familiar from algebra. The Taylor series for many functions can be found by other means, using the special properties of power series that allow their manipulation. Some of these properties are presented in the worked examples. We have already found the geometric series for $1/(1-z)$ and the series for $\log(1+z)$ in Worked Examples 3.1.12 and 3.1.14. This result for geometric series will be used in the following proof.

Proof of Taylor's Theorem Let $0 < \sigma < r$ and let γ be the circle $\gamma(t) = z_0 + \sigma e^{it}, 0 \le t \le 2\pi$, of radius σ centered at z_0. If z is any point inside γ, Cauchy's

Integral Formula 2.4.4 gives

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta.$$

The plan is to use the geometric series to expand the integrand as a power series in $z - z_0$ and then use Proposition 3.1.9 to integrate term by term. Finally the coefficients of the resulting integrated series are recognized to be those of the Taylor series by the Cauchy Integral Formula for Derivatives 2.4.6.

Since z is inside the circle γ and ζ is on its boundary, we have the inequality $|(z - z_0)/(\zeta - z_0)| < 1$. The geometric series of Worked Example 3.1.12 allows the following expansion:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\cdot\frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^n,$$

so

$$f(z) = \frac{1}{2\pi i}\int_\gamma\left[\frac{f(\zeta)}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^n\right]d\zeta = \frac{1}{2\pi i}\int_\gamma\left[\sum_{n=0}^{\infty}\frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^{n+1}}\right]d\zeta.$$

Furthermore, since the curve γ stays away from the boundary of the disk of convergence, Worked Example 3.1.12 also shows that the convergence of the series

$$\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^n$$

is uniform in ζ as ζ goes around the circle γ with z fixed. Also, $f(\zeta)/(\zeta - z_0)$ is a continuous function of ζ around the circle γ, so it is bounded there. It follows that the series

$$\sum_{n=0}^{\infty}\frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^{n+1}}$$

converges uniformly on γ to $f(\zeta)/(\zeta - z)$. (The first series satisfies the Cauchy Criterion uniformly in ζ, so it still satisfies it after being multiplied by something that remains bounded. The student is asked to supply the details in Exercise 21. By Proposition 3.1.9, we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^{n+1}}d\zeta \\ &= \sum_{n=0}^{\infty}\left[(z - z_0)^n\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta\right] = \sum_{n=0}^{\infty}\left[(z - z_0)^n\frac{f^{(n)}(z_0)}{n!}\right], \end{aligned}$$

as desired. The last equality is the Cauchy Integral Formula for Derivatives 2.4.6. Since the radius of the circle γ was arbitrary, so long as it fit inside the region of analyticity, this representation of $f(z)$ is valid in the largest open disk centered at z_0 which is contained in the region A. ■

The following consequence of this theorem was mentioned informally at the beginning of this section.

Corollary 3.2.8 *Let A be a region in $\mathbb{C}$ and let f be a complex-valued function defined on A. Then f is analytic on A if and only if for each z_0 in A there is a number $r > 0$ such that the disk $D(z_0; r) \subset A$ and f equals a convergent power series on $D(z_0; r)$.*

Proof Taylor's Theorem shows that every analytic function is equal to a power series, in fact to its Taylor series, on every disk in A. On the other hand, if $f(z)$ is equal to a convergent power series on $D(z_0; r)$, then $D(z_0; r)$ must be in the interior of the circle of convergence of the series and so f must be analytic on $D(z_0; r)$. Since there is such a disk and convergent power series for each z_0 in A, it follows from the analyticity of power series (3.2.3) that f is analytic on A. ■

The condition in this corollary may thus be taken as an alternative definition of "analytic". We have shown that the notions of differentiability on a region and analyticity on a region coincide for functions of a complex variable. (Keep in mind that they do *not* coincide for real variables.) Cauchy's Theorem, Cauchy's Integral Formulas, and Taylor's Theorem are among the most fundamental theorems of complex analysis.

In specific examples, the higher order derivatives of f may be complicated expressions, and finding the Taylor series may be made easier by searching directly for a convergent series that represents f rather than computing the derivatives. By Corollary 3.2.5, if $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ and the series converges, then it must be the Taylor series. In fact we can sometimes then use Taylor's Theorem to tell us formulas for the derivatives, having found the series by other means. Some of these tricks for manipulating series and applications are found in the worked examples.

Zeros of Analytic Functions Suppose that $f : \Omega \to \mathbb{C}$ is analytic on an open set Ω in $\mathbb{C}$ and that $c \in \Omega$. We know that the Taylor series for $f(z)$ centered at c converges to $f(z)$ at least in the largest open disk $D(c; r)$ centered at c and contained in Ω. Therefore, we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(z - c)^k \qquad \text{for} \qquad |z - c| < r$$

If $f^{(k)}(c) = 0$ for all k, then $f(z)$ is identically 0 in $D(c; r)$. If not, then there is a smallest nonnegative integer n with $f^{(n)}(c) \neq 0$. If $n = 0$, then $f(c) \neq 0$. If $n > 0$, then $f(c) = f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0$ but $f^{(n)}(c) \neq 0$. In this case we say that f has a ***zero of order** n* at c. From algebra we know that a polynomial has a zero at c if and only if $z - c$ is a factor of that polynomial. The zero has

order or ***multiplicity*** n if the factor $z-c$ occurs exactly n times. Consideration of the Taylor series shows that an analytic function has a zero at c if and only if $z-c$ is a factor of that function and then the multiplicity is defined in the same way. For z in $D(c;r)$ we have

$$\begin{aligned} f(z) &= a_n(z-c)^n + a_{n+1}(z-c)^{n+1} + a_{n+2}(z-c)^{n+2} + \dots \\ &= (z-c)^n \left[a_n + a_{n+1}(z-c) + a_{n+2}(z-c)^2 + \dots\right] \end{aligned}$$

where $a_k = f^{(k)}(c)/k!$. The power series in square brackets converges in $D(c;r)$ to an analytic function $\varphi(z)$ with $\varphi(c) = a_n = f^{(n)}(c)/n! \neq 0$. Thus f has a zero of order n at c if and only if $f(z)$ can be factored in a neighborhood of c as $f(z) = (z-c)^n\varphi(z)$ where φ is analytic in a neighborhood of c and $\varphi(c) \neq 0$.

Isolation of Zeros A closer examination of the factorization in the last paragraph gives another valuable conclusion. The factor $\varphi(z)$ is analytic and so is continuous on $D(c;r)$. Since $\varphi(c) \neq 0$, there is a radius ρ with $0 < \rho < r$ such that $\varphi(z)$ is never 0 for $|z-c| < \rho$. Since $f(z) = (z-c)^n\varphi(z)$ for all z in $D(c;\rho)$, we conclude that f has no zeros other than c in that disk. In this sense the zeros of f are ***isolated***. This analysis is summarized in the following proposition and corollary.

Proposition 3.2.9 *Suppose $f : \Omega \to \mathbb{C}$ is analytic on an open set Ω in $\mathbb{C}$ and that $c \in \Omega$. Let $D(c;r)$ be an open disk centered at c and contained in Ω and suppose $f(c) = 0$. Then exactly one of two things must occur.*

(i) *$f(z) = 0$ for every z in $D(c;r)$.*

(ii) *There is an integer n such that*

$$f(c) = f'(c) = f''(c) = \dots f^{(n-1)}(c) = 0 \quad \textit{and} \quad f^{(n)}(c) \neq 0.$$

In the latter case there is a function $\varphi(z)$ analytic in $D(c;r)$ with $\varphi(c) \neq 0$ and $f(z) = (z-c)^n\varphi(z)$ for all z in $D(c;r)$ and a radius $\rho > 0$ such that $f(z)$ is 0 only at c in the disk $D(c;\rho)$.

Corollary 3.2.10 (Local Isolation of Zeros) *Suppose $f : \Omega \to \mathbb{C}$ is analytic on an open set Ω in $\mathbb{C}$ and that $c \in \Omega$. If there is a sequence $z_1, z_2, z_3, \dots$ of distinct points in Ω such that $z_k \to c$ as $k \to \infty$ and $f(z_k) = 0$ for each k, then $f(z) = 0$ for each z in the largest open disk centered at c and contained in Ω.*

Worked Examples

Example 3.2.11 *Use the series expansion given in Table 3.2.1 to confirm the identity $e^{iz} = \cos z + i \sin z$ for all z.*

Solution Using the series for $\cos z$ and $\sin z$ from the table gives

$$\begin{aligned}
\cos z + i\sin z &= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!} + i\sum_{n=1}^{\infty}\frac{(-1)^{n+1} z^{2n-1}}{(2n-1)!} \\
&= \sum_{n=0}^{\infty}\frac{(iz)^{2n}}{(2n)!} + \sum_{n=1}^{\infty}\frac{(-i)i^{2n} z^{2n-1}}{(2n-1)!} \\
&= \sum_{n=0}^{\infty}\frac{(iz)^{2n}}{(2n)!} + \sum_{n=1}^{\infty}\frac{(iz)^{2n-1}}{(2n-1)!} = \sum_{k=0}^{\infty}\frac{(iz)^k}{k!} = e^{iz},
\end{aligned}$$

as desired.

Example 3.2.12 *Can a power series $\sum a_n(z-2)^n$ converge at $z=0$ but diverge at $z=3$?*

Solution No. If it converges at $z=0$ this implies, by the Power Series Convergence Theorem 3.2.1, that the radius of convergence R satisfies $R \geq 2$. But $z=3$ lies inside that circle, so the series would converge there (see Figure 3.2.3).

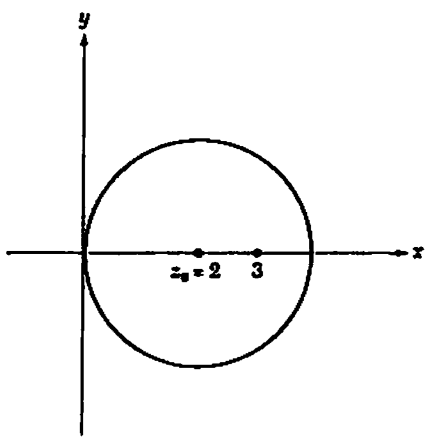


Figure 3.2.3: The circle of convergence for the power series in Worked Example 3.2.12 must be at least this big.

Example 3.2.13 *Find the Taylor series around $z_0 = 0$ for $f(z) = 1/(4+z^2)$ and calculate the radius of convergence.*

Solution Write

$$f(z) = \frac{1}{4}\left[\frac{1}{1-(-z^2/4)}\right].$$

We know that so long as $|w| < 1$, then $1/(1-w) = \sum_{n=0}^{\infty} w^n$. Replacing w by $-z^2/4$ gives

$$f(z) = \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{z^2}{4}\right)^n = \sum_{n=0}^{\infty}(-1)^n 4^{-(n+1)} z^{2n},$$

as long as $|(-z^2/4)| < 1$; that is, as long as $|z| < 2$. Therefore the radius of convergence is 2. Notice that this is the largest disk around $z_0 = 0$ on which f is analytic, since analyticity fails at $z = \pm 2i$.

Example 3.2.14 *Find the Taylor series of* $\log(1+z)$ *around* $z = 0$ *and give its radius of convergence (see Table 3.2.1).*

First Solution We have already done this problem as Worked Example 3.1.14 using the geometric series and term-by-term integration.

Second Solution We use the principal branch of log so that the function $f(z) = \log(1+z)$ is defined at $z = 0$. Since f is analytic on the region $A = \mathbb{C}\backslash\{x+iy \mid y = 0, x \le -1\}$ shown in Figure 3.2.4, the radius of convergence of the Taylor series will be ≥ 1 by Taylor's Theorem (3.2.7). That it is *exactly* 1 can be shown as follows. We know that

$$f(0) = \log 1 = 0,$$

$$f'(z) = \frac{1}{z+1}, \quad \text{so} \quad f'(0) = 1,$$

$$f''(z) = -\frac{1}{(z+1)^2}, \quad \text{so} \quad f''(0) = -1,$$

and

$$f'''(z) = \frac{2}{(z+1)^3}, \quad \text{so} \quad f'''(0) = 2.$$

Inductively, we see that

$$f^{(n)}(z) = \frac{(n-1)!(-1)^{n-1}}{(z+1)^n},$$

so $f^{(n)}(0) = (n-1)!(-1)^{n-1}$. Thus the Taylor series is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}z^n$$

(in agreement with Table 3.2.1). When $z = -1$, it is the harmonic series which diverges, so the radius of convergence is ≤ 1 and thus is exactly 1. (A general procedure to follow for determining the exact radius of convergence without computing the series is found in Exercise 19.)

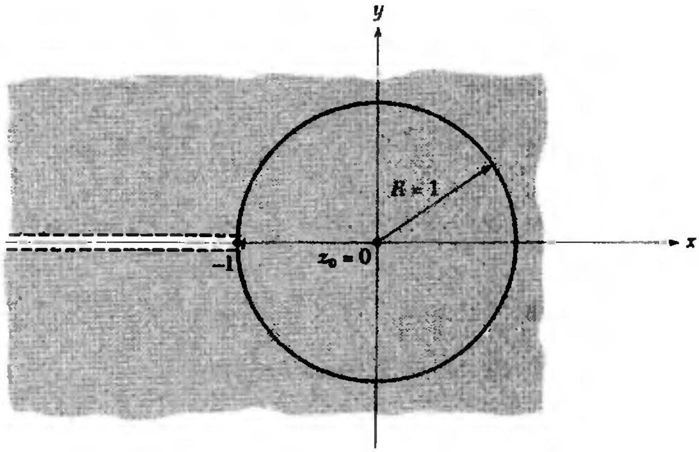


Figure 3.2.4: Taylor series of $\log(1+z)$.

Example 3.2.15 *Suppose that $\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergence $\geq r_0$. Define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Prove that $\sum c_n z^n$ has radius of convergence $\geq r_0$ and that inside this circle of radius r_0 we have*

$$\sum_{n=0}^{\infty} c_n z^n = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right).$$

Solution This way of multiplying out two power series is a generalization of the manner in which polynomials are multiplied. A direct proof can be given but would be somewhat lengthy. If we use Taylor's Theorem, the proof is fairly simple. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n, g(z) = \sum_{n=0}^{\infty} b_n z^n$, and let $A = \{z \text{ such that } |z| < r_0\}$. Then f and g are analytic on A, so fg is also analytic on A. By Taylor's Theorem we can write

$$(f \cdot g)(z) = \sum_{n=0}^{\infty} \frac{(f \cdot g)^{(n)}(0)}{n!} z^n$$

for all z in A. It is simple exercise (as in calculus) to show by induction that the nth derivative of the product $f(z)g(z)$ is given by

$$(f \cdot g)^{(n)}(z) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z), \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Hence,

$$\frac{(f \cdot g)^{(n)}(0)}{n!} = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} f^{(k)}(0) g^{(n-k)}(0) = \sum_{k=0}^{n} a_k b_{n-k}.$$

Thus, $\sum_{n=0}^{\infty} c_n z^n$ converges on A (and therefore, by Taylor's Theorem, the radius of convergence is $\geq r_0$) and on A

$$\sum_{n=0}^{\infty} c_n z^n = (f \cdot g)(z) = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right).$$

Example 3.2.16 *Compute the Taylor series around $z = 0$ and give the radii of convergence for*

(a) $z/(z-1)$

(b) $e^z/(1-z)$ *(Compute the first few terms only.)*

Solution

(a) The geometric series $(1-z)^{-1} = 1+z+z^2+\ldots$ is valid for $|z| < 1$. Therefore, for such z,

$$\frac{z}{z-1} = -z(1+z+z^2+\ldots) = -z-z^2-z^3-z^4-\ldots.$$

By the uniqueness of representation by power series, this is the Taylor series of $z/(z-1)$ around 0. By observing that $z/(z-1)$ is analytic on the open disk $|z| < 1$, we know by the Taylor theorem that the Taylor series must have a radius of convergence ≥ 1. Of course, a close analysis of the series $-z-z^2-z^3-z^4-\ldots$, using the ratio test or the root test, shows that the radius of convergence is exactly 1.

(b) $1/(1-z) = 1+z+z^2+\ldots$ for $|z| < 1$ and $e^z = 1+z+z^2/2+\ldots$ for all z. Thus by Worked Example 3.2.15 we get the series for the product by formally multiplying the two series out as if they were polynomials; the result must still converge for $|z| < 1$. We get

$$\begin{aligned}
\frac{e^z}{1-z} &= (1+z+z^2+z^3+\ldots)\left(1+z+\frac{z^2}{2}+\frac{z^3}{3!}+\ldots\right)\\
&= 1+(z+z)+\left(\frac{z^2}{2}+z^2+z^2\right)+\left(\frac{z^3}{6}+\frac{z^3}{2}+z^3+z^3\right)+\ldots\\
&= 1+2z+\frac{5z^2}{2}+\frac{8z^3}{3}+\ldots.
\end{aligned}$$

In the last series the general term has no simple form. Note that this method is faster than computing $f^{(k)}(0)$ for moderately large k.

Example 3.2.17 *This example deals with the application of series to differential equations. Find a function $f(x)$ such that $f(0) = 0$ and $f'(x) = 3f(x) + 2$ for all x.*

Solution Suppose that there is a solution f that is the restriction to the real axis of a function that is analytic in $\mathbf{C}$. Therefore, it will have a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Since $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$, we must have

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = 3\left(\sum_{n=0}^{\infty} a_n z^n\right) + 2,$$

that is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n = (2+3a_0) + \sum_{n=1}^{\infty} 3a_n z^n.$$

Thus,

$$0 = (2+3a_0 - a_1) + \sum_{n=1}^{\infty}[3a_n - (n+1)a_{n+1}]z^n.$$

We know that $a_0 = f(0) = 0$. Therefore $a_1 = 2$. For $n \geq 1, a_{n+1} = 3a_n/(n+1)$, so

$$a_2 = 3a_1/2, \quad a_3 = 3^2 a_1/(3)(2), \quad a_4 = 3^3 a_1/(4)(3)(2), \ldots,$$

and in general,

$$a_n = 3^{n-1}a_1/n! = 3^n\left(\frac{2}{3}\right)/n!.$$

Notice that this formula also gives $a_1 = 2$.) Thus if there is a power series that represents a solution it must be

$$f(z) = \frac{2}{3}\sum_{n=1}^{\infty}\frac{3^n}{n!}z^n.$$

Taking derivatives term by term confirms that this is a solution. In this case we can even recognize the function the series represents:

$$f(z) = \frac{2}{3}\sum_{n=1}^{\infty}\frac{(3z)^n}{n!} = \frac{2}{3}\left[\left(\sum_{n=0}^{\infty}\frac{(3z)^n}{n!}\right) - 1\right] = \frac{2}{3}(e^{3z} - 1).$$

The reader should check that this does solve the original problem.[4]

Example 3.2.18 (Generating Function for the Hermite Polynomials) *The function $f(z) = e^{2tz - z^2}$ is analytic everywhere and so has a power series expansion in powers of z whose coefficients depend on t. If we put $f(z) = \sum_{n=0}^{\infty} H_n(t)z^n/n!$, then the functions $H_n(t)$ are called the* ***Hermite polynomials****. (One needs to check that they are in fact polynomials in t.) The function f is called a* ***generating function****. Compute $H_0(t), H_1(t)$, and $H_2(t)$.*

[4] For additional applications of power series to differential equations, see, for example, J. Marsden and A. Weinstein, *Calculus II* (New York: Springer-Verlag, 1985), §12.6, or virtually any text on differential equations.

Solution From $f(z) = \sum_{n=0}^{\infty} H_n(t)z^n/n!$, the following hold:

$$H_0(t) = f(0) = 1$$

$$H_1(t) = f'(0) = (2t - 2z)e^{2tz-z^2}\Big|_{z=0} = 2t$$

$$H_2(t) = f''(0) = \left[-2e^{2tz-z^2} + (2t-2z)^2 e^{2tz-z^2}\right]_{z=0} = 4t^2 - 2$$

Proceeding inductively, we see that $f^{(k)}(z)$ is a polynomial in t and z multiplied by e^{2tz-z^2}, and so evaluation at $z = 0$ will always produce a polynomial in t.

Example 3.2.19 *Discuss the zero of* $f(z) = 1 - \cos z$ *at* 0.

Solution If $f(z) = 1 - \cos z$, then $f'(z) = \sin z$ and $f''(z) = \cos z$. Thus, $f(0) = 0$ and $f'(0) = 0$, but $f''(0) = 1 \neq 0$. Thus f has a zero of order 2 at 0. The Taylor series for f centered at 0 is

$$f(z) = \frac{1}{2!}z^2 - \frac{1}{4!}z^4 + \frac{1}{6!}z^6 - + \cdots = z^2\left[\frac{1}{2} - \frac{1}{24}z^2 + \frac{1}{720}z^4 - + \ldots\right].$$

Thus, $f(z) = z^2\varphi(z)$ where $\varphi(z) = (1/2) - (1/24)z^2 + (1/720)z^4 - + \ldots$. In particular, $\varphi(0) = 1/2 \neq 0$. Since $\varphi(z) = (1-\cos z)/z^2$, the only zeros of φ are at the points where $\cos z = 1$. The closest of these to 0 are $\pm 2\pi$.

Exercises

1. Find the radius of convergence of each of the following power series:

   (a) $\displaystyle\sum_{n=0}^{\infty} nz^n$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{e^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} n!\frac{z^n}{n^n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$

2. Find the radius of convergence of each of the following power series:

   (a) $\displaystyle\sum_{n=0}^{\infty} n^2 z^n$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{z^{2n}}{4^n}$

(c) $\sum_{n=0}^{\infty} n!z^n$

(d) $\sum_{n=0}^{\infty} \frac{z^n}{1+2^n}$

3. Compute the Taylor series of the following functions around the indicated points and determine the set on which the series converges:

   (a) $e^z, z_0 = 1$

   (b) $1/z, z_0 = 1$

4. Establish the Taylor series for $\sin z, \cos z$, and $(1+z)^\alpha$ in Table 3.2.1.

5. Compute the Taylor series of the following. (Give only the first few terms where appropriate.)

   (a) $(\sin z)/z, z_0 = 1$

   (b) $z^2 e^z; z_0 = 0$

   (c) $e^z \sin z, z_0 = 0$

6. Compute the first four terms of the Taylor series of $1/(1+e^z)$ around $z_0 = 0$. What is the radius of convergence?

7. Compute the Taylor series of the following around the indicated points:

   (a) $e^{z^2}, z_0 = 0$

   (b) $1/(z-1)(z-2), z_0 = 0$

8.• Compute the Taylor series of the following around the indicated point:

   (a) $\sin z^2, z_0 = 0$

   (b) $e^{2z}, z_0 = 0$

9. Compute the first few terms in the Taylor expansion of $\sqrt{z^2-1}$ around 0.

10. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ converge for $|z| < R$. For $|z| < R^2$, define $F(z)$ by selecting r with $|z|/R < r < R$ and setting

$$F(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta} g\left(\frac{z}{\zeta}\right) d\zeta,$$

where γ is the circle of radius r centered at the origin.

   (a) Show that the value of $F(z)$ does not depend on r so long as $|z|/R < r < R$.

   (b) Show that $F(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. *Hint*: Use Worked Example 2.4.15.

11. Establish the following:

$$\sinh z = \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n-1)!} \quad \text{and} \quad \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

12. What is the flaw in the following reasoning? Since $e^z = \sum_{n=0}^{\infty} z^n/n!$, we get $e^{1/z} = \sum_{n=0}^{\infty} 1/(n!z^n)$. Since this converges (because e^z is entire) and since the Taylor expansion is unique, the Taylor expansion of $f(z) = e^{1/z}$ around $z = 0$ is $\sum_{n=0}^{\infty} z^{-n}/n!$.

13. Differentiate the series for $1/(1-z)$ to obtain expansions for

$$\frac{1}{(1-z)^2} \quad \text{and} \quad \frac{1}{(1-z)^3}.$$

Give the radius of convergence.

14.* Let $f(z) = \sum a_n z^n$ have radius of convergence R and let $A = \{z \text{ such that } |z| < R\}$. Let $z_0 \in A$ and $\tilde{R}$ be the radius of convergence of the Taylor series of f around z_0. Prove that $R - |z_0| \le \tilde{R} \le R + |z_0|$.

15. If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, show that $\sum_{n=0}^{\infty} (\operatorname{Re} a_n) z^n$ has radius of convergence $\ge R$.

16. Let $f(z) = \sum a_n z^n$ be a power series with radius of convergence $R > 0$. Show that $\int_\gamma f = 0$ for each closed curve γ in the disk $A = \{z \text{ such that } |z| < R\}$ by either of the following two methods.

    (a) using Cauchy's Theorem

    (b) justifying term-by-term integration

17. In what region does

$$\sum_{n=1}^{\infty} \frac{\sin nz}{2^n}$$

represent an analytic function? What about

$$\sum_{n=1}^{\infty} \frac{\sin nz}{n^2}?$$

18. Find the first few terms of the Taylor expansion of $\tan z = (\sin z)/(\cos z)$ around $z = 0$. *Hint:* We know that such an expansion exists. Write

$$\frac{\sin z}{\cos z} = a_0 + a_1 z + a_2 z^2 + \dots .$$

Multiply by

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots ,$$

and use Worked Example 3.2.15 to solve for a_0, a_1, a_2.

19.* Let f be analytic on the region A, let $z_0 \in A$, and let D be the largest open disk centered at z_0 and contained in A.

(a) If f is unbounded on D, show that the radius of D equals the radius of convergence of the Taylor series for f at z_0.

(b) If there exists no analytic extension of f (that is, if there are no $\tilde{f}$ and A' such that $\tilde{f}$ is analytic on $A', A' \supset A, A' \neq A$, and $f = \tilde{f} \mid A$), show by an example that the radius of convergence of the Taylor series of f at z_0 can still be greater than the radius of D. *Hint*: Use the principal branch of $\log(1+z)$ with $z_0 = -2+i$.

20. Prove: A power series converges absolutely everywhere or nowhere on its circle of convergence. Give an example to show that each case can occur.

21.* If $\sum g_n(z)$ converges uniformly on a set $B \subset \mathbf{C}$ and $h(z)$ is a bounded function on B, prove that $\sum[h(z)g_n(z)]$ converges uniformly on B to $h(z)[\sum g_n(z)]$.

22. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converge for $|z| < R$. If $0 < r < R$, show that $f(z) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$, where $z = re^{i\theta}$ and

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(re^{i\theta})e^{-in\theta}\,d\theta.$$

Also show

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

The second equation is referred to as ***Parseval's theorem***, and we say that $f(z) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$ expresses the Taylor series as a ***Fourier series***. *Hint:* Use the Cauchy Integral Formula 2.4.6 for a_n and expand $f\bar{f}$ in a series, and then integrate term by term.

23. Let $H_n(x)$ be the Hermite polynomials introduced in Worked Example 3.2.18. Show that $H_1(x) = 2xH_0(x)$ and that for $n \geq 1, H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.

24.* Compute the Taylor expansion of $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ around $z = 2$ (see Worked Example 3.1.15).

25. Find a function such that $f(0) = 1$ and $f'(x) = x + 2f(x)$ for all x (see Worked Example 3.2.17).

26. Find a function f such that $f(0) = 1$ and $f'(x) = xf(x)$ for all x.

3.3 Laurent Series and Classification of Singularities

The Taylor series enables us to find a convergent power series expansion of a function $f(z)$ around a point z_0 when f is analytic in a whole disk around z_0. Thus, the Taylor expansion does not apply to functions like $f(z) = 1/z$ or e^z/z^2 around $z_0 = 0$ because they fail to be analytic at $z = 0$. For such functions there is another expansion, called the ***Laurent expansion*** (formulated in approximately 1840), that uses *inverse powers* of z as well as powers of z. This expansion is particularly important in the study of singular points of functions and leads to another fundamental result of complex analysis, the residue theorem, which is studied in Chapter 4.

Theorem 3.3.1 (Laurent Expansion Theorem) *Let $0 \le r_1 < r_2$, and $z_0 \in \mathbf{C}$, and consider the region $A = \{z \in \mathbf{C} \mid r_1 < |z - z_0| < r_2\}$* (see Figure 3.3.1). *We allow $r_1 = 0$ or $r_2 = \infty$ (or both). Let f be analytic on the region A. Then we can write*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where both series on the right side of the equation converge absolutely on A and uniformly in sets of the form $B_{\rho_1,\rho_2} = \{z \mid \rho_1 \le |z - z_0| \le \rho_2\}$, where $r_1 < \rho_1 < \rho_2 < r_2$. This series for f is called the ***Laurent series*** *or* ***Laurent expansion*** *around z_0 in the annulus A.*

If γ is a circle around z_0 with radius r, where $r_1 < r < r_2$, then the coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \qquad n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{1}{2\pi i} \int_\gamma f(\zeta)(\zeta - z_0)^{n-1} d\zeta \qquad n = 1, 2, \ldots.$$

(If we set $b_n = a_{-n}$, then the first formula covers both cases.) Any pointwise convergent expansion of f of this form equals the Laurent expansion; in other words, the Laurent expansion is unique.

The equations for the coefficients a_n and b_n in the Laurent series are not very practical for computing the Laurent series of a given function f. Notice that we *cannot* set $a_n = f^{(n)}(z_0)/n!$ as we did with the Taylor expansion. Indeed, $f^{(n)}(z_0)$ is not even defined, since $z_0 \notin A$.

There are a few tricks that can be used to obtain expansions of the desired form and the uniqueness of the expansion guarantees it must be the desired one. A few such techniques are given in the following text and in the worked examples.

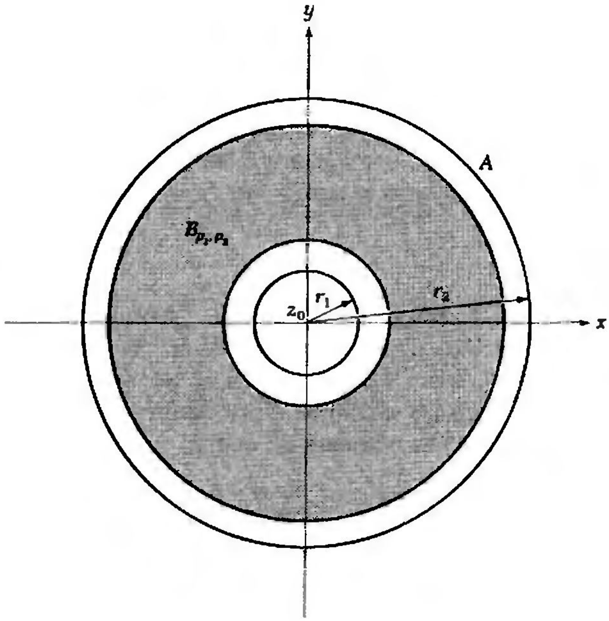


Figure 3.3.1: Laurent series, with $z_0 = 0$.

In the following proof we shall see that the power series part of f, namely the series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

converges and so is analytic *inside* the circle $|z - z_0| = r_2$, whereas the singular part,

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

converges *outside* $|z - z_0| = r_1$. The sum therefore converges *between* these circles.

Example In this example we show that uniqueness is dependent on the choice of A. Let $A = \{z \text{ such that } |z| > 1\}$ and $f(z) = 1/[z(z-1)]$. In this situation, f has the Laurent expansion

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z}\left[\frac{1}{z\left(1 - \frac{1}{z}\right)}\right] = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right) = \frac{1}{z^2} + \frac{1}{z^3} + \ldots$$

(valid if $|z| > 1$), whereas on the choice $A = \{z \text{ such that } 0 < |z| < 1\}$, it has the different expansion

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z}(1 + z + z^2 + \ldots) = -\left(\frac{1}{z} + 1 + z + z^2 + \ldots\right),$$

(valid for $0 < |z| < 1$). By uniqueness these are *the* Laurent expansions for the appropriate regions.

Proof of the Laurent Expansion Theorem As with the proof of Taylor's Theorem, we begin with Cauchy's Integral Formula. We will first show uniform convergence of the stated series on B_{ρ_1,ρ_2}, where a_n and b_n are defined in the theorem. Since all the circles γ of radius r are homotopic to each other in A as long as $r_1 < r < r_2$ (Why?), the numbers a_n and b_n are independent of r, so

$$a_n = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta$$

and

$$b_n = \frac{1}{2\pi i}\int_{\gamma_2} f(\zeta)(\zeta - z_0)^{n-1}d\zeta,$$

where γ_1 is a circle of radius $\tilde{\rho}_1$ and γ_2 is a circle of radius $\tilde{\rho}_2$ and where $r_1 < \tilde{\rho}_1 < \rho_1 < \rho_2 < \tilde{\rho}_2 < r_2$ (see Figure 3.3.2).

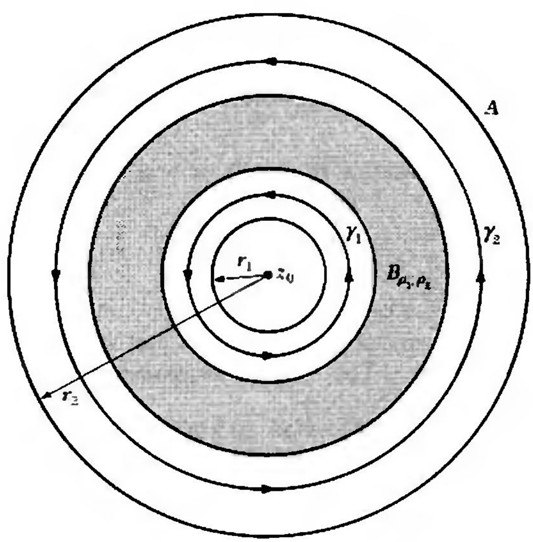


Figure 3.3.2: Construction of the curves γ_1 and γ_2.

For $z \in B_{\rho_1,\rho_2}$, we have

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta - z}d\zeta$$

by Cauchy's Integral Formula (see Exercise 5).

As in Taylor's Theorem, for ζ on γ_2 (and z fixed inside γ_2), we write the series

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} + \frac{z - z_0}{(\zeta - z_0)^2} + \frac{(z-z_0)^2}{(\zeta - z_0)^3} + \dots,$$

which converges uniformly in ζ on γ_2.

Substituting the series in the preceding equation for f, we may integrate term by term (by Proposition 3.1.9 and the fact that $f(\zeta)$ is bounded—see Exercise 21, 3.2) and thus obtain

$$\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta - z}d\zeta = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\left[\int_{\gamma_2}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta\right](z-z_0)^n = \sum_{n=0}^{\infty}a_n(z-z_0)^n.$$

Since this power series converges for z inside γ_2, it converges uniformly on strictly smaller disks (in particular, on B_{ρ_1,ρ_2}). Similarly the series

$$\frac{-1}{\zeta - z} = \frac{1}{(z-z_0)\left(1 - \frac{\zeta - z_0}{z - z_0}\right)} = \frac{1}{z - z_0} + \frac{\zeta - z_0}{(z-z_0)^2} + \frac{(\zeta - z_0)^2}{(z-z_0)^3} + \dots$$

converges uniformly with respect to ζ on γ_1. Thus,

$$\frac{-1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta - z}d\zeta = \sum_{n=1}^{\infty}\frac{1}{2\pi i}\left[\int_{\gamma_1}f(\zeta)\cdot(\zeta - z_0)^{n-1}d\zeta\right]\cdot\frac{1}{(z-z_0)^n} = \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}.$$

This series converges for z outside γ_1, so the convergence is uniform outside any strictly larger circle. This fact can be proved in the same way as the analogous fact for power series by using the Abel-Weierstrass Lemma 3.2.2. The student is asked to do this in Exercise 15. (Another method is to make the transformation $w = 1/(z - z_0)$ and apply the power series result to $\sum_{n=1}^{\infty} b_n w^n$.)

We have now proved the existence of the Laurent expansion. To show uniqueness, let us suppose that we have an expansion for f:

$$f(z) = \sum_{n=0}^{\infty}a_n(z-z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}.$$

If this converges in A it will, by the preceding remarks, do so uniformly on the circle γ, so we can form

$$\frac{f(z)}{(z-z_0)^{k+1}} = \sum_{n=0}^{\infty}a_n(z-z_0)^{n-k-1} + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^{n+k+1}},$$

which also converges uniformly. We then integrate term by term. By Worked Example 2.1.12, we have

$$\int_\gamma (z-z_0)^m dz = \begin{cases} 0 & m \neq -1 \\ 2\pi i & m = -1 \end{cases}.$$

Thus, if $k \geq 0$, each term of the second series and all those of the first except that with $n = k$ integrate to 0 around γ. Hence,

$$\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}} dz = 2\pi i a_k.$$

Similarly, if $k \leq -1$, all terms integrate to 0 except that in the second series with $n = -k$, so

$$\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}} dz = 2\pi i b_{-k} \quad \text{i.e.,} \quad b_n = \frac{1}{2\pi i}\int_\gamma f(z)(z-z_0)^{n-1} dz \quad \text{for} \quad n \geq 1.$$

Thus, the coefficients a_n, b_n are uniquely determined by f and the proof is complete. ■

Isolated Singularities: Classification of Singular Points We want to look in more detail at the special case of the Laurent Expansion Theorem 3.3.1 when $r_1 = 0$. In this case, f is analytic on $\{z \mid 0 < |z - z_0| < r_2\}$, which is the deleted r_2 neighborhood of z_0 (see Figure 3.3.3), and we say that z_0 is an ***isolated singularity*** of f. Thus we can expand f in a Laurent series as follows:

$$f(z) = \ldots + \frac{b_n}{(z-z_0)^n} + \ldots + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

(valid for $0 < |z - z_0| < r_2$).

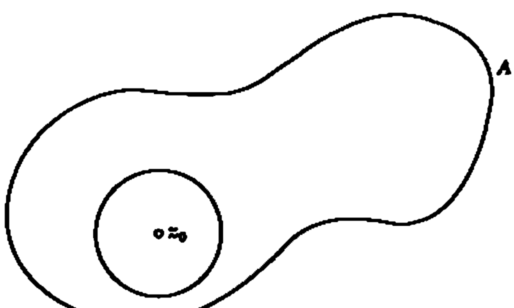


Figure 3.3.3: Isolated singularity.

Definition 3.3.2 *If f is analytic on a region A that contains some deleted ϵ neighborhood of z_0, then z_0 is called an* ***isolated singularity****. (Thus, the preceding Laurent expansion is valid in such a deleted ϵ neighborhood.)*

1. *If z_0 is an isolated singularity of f and if all but a finite number of the b_n are zero, then z_0 is called a* ***pole*** *of f. If k is the highest integer such that $b_k \neq 0$, z_0 is called a* ***pole of order*** *k. (To emphasize that $k \neq \infty$, we sometimes say "a pole of finite order k.") If z_0 is a first-order pole, we also say it is a* ***simple pole****.*

2. *If an infinite number of b_k's are nonzero, z_0 is called an* ***essential singularity****. (Sometimes this z_0 is called a pole of infinite order.) "Pole" shall always mean a pole of finite order.*

3. *We call b_1 the* ***residue*** *of f at z_0.*

4. *If all the b_k's are zero, we say that z_0 is a* ***removable singularity****.*

5. *A function that is analytic in a region A, except for poles in A, is called* ***meromorphic*** *in A. The phrase "f is a meromorphic function" means that f is meromorphic in $\mathbb{C}$.*

Thus, f has a pole of order k at z_0 if and only if its Laurent expansion in a deleted neighborhood about z_0 has the form

$$\frac{b_k}{(z-z_0)^k} + \ldots + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \ldots .$$

The part

$$\frac{b_k}{(z-z_0)^k} + \ldots + \frac{b_1}{z-z_0},$$

often called the ***principal part*** of f at z_0, tells "how singular" f is at z_0.

If f has a removable singularity, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

is a convergent power series; in this case, if we set $f(z_0) = a_0$, f will be analytic at z_0. In other words, *f has a removable singularity at z_0* iff *f can be defined at z_0 in such a way that f becomes analytic at z_0.*

As we shall see in Chapter 4, finding the Laurent expansion is not as important as being able to compute the residue b_1, and this computation can often be done without computing the Laurent series. Techniques for doing so will be studied in §4.1. The important property of b_1 not shared by other coefficients is stated in the following result.

Proposition 3.3.3 *Let f be analytic on a region A and have an isolated singularity at z_0 with residue b_1 at z_0. If γ is any circle around z_0 in A whose interior, except for the point z_0, lies in A, then*

$$\int_\gamma f(z)dz = b_1 \cdot 2\pi i.$$

This conclusion follows from the formula for b_1 in the Laurent Expansion Theorem 3.3.1. The point is that we can compute b_1 by methods other than the preceding integral and therefore we can use Proposition 3.3.3 to *compute* $\int_\gamma f$. For example, if $z \neq 0$, then

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

(Why?), so $e^{1/z}$ has an essential singularity at $z = 0$ and $b_1 = 1$. Thus $\int_\gamma e^{1/z}dz = 2\pi i$ for any circle γ around 0.

The following proposition characterizes the various types of singularities.

Proposition 3.3.4 *Let f be analytic on a region A and have an isolated singularity at z_0.*

1. *z_0 is a removable singularity iff any one of the following conditions holds:*
   (a) *f is bounded in a deleted neighborhood of z_0.*
   (b) $\lim_{z\to z_0} f(z)$ *exists.*
   (c) $\lim_{z\to z_0}(z - z_0)f(z) = 0$.

2. *z_0 is a simple pole iff* $\lim_{z\to z_0}(z - z_0)f(z)$ *exists and is unequal to zero. The limit equals the residue of f at z_0.*

3. *z_0 is a pole of order $\leq k$ (or possibly a removable singularity) iff any one of the following conditions holds:*
   (a) *There are a constant $M > 0$ and an integer $k \geq 1$ such that*
   $$|f(z)| \leq \frac{M}{|z - z_0|^k}$$
   in a deleted neighborhood of z_0.
   (b) $\lim_{z\to z_0}(z - z_0)^{k+1}f(z) = 0$.
   (c) $\lim_{z\to z_0}(z - z_0)^k f(z)$ *exists.*

4. *z_0 is a pole of order $k \geq 1$ iff there is an analytic function ϕ defined on a neighborhood U of z_0 such that $U\backslash\{z_0\} \subset A, \phi(z_0) \neq 0$, and*
   $$f(z) = \frac{\phi(z)}{(z - z_0)^k}$$
   for all $z \in U, z \neq z_0$.

Proof

1. If z_0 is a removable singularity, then in a deleted neighborhood of z_0 we have $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. Since this series represents an analytic function in an undeleted neighborhood of z_0, obviously conditions (a), (b) and (c) hold. Conditions (a) and (b) each obviously imply condition (c), so it remains to be shown that condition (c) implies that z_0 is a removable singularity for f. We must prove that each b_k in the Laurent expansion of f around z_0 is 0. Now

$$b_k = \frac{1}{2\pi i}\int_{\gamma_r} f(\zeta)(\zeta - z_0)^{k-1}d\zeta,$$

where γ_r is a circle in A whose interior (except for z_0) lies in A. Let $\epsilon > 0$ be given. By condition (c) we can choose r with $0 < r < 1$ such that on γ_r we have the estimate $|f(\zeta)| < \epsilon/|\zeta - z_0| = \epsilon/r$. Then

$$\begin{aligned} |b_k| &\le \frac{1}{2\pi}\int_{\gamma_r} |f(\zeta)||\zeta - z_0|^{k-1}|d\zeta| \le \frac{1}{2\pi}\frac{\epsilon}{r}r^{k-1}\int_{\gamma_r}|d\zeta| \\ &= \frac{1}{2\pi}\frac{\epsilon}{r}r^{k-1}2\pi r = \epsilon r^{k-1} \le \epsilon. \end{aligned}$$

Thus, $|b_k| \le \epsilon$. Since ϵ was arbitrary, $b_k = 0$.

We shall use part 3 to prove part 2, so we prove part 3 next.

2. If z_0 is a simple pole, then in a deleted neighborhood of z_0,

$$f(z) = \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n(z-z_0)^n = \frac{b_1}{z-z_0} + h(z),$$

where h is analytic at z_0 and where $b_1 \neq 0$ by the Laurent expansion. Hence

$$\lim_{z\to z_0}(z - z_0)f(z) = \lim_{z\to z_0}[b_1 + (z - z_0)h(z)] = b_1.$$

On the other hand, suppose that $\lim_{z\to z_0}(z-z_0)f(z)$ exists and is unequal to zero. Thus, $\lim_{z\to z_0}(z - z_0)^2 f(z) = 0$. By the result obtained in part 3, this shows that

$$f(z) = \frac{b_1}{(z - z_0)} + \sum_{n=0}^{\infty} a_n(z - z_0)^n = \frac{b_1}{z - z_0} + h(z)$$

for some constant b_1 and analytic function h where b_1 may or may not be zero. But then $(z-z_0)f(z) = b_1 + (z - z_0)h(z)$, so $\lim_{z\to z_0}(z - z_0)f(z) = b_1$. Thus, in fact, $b_1 \neq 0$, and therefore f has a simple pole at z_0.

3. This statement follows by applying part 1 to the function $(z-z_0)^k f(z)$, which is analytic on A. (The student should write out the details.)

4. By definition, z_0 is a pole of order $k \geq 1$ iff

$$
\begin{aligned}
f(z) & \\
= & \frac{b_k}{(z-z_0)^k} + \frac{b_{k-1}}{(z-z_0)^{k-1}} + \ldots + \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n \\
= & \frac{1}{(z-z_0)^k}\left[b_k + b_{k-1}(z-z_0) + \ldots + b_1 (z-z_0)^{k-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{n+k}\right]
\end{aligned}
$$

(where $b_k \neq 0$). This expansion is valid in a deleted neighborhood of z_0. Let

$$\phi(z) = b_k + b_{k-1}(z-z_0) + \ldots + b(z-z_0)^{k-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{n+k}.$$

Then $\phi(z)$ is analytic in the corresponding undeleted neighborhood (since it is a convergent power series) and $\phi(z_0) = b_k \neq 0$. Conversely, given such a ϕ, we can retrace these steps to show that z_0 is a pole of order $k \geq 1$. ∎

Zeros and Poles Near a zero of order n, an analytic function acts much like z^n near 0. Near a pole of order n, it acts much like $1/z^n$. This suggests that a reciprocal of a function with a zero should have a pole and conversely. This is, in fact, correct. Suppose A is an open set in $\mathbb{C}$ and $z_0 \in A$. We know that a function with a zero of order n at z_0 can be factored as $(z-z_0)^n \varphi(z)$ where $\varphi(z)$ is analytic and nonzero in a neighborhood of z_0. A function has a pole of order m if and only if it factors as $(z-z_0)^{-m}\psi(z)$ where ψ is analytic and nonzero in a neighborhood of z_0. With $\psi = 1/\varphi$, we see that the reciprocal of a function with a zero of order n has a pole of order n at that point. The reciprocal of a function with a pole of order n has a removable singularity at that point, and when the singularity is removed, it becomes a zero of order n. If the numerator and denominator of a fraction both have zeros, we can factor and cancel.

Proposition 3.3.5 *Suppose f and g are analytic in a neighborhood of z_0 with zeros there of order n and k respectively. (Take the order to be 0 if the function is not 0 at z_0). Let $h(z) = f(z)/g(z)$. Then*

1. *if $k > n$, then h has a pole of order $k-n$ at z_0.*

2. *if $k = n$, then h has a removable singularity with nonzero limit at z_0.*

3. *if $k < n$, then h has a removable singularity at z_0, and setting $h(z_0) = 0$ produces an analytic function with a zero of order $n-k$ at z_0.*

Proof We know that there is neighborhood $D = \{z \in \mathbb{C} \mid |z-z_0| < r\}$ on which f and g factor as $f(z) = (z-z_0)^n \varphi(z)$ and $g(z) = (z-z_0)^k \psi(z)$ where φ and ψ are analytic and neither is ever 0 on D. The function $H(z) = \varphi(z)/\psi(z)$ is analytic and never equal to 0 on D. Thus, for z in the deleted neighborhood $U = D\setminus\{z_0\}$, we have

$$h(z) = \frac{f(z)}{g(z)} = \frac{(z-z_0)^n\varphi(z)}{(z-z_0)^k\psi(z)} = (z-z_0)^{n-k}H(z).$$

Our conclusions now follow from Propositions 3.2.9 and 3.3.4. ■

Essential Singularities In many examples, the singularities are poles. It is not hard to show that if $f(z)$ has a pole (of finite order k) at z_0, then $|f(z)| \to \infty$ as $z \to z_0$ (see Exercise 7). However, in case of an essential singularity, $|f|$ will not, in general, approach ∞ as $z \to z_0$. The following result, proved by C. E. Picard in 1879, answers this question.

Theorem 3.3.6 (Picard Theorem) *Let f have an essential singularity at z_0 and let U be any (arbitrarily small) deleted neighborhood of z_0. Then for all $w \in \mathbb{C}$, except perhaps one value, the equation $f(z) = w$ has infinitely many solutions z in U.*

This theorem actually belongs in a more advanced course.[5] However, we can easily prove a simpler version, which in any case is the jumping-off point for Picard's Theorem.

Theorem 3.3.7 (Casorati-Weierstrass Theorem) *Let f have an (isolated) essential singularity at z_0 and let $w \in \mathbb{C}$. Then there is a sequence $z_1, z_2, z_3, \ldots$ in U such that $z_n \to z_0$ and $f(z_n) \to w$.*

Proof If there were no such sequence, then there would be an $\epsilon > 0$ and a $\delta > 0$ such that $|f(z) - w| > \epsilon$ for all z in the deleted neighborhood $U = \{z \in \mathbb{C} \mid 0 < |z - z_0| < \delta\}$. In particular, $f(z)$ is never equal to w in U, so the function $g(z) = 1/(f(z) - w)$ is analytic on U and $|g(z)| < 1/\delta$ there. Thus, any singularity of g at z_0 is removable. The values of g cannot be constantly 0 near z_0 since f is not constantly infinite. (The singularity is isolated.) From Corollary 3.2.8, any zero of g at z_0 is isolated and has a finite order k. Therefore, $f(z) = w + 1/g(z)$ would either be analytic (if $k = 0$) or have a pole of order k at z_0 by Proposition 3.3.5. This would contradict the hypothesis that f has an essential singularity at z_0. ■

See Exercise 20 for another interpretation of this result.

Worked Examples

Example 3.3.8 *Find the Laurent expansions of the following functions (with z_0, r_1, r_2 as indicated):*

(a) $(z+1)/z; z_0 = 0, r_1 = 0, r_2 = \infty$

(b) $z/(z^2+1); z_0 = i, r_1 = 0, r_2 = 2$

[5]See E. C. Titchmarsh, *The Theory of Functions*, Second Edition (New York: Oxford University Press, 1939), p. 283, corrected reprinting 1968.

Solution Write

(a) $\dfrac{z+1}{z} = \dfrac{1}{z} + 1$. This equation is in the form of the Laurent expansion, so, by uniqueness, it equals it; that is, $b_k = 0$ for $k > 1, b_1 = 1, a_0 = 1, a_k = 0$ for $k \geq 1$.

(b) A partial-fraction expansion gives

$$\frac{z}{z^2+1} = \frac{z}{(z+i)(z-i)} = \frac{1}{2}\frac{1}{z-i} + \frac{1}{2}\frac{1}{z+i}.$$

Because $1/(z+i)$ is analytic near $z = i$, it can be expanded as a power series in $z - i$ by using the geometric series (see Figure 3.3.4):

$$\begin{aligned} \frac{1}{z+i} &= \frac{1}{2i+(z-i)} = \frac{1}{2i}\frac{1}{1-\left(-\frac{z-i}{2i}\right)} \\ &= \frac{1}{2i}\sum_{n=0}^{\infty}\left(-\frac{z-i}{2i}\right)^n = \sum_{n=0}^{\infty} i^{n-1}2^{-n-1}(z-i)^n. \end{aligned}$$

Thus, the Laurent expansion is

$$\frac{z}{z^2+1} = \frac{1}{2}(z-i)^{-1} + \sum_{n=0}^{\infty} i^{n-1}2^{-n-2}(z-i)^n.$$

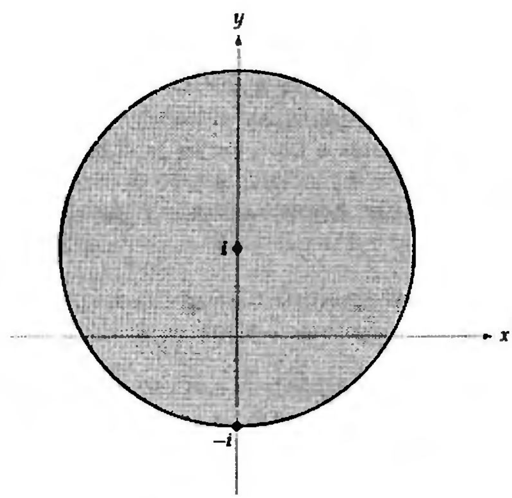


Figure 3.3.4: Region of convergence for the expansion of $1/(z+i)$.

Example 3.3.9 *Determine the order of the pole of each of the following functions at the indicated singularity:*

a) $(\cos z)/z^2, z_0 = 0$

b) $(e^z - 1)/z^2, z_0 = 0$

c) $(z+1)/(z-1), z_0 = 0$

Solution

a) The function z^2 has a zero of order 2 and $\cos 0 = 1$, so $(\cos z)/z^2$ has a pole of order 2 by Proposition 3.3.5. Alternatively,

$$\frac{\cos z}{z^2} = \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right) = \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - \ldots,$$

so again the pole is of order 2.

b) The numerator has a zero of order 1 at 0 (why?) and the denominator a zero of order 2. The quotient thus has a simple pole by Proposition 3.3.5. Alternatively,

$$\frac{e^z - 1}{z^2} = \frac{1}{z^2}\left[\left(1 + z + \frac{z^2}{2!} + \ldots\right) - 1\right] = \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \ldots,$$

so the pole is simple.

(c) There is no pole since the function is analytic at 0.

Example 3.3.10 *Determine which of the following functions have removable singularities at $z_0 = 0$:*

(a) $(\sin z)/z$

(b) e^z/z

(c) $(e^z - 1)^2/z^2$

(d) $z/(e^z - 1)$

Solution

(a) $\lim_{z\to 0} z \cdot (\sin z)/z = \lim_{z\to 0} \sin z = 0$, so the singularity is removable (by Proposition 3.3.4.). Alternatively,

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \ldots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots.$$

(b) $\lim_{z\to 0} z \cdot e^z/z = 1$, so the pole is simple (the singularity is not removable).

(c) $(e^z - 1)/z$ has a removable singularity, since $\lim_{z\to 0} z \cdot (e^z - 1)/z = 0$, so $[(e^z - 1)/z]^2$ also has a removable singularity.

(d) $\lim_{z\to 0} z/(e^z - 1) = 1$, because $(e^z - 1)/z = 1 + z/2 + z^2/3! + \ldots \to 1$ as $z \to 0$. Thus, $z/(e^z - 1)$ has a removable singularity.

Example 3.3.11 *Show that the function*

$$f(z) = \frac{1 - \cos(z^5)}{\sin(z^3)}$$

has a removable singularity at $z_0 = 0$ and that when the singularity is "removed", the resulting function has a zero of order 7.

Solution The Taylor series expansion for numerator and denominator are

$$1 - \cos(z^5) = \frac{1}{2!}z^{10} - \frac{1}{4!}z^{20} + \frac{1}{6!}z^{30} - \ldots = z^{10}\left(\frac{1}{2!} - \frac{1}{4!}z^{10} + \ldots\right)$$

and

$$\sin(z^3) = z^3 - \frac{1}{3!}z^9 + \frac{1}{5!}z^{15} - \ldots = z^3\left(1 - \frac{1}{3!}z^6 - \frac{1}{5!}z^{12} + \ldots\right).$$

The numerator has a zero of order 10 and the denominator has a zero of order 3 at $z_0 = 0$. By Proposition 3.3.5, the quotient has a removable singularity there and extending the function to have value 0 at z_0 results in a zero of order $10 - 3 = 7$. In fact, for $0 < |z| < \sqrt[3]{\pi}$, we have

$$f(z) = z^7 \frac{\frac{1}{2} - \frac{1}{4!}z^{10} + \frac{1}{6!}z^{20} - \ldots}{1 - \frac{1}{3!}z^6 + \frac{1}{5!}z^{12} - \ldots}.$$

This function equals z^7 multiplied by an analytic function with value $1/2$ at $z_0 = 0$.

Exercises

1. Find the Laurent series expansions of the following functions around $z_0 = 0$ in the regions indicated:

   (a)• $\sin(1/z), 0 < |z| < \infty$

   (b)• $1/z(z + 1), 0 < |z| < 1$

   (c) $z/(z + 1), 0 < |z| < 1$

   (d) $e^z/z^2, 0 < |z| < \infty$

2. Find the Laurent series expansion of $1/z(z + 1)$ around $z_0 = 0$ valid in the region $1 < |z| < \infty$.

3. Find the Laurent series expansion of $z/(z+1)$ around $z_0 = 0$ valid in the region $1 < |z| < \infty$.

4. Expand $\dfrac{1}{z(z-1)(z-2)}$ in a Laurent series in the following regions:

   (a) $0 < |z| < 1$

   (b) $1 < |z| < 2$

5.* Let γ_1 and γ_2 be two concentric circles around z_0 of radii R_1 and R_2, $R_1 < R_2$. If z lies between the circles and f is analytic on a region containing γ_1, γ_2, and the region between them, show that

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}d\zeta.$$

6. Suppose the Laurent series of $f(z) = e^{1/z}/(1-z)$ valid for $0 < |z| < 1$ is $\sum_{n=-\infty}^{\infty} c_n z^n$. Compute c_{-2}, c_{-1}, c_0, c_1, and c_2.

7. Let f have a pole at z_0 of order $k \geq 1$. Prove that $f(z) \to \infty$ as $z \to z_0$. *Hint*: Use part 4 of Proposition 3.3.4.

8. Prove, using the Taylor series, the following complex version of *l'Hôpital's rule*: Let $f(z)$ and $g(z)$ be analytic, both having zeros of order k at z_0. Then $f(z)/g(z)$ has a removable singularity and

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(k)}(z_0)}{g^{(k)}(z_0)}.$$

9. Which of the following functions have removable singularities at the indicated points?

   (a) $\dfrac{\cos(z-1)}{z^2}, z_0 = 0$

   (b) $z/(z-1), z_0 = 1$

   (c) $f(z)/(z-z_0)^k$ if f has a zero at z_0 of order k

10. If f is analytic on a region containing a circle γ and its interior and has a zero of order 1 only at z_0 inside or on γ, show that

$$z_0 = \frac{1}{2\pi i}\int_{\gamma} \frac{zf'(z)}{f(z)}dz.$$

Hint: Let $f(z) = (z - z_0)\phi(z)$ and apply the Cauchy Integral Formula.

11. Find the first few terms in the Laurent expansion of $1/(e^z - 1)$ around $z = 0$. *Hint:* Show that since $1/(e^z - 1)$ has a simple pole, we can write

$$\frac{1}{e^z - 1} = \frac{b_1}{z} + a_0 + a_1 z + a_2 z^2 + \ldots.$$

Then cross multiply (using Worked Example 3.2.15) and solve for b_1, a_0, a_1.

12.* For f as in the Laurent Expansion Theorem 3.3.1, show that if $r_1 < r < r_2$, then

$$\int_0^{2\pi} |f(z_0 + re^{i\theta})|^2 d\theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n} + 2\pi \sum_{n=1}^{\infty} |b_n|^2 r^{-2n}.$$

13. Use the method of Exercise 11 to find the first few terms in the Laurent expansion of $\cot z = (\cos z)/(\sin z)$ around $z = 0$.

14. Define a branch of $\sqrt{z^2 - 1}$ that is analytic except for the segment $[-1, 1]$ on the real axis. Determine the first few terms in the Laurent expansion that is valid for $|z| > 1$.

15. If the series

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

converges for $|z - z_0| > R$, prove that it necessarily converges *uniformly* on the set $F_r = \{z \text{ such that } |z - z_0| > r\}$ for $r > R$. *Hint:* Adapt the Abel-Weierstrass Lemma and the Weierstrass M Test to this case.

16.* Let f have a zero at z_0 of multiplicity k. Show that the residue of f'/f at z_0 is k.

17. Discuss the singularities of $1/\cos(1/z)$.

18. Evaluate $\int_\gamma z^n e^{1/z} dz$, where γ is the circle of radius 1 centered at 0 and traveled once in the counterclockwise direction.

19. Find the residues of the following functions at the indicated points:

   (a) $1/(z^2 - 1), z = 1$
   (b) $z/(z^2 - 1), z = 1$
   (c) $(e^z - 1)/z^2, z = 0$
   (d) $(e^z - 1)/z, z = 0$

20. (a) Let z_0 be an essential singularity of f and let U be any deleted neighborhood of z_0. Prove that the closure of $f(U)$ is $\mathbb{C}$.
   (b) Assuming the Picard Theorem 3.3.6, derive the "Little Picard Theorem" *The image of an entire nonconstant function misses at most one point of* $\mathbb{C}$.

Review Exercises for Chapter 3

1. Find the Taylor expansion of $\log z$ (the principal branch of the logarithm) around $z = 1$.

2.• Where are the poles of $1/\cos z$ and what are their orders?

3. Find the Laurent expansion of $1/(z^2 + z^3)$ around $z = 0$.

4. The 68th derivative of $f(z) = e^{z^2}$ at $z = 0$ is given by $(68)!/(34)!$. Prove this without actually computing the 68th derivative.

5. Expand $z^2 \sin z^2$ in a Taylor series around $z = 0$.

6. Let

$$f(z) = \frac{\pi z(1 - z^2)}{\sin(\pi z)}.$$

(a) Identify all singularities of f in $\mathbb{C}$ and classify each as removable, a pole (of what order), or essential.

(b) (i) How do you know that $f(z)$ has a series expansion $\sum_{k=-\infty}^{\infty} c_k z^k$ valid for z near 0?

(ii) What can you say about c_k for $k < 0$?

(iii) Find c_0, c_1, and c_2.

(c) For what set of values of z is the series expansion in part (b) valid? (You need not determine whether the series converges on the boundary, but give the interior of the region.)

(d) Let the function g be defined to be $f(z)$ for $z \neq -1, 0, 1$, let $g(0) = 1$ and let $g(z) = 2$ if $z = \pm 1$. Discuss the relationship between the function g and your answers to parts (a), (b), and (c).

7. Verify the Picard Theorem for the function $e^{1/z}$.

8. Let $\exp[t(z - 1/z)/2] = \sum_{n=-\infty}^{\infty} J_n(t) z^n$ be the Laurent expansion for each fixed $t \in \mathbb{R}$. The function $J_n(t)$ is called the ***Bessel function*** of order n. Show that

(a) $J_n(t) = \dfrac{1}{\pi} \displaystyle\int_0^{\pi} \cos(t \sin\theta - n\theta)\, d\theta$

(b) $J_{-n}(t) = (-1)^n J_n(t)$

9. Find the radii of convergence of

(a)• $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n^2} z^n$

(b) $\displaystyle\sum_{n=0}^{\infty} z^{n!}$

10. Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with radius of convergence $R > 0$. If $0 < r < R$, show that there is a constant M such that $|a_n| \le Mr^{-n}, n = 0, 1, 2, \ldots$.

11. Let f be analytic on $\mathbb{C}\backslash\{0\}$. Show that the Laurent expansions of f valid in the regions $\{z$ such that $|z| > 0\}$ and $\{z$ such that $|z| > 1\}$ are the same.

12.• Suppose that f is analytic on the open unit disk $|z| < 1$ and that there is a constant M such that $|f^{(k)}(0)| \le M^k$ for all k. Show that f can be extended to an entire function.

13. Suppose that f is analytic in a region containing the closed unit disk $|z| \le 1$, that $f(0) = 0$, and that $|f(z)| < 1$ if $|z| = 1$. Show that there are no $z \neq 0$ with $|z| < 1$ and $f(z) = z$. *Hint:* Use the Schwarz Lemma.

14. What is the radius of convergence of the Taylor expansion of

$$f(z) = \frac{e^z}{(z-1)(z+1)(z-2)(z-3)}$$

when expanded around $z = i$?

15.• Evaluate

$$\int_\gamma \frac{z^2 + e^z}{z(z-3)} dz$$

where γ is the unit circle.

16. Suppose that the complex series $\sum_{n=0}^{\infty} a_n$ converges but that $\sum_{n=0}^{\infty} |a_n|$ diverges. Show that the power series $\sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence equal to 1. Answer the same question but assume that the series $\sum_{n=0}^{\infty} a_n$ converges and that $\sum_{n=0}^{\infty} n|a_n|$ diverges.

17. Find the Laurent expansion of

$$f(z) = \frac{1}{z(z^2+1)}$$

that is valid for

(a) $0 < |z| < 1$

(b) $1 < |z|$

18. Find the Laurent series expansion of $f(z) = 1/(1+z^2) + 1/(3-z)$ valid in each of the following regions:

(a) $\{z$ such that $|z| < 1\}$

(b) $\{z$ such that $1 < |z| < 3\}$

(c) $\{z \text{ such that } |z| > 3\}$

19. Let f be entire and let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R. Can you find another power series $\sum b_n z^n$ with radius of convergence $\geq R$ such that

$$\sum_{n=0}^{\infty} b_n z^n = f\left(\sum_{n=0}^{\infty} a_n z^n\right).$$

20.• Let f be entire and suppose that $f(z) \to \infty$ as $z \to \infty$. Prove that f is a polynomial. *Hint:* Show that $f(1/z)$ has a pole of finite order at $z = 0$.

21. Let f have an isolated singularity at z_0. Show that if $f(z)$ is bounded in a deleted neighborhood of z_0, then $\lim_{z \to z_0} f(z)$ exists.

22. Let f be analytic on $|z| < 1$. Show that the inequality $|f^{(k)}(0)| \geq k!5^k$ cannot hold for all k.

23. Evaluate the definite integral $\displaystyle\int_0^{2\pi} e^{e^{i\theta}}\, d\theta$.

24.• Let $f(z)$ be entire and satisfy these two conditions:

(i) $f'(z) = f(z)$

(ii) $f(0) = 1$

Show that $f(z) = e^z$. If you replace (i) by $f(z_1 + z_2) = f(z_1)f(z_2)$, show that $f(z) = e^{az}$ for some constant a.

25. Determine the order of the poles of the following functions at their singularities:

(a) $\dfrac{e^z(z-3)}{(z-1)(z-5)}$

(b) $(e^z - 1)/z$

(c) $(e^z - 2)/z$

(d) $(\cos z)/(1 - z)$

26. Identify the singularities of $f(z) = z/(e^z - 1)(e^z - 2)$ and classify each as removable, essential, or a pole of specified order.

27. Evaluate $\displaystyle\int_\gamma (e^z/z^2)\, dz$ where γ is the unit circle.

28.* (a) Show by example that the mean value theorem for analytic functions is not true. In other words, let f be defined on the region A and let $z_1, z_2 \in A$ be such that the straight line joining z_1 to z_2 lies in A. Show that there need not be a z_0 on this straight line such that

$$f'(z_0) = \frac{f(z_1) - f(z_2)}{z_1 - z_2}.$$

(b) If, however, $|f'(z_0)| \le M$ on this line, prove that $|f(z_1) - f(z_2)| \le M|z_1 - z_2|$ and generally that if $|f'(z_0)| \le M$ on a curve γ joining z_1 to z_2, then $|f(z_1) - f(z_2)| \le Ml(\gamma)$.

29. Let $f(z) = (z^2 - 1)/[\cos(\pi z) + 1]$ have the series expansion $\sum_{n=0}^{\infty} a_n z^n$ near $z = 0$.

(a) Compute a_0, a_1, and a_2.

(b) Identify the singularities of f and classify each as essential or a pole of specified order.

(c) What is the radius of convergence of the series?

30. If $f(z) = f(-z)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is convergent on a disk $|z| < R, R > 0$. show that $a_n = 0$ for $n = 1, 3, 5, 7, \ldots$.

31. If f is entire and is bounded on the real axis, then f is constant. Prove or give a counterexample.

32. Let f be analytic on a region A containing the disk $\{z \in \mathbb{C} \mid |z - z_0| \le R\}$. so that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Let $R_n(z)$ equal $f(z)$ minus the nth partial sum. (R_n is thus the remainder . Let $\rho < R$ and let M be the maximum of f on $\{z$ such that $|z - z_0| = R$ ·. Show that $|z - z_0| \le \rho$ implies that

$$|R_n(z)| \le M \left(\frac{\rho}{R}\right)^{n+1} \frac{1}{1 - \rho/R}.$$

33.* The ***Bernoulli numbers*** B_n are defined to be the coefficients of the power series of $z/(e^z - 1)$:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

(a) Determine the radius of convergence of this series.

(b) Using the Cauchy Integral Formulas and the contour $|z| = 1$, find an integral expression for B_n of the form

$$B_n = \int_0^{2\pi} g_n(\theta)\, d\theta,$$

for suitable functions $g_n(\theta)$. where $0 \le \theta \le 2\pi$.

34. The ***Legendre polynomials*** $P_n(\alpha)$ are defined to be the coefficients of z^n in the Taylor development

$$(1 - 2\alpha z + z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(\alpha) z^n.$$

Prove that $P_n(\alpha)$ is a polynomial of degree n and find P_1, P_2, P_3, P_4.

35. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} 2^n z^{n^2}$.

36.* Prove:

(a) $\displaystyle\left(\frac{z^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_\gamma \frac{z^n e^{zt}}{n! t^n}\frac{dt}{t}$, where γ is the unit circle

(b) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{z^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2z\cos\theta}\, d\theta$

37. Find a power series which solves the functional equation $f(z) = z + f(z^2)$ and show that there is only one power series which solves the equation with $f(0) = 0$.

38. What is wrong with the following argument? Consider

$$f(z) = \ldots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \ldots.$$

Note that

$$z + z^2 + \ldots = \frac{z}{1-z}$$

whereas

$$1 + \frac{1}{z} + \frac{1}{z^2} + \ldots = \frac{1}{1 - 1/z} = \frac{-z}{1-z}.$$

Hence $f(z) = 0$. Is f in fact the zero function?

39. Suppose f is an entire function and that $|f^{(k)}(0)| \le 1$ for all $k \ge 0$. Show that $|f(z)| \le e^{|z|}$ for all $z \in \mathbf{C}$.

40. Let

$$f(z) = \frac{(z-1)^2(z+3)}{1-\sin(\pi z/2)}.$$

(a) Find all the singularities of f and identify each as a removable singularity, a pole (give the order), or an essential singularity.

(b) If $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$ is the Taylor expansion of f centered at 0, find a_0, a_1, and a_2.

(c) What is the radius of convergence of the series in (b)?

Chapter 4

Calculus of Residues

This chapter focuses on the Residue Theorem, which states that the integral of an analytic function f around a closed contour equals $2\pi i$ times the sum of the residues of f inside the contour. We shall use this theorem in our first main application of complex analysis, the evaluation of definite integrals. The chapter begins with some techniques for computing residues of functions at isolated singularities.

4.1 Calculation of Residues

We recall from §3.3 that if f has an isolated singularity at z_0, then f admits a Laurent expansion that is valid in a deleted neighborhood of z_0:

$$f(z) = \ldots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + \ldots,$$

where b_1 is called the ***residue*** of f at z_0. This is written

$$b_1 = \text{Res}(f; z_0).$$

We want to develop techniques for computing the residue without having to find the whole Laurent expansion. Of course, if the Laurent expansion is known, there is no problem. For example, since

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \ldots + \frac{1}{n!z^n} + \ldots,$$

the coefficient of $1/z$ is 1, so $f(z) = e^{1/z}$ has residue 1 at $z_0 = 0$.

If we are given f defined on a region A with an isolated singularity at z_0, then we proceed in the following way to find the residue. First we decide whether we can easily find the first few terms in the Laurent expansion about z_0. If so, the residue of f at z_0 will be the coefficient of $1/(z-z_0)$ in that expansion. If not, then we guess the order of singularity, verify it according to the rules that will be developed in this section (some rules were already developed in Proposition 3.3.4). and calculate the residue according to these rules. (The rules are summarized in Table 4.1.1 given later in this section). If we have any doubt as to what order to guess, we should work systematically by first guessing removable singularity, then simple pole, and so on, checking against Table 4.1.1 until we obtain a verified answer.

Removable Singularities Let f be analytic in a deleted neighborhood of z_0. Recall from §3.3 that f has a removable singularity at z_0 iff $\lim_{z \to z_0}(z-z_0)f(z) = 0$. The following theorem covers many important cases and is sometimes the easiest to use.

Proposition 4.1.1 *If $g(z)$ and $h(z)$ are analytic and have zeros at z_0 of the same order, then $f(z) = g(z)/h(z)$ has a removable singularity at z_0.*

Proof By Proposition 3.3.4, we can write $g(z) = (z - z_0)^k \tilde{g}(z)$, where $\tilde{g}(z_0) \neq 0$ and $h(z) = (z - z_0)^k \tilde{h}(z)$, where $\tilde{h}(z_0) \neq 0$ and $\tilde{g}$ and $\tilde{h}$ are analytic and nonzero at z_0. Thus, $f(z) = \tilde{g}(z)/\tilde{h}(z)$ is analytic at z_0. ■

Likewise, if g has a zero at z_0 of order greater than h, then g/h has a removable singularity at z_0.

Examples

(i) $e^z/(z-1)$ has no singularity at $z_0 = 0$.

(ii) $(e^z - 1)/z$ has a removable singularity at 0 because $e^z - 1$ and z have zeros of order 1. (They vanish at $z = 0$ but their derivatives do not.)

(iii) $z^2/\sin^2 z$ has a removable singularity at $z_0 = 0$ because both the numerator and the denominator have zeros of order 2. ■

The preceding discussion is summarized in lines 1 and 2 of Table 4.1.1.

Simple Poles By Proposition 3.3.4, if $\lim_{z \to z_0}(z - z_0)f(z)$ exists and is nonzero, then f has a simple pole at z_0 and this limit equals the residue. Let us apply this result to obtain a useful method for computing residues.

Proposition 4.1.2 *Let g and h be analytic at z_0 and assume that $g(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$. Then $f(z) = g(z)/h(z)$ has a simple pole at z_0 and*

$$\operatorname{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}.$$

Proof Since $h(z_0) = 0$, the definition of the derivative gives

$$\lim_{z\to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{h(z)}{z - z_0} = h'(z_0) \neq 0,$$

and therefore

$$\lim_{z\to z_0} \frac{z - z_0}{h(z)} = \frac{1}{h'(z_0)}.$$

Thus,

$$\lim_{z\to z_0} (z - z_0)\frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}$$

exists and therefore equals the residue. ■

Alternative Proof Since $h(z_0) = 0$ and $h'(z_0) \neq 0$, we can find a function $\tilde{h}$ that is analytic at z_0 such that $h(z) = \tilde{h}(z)(z - z_0)$. Note that $\tilde{h}(z_0) = h'(z_0) \neq 0$. Thus, we can write $g(z)/h(z) = g(z)/[\tilde{h}(z)(z - z_0)]$ and $g(z)/\tilde{h}(z)$ is analytic at z_0. Hence we can write a Taylor series

$$\frac{g(z)}{\tilde{h}(z)} = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

where $a_0 = g(z_0)/\tilde{h}(z_0)$. Therefore,

$$\frac{g(z)}{\tilde{h}(z)(z - z_0)} = \sum_{n=0}^{\infty} a_n(z - z_0)^{n-1}$$

is the Laurent expansion of g/h and thus $a_0 = g(z_0)/\tilde{h}(z_0) = g(z_0)/h'(z_0)$ is the residue at z_0. ■

As we have seen, if $g(z)$ has a zero of order k and $h(z)$ has has a zero of order l at z_0, with $l > k$, then $g(z)/h(z)$ has a pole of order $l - k$ at z_0. If $l = k + 1$, we have a simple pole and can obtain the residue from the next proposition.

Proposition 4.1.3 *Suppose that $g(z)$ has a zero of order k at z_0 and that $h(z)$ has a zero of order $k + 1$. Then $g(z)/h(z)$ has a simple pole with residue given by*

$$\operatorname{Res}\left(\frac{g}{h}; z_0\right) = (k + 1)\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)}.$$

Proof By Taylor's theorem and the fact that $g(z_0) = 0, \ldots, g^{(k-1)}(z_0) = 0$, we can write

$$g(z) = \frac{(z - z_0)^k}{k!} g^{(k)}(z_0) + (z - z_0)^{k+1}\tilde{g}(z),$$

where $\tilde{g}$ is analytic. Similarly,

$$h(z) = \frac{(z - z_0)^{k+1}}{(k+1)!} h^{(k+1)}(z_0) + (z - z_0)^{k+2}\tilde{h}(z).$$

Thus,

$$(z - z_0)\frac{g(z)}{h(z)} = \frac{[g^{(k)}(z_0)/k!] + (z - z_0)\tilde{g}(z)}{[h^{(k+1)}(z_0)/(k+1)!] + (z - z_0)\tilde{h}(z)}.$$

As $z \to z_0$, this converges (by the quotient theorem for limits) to

$$(k+1)\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)},$$

which proves our assertion. ■

Another proof that

$$\lim_{z \to z_0} (z - z_0)\frac{g(z)}{h(z)} = (k+1)\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)}$$

is obtained by using l'Hôpital's rule (see Exercise 8, §3.3) and observing that both $(z - z_0)g(z)$ and $h(z)$ are analytic at z_0 with z_0 a zero of order $(k+1)$.

Examples

(i) e^z/z at $z = 0$. In this case 0 is not a zero of e^z but is a first-order zero of z, so the residue at 0 is $1 \cdot e^0/1 = 1$. Clearly, Proposition 4.1.2 also applies.

(ii) $e^z/\sin z$ at 0. Here, e^z is not zero at the point $z = 0$, and, since $\sin' 0 = \cos 0 = 1, 0$ is a first-order zero of $\sin z$. Thus the residue is $e^0/\cos 0 = 1$.

(iii) $z/(z^2+1)$ at $z = i$. Here $g(z) = z, h(z) = z^2 + 1$. Therefore, $g(i) = i \neq 0$ and $h(i) = 0, h'(i) = 2i \neq 0$. Thus the residue at i is $g(i)/h'(i) = 1/2$.

(iv) $z/(z^4 - 1)$ at $z = 1$. Here $g(z) = z$, and $h(z) = z^4 - 1$. Thus $g(1) = 1 \neq 0$ and $h(1) = 0, h'(1) = 4 \neq 0$, and so the residue is $1/4$.

(v) $z/(1 - \cos z)$ at $z = 0$. Here $g(0) = 0$ and $g'(z) = 1 \neq 0$, so 0 is a simple zero of g. Also $h(0) = 0, h'(0) = \sin 0 = 0$, and $h''(0) = \cos 0 = 1 \neq 0$, so 0 is a double zero of h. Thus, by Proposition 4.1.3 (see also line 5 of Table 4.1.1) the residue at 0 is

$$2 \cdot \frac{g'(0)}{h''(0)} = 2 \cdot \frac{1}{1} = 2. \quad \blacklozenge$$

Double Poles The formulas for residues at double poles become more complicated and the residues a little more laborious to obtain. Probably the most useful formula for finding the residue in this case is the following result.

Proposition 4.1.4 *Let g and h be analytic at z_0 and let $g(z_0) \neq 0, h(z_0) = h'(z_0) = 0$, and $h''(z_0) \neq 0$. Then $g(z)/h(z)$ has a second-order pole at z_0 and the residue is*

$$\operatorname{Res}\left(\frac{g}{h}; z_0\right) = 2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}.$$

Proof Since g has no zero and h has a second-order zero, we know that the pole is of second order (see the remark preceding Proposition 4.1.3). Thus we may write the Laurent series in the form

$$\frac{g(z)}{h(z)} = \frac{b_2}{(z-z_0)^2} + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

and we want to compute b_1. We can write

$$g(z) = g(z_0) + g'(z_0)(z-z_0) + \frac{g''(z_0)}{2}(z-z_0)^2 + \ldots$$

and

$$h(z) = \frac{h''(z_0)}{2}(z-z_0)^2 + \frac{h'''(z_0)}{6}(z-z_0)^3 + \ldots.$$

Therefore,

$$\begin{aligned} g(z) &= h(z)\left[\frac{b_2}{(z-z_0)^2} + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \ldots\right] \\ &= \left[\frac{h''(z_0)}{2} + \frac{h'''(z_0)}{6}(z-z_0) + \ldots\right] \cdot [b_2 + b_1(z-z_0) + a_0(z-z_0)^2 + \ldots]. \end{aligned}$$

We can multiply out these two convergent power series as if they were polynomials (see Worked Example 3.2.15). The result is

$$g(z) = \frac{b_2 h''(z_0)}{2} + \left[\frac{b_2 h'''(z_0)}{6} + \frac{b_1 h''(z_0)}{2}\right](z-z_0) + \ldots.$$

Since these two power series are equal we can conclude that the coefficients are equal. Therefore,

$$g(z_0) = \frac{b_2 h''(z_0)}{2} \quad \text{and} \quad g'(z_0) = \frac{b_2 h'''(z_0)}{6} + \frac{b_1 h''(z_0)}{2}.$$

Solving for b_1 yields the theorem. ■

Observe that for a second-order pole of the form $g(z)/(z-z_0)^2$ where $g(z_0) \neq 0$, the formula in Proposition 4.1.4 simplifies to $g'(z_0)$, as it should (Why?).

The following result may be proved in an analogous manner.

Proposition 4.1.5 *Let g and h be analytic at z_0 and let $g(z_0) = 0, g'(z_0) \neq 0, h(z_0) = 0, h'(z_0) = 0, h''(z_0) = 0$, and $h'''(z_0) \neq 0$. Then g/h has a second-order pole at z_0 with residue*

$$\frac{3g''(z_0)}{h'''(z_0)} - \frac{3}{2}\frac{g'(z_0)h^{(iv)}(z_0)}{[h'''(z_0)]^2}.$$

The proof is left for Exercise 4.

Examples

(i) $e^z/(z-1)^2$ has a second-order pole at $z_0 = 1$; here we choose $g(z) = e^z, h(z) = (z-1)^2$ and note that $g(1) = e \neq 0, h(z_0) = 0, h'(z_0) = 2(z_0 - 1) = 0$, and $h''(z_0) = 2 \neq 0$. Therefore, by Proposition 4.1.4, the residue at 1 is $[(2 \cdot e)/2] - (2/3) \cdot [(e \cdot 0)/2^2] = e$.

(ii) $(e^z - 1)/\sin^3 z$ with $z_0 = 0$. Here we choose $g(z) = e^z - 1, h(z) = \sin^3 z$ and then note that $g(0) = 0, g'(0) \neq 0, h(0) = 0, h'(z) = 3(\sin^2 z)(\cos z)$, so $h'(0) = 0, h''(z) = 6\sin z \cdot \cos^2 z - 3\sin^3 z$ (which is equal to zero at 0), and finally $h'''(z) = 6\cos^3 z - 12\sin^2 z \cdot \cos z - 9\sin^2 z \cdot \cos z$ (which is 6 at $z = 0$). We also compute $h^{(iv)}(0) = 0$. Thus by Proposition 4.1.5, the residue is $3 \cdot (1/6) = 1/2$. ♦

Higher-Order Poles For poles of order greater than 2 we could develop formulas in the same manner in which we developed the preceding ones, but they would be more complicated. Instead, two general methods can be used. The first is described in the next proposition.

Proposition 4.1.6 *Let f have an isolated singularity at z_0 and let k be the smallest integer ≥ 0 such that $\lim_{z\to z_0}(z - z_0)^k f(z)$ exists. Then $f(z)$ has a pole of order k at z_0 and, if we let $\phi(z) = (z - z_0)^k f(z)$, then ϕ can be defined uniquely at z_0 so that ϕ is analytic at z_0 and*

$$\operatorname{Res}(f; z_0) = \frac{\phi^{(k-1)}(z_0)}{(k-1)!}.$$

Proof Since $\lim_{z\to z_0}(z - z_0)^k f(z)$ exists, $\phi(z) = (z - z_0)^k f(z)$ has a removable singularity at z_0, by Proposition 3.3.4. Thus in a neighborhood of z_0,

$$\phi(z) = (z - z_0)^k f(z) = b_k + b_{k-1}(z - z_0) + \ldots + b_1(z - z_0)^{k-1} + a_0(z - z_0)^k + \cdots,$$

so

$$f(z) = \frac{b_k}{(z - z_0)^k} + \frac{b_{k-1}}{(z - z_0)^{k-1}} + \ldots + \frac{b_1}{(z - z_0)} + a_0 + a_1(z - z_0) + \ldots.$$

If $b_k = 0$, then $\lim_{z\to z_0}(z - z_0)^{k-1}f(z)$ exists, which contradicts the hypothesis about k. Thus, z_0 is a pole of order k. Finally, consider the expansion for $\phi(z)$, and differentiate it $k - 1$ times at z_0 to obtain $\phi^{(k-1)}(z_0) = [(k-1)!]b_1$. ■

In this theorem it is the residue formula that is important rather than the test for the order. It may be easier to test the order by writing (if possible) $f = g/h$ and showing that h has a zero of order k greater than that of g. Then we have a pole of order k (as was explained in the text preceding Proposition 4.1.3).

Let us now suppose that the form of f makes application of Proposition 4.1.6 inconvenient. (For example, consider $e^z/\sin^4 z$ with $z_0 = 0$. Here $k = 4$, since the numerator has no zero and the denominator has a zero of order 4.) There is an alternative method that generalizes Proposition 4.1.4. Suppose that $f(z) = g(z)/h(z)$ and that $h(z)$ has a zero of order k more than g at z_0; therefore, f has a pole of order k. We write

$$\frac{g(z)}{h(z)} = \frac{b_k}{(z-z_0)^k} + \ldots + \frac{b_1}{(z-z_0)} + \rho(z),$$

where ρ is analytic. Also, suppose that z_0 is a zero of order m for $g(z)$ and a zero of order $m + k$ for $h(z)$. Then

$$g(z) = \sum_{n=m}^{\infty} \frac{g^{(n)}(z_0)(z-z_0)^n}{n!} \quad \text{and} \quad h(z) = \sum_{n=m+k}^{\infty} \frac{h^{(n)}(z_0)(z-z_0)^n}{n!}.$$

Thus, we can write

$$\sum_{n=m}^{\infty} \frac{g^{(n)}(z_0)(z-z_0)^n}{n!} = \left[\sum_{n=m+k}^{\infty} \frac{h^{(n)}(z_0)(z-z_0)^n}{n!}\right] \cdot \left[\frac{b_k}{(z-z_0)^k} + \ldots + \frac{b_1}{(z-z_0)} + \rho(z)\right].$$

We can then multiply out the right side of the equation as if the factors were polynomials (because of Worked Example 3.2.15) and compare the coefficients of $(z - z_0)^m, (z - z_0)^{m+1}, \ldots, (z - z_0)^{m+k-1}$ to obtain k equations in $b_1, b_2, \ldots, b_k$. Finally, we can solve these equations for b_1. This method is sometimes more practical than that of Proposition 4.1.6. When $m = 0$ (that is, when $g(z_0) \neq 0$), the explicit formula contained in the following proposition can be used. (The student should prove this result by using the procedure just described.)

Proposition 4.1.7 *Let g and h be analytic at z_0, with $g(z_0) \neq 0$, and assume $h(z_0) = 0 = \ldots = h^{(k-1)}(z_0)$ and $h^{(k)}(z_0) \neq 0$. Then g/h has a pole of order k and* *the residue at* z_0, $\mathrm{Res}(g/h; z_0)$ *is given by*

$$\mathrm{Res}(g/h; z_0) = \left[\frac{k!}{h^{(k)}(z_0)}\right]^k \times$$
$$\begin{vmatrix} \frac{h^{(k)}(z_0)}{k!} & 0 & 0 & \cdots & 0 & g(z_0) \\ \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & 0 & \cdots & 0 & g^{(1)}(z_0) \\ \frac{h^{(k+2)}(z_0)}{(k+2)!} & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & \cdots & 0 & \frac{g^{(2)}(z_0)}{2!} \\ \vdots & \vdots & \vdots & & & \vdots \\ \frac{h^{(2k-1)}(z_0)}{(2k-1)!} & \frac{h^{(2k-2)}(z_0)}{(2k-2)!} & \frac{h^{(2k-3)}(z_0)}{(2k-3)!} & \cdots & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{g^{(k-1)}(z_0)}{(k-1)!} \end{vmatrix},$$

where the vertical bars denote the determinant of the enclosed $k \times k$ *matrix.*

Table 4.1.1 Techniques for Finding Residues

In this table g and h are analytic at z_0 and f has an isolated singularity. The most useful and common tests are indicated by an asterisk.

	Function	Test	Type of Singularity	Residue at z_0
1.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z) = 0$	removable	0
*2.	$\frac{g(z)}{h(z)}$	g and h have zeros of same order	removable	0
*3.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z) = 0$ exists and is $\neq 0$	simple pole	$\lim_{z\to z_0}(z-z_0)f(z$ [illegible]
*4.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0) = 0$, $h'(z_0)\neq 0$	simple pole	$\frac{g(z_0)}{h'(z_0)}$
5.	$\frac{g(z)}{h(z)}$	g has zero of order k, h has zero of order $k+1$	simple pole	$(k+1)\frac{g^{(k)}(z_0}{h^{(k+1)}(z}$ [illegible]
*6.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0$ $h(z_0) = 0 = h'(z_0)$ $h''(z_0)\neq 0$	second-order pole	$2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0}{h}$ [illegible]
*7.	$\frac{g(z)}{(z-z_0)^2}$	$g(z_0)\neq 0$	second-order pole	$g'(z_0)$
*8.	$\frac{g(z)}{h(z)}$	$g(z_0) = 0, g'(z_0)\neq 0$, $h(z_0) = 0 = h'(z_0)$ $= h''(z_0), h'''(z_0)\neq 0$	second-order pole	$3\frac{g''(z_0)}{h'''(z_0)} - \frac{3}{2}\frac{g}{}$ [illegible]
9.	$f(z)$	k is the smallest integer such that $\lim_{z\to z_0}\phi(z_0)$ exists where $\phi(z) = (z-z_0)^k f(z)$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z}{(k-1)!}$
*10.	$\frac{g(z)}{h(z)}$	g has zero of order l, h has zero of order $k+l$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1}(z}{(k-1}$ where $\phi(z) =$ [illegible]
11.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0) = \ldots = h^{k-1}(z_0) = 0, h^k(z_0)\neq 0$	pole of order k	see Proposition [illegible]

Examples

(i) Find the residue of $\dfrac{z^2}{(z-1)^3(z+1)}$ at $z_0 = 1$.

The pole is of order 3. We use Proposition 4.1.6. In this case,

$$\phi(z) = \frac{z^2}{z+1},$$

so

$$\phi'(z) = \frac{(z+1)\cdot 2z - z^2}{(z+1)^2} = \frac{z^2+2z}{(z+1)^2} = 1 - \frac{1}{(z+1)^2}$$

and

$$\phi''(z) = \frac{2}{(z+1)^3} \quad \text{so that} \quad \phi''(1) = \frac{1}{4}.$$

Since $k = 3$, the residue is $(1/2)(1/4) = 1/8$.

(ii) Find the residue of $e^z/\sin^3 z$ at $z = 0$.

Here $k = 3$ and we shall use Proposition 4.1.7, with $g(z) = e^z$ and $h(z) = \sin^3 z$. We need to compute $h'''(0), h^{(iv)}(0)$, and $h^{(v)}(0)$. These are, by straightforward computation, $h'''(0) = 6, h^{(iv)}(0) = 0$, and $h^{(v)}(0) = -60$; thus, $h'''/3! = 1, h^{(iv)}/4! = 0$, and $h^{(v)}/5! = -1/2$. Also $g^{(l)}(0)/l! = 1/l!$, so the residue is

$$\left(\frac{3!}{6}\right)^3 \times \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & \frac{1}{2} \end{vmatrix} = 1.$$

(The last column is subtracted from the first.) ♦

Essential Singularities In the case of an essential singularity there are no simple formulas like the preceding ones, so we must rely on our ability to find the Laurent expansion. For example, consider

$$f(z) = e^{(z+1/z)} = e^z \cdot e^{1/z} = \left(1 + z + \frac{z^2}{2!} + \ldots\right)\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots\right).$$

Gathering terms involving $1/z$, we get

$$\frac{1}{z}\left(1 + \frac{1}{2!} + \frac{1}{2!3!} + \frac{1}{3!4!} + \ldots\right).$$

(We multiply out as in the procedure of Worked Example 3.2.15, a method that is justified by a more general result that is outlined in Exercise 12.) The residue is thus

$$\operatorname{Res}(f;0) = 1 + \frac{1}{2!} + \frac{1}{2!3!} + \frac{1}{3!4!} + \ldots.$$

We do not attempt to sum the series explicitly.

Worked Examples

Example 4.1.8 *Compute the residue of* $\dfrac{z^2}{\sin^2 z}$ *at* $z = 0$.

Solution Since both numerator and denominator have a zero of order 2, the singularity is removable, and so the residue is zero.

Example 4.1.9 *Find the residues at all singularities of*

$$\tan z = \frac{\sin z}{\cos z}.$$

Solution 1 The singularities of $\tan z$ occur when $\cos z = 0$. The zeros of $\cos z$ are

$$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots,$$

and these are the only zeros of $\cos z$. We conclude that the singularities of $\tan z$ occur at the points $z_n = (2n+1)\pi/2$, where n is an integer. We choose $g(z) = \sin z$ and $h(z) = \cos z$. At any $z_n, h'(z_n) = \pm 1 \neq 0$, so each z_n is a simple pole of $\tan z$. Thus we may use formula 4 of Table 4.1.1 to obtain

$$\operatorname{Res}(\tan z; z_n) = \frac{g(z_n)}{h'(z_n)} = -1.$$

Solution 2 We know that

$$\begin{aligned}
\sin z &= (-1)^n \sin(z - \pi n) = (-1)^{n+1} \cos\left(z - \pi n - \frac{\pi}{2}\right) \\
&= (-1)^{n+1} \cos(z - z_n) = (-1)^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(z - z_n)^{2k}
\end{aligned}$$

and

$$\begin{aligned}
\cos z &= (-1)^n \cos(z - \pi n) = (-1)^{n+1} \sin\left(z - \pi n - \frac{\pi}{2}\right) \\
&= (-1)^n \sin(z - z_n) = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(z - z_n)^{2k+1}.
\end{aligned}$$

As before, the poles are simple, so the series for $\tan z$ is of the form

$$\tan z = \frac{b_1}{z - z_n} + \sum_{k=0}^{\infty} a_k (z - z_n)^k,$$

so $\sin z = \tan z \cos z$ becomes

$$\begin{aligned}
&(-1)^{n+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}(z-z_n)^{2k}\\
&= \left[\frac{b_1}{z-z_n}+\sum_{k=0}^{\infty}a_k(z-z_n))^k\right]\left[(-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(z-z_n)^{2k+1}\right].
\end{aligned}$$

Cancelling $(-1)^n$ from each side, this becomes

$$\begin{aligned}
-\left[1-\frac{(z-z_n)^2}{2}+\ldots\right] &= \left[\frac{b_1}{z-z_n}+a_0+a_1(z-z_n)+\ldots\right]\\
&\quad\times\left[(z-z_n)-\frac{(z-z_n)^3}{6}+\ldots\right]\\
&= b_1+a_0(z-z_n)+(a_1-\frac{1}{6})(z-z_n)^2+\ldots.
\end{aligned}$$

Comparing the first terms, we get $b_1 = -1$.

Example 4.1.10 *Evaluate the residue of* $\dfrac{z^2-1}{(z^2+1)^2}$ *at* $z=i$.

Solution 1 $(z^2+1)^2$ has a zero of order 2 at i and $i^2-1\neq 0$, so $(z^2-1)/(z^2+1)^2$ has a pole of order 2; thus, to find residue we use formula 6 of Table 4.1.1. We choose $g(z) = z^2-1$, which satisfies $g(i) = -2$ and $g'(i) = 2i$. We also take $h(z) = (z^2+1)^2$ and note that $h'(z) = 4z(z^2+1)$, so $h(i) = h'(i) = 0$. Also, $h''(z) = 4(z^2+1)+8z^2 = 12z^2+4$, so $h''(i) = -8$ and $h'''(z) = 24z$; therefore, $h'''(i) = 24i$. Thus, the residue is

$$\frac{2\cdot 2i}{-8}-\frac{2}{3}\cdot\frac{(-2)\cdot 24i}{64}=0.$$

Solution 2 We know from algebra (or integration techniques from calculus) that $(z^2-1)/(z^2+1)^2$ has a partial-fraction expansion of the form

$$\frac{z^2-1}{(z^2+1)^2}=\frac{Az+B}{(z-i)^2}+\frac{a}{(z-i)}+\frac{Cz+D}{(z+i)^2}+\frac{b}{(z+i)}.$$

Solving for the coefficients gives the identity

$$\frac{z^2-1}{(z^2+1)^2}=\frac{1}{2}\frac{1}{(z-i)^2}+\frac{1}{2}\frac{1}{(z+i)^2}.$$

The second term is analytic at $z=i$, so the Laurent series is of the form

$$\frac{z^2-1}{(z^2+1)^2}=\frac{1}{2}\frac{1}{(z-i)^2}+\sum_{n=0}^{\infty}a_n(z-i)^n.$$

The residue is the coefficient of $(z-i)^{-1}$, which, since this term is missing, is 0.

Exercises

1. Find the residues of the following functions at the indicated points:

   (a) $\dfrac{e^z - 1}{\sin z}, z_0 = 0$

   (b) $\dfrac{1}{e^z - 1}, z_0 = 0$

   (c) $\dfrac{z+2}{z^2 - 2z}, z_0 = 0$

   (d) $\dfrac{1 + e^z}{z^4}, z_0 = 0$

   (e) $\dfrac{e^z}{(z^2 - 1)^2}, z_0 = 1$

2. Find the residues of the following functions at the indicated points:

   (a)$^\bullet$ $\dfrac{e^{z^2}}{z - 1}, z_0 = 1$

   (b) $\dfrac{e^{z^2}}{(z-1)^2}, z_0 = 0$

   (c)$^\bullet$ $\left(\dfrac{\cos z - 1}{z}\right)^2, z_0 = 0$

   (d) $\dfrac{z^2}{z^4 - 1}, z_0 = e^{i\pi/2}$

3. If $f(z)$ has residue b_1 at $z = z_0$, show by example that $[f(z)]^2$ need not have residue b_1^2 at $z = z_0$.

4. Deduce Proposition 4.1.5 from Proposition 4.1.4.

5.$^\bullet$ Explain what is wrong with the following reasoning. Let

$$f(z) = \frac{1 + e^z}{z^2} + \frac{1}{z}.$$

   Since $f(z)$ has a pole at $z = 0$, the residue at that point is the coefficient of $1/z$, namely 1. Compute the residue correctly.

6. Complete the proof of Proposition 4.1.7.

7. Find all singular points of the following three functions and compute the residues at those points:

   (a) $\dfrac{1}{z^3(z+4)}$

   (b) $\dfrac{1}{z^2 + 2z + 1}$

(c) $\dfrac{1}{z^3 - 3}$

8.• Find all singular points of the following functions and compute the residues at those points:

(a) $\dfrac{1}{e^z - 1}$

(b) $\sin \dfrac{1}{z}$

9. Find the residue of $1/(z^2 \sin z)$ at $z = 0$.

10. If f_1 and f_2 have residues r_1 and r_2 at z_0, show that the residue of $f_1 + f_2$ at z_0 is $r_1 + r_2$.

11.• If f_1 and f_2 have simple poles at z_0, show that $f_1 f_2$ has a second-order pole at z_0. Derive a formula for the residue.

12. Let

$$f(z) = \ldots + \frac{b_k}{(z - z_0)^k} + \ldots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \ldots$$

and

$$g(z) = \ldots + \frac{d_k}{(z - z_0)^k} + \ldots + \frac{d_1}{z - z_0} + c_0 + c_1(z - z_0) + \ldots$$

be Laurent expansions for f and g valid for $0 < |z - z_0| < r$. Show that the Laurent expansion for fg is obtained by formally multiplying these series. Do this by proving the following result: If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent, then

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n,$$

where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$; moreover, the series $\sum_{n=0}^{\infty} c_n$ is absolutely convergent. *Hint*: Show that

$$\sum_{j=0}^{n} |c_j| \le \sum_{j=0}^{n}\sum_{k=0}^{n} |a_j||b_{k-j}| \le \left(\sum_{j=0}^{n} |a_j|\right)\left(\sum_{k=0}^{n} |b_k|\right)$$

and use this to deduce that $\sum c_n$ converges absolutely. Estimate the error between

$$\sum_{k=0}^{n} c_k, \quad \left(\sum_{j=0}^{n} a_j\right)\left(\sum_{k=0}^{n} b_k\right), \quad \text{and} \quad \left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{k=0}^{\infty} b_k\right).$$

13. Compute the residue of the following functions at their singularities:

(a) $\dfrac{1}{(1-z)^3}$

(b) $\dfrac{e^z}{(1-z)^3}$

(c) $\dfrac{1}{z(1-z)^3}$

(d) $\dfrac{e^z}{z(1-z)^3}$

14. Find the residues of $(z^2-1)/[\cos(\pi z)+1]$ at each of its singularities. (See Review Exercise 29 of Chapter 3.)

4.2 Residue Theorem

The Residue Theorem, which is proved in this section, includes Cauchy's Theorem and Cauchy's Integral Formula as special cases. It is one of the main results of complex analysis and leads quickly to interesting applications, some of which are considered in §4.3. The main tools needed to prove the theorem are Cauchy's Theorem (2.2.1 and 2.3.14) and the Laurent Expansion Theorem 3.3.1.

The precise proof of the Residue Theorem is preceded by two intuitive proofs that only use the material in §2.2 and the following property of the residue at z_0:

$$2\pi i \operatorname{Res}(f;z_0) = \int_\gamma f(z)dz$$

where γ is a small circle around z_0 (see Proposition 3.3.3). For most practical examples the intuitive proofs are perfectly adequate, but, as was evident in §2.2, it is difficult to formulate a general theorem to which the argument rigorously applies.

Statement of the Residue Theorem We begin with a statement of the Residue Theorem followed by two intuitive proofs and then a precise proof.

Theorem 4.2.1 (Residue Theorem) *Let A be a region and let $z_1,\ldots,z_n\in A$ be n distinct points in A. Let f be analytic on $A\backslash\{z_1,z_2,\ldots,z_n\}$; that is, let f be analytic on A except for isolated singularities at $z_1,\ldots,z_n$. Let γ be a closed curve in A homotopic to a point in A. Assume that no z_i lies on γ. Then*

$$\int_\gamma f(z)dz = 2\pi i\sum_{i=1}^{n}[\operatorname{Res}(f;z_i)]I(\gamma;z_i), \tag{4.2.1}$$

where $\operatorname{Res}(f;z_i)$ is the residue of f at z_i and $I(\gamma;z_i)$ is the index (winding number) of γ with respect to z_i (see §2.4).

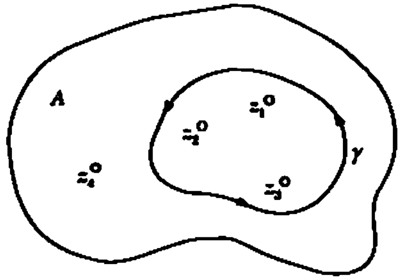


Figure 4.2.1: Residue theorem: $\int_\gamma f = 2\pi i[\text{Res}(f; z_1) + \text{Res}(f; z_2) + \text{Res}(f; z_3)]$.

In most practical examples, γ will be a simple closed curve traversed counterclockwise, and thus $I(\gamma; z_i)$ will be 1 or 0 according to whether z_i lies inside γ or outside γ. This is illustrated in Figure 4.2.1.

The same policy regarding computation of the index $I(\gamma; z)$ used in §2.4 will be followed in this section. An intuitive proof is acceptable as long as such statements can be substantiated with homotopy arguments when asked for. For example, $I(\gamma; z_0) = +1$ provided γ can be shown to be homotopic in $\mathbb{C}\backslash\{z_0\}$ to a circle $\gamma(t) = z_0 + re^{i\theta}, 0 \le \theta \le 2\pi$.

A general formulation of the Residue Theorem for simple closed curves may be stated as follows:

> *If γ is a simple closed curve in the region A whose inside lies in A and if f is analytic on $A\backslash\{z_1, \ldots, z_n\}$, then $\int_\gamma f$ is $2\pi i$ times the sum of the residues of f inside γ when γ is traversed in the counterclockwise direction.*

This is the classical way of stating the Residue Theorem, but our original statement (Theorem 4.2.1) is preferred by some because it does not restrict us to simple closed curves and does not rely on the difficult Jordan curve theorem (see Definition 2.4.1).

Two short intuitive proofs of the Residue Theorem are now given for simple closed curves. They will be illustrated by an example showing that, in practical cases, such proofs can be made quite precise.

First Intuitive Proof of the Residue Theorem for Simple Closed Curves Since γ is contractible in A to a point in A, the inside of γ lies in A. Suppose that each z_i lies in the inside of γ. Around each z_i draw a circle γ_i small enough to be inside γ and surround none of the other z_k. Apply Worked Example 2.2.9 (the Generalized Deformation Theorem) to obtain $\int_\gamma f = \sum_{i=1}^n \int_{\gamma_i} f$, since f is analytic in $\gamma, \gamma_1, \ldots, \gamma_n$ and the region between them (Figure 4.2.2).

Suppose that $\gamma, \gamma_1, \ldots, \gamma_n$ are all traversed in the counterclockwise direction. As shown in Proposition 3.3.3, $\int_{\gamma_i} f = 2\pi i \text{ Res}(f; z_i)$, so $\int_\gamma f = 2\pi i \sum_{i=1}^n \text{Res}(f; z_i)$,

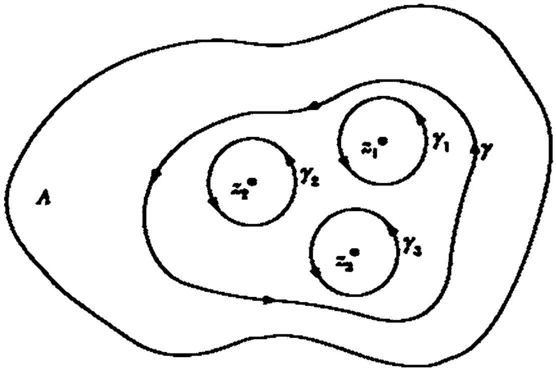


Figure 4.2.2: First intuitive proof of the Residue Theorem.

which is the statement of the Residue Theorem since $I(\gamma; z) = 1$ for z inside γ and $I(\gamma; z) = 0$ for z outside γ. ∎

Second Intuitive Proof of the Residue Theorem for Simple Closed Curves This proof proceeds in the same manner as the preceding one except that a different justification is given that $\int_\gamma f = \sum_{i=1}^n \int_{\gamma_i} f$. The circles are connected as shown in Figure 4.2.3, to obtain a new curve $\tilde{\gamma}$.

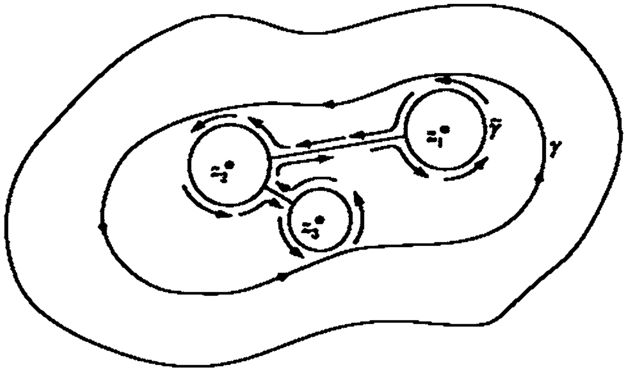


Figure 4.2.3: Second intuitive proof of the Residue Theorem.

Thus, γ and $\tilde{\gamma}$ are homotopic in $A\{z_1, \ldots, z_n\}$, and so, by the Deformation Theorem, $\int_\gamma f = \int_{\tilde{\gamma}} f$. But $\int_{\tilde{\gamma}} f = \sum_{i=1}^n \int_{\gamma_i} f$, since the portions along the connecting curves cancel out. ∎

Why do these proofs fail to be precise? First, we assumed that γ is simple. Second, we use the Jordan curve theorem (which we did not prove) to be able to discuss the inside and outside of γ and the fact that $I(\gamma; z) = 1$ for z inside γ and $I(\gamma; z) = 0$ for z outside γ. Finally, in the first intuitive proof, we used Worked Example 2.2.9, which was established only informally, to justify that $\int_\gamma f = \sum_{i=1}^n \int_{\gamma_i} f$.

Example *Evaluate* $\displaystyle\int_\gamma \frac{dz}{z^2-1}$, *where* γ *is a circle with center* 0 *and radius* 2.

The function $1/(z^2-1)$ has simple poles at $-1, 1$. We evaluate the integral using the Residue Theorem:

$$\begin{aligned}\int_\gamma \frac{dz}{z^2-1} &= 2\pi i\left[\operatorname{Res}\left(\frac{1}{z^2-1}; -1\right) + \operatorname{Res}\left(\frac{1}{z^2-1}; 1\right)\right] \\ &= 2\pi i\left[\frac{1}{2(-1)} + \frac{1}{2\cdot 1}\right] = 0.\end{aligned}$$

In this example it is clear what we mean by the inside and outside of γ, and we know that -1 and 1 have an index $+1$ with respect to γ.

The figure in this example corresponding to Figure 4.2.2 in the general discussion is obtained by drawing two circles γ_1 and γ_2 of radius $1/4$, say, around -1 and 1, respectively, as in Figure 4.2.4.

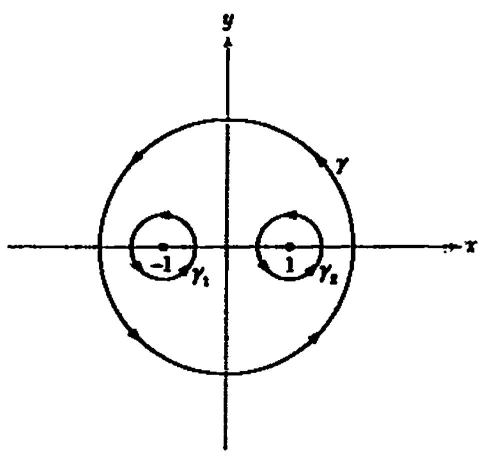


Figure 4.2.4: Justifying that $\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f$.

The only statement in the preceding proofs of the Residue Theorem that was not precise was

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

In this example, this may be justified by considering the curve in Figure 4.2.5(i) and showing that it is homotopic in $\mathbb{C}\backslash\{1,-1\}$ to a point. This is geometrically clear; a homotopy is indicated in Figure 4.2.5(ii) and (iii). ♦

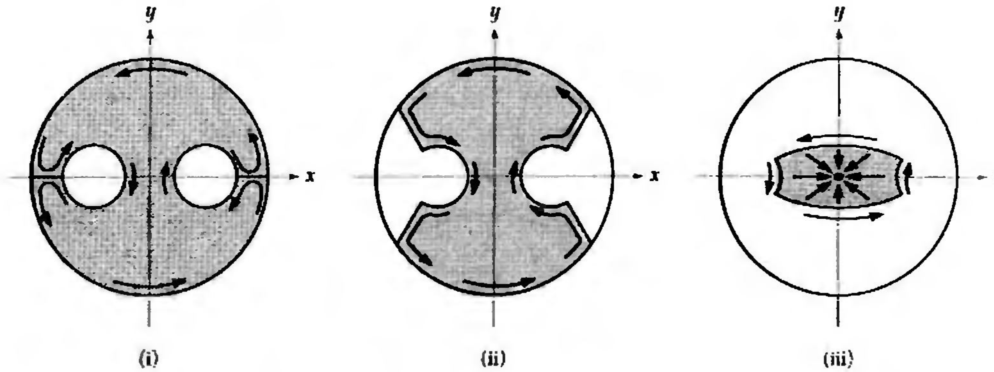


Figure 4.2.5: A curve that is homotopic to a point.

Precise Proof of the Residue Theorem Since z_i is an isolated singularity of f, we can write a convergent Laurent series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_i)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z - z_i)^m}$$

in some deleted neighborhood of z_i of the form $\{z \mid r > |z - z_i| > 0\}$ for some $r > 0$. By Proposition 3.3.2 and Exercise 15 of §3.3, the singular part of the Laurent series expansion

$$S_i(z) = \sum_{m=1}^{\infty} \frac{b_m}{(z - z_i)^m}$$

converges on $\mathbb{C}\backslash\{z_i\}$, uniformly outside any circle $|z - z_i| = \epsilon > 0$. Hence $S_i(z)$ is analytic on $\mathbb{C}\backslash\{z_i\}$ (see Theorem 3.1.8).

Consider the function

$$g(z) = f(z) - \sum_{i=1}^{n} S_i(z).$$

Since f is analytic on $A\backslash\{z_1, \ldots z_n\}$ and since each $S_i(z)$ is analytic on $\mathbb{C}\backslash\{z_i\}$, g is analytic on $A\backslash\{z_1, \ldots, z_n\}$.

All the z_i's are removable singularities of g because on a deleted neighborhood $\{z \mid r > |z - z_i| > 0\}$, which does not contain any of the singularities, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_i)^n + S_i(z),$$

so

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_i)^n - \sum_{j=1}^{i-1} S_j(z) - \sum_{j=i+1}^{n} S_j(z).$$

Since the functions $S_j, j \neq i$, are analytic on $\mathbf{C}\backslash\{z_j\}$, we know that $\lim_{z\to z_i} g(z)$ exists and equals $a_0 - \sum_{j=1;j\neq i}^{n} S_j(z_i)$. Consequently, z_i is a removable singularity of g.

Because g can be defined at the points z_i in such a way that g is analytic on all of A, we can apply the Cauchy Theorem 2.3.14 to obtain $\int_\gamma g = 0$. Hence

$$\int_\gamma f = \sum_{i=1}^{n} \int_\gamma S_i.$$

Next consider the integral $\int_\gamma S_i$. The function $S_i(z)$ is of the form

$$\sum_{m=1}^{\infty} \frac{b_m}{(z - z_i)^m},$$

which, as we have noted, converges uniformly outside a small disk centered at z_i. Thus the convergence is uniform on γ. (Since $\mathbf{C}\backslash\{\gamma([a,b])\}$ is an open set, each z_i has a small disk around it not meeting γ.) By Proposition 3.1.9,

$$\int_\gamma S_i = \sum_{m=1}^{\infty} \int_\gamma \frac{b_m}{(z - z_i)^m} dz.$$

But for $m > 1$ and $z \neq z_i$,

$$\frac{1}{(z - z_i)^m} = \frac{d}{dz}\left[\frac{(z - z_i)^{1-m}}{1 - m}\right],$$

so by Proposition 2.1.7 and the fact that γ is a closed curve, all terms are zero except the term in which $m = 1$. Thus,

$$\int_\gamma S_i = \int_\gamma \frac{b_1}{z - z_i} dz = b_1 \int_\gamma \frac{1}{z - z_i} dz.$$

By definition of the index, this is equal to $b_1 \cdot 2\pi i \cdot I(\gamma; z_i) = 2\pi i[\mathrm{Res}(f; z_i)]I(\gamma; z_i)$. Thus,

$$\int_\gamma f = \sum_{i=1}^{n} \int_\gamma S_i = \sum_{i=1}^{n} 2\pi i[\mathrm{Res}(f; z_i)]I(\gamma; z_i)$$

and the theorem is proved. ■

Residues and Behavior at Infinity If a function f is analytic for all large enough z (that is, outside some large circle), then it is analytic in a deleted neighborhood of ∞ in the sense of the Riemann sphere and the point at ∞ as defined in §1.4. We can think of ∞ as an isolated singularity of f, perhaps removable. Let $F(z) = f(1/z)$. If $z = 0$, we set $1/z = \infty$. (Equivalently, $1/z \to \infty$ as $z \to 0$). Thus, it makes sense to discuss the behavior of f at ∞ in terms of the behavior of F at 0.

Definition 4.2.2 *Let $F(z) = f(1/z)$. Then we say that*

(i) *f has a pole of order k at ∞ if F has a pole of order k at* 0;

(ii) *f has a zero of order k at ∞ if F has a zero of order k at* 0.

(iii) *We define* $\operatorname{Res}(f;\infty) = -\operatorname{Res}((1/z^2)F(z);0)$.

Notice in particular that a polynomial of degree k has a pole of order k at ∞. This agrees with what we saw in the proof of the Fundamental Theorem of Algebra in §2.4. As $z \to \infty$, a polynomial of degree k behaves much like z^k. See also Worked Example 4.2.7. The definition of residue at ∞ may seem a bit strange, but it is designed to make the next two propositions work out correctly.

Proposition 4.2.3 *Suppose there is an $R_0 > 0$ such that f is analytic on the set $\{z \in \mathbb{C}$ such that $|z| > R_0\}$. If $R > R_0$, and Γ denotes the circle of radius R centered at* 0 *traversed once counterclockwise, then $\int_\Gamma f = -2\pi i \operatorname{Res}(f;\infty)$.*

Proof Let $r = 1/R$, and let γ be the circle of radius r centered at 0, and traversed counterclockwise. If z is inside γ, then $1/z$ is outside Γ, so the function $g(z) = f(1/z)/z^2$ is analytic everywhere inside γ except at 0. Thus,

$$\begin{aligned} 2\pi i \operatorname{Res}(g;0) &= \int_\gamma [f(1/z)/z^2]dz = \int_0^{2\pi} f(r^{-1}e^{-it})r^{-2}e^{-2it}re^{it}dt \\ &= \int_0^{2\pi} f(Re^{-it})Re^{-it}dt = \int_{-2\pi}^{0} f(Re^{is})Re^{is}ds \\ &= \int_0^{2\pi} f(Re^{is})Re^{is}ds = \int_\Gamma f. \end{aligned}$$

The next-to-last equality comes from the 2π periodicity of e^{is}. ■

The choice of the minus sign comes from the fact that as we proceed along a simple closed curve in $\mathbb{C}$ in the counterclockwise direction, the region we normally think of as the inside lies to the left. (Look at any of the figures in this section. The point at ∞ lies to the left if we proceed in the opposite direction along the curve. Hence the minus sign. For the curves in the last proof, if z proceeds in the counterclockwise direction along Γ, then $1/z$ proceeds in the clockwise direction along γ. Since f is analytic outside Γ except possibly at ∞, Proposition 4.2.3 may be interpreted as saying that $(1/2\pi i)\int_\Gamma f$ is the negative of the residue of f outside Γ. This is correct more generally.

Proposition 4.2.4 *Let γ be a simple closed curve in $\mathbb{C}$ traversed once counterclockwise. Let f be analytic along γ and have only finitely many singularities outside γ. Then*

$$\int_\gamma f = -2\pi i \sum \{\textit{residues of } f \textit{ outside } \gamma \textit{ including at } \infty\}.$$

Idea of the Proof Apply the Residue Theorem to a composite curve such as that in Figure 4.2.6. Choose Γ to be a circle large enough to contain γ and all the finite singularities of f in its interior. The reader is asked to supply the remaining details of an informal proof in Exercise 14. ■

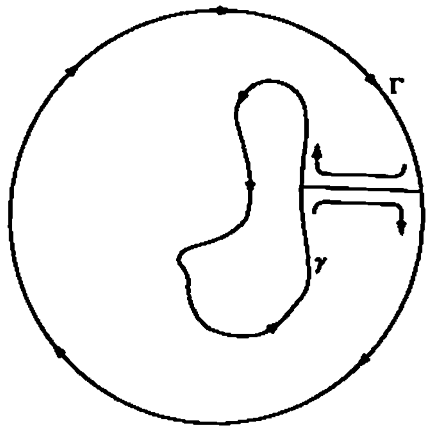


Figure 4.2.6: Curve used in the proof of the Residue Theorem for the exterior of a curve.

Worked Examples

Example 4.2.5 *Evaluate the integral*

$$\int_\gamma \frac{dz}{z^4+1},$$

where γ consists of the portion of the x axis from -2 to $+2$ and the semicircle in the upper half plane from 2 to -2 centered at 0.

Solution The singular points of the integrand occur at the fourth roots of -1, namely,

$$e^{\pi i/4}, \quad e^{(\pi+2\pi)i/4} = e^{3\pi i/4}, \quad e^{(\pi+4\pi)i/4} = e^{5\pi i/4} \quad \text{and} \quad e^{(\pi+6\pi)i/4} = e^{7\pi i/4}$$

(see Figure 4.2.7).

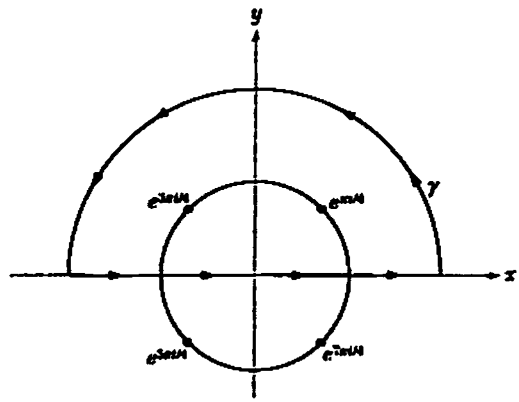


Figure 4.2.7: The curve γ in Example 4.2.5.

By the Residue Theorem, the required integral equals

$$2\pi i\left[\operatorname{Res}\left(\frac{1}{z^4+1}, e^{\pi i/4}\right) I\left(\gamma, e^{\pi i/4}\right)+\operatorname{Res}\left(\frac{1}{z^4+1}, e^{3\pi i/4}\right) I\left(\gamma, e^{3\pi i/4}\right)\right.$$
$$\left.+\operatorname{Res}\left(\frac{1}{z^4+1}, e^{5\pi i/4}\right) I\left(\gamma, e^{5\pi i/4}\right)+\operatorname{Res}\left(\frac{1}{z^4+1}, e^{7\pi i/4}\right) I\left(\gamma, e^{7\pi i/4}\right)\right].$$

It is intuitively clear that

$$I(\gamma; e^{\pi i/4}) = 1 \quad \text{and} \quad I(\gamma; e^{3\pi i/4}) = 1,$$

whereas the other two indexes are zero. This can be more carefully justified as follows: γ is homotopic to a circle $\tilde{\gamma}$ around $e^{\pi i/4}$ traversed counterclockwise. To see this, reparametrize γ so that it is defined on the interval $[0, 2\pi]$. A suitable homotopy is then $H(s,t) = (1-t)\gamma(s) + t\tilde{\gamma}(s)$, which is illustrated in Figure 4.2.8.

We know that $I(\tilde{\gamma}; e^{\pi i/4}) = 1$ by Worked Example 2.1.12, and that $I(\tilde{\gamma}; e^{\pi i/4}) = I(\gamma; e^{\pi i/4})$ by the Deformation Theorem. Thus $I(\gamma; e^{\pi i/4}) = 1$, and, similarly, $I(\gamma; e^{3\pi i/4}) = 1$. Furthermore, γ can be contracted to the origin along the radii of the semicircle, so by Cauchy's Theorem, $I(\gamma; e^{5\pi i/4}) = 0$ and $I(\gamma; e^{7\pi i/4}) = 0$.

To calculate

$$\operatorname{Res}\left(\frac{1}{z^4+1}; e^{\pi i/4}\right),$$

observe that $e^{\pi i/4}$ is a simple pole of the function $1/(z^4+1)$, so we can use formula 4 of Table 4.1.1 to obtain

$$\operatorname{Res}\left(\frac{1}{z^4+1}; e^{\pi i/4}\right) = \frac{1}{4(e^{\pi i/4})^3} = \frac{e^{\pi i/4}}{4e^{\pi i}} = -\frac{e^{\pi i/4}}{4}.$$

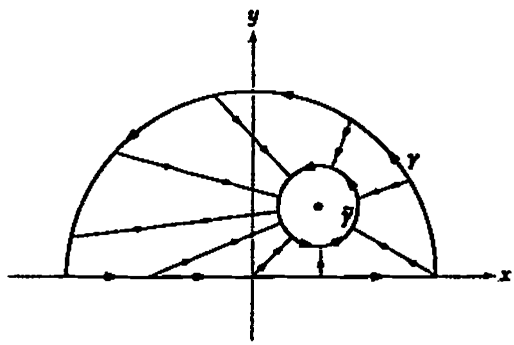


Figure 4.2.8: Homotopy between γ and $\tilde{\gamma}$.

Similarly,

$$\operatorname{Res}\left(\frac{1}{z^4+1}; e^{3\pi i/4}\right) = \frac{1}{4(e^{3\pi i/4})^3} = \frac{e^{-\pi i/4}}{4}.$$

Therefore,

$$\oint_\gamma \frac{dz}{z^4+1} = \frac{2\pi i}{4}(e^{-\pi i/4} - e^{\pi i/4}) = \pi \sin\frac{\pi}{4} = \frac{\pi\sqrt{2}}{2}.$$

We do not actually have to use such a detailed method to calculate the indexes (winding numbers). We simply use our intuition to calculate the number of times the curve in question winds around the given point in the counterclockwise direction. Keep in mind that the justification for this intuition consists of an argument like the preceding one.

Example 4.2.6 *Evaluate*

$$\oint_\gamma \frac{1+z}{1-\cos z}\,dz,$$

where γ is the circle of radius 7 around zero.

Solution The singularities of $(1+z)/(1-\cos z)$ occur where $1-\cos z = 0$. But $(e^{iz}+e^{-iz})/2 = 1$ implies that $(e^{iz})^2 - 2(e^{iz}) + 1 = 0$, that is, that $(e^{iz}-1)^2 = 0$, and hence $e^{iz} = 1$. Therefore, the singularities occur at $z = 2\pi n$ for $n = \ldots, -2, -1, 0, 1, 2, 3, \ldots$. The only singularities of $(1+z)/(1-\cos z)$ that lie inside the circle of radius 7 are $z_1 = 0, z_2 = 2\pi$, and $z_3 = -2\pi$ (see Figure 4.2.9). Also, $d(1-\cos z)/dz = \sin z$, which is zero at $0, -2\pi, 2\pi$; and $d^2(1-\cos z)/dz^2 = \cos z$, which is nonzero at $0, -2\pi$, and 2π, so these singularities are poles of order 2.

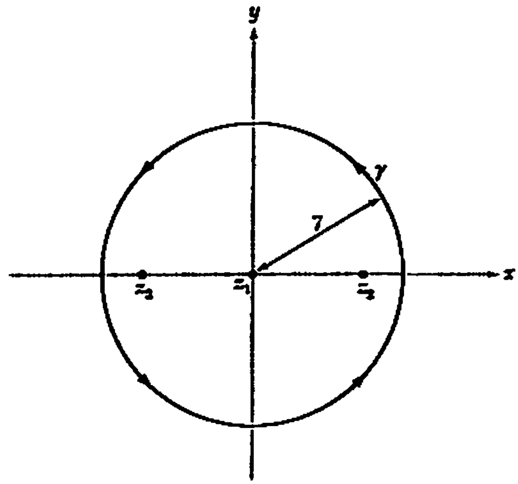


Figure 4.2.9: The curve γ contains three singularities.

The residue at one of these poles z_0 is, by formula 6 of Table 4.4.1,

$$2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}.$$

In this case, $g(z) = 1 + z$ so that $g'(z) = 1$; also, $h(z) = 1 - \cos z$ so that $h'(z) = \sin z, h''(z) = \cos z$, and $h'''(z) = -\sin z$. Thus, $h'''(z) = 0$ for $z = z_1, z_2, z_3$, so the formula for the residue becomes $2g'(z_0)/h''(z_0)$. Hence,

$$\operatorname{Res}(f; z_1) = \frac{2}{\cos 0} = 2, \quad \operatorname{Res}(f; z_2) = \frac{2}{\cos(2\pi)} = 2, \quad \operatorname{Res}(f; z_3) = \frac{2}{\cos(-2\pi)} = 2.$$

Thus, by the Residue Theorem,

$$\int_\gamma \frac{1+z}{1-\cos z}dz = 2\pi i[\operatorname{Res}(f; z_1) + \operatorname{Res}(f; z_2) + \operatorname{Res}(f; z_3)] = 12\pi i.$$

Notice that we have implicitly used the fact that $I(\gamma; z) = 0$ for z outside γ and $I(\gamma; z) = 1$ for z inside γ.

Example 4.2.7 *Show that if $p(z)$ is a polynomial of degree at least 2, then the sum of the residues of $1/p(z)$ at all the zeros of p must be 0.*

Solution 1 Suppose the degree of p is n so that we can write

$$p(z) = \sum_{k=0}^{n} a_k z^k \quad \text{with} \quad a_n \neq 0, \quad \text{where} \quad n \geq 2.$$

We know p can have at most n different zeros, so if γ is a circle of large enough radius R centered at 0, it surrounds all the finite singularities of $1/p(z)$. Thus,

$$\int_\gamma \frac{1}{p(z)}dz = 2\pi i \sum \left(\text{residues of } \frac{1}{p}\right),$$

and this holds for all large enough R. But for large R,

$$\frac{|a_{n-1}|}{R} + \frac{|a_{n-2}|}{R^2} + \ldots + \frac{|a_0|}{R^n} < \frac{|a_n|}{2},$$

so

$$\begin{aligned} |p(z)| &= \left|z^n \sum_{k=0}^{n} a_k z^{k-n}\right| \geq R^n \left(|a_n| - \left|\frac{a_{n-1}}{R} + \frac{a_{n-2}}{R^2} + \ldots + \frac{a_0}{R^n}\right|\right) \\ &\geq R^n \left[|a_n| - \left(\frac{|a_{n-1}|}{R} + \frac{|a_{n-2}|}{R^2} + \ldots + \frac{|a_0|}{R^n}\right)\right] \geq R^n \frac{|a_n|}{2} \end{aligned}$$

and so

$$\left|\int_\gamma \left[\frac{1}{p(z)}\right] dz\right| \leq \frac{2\pi R}{R^n |a_n|/2} = \frac{4\pi}{R^{n-1}|a_n|}.$$

Thus,

$$\left|\sum \left(\text{residues of } \frac{1}{p}\right)\right| \leq \frac{2}{R^{n-1}|a_n|}.$$

Letting $R \to \infty$, we obtain $|\sum$ (residues of $1/p)| \leq 0$. Therefore, the sum is 0.

Solution 2 (This solution makes use of residues at infinity.) With γ as before, there are no finite singularities of $1/p$ outside γ, so

$$\int_\gamma \frac{1}{p(z)}dz = -2\pi i \operatorname{Res}\left(\frac{1}{p};\infty\right) = 2\pi i \operatorname{Res}\left(\frac{1}{z^2}\frac{1}{p(1/z)};0\right).$$

But

$$\frac{1}{p(1/z)} \cdot \frac{1}{z^2} = \frac{1}{a_0 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n}} \cdot \frac{1}{z^2} = \frac{z^n}{a_0 z^n + \ldots + a_n} \cdot \frac{1}{z^2}.$$

Since $n \geq 2$, the singularity at $z = 0$ is removable, so the residue is 0 and hence the integral is 0. But the integral is equal to the sum of the residues of $1/p$ at the zeros of p.

Exercises

1. Evaluate $\int_\gamma \frac{dz}{(z+1)^3}$, where (a) γ is a circle of radius 2, center 0, and (b) γ is a square with vertices $0, 1, 1+i, i$.

2.• Deduce Cauchy's Integral Formula from the Residue Theorem.

3. Evaluate $\int_\gamma \frac{z}{z^2+2z+5}dz$, where γ is the unit circle.

4. Evaluate $\int_\gamma \frac{1}{e^z-1}dz$, where γ is the circle of radius 9 and center 0.

5. Evaluate $\int_\gamma \tan z\, dz$, where γ is the circle of radius 8 centered at 0.

6.• Show that $\int_\gamma \frac{5z-2}{z(z-1)}dz = 10\pi i$, where γ is any circle of radius greater than 1 and center 0.

7. Evaluate the contour integral $\int_\gamma \frac{e^{-z^2}}{z^2}dz$, where (a) γ is the square with the four vertices $-1-i, 1-i, 1+i$ and $-1+i$ and (b) γ is the ellipse $\gamma(t) = a\cos t + ib\sin t$, where $a, b > 0$, and $0 \le t \le 2\pi$.

8. Let f be analytic on $\mathbb{C}$ except for poles at 1 and -1. Assume that $\text{Res}(f;1) = -\text{Res}(f;-1)$. Let $A = \{z \mid z \notin [-1,1]\}$. Show that there is an analytic function h on A such that $h'(z) = f(z)$.

9. Evaluate the following integrals:

   (a) $\displaystyle\int_{|z|=\frac{1}{2}} \frac{dz}{z(1-z)^3}$

   (b) $\displaystyle\int_{|z|=\frac{1}{2}} \frac{e^z dz}{z(1-z)^3}$

10.• Evaluate the following integrals:

   (a) $\displaystyle\int_{|z|=\frac{1}{2}} \frac{dz}{(1-z)^3}$

   (b) $\displaystyle\int_{|z+1|=\frac{1}{2}} \frac{dz}{(1-z)^3}$

   (c) $\displaystyle\int_{|z-1|=\frac{1}{2}} \frac{dz}{(1-z)^3}$

   (d) $\displaystyle\int_{|z-1|=\frac{1}{2}} \frac{e^z}{(1-z)^3}dz$

11. Let $f : A \to B$ be analytic, one-to-one, and onto, and let $f'(z) \neq 0$ for $z \in A$. Let γ be a curve in A and let $\tilde{\gamma} = f \circ \gamma$. Also let g be continuous on $\tilde{\gamma}$. Show that

$$\int_\gamma (g \circ f) \cdot f' = \int_{\tilde{\gamma}} g.$$

What does this result become in the case where $f(z) = 1/z$?

12. Show that if $\lim_{z\to\infty}[-zf(z)]$ exists, it equals the residue of f at ∞.

13. (a) Find the residue of $(z-1)^3/z(z+2)^3$ at $z = \infty$.
    (b) Give two methods of evaluating

$$\int_\gamma \frac{(z-1)^3}{z(z+2)^3}\,dz,$$

where γ is the circle with center 0 and radius 3.

14. Show informally that if γ is a simple closed curve traveled counterclockwise, then

$$\int_\gamma f = -2\pi i \sum \{\text{residues of } f \text{ outside } \gamma \text{ including } \infty\}.$$

15. Choose a branch of $\sqrt{z^2-1}$ that is analytic on $\mathbb{C}$ except for the segment $[-1, 1]$ on the real axis. Evaluate

$$\int_\gamma \sqrt{z^2-1}\,dz,$$

where γ is the circle of radius 2 centered at 0.

4.3 Evaluation of Definite Integrals

The Residue Theorem says that an integral around a closed curve can often be evaluated by computations involving the integrand at a few points inside the curve. The Deformation Theorem then says that the resulting value does not change as the curve is shifted so long as no singularities of the integrand are crossed in the process. These two results make the calculus of residues a powerful tool for the evaluation of certain definite integrals, some of which may have no obvious connection to complex analysis.

For example, the change of variable $z = e^{i\theta}$ might convert an integral over the real interval $-\pi \leq \theta \leq \pi$ into one around the unit circle in the complex plane. In this section we will apply residue calculus to this type of integral and to improper integrals of the forms $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^\infty f(x)\,dx$. Some examples and devices for evaluating integrals involving "multiple-valued" functions such as roots and logarithms are also given. The techniques developed throughout this section are summarized in Table 4.3.1 toward the end of the section. This table should not be used too literally in all cases. Indeed, it is valuable to *understand* the techniques and estimates used to establish the formulas since the same ideas can often be used when the formulas obtained here do not directly apply. Miscellaneous examples working out important special cases and illustrating how the methods may be modified to handle nonstandard problems are given in the Student Guide and the Instructors Supplement.

Rational Functions of Sine and Cosine Perhaps the most straightforward type of real definite integral to which we may apply the residue methods are those for which a simple change of variable converts the integral to one over a closed curve. The interval of integration becomes a parameter interval for the curve. This method applies particularly well to integrands involving sine and cosine over a period interval such as $[-\pi,\pi]$ or $[0,2\pi]$. The change of variable $z=e^{i\theta}$ converts each of these to an integral over the unit circle, while $\sin\theta$ and $\cos\theta$ become the fractions

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{z-(1/z)}{2i}\quad\text{and}\quad\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{z+(1/z)}{2}.$$

We now give an example to show how this procedure works.

Example 4.3.1 *Let a be a positive real constant not equal to 1 and evaluate*

$$I=\int_0^{2\pi}\frac{d\theta}{1+a^2-2a\cos\theta}.$$

Solution Let $z=e^{i\theta}$ for $0\le\theta\le 2\pi$, and let γ be the unit circle centered at the origin. Then $dz=ie^{i\theta}d\theta=izd\theta$. The change of variables suggested above gives

$$\begin{aligned}I&=\int_\gamma\frac{1}{1+a^2-\frac{2a}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz}=\frac{1}{i}\int_\gamma\frac{dz}{z+a^2z-az^2-a}\\&=\int_\gamma\frac{idz}{(z-a)(az-1)}=\int_\gamma\frac{idz}{a(z-a)(z-(1/a))}.\end{aligned}$$

The integrand has simple poles at $z=a$ and at $z=1/a$. The residues at these points are

$$\begin{aligned}\operatorname{Res}\left(\frac{i}{a(z-a)(z-(1/a))};a\right)&=\left.\frac{i}{az-1}\right|_{z=a}=\frac{i}{a^2-1}\\\operatorname{Res}\left(\frac{i}{a(z-a)(z-(1/a))};\frac{1}{a}\right)&=\left.\frac{i}{a(z-a)}\right|_{z=1/a}=\frac{i}{1-a^2}.\end{aligned}$$

If $0<a<1$, then a is inside γ and $1/a$ is outside. Thus,

$$I=2\pi i\left(\frac{i}{a^2-1}\right)=\frac{2\pi}{1-a^2}.$$

If $a > 1$, then a is outside γ and $1/a$ is inside. So

$$I = 2\pi i \left(\frac{i}{1-a^2} \right) = \frac{2\pi}{a^2 - 1}. \quad \blacklozenge$$

Note that there are no values of θ for which the denominator in this example vanishes. In these situations, we can always do such a change of variable. The method may be formulated in general terms as follows.

Proposition 4.3.2 *Let $R(x, y)$ be a rational function of x and y whose denominator does not vanish on the unit circle. Then*

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = 2\pi i \sum [\text{residues of } f(z) \text{ inside the unit circle}],$$

where

$$f(z) = \frac{1}{iz} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right) \right).$$

Proof Since R is a rational function, so is f. Therefore, there are a finite number of poles and no other singularities. The hypothesis on R ensures that none of them are on the unit circle, and the same change of variable as in the example shows that

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \int_\gamma f(z)\, dz,$$

where γ is the unit circle centered at the origin traveled once counterclockwise. The proposition follows from the Residue Theorem. ∎

If one forgets the formula for f in this proposition, one can always proceed as in the example writing $\sin\theta$ and $\cos\theta$ in terms of $e^{i\theta}$ and then making the change of variable.

Improper Integrals Improper integrals of the types mentioned in the introduction to §1.1 are defined in calculus; we now recall these definitions. First of all, those along half lines are defined as limits:

$$\int_a^\infty f(x)\, dx = \lim_{B\to\infty} \int_a^B f(x)\, dx \quad \text{and} \quad \int_{-\infty}^b f(x)\, dx = \lim_{A\to\infty} \int_{-A}^b f(x)\, dx.$$

Integrals over the whole line are defined by splitting the line into two rays and requiring that both portions of the integral exist as finite numbers.

$$\begin{aligned} \int_{-\infty}^\infty f(x)\, dx &= \int_{-\infty}^0 f(x)\, dx + \int_0^\infty f(x)\, dx \\ &= \lim_{A\to\infty} \int_{-A}^0 f(x)\, dx + \lim_{B\to\infty} \int_0^B f(x)\, dx. \end{aligned}$$

An important point is that both limits are assumed to exist independently.

The existence of such integrals (i.e., the existence of these limits as finite numbers) may often be established by comparison to simpler integrals. This is also a calculus (or real analysis) theorem.

Proposition 4.3.3 *Suppose $|f(x)| \le g(x)$ for all $x \ge a$ and that $\int_a^\infty g(x)\,dx$ converges to a finite number G. Then the integral $\int_a^\infty f(x)\,dx$ also converges and $|\int_a^\infty f(x)\,dx| \le G$. Similar conclusions hold for integrals along a left half line or over the whole real line.*

Idea of the Proof The integral $\int_a^\infty |f(x)|\,dx$ converges to a number no larger than G since the numbers $\int_a^B |f(x)|\,dx$ are increasing as B increases and remain bounded above by G. Having thus established the result for positive functions, the convergence of $\int_a^\infty f(x)\,dx$ then follows in much the same way that absolute convergence of a series implies its convergence. The idea is that if f is *real* valued, then $|f|$ and $|f| - f$ are both positive functions dominated by $2g$, and $\int f = \int |f| - \int(|f| - f)$. If f is *complex* valued, we can work with its real and imaginary parts since $|\mathrm{Re}(f)| \le |f| \le g$ and $|\mathrm{Im}(f)| \le |f| \le g$. The final inequality follows since

$$\left|\int_a^B f(x)\,dx\right| \le \int_a^B |f(x)|\,dx \le \int_a^B g(x)\,dx = G$$

for each B. ∎

Once we know that the limits for the half line integrals exist independently, then we can use any convenient special form of the limit to evaluate it.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \lim_{A\to\infty}\int_{-A}^{0} f(x)\,dx + \lim_{B\to\infty}\int_0^B f(x)\,dx \\ &= \lim_{A\to\infty, B\to\infty}\int_{-A}^{B} f(x)\,dx \\ &= \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx\end{aligned}$$

The symmetric limit in the last line is a very convenient form, as we shall soon see. It is important to remember that the existence of this special symmetric form of the limit does not imply that of the more general one—a lot of cancellation might occur. However, if the general one exists, then this special one must as well and be equal to it. The intermediate form $\int_{-A}^{B} f(x)\,dx$ is more subtle. At first glance it looks the same as the general defining form, but it is not quite. It is a nontrivial observation that they are equivalent. We prove the following lemma in the Internet Supplement.

Lemma 4.3.4 *If $\lim_{A\to\infty, B\to\infty}\int_{-A}^{B} f(x)\,dx$ exists, then $\int_{-\infty}^{\infty} f(x)\,dx$ exists and is equal to this limit.*

As we shall see in the following subsections, these observations about improper integrals allow us to evaluate them by building the appropriate interval, $[-R, R]$ or $[-A, B]$, into a closed curve and then letting the ends tend to infinity.

Integrals on the Whole Real Line Now we consider integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$ in which the integrand is well enough behaved to allow the application of residue methods. For the integral to converge, the integrand must approach 0 as x approaches infinity in both directions along the real axis. With restraints on the growth of f in other directions, we can make progress. We will illustrate this by examples that will motivate some general methods.

Example 4.3.5 *Evaluate* $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx$.

Solution First of all, observe that the improper integral $\int_{-\infty}^{\infty}[x^2/(1+x^4)]\,dx$ converges. One way to see this is to note first that the integrand is even and continuous. So it suffices to show that $\int_1^{\infty}[x^2/(1+x^4)]\,dx$ converges. This integral converges by comparison with $\int_1^{\infty}[1/x^2]\,dx = \lim_{B\to\infty}(-(1/B)+1) = 1$. Since our integral converges we can use a symmetric limit to evaluate it; let the desired integral be denoted I and write

$$I = \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{x^2}{1+x^4}\,dx.$$

The interval of integration, $[-R, R]$, may be considered as a path γ_R along that part of the real axis in $\mathbf{C}$ from $-R$ to R. It may be extended to a simple closed curve $\Gamma_R = \gamma_R + \mu_R$ by returning from R to $-R$ along the semicircle $\mu_R = \{z \in \mathbf{C} \mid \mathrm{Im}(z) \geq 0 \text{ and } |z| = R\}$ in the upper half plane as sketched in Figure 4.3.1.

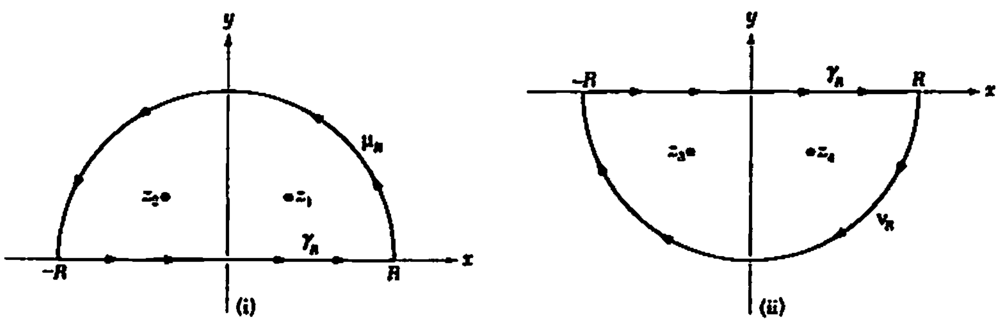


Figure 4.3.1: Curves for two solutions to Example 4.3.5

Along γ_R, the integrand is the same as the function $f(z) = z^2/(1+z^4)$ which is analytic everywhere except for simple poles at the fourth roots of -1

$$z_1 = e^{\pi i/4}\,; \qquad z_2 = e^{3\pi i/4}\,; \qquad z_3 = e^{5\pi i/4}\,; \qquad z_4 = e^{7\pi i/4}$$

Of these, z_1 and z_2 are in the upper half plane and z_3 and z_4 are in the lower. If $R > 1$, then z_1 and z_2 are inside Γ_R while z_3 and z_4 are outside. The residues at these points may be evaluated from

$$\mathrm{Res}(f; z_k) = \frac{z_k^2}{4z_k^3} = \frac{1}{4z_k}.$$

The Residue Theorem says that

$$\begin{aligned}\int_{\Gamma_R} f(z)\,dz &= 2\pi i\,(\mathrm{Res}(f;z_1) + \mathrm{Res}(f;z_2)) = \frac{2\pi i}{4}\left(e^{-\pi i/4} + e^{-3\pi i/4}\right)\\ &= \frac{2\pi i}{4}e^{-\pi i/2}\left(e^{\pi i/4} + e^{-\pi i/4}\right) = \frac{2\pi i}{4}(-i)2\cos(\pi/4) = \frac{\pi}{\sqrt{2}}.\end{aligned}$$

This is true for every $R > 1$. For the two pieces of the path we have

$$\int_{\gamma_R} f(z)\,dz = \int_{-R}^{R} \frac{x^2}{1+x^4}\,dx,$$

while along μ_R we have $|z| = R > 1$ so that

$$|f(z)| = \frac{R^2}{|1+z^4|} \le \frac{R^2}{R^4 - 1}.$$

Thus,

$$\left|\int_{\mu_R} f(z)\,dz\right| \le \frac{R^2}{R^4-1}\,\mathrm{length}(\mu_R) = \frac{\pi R^3}{R^4-1}.$$

This last quantity tends to 0 as $R \to \infty$, so $\lim_{R\to\infty}\int_{\mu_R} f(z)\,dz = 0$. Putting the pieces together we find that

$$\begin{aligned}\frac{\pi}{\sqrt{2}} &= \lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz = \lim_{R\to\infty}\int_{\mu_R} f(z)\,dz + \lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz\\ &= 0 + \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{1+x^4}\,dx = \int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx.\end{aligned}$$

We conclude that

$$\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}}.$$

The path γ_R could also be extended to a simple closed curve $\Lambda_R = \gamma_R + \nu_R$ by returning from R to $-R$ through the lower half plane along the semicircle $\nu_R = \{z \in \mathbf{C} \mid \mathrm{Im}(z) \le 0 \text{ and } |z| = R\}$. If $R > 1$ we now have z_3 and z_4 inside the contour Λ_R and z_1 and z_2 outside. Notice however that the curve goes around these points in a clockwise (negative) orientation.

The Residue Theorem 4.2.1 gives

$$\begin{aligned}\int_{\Lambda_R} f(z)\,dz &= -2\pi i\,(\operatorname{Res}(f;z_3)+\operatorname{Res}(f;z_4)) = -\frac{2\pi i}{4}\left(e^{-5\pi i/4}+e^{-7\pi i/4}\right)\\ &= -\frac{2\pi i}{4}e^{-3\pi i/2}\left(e^{\pi i/4}+e^{-\pi i/4}\right)\\ &= -\frac{2\pi i}{4}(i)2\cos(\pi/4) = \frac{\pi}{\sqrt{2}}.\end{aligned}$$

The same estimates apply to $|f(z)|$ along ν_R as along μ_R and the length of this semicircle is also πR. The same argument shows that the integral along this arc tends to 0 as $R \to \infty$. Therefore, we have

$$\begin{aligned}\frac{\pi}{\sqrt{2}} &= \lim_{R\to\infty}\int_{\Lambda_R} f(z)\,dz = \lim_{R\to\infty}\int_{\nu_R} f(z)\,dz + \lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz\\ &= 0 + \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{1+x^4}\,dx = \int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx.\end{aligned}$$

We conclude that

$$\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}},$$

just as before. ♦

Some simple checks, such as determining that in this example the integral must be real and positive since the integrand is nonnegative and real on the real axis, can often detect computational errors. This example could also have been done by the method of partial fractions decomposition of the integrand, but many of the integrals we will meet later cannot be done by such techniques.

The key elements needed to make this evaluation work were

1. An estimate to establish convergence of the improper integral so that a symmetric limit could be used for its evaluation

2. Finitely many singularities in the upper (or lower) half plane with none on the real axis so that all could be enclosed in Γ_R (or Λ_R) for large enough R and the residues evaluated

3. An estimate showing that the integral along the upper (or lower) semicircle tends to 0 as R tends to ∞

The same argument can be carried through provided we have these elements.

Proposition 4.3.6 (i) *Suppose f is analytic on an open set containing the closed upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}$ except for a finite number of isolated singularities none of which lie on the real axis, and that there are positive* *real constants M, p, and R_0 with $p > 1$ and $|f(z)| \le M/|z|^p$ whenever $z \in \mathcal{H}$ and $|z| \ge R_0$. Then*

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{H}\}.$$

(ii) *If the conditions of (i) hold with $\mathcal{H}$ replaced by the closed lower half plane $\mathcal{L} = \{z \in \mathbf{C} \mid \operatorname{Im}(z) \le 0\}$, then*

$$\int_{-\infty}^{\infty} f(x)\,dx = -2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{L}\}.$$

(iii) *Both of these formulas hold if $f = P/Q$ where P and Q are polynomials, the degree of Q is at least 2 greater than that of P, and Q has no real zeros.*

Proof The arguments for (i) and (ii) go exactly as in the example. In (i), work with the curve $\Gamma_R = \gamma_R + \mu_R$ for $R > R_0$. In (ii) use $\Lambda_R = \gamma_R + \nu_R$. There is no trouble between $-R_0$ and R_0, and the improper integrals from $-\infty$ to $-R_0$ and from R_0 to ∞ converge by comparison to the convergent improper integral $\int_{R_0}^{\infty}(1/x^p)\,dx$. For $R > R_0$ we have

$$\left|\int_{\mu_R} f(z)\,dz\right| \le \frac{\pi R}{R^p} \quad\text{and}\quad \left|\int_{\nu_R} f(z)\,dz\right| \le \frac{\pi R}{R^p}.$$

Each of these tends to 0 as $R \to \infty$ since $p > 1$. The rest of the arguments go exactly as in the example.

Finally suppose that $f = P/Q$ as in (iii). We shall complete the proof by establishing an inequality $|f(z)| \le M/|z|^2$ for $|z|$ large. If P is of degree n and Q is of degree $n + p$ with $p \ge 2$, then we know that there is an $M_1 > 0$ such that $|P(z)| \le M_1|z|^n$ for $|z| \ge 1$ and an $M_2 > 0$ and $R_0 > 1$ such that $|Q(z)| \ge M_2|z|^{n+p}$ for $|z| \ge R_0$. (See the proof of the Fundamental Theorem of Algebra (2.4.9).) Thus

$$\left|\frac{P(z)}{Q(z)}\right| \le \frac{M_1}{M_2}\cdot\frac{1}{|z|^p} \le \frac{M_1}{M_2}\cdot\frac{1}{|z|^2}$$

whenever $|z| \ge R_0 \ge 1$ since $p \ge 2$. The proof is completed using the choice $M = M_1/M_2$. ■

Rational functions such as that of Example 4.3.5 are probably the most readily recognized integrands to which Proposition 4.3.6 or related methods apply, but there are others. Consider the following example.

Example 4.3.7 *Let a be a nonzero real constant and evaluate*

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1 + x^2}\,dx.$$

At first glance it may be "obvious" that everything should work. The numerator of the integrand is bounded by 1 along the real axis, so the improper integral converges by comparison to the arctangent integral $\int_{-\infty}^{\infty}(1/(1+x^2))\,dx = \pi$. This much is correct. However, if we attempt to extend into the complex plane using $(\cos az)/(1+z^2)$ as the integrand, we do not have the needed growth conditions in either half plane. The function $\cos az = (e^{iaz}+e^{-iaz})/2$ grows exponentially as we proceed outward along either the positive or negative imaginary axes, $\cos iay = (e^{-ay}+e^{ay})/2$. If $a>0$, the first term shrinks exponentially along the positive imaginary axis while the second grows and the opposite occurs along the negative imaginary axis. If $a<0$, the situation is reversed. This gives the clue as to how to proceed. Convert the problem to one involving complex exponentials first.

$$\int_{-\infty}^{\infty}\frac{\cos ax}{1+x^2}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\operatorname{Re}\left(\frac{e^{iax}}{1+x^2}\right)dx = \operatorname{Re}\left(\int_{-\infty}^{\infty}\frac{e^{iax}}{1+x^2}\,dx\right)$$

This is an integral to which we can apply Proposition 4.3.6. The integrand, $g(z) = e^{iaz}/(1+z^2)$ is analytic on the whole plane except for simple poles at $z_1 = i$ and $z_2 = -i$. Of these, i is in the upper half plane and $-i$ in the lower. The residues at these points may be evaluated from

$$\operatorname{Res}(g;z_k) = \frac{e^{iaz_k}}{2z_k}.$$

Therefore,

$$\operatorname{Res}(g;i) = \frac{e^{iai}}{2i} = \frac{e^{-a}}{2i} \quad\text{and}\quad \operatorname{Res}(g;-i) = \frac{e^{-iai}}{-2i} = \frac{e^{a}}{-2i}.$$

With $y = \operatorname{Im}(z)$, we have

$$\left|e^{iaz}\right| = \left|e^{iax-ay}\right| = e^{-ay},$$

so

$$|g(z)| = \frac{e^{-ay}}{|1+z^2|} \le \frac{e^{-ay}}{R^2-1} \le \frac{2e^{-ay}}{R^2}$$

provided $|z|\ge R\ge\sqrt{2}$. The numerator is bounded by 2 if the exponent is negative. Thus we have the conditions of Proposition 4.3.6 in the upper half plane if $a>0$ and in the lower half plane if $a<0$.

For $a>0$ we conclude

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\cos ax}{1+x^2}\,dx &= \operatorname{Re}\left(\int_{-\infty}^{\infty}\frac{e^{iax}}{1+x^2}\,dx\right) = \operatorname{Re}\left(2\pi i\operatorname{Res}(g;i)\right) = \operatorname{Re}\left(2\pi i\frac{e^{-a}}{2i}\right)\\ &= \frac{\pi}{e^{a}}.\end{aligned}$$

For $a<0$ we conclude

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\cos ax}{1+x^2}\,dx &= \operatorname{Re}\left(\int_{-\infty}^{\infty}\frac{e^{iax}}{1+x^2}\,dx\right) = \operatorname{Re}\left(-2\pi i\operatorname{Res}(g;-i)\right) = \operatorname{Re}\left(-2\pi i\frac{e^{a}}{-2i}\right)\\ &= \pi e^{a}.\end{aligned}$$

We could also handle the case $a < 0$ by using the fact that cosine is an even function and appealing to the case for positive constants

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos(-ax)}{1+x^2}\,dx = \frac{\pi}{e^{-a}} = \pi e^{a}$$

as before. In either case, our final result is

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,dx = \frac{\pi}{e^{|a|}}. \quad \blacklozenge$$

Fourier Transforms The ***Fourier transform*** of a function $f : \mathbf{R} \to \mathbf{C}$ is a new function defined for a number ω, either real or complex, by the improper integral

$$\hat{f}(\omega) := (\mathcal{F}f)(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx.$$

The Fourier transform is important in differential equations (see Exercise 24), theoretical physics, quantum mechanics, and many other areas of mathematics and science, and there is an enormous body of literature concerning it. (There are variations in its definition. The factor of $1/\sqrt{2\pi}$ may be missing and the exponent might be $-2\pi i\omega x$.) Thus, integrals of the form $\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$ are of definite interest. If ω is real and $f(x)$ is real for real x, then the real and imaginary parts form the ***Fourier cosine*** and ***sine transforms*** of f:

$$\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx = \operatorname{Re}\left(\hat{f}(\omega)\right)$$
$$\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx = -\operatorname{Im}\left(\hat{f}(\omega)\right).$$

Example 4.3.7 showed how residue methods can be used to evaluate integrals of this type. The key to the argument there was that with a real the exponential factor e^{iax} remained bounded in either the upper or lower half plane. However, somewhat more is true. The absolute value of this factor is e^{-ay}, which shrinks exponentially as we move away from the real axis into the upper half plane if $a > 0$ and the lower half plane if $a < 0$. If we take advantage of this we can evaluate the integral with much more mild conditions on f. To do so we will use a rectangular contour instead of a semicircle. Lemma 4.3.4 will be used to establish the convergence of the integral and to evaluate it as the limit of $\int_{-A}^{B}$ as A and B tend independently to infinity.

Example 4.3.8 *Let ω be a nonzero real constant and evaluate*

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{1+ix}\,dx.$$

Solution This is an integral of the Fourier transform type with $f(z) = 1/(1+iz)$. The integrand is $g(z) = f(z)e^{-i\omega z}$. Along the real axis we have

$$|g(x)| = \left|e^{-i\omega x}\right| / |1+ix| = 1/\sqrt{1+x^2}.$$

This decreases too slowly to apply Proposition 4.3.6, and, in fact, the integral $\int_{-\infty}^{\infty} |g(x)|\,dx$ diverges. Therefore, a simple comparison test will not suffice to establish convergence of our integral. Instead we will appeal to Lemma 4.3.4 and obtain our result as

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{1+ix}\,dx = \lim_{A\to\infty, B\to\infty} \int_{-A}^{B} \frac{e^{-i\omega x}}{1+ix}\,dx.$$

Since $|f(z)| = 1/|1+iz| \le 1/(|z|-1)$ for $|z| > 1$, this factor does shrink toward 0. For each $\epsilon > 0$, there is an $R(\epsilon)$ such that $|f(z)| < \epsilon$ whenever $|z| \ge R(\epsilon)$. As in Example 4.3.7, the exponential factor will behave well on a half plane. Which half plane depends on the sign of ω. If $z = x + iy$ with x and y real, then $\left|e^{-i\omega z}\right| = \left|e^{-i\omega x+\omega y}\right| = e^{\omega y}$. Therefore,

$$\begin{array}{lll} \omega < 0 & \text{implies} \quad \left|e^{-i\omega z}\right| = e^{\omega y} \le 1 & \text{in the upper half plane } \mathcal{H} \\ \omega > 0 & \text{implies} \quad \left|e^{-i\omega z}\right| = e^{\omega y} \le 1 & \text{in the lower half plane } \mathcal{L}. \end{array}$$

The integrand is analytic except for a simple pole at i where the residue is

$$\operatorname{Res}(g; i) = \operatorname{Res}\left(\frac{e^{-i\omega z}}{i(z-i)}; i\right) = \left.\frac{e^{-i\omega z}}{i}\right|_{z=i} = \frac{e^{\omega}}{i}.$$

Let

$$\begin{aligned} \Sigma_{\mathcal{H}} &= \text{ the sum of the residues in } \mathcal{H} = \frac{e^{\omega}}{i} \\ \Sigma_{\mathcal{L}} &= \text{ the sum of the residues in } \mathcal{L} = 0. \end{aligned}$$

If A and B are both larger that $R(\epsilon)$ and larger that 1, we can consider the rectangular paths indicated in Figure 4.3.2.

In each case the portion γ is the segment of the real axis from $-A$ to B. This is closed as a rectangle $\Gamma = \gamma + \mu_1 + \mu_2 + \mu_3$ counterclockwise through the upper half plane and as a rectangle $\Lambda = \gamma + \nu_1 + \nu_2 + \nu_3$ clockwise through the lower half plane. In each case the distance C from the real axis will be selected larger than $R(\epsilon)$ and depending appropriately on A and B. So long as it is larger than 1 we will have

$$\int_{\Gamma} g = 2\pi i \Sigma_{\mathcal{H}} = 2\pi e^{\omega} \quad \text{and} \quad \int_{\Lambda} g = -2\pi i \Sigma_{\mathcal{L}} = 0.$$

Thus,

$$\begin{aligned} \int_{-A}^{B} \frac{e^{-i\omega x}}{1+ix}\,dx = \int_{\gamma} g &= 2\pi i \Sigma_{\mathcal{H}} - \left(\int_{\mu_1} g + \int_{\mu_2} g + \int_{\mu_3} g\right) \\ &= -2\pi i \Sigma_{\mathcal{L}} - \left(\int_{\nu_1} g + \int_{\nu_2} g + \int_{\nu_3} g\right). \end{aligned}$$

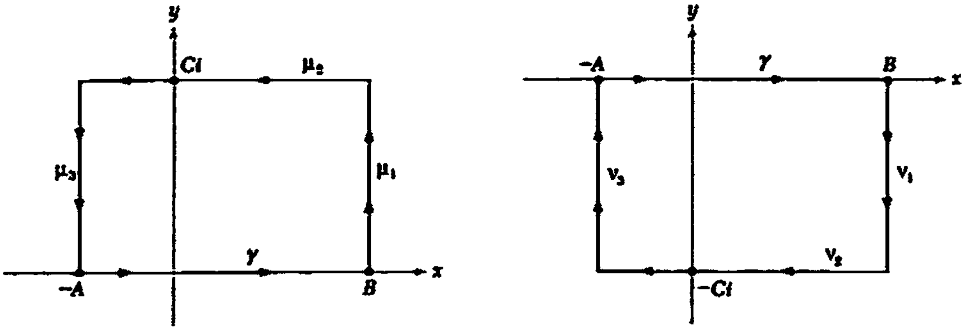


Figure 4.3.2: Paths for Example 4.3.8.

Once A, B, and C are large enough so that further increases enclose no more singularities, then $\Sigma_{\mathcal{H}}$ and $\Sigma_{\mathcal{L}}$ no longer change. The existence of the limit can be established and the integral evaluated by showing that one or the other of the quantities in parentheses tends to 0.

First suppose $\omega < 0$. In that case we have good behavior in the upper half plane, so we use Γ. We estimate the contribution along each of the pieces. Along μ_1, $z = B + iy$, so

$$\left|\int_{\mu_1} g\right| = \left|\int_0^C f(B+iy)e^{-i\omega(B+iy)}\, i\, dy\right| \le \int_0^C \epsilon e^{\omega y}\, dy = \frac{\epsilon}{\omega}(e^{C\omega} - 1) \le \frac{\epsilon}{|\omega|}.$$

Similarly,

$$\left|\int_{\mu_3} g\right| = \left|\int_C^0 f(-A+iy)e^{-i\omega(-A+iy)}\, i\, dy\right| \le \int_0^C \epsilon e^{\omega y}\, dy \le \frac{\epsilon}{|\omega|}.$$

Thus, these two contributions are small if A and B are larger than $R(\epsilon)$. Having set A and B, we now adjust C to make the horizontal contribution small also. Along μ_2 we have $z = x + iC$. Requiring $C > R(\epsilon)$, we have

$$\left|\int_{\mu_2} g\right| = \left|\int_B^{-A} f(x+iC)e^{-i\omega(x+iC)}\, dx\right| \le \int_{-A}^B \epsilon e^{C\omega}\, dx \le \epsilon(A+B)\, e^{\omega C}.$$

Recalling that $\omega < 0$, we can select C larger than 1, larger than $R(\epsilon)$ and large enough so that $(A+B)\, e^{\omega C} < 1$. We find that for A and B larger than 1 and larger than $R(\epsilon)$,

$$\left|\int_{-A}^B g(x)\, dx - 2\pi i \Sigma_{\mathcal{H}}\right| \le \left|\int_{\mu_1} g\right| + \left|\int_{\mu_2} g\right| + \left|\int_{\mu_3} g\right| \le \left(\frac{2}{|\omega|} + 1\right)\epsilon.$$

Thus, the limit exists and for $\omega < 0$ and we have

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{1+ix}\, dx = \lim_{A\to\infty, B\to\infty} \int_{-A}^B \frac{e^{-i\omega x}}{1+ix}\, dx = 2\pi i \Sigma_{\mathcal{H}} = 2\pi e^{\omega}.$$

The computation is quite similar for $\omega > 0$ except that now we use the rectangle Λ in the lower half plane.

$$\left|\int_{\nu_1} g\right| = \left|\int_0^{-C} f(B+iy)e^{-i\omega(B+iy)}\,i\,dy\right| \le \int_{-C}^0 \epsilon e^{\omega y}\,dy = \frac{\epsilon}{\omega}(1-e^{-C\omega}) \le \frac{\epsilon}{\omega}.$$

Similarly,

$$\left|\int_{\nu_3} g\right| = \left|\int_0^{-C} f(-A+iy)e^{-i\omega(-A+iy)}\,i\,dy\right| \le \int_{-C}^0 \epsilon e^{\omega y}\,dy \le \frac{\epsilon}{\omega}.$$

and

$$\left|\int_{\nu_2} g\right| = \left|\int_B^{-A} f(x-iC)e^{-i\omega(x-iC)}\,dx\right| \le \int_{-A}^B \epsilon e^{-C\omega}\,dx \le \epsilon(A+B)\,e^{-\omega C}.$$

Recalling that $\omega > 0$, we can select C larger than 1, larger than $R(\epsilon)$, and large enough so that $(A+B)\,e^{-\omega C} < 1$. We find that for A and B larger than 1 and larger than $R(\epsilon)$,

$$\left|\int_{-A}^B g(x)\,dx + 2\pi i\Sigma_{\mathcal{L}}\right| \le \left|\int_{\nu_1} g\right| + \left|\int_{\nu_2} g\right| + \left|\int_{\nu_3} g\right| \le \left(\frac{2}{|\omega|}+1\right)\epsilon.$$

Thus, the limit exists and for $\omega > 0$ we have

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{1+ix}\,dx = \lim_{A\to\infty,B\to\infty} \int_{-A}^B \frac{e^{-i\omega x}}{1+ix}\,dx = -2\pi i\Sigma_{\mathcal{L}} = 0.$$

Thus, our final result is

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{1+ix}\,dx = \begin{cases} 2\pi e^{\omega}, & \text{if } \omega < 0 \\ 0, & \text{if } \omega > 0 \end{cases}. \quad \blacklozenge$$

Notice that the transform integral we have just computed depends discontinuously on ω with a jump discontinuity at $\omega = 0$. Some insight may be gained by considering the real and imaginary parts of the integrand at that point:

$$\int_{-A}^B \frac{1}{1+ix}\,dx = \int_{-A}^B \frac{1}{1+x^2}\,dx - i\int_{-A}^B \frac{x}{1+x^2}\,dx$$

For the first integral we have

$$\int_{-A}^B \frac{1}{1+x^2}\,dx = \arctan(B) - \arctan(-A) \to \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

as $A \to \infty$ and $B \to \infty$. In the second integral, the portions over the left half line and the right half line both diverge, the first to $-\infty$ and the second to $+\infty$.

Thus, the improper integral $\int_{-\infty}^{\infty}(x/(1+x^2))\,dx$ does not converge. However, the integrand is an odd function of x, so the symmetric limit $\lim_{R\to\infty}\int_{-R}^{R}(x/(1+x^2))\,dx$ does exist and is 0. In this sense we get a *principal value* for our divergent integral. That value is π, which is exactly the midpoint of the jump discontinuity. We will see further variations on the principal value idea in the next subsection where we examine what to do with singularities on the path of integration.

The method of the last example can be employed whenever one has integrals of the type $\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$ with fairly mild restrictions on f. The key ingredients of the argument were

1. $f(z)$ is analytic on an open set containing the upper half plane if $\omega < 0$ and the lower half plane if $\omega > 0$ except for finitely many isolated singularities, none of which are on the real axis.

2. $f(z) \to 0$ as $z \to 0$ in that half plane in the sense that for each $\epsilon > 0$ there is an $R(\epsilon)$ such that $|f(z)| < \epsilon$ whenever $|z| \geq R(\epsilon)$ and z is in that half plane.

Other than these properties, we used nothing special about f to show that the value of the integral was $2\pi i$ times the sum of the residues of the integrand in the appropriate half plane. In our example, the conditions held in both half planes. The argument carries through with no change to give the general result.

Proposition 4.3.9 *Under either of the situations (i) or (ii) described below, the improper integral $\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$ converges to the value given by the corresponding formula. If $f(x)$ is real for real x, then the integrals $\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx$ and $\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx$ are equal respectively to its real part and the negative of its imaginary part.*

(i) $\omega < 0$: *Suppose that f is analytic on an open set containing the closed upper half plane $\mathcal{H} = \{z \in \mathbf{C} \mid \operatorname{Im}(z) \geq 0\}$ except for a finite number of isolated singularities none of which are on the real axis. Suppose also that $f(z) \to 0$ as $z \to \infty$ in that half plane in the sense that for each $\epsilon > 0$ there is an $R(\epsilon)$ such that $|f(z)| < \epsilon$ whenever $|z| \geq R(\epsilon)$ and $z \in \mathcal{H}$. Then*

$$\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = 2\pi i\Sigma\,\{\text{residues of } f(z)e^{-i\omega z} \text{ in } \mathcal{H}\}$$

(ii) $\omega > 0$: *If the conditions of (i) hold with $\mathcal{H}$ replaced by the closed lower half plane $\mathcal{L} = \{z \in \mathbf{C} \mid \operatorname{Im}(z) \leq 0\}$, then*

$$\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = -2\pi i\Sigma\,\{\text{residues of } f(z)e^{-i\omega z} \text{ in } \mathcal{L}\}$$

(iii) *Both (i) and (ii) are valid if $f = P/Q$ where P and Q are polynomials, the degree of Q is greater than that of P, and Q has no zeros on the real axis.*

The constructions, estimates, and arguments for (i) and (ii) are exactly like those in the example. That example is an illustration of (iii), and the general argument for (iii) uses the same idea as that for part (iii) of Proposition 4.3.6 except that now the degree of Q need only be at least 1 more than that of P. We can therefore omit the details of the general argument.

A caution in the use of these theorems is needed. Note that $\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx$ is *not* the sum of residues of $f(z)\cos(\omega z)$ in an appropriate half plane. This formula is false. As was pointed out in the discussion of Example 4.3.7, the hypotheses of the theorems need not apply, even if $|f(z)| \le M/|z^2|$.

Semicircular paths could also be used for establishing the result about Fourier integrals if one first establishes a lemma about them.

Jordan's Lemma *Suppose $f(z) \to 0$ as $|z| \to \infty$ uniformly in* $\arg z$ *for* $0 \le \arg z \le \pi$ *and that there is a positive constant c such that $f(z)$ is analytic for $|z| > c$ and* $0 \le \arg z \le \pi$. *If $\omega < 0$, then $\int_{\gamma_\rho} e^{-i\omega z} f(z)\,dz \to 0$ as $\rho \to \infty$ where γ_ρ is the semicircle $\gamma_\rho(\theta) = \rho e^{i\theta}$ for $0 \le \theta \le \pi$.*

See, for example, E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Fourth Edition (London: Cambridge University Press, 1927), p. 115, for a proof.

Cauchy Principal Value So far we have required that our integrands be analytic on the path of integration. There are two situations in which we might want to relax this. In the first there are singularities on the path, and in the second, branch cuts are involved. As an example of the first kind, suppose we slightly alter Example 4.3.5 and ask for the value of

$$\int_{-\infty}^{\infty} \frac{x}{x^3+1}\,dx.$$

Now the integrand has simple poles at the cube roots of -1, one of which is on the real axis. Arguments very much like those for Propositions 4.3.6 and 4.3.9 will still work, but the paths must be altered to accommodate singularities on the real axis, and they will contribute to the value of the integral.

If $f(x)$ is continuous on the real line except at x_0, the integral $\int_{-\infty}^{\infty} f(x)\,dx$ need not be defined. Consider

$$\int_{-\infty}^{x_0-\epsilon} f(x)\,dx + \int_{x_0+\eta}^{\infty} f(x)\,dx \qquad \text{with } \epsilon > 0 \quad \text{and } \eta > 0.$$

In the usual definition from calculus for the improper integral, both of these improper integrals must exist, and then the limits must exist separately as $\epsilon \to 0$ and $\eta \to 0$. In that case we say the improper integral is ***convergent***. Some care is in order. For example,

$$\int_{-\infty}^{-\epsilon} \frac{1}{x^3}\,dx + \int_{\epsilon}^{\infty} \frac{1}{x^3}\,dx = -\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon^2} = 0$$

while

$$\int_{-\infty}^{-\epsilon} \frac{1}{x^3}\,dx + \int_{2\epsilon}^{\infty} \frac{1}{x^3}\,dx = -\frac{1}{2\epsilon^2} + \frac{1}{2(2\epsilon)^2} = -\frac{3}{8\epsilon^2} \to -\infty \qquad \text{as} \quad \epsilon \to 0.$$

Thus, the value of the integral can depend on how we let ϵ and η approach 0. In this example the separate integrals $\int_0^\infty (1/x^3)\,dx$ and $\int_{-\infty}^0 (1/x^3)\,dx$ are not convergent.

In cases such as these it is sometimes useful to define a ***principal value*** for the integral by using the symmetric way of letting ϵ and η approach 0. We keep $\epsilon = \eta$ as in the first evaluation above. By doing so we can apply the Residue Theorem to the evaluation of such integrals. We allow for a finite number of discontinuities on the real axis by requiring f to be continuous on $\mathbb{R}$ except for a finite number of points $x_1 < x_2 < x_3 < \cdots < x_n$. If $\int_{-\infty}^{x_1-\epsilon} f(x)\,dx$ and $\int_{x_n+\epsilon}^{\infty} f(x)\,dx$ each converge for every $\epsilon > 0$, and if

$$\lim_{\epsilon\to 0}\left[\int_{-\infty}^{x_1-\epsilon} f(x)\,dx + \int_{x_1+\epsilon}^{x_2-\epsilon} f(x)\,dx + \cdots + \int_{x_{n-1}+\epsilon}^{x_n-\epsilon} f(x)\,dx + \int_{x_n+\epsilon}^{\infty} f(x)\,dx\right]$$

exists and is finite, then we shall call this limit the ***Cauchy principal value*** and denote it by P. V. $\int_{-\infty}^{\infty} f(x)\,dx$. Observe that if the integral is convergent at all these points, we recover the usual value for the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$. However, as we have seen, the Cauchy principal value can exist even when the integrals are not convergent in the usual sense.

To apply residue methods to such integrals, we require that $f(z)$ be analytic on an appropriate half plane except for a finite number of isolated singularities some of which may lie on the real axis, say at $x_1 < x_2 < x_3 < \cdots < x_n$. We modify the curves used earlier as shown in Figure 4.3.3.

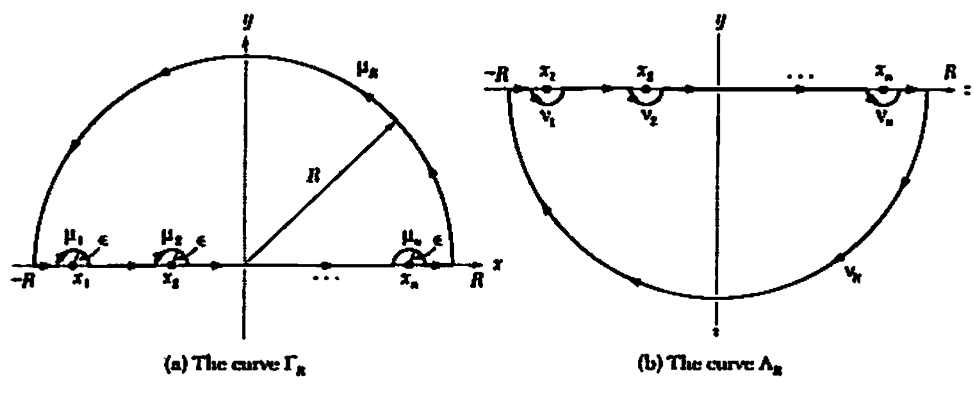


Figure 4.3.3: Modification of paths to allow for singularities on the axis.

The radius R of the large semicircle is chosen sufficiently large and the radii r of the small semicircles sufficiently small so that the semicircles do not overlap and all singularities off the axis in the appropriate half plane are enclosed so that further shrinking r or increasing R leaves the integral around the whole path equal to $\pm 2\pi i$ times the sum of the residues in that half plane off the axis. Conditions on f such as those in Proposition 4.3.6 will then guarantee that the integral along the large semicircle will tend to 0 as $R \to \infty$. The rectangular paths used for integrals of Fourier type are modified in a similar way, and conditions such as those in Proposition 4.3.9 will guarantee that the integral along the three sides of the rectangle in the half plane tend to 0 as those sides are pushed out toward infinity in the same way that they were there. If we can ensure that the limits of the integrals along the small semicircles exist and are finite as r tends to 0, then the principal value will exist and we can calculate its value. In many cases these limits are handled by the following lemma.

Lemma 4.3.10 *Let $f(z)$ be analytic with a simple pole at z_0 and γ_r be an arc of a circle of radius r and angle α centered at z_0. (See Figure 4.3.4.) Then*

$$\lim_{r \to 0} \int_{\gamma_r} f = \alpha i \operatorname{Res}(f; z_0).$$

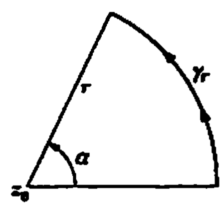


Figure 4.3.4: Integrate part way around a simple pole.

Proof For z near z_0 we have $f(z) = b_1/(z - z_0) + h(z)$ where h is analytic and $b_1 = \operatorname{Res}(f; z_0)$. Therefore,

$$\int_{\gamma_r} f(z)\, dz = \int_{\gamma_r} \frac{b_1}{z - z_0}\, dz + \int_{\gamma_r} h(z)\, dz.$$

Since h is analytic, it is certainly bounded near z_0. We write this as $|h(z)| \le M$ for $|z - z_0| < R$. Provided $r < R$, we have

$$\left| \int_{\gamma_r} h(z)\, dz \right| \le M \operatorname{length}(\gamma_r) = M\alpha r \to 0 \qquad \text{as} \quad r \to 0.$$

This leaves the first integral on the right in which we may put $z = z_0 + re^{i\theta}$ for $\alpha_0 \le \theta \le \alpha_o + \alpha$ along γ_r to find

$$\int_{\gamma_r} \frac{b_1}{z - z_0}\, dz = \int_{\alpha_o}^{\alpha_o + \alpha} \frac{b_1}{re^{i\theta}} ire^{i\theta}\, d\theta = b_1 \alpha i$$

independent of r for small r. Thus $\lim_{r\to 0}\int_{\gamma_r} f = \alpha i \operatorname{Res}(f; z_0)$, as claimed. ■

With this lemma in hand we have all the pieces we need for the required modifications to Propositions 4.3.6 and 4.3.9. With appropriate growth conditions in the upper half plane we have the following.

Proposition 4.3.11 *Let $\mathcal{H}_0$ be the open upper half plane $\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$, and let f be analytic on an open set containing its closure $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}$ except for finitely many isolated singularities. Suppose that of these $x_1, \ldots, x_n$ are on the real axis and are simple poles. Then if either*

(i) *f satisfies the conditions of part (i) of Proposition 4.3.6 (except for the poles on the axis) or*

(ii) *$f(z) = e^{-i\omega z} g(z)$ with $\omega < 0$ and g satisfying part (i) of Proposition 4.3.9,*

then the principal value integral exists and

$$\text{P.V.}\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \sum \{\textit{residues of } f \textit{ in } \mathcal{H}_0\} + \pi i \sum_{j=1}^{n} \operatorname{Res}(f; x_j)$$

Proof For part (i), let $\Gamma_R = \mu_R + \mu_1 + \cdots + \mu_n + \gamma_R$ be the closed curve indicated in the left sketch in the figure. Here μ_R is the large semicircle of radius R from R to $-R$ through the upper half plane, μ_j for $1 \leq j \leq n$ are the small semicircles of radius r around the poles on the axis, and γ_R consists of the straight line portions along the real axis. The radius R is selected large enough and r small enough so that the semicircles do not overlap and Γ_R surrounds all of the singularities in the open upper half plane. Notice that Γ_R surrounds these singularities in the counterclockwise sense but that the singularities on the axis are outside the curve and the small semicircles are oriented clockwise with respect to the poles at their centers. As a result the lemma gives $\lim_{r\to 0}\int_{\mu_j} f = -\pi i \operatorname{Res}(f; x_j)$. The improper integrals along each end of the axis, $\int_{-\infty}^{x_1 - r} f(x)\,dx$ and $\int_{x_n + r}^{\infty} f(x)\,dx$, each converge for every $r > 0$ by comparison because of the estimate $|f(z)| \leq M/|z|^p$ for large z. The same estimate shows that $\int_{\mu_R} f \to 0$ as $R \to \infty$ just as in Proposition 4.3.6. From the Residue Theorem we have

$$2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{H}_0\} = \int_{\Gamma_R} f = \int_{\mu_R} f + \int_{\gamma_R} f + \sum_{j=1}^{n} \int_{\mu_j} f.$$

Thus,

$$\begin{aligned}\int_{\gamma_R} f &= 2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{H}_0\} - \int_{\mu_R} f - \sum_{j=1}^{n} \int_{\mu_j} f \\ &\to 2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{H}_0\} - 0 + \pi i \sum_{j=1}^{n} \operatorname{Res}(f; x_j)\end{aligned}$$

as $R \to \infty$ and $r \to 0$. Thus, the principal value limit exists with the value claimed.

The argument for (ii) is analogous except that the rectangular paths of Proposition 4.3.9, modified by appropriate small semicircles, are used. The convergence of the integral along each of the infinite segments along the axis follows as it did in Proposition 4.3.9 by a slight modification of Lemma 4.3.4. ■

The result for the lower half plane is similar except for some minus signs.

Proposition 4.3.12 *Let $\mathcal{L}_0$ be the open lower half plane $\{z \in \mathbb{C} \mid \operatorname{Im}(z) < 0\}$, and let f be analytic on an open set containing its closure $\mathcal{L} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \leq 0\}$ except for finitely many isolated singularities. Suppose that of these $x_1, \ldots, x_n$ are on the real axis and are simple poles. Then if either*

(i) *f satisfies the conditions of part (ii) of Proposition 4.3.6 (except for the poles on the axis) or*

(ii) *$f(z) = e^{-i\omega z}g(z)$ with $\omega > 0$ and g satisfying part (ii) of Proposition 4.3.9,*

then the principal value integral exists and

$$\text{P.V.} \int_{-\infty}^{\infty} f(z)\,dz = -2\pi i \sum \{\textit{residues of } f \textit{ in } \mathcal{L}_0\} - \pi i \sum_{j=1}^{n} \operatorname{Res}(f; x_j)$$

The proof of this is basically the same as that for Proposition 4.3.11 except that the curve through the lower half plane is used as indicated in the right sketch in the figure. Notice that now the curve as a whole is negatively oriented (clockwise) with respect to the singularities in its interior, but the small semicircles around the poles on the axis proceed *counterclockwise* with respect to their centers. The portion γ_R along the axis is the same as before and still proceeds from left to right. Thus,

$$\int_{\gamma_R} f = -2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{L}_0\} - \int_{\nu_R} f - \sum_{j=1}^{n} \int_{\nu_j} f,$$

which converges to

$$-2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{L}_0\} - 0 - \pi i \sum_{j=1}^{n} \operatorname{Res}(f; x_j)$$

as $R \to \infty$ and $r \to 0$. Thus, the principal value limit exists with the value claimed.

The argument for (ii) is modified in the same way. ■

The challenge problem with which we began this subsection illustrates both of these propositions.

Example 4.3.13 *Discuss and find the principal value for the improper integral*

$$\int_{-\infty}^{\infty} \frac{x}{x^3 + 1}\,dx.$$

Solution The integrand $f(z) = z/(z^3+1)$ behaves like $1/x^2$ for large x, so there is no trouble with the integral for large x. However, there are simple poles at the cube roots of -1. one of which, -1, lies on the axis. Near that point the function behaves like $-1/(3(x+1))$. So the improper integral as such diverges. The change of sign across -1 suggests cancellation, which might make the principal value limit exist. The last two propositions say it does. The poles are at $z_1 = -1$, $z_2 = e^{\pi i/3}$, and $z_3 = e^{-\pi i/3}$. They are simple, and the residues are given by

$$\operatorname{Res}(f; z_k) = \left.\frac{z}{3z^2}\right|_{z=z_k} = \frac{1}{3z_k}.$$

Therefore,

$$\operatorname{Res}(f;-1) = -\frac{1}{3}\,; \quad \operatorname{Res}(f;e^{\pi i/3}) = \frac{1}{3}e^{-\pi i/3}\,; \quad \operatorname{Res}(f:e^{-\pi i/3}) = \frac{1}{3}e^{\pi i/3}.$$

The integrand is a rational function with the degree of the denominator 2 larger than that of the numerator, so we have the growth condition required by Proposition 4.3.6 in both half planes. Using Proposition 4.3.11 in the upper half plane, the principal value is

$$\begin{aligned}\text{P.V.}\int_{-\infty}^{\infty}\frac{x}{x^3+1}\,dx &= 2\pi i\operatorname{Res}(f;e^{\pi i/3}) + \pi i\operatorname{Res}(f;-1) = 2\pi i\frac{1}{3}e^{-\pi i/3} - \frac{\pi i}{3}\\ &= \frac{\pi i}{3}\left(2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right) - \frac{\pi i}{3} = \frac{\pi}{\sqrt{3}}.\end{aligned}$$

Using Proposition 4.3.12 in the lower half plane, we get the same result.

$$\begin{aligned}\text{P.V.}\int_{-\infty}^{\infty}\frac{x}{x^3+1}\,dx &= -2\pi i\operatorname{Res}(f;e^{-\pi i/3}) - \pi i\operatorname{Res}(f;-1) = -2\pi i\frac{1}{3}e^{\pi i/3} + \frac{\pi i}{3}\\ &= -\frac{\pi i}{3}\left(2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right) + \frac{\pi i}{3} = \frac{\pi}{\sqrt{3}}. \quad \blacklozenge\end{aligned}$$

The next example shows an interesting application of these ideas.

Example 4.3.14 *Show that the following improper integral converges and find its value.*

$$\int_0^{\infty}\frac{\sin x}{x}\,dx$$

Solution Let $h(x)$ be defined by $h(x) = (\sin x)/x$ for $x \neq 0$ and $h(0) = 1$. Then h is continuous everywhere and in particular at 0. Since the integrand is even, we will have $\int_0^{\infty}[(\sin x)/x]\,dx = (1/2)\int_{-\infty}^{\infty}h(x)\,dx$ provided we can show that the latter integral converges. Except at 0, the integrand is the imaginary part of e^{iz}/z. According to Proposition 4.3.11, the principal value integral of this exists, and

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx = \pi i\operatorname{Res}\left(\frac{e^{iz}}{z};0\right) = \pi i.$$

Since this limit exists, so does that of its imaginary part

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx = \text{P.V.}\int_{-\infty}^{\infty} h(x)\,dx = \pi.$$

The proof of the existence of this principal value integral included the fact that the improper integrals $\int_r^{\infty}[(\sin x)/x]\,dx$ and $\int_{-\infty}^{-r}[(\sin x)/x]\,dx$ converge for each positive r. There is no problem with the integral at infinity. Since h is continuous at 0, we conclude that the improper integral on the whole line converges and must be equal to the principal value. Thus,

$$\int_0^{\infty}\frac{\sin x}{x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} h(x)\,dx = \frac{\pi}{2}. \quad \blacklozenge$$

Integrals Involving Branch Cuts The values of the integrals we have studied so far in this chapter resulted from the residues of the integrand. Integrands involving noninteger powers or logarithms may also bring in the change in value from one side of a branch cut to the other. The next two examples illustrate this phenomenon.

The integral $\int_0^{\infty} x^{a-1} f(x)\,dx$ is referred to as a ***Mellin transform***. One uses the phrase "*transform of f*" since it can be considered as changing the function f to a new function of the variable a on the range of a for which the improper integral converges. For example, with $a = 4/3$ and $f(x) = 1/(1+x^2)$, the integral becomes

$$\int_0^{\infty}\frac{\sqrt[3]{x}}{1+x^2}\,dx$$

This can be evaluated by residue calculus and the solution illustrates the general method.

Example 4.3.15 *Evaluate the integral* $\displaystyle\int_0^{\infty}\frac{\sqrt[3]{x}}{1+x^2}\,dx.$

Solution The integral is along the positive real axis. The idea is to make this a branch cut for the cube root and take advantage of the differing values on opposite sides of the cut. For integrand we take $g(z) = \sqrt[3]{z}/(1+z^2)$ with the root defined by $\sqrt[3]{\rho e^{i\theta}} = \rho^{1/3}e^{i\theta/3}$ for $0 < \theta < 2\pi$. This is analytic off the positive real axis with simple poles at $\pm i$. The residues are

$$\text{Res}(g; i) = \frac{\sqrt[3]{i}}{2i} = \frac{e^{\pi i/6}}{2i} \quad\text{and}\quad \text{Res}(g; -i) = \frac{\sqrt[3]{-i}}{-2i} = -\frac{e^{3\pi i/6}}{2i}.$$

Letting Σ denote the sum of the residues, we get

$$\Sigma = \frac{1}{2i}\left(e^{\pi i/6} - e^{3\pi i/6}\right) = -\frac{e^{\pi i/3}}{2i}\left(e^{\pi i/6} - e^{-\pi i/6}\right) = -e^{\pi i/3}\sin(\pi/6) = -e^{\pi i/3}/2.$$

For $0 < r < 1$ and $R > 1$ we can form a curve $\Gamma = \gamma_1 + \gamma_R + \gamma_2 + \gamma_r$ enclosing these poles as indicated in Figure 4.3.5. The left figure shows a preliminary version and the right a refinement.

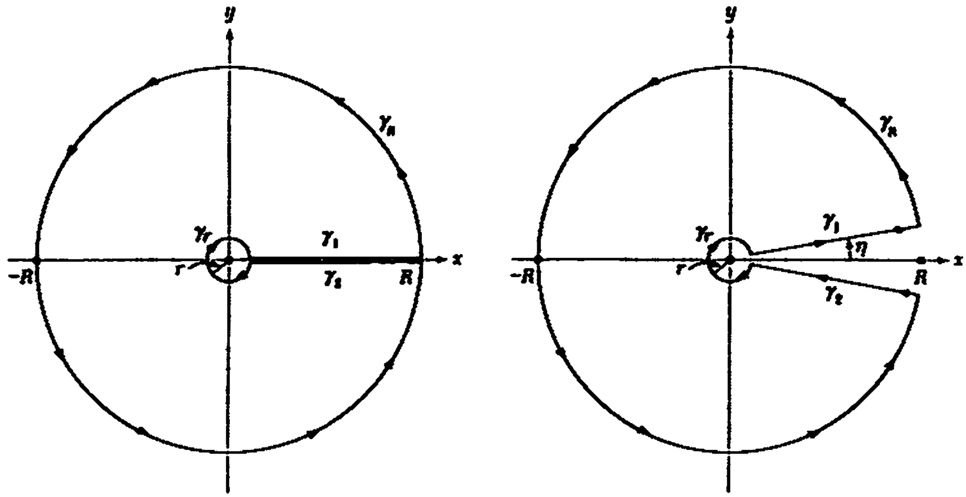


Figure 4.3.5: Contours for the Mellin transform integral.

- γ_1 (preliminary) proceeds from r to R along the "top side of" the positive real axis. That is, $z = x$, $r \le x \le R$.
- γ_R (preliminary) proceeds from R around to R counterclockwise along the circle of radius R centered at 0.
- γ_2 (preliminary) proceeds from R back to r along the "bottom side of" the positive real axis. That is, $z = xe^{2\pi i}$.
- γ_r (preliminary) proceeds from r around to r clockwise along the circle of radius r centered at 0.

The paths γ_1 and γ_2 incorporate the integral along the positive real axis, which we want. They do not cancel out since the argument of z is taken to be 0 along the first and 2π on the other. Taking cube roots creates different values. The computation is much easier to understand with the curves in this position but is open to the objection that the curve lies on the boundary of the region of analyticity and not within it. This objection will be overcome by switching to the refined curve in which γ_1 and γ_2 proceed along the rays $xe^{\eta i}$ and $xe^{(2\pi-\eta)i}$ for a small angle η instead of exactly along the axis. The arcs γ_R and γ_r are as before except for omitting a short arc across the real axis. We will then take a limit as $\eta \to 0$.

Begin by looking at the intuitive, unrefined, computation. Along the component curves we compute as follows. On γ_1, $z = x$ and

$$\int_{\gamma_1} g = \int_r^R \frac{x^{1/3}}{1+x^2}\,dx \to \int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\,dx \qquad \text{as } r \to 0 \text{ and } R \to \infty.$$

On γ_R, $|z| = R$, and

$$\left|\int_{\gamma_R} g\right| \le \frac{R^{1/3} 2\pi R}{R^2 - 1} \to 0 \quad \text{as} \quad R \to \infty.$$

Thus,

$$\int_{\gamma_R} g \to 0 \quad \text{as } R \to \infty.$$

On γ_2: $z = xe^{2\pi i}$ and

$$\int_{\gamma_2} g = \int_R^r \frac{x^{1/3} e^{2\pi i/3}}{1 + x^2 e^{4\pi i}} e^{2\pi i} dx.$$

This converges to $-e^{2\pi i/3} \int_0^\infty \frac{\sqrt[3]{x}}{1+x^2} dx$ as $r \to 0$ and $R \to \infty$. On γ_r, $|z| = r$, and

$$\left|\int_{\gamma_r} g\right| \le \frac{r^{1/3} 2\pi r}{1 - r^2} \to 0 \quad \text{as} \quad r \to 0.$$

Therefore, $\int_{\gamma_r} g \to 0 \quad$ as $r \to 0$. As $r \to 0$ and $R \to \infty$, we thus find that the integral $\int_\Gamma g$ converges to

$$\begin{aligned} 0 + 0 + (1 - e^{2\pi i/3}) \int_0^\infty \frac{\sqrt[3]{x}}{1+x^2} dx &= e^{\pi i/3}(e^{-\pi i/3} - e^{\pi i/3}) \int_0^\infty \frac{\sqrt[3]{x}}{1+x^2} dx \\ &= -2ie^{\pi i/3} \sin(\pi/3) \int_0^\infty \frac{\sqrt[3]{x}}{1+x^2} dx. \end{aligned}$$

But we know that $\int_\Gamma g = 2\pi i \Sigma$ for $0 < r < 1$ and $R > 1$. Thus,

$$\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2} dx = -\frac{\pi e^{-\pi i/3}}{\sin(\pi/3)} \Sigma.$$

This is almost the formulation that will be taken by the more general result. To bring it into a form using the parameter a (here $4/3$), it is convenient to introduce minus signs in numerator and denominator and present it as

$$\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2} dx = -\frac{\pi e^{-4\pi i/3}}{\sin(4\pi/3)} \Sigma.$$

For our current example we find

$$\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2} dx = \left(-\frac{\pi e^{-\pi i/3}}{\sqrt{3}/2}\right)\left(-\frac{e^{\pi i/3}}{2}\right) = \frac{\pi}{\sqrt{3}}.$$

To handle the objection that γ_1 and γ_2 lie on the boundary, consider the modified curves involving the small angle η. There we argue as follows.

On γ_1, $z = xe^{\eta i}$ and

$$\int_{\gamma_1} g = \int_r^R \frac{x^{1/3}e^{\eta i/3}}{1+x^2e^{2\eta i}}\, e^{\eta i}dx = e^{4\eta i/3}\int_r^R \frac{\sqrt[3]{x}}{1+x^2e^{2\eta i}}\, dx.$$

On γ_2, $z = xe^{(2\pi-\eta)i}$ and

$$\int_{\gamma_2} g = \int_R^r \frac{x^{1/3}e^{(2\pi-\eta)i/3}}{1+x^2e^{2(2\pi-\eta)i}}\, e^{(2\pi-\eta)i}dx = -e^{4(2\pi-\eta)i/3}\int_r^R \frac{\sqrt[3]{x}}{1+x^2e^{-2\eta i}}\, dx.$$

For each $0 < r < 1$, $R > 1$, and $0 < \eta < \pi/2$, we know

$$2\pi i\Sigma = \int_{\gamma_1} g + \int_{\gamma_R} g + \int_{\gamma_2} g + \int_{\gamma_r} g$$

Let $\epsilon > 0$. The estimates on the integrals along γ_R and γ_r are still valid and are independent of η since the only changes have been to shorten the curves somewhat. Also, we know that the improper integral we are studying converges absolutely (Use the comparison test with the convergent integral $\int_0^1 \sqrt[3]{x}\, dx + \int_1^\infty x^{-5/3}\, dx$.) Thus, we can select r and R so that

$$\left|\int_{\gamma_r} g\right| < \epsilon; \quad \left|\int_{\gamma_R} g\right| < \epsilon; \quad \left|\int_r^R \frac{\sqrt[3]{x}}{1+x^2}\, dx - \int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\, dx\right| < \epsilon$$

with the first two estimates valid for every η between 0 and $\pi/2$. On the resulting interval $[r, R]$, the integrands on γ_1 and γ_2 converge uniformly as $\eta \to 0$. Therefore we can select η small enough so that

$$\left|\int_{\gamma_1} g - \int_r^R \frac{\sqrt[3]{x}}{1+x^2}\, dx\right| < \epsilon \quad \text{and} \quad \left|\int_{\gamma_2} g - \left(-e^{8\pi i/3}\int_r^R \frac{\sqrt[3]{x}}{1+x^2}\, dx\right)\right| < \epsilon.$$

We are left with

$$\begin{aligned}\left|2\pi i\Sigma - (1-e^{8\pi i/3})\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\, dx\right| &\le \left|\int_{\gamma_R} g\right| + \left|\int_{\gamma_r} g\right| + \left|\int_{\gamma_1} g - \int_r^R \frac{\sqrt[3]{x}}{1+x^2}\, dx\right| \\ &+ \left|\int_r^R \frac{\sqrt[3]{x}}{1+x^2}\, dx - \int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\, dx\right| + \left|\int_{\gamma_2} g - \left(-e^{8\pi i/3}\int_r^R \frac{\sqrt[3]{x}}{1+x^2}\, dx\right)\right| \\ &+ \left|\left(-e^{8\pi i/3}\int_r^R \frac{\sqrt[3]{x}}{1+x^2}\, dx\right) - \left(-e^{8\pi i/3}\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\, dx\right)\right| \le 6\epsilon.\end{aligned}$$

Since this can be done for every $\epsilon > 0$, we conclude that

$$2\pi i\Sigma = (1-e^{8\pi i/3})\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\, dx = -2ie^{\pi i/3}\sin(\pi/3)\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\, dx.$$

as before. ♦

In the example just completed, we were working with $a = 4/3$ and $f(x) = 1/(1+x^2)$. The general result is as follows.

Proposition 4.3.16 *Let f be analytic on $\mathbb{C}$ except for a finite number of isolated singularities none of which lie on the strictly positive real axis (that is, all lie in the complement of the set $\{x + iy \mid y = 0 \text{ and } x > 0\}$). Let $a > 0$ with the restriction that it is not an integer, and suppose both of the following conditions hold:*

(i) *There are constants $M_1 > 0$, $R_1 > 0$, and $b > a$ such that $|f(z)| \leq M_1/|z|^b$ for $|z| \geq R_1$.*

(ii) *There are constants $M_2 > 0$, $R_2 > 0$, and d with $0 < d < a$ such that $|f(z)| \leq M_2/|z|^d$ for $0 < |z| \geq R_2$.*

Then the integral $\int_0^\infty x^{a-1} f(x)\,dx$ exists in the sense of being absolutely convergent and

$$\int_0^\infty x^{a-1} f(x)\,dx = -\frac{\pi e^{-\pi a i}}{\sin \pi a} \sum \{\text{residues of } z^{a-1} f(z)\}.$$

The sum is over the singularities of f excluding the residue at 0 and $z^{a-1} = e^{(a-1)\log z}$ using the branch with $0 < \arg z < 2\pi$.

The proof follows the solution of the example and is a typical approach for dealing with branch points. The estimates $|f(z)| \leq M_1/|z|^b$ for large $|z|$ and $|f(z)| \leq M_2/|z|^d$ for small $|z|$ serve to make the improper integral absolutely convergent and to establish the necessary estimates along γ_R and γ_r. The following corollary, which the reader should verify, discusses the case in which f is a rational function as it was in the example.

Corollary 4.3.17 *The hypotheses of Proposition 4.3.16 hold if $f(z) = P(z)/Q(z)$ for polynomials P of degree p and Q of degree q satisfying both of the following conditions:*

(i) $0 < a < q - p$.

(ii) *If n_Q is the order of the zero of Q at 0 (with the convention that $n_Q = 0$ if $Q(0) \neq 0$), and if n_P is the order of the zero of P at 0, then $n_Q - n_P < a$. (This condition holds, for instance, if $n_Q = 0$.)*

Logarithms The curve used above worked well for roots since the values on opposite sides of the cut differed by a multiplicative constant. For logarithms it may not work as well since the values on opposite sides of the cut differ by an *additive* constant of $2\pi i$. The terms involving $\log|z|$ are likely to be the same and cancel. Variations on the idea may work.

Example 4.3.18 *Show that for $p > 0$ and $q > 0$ we have*

$$\int_0^\infty \frac{\log(px)}{q^2 + x^2}\,dx = \frac{\pi}{2q}\log(pq).$$

Solution We can try working with the integrand $g(z) = \log(pz)/(q^2+z^2)$. Logarithm can be defined by $\log(\rho e^{i\theta}) = \ln(\rho) + i\theta$ with the argument θ taken in any convenient interval of length 2π. If we select $0 \le \theta < 2\pi$, then g is analytic off the positive real axis, which is a branch cut. If we try the curves used above, the terms involving the logarithm along the positive real axis cancel and we are left without a solution. If we use half circles instead as indicated in the figure, we do better. The integrand has a simple pole at iq in the upper half plane and the residue there is

$$\operatorname{Res}(g;iq) = \frac{\log(ipq)}{2iq} = \frac{\log(pq) + (\pi i/2)}{2iq} = \frac{\log(pq)}{2qi} + \frac{\pi}{4q}.$$

With $0 < r < q < R$, build a simple closed curve $\gamma = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}$ as in Figure 4.3.6 enclosing only this pole using the following segments:

- I proceeds from r to R along the "top side of" the positive real axis. That is. $z = x$, $r \le x \le R$.
- II proceeds from R around to $-R$ counterclockwise along the semicircle of radius R centered at 0.
- III proceeds from $-R$ back to $-r$ along the negative real axis. That is. $z = te^{\pi i}$.
- IV proceeds from $-r$ around to r clockwise along the semicircle of radius r centered at 0.

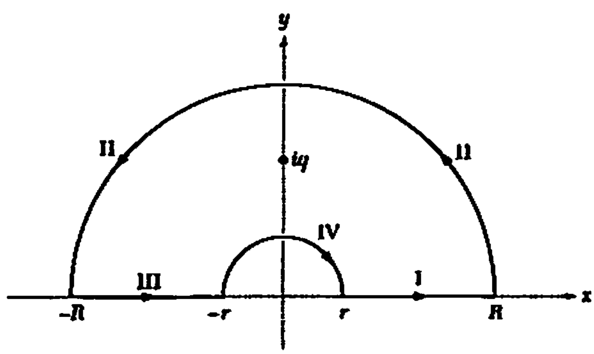


Figure 4.3.6: Contour used for Example 4.3.18.

From the Residue Theorem

$$\int_\gamma g = 2\pi i \operatorname{Res}(g;iq) = \frac{\pi}{q}\log(pq) + \frac{\pi^2}{2q}i.$$

Along the segments I and III we have

$$\int_{\mathrm{I}} g = \int_r^R \frac{\log(px)}{q^2+x^2}\,dx,$$

which tends to

$$\int_0^\infty \frac{\log(px)}{q^2+x^2}\,dx \quad \text{as } r \to 0 \text{ and } R \to \infty,$$

and

$$\int_{\mathrm{III}} g = \int_{-R}^{-r} \frac{\log(px)}{q^2+x^2}\,dx = \int_R^r \frac{\log(-px)}{q^2+(-x)^2}\,(-dx) = \int_r^R \frac{\log(px)+\pi i}{q^2+x^2}\,dx,$$

which tends to

$$\int_0^\infty \frac{\log(px)}{q^2+x^2}\,dx + \pi i \int_0^\infty \frac{1}{q^2+x^2}\,dx \quad \text{as} \quad r \to 0 \quad \text{and} \quad R \to \infty.$$

Along the arcs we have

$$\left|\int_{\mathrm{II}} g\right| = \left|\int_0^\pi \frac{\log(Re^{i\theta})}{q^2+R^2e^{2i\theta}} iRe^{i\theta}\,d\theta\right| = \left|\int_0^\pi \frac{\ln R + i\theta}{q^2+R^2e^{2i\theta}} iRe^{i\theta}\,d\theta\right| \le \frac{\ln R+\pi}{R^2-q^2}\,\pi R,$$

which tends to zero as $R \to \infty$, and

$$\left|\int_{\mathrm{IV}} g\right| = \left|\int_\pi^0 \frac{\log(re^{i\theta})}{q^2+r^2e^{2i\theta}} ire^{i\theta}\,d\theta\right| = \left|\int_0^\pi \frac{\ln r + i\theta}{q^2+r^2e^{2i\theta}} ire^{i\theta}\,d\theta\right|$$
$$\le \frac{|\ln r| + \pi}{q^2-r^2}\,\pi r,$$

which also tends to zero as $r \to 0$. We actually used L'Hôpital's rule to evaluate the limits. Putting the preceding pieces together we find that

$$\frac{\pi}{q}\log(pq) + \frac{\pi^2}{2q}i = \int_\gamma g = \int_{\mathrm{I}} g + \int_{\mathrm{II}} g + \int_{\mathrm{III}} g + \int_{\mathrm{IV}} g,$$

which tends to

$$\int_0^\infty \frac{\log(px)}{q^2+x^2}\,dx + 0 + \int_0^\infty \frac{\log(px)}{q^2+x^2}\,dx + \pi i \int_0^\infty \frac{1}{q^2+x^2}\,dx + 0$$

as $r \to 0$ and $R \to \infty$. Thus,

$$\frac{\pi}{q}\log(pq) + \frac{\pi^2}{2q}i = 2\int_0^\infty \frac{\log(px)}{q^2+x^2}\,dx + \pi i \int_0^\infty \frac{1}{q^2+x^2}\,dx.$$

Comparing real and imaginary parts we find that

$$\int_0^\infty \frac{\log(px)}{q^2+x^2}\,dx = \frac{\pi}{2q}\log(pq)$$

and

$$\int_0^\infty \frac{1}{q^2+x^2}\,dx = \frac{\pi}{2q}. \qquad \blacklozenge$$

The techniques we have developed so far are summarized in Table 4.3.1.

Table 4.3.1 **Evaluation of Definite Integrals**

1. Formula $\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \{\text{residues of } f \text{ in upper half plane}\}$

Condition No poles of $f(z)$ on real axis; finite number of poles in $\mathbb{C}$; $|f(z)| \leq M/|z|^2$ for large $|z|$

2. Formula $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx = 2\pi i \sum \{\text{residues of } P/Q \text{ in upper half plane}\}$

Condition P, Q polynomials; $\deg Q \geq 2 + \deg P$; no real zeros of Q.

3a. Formula $\int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx = I = 2\pi i \sum \{\text{residues of } e^{-i\omega z} f(z) \text{ in upper half plane}\}$

Condition $\omega < 0$; $|f(z)| \leq M/|z|$ for $|z|$ large and no poles of f on real axis; or $f(z) = P(z)/Q(z)$ where $\deg Q(z) \geq 1 + \deg P(z)$ and Q has no real zero

If $\omega > 0$, use $-\sum\{\text{residues in lower half plane}\}$

b. Formula $\int_{-\infty}^{\infty} \cos(\omega x) f(x)dx = \operatorname{Re} I$; $\int_{-\infty}^{\infty} \sin(\omega x) f(x)dx = -\operatorname{Im} I$

If $\omega > 0$, use lower half plane as above.

Condition f real on real axis.

4. Formula $\int_{0}^{2\pi} R(\cos\theta, \sin\theta)d\theta = 2\pi i \sum \{\text{residues of } f \text{ inside unit circle}\}$

$f(z) = \frac{1}{iz} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)$

Condition R rational and $R(\cos\theta, \sin\theta)$ continuous in θ. (No poles on unit circle.)

5. Formula $\int_{0}^{\infty} x^{a-1} f(x)dx = \frac{-\pi e^{-\pi a i}}{\sin(\pi a)} \sum \left\{ \begin{array}{l} \text{residues of } z^{a-1} f(z) \text{ at} \\ \text{poles of } f \text{ excluding } 0 \end{array} \right\}$

using the branch $0 < \arg z < 2\pi$.

Condition $a > 0$ and f has finite number of poles, none on positive real axis; $|f(z)| \leq M/|z|^b$, $b > a$, for $|z|$ large; and $|f(z)| \leq M/|z|^d$, $d < a$, for $|z| \to 0$

or

$f = P/Q$, and Q has no zeros on positive real axis.

$0 < a < \deg Q - \deg P$ and $n_Q - n_P < a$, where n_Q = order of the zero of Q at 0 and n_P = order of the zero of P at 0.

6. Formula $\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \left\{ \begin{array}{l} \text{residues in upper} \\ \text{half plane} \end{array} \right\} + \pi i \sum \{\text{residues on } x \text{ axis}\}$

Condition Same as entry 1 except that simple poles are allowed on x axis.

7. Formula $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx = 2\pi i \sum \left\{ \begin{array}{l} \text{residues in upper} \\ \text{half plane} \end{array} \right\} + \pi i \sum \{\text{residues on } x \text{ axis}\}$

Condition Same as entry 2 except that simple poles are allowed on x axis.

8a. Formula (i) $\omega < 0$: $\int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx = I =$

$$2\pi i \sum \left\{ \begin{array}{l} \text{residues of } e^{-i\omega z} f(z) \\ \text{in upper half plane} \end{array} \right\} + \pi i \sum \left\{ \begin{array}{l} \text{residues of } e^{-i\omega z} f(z) \\ \text{on } x \text{ axis} \end{array} \right\}$$

(ii) $\omega > 0$: $I =$

$$-2\pi i \sum \left\{ \begin{array}{l} \text{residues of } e^{-i\omega z} f(z) \\ \text{in lower half plane} \end{array} \right\} - \pi i \sum \left\{ \begin{array}{l} \text{residues of } e^{-i\omega z} f(z) \\ \text{on } x \text{ axis} \end{array} \right\}$$

Condition Same as entry 3 except that simple poles are allowed on x axis.

b. Formula $\int_{-\infty}^{\infty} \cos(\omega x) f(x)dx = \operatorname{Re} I$ $\int_{-\infty}^{\infty} \sin(\omega x) f(x)dx = -\operatorname{Im} I$

If $\omega > 0$, use lower half plane as in entry 8a.

Condition f real on real axis; simple poles allowed on real axis in entry 8a.

There are many improper integrals which converge but which do not fit the standard patterns we have developed in this section. We will see in the worked examples a few ways of modifying these techniques to meet special cases.

One very important special example is the normal probability function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

This function behaves very well along the real line, shrinking faster for large x than any rational function. But e^{-z^2} does not have good limiting behavior for large z in either half plane. Along the 45° lines, its absolute value is constantly 1, while it grows faster than any polynomial in both directions along the imaginary axis. Nevertheless, we can evaluate the integral.

Proposition 4.3.19 (Gaussian Integral) *We have*

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}.$$

This formula is important in probability and statistics and in other areas of mathematics and applications. We will meet it again in Chapter 7 where we will see a method of evaluating it using the gamma function. The most elementary method for establishing the preceding formula uses a double integral and polar coordinates. See Exercise 21 of this section or Chapter 9 of J. Marsden and M. Hoffman, *Elementary Classical Analysis*, Second Edition (New York: W. H. Freeman and Company, 1993). A method using residues is outlined in Exercise 25.[1] In the worked examples we will use this result and the methods of this section to see that the normal probability function is equal to its own Fourier transform.

Worked Examples

Some of the following integrals cannot be directly evaluated by any of the formulas that we have so far developed. The basic techniques used in the solution of this problem, however, are similar to those we have already applied.

Example 4.3.20 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{1}{1+x^{2n}}\, dx,$$

where $n \geq 1$ is a positive integer.

Solution This integral could be evaluated using Proposition 4.3.6, but we would have to consider all the poles in the upper half plane. If we use instead the contour indicated in Figure 4.3.7, we need consider only one pole.

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[1] Another residue method evaluates it by relating it to the Fresnel integrals, $\int_{-\infty}^{\infty} \sin(x^2)\, dx$ and $\int_{-\infty}^{\infty} \cos(x^2)\, dx$. This method and several others, together with historical comments, are discussed in the Internet Supplement and also in D. Mitrinovic and J. Keckic, *The Cauchy Method of Residues* (Dortrecht, The Netherlands: D. Reidel Publishing Company, 1984), pp. 158–164.

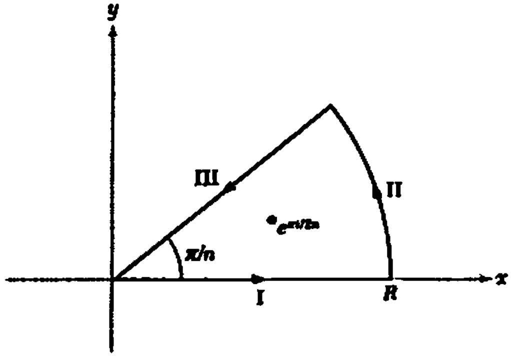

Figure 4.3.7: Contour for Worked Example 4.3.20.

The only singularity of $f(z) = 1/(1 + z^{2n})$ inside this contour is a simple pole at $e^{\pi i/2n}$, where the residue is $-e^{\pi i/2n}/2n$. Thus,

$$\begin{aligned} -\frac{\pi i}{n}e^{\pi i/2n} &= \int_{\mathrm{I}} f + \int_{\mathrm{II}} f + \int_{\mathrm{III}} f \\ &= \int_0^R \frac{1}{1+x^{2n}}dx + \int_0^{\pi/n} \frac{1}{1+R^{2n}e^{2ni\theta}} iRe^{i\theta}\,d\theta + \int_R^0 \frac{1}{1+r^{2n}e^{2\pi i}} e^{\pi i/n}\,dr \\ &= \left(1 - e^{\pi i/n}\right)\int_0^R \frac{1}{1+x^{2n}}dx + iR\int_0^{\pi/n} \frac{1}{1+R^{2n}e^{2ni\theta}} e^{i\theta}\,d\theta. \end{aligned}$$

The second integral is no larger in absolute value than $(\pi/n)R/(R^{2n} - 1)$, which goes to 0 as $R \to \infty$. Letting $R \to \infty$, we obtain

$$\int_0^\infty \frac{1}{1+x^{2n}}dx = -\frac{\pi i}{n}\frac{e^{\pi i/2n}}{1 - e^{\pi i/n}} = \frac{\pi}{2n}\csc\frac{\pi}{2n}.$$

Example 4.3.21 *Use residues to prove that*

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \frac{\pi}{2}.$$

Solution Recall that a suitable domain of $\sqrt{z^2-1}$ consists of $\mathbb{C}$ minus the half lines $x \geq 1$ and $x \leq -1$. Consider the curve γ in Figure 4.3.8, consisting of the incomplete circles of radius r around 0 and radius ϵ around 1 and -1 and horizontal lines a distance δ from the real axis.

The function $1/(z\sqrt{z^2-1})$ is defined and analytic in the region $\mathbb{C}$ minus the half lines $x \geq 1$ and $x \leq -1$ except for a simple pole at 0. To see this, consider $\sqrt{z^2-1}$ written as the product $\sqrt{z-1}\sqrt{z+1}$ in which the first factor uses a branch of the square root defined with a branch cut from $+1$ to $-\infty$ by

$$f(z) = \sqrt{z-1} = \sqrt{|z-1|}e^{i[\arg(z-1)]/2}$$

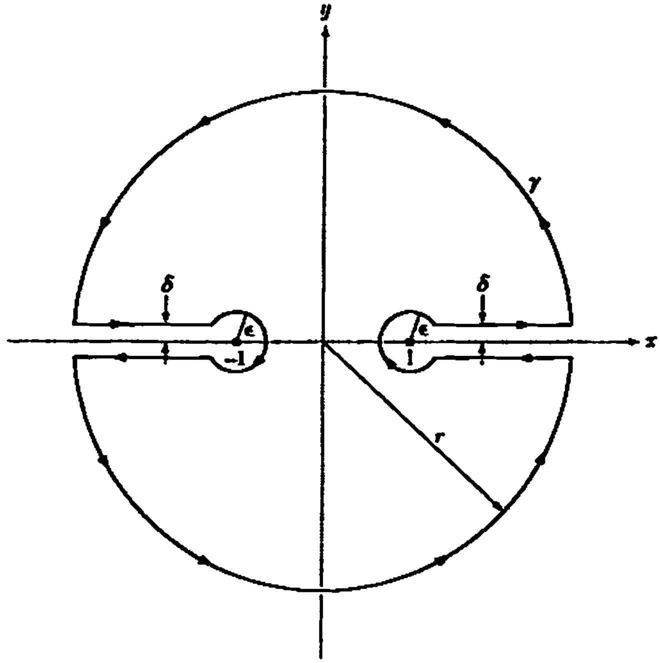

Figure 4.3.8: The curve γ used for Worked Example 4.3.21.

for $-\pi < \arg(z-1) \le \pi$ and the second factor uses a branch of square root defined with a branch cut from -1 to $+\infty$ by

$$g(z) = \sqrt{z+1} = \sqrt{|z+1|}e^{i[\arg(z+1)]/2}$$

for $0 < \arg(z+1) < \pi$ (see Figure 4.3.9).

The product $f(z)g(z)$ gives a square root for z^2-1 that appears to be analytic only on the plane with the whole real axis deleted. Crossing the branch cut for either factor changes the sign of that factor. Thus, the product changes sign if we cross the axis at a point x with $|x| > 1$. However, crossing in the region $-1 < x < 1$ changes both factors, so the product does not change but is continuous across this segment. Thus, it is analytic across this segment by the corollary to Morera's theorem established in Worked Example 2.4.17. We may use this function to define our integrand in a way that is analytic on the set $\mathbb{C}\backslash\{z \mid \operatorname{Im} z = 0 \text{ and } |\operatorname{Re} z| \ge 1\}$. By the Residue Theorem,

$$\int_\gamma \frac{dz}{z\sqrt{z^2-1}} = 2\pi i \operatorname{Res}\left(\frac{1}{z\sqrt{z^2-1}}; 0\right) = 2\pi.$$

The student should verify that (a) the integral over the incomplete circle of radius r approaches zero as $r \to \infty$ (the integrand is less than or equal to $M/|z|^2$

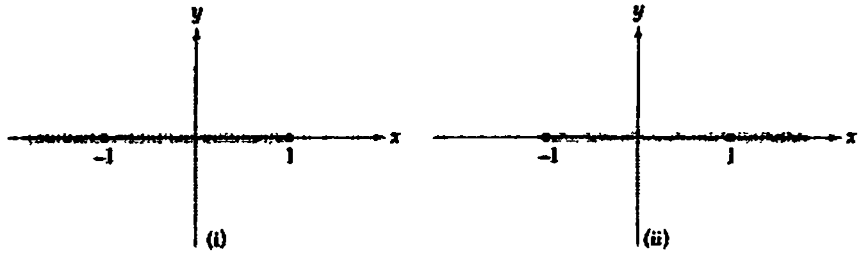

Figure 4.3.9: Branch cuts needed for $\sqrt{z^2-1}$: (i) for $\sqrt{z-1}$; (ii) for $\sqrt{z+1}$.

for $|z|$ large), (b) the integral over the incomplete circles of radius ϵ approaches zero as $\epsilon \to 0$ (the integral is bounded by a constant times $\epsilon/\sqrt{\epsilon} = \sqrt{\epsilon}$ on those circles), and (c) for fixed ϵ and r the integral over the horizontal lines approaches

$$4\int_{1+\epsilon}^{r} \frac{dx}{x\sqrt{x^2-1}}.$$

These three facts, together with the previously established fact

$$\int_\gamma \frac{dz}{z\sqrt{z^2-1}} = 2\pi,$$

show that

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \pi/2. \quad \blacklozenge$$

Our final worked example is an important Fourier transform which does not follow directly from the methods we have developed, but does from a very similar construction.

Example 4.3.22 *Show that the normal probability function,*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

is equal to its own Fourier transform. That is, $\hat{f}(\omega) = f(\omega)$ for all real ω.

Solution We can manipulate the integral giving the Fourier transform of f by competing the square in the exponent:

$$\begin{aligned}\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{-i\omega x}\,dx = e^{-\omega^2/2}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(x+i\omega)^2/2}\,dx \\ &= f(\omega)\frac{1}{\sqrt{2\pi}} I(\omega),\end{aligned}$$

where $I(\omega) = \int_{-\infty}^{\infty} e^{-(x+i\omega)^2/2}\,dx$. We know from the Gaussian integral that $I(0) = \sqrt{2\pi}$, so our conclusion will follow if we can show that $I(\omega) = I(0)$ for every real ω. To see this, consider the integral of $g(z) = e^{-z^2/2}$ around a rectangle $\Gamma = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}$ such as that shown in Figure 4.3.10.

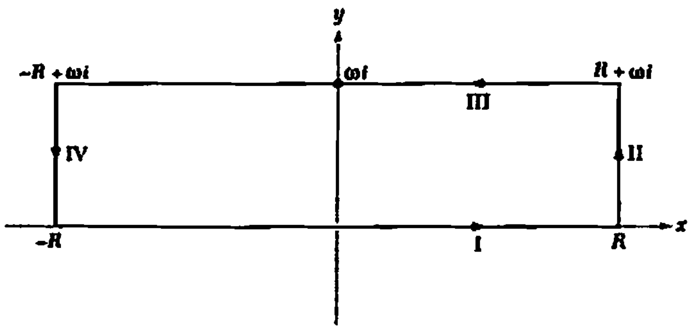

Figure 4.3.10: Contour for the Fourier transform of the normal probability function.

We know that $0 = \int_\Gamma g = \int_{\mathrm{I}} g + \int_{\mathrm{II}} g + \int_{\mathrm{III}} g + \int_{\mathrm{IV}} g$ since g is an entire function. Along the horizontal sides where $z = x$ and $z = x + i\omega$, we have $\int_{\mathrm{I}} g \to I(0)$ and $\int_{\mathrm{III}} g \to -I(\omega)$ as $R \to \infty$. Our conclusion will follow as soon as we show that $\int_{\mathrm{II}} g$ and $\int_{\mathrm{IV}} g$ tend to 0 as $R \to \infty$. We do that for the right side, II, and for $\omega > 0$. The other cases are similar.

$$\begin{aligned}\left|\int_{\mathrm{II}} g\right| &= \left|\int_0^\omega e^{-(R+iy)^2/2} i\,dy\right| = \left|\int_0^\omega e^{-R^2/2} e^{-iRy} e^{y^2/2} i\,dy\right| \\ &\le \int_0^\omega e^{-R^2/2} e^{\omega^2/2}\,dy = \omega e^{-R^2/2} e^{\omega^2/2} \to 0 \quad \text{as } R \to \infty.\end{aligned}$$

This establishes our assertion. ♦

Exercises

1. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 - 2x + 4}$.

2. Prove that $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$. *Hint:* Consider $\displaystyle\int_{-\infty}^{\infty} \frac{1 - e^{2ix}}{x^2}\,dx$ and apply Proposition 4.3.11.

3. Evaluate $\displaystyle\int_0^{\pi} \frac{d\theta}{(a + b\cos\theta)^2}$ for $0 < b < a$.

4.* Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{1 + x^6}$.

5. Evaluate $\int_0^\infty \frac{\cos mx}{1+x^4}dx$.

6. Evaluate $\int_0^\infty \frac{x^{a-1}}{1+x^3}dx$ for $0 < a < 3$.

7. Evaluate $\int_0^\infty \frac{x\sin x}{1+x^2}dx$.

8.• (a) Prove that

$$\int_{-\infty}^{\infty} \frac{\cos x}{e^x+e^{-x}}dx = \frac{\pi}{e^{\pi/2}+e^{-\pi/2}}$$

by integrating the function $e^{iz}/(e^z+e^{-z})$ around the rectangle with vertices $-r, r, r+\pi i, -r+\pi i$; let $r \to \infty$.

(b) Use the same technique to show that $\int_{-\infty}^{\infty} \frac{e^{-x}}{1+e^{-2\pi x}}dx = \frac{1}{2\sin\frac{1}{2}}$.

9. Evaluate P.V. $\int_{-\infty}^{\infty} \frac{dx}{(x-a)^2(x-1)}$ where $\operatorname{Im} a > 0$.

10. Show that $\int_0^\pi \sin^{2n}\theta d\theta = \frac{\pi(2n)!}{(2^n n!)^2}$.

11. Show that for $a > 0, b > 0$, $\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}dx = \frac{\pi}{4b^3}(1+ab)e^{-ab}$.

12.• Show that for $0 < b < 1$, $\int_0^\infty \frac{1}{x^b(x+1)}dx = \frac{\pi}{\sin(b\pi)}$.

13. Find P.V. $\int_{-\infty}^{\infty} \frac{dx}{x(x^2-1)}$.

14. Prove that $\int_0^\infty \frac{\log x}{(x^2+1)^2}dx = -\frac{\pi}{4}$.

15. Find $\int_0^1 \frac{dx}{\sqrt{x^2-1}}$, by (a) changing variables to $y = 1/(x+\sqrt{x^2-1})$ and (b) considering the curve in Figure 4.3.11 and finding the residue of a branch of $1/\sqrt{z^2-1}$ at ∞.

16.• Let $P(z)$ and $Q(z)$ be polynomials with $\deg Q(z) \geq 2 + \deg P(z)$. Show that the sum of the residues of $P(z)/Q(z)$ is zero.

17. Evaluate $\int_{-\infty}^{\infty} \frac{\cos bx}{x^2+a^2}dx$ for $a > 0, b > 0$.

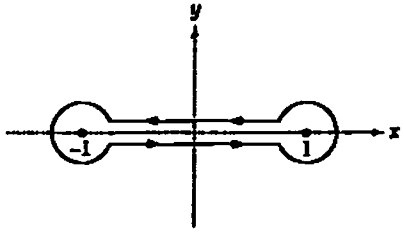

Figure 4.3.11: Contour for evaluating $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^2-1}}$.

18. Let $f(z)$ be as in formula 5 of Table 4.3.1 except allow f to have a finite number of simple poles on the (strictly) positive axis. Show that

$$\begin{aligned}\text{P.V.}\int_0^\infty x^{a-1}f(x)dx &= \frac{-\pi e^{-\pi a i}}{\sin(\pi a)}\sum \begin{array}{l}\{\text{residues of } z^{a-1}f(z) \text{ at poles}\\ \text{of } f \text{ off the nonnegative real axis}\}\end{array}\\ &+\frac{\pi e^{-\pi a i}\cos\pi a}{\sin\pi a}\sum \begin{array}{l}\{\text{residues of } (-z)^{a-1}f(z) \text{ at poles}\\ \text{of } f \text{ on the positive real axis}\}\end{array}\end{aligned}$$

19. Use Exercise 18 to show that $\text{P.V.}\displaystyle\int_0^\infty \frac{x^{a-1}}{1-x}dx = \pi\cot(\pi a)$ for $0 < a < 1$.

20.* Establish the following formulas:

(a) $\displaystyle\int_0^\pi \frac{d\theta}{1+\sin^2\theta} = \frac{\pi}{\sqrt{2}}$

(b) $\displaystyle\int_0^\infty \frac{x^2dx}{(x^2+a^2)^2} = \frac{\pi}{4a}$ for $a > 0$

(c) $\displaystyle\int_{-\infty}^\infty \frac{x^3\sin x}{(x^2+1)^2}dx = \frac{\pi}{2}e^{-1}$

(d) $\displaystyle\int_0^\infty \frac{x\sin x}{x^4+1}dx = \frac{\pi}{2}e^{-1/\sqrt{2}}\sin\frac{1}{\sqrt{2}}$

21. Prove Proposition 4.3.19 by evaluating a double integral over the whole plane in polar coordinates.

22. In Worked Example 4.3.20, can the exponent $2n$ be replaced by any other power $p \geq 2$?

23. Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{2+\cos\theta}\cos(4\theta)d\theta$ by considering the real part of the integral $\displaystyle\int_0^{2\pi} \frac{1}{2+\cos\theta}e^{4i\theta}d\theta$ and then converting to an integral around the unit circle.

24. Recall that the Fourier transform of a function $g(x)$ is defined as

$$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\omega x}\, dx.$$

Show that if f is differentiable and the integrals for $\hat{f}$ *and* $(f')\hat{}$ converge, then

$$(f')\hat{}(\omega) = \frac{1}{i\omega} \hat{f}(\omega)$$

Hint: $f(x)$ must go to 0 in both directions along the x axis. Try integrating by parts.

25. (a) Evaluate

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{-z^2+\sqrt{\pi i}z}}{e^{2\sqrt{\pi i}z}-1} dz$$

where $\sqrt{\pi i} = \sqrt{\pi} e^{i\pi/4}$ and γ_R is as shown in Figure 4.3.12.

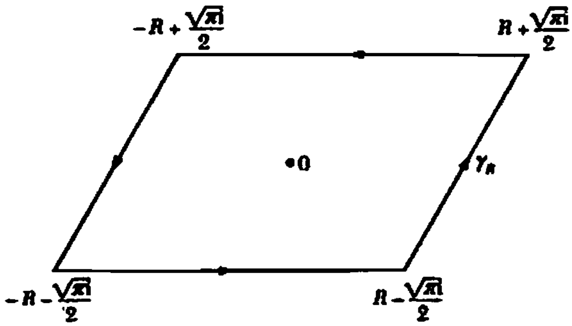

Figure 4.3.12: The contour used for $\int_{-\infty}^{\infty} e^{-x^2} dx$.

(b) Show that the integrals along the horizontal parts partially cancel to give a multiple of $\int_{-\infty}^{\infty} e^{-x^2} dx$. Use this to show $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

4.4 Evaluation of Infinite Series and Partial-Fraction Expansions

In §4.3 we saw how to use sums of residues to evaluate integrals. In this section we give a brief discussion of some applications in the other direction: using integrals to evaluate sums. For instance, we shall see that by applying these theorems we can prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This is a famous formula of Leonhard Euler, who discovered it in the eighteenth century by using other techniques.

Infinite Series We shall develop a general method for evaluating series of the form $\sum_{n=-\infty}^{\infty} f(n)$, where f is a given function. We assume f is a meromorphic function with a finite number of poles, none of which are integers. Suppose that $G(z)$ is a meromorphic function whose only poles are simple poles at the integers, where the residues are all 1. Thus at the integers, the residues of $f(z)G(z)$ are $f(n)$. Then if γ is a closed curve enclosing $-N, -N+1, \ldots, 0, 1, \ldots, N$, the Residue Theorem gives

$$\int_\gamma G(z)f(z)dz = 2\pi i \left\{ \left[\sum_{n=-N}^{N} f(n) \right] + \sum \{\text{residues of } G(z)f(z) \text{ at poles of } f\} \right\}.$$

If $\int_\gamma G(z)f(z)dz$ has a controllable limiting behavior as γ becomes large, we will have information about the limiting behavior of $\sum_{n=-N}^{N} f(n)$ as $N \to \infty$ in terms of the residues of $G(z)f(z)$ at the poles of f. A suitable $G(z)$ is $\pi \cot \pi z$.

Of course, we always have

$$\int_\gamma G(z)f(z)dz = 2\pi i \sum \{\text{all residues of } G(z)f(z) \text{ inside } \gamma\}$$

so that if some of the poles of f happen to be at integers, we need only move terms around

$$\begin{aligned} \int_\gamma G(z)f(z)dz &= 2\pi i \left\{ \sum_{n=-N}^{N} \{f(n) \mid n \text{ is not a singularity of } f\} \right. \\ &\qquad \left. + \sum \{\text{residues of } G(z)f(z) \text{ at singularities of } f\} \right\}. \end{aligned}$$

These considerations lead to the following.

Theorem 4.4.1 (Summation Theorem) *Let f be analytic in $\mathbb{C}$ except for finitely many isolated singularities. Let C_N be a square with vertices at $(N+\frac{1}{2}) \times (\pm 1 \pm i)$, $N = 1, 2, 3, \ldots$ (Figure 4.4.1). Suppose that $\int_{C_N} (\pi \cot \pi z) f(z)dz \to 0$ as $N \to \infty$. Then we have the summation formula*

$$\begin{aligned} &\lim_{N\to\infty} \sum_{-N}^{N} \{f(n) \mid n \text{ is not a singularity of } f\} \\ &\qquad = -\sum \{\text{residues of } (\pi \cot \pi z)\, f(z) \text{ at the singularities of } f\}. \end{aligned}$$

If none of the singularities of f are at integers, then $\lim_{N\to\infty} \sum_{n=-N}^{N} f(n)$ exists, is finite, and

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f(n) = -\sum \{\text{residues of } \pi \cot \pi z f(z) \text{ at singularities of } f\}.$$

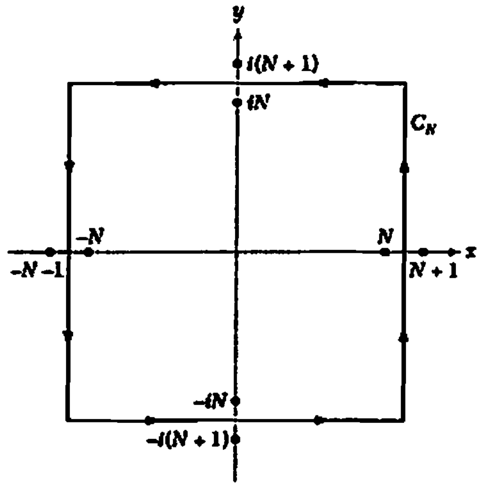

Figure 4.4.1: Contour for evaluating $\sum_{-\infty}^{\infty} f(n)$.

Proof In this argument we assume none of the singularities of f are at integers. (For the more general case simply insert the qualifying phrase in the first sum and move appropriate terms—there can be only finitely many of them.) By the Residue Theorem,

$$\begin{aligned}\int_{C_N} (\pi\cot\pi z)\, f(z)dz &= 2\pi i \sum \left\{ \begin{array}{l} \text{residues of } (\pi\cot\pi z)\, f(z) \text{ at the} \\ \text{integers } -N, -N+1, \ldots, 0, 1, \ldots, N \end{array} \right\} \\ &\quad + 2\pi i \sum \left\{ \begin{array}{l} \text{residues of } (\pi\cot\pi z)\, f(z) \\ \text{at the singularities of } f \end{array} \right\}\end{aligned}$$

for N sufficiently large so that C_N encloses all of the singularities of f. Since $\cot\pi z = (\cos\pi z)/(\sin\pi z)$ and $(\sin\pi z)' \neq 0$ at $z = n$, we see that n is a simple pole of $\cot\pi z$ and that $\operatorname{Res}(\cot\pi z; n) = (\cos\pi n)/(\pi\cos\pi n) = 1/\pi$ (use formula 4 of Table 4.1.1). Therefore, $\operatorname{Res}((\pi\cot\pi z)\, f(z); n) = \pi f(n) \operatorname{Res}(\cot\pi z; n) = f(n)$. Thus, $\sum\{\text{residues of } (\pi\cot\pi z)\, f(z) \text{ at the integers } -N, -N+1, \ldots, 0, 1, \ldots, N\} = \sum_{n=-N}^{N} f(n)$. Taking limits on both sides of the preceding displayed equation for the integral $\int_{C_N} (\pi\cot\pi z)\, f(z)dz$ and using the fact that $\int_{C_N} (\pi\cot\pi z)\, f(z)dz \to 0$ as $N \to \infty$, we obtain

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f(n) = -\sum \{\text{residues of } (\pi\cot\pi z)\, f(z) \text{ at the singularities of } f\}. \quad ■$$

It is important to notice that what we have obtained is a formula for the limit of the symmetric partial sums of $\sum_{-\infty}^{\infty} f(n)$. This is not the same as the doubly infinite series itself, which demands that the upper and lower limits converge independently:

$$\sum_{-\infty}^{\infty} f(n) = \lim_{N,M\to\infty} \sum_{-M}^{N} f(n) = \lim_{M\to\infty} \sum_{n=-M}^{-1} f(n) + \lim_{N\to\infty} \sum_{n=0}^{N} f(n).$$

If the doubly infinite series is known to converge, then our limit must give the same answer, but $\lim_{N\to\infty} \sum_{-N}^{N} f(n)$ may exist when the more general limit does not. The situation is somewhat analogous to the computation of an improper integral by a Cauchy principal value. We may check independently that the double limit exists, or as in our first example, we may be interested in a singly infinite series. Note that if f is an even function, then $\sum_{-N}^{N} f(n) = f(0) + 2\sum_{n=1}^{N} f(n)$.

The cotangent function is not the only candidate for a useful function for G. Others are $2\pi i/(e^{2\pi i z} - 1)$ and $-2\pi i/(e^{-2\pi i z} - 1)$. We indicate in the exercises a way of using $\pi \csc \pi z$ that is particularly useful for alternating series.[2]

Next we establish a criterion by which f can be judged to satisfy the hypotheses of the summation theorem (4.4.1).

Proposition 4.4.2 *Suppose f is analytic on $\mathbb{C}$ except for isolated singularities. If there are constants R and $M > 0$ such that $|zf(z)| \le M$ whenever $|z| \ge R$, then the hypotheses of the summation theorem* (4.4.1) *are satisfied.*

Proof Since $|zf(z)|$ is bounded outside R, all singularities of f are in the region $|z| \le R$. Since they are isolated, there must be a finite number of them (Why?). Furthermore, $|f(1/z)/z|$ is bounded by M in the region $|z| < 1/R$, and so 0 is a removable singularity of $f(1/z) \cdot 1/z$ and we can therefore write $f(1/z) \cdot 1/z = a_0 + a_1 z + a_2 z^2 + \ldots$ for $|z| < 1/R$; hence

$$f(z) = \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{a_2}{z^3} + \cdots$$

for $|z| > R$. By the Residue Theorem,

$$\begin{aligned}\int_{C_N} \frac{\pi \cot \pi z}{z} dz &= 2\pi i \left\{ \text{residue of } \frac{\pi \cot \pi z}{z} \text{ at } z = 0 \right\} \\ &\quad + 2\pi i \sum \left\{ \begin{array}{c} \text{residues of } \dfrac{\pi \cot \pi z}{z} \text{ at} \\ n = \pm 1, \pm 2, \ldots, \pm N \end{array} \right\}.\end{aligned}$$

Since the pole at 0 is of order 2, we can write

$$\frac{\pi \cot \pi z}{z} = \frac{b_{-2}}{z^2} + \frac{b_{-1}}{z} + b_0 + b_1 z + b_2 z^2 + \ldots.$$

[2] A more complete exposition and extensive references may be found in D. S. Mitrinovic and J. D. Kečkić, *The Cauchy Method of Residues* (Dordrecht, The Netherlands: D. Reidel Publishing Company, 1984).

The function $(\pi \cot \pi z)/z$ is even, that is, $[\pi \cot \pi(-z)]/(-z) = (\pi \cot \pi z)/z$, so uniqueness of the Laurent expansion shows that coefficients of odd powers of z are zero; in particular, $b_{-1} = 0$. But $b_{-1} = \operatorname{Res}[(\pi \cot \pi z)/z; 0]$. (Instead of this trick we could have used formula 9 of Table 4.1.1.) Also, $\operatorname{Res}[(\pi \cot \pi z)/z; n] = 1/n$ for $n = \pm 1, \pm 2, \ldots, \pm N$ (Why?), so

$$\sum \{\text{residues of } (\pi \cot \pi z)/z \text{ at } n = \pm 1, \pm 2, \ldots, \pm N\} = 0.$$

Consequently,

$$\int_{C_N} \frac{\pi \cot \pi z}{z} dz = 0.$$

Thus, we can write

$$\int_{C_N} (\pi \cot \pi z) f(z) dz = \int_{C_N} \pi(\cot \pi z) \left[f(z) - \frac{a_0}{z} \right] dz.$$

To estimate this integral, we observe that

$$f(z) - \frac{a_0}{z} = \frac{a_1}{z^2} + \frac{a_2}{z^3} + \ldots$$

for $|z| > R$. Since $a_1 + a_2 w + a_3 w^2 + \ldots$ represents an analytic function for $|w| < 1/|R|$, it is bounded, say, by M' on the closed disk $|w| \leq 1/R'$, where $R' > R$. This implies that

$$\left| f(z) - \frac{a_0}{z} \right| \leq \frac{M'}{|z|^2}$$

for $|z| \geq R'$. Suppose that N is sufficiently large that all points on C_N satisfy $|z| \geq R'$. Then

$$\left| \int_{C_N} \pi(\cot \pi z) \left[f(z) - \frac{a_0}{z} \right] dz \right| \leq \frac{\pi M' \cdot 8(N + \frac{1}{2})}{(N + \frac{1}{2})^2} \left(\sup_{z \in C_N} |\cot \pi z| \right).$$

It is readily verified that

$$\sup\{|\cot \pi z| \text{ such that } z \text{ lies on } C_N\} = \frac{e^{2\pi(N+1/2)} + 1}{e^{2\pi(N+1/2)} - 1}$$

(note that on the vertical sides, $|\cot \pi z| \leq 1$; on the horizontal sides, the maximum occurs at $x = 0$). Hence for all N sufficiently large we obtain the inequality $\sup_{z \text{ on } C_N} |\cot \pi z| \leq 2$. The previous inequality then shows that

$$\int_{C_N} \pi(\cot \pi z) \left[f(z) - \frac{a_0}{z} \right] dz$$

approaches zero as $N \to \infty$, which in turn shows that

$$\int_{C_N} (\pi \cot \pi z) f(z) dz \to 0 \quad \text{as} \quad N \to \infty. \quad \blacksquare$$

One learns in calculus that the p series $\sum_{n=1}^{\infty}(1/n^p)$ converges if $p > 1$ and diverges if $p \leq 1$, but usually with no indication of just what that sum might be. We encountered this sum in Chapter 3 as $\zeta(p)$ where ζ is the Riemann zeta function, an important ingredient in number theory. The case $p = 2$ is interesting and there are many ways to evaluate $\zeta(2)$, which is a series first summed by Euler.

Proposition 4.4.3 *The following summation formula holds:*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof We apply the summation theorem (or its corollary) with $f(z) = 1/z^2$. Since $\tan z$ has a simple zero at $z = 0$, $\cot z$ has a simple pole there. If the Laurent expansion is $\cot z = b_1/z + a_0 + a_1 z + \ldots$, then

$$\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right) = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)\left(\frac{b_1}{z} + a_0 + a_1 z + \ldots\right).$$

Multiplying, collecting terms, and comparing coefficients, we find $b_1 = 1, a_0 = 0$, and $a_1 = -\frac{1}{3}$. Thus,

$$\frac{\pi \cot \pi z}{z^2} = \frac{\pi(1/\pi z - \pi z/3 + \ldots)}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{z} \cdot \frac{1}{3} + \ldots,$$

so

$$\text{Res}\left(\frac{\pi \cot \pi z}{z^2}; 0\right) = \frac{-\pi^2}{3}.$$

Since the only singularity of f is at $z = 0$, the summation formula becomes

$$\lim_{N\to\infty}\left(\sum_{n=-N}^{-1} \frac{1}{n^2} + \sum_{n=1}^{N} \frac{1}{n^2}\right) = \frac{\pi^2}{3}$$

and, since $1/(-n)^2 = 1/n^2$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2} = \frac{\pi^2}{6}. \quad \blacksquare$$

Partial-Fraction Expansions If $f(z) = p(z)/q(z)$ is a rational function, we know a trick from algebra that is often useful in calculus: The function f can be expanded in "partial fractions" in terms of the zeros of the denominator. A meromorphic function can sometimes be thought of as somewhat like a rational function with possibly infinitely many zeros in the denominator, and one might wonder if a similar expansion is possible. Although one should not take this analogy too seriously, something along these lines can be done. First we give a specific example that shows how the summation theorem can be used and that will be used in Chapter 7. Then we will give a somewhat more general result.

Proposition 4.4.4 *Let z be any complex number not equal to an integer; then both*

$$\sum_{n=1}^{\infty}\left(\frac{1}{z-n}+\frac{1}{n}\right) \quad \textit{and} \quad \sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)$$

are absolutely convergent series and

$$\pi\cot\pi z = \frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z-n}+\frac{1}{n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right).$$

This equation can also be written

$$\pi\cot\pi z = \frac{1}{z}+\sum_{n=-\infty}^{\infty}{}'\left(\frac{1}{z-n}+\frac{1}{n}\right),$$

where the prime indicates that the term corresponding to $n=0$ is omitted.

Proof For n sufficiently large, $|z-n| > n/2$. Therefore,

$$\left|\frac{1}{z-n}+\frac{1}{n}\right| = \left|\frac{z}{(z-n)n}\right| \le \frac{2|z|}{n^2}.$$

By comparison with the convergent series

$$2|z|\cdot\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots\right),$$

we see that

$$\sum_{n=1}^{\infty}\left(\frac{1}{z-n}+\frac{1}{n}\right)$$

is absolutely convergent. Similarly,

$$\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)$$

is absolutely convergent. Fix z and consider the function $f(w) = 1/(w-z)$. This function is meromorphic; its only pole is at z, which is not an integer, and it is easy to see that $|wf(w)|$ is bounded for w sufficiently large (as in Proposition 4.3.6). By Proposition 4.4.2, we see that the hypotheses of the summation theorem are satisfied, so

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{n-z} = -\left\{\text{residue of } \frac{\pi \cot \pi w}{w-z} \text{ at } w = z\right\} = -\pi \cot \pi z.$$

We note that

$$\sum_{n=-N}^{N} \frac{1}{z-n} = \frac{1}{z} + \sum_{n=1}^{N} \left(\frac{1}{z-n} + \frac{1}{n}\right) + \sum_{n=1}^{N} \left(\frac{1}{z+n} - \frac{1}{n}\right),$$

so

$$\frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right) = \pi \cot \pi z. \quad \blacksquare$$

We could also have obtained the expansion for cotangent from the following theorem.

Theorem 4.4.5 (Partial-Fraction Theorem) *Suppose that f is meromorphic with simple poles at $a_1, a_2, a_3, \ldots$ with $0 < |a_1| \le |a_2| \le \ldots$ and residues b_k at a_k. (We are assuming f is analytic at 0.) Suppose there is a sequence $R_1, R_2, R_3, \ldots$ with the property $\lim_{n\to\infty} R_n = \infty$ and there are simple closed curves C_N satisfying*

(i) *$|z| \ge R_N$ for all z on C_N.*

(ii) *There is a constant S with length $(C_N) \le SR_N$ for all N.*

(iii) *There is a constant M with $|f(z)| \le M$ for all z on C_N and for all N. (The same M should work for all N.)*

Then

$$f(z) = f(0) + \sum_{n=1}^{\infty} \left(\frac{b_n}{z-a_n} + \frac{b_n}{a_n}\right).$$

Proof If $z_0 \ne 0$ is not a pole of f, let $F(z) = f(z)/(z-z_0)$. Then F has simple poles at z_0 and at $a_1, a_2, a_3, \ldots$. Clearly

$$\operatorname{Res}(F; z_0) = \lim_{z\to z_0} (z-z_0)F(z) = f(z_0)$$

and

$$\operatorname{Res}(F; a_n) = \lim_{z\to a_n} (z-a_n)\frac{f(z)}{z-z_0} = \frac{b_n}{a_n - z_0}.$$

By the Residue Theorem,

$$\frac{1}{2\pi i}\int_{C_N}\frac{f(z)}{z-z_0}dz = f(z_0) + \sum\left\{\frac{b_n}{a_n - z_0} \mid a_n \text{ is inside } C_N\right\}$$

and, in particular,

$$\frac{1}{2\pi i}\int_{C_N}\frac{f(z)}{z}dz = f(0) + \sum\left\{\frac{b_n}{a_n} \mid a_n \text{ is inside } C_N\right\}.$$

Subtracting these last two equations,

$$\frac{z_0}{2\pi i}\int_{C_N}\frac{f(z)}{z(z-z_0)}dz = f(z_0) - f(0) + \sum\left\{\frac{b_n}{a_n - z_0} - \frac{b_n}{a_n} \mid a_n \text{ is inside } C_N\right\}.$$

Along C_N, $|z| \geq R_N$ and $|z - z_0| \geq |R_N - |z_0||$, and so the integral in the last equality is bounded above by

$$\frac{|z_0|}{2\pi}\frac{M}{R_N|R_N - |z_0||}[\text{ length } (C_N)] \leq \frac{|z_0|MS}{2\pi|R_N - |z_0||}.$$

This goes to 0 as $N \to \infty$, and each of the a_n is eventually inside C_N. Therefore,

$$\begin{aligned} f(z_0) &= f(0) - \lim_{N\to\infty}\left(\sum\left\{\frac{b_n}{a_n - z_0} - \frac{b_n}{a_n} \mid a_n \text{ is inside } C_N\right\}\right) \\ &= f(0) - \sum_{n=1}^{\infty}\left(\frac{b_n}{a_n - z_0} - \frac{b_n}{a_n}\right) = f(0) + \sum_{n=1}^{\infty}\left(\frac{b_n}{z_0 - a_n} + \frac{b_n}{a_n}\right). \end{aligned}$$

Since this formula holds at all z_0 for which f is analytic, we have established the theorem. ■

Contours commonly used for the C_N are circles of radius R_N or large squares such as those in Figure 4.4.1. The expansion given in the partial-fraction theorem is a special case of a more general result known as the Mittag-Leffler theorem[3] named after the famous Swedish mathematician Gösta Mittag-Leffler (1846–1927).

Exercises

1.* Show that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

2. Show that $\displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^{n-1}}{(2n-1)^3}\right] = \frac{\pi^3}{32}$. (You may use the answer to Exercise 3 below).

[3] It may be found in P. Henrici, *Applied and Computational Complex Analysis*, Vol. 1 (New York: Wiley-Interscience, 1974), pp. 655–660, and (New York: Springer-Verlag, 1986).

3. Show that $\sum_{n=0}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a}\coth \pi a + \frac{1}{2a^2}$ for $a > 0$.

4. Show that $\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$. *Hint*: Start with the expansion for $\pi \cot \pi z$.

5. Develop a method for evaluating series of the form $\sum_{n=-\infty}^{\infty}(-1)^n f(n)$ where f is a meromorphic function in $\mathbb{C}$ with a finite number of poles none of which lie at the integers. In other words, develop theorems analogous to the summation theorem (4.4.1) and Proposition 4.4.2. *Hint*: $\pi/\sin \pi z$ has poles at the integers with $\operatorname{Res}(\pi/\sin \pi z; n) = (-1)^n$. Discuss how you would handle the summation if some of the poles of f did lie at the integers; see Proposition 4.4.4.

6. Show that if $2z-1$ is not an integer, then

$$\frac{1}{\cos \pi z} = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{2z-1}{(2z-1)^2-4n^2} + \frac{4}{1-4n^2}\right].$$

Hint: $\cos(x+iy) = \cos x \cosh y + i \sin x \sinh y$. Use the square with corners $\pm N \pm Ni$ for C_N given in the partial-fraction theorem (4.4.5). Finally, combine the n and $-n$ terms.

7.* Use the partial-fraction theorem to show that

$$\cot z = \frac{1}{z} + \sum{}' \left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2};$$

where $\sum'$ means the sum is over all $n \neq 0$.

8. Prove that $1 - 1/2^2 + 1/3^2 - 1/4^2 + \ldots = \pi^2/12$.

9.* Try to evaluate the sum $\sum_{n=1}^{\infty}(1/n^3)$. (This problem is a bit open ended; don't be discouraged if you are not successful.)

Review Exercises for Chapter 4

1.* Evaluate $\int_0^{2\pi} \frac{d\theta}{2-\sin\theta}$.

2. Evaluate $\int_\gamma \frac{1}{(z-1)(z-2)}dz$ where

(a) γ is the circle with center 0 and radius $1/2$ traveled once counterclockwise.

(b) Same as (a) but radius $3/2$.

(c) Same as (a) but radius 5/2.

3. Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$.

4. Evaluate $\int_C z^n e^{1/z} dz$ if C is the unit circle centered at 0 and n is a positive integer.

5. Compute P. V. $\int_{-\infty}^{\infty} \frac{\sin x}{x(x+1)(x^2+1)} dx$.

6. Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^6+1} dx$.

7. Evaluate $\int_0^{\pi} \frac{d\theta}{2\cos\theta + 3}$.

8.* Let f be analytic on a region containing the closed upper half plane $\{z \mid \operatorname{Im} z \geq 0\}$. Suppose that for some constant $a > 0$, $|f(z)| \leq M/|z|^a$ for $|z|$ large. Show that for $\operatorname{Im} z > 0$,

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x-z} dx.$$

9. Evaluate $\int_0^{2\pi} \exp(e^{i\theta}) d\theta$.

10. Show that $\frac{2}{\pi} \int_0^{\infty} \frac{\sin kt}{t} dt$ is 1 if $k > 0$, is zero if $k = 0$, and is -1 if $k < 0$.

11. Evaluate $\int_{|z|=1} \frac{\cos(e^{-z})}{z^2} dz$.

12.* Show that $\int_0^{\infty} \frac{x^{m-1}}{1+x^n} dx = \frac{\pi}{n \sin(m\pi/n)}$, where $0 < m < n$.

13. Find the Laurent expansions of $f(z) = \frac{1}{(z-1)(z-2)}$ that are valid for (a) $0 < |z| < 1$ and (b) $|z| > 2$. Choose $z_0 = 0$.

14.* Show that $\int_0^{\infty} \operatorname{sech} x \, dx = \frac{\pi}{2}$. *Hint:* Consider the rectangle with corners at $(\pm R, \pm R + \pi i)$.

15. What is the radius of convergence of the Taylor series of $1/\cos z$ around $z = 0$?

16. Explain what is wrong with the following reasoning. We know that $a^z = e^{z \log a}$, so $d(a^z)/dz = (\log a)a^z$. On the other hand, $d(a^z)/dz = za^{z-1}$. Thus $za^{z-1} = a^z(\log a)$, so $z = a \log a$.

17. Find the residues of the following at each singularity:

 (a) $\dfrac{z}{1-e^{z^2}}$

 (b) $\dfrac{\sin(z^2)}{(\sin z)^2}$

 (c) $\sin(e^{1/z})$

18.* Where is $\sum_{n=0}^{\infty} z^n e^{-izn}$ analytic?

19. Let $f(z)$ have a zero of order k at z_0. Show that $\operatorname{Res}(f'/f; z_0) = k$. Find $\operatorname{Res}(f''/f'; z_0)$ and $\operatorname{Res}(f''/f; z_0)$.

20. Let f be entire and suppose that Re f is a polynomial in x, y. Prove that f is a polynomial.

21. Explain what is wrong with the following argument, then compute the residue correctly. The expansion

$$\frac{1}{z(z-1)^2} = \frac{1}{(z-1)^2} \cdot \frac{1}{1+(z-1)} = \cdots + \frac{1}{(z-1)^5} - \frac{1}{(z-1)^4} + \frac{1}{(z-1)^3}$$

is the Laurent expansion; since there is no term in $1/(z-1)$, the residue at $z = 1$ is zero.

22. Verify the maximum principle for harmonic functions and the minimum principle for harmonic functions for the harmonic function $u(x, y) = x^2 - y^2$ on $[0, 1] \times [0, 1]$.

23. Evaluate the integral $\displaystyle\int_\gamma \frac{1}{z(z-1)(z-2)} dz$, where γ is the circle centered at 0 with radius 3/2.

24. Repeat Exercise 23 but with radius 1/2.

25. Determine the radius of convergence of the following series:

 (a) $\displaystyle\sum_1^{\infty} \frac{\log(n^n)}{n!} z^n$

 (b) $\displaystyle\sum_1^{\infty} \left(1 - \frac{1}{n}\right)^n z^n$

26.* Establish the following:

$$\int_0^{\infty} \frac{\sinh ax}{\sinh \pi x} dx = \frac{1}{2} \tan \frac{a}{2} \quad \text{for} \quad -\pi < a < \pi.$$

Hint: Integrate $e^{az}/\sinh(\pi z)$ over a "square" with sides $y = 0, y = 1, x = -R, x = +R$, and circumvent the singularities at $0, i$.

27. Expand the following in Laurent series as indicated: $f(z) = \left(\frac{1}{1-z}\right)^3$

   (a) for $|z| < 1; z_0 = 0$
   (b) for $|z| > 1; z_0 = 0$
   (c) for $|z+1| < 2; z_0 = -1$
   (d) for $0 < |z-1| < \infty; z_0 = 1$

28. Show that

$$\int_0^{\pi/2} \frac{d\theta}{(a+\sin^2\theta)^2} = \frac{\pi(2a+1)}{4(a^2+a)^{3/2}} \quad \text{for} \quad a > 0.$$

29. Establish the following formulas:

   (a)* $\displaystyle\int_0^\infty \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$

   (b) $\displaystyle\int_0^\infty \frac{x^a}{x^2+b^2}\,dx = \frac{\pi b^{a-1}}{2\cos(\pi a/2)} \quad \text{for} \quad -1 < a < 1$

30. Prove that

$$\tan z = 2z\sum_0^\infty \frac{1}{(n+\frac{1}{2})^2\pi^2 - z^2} \quad \text{for} \quad z \neq (n+\frac{1}{2})\pi.$$

Hint: Start with the identity for $\cot z$ in Proposition 4.4.4 and use $\tan z = \cot z - 2\cot 2z$.

31. Prove that

$$\sum_{-\infty}^{\infty} \frac{(-1)^n}{(a+n)^2} = \pi^2 \csc(\pi a)\cot(\pi a).$$

32. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$.

33. Explain what is wrong with the following reasoning:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{\gamma_R} \frac{\sin z}{z}\,dz,$$

where γ_R is the x axis from $-R$ to R plus the circumference $\{z = Re^{i\theta} \mid 0 \leq \theta \leq \pi\}$. But $(\sin z)/z$ is analytic everywhere, including zero, so by Cauchy's Theorem,

$$\int_{\gamma_R} \frac{\sin z}{z}\,dz = 0 \quad \text{and so} \quad \int_0^\infty \frac{\sin x}{x}\,dx = 0.$$

34.* Let $f(z)$ be analytic inside and on a simple closed contour γ. For z_0 on γ, and γ differentiable near z_0, show that

$$f(z_0) = \frac{1}{\pi i}\,\text{P.V.}\int_\gamma \frac{f(\zeta)}{\zeta - z_0}\,d\zeta.$$

35. Use Exercise 34 to find sufficient conditions under which

$$f(x,0) = \frac{1}{\pi i}\,\text{P.V.}\int_{-\infty}^{\infty} \frac{f(\zeta,0)}{\zeta - x}\,d\zeta$$

for $f(x,y) = f(z)$ analytic. Deduce that

$$u(x,0) = \frac{1}{\pi}\,\text{P.V.}\int_{-\infty}^{\infty} \frac{v(\zeta,0)}{\zeta - x}\,d\zeta \quad \text{and} \quad v(x,0) = -\frac{1}{\pi}\,\text{P.V.}\int_{-\infty}^{\infty} \frac{u(\zeta,0)}{\zeta - x}\,d\zeta.$$

The functions u and v are called ***Hilbert transforms*** of one another.

36. Show that

$$\frac{1}{\sin z} = \frac{1}{z} + \sum{}'(-1)^n\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right) = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{(-1)^n}{z^2 - n^2\pi^2}$$

where $\sum'$ means the sum is taken over all $n \neq 0$.

37. Evaluate $\displaystyle\int_0^\infty \frac{\sin\omega x}{\sinh bx}\,dx$.

38. When a nonlinear oscillator is forced with a frequency ω, a measure of the oscillator's "chaotic response" is given by[4] $M = \displaystyle\int_0^\infty \operatorname{sech} bt\cos\omega t\,dt$. Show that $M = (\pi/2b)\operatorname{sech}(\omega\pi/2b)$.

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[4]See J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (New York: Springer-Verlag, 1983), §4.5.

Chapter 5

Conformal Mappings

Chapter 1 included a brief investigation of some geometric aspects of analytic functions. Now we return to this topic to develop some further techniques and applications. In the first section of the chapter, we seek to map a given region of the complex plane to another given region by a one-to-one, onto, analytic function. That such mappings always exist (under suitable hypotheses on the regions in question), at least in theory, is the statement of the famous Riemann Mapping Theorem, which is discussed in this section; the proof is available in the Internet Supplements. Using this theory, §5.2 discusses several concrete cases for which such mappings can be written explicitly.

The theory of conformal mappings has several important applications to the Dirichlet problem and to harmonic functions. These applications are used in problems of heat conduction, electrostatics, and hydrodynamics, which will be discussed in §5.3. The basic idea of such applications is that a conformal mapping can be used to map a given region to a simpler region on which the problem can be solved by inspection. By transforming back to the original region, the desired answer is obtained.

5.1 Basic Theory of Conformal Mappings

Conformal Transformations The following definition was presented in §1.5: A mapping $f : A \to B$ is called ***conformal*** if, for each $z_0 \in A$, f rotates tangent vectors to curves through z_0 by a definite angle θ and stretches them by a definite factor r. Let us also recall the following theorem proved in §1.5.

Theorem 5.1.1 (Conformal Mapping Theorem) *Let $f : A \to B$ be analytic and let $f'(z_0) \neq 0$ for each $z_0 \in A$. Then f is conformal.*

Actually, if f merely preserves angles and if certain conditions of regularity hold, then f must be analytic and $f'(z_0) \neq 0$ (see Exercise 8). Therefore, we can say that "conformal" means *analytic with a nonzero derivative.*

As an example, let $A = \{z \mid \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0\}$ and $B = \{z \mid \operatorname{Im} z > 0\}$. The map $f : A \to B$ defined by $z \mapsto z^2$ is conformal, since it is analytic and has the nonzero derivative $f'(z_0) = 2z_0$ on A. Figure 5.1.1 illustrates the theorem by showing preservation of angles in this case. If $f'(z_0) = 0$, angles need not be preserved. For instance, for the map $z \mapsto z^2$, the x and y axes intersect at an angle $\pi/2$ but the images intersect at an angle π. Such a point where $f'(z_0) = 0$ for an analytic function f is called a ***singular point***. Singular points are studied in greater detail in Chapter 6.

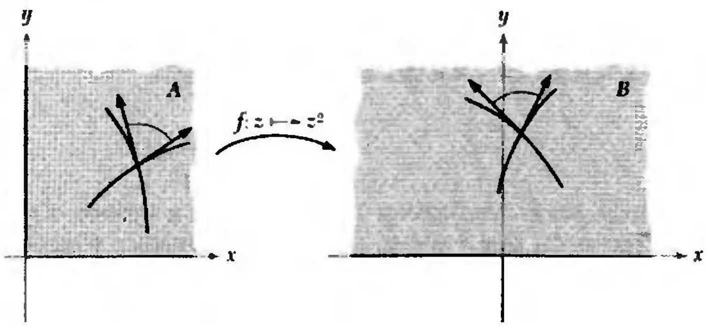


Figure 5.1.1: A conformal map.

Proposition 5.1.2

(i) *If $f : A \to B$ is conformal and bijective (one-to-one and onto), then $f^{-1} : B \to A$ is also conformal.*

(ii) *If $f : A \to B$ and $g : B \to C$ are conformal and bijective, then $g \circ f : A \to C$ is conformal and bijective.*

Proof

(i) Since f is bijective, the mapping f^{-1} exists. By the Inverse Function Theorem (1.5.10), f^{-1} is analytic with $df^{-1}(w)/dw = 1/[df(z)/dz]$ where $w = f(z)$. Thus $df^{-1}(w)/dw \neq 0$, so f^{-1} is conformal.

(ii) Certainly $g \circ f$ is bijective and analytic, since f and g are. (The inverse of $g \circ f$ is $f^{-1} \circ g^{-1}$.) The derivative of $g \circ f$ at z is $g'(f(z)) \cdot f'(z) \neq 0$. Therefore, $g \circ f$ is conformal by definition. ■

Because of the two properties in Proposition 5.1.2 (and the obvious fact that the identity map $z \mapsto z$ is conformal), we refer to the set of bijective conformal maps of a fixed region to itself as a ***group***.

Property (i) can be used to solve various problems (such as the Dirichlet problem) associated with a given region A. The method will be to find a bijective conformal map $f : A \to B$ where B is a simpler region on which the problem can be solved. To obtain the answer on A we then transform our answer from B to A by f^{-1}. The Dirichlet problem involves harmonic functions, so we should check that harmonic functions remain harmonic when we compose them with a conformal map. To do so, we prove the following result.

Proposition 5.1.3 *Let u be harmonic on a region B and let $f : A \to B$ be analytic. Then $u \circ f$ is harmonic on A.*

Proof Let $z \in A$ and $w = f(z)$. Let U be an open disk in B around w and let $V = f^{-1}(U)$. It suffices to show that $u \circ f$ is harmonic on V (Why?). By Proposition 2.5.8, there is an analytic function g on U such that $u = \mathrm{Re}\, g$. Then $u \circ f = \mathrm{Re}(g \circ f)$ (Why?), and we know that $g \circ f$ is analytic by the chain rule. Thus $\mathrm{Re}(g \circ f)$ is harmonic. ■

Riemann Mapping Theorem There is a basic but more sophisticated theorem that guarantees the existence of conformal mappings between two given regions A and B. The validity of this theorem in several special cases is verified in §5.2. The general theorem is not always of immediate practical value, because it does not tell us explicitly how to find conformal maps. Nevertheless, it is an important theorem that we should be aware of. We shall prove uniqueness here but leave existence to the Internet Supplement.[1]

Theorem 5.1.4 (Riemann Mapping Theorem) *Let A be a connected and simply connected region other than the whole complex plane. Then there exists a bijective conformal map $f : A \to D$ where $D = \{z \text{ such that } |z| < 1\}$. Furthermore, for any fixed $z_0 \in A$, we can find an f such that $f(z_0) = 0$ and $f'(z_0) > 0$. With such a specification, f is unique.*

From this result we see that if A and B are any two simply connected regions with $A \neq \mathbb{C}, B \neq \mathbb{C}$, then there is a bijective conformal map $g : A \to B$. Indeed, if $f : A \to D$ and $h : B \to D$ are conformal, we can set $g = h^{-1} \circ f$ (see Figure 5.1.2). Two regions A and B are called ***conformally equivalent*** if there is a bijective conformal map from A to B. Thus, the Riemann Mapping Theorem implies that two simply connected regions (unequal to $\mathbb{C}$) are conformally equivalent.

Proof of Uniqueness in Theorem 5.1.4 Suppose f and g are bijective conformal maps of A onto D with $f(z_0) = g(z_0) = 0, f'(z_0) > 0$, and $g'(z_0) > 0$. We want to show that $f(z) = g(z)$ for all z in A. To do this, define h on D by $h(w) = g(f^{-1}(w))$ for $w \in D$. Then $h : D \to D$ and $h(0) = g(f^{-1}(0)) = g(z_0) = 0$. By the Schwarz Lemma 2.5.7, $|h(w)| \leq |w|$ for all $w \in D$. Exactly the same argument applies to $h^{-1} = f \circ g^{-1}$, so that $|h^{-1}(\zeta)| \leq |\zeta|$ for all $\zeta \in D$. With $\zeta = h(w)$,

[1] See also E. Hille, *Analytic Function Theory*, Vol. II (Boston: Ginn and Company, 1959) p. 322, or L. Ahlfors, *Complex Analysis* (New York: McGraw-Hill, 1966), p. 222.

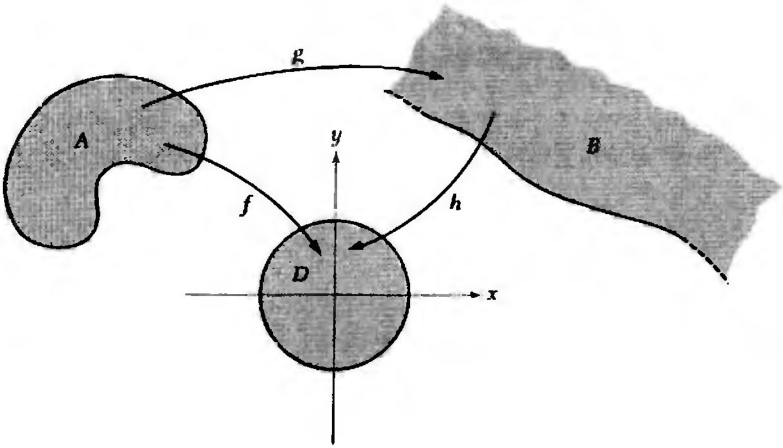


Figure 5.1.2: To transform A to B we compose h^{-1} with f.

this gives $|w| \leq |h(w)|$. Combining these inequalities, we get $|h(w)| = |w|$ for all $w \in D$. The Schwarz Lemma now tells us that $h(w) = cw$ for a constant c with $|c| = 1$. Thus, $cw = g(f^{-1}(w))$. With $z = f^{-1}(w)$ we obtain $cf(z) = g(z)$ for all $z \in A$. In particular, $cf'(z_0) = g'(z_0)$. Since both $f'(z_0)$ and $g'(z_0)$ are positive real numbers, so is c. Thus, $c = 1$ and so $f(z) = g(z)$, as desired. ∎

The condition $f'(z_0) > 0$ is equivalent to saying that $f'(z_0)$ lies on the positive real axis; that is, $\arg f'(z_0) = 0$. Using the preceding argument, one can modify the uniqueness assertion so that $f(z_0)$ and $\arg f'(z_0)$ are specified. The student is asked to prove this in Exercise 7.

Here is another useful fact about conformal maps. Let A and B be two (connected) regions with boundaries $\mathrm{bd}(A)$ and $\mathrm{bd}(B)$. Suppose that $f : A \to f(A)$ is conformal.

> *If $f(A)$ has boundary $\mathrm{bd}(B)$ and if, for some $z_0 \in A$, we have $f(z_0) \in B$, then $f(A) = B$. In other words, to determine the image of a conformal map, we merely need to check the boundaries and a single point inside.*

To prove this we argue as follows. Since B is open, $B \cap \mathrm{bd}(B) = \varnothing$. The closure of B is $B \cup \mathrm{bd}(B)$, so we can decompose the plane as a disjoint union $\mathbb{C} = B \cup \mathrm{bd}(B) \cup \mathrm{ext}\, B$, where $\mathrm{ext}\, B$ is open. Since f' never vanishes on A, the Inverse Function Theorem shows that $f(A)$ is open. Thus, $f(A) \cap \mathrm{bd}(f(A)) = \varnothing$. But $\mathrm{bd}(f(A)) = \mathrm{bd}(B)$, so $f(A)$ is contained in the union of the disjoint open sets B and $\mathrm{ext}\, B$. Since f is continuous on the connected set A, the set $f(A)$ is connected. Therefore from the definition of connectedness (see Definition 1.4.12), either $f(A) \subset B$ or $f(A) \subset \mathrm{ext}\, B$. Because $f(z_0) \in B$, we must have $f(A) \subset B$. Since $f(A)$ is open, it is open relative to B. Finally,

$$\begin{aligned} f(A) &= f(A) \cap B = [f(A) \cap B] \cup [\mathrm{bd}(B) \cap B] \\ &= [f(A) \cap B] \cup [\mathrm{bd}(f(A)) \cap B] \\ &= [f(A) \cup \mathrm{bd}(f(A)] \cap B = \mathrm{cl}(f(A)) \cap B \end{aligned}$$

so that $f(A)$ is closed relative to B. Since B is connected, $f(A) = B$ (see Proposition 1.4.13).

Simple connectivity is an essential hypothesis in the Riemann Mapping Theorem. It is easy to show (see Worked Example 5.1.7) that only a simply connected region can be mapped bijectively by an analytic map onto D. On the other hand, the annuli $0 < |a| < 1$ and $1 < |z| < 2$ are *not conformally equivalent*; see Worked Example 6.1.14.

Behavior on the Boundary The Riemann Mapping Theorem and most of our other remarks about conformal maps have discussed behavior on *regions* that are *open, connected sets*. In particular the Riemann Mapping Theorem does not say what happens on the boundary of A or of D. Many of the applications, however, involve determining behavior inside a region from information on the boundary.

In §5.2 we will look at many concrete examples involving such regions as disks, half planes, quarter planes, and so on, and the maps will usually be well behaved on the boundary. This is no accident, as the next theorem shows, but it is not automatic.

The connected and simply connected open sets to which the Riemann Mapping Theorem applies can be rather complicated. For example, consider the set A obtained by deleting from the square

$$S = \{z \mid 0 < \mathrm{Re}\, z < 2 \quad \text{and} \quad 0 < \mathrm{Im}\, z < 2\}$$

the vertical segments

$$J_n = \{z = 1/n + yi \mid 0 \le y \le 1\},$$

$n = 1, 2, 3, \ldots$ (see Figure 5.1.3). The Riemann Mapping Theorem guarantees that there is a conformal map of A onto D, but attempting to extend it continuously to the boundary of A, particularly to 0, creates problems.[2] For well-behaved regions there is a nice result, which we state without proof.

Theorem 5.1.5 (Osgood-Caratheodory Theorem) *If A_1 and A_2 are bounded simply connected regions whose boundaries γ_1 and γ_2 are simple continuous closed curves, then any conformal map of A_1 one-to-one onto A_2 can be extended to a continuous map of $A_1 \cup \gamma_1$ one-to-one onto $A_2 \cup \gamma_2$.*

[2] A detailed description of the boundary behavior of conformal maps may be found in A. I. Markushevich, *Theory of Functions of a Complex Variable*, Volume 3 (New York: Chelsea Publishing Company, 1977), Chapter 2.

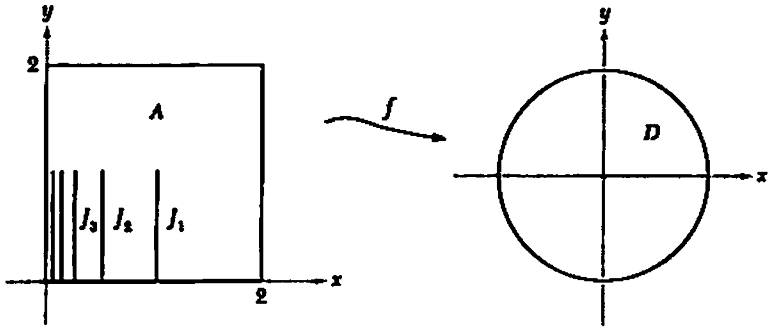


Figure 5.1.3: Even though A has a wild boundary, it can be mapped conformally to D.

Once the boundaries are known to be mapped continuously, we can get information about the regions themselves. The next theorem outlines such a procedure. The conditions are restrictive enough that we do not need to check a point $z_0 \in A$.

Theorem 5.1.6 *Let A be a bounded region with $f : A \to \mathbb{C}$ a bijective conformal map onto its image $f(A)$. Suppose that f extends to be continuous on $\mathrm{cl}(A)$ and that f maps the boundary of A onto a circle of radius R. Then $f(A)$ equals the inside of that circle. More generally, if B is a bounded region that, together with its boundary, can be mapped conformally onto the unit disk and its boundary and if f maps* $\mathrm{bd}(A)$ *onto* $\mathrm{bd}(B)$, *then $f(A) = B$.*

Proof By composing f with the conformal map h that takes B to the unit disk it is sufficient to consider the special case in which B equals $D = \{z$ such that $|z| < 1\}$. On $\mathrm{bd}(A), |f(z)| = 1$, so by the Maximum Modulus Theorem, $|f(z)| \leq 1$ on A. Since f cannot be constant, $f(A) \subset D$. In other words, at no $z \in A$ is the maximum $|f(z)| = 1$ reached. We have assumed that $f(\mathrm{bd}(A)) = \mathrm{bd}(D)$, but this is also equal to $\mathrm{bd}(f(A))$. To see this, use compactness of $\mathrm{cl}(A)$, continuity of f. and $D \cap \mathrm{bd}(D) = f(A) \cap \mathrm{bd}(f(A)) = \varnothing$. Thus our earlier argument applies to show that $f(A) = D$. ■

Worked Examples

Example 5.1.7 *Find a bijective conformal map that takes a bounded region to an unbounded region. Can you find one that takes a simply connected region to a region that is not simply connected?*

Solution Consider $f(z) = 1/z$ on $A = \{z \mid 0 < |z| < 1\}$. Clearly, A is bounded. Also, $B = f(A) = \{z$ such that $|z| > 1\}$; f is conformal from A to B and has an inverse $g^{-1}(w) = 1/w$. But B is unbounded.

The answer to the second part of the question is no. If A is simply connected and $f : A \to B$ is a bijective conformal map, then B must be simply connected. To show this, let γ be a closed curve in B and let $\tilde{\gamma} = f^{-1} \circ \gamma$. Then if $H(t, s)$ is a homotopy shrinking $\tilde{\gamma}$ to a point, $f \circ H(t, s)$ is a homotopy shrinking γ to a point.

Example 5.1.8 *Consider the harmonic function* $u(x, y) = x + y$ *on the region* $A = \{z \mid 0 < \operatorname{Im} z < 2\pi\}$. *What is the corresponding harmonic function in* $B = \mathbb{C}\backslash$*(positive real axis) when* A *is transformed by* $z \mapsto e^z$*?*

Solution Let $f(z) = e^z$. We know from Chapter 1 that f is one-to-one, onto B, and that $f'(z) = e^z \neq 0$. Thus f is conformal from A to B, and therefore, by Proposition 5.1.3, the corresponding function on B is harmonic. This function is

$$\begin{aligned}\varphi(x, y) &= u(f^{-1}(x, y)) = u(\log(x + iy)) \\ &= u\left(\log \sqrt{x^2 + y^2} + i \tan^{-1} \frac{y}{x}\right) \\ &= \log \sqrt{x^2 + y^2} + \tan^{-1} \frac{y}{x},\end{aligned}$$

where $\tan^{-1}(y/x) = \arg(x + iy)$ lies in $]0, 2\pi[$. Note that to check directly that φ is harmonic would be slightly tedious, but we know it must be so by Proposition 5.1.3.

Example 5.1.9 *What is the image of the region*

$$A = \{z \mid (\operatorname{Re} z)(\operatorname{Im} z) > 1 \textit{ and } \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$$

under the transformation $z \mapsto z^2$*?*

Solution On the right half plane $\{z \mid \operatorname{Re} z > 0\}$, we know that $f(z) = z^2$ is conformal (Why?). To find the image of A we first find the image of the curve $xy = 1$. Let $w = z^2 = u + iv$. Then $u = x^2 - y^2, v = 2xy$. Thus the image of $xy = 1$ is the curve $v = 2$. We must check the location of the image of a point in A, say, $z = 2 + 2i$. Here $z^2 = 8i$, and therefore the image is the shaded region B in Figure 5.1.4.

Exercises

1. What is the image of the first quadrant under the mapping $z \mapsto z^3$?

2. Consider $f = u + iv$ where $u(x, y) = 2x^2 + y^2$ and $v = y^2/x$. Show that the curves $u =$ constant and $v =$ constant intersect orthogonally but that f is not analytic.

3.* Near what points are the following maps conformal?

 (a) $f(z) = z^3 + z^2$

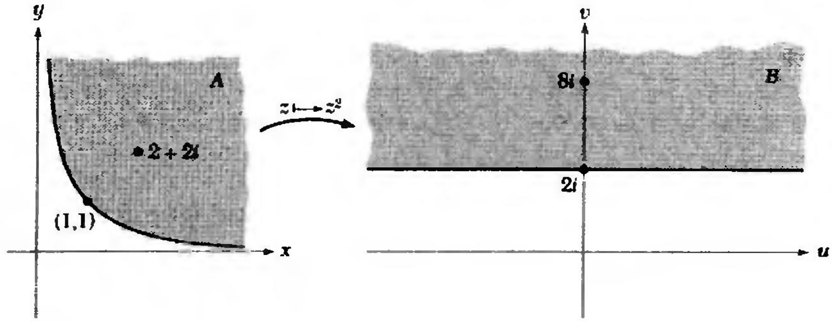


Figure 5.1.4: Image of the set A under the conformal map $z \mapsto z^2$.

(b) $f(z) = z/(1+5z)$

4. Near what points are the following maps conformal?

   (a) $f(z) = \bar{z}$

   (b) $f(z) = (\sin z)/(\cos z)$

5. Consider the harmonic function $u(x,y) = 1 - y + x/(x^2+y^2)$ on the upper half plane $y > 0$. What is the corresponding harmonic function on the first quadrant $x > 0, y > 0$, under the transformation $z \mapsto z^2$?

6. Let A and B be regions whose boundaries are smooth arcs. Let f be conformal on a region including $A \cup \mathrm{bd}(A)$ and map A onto B and $\mathrm{bd}(A)$ onto $\mathrm{bd}(B)$. Let u be harmonic on B and $u = h(z)$ for z on the boundary of B. Let $v = u \circ f$ so that v equals $h \circ f$ on the boundary of A. Prove that $\partial v/\partial n = 0$ at z_0 iff $\partial u/\partial n = 0$ at $f(z_0)$ where $z_0 \in \mathrm{bd}(A)$ and $\partial/\partial n$ denotes the derivative in the normal direction to the boundary.

7.• Let A and B be regions as in the Riemann Mapping Theorem. Given $z_0 \in A, w_0 \in B$, and an angle θ_0, and by assuming that theorem, show that there exists a conformal map $f : A \to B$ with $f(z_0) = w_0$ and $\arg f'(z_0) = \theta_0$; also show that such an f is unique.

8. Let $f : A \to B$ be a function such that $\partial f/\partial x$ and $\partial f/\partial y$ exist and are continuous. Suppose that f is one-to-one and onto and preserves angles. Prove that f is analytic and conformal. Can the map in Exercise 2 preserve all angles? *Hint*: Let $c(t)$ be a curve with $c(0) = z_0$ and let $d(t) = f(c(t))$. Prove

$$d'(t) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)c'(t) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\overline{c'(t)}$$

and examine the statement that $d'(0)/c'(0)$ has constant argument in order to establish the Cauchy-Riemann equations for f.

9. If $f : A \to B$ is bijective and analytic with an analytic inverse, prove that f is conformal.

10. Let a and b be two fixed complex numbers and let $f : \mathbb{C} \to \mathbb{C}, z \mapsto az + b$. Show that f can be written as a rotation followed by a magnification followed by a translation.

11. The Riemann Mapping Theorem explicitly excludes the case $A = \mathbb{C}$ from consideration.

    (a) Is there a conformal map of $\mathbb{C}$ one-to-one onto the unit disk D?

    (b) Is there a conformal map of D one-to-one onto $\mathbb{C}$?

12. Show that every bijective conformal transformation of $\mathbb{C}$ onto $\mathbb{C}$ is of the type described in Exercise 10.

13.• Suppose that f is a conformal map from a bounded region A onto an unbounded region B. Show that f cannot be extended in such a way as to be continuous on $A \cup \mathrm{bd}(A)$. *Note*: The full force of conformality is not needed in this problem.

5.2 Fractional Linear and Schwarz-Christoffel Transformations

This section investigates some ways of obtaining *specific* conformal maps between two given regions. No general prescription can be given for obtaining these maps; however, after a little practice the student will be able to combine fractional linear transformations (studied in this section) with other familiar transformations (like z^2, e^z, or $\sin z$) and thus be able to handle many useful situations. To aid in this effort, some common transformations are illustrated in Figures 5.2.10 and 5.2.11 at the end of this section. In addition, the Schwarz-Christoffel Formula will be studied briefly, even though it yields answers that usually can be given only in terms of integrals.

Fractional Linear Transformations The simplest and one of the most useful conformal mappings will be discussed first. A ***fractional linear transformation*** (also called a bilinear transformation or Möbius transformation) is a mapping of the form

$$T(z) = \frac{az + b}{cz + d}$$

where a, b, c, d are fixed complex numbers. We shall assume that $ad - bc \neq 0$ because otherwise T would be a constant (Why?) and we want to omit that case. Some properties of these transformations will be developed in the next four propositions.

Proposition 5.2.1 *The map T defined in the preceding display is bijective and conformal from the set*

$$A = \{z \mid cz + d \neq 0, \quad \text{that is,} \quad z \neq -d/c\} \quad \text{onto} \quad B = \{w \mid w \neq a/c\}.$$

The inverse of T is also a fractional linear transformation given by

$$T^{-1}(w) = \frac{-dw + b}{cw - a}.$$

Proof Certainly T is analytic on A and $S(w) = (-dw + b)/(cw - a)$ is analytic on B. The map T will be bijective if we can show that $T \circ S$ and $S \circ T$ are the identities since this means that T has S as its inverse. Indeed, this is seen in this computation:

$$\begin{aligned} T(S(w)) &= \frac{a\left(\dfrac{-dw + b}{cw - a}\right) + b}{c\left(\dfrac{-dw + b}{cw - a}\right) + d} \\ &= \frac{-adw + ab + bcw - ab}{-cdw + bc + dcw - da} \\ &= \frac{(bc - ad)w}{bc - ad} = w. \end{aligned}$$

We can cancel because $cw - a \neq 0$ and $bc - ad \neq 0$. Similarly, $S(T(z)) = z$. Finally, $T'(z) \neq 0$ because

$$1 = \frac{d}{dz}[z] = \frac{d}{dz}[S(T(z))] = S'(T(z)) \cdot T'(z),$$

so $T'(z) \neq 0$. ■

It is sometimes convenient to adopt the convention that $T(-d/c) = \infty$ (although we must, as always, be careful to avoid the erroneous answers that we would obtain if we cancelled ∞/∞ or $0/0$). In fact, we can show that all fractional linear transformations are conformal maps of the extended plane $\bar{\mathbb{C}}$ to itself. Some special cases should be noted. For example, if $a = 1, c = 0$, and $d = 1$, we get $T(z) = z + b$, which is a translation or "shift" that merely translates by the vector b (see Figure 5.2.1). In case $b = c = 0, d = 1, T$ becomes $T(z) = az$.

This map, multiplication by a, is a rotation by $\arg a$ and magnification by $|a|$. The student should review the geometric meaning in this case. Finally, $T(z) = 1/z$ is an inversion. It is pictured in Figure 5.2.2.

Proposition 5.2.2 *Any conformal map of $D = \{z \text{ such that } |z| < 1\}$ onto itself is a fractional linear transformation of the form*

$$T(z) = e^{i\theta}\frac{z - z_0}{1 - \bar{z}_0 z}$$

for some fixed $z_0 \in D$ and $\theta \in [0, 2\pi[$; moreover, any T of this form is a conformal map of D onto D.

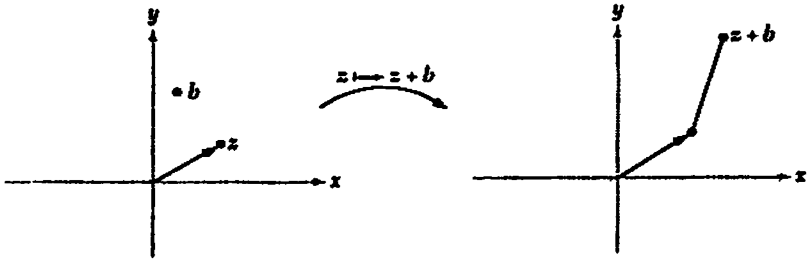


Figure 5.2.1: Translation.

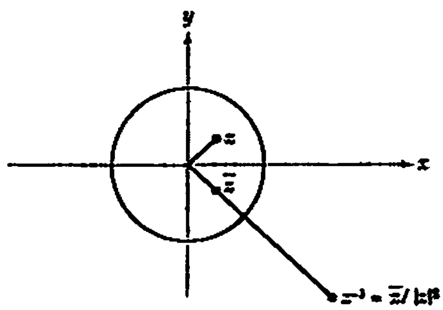


Figure 5.2.2: Inversion.

Proof First we check that for T of this form, $|z| = 1$ implies that $|T(z)| = 1$. Indeed,

$$|T(z)| = \left|\frac{z - z_0}{1 - \bar{z}_0 z}\right| = \frac{|z - z_0|}{|z||z^{-1} - \bar{z}_0|}.$$

But $|z| = 1$ and so $z^{-1} = \bar{z}$. Hence we get

$$|T(z)| = \frac{|z - z_0|}{|\bar{z} - \bar{z}_0|} = 1$$

since $|w| = |\bar{w}|$. The only singularity of T is at $z = \bar{z}_0^{-1}$, which lies outside the unit circle. Thus by the Maximum Modulus Theorem 2.5.6, T maps D to D. But by Proposition 5.2.1,

$$T^{-1}(w) = e^{-i\theta}\left[\frac{w - (-e^{i\theta} z_0)}{1 - (-e^{-i\theta}\bar{z}_0)w}\right],$$

which, since it has the same form as T, is also a map from D to D. Thus, T is conformal from D onto D.

Let $R : D \to D$ be any conformal map. Let $z_0 = R^{-1}(0)$ and let $\theta = \arg R'(z_0)$. The map T defined in Proposition 5.2.2 also has $T(z_0) = 0$ and $\theta = \arg T'(z_0)$; indeed,

$$T'(z) = e^{i\theta}\left[\frac{1-|z_0|^2}{(1-\bar{z}_0 z)^2}\right],$$

which, at $z = z_0$, equals

$$e^{i\theta}\left(\frac{1}{1-|z_0|^2}\right),$$

a real constant times $e^{i\theta}$. Thus, by uniqueness of conformal maps (see the Riemann Mapping Theorem 5.1.4 and Exercise 7, §5.1), $R = T$. ∎

The result of this is that the only way to map a disk onto itself conformally is by means of a fractional linear transformation. These transformations have two additional properties, as will be shown in the two results that follow.

Proposition 5.2.3 *Let T be a fractional linear transformation. If $L \subset \mathbb{C}$ is a straight line and $S \subset \mathbb{C}$ is a circle, then $T(L)$ is either a straight line or a circle and $T(S)$ is either a straight line or a circle.*

A line can map either to a circle or to a line. If we regard lines as circles of infinite radius, then this result can be summarized by saying that *circles transform into circles.*

Proof We can write $T = T_4 \circ T_3 \circ T_2 \circ T_1$, where

$$T_1(z) = z + d/c, T_2(z) = 1/z, T_3(z) = (bc - ad)z/c^2 \quad \text{and} \quad T_4(z) = z + a/c.$$

If $c = 0$, we merely write $T(z) = (a/d)z + b/d$. That we can write T this way is easily verified (see Exercise 11). It is obvious that T_1, T_3, and T_4 map lines to lines and circles to circles. Thus if we can verify the conclusion for $T(z) = 1/z$, the proof will be complete. We know from analytic geometry that a line or circle is determined by the equation

$$Ax + By + C(x^2 + y^2) = D$$

for constants A, B, C, D, with not all A, B, C zero. Let $z = x + iy$, suppose that $z \neq 0$, and let $1/z = u + iv$ so that $u = x/(x^2 + y^2)$ and $v = -y/(x^2 + y^2)$. Thus the preceding equation is equivalent to

$$Au - Bv - D(u^2 + v^2) = -C,$$

which is also a line or a circle. ∎

Another property of fractional linear transformations is described in the next result.

Proposition 5.2.4 (Cross Ratios) *Given two sets of distinct points z_1, z_2, z_3 and w_1, w_2, w_3 (that is, $z_1 \neq z_2, z_1 \neq z_3, z_2 \neq z_3$ and $w_1 \neq w_2, w_2 \neq w_3, w_1 \neq w_3$, but we could have $z_1 = w_2$, and so on), there is a unique fractional linear transformation T taking $z_i \mapsto w_i, i = 1, 2, 3$. In fact, if $T(z) = w$, then*

$$\frac{w - w_1}{w - w_2} \cdot \frac{w_3 - w_2}{w_3 - w_1} = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}.$$

Instead of trying to remember the equation in this proposition, it is often easier to proceed directly (see Worked Example 5.2.12).

Proof The stated equation defines a fractional linear transformation $w = T(z)$ (Why?). By direct substitution we see that it has the desired properties $T(z_i) = w_i, i = 1, 2, 3$. (See Exercise 20.) Let us show that it is unique. Define

$$S(z) = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}.$$

Then S is a fractional linear transformation taking z_1 to 0, z_3 to 1, and z_2 to ∞. (z_2 is the singularity of S.) Let R be any other fractional linear transformation $R(z) = (az + b)/(cz + d)$ with $R(z_1) = 0, R(z_3) = 1$, and $R(z_2) = \infty$ (that is, $cz_2 + d = 0$). Then $az_1 + b = 0, cz_2 + d = 0$, and $(az_3 + b)/(cz_3 + d) = 1$. Thus we get $a = -b/z_1$ and $c = -d/z_2$, so the last condition gives $b(z_1 - z_3)/z_1 = d(z_2 - z_3)/z_2$. Substituting in R we see, after simplification (that the student should do), that $R = S$.

We use this result to prove that T is unique as follows. Let T be any fractional linear transformation taking z_i to $w_i, i = 1, 2, 3$. The fractional linear transformation ST^{-1} takes $w_1 = Tz_1$ to $0, w_3 = Tz_3$ to 1, and $w_2 = Tz_2$ to ∞. Therefore ST^{-1} is uniquely determined by the preceding computation. Hence T is uniquely determined since $T = (ST^{-1})^{-1}S$. ∎

It follows that we can use a fractional linear transformation to map any three points to any other three. Three points lie on a unique circle or line, so by Proposition 5.2.3, the transformation takes the circle (or line) through z_1, z_2, z_3 to the circle (or line) through w_1, w_2, w_3. For example, we could have the situation depicted in Figure 5.2.3.

The inside of the disk maps to one of the two half planes. To determine which, one can check to see where the center of the circle goes (or any other point, especially if the center happens to go to ∞). Another way to do this is by checking orientation. As we proceed from z_1 through z_2 to z_3, located as in the figure, we go clockwise around the circle with the disk to the right. The image must proceed from w_1 through w_2 to w_3 along the line with the image of the disk to the right as shown. The half plane that is the image can be switched by interchanging z_1 and z_3. Suppose z_1, z_2, z_3 and w_1, w_2, w_3 determine circles C_1 and C_2 bounding disks D_1 and D_2. If the fractional linear transformation taking z_1, z_2, and z_3 to w_1, w_2, and w_3 is analytic on D_1 then it must map D_1 onto D_2 and the exterior of C_1 onto

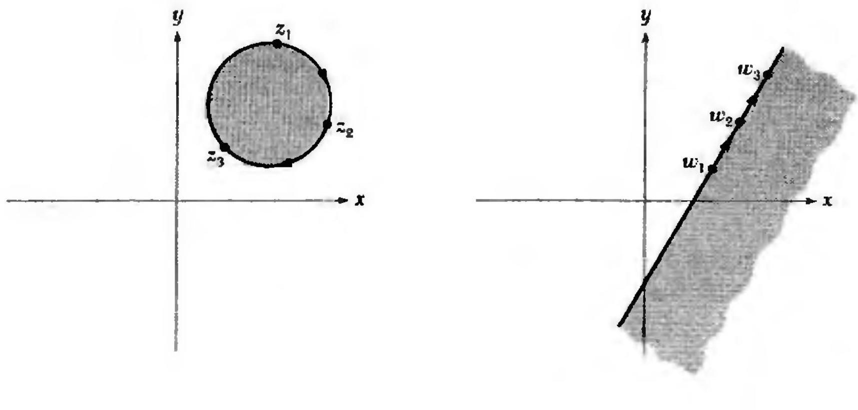


Figure 5.2.3: Effect of a fractional linear transformation.

the exterior of C_2. If the zero of the denominator is in D_1 then it maps D_1 to the exterior of C_2 together with the point at infinity. Again the situation may be determined by examining the orientation of the points along the circles and may be reversed by changing the orientation of one of the triples of points. Many of these ideas and techniques are illustrated in Worked Example 5.2.15.

As mentioned earlier, fractional linear transformations can be combined with other transformations to obtain a fairly large class of conformal maps. This is also illustrated in the worked examples.

Reflection in a Circle The idea of reflection in a circle, which was used in the proof of the Poisson formula in §2.5, can readily be generalized to circles with centers other than 0. It can be discussed purely geometrically and works well with fractional linear transformations. In the spirit of this section, straight lines can be thought of as circles of infinite radius. In this case the new notion of reflection becomes the usual reflection. In particular, reflection in the real axis is complex conjugation. The key proposition is a nice illustration of the use of complex analysis in an apparently completely geometric setting.

Proposition 5.2.5 *Let C be a circle (or straight line) and z a point not on C. Then all the circles (or lines) through z which cross C at right angles intersect each other at a single point $\tilde{z}$. (If z happens to be the center of C, then $\tilde{z}$ is the point at infinity.)*

Proof While reading this proof, please refer to Figure 5.2.4.

Let f be any fractional linear transformation that takes C to the real line and the interior of C to the upper half plane. The family of circles passing through z and crossing C at right angles must map to the family of circles that pass through $w = f(z)$ and cross the real axis at right angles, since f maps circles to circles and

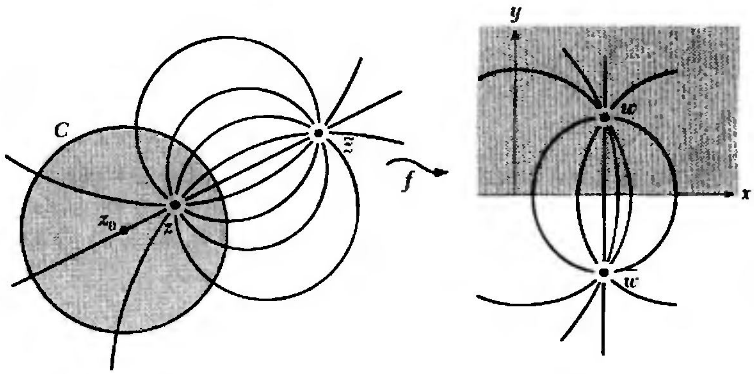


Figure 5.2.4: The circles emanating from z cross C at right angles and all pass through the point $\tilde{z}$.

preserves angles. But the latter family clearly all intersect at $\bar{w}$. Thus the first family must all cross each other at the single point $\tilde{z} = f^{-1}(\bar{w})$. ■

Definition 5.2.6 *Let C be a circle or straight line and z a point not on C. The unique point $\tilde{z}$ obtained in Proposition 5.2.5 is called the* **reflection** *of z in C. If z is on C, put $\tilde{z} = z$.*

Since fractional linear transformations take circles to circles and preserve angles, the next assertion should not be surprising.

Proposition 5.2.7 *If g is a fractional linear transformation and C is a circle (or line), then g takes the reflection of z in C to the reflection of $g(z)$ in $g(C)$.*

This assertion may be paraphrased in the following somewhat imprecise but easily remembered form: *A fractional linear transformation preserves reflection in circles*; that is

$$g(\tilde{z}) = \widetilde{g(z)}.$$

Proof The family of circles through z orthogonal to C is carried over to the family of circles through $g(z)$ orthogonal to $g(C)$ since g takes circles to circles and preserves angles. Thus the intersection of the first family, which is $\tilde{z}$, must map to the intersection of the second family, which is $\widetilde{g(z)}$. ■

In fact, reflection is "almost" a fractional linear transformation itself.

Proposition 5.2.8 *If C is a circle (or line), then the map $z \mapsto \tilde{z}$ of reflection in C is a composition of fractional linear transformations and complex conjugation.*

If C is the circle with center z_0 and radius R, then

$$\tilde{z} = \overline{\left(\frac{\bar{z}_0 z + R^2 - |z_0|^2}{z - z_0}\right)}.$$

Proof As in the proof of Proposition 5.2.5, we transfer the problem to the upper half plane $\mathcal{H}$. The idea is to make use of the fact that for $w \in \mathcal{H}$, inversion across the real axis is the same as complex conjugation; that is, $\tilde{w} = \bar{w}$. The fractional linear transformation

$$w = f(z) = i\frac{R + z - z_0}{R - z + z_0}$$

takes the circle C to the real line and its interior to $\mathcal{H}$. This map may be found by composing the map $z \mapsto (z - z_0)/R$ of C to the unit circle with the map $\zeta \mapsto -i(\zeta+1)/(\zeta-1)$ of the unit disk to $\mathcal{H}$ (see Figure 5.2.10(vi)). Using Proposition 5.2.7 and the observation that $\tilde{w} = \bar{w}$, we find that $f(\tilde{z}) = \overline{f(z)}$. Thus, $\tilde{z} = f^{-1}\left(\overline{f(z)}\right)$. Since f^{-1} is also a fractional linear transformation, this gives the general assertion. To obtain the explicit formula, we solve

$$i\frac{R + \tilde{z} - z_0}{R - \tilde{z} + z_0} = f(\tilde{z}) = \overline{f(z)} = -i\frac{R + \bar{z} - \bar{z}_0}{R - \bar{z} + \bar{z}_0}$$

for $\tilde{z}$ and get the result stated. ■

From the formula of Proposition 5.2.8 we can readily calculate another geometric description of $\tilde{z}$.

Proposition 5.2.9 *If C is a circle with center z_0 and radius R and if $z \neq z_0$, then $\tilde{z}$ is the point on the same ray from z_0 as z and is such that the product of the distances from z_0 is R^2, that is,*

$$|z - z_0| \cdot |\tilde{z} - z_0| = R^2.$$

Proof Use the fact that $|\tilde{z} - z_0| = |\bar{\tilde{z}} - \bar{z}_0|$ and compute $|z - z_0| \cdot |\bar{\tilde{z}} - \bar{z}_0|$ using the formula of Proposition 5.2.8. (See Exercise 23 and Figure 5.2.5) ■

From the preceding characterizations, most of the following proposition should now be clear.

Proposition 5.2.10

(i) $\tilde{\tilde{z}} = z$.

(ii) *The map $z \mapsto \tilde{z}$ is not conformal, but angles are preserved in magnitude and reversed in direction (just as in complex conjugation).*

(iii) *If C is a straight line, $\tilde{z}$ is the point on the line perpendicular to C through z and at an equal distance on the opposite side of C.*

(iv) *The map $z \mapsto \tilde{z}$ takes circles to circles (straight lines count as circles of infinite radius).*

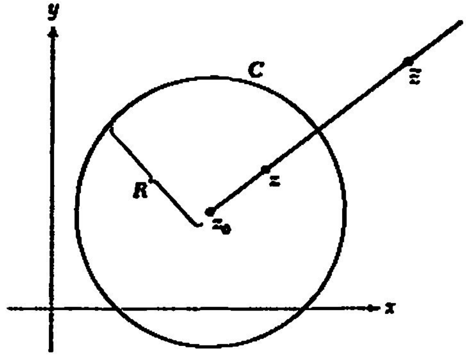


Figure 5.2.5: Reflection of z in C.

Schwarz-Christoffel Formula The Schwarz-Christoffel Formula gives an integral expression for mapping the upper half plane or unit circle to the interior of a given polygon. The case of the upper half plane will be discussed here; the case of a circle is left as an exercise.

Proposition 5.2.11 (Schwarz-Christoffel Formula) *Suppose that P is a polygon in the w plane with vertices at $w_1, w_2, \ldots, w_n$ and with exterior angles $\pi\alpha_i$, where $-1 < \alpha_i < 1$ (see Figure 5.2.6).*

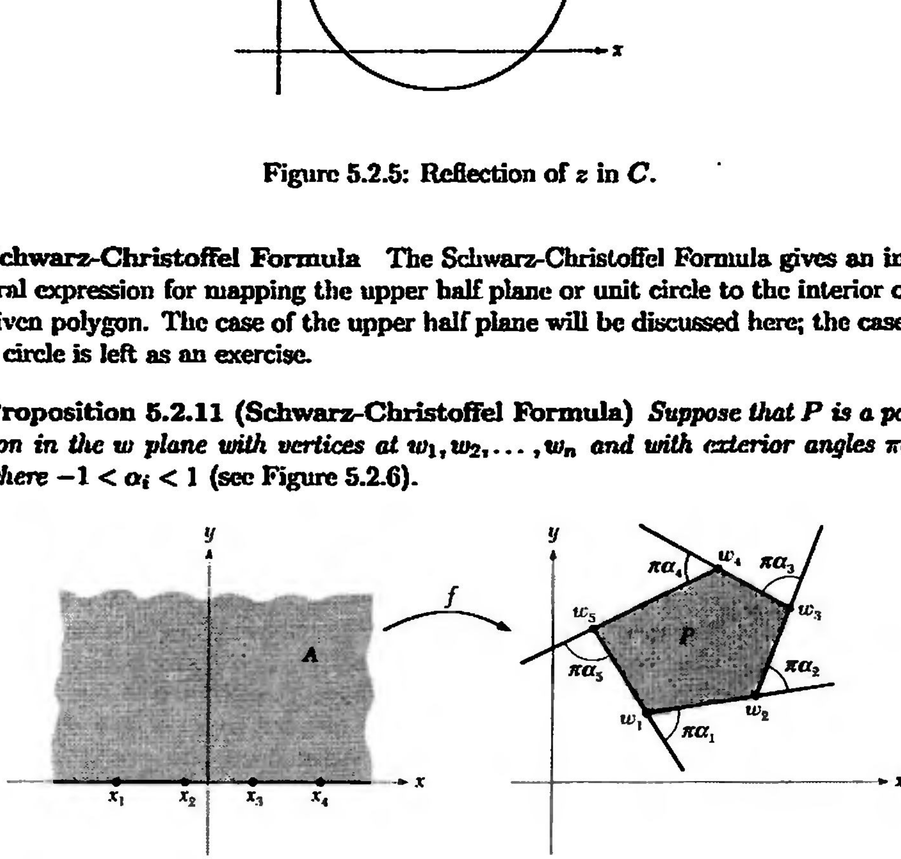


Figure 5.2.6: Schwarz-Christoffel Formula.

Then conformal maps from $A = \{z \mid \operatorname{Im} z > 0\}$ onto B, the interior of P, have the form

$$f(z) = a\left(\int_{z_0}^{z} (\zeta - x_1)^{-\alpha_1} \ldots (\zeta - x_{n-1})^{-\alpha_{n-1}} d\zeta\right) + b$$

where a and b are constants and the integration is along any path in A joining $z_0 \in A$

to z; the principal branch is used for the powers in the integrand. Furthermore, each of the following hold.

(i) *Two of the points* $x_1, \ldots, x_{n-1}$ *may be chosen arbitrarily.*

(ii) *a and b determine the size and position of P.*

(iii) $f(x_i) = w_i, i = 1, \ldots, n-1$.

(iv) *f takes the point at infinity to* w_n.

The geometric meaning of the constants a and b here is explained in more detail in the following proof. It can be shown that the function f can be extended to be continuous on the x axis and that it maps the x axis to the polygon P. However, the function f is not analytic on the x axis, because it does not preserve angles at x_i. But f will be analytic on A itself. Only the main ideas of the proof of the Schwarz-Christoffel Formula will be given here, because to make the proof absolutely precise would be rather tedious.

Sketch of Proof of Schwarz-Christoffel Formula The first step is to show that if $x_1, \ldots, x_{n-1}$ have already been chosen, then f maps the real axis to a polygon having the correct angles. Let

$$g(z) = a(z - x_1)^{-\alpha_1} \ldots (z - x_{n-1})^{-\alpha_{n-1}}$$

so that on $A, f'(z) = g(z)$. Then

$$\arg f'(z) = \arg g(z) = \arg a - \alpha_1 \arg(z - x_1) - \ldots - \alpha_{n-1} \arg(z - x_{n-1}).$$

At a point where $f'(z)$ exists, $\arg f'(z)$ represents the amount f rotates tangent vectors. Thus, as z moves along the real axis, $f(z)$ moves along a straight line for z on each of the segments $]-\infty, x_1[, \ldots,]x_i, x_{i+1}[, \ldots,]x_{n-1}, \infty[$. As z crosses $x_i, \arg f(z)$ jumps by an amount $\alpha_i \pi$. (If $z - x_i < 0, \arg(z - x_i) = \pi$; if $z - x_i > 0, \arg(z - x_i) = 0$.) Thus the real axis is mapped to a polygon with the correct angles. The last angle of the polygon is determined, since we must have $\sum_{i=1}^{n} \alpha_i \pi = 2\pi$.

Next, we adjust this polygon to obtain P. Equality of angles forces similarity of polygons only for triangles. (For example, not all rectangles are squares.) This is the basic reason why two of the points x_i may be chosen arbitrarily (three if we count the point at infinity). The positions of the other points relative to these points control the ratios of the lengths of the sides of the image polygon. By choosing the x_i correctly we thus obtain a polygon similar to P. Another way to understand this problem is to consider mapping the upper half plane to a disk. We know that this can be accomplished by a fractional linear transformation and that this transformation is completely determined by its value at three of the boundary points. (Two of the finite points and the value at infinity are specified.) Choosing a and b properly means performing a scaling, a rotation, and a translation to bring this polygon to P. ■

Worked Examples

Example 5.2.12 *Find a conformal map taking the set*

$$A = \{z \mid 0 < \arg z < \pi/2, 0 < |z| < 1\}$$

to the set

$$D = \{z \quad \text{such that } |z| < 1\}.$$

Solution The answer is not given by $z \mapsto z^4$, because this map does not map A onto D; its image omits the positive real axis.

First consider $z \mapsto z^2$. This maps A to B where B is the intersection of D and the upper half plane (Figure 5.2.7). Consulting Figure 5.2.10(iv), we next map B to the first quadrant by $z \mapsto (1+z)/(1-z)$ and square to get the upper half plane, then map $z \mapsto (z-i)/(z+i)$ to give the unit circle.

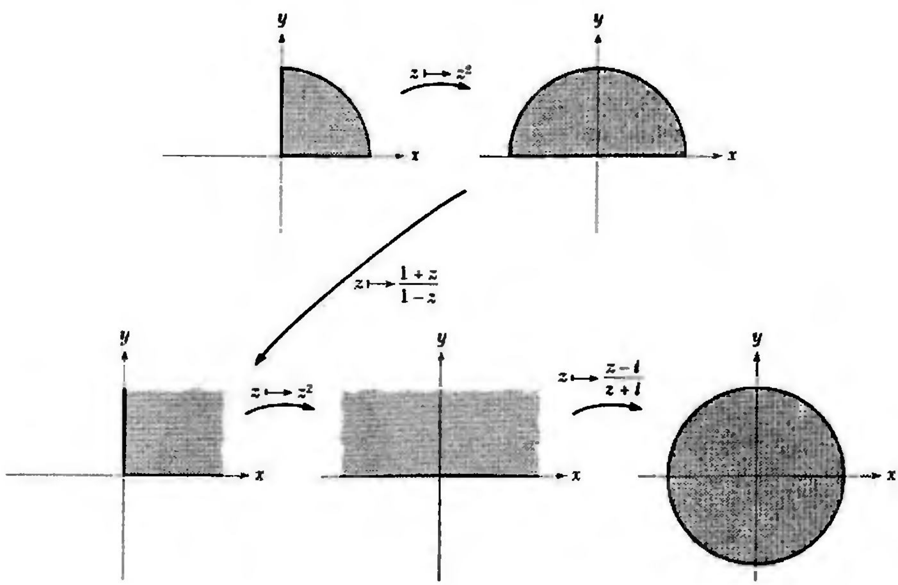


Figure 5.2.7: Successive transformations taking the quarter circle to a full circle.

Thus we obtain our transformation by successive substitution:

$$w_1 = z^2; \qquad w_2 = \frac{1+w_1}{1-w_1} = \frac{1+z^2}{1-z^2}; \quad w_3 = w_2^2 = \left(\frac{1+z^2}{1-z^2}\right)^2;$$

$$w_4 = \frac{w_3 - i}{w_3 + i} = \frac{\left(\frac{1+z^2}{1-z^2}\right)^2 - i}{\left(\frac{1+z^2}{1-z^2}\right)^2 + i}.$$

Therefore,

$$f(z) = \frac{(1+z^2)^2 - i(1-z^2)^2}{(1+z^2)^2 + i(1-z^2)^2}$$

is the required transformation.

Example 5.2.13 *Verify Figure 5.2.10(vi).*

Solution We seek a fractional linear transformation $T(z) = (az+b)/(cz+d)$ such that $T(-1) = i, T(0) = -1, T(1) = -i$. Thus, $(-a+b)/(-c+d) = i, b = -d$, and $(a+b)/(c+d) = -i$. Solving gives $-a-d = i(-c+d), b = -d, a-d = -i(c+d)$. Adding the first and last equations, we get $-2d = i(-2c)$ or $d = ic$, and subtracting gives us $a = -id$. We can set, say, $b = 1$ (because numerator and denominator can be multiplied by a constant) so that $d = -1, a = i, c = i$, and thus

$$T(z) = \frac{iz+1}{iz-1} = \frac{z-i}{z+i}.$$

We must check that $T(i)$ lies inside the unit circle. This is true because $T(i) = 0$. (If it lay outside, we would interchange $A = -1$ and $B = 0$.)

Example 5.2.14 *Find a conformal map that takes the half plane shown in Figure 5.2.8 onto the unit disk.*

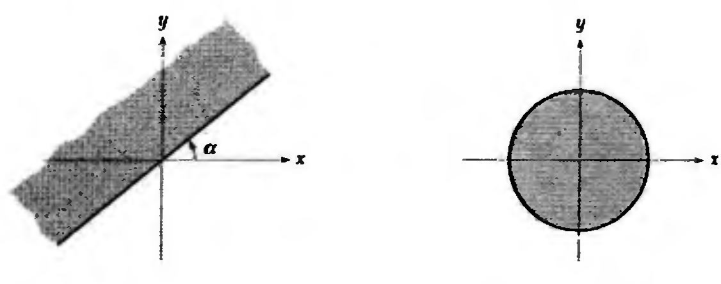


Figure 5.2.8: Mapping a rotated half plane to the disk.

Solution Consider $S(z) = e^{-i\alpha}z$. This maps the region A to the upper half plane (Why?). Then, using Figure 5.2.10(vi), we get

$$T(z) = \frac{e^{-i\alpha}z - i}{e^{-i\alpha}z + i}$$

as the required transformation.

Example 5.2.15 *Study the action of the functions $f(z) = (z-1)/(z-3)$ and $g(z) = (z+1)/(3z+1)$ on the unit circle, the unit disk, and the real axis.*

Solution First we compute the images of a few likely points:

(i) $f(1) = 0 \quad g(1) = \frac{1}{2}$ (ii) $f(i) = \frac{2}{5} - \frac{1}{5}i \quad g(i) = \frac{2}{5} - \frac{1}{5}i$

(iii) $f(-1) = \frac{1}{2} \quad g(-1) = 0$ (iv) $f(0) = \frac{1}{3} \quad g(0) = 1$

(v) $f(3) = \infty \quad g(-\frac{1}{3}) = \infty$

Thus, f takes the unit circle through $1, i, -1$ to the circle through $0, (2/5) - (i/5), 1/2$. The map g takes the unit circle to the same circle but with the orientation reversed. f takes the unit disk to the interior of the image circle while g takes it to the exterior. This can be determined by examining the images of 0 or by noticing that $g(-1/3) = \infty$.

It may not be obvious what the image circle is, but it is easier after noticing that both f and g take the real axis onto the real axis. (The line through $-1, 0, 1$ goes to the line through $1/2, 1/3, 0$ in the case of f and through $0, 1, 1/2$ in the case of g. Think about where the various pieces of the line go.) The unit circle crosses the real axis at right angles at ± 1, so the image circle must cross the axis at right angles at 0 and $1/2$. Thus, it is the circle of radius $1/4$ centered at $1/4$; check that this goes through $(2/5) - (1/5)i$. The effects of these maps are indicated in Figure 5.2.9.

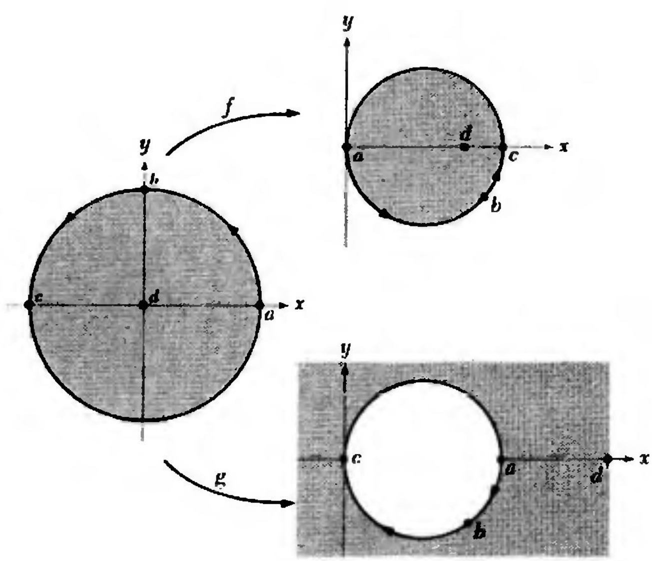


Figure 5.2.9: The maps for Worked Example 5.2.15.

Figures 5.2.10 and 5.2.11 summarize some of the common transformations for reference purposes.

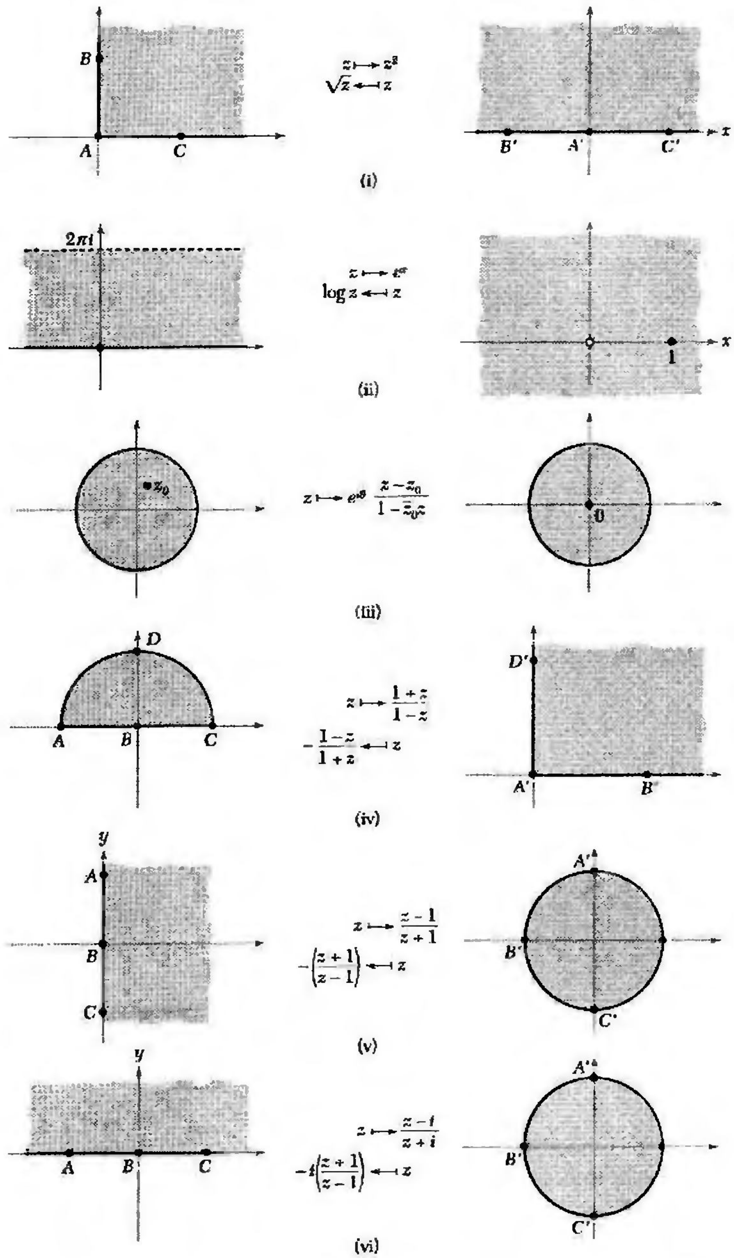


Figure 5.2.10: Some common transformations.

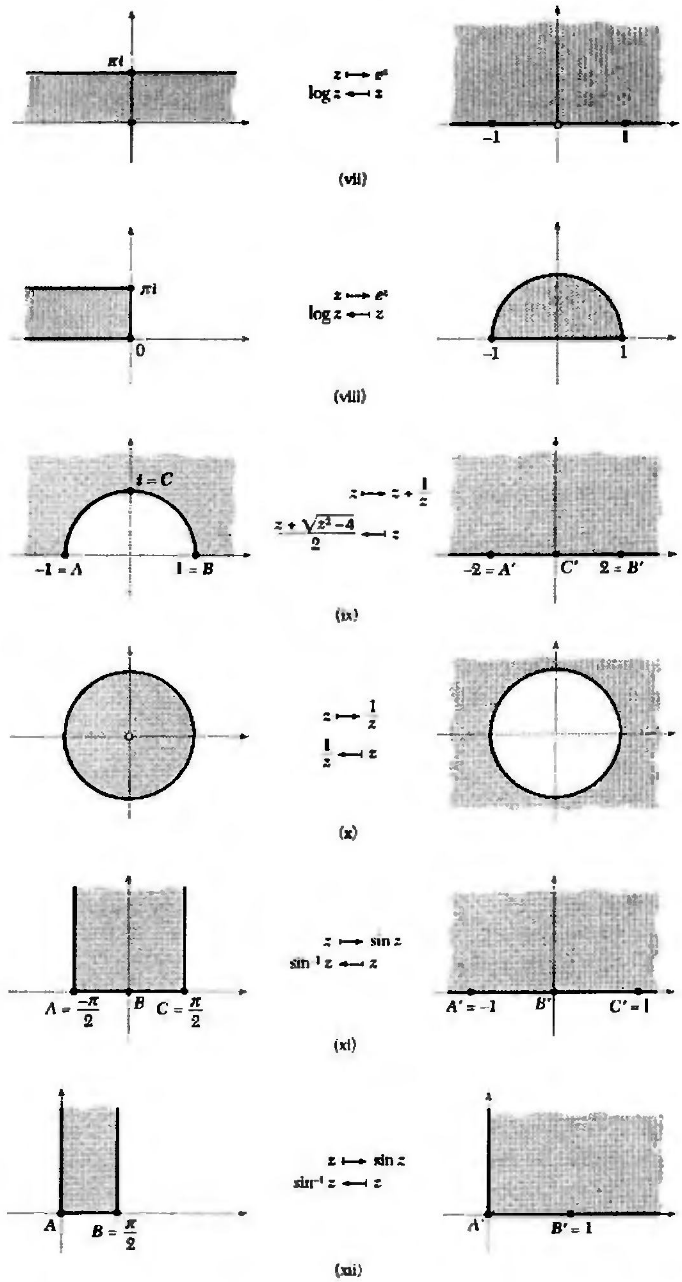


Figure 5.2.11: More common transformations.

Exercises

1. Let $f(z) = (z-1)/(z+1)$. What is the image under f of
   (a) the real line?
   (b) the circle with center 0 and radius 2?
   (c) the circle with center 0 and radius 1?
   (d) the imaginary axis?

2. Let $f(z) = (z-i)/(z+i)$. What is the image under f of
   (a) the real line?
   (b) the circle with center 0 and radius 2?
   (c) the circle with center 0 and radius 1?
   (d) the imaginary axis?

3. Find fractional linear transformations f satisfying $f(z_i) = w_i$ for $i = 1, 2, 3$ if
   (a) $z_1 = -1, z_2 = 1, z_3 = 2; w_1 = 0, w_2 = -1, w_3 = -3$.
   (b) $z_1 = -1, z_2 = 1, z_3 = 2; w_1 = -3, w_2 = -1, w_3 = 0$.

4. Find fractional linear transformations f satisfying $f(z_i) = w_i$ for $i = 1, 2, 3$ if
   (a) $z_1 = i, z_2 = 0, z_3 = -1; w_1 = 0, w_2 = -i, w_3 = \infty$.
   (b) $z_1 = i, z_2 = 0, z_3 = -1; w_1 = -i, w_2 = 0, w_3 = \infty$.

5. Find a fractional linear transformation that takes the unit disk to the upper half plane with $f(0) = 2 + 2i$.

6. Find a fractional linear transformation that takes the unit disk to the right half plane with $f(0) = 3$.

7. Find a conformal map of the unit disk onto itself that takes $\frac{1}{2}$ to $\frac{1}{3}$.

8. Find a conformal map of the unit disk onto itself that takes $\frac{1}{4}$ to $-\frac{1}{3}$.

9. Find a conformal map of the set $A = \{z \mid |z-1| < \sqrt{2} \text{ and } |z+1| < \sqrt{2}\}$ one-to-one onto the open first quadrant.

10. Map the region in Exercise 9 to the upper half plane.

11. Prove: Any fractional linear transformation with $c \neq 0$ can be written $T = T_4 \circ T_3 \circ T_2 \circ T_1$, where

$$T_1(z) = z + \frac{d}{c}, \quad T_2(z) = \frac{1}{z}, \quad T_3(z) = \frac{bc - ad}{c^2} z, \quad \text{and} \quad T_4(z) = z + \frac{a}{c}$$

    Interpret this result geometrically.

12.* Prove that if both T and R are fractional linear transformations, then so is $T \circ R$.

13. Find a conformal map of the unit disk onto itself that maps $1/2$ to 0.

14. Show that $K(z) = z/(1-z)^2$ takes the open unit disk one-to-one onto the set $\mathbb{C}\backslash\{z \mid \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z < -\frac{1}{4}\}$.

15. Find all conformal maps that take the disk of radius R and center 0 onto the unit disk.

16. Establish parts (iii), (iv), and (v) of Figure 5.2.10.

17. Prove: The most general conformal transformation that takes the upper half plane onto the unit disk is

$$T(z) = e^{i\theta}\left(\frac{z-\lambda}{z-\bar{\lambda}}\right),$$

where $\operatorname{Im} \lambda > 0$.

18.* Suppose that a, b, c, d are real and that $ad > bc$; show that $T(z) = (az + b)/(cz+d)$ leaves the upper half plane invariant. Show that every conformal map of the upper half plane onto itself is of this form.

19. Find a conformal map that takes $\{z \mid 0 < \arg z < \pi/8\}$ onto the unit disk.

20.* The ***cross ratio*** of four distinct points z_1, z_2, z_3, z_4 is defined by

$$[z_1, z_2, z_3, z_4] = \frac{z_4 - z_1}{z_4 - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}.$$

Show that every fractional linear transformation has the property that

$$[T(z_1), T(z_2), T(z_3), T(z_4)] = [z_1, z_2, z_3, z_4].$$

Hint: Use Exercise 11.

21. Let γ_1 and γ_2 be two circles that intersect orthogonally. Let T be a fractional linear transformation. What can be said about $T(\gamma_1)$ and $T(\gamma_2)$?

22.* Show that the cross ratio $[z_1, z_2, z_3, z_4]$ defined in Exercise 20 is real iff z_1, z_2, z_3, z_4 lie on a line or circle. Use this to give another proof of Proposition 5.2.3.

23. Complete the calculation in the proof of Proposition 5.2.9.

24. Show that a fractional linear transformation T that is not the identity map has at most two fixed points (that is, points z for which $T(z) = z$). Give an example to show that T need not have any fixed points. Find the fixed points of $T(z) = z/(z+1)$.

25. Conformally map the region $A = \{z \mid \operatorname{Re} z < 0, 0 < \operatorname{Im} z < \pi\}$ onto the first quadrant.

26. Conformally map $A = \{z \text{ such that } |z-1| < 1\}$ onto $B = \{z \mid \operatorname{Re} z > 1\}$.

27. Conformally map the region $\mathbb{C}\backslash\{\text{nonpositive real axis}\}$ onto the region $A = \{z \mid -\pi < \operatorname{Im} z < \pi\}$.

28.• Argue that the conformal maps that take $|z| < 1$ to the interior of a polygon with vertices $w_1, \ldots, w_n$ and points $z_1, \ldots, z_n$ on the unit circle $|z| = 1$ to the points $w_1, \ldots, w_n$ are given by

$$f(z) = a\left[\int_0^z (\zeta - z_1)^{-\alpha_1} \ldots (\zeta - z_n)^{-\alpha_n} d\zeta\right] + b,$$

where the α_i's are as in the Schwarz-Christoffel Formula.

29. Show that

$$f(z) = \int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta-1)(\zeta-c)}}$$

(where c is a positive real constant) maps the upper half plane to a rectangle. (The integrand is called an ***elliptic integral*** and generally cannot be computed explicitly.)

30. Verify part (ix) of Figure 5.2.11.

31. Is it possible to map the upper half plane conformally to a triangle using fractional linear transformations? Devise a formula that is based on the Schwarz-Christoffel Formula.

32.• Verify from the Schwarz-Christoffel Formula 5.2.11 that a conformal map from the upper half plane to $\{z \mid \operatorname{Im} z > 0 \text{ and } -\pi/2 < \operatorname{Re} z < \pi/2\}$ is $z \mapsto \sin^{-1} z$.

33. Show that $f(z) = 4/z$ maps the region $A = \{z \text{ such that } |z-1| > 1 \text{ and } |z-2| < 2\}$ one-to-one onto the strip $B = \{z \mid 1 < \operatorname{Re} z < 2\}$.

34. Suppose C_1 and C_2 are two tangent circles with C_2 in the interior of C_1. Show that an infinite number of circles can be placed in the region between C_1 and C_2, each tangent to C_1 and C_2 and each tangent to the next as shown in Figure 5.2.12. Show that the points of tangency of these circles each with the next lie on a circle. *Hint*: Consider Exercise 33.

35. Consider a fractional linear transformation of the form

$$f(z) = a\left(\frac{z-b}{z-d}\right).$$

Show that

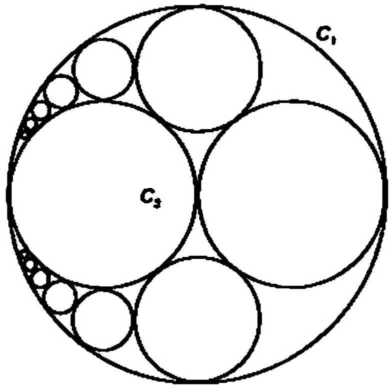


Figure 5.2.12: A fractional linear transformation can be used to pack the disk with circles.

(a) Circles through the points b and d are mapped to lines through the origin.

(b) The *circles of Apollonius* with equation $|(z - b)/(z - d)| = r/|a|$ are mapped to circles with center 0, radius r.

(c) The circles in (a) and (b), when located in the z plane, are called *Steiner circles*. Sketch them and verify that both these circles and their images meet in right angles.

5.3 Applications of Conformal Mappings to Laplace's Equation, Heat Conduction, Electrostatics, and Hydrodynamics

We are now in a position to apply the theory of conformal maps to some physical problems. In doing so we will solve the Dirichlet problem[3] and related problems for several types of two-dimensional regions. We will then apply these results to the three classes of physical problems mentioned in the title of this section. Only a meager knowledge of elementary physics is needed to understand these examples. The student is cautioned that the variety of problems that can explicitly be solved in this way is somewhat limited and that the methods discussed apply only to two-dimensional problems.

[3]The problem of finding a harmonic function on a region A whose values are specified on the boundary of A is called a *Dirichlet problem*. This problem was discussed in §2.5.

Dirichlet and Neumann Problems Recall that a function $u(x, y)$ is said to satisfy ***Laplace's equation*** (or be ***harmonic***) on a region A when

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

In addition to this condition, some boundary behavior determined by the physical problem to be solved is generally specified. These boundary conditions usually determine u uniquely. For example, the uniqueness theorem for the Dirichlet problem (2.5.12) showed that a harmonic function $u(x, y)$ whose value on the boundary of A is specified, is uniquely determined. We shall also have occasion to consider the boundary condition in which $\partial u/\partial n = \operatorname{grad} u \cdot \mathbf{n}$, the derivative in the direction normal to $\mathrm{bd}(A)$, is specified on the boundary. For $\partial u/\partial n$ to be well defined, the boundary of A should be at least piecewise smooth, so that it has a well defined normal direction. We accept as clear what is meant by the outward unit normal, $\mathbf{n}$ (see Figure 5.3.1).

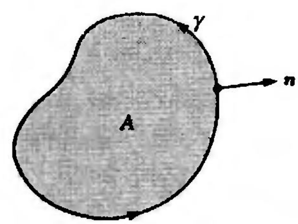


Figure 5.3.1: The Neumann problem.

The problem of finding a harmonic function u with $\partial u/\partial n$ specified on the boundary is called the ***Neumann problem***. We cannot specify $\phi = \partial u/\partial n$ arbitrarily because if such a u exists, then we claim that

$$\int_\gamma \frac{\partial u}{\partial n} = 0$$

where γ is the boundary of A. To prove this, we apply Green's theorem (see §2.2), which can be written in the divergence form (often called Gauss' Theorem)

$$\int_\gamma \mathbf{X} \cdot \mathbf{n}\, ds = \int_A \operatorname{div} \mathbf{X}\, dx\, dy$$

where X is a given vector function with components (X^1, X^2) and where the divergence of $\mathbf{X}$ is defined by

$$\operatorname{div} \mathbf{X} = \frac{\partial X^1}{\partial x} + \frac{\partial X^2}{\partial y}.$$

Applying this formula to $\mathbf{X} = \operatorname{grad} u$ gives

$$\int_\gamma \frac{\partial u}{\partial n} ds = \int_\gamma (\operatorname{grad} u) \cdot \mathbf{n}\, ds = \int_A \operatorname{div} \operatorname{grad} u\, dx\, dy = \int_A \nabla^2 u\, dx\, dy = 0$$

because $\operatorname{div} \operatorname{grad} u = \nabla^2 u = 0$. This proves our claim.

If we are given a boundary condition ϕ on γ with $\int_\gamma \phi = 0$, then it can be shown that the Neumann problem indeed has a solution. We can prove uniqueness: *The solution of the Neumann problem on a bounded simply connected region is unique up to the addition of a constant.* Let u_1 and u_2 be two solutions with $\partial u_1/\partial n = \partial u_2/\partial n$ on $\gamma = \operatorname{bd}(A)$. Let v_1 and v_2 be harmonic conjugates of u_1 and u_2 and set $u = u_1 - u_2. v = v_1 - v_2$. Now $\partial u/\partial n = 0$, and so, by Proposition 1.5.13, v is constant along γ. Thus by uniqueness of the solution to the Dirichlet problem, v is constant on A. Therefore, since $-u$ is the harmonic conjugate of v, u is constant on A as well. This proves our claim.

If the boundary values specified in the Dirichlet and Neumann problems are not continuous, the uniqueness results are still valid, in a sense, but are more difficult to obtain. However, the student is cautioned that on an unbounded region we do not have uniqueness. For example, let A be the upper half plane. Then $u_1(x, y) = x$ and $u_2 = x + y$ have the same boundary values (at $y = 0$) and are harmonic but are not equal. To recover uniqueness for unbounded regions, a "condition at ∞" must also be specified: "u bounded on all of A" is such a condition. Some of these conditions will be illustrated in the examples that are integrated into this section.

The Dirichlet and Neumann problems can also be combined; for example, u can be specified on one part of the boundary and $\partial u/\partial n$ can be specified on another.

Method of Solution The basic method for solving the Dirichlet and Neumann problems in a given region A is as follows. Take the given region A and transform it by a conformal map to a "simpler" region B on which the problem can be solved. This procedure is justified by the fact that under a conformal map f, harmonic functions are transformed again into harmonic functions (see Proposition 5.1.3). When we have solved the problem on B, we can transform the answer back to A.

For the Dirichlet problem, we are given the boundary values on $\operatorname{bd}(A)$, which get mapped to the corresponding boundary values on B. (We assume that the conformal map f is defined on the boundary.) The specification of $\partial u/\partial n$ is a bit more complicated. However, the special case $\partial u/\partial n = 0$ is easy to understand. Let $u \circ f = u_0$ be the solution we seek; that is, $u_0(x, y) = u(f(x, y))$ (see Figure 5.3.2). Then we claim that $\partial u_0/\partial n = 0$ iff $\partial u/\partial n = 0$ *on corresponding portions.* This follows because $\partial u_0/\partial n = 0$ and $\partial u/\partial n = 0$ mean that the conjugates are constant on those portions, and if v is the conjugate of u, then $v \circ f = v_0$ is the conjugate of u_0, which proves the claim. These are the only types of boundary conditions for $\partial u/\partial n$ that will be dealt with in this text.

To use this method we need to be able to solve the problem in some simple region B. We already saw in §2.5 that the unit disk is suitable for this purpose because in that case we have the Poisson formula for the solution of the Dirichlet

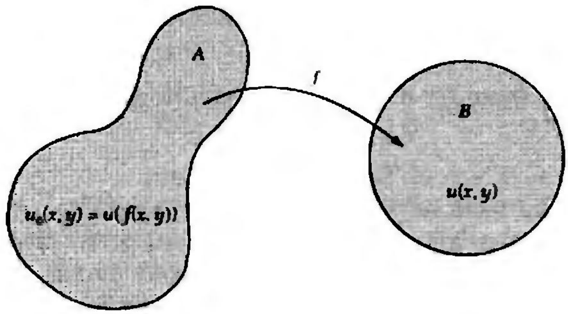


Figure 5.3.2: Transformation of harmonic functions.

problem. However, we can sometimes get more explicit solutions than those yielded by that formula.

The following situation is used to illustrate the method and will be used in subsequent examples. We consider the upper half plane H and the problem of finding a harmonic function that takes the constant boundary values c_0 on the interval $]-\infty, x_1[$, the value c_1 on $]x_1, x_2[, \ldots,$ and c_n on $]x_n, \infty[$ where $x_1 < x_2 < \ldots < x_n$ are points on the real axis. We claim that a solution is given by the ***standard upper half-plane solution***:

$$u(x,y) = c_n + \frac{1}{\pi}[(c_{n-1} - c_n)\theta_n + \ldots + (c_0 - c_1)\theta_1]$$

where $\theta_1, \ldots, \theta_n$ are as indicated in Figure 5.3.3; $0 \le \theta_i \le \pi$.

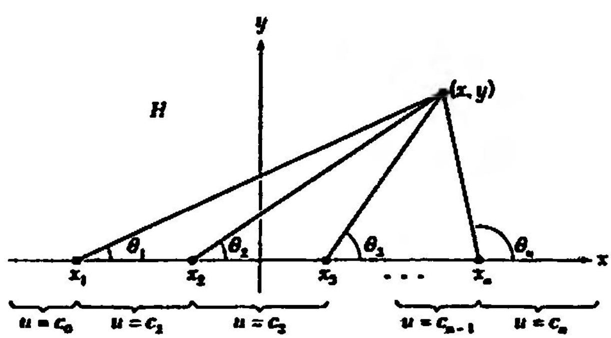


Figure 5.3.3: Solving this Dirichlet problem in the upper half plane.

To see this, note that u is the real part of the analytic function

$$c_n + \frac{1}{\pi i}[(c_{n-1} - c_n)\log(z - x_n) + \ldots + (c_0 - c_1)\log(z - x_1)].$$

Also, on $]x_i, x_{i+1}[$, u reduces to c_i. (The student should check this.) As mentioned previously, the Dirichlet problem does not have a unique solution, so the question arises: Why was *this* solution chosen? Another solution could have been obtained by adding $u(x, y) = y$ to the previous solution. The answer is that the u that is given by the standard upper half plane solution is bounded (Why?) and this answer is physically reasonable in many examples.

Thus if a problem can be transformed to the one described by Figure 5.3.3, we can use the standard solution. This will be done in the examples in this section.

Heat Conduction Physical laws tell us that if a two-dimensional region is maintained at a steady temperature T (that is, a temperature not changing in time, accomplished by fixing the temperature at the walls, or by insulating them), then T should be harmonic.[4]

The negative of the gradient of T represents the direction in which heat flows. Thus, by using Proposition 1.5.13, we can *interpret the level surfaces of the harmonic conjugate ϕ of T as the lines along which heat flows and the temperature is decreasing*. Lines of constant T are called ***isotherms***; lines on which the conjugate ϕ are constant are called ***flux lines*** (Figure 5.3.4).

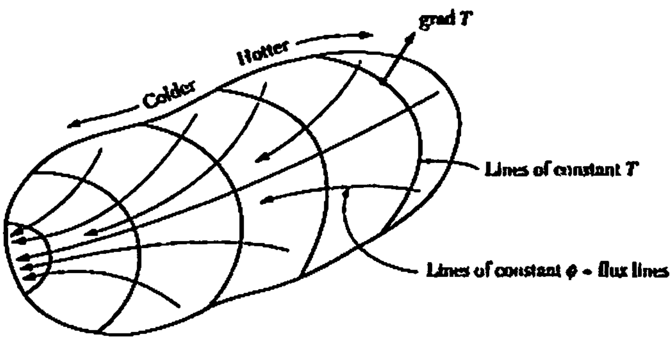


Figure 5.3.4: Heat conduction.

[4]This is a consequence of conservation of energy and Gauss' Theorem. The heat flows in the direction of the vector field $= \kappa\nabla T$(κ = conductivity) and the energy density is $c\rho T$(c = specific heat, ρ = density). Then the law of conservation of energy states that the rate of change of energy in any planar region V equals the rate at which energy enters V; that is,

$$\frac{d}{dt}\int_V c\rho T\,dx\,dy = -\int_{\mathrm{bd}(V)} -\kappa\nabla T\cdot \mathbf{n}\,ds.$$

By Gauss' Theorem, this condition is equivalent to

$$\frac{\partial}{\partial t}(c\rho T) = \kappa\nabla^2 T.$$

If c, ρ, T are independent of t, we conclude that T is harmonic: $\nabla^2 T = 0$.

Thus, to say that T is prescribed on a portion of the boundary means that the portion is maintained at a preassigned temperature (for example, by a heating device). The condition $\partial T/\partial n = 0$ means that the flux line (or $-\operatorname{grad} T$) is parallel to the boundary; in other words, the boundary is ***insulated***. (No heat flows across the boundary.)

Example 5.3.1 *Let A be the first quadrant; the x axis is maintained at $T = 0$ while the y axis is maintained at $T = 100$. Find the temperature distribution in A. (Physically, the region may be approximated by a thin metal sheet.)*

Solution We map the first quadrant to the upper half plane by $z \mapsto z^2$ (Figure 5.3.5).

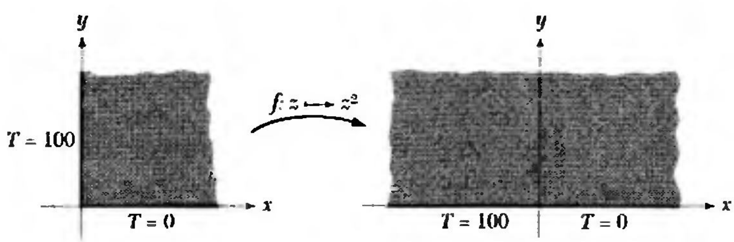


Figure 5.3.5: Map the region A to the upper half plane.

It is physically reasonable that the temperature should be a bounded function: otherwise we would obtain arbitrarily high (or low) temperatures. Therefore, we use the standard solution in the upper half plane:

$$u(x,y) = \frac{1}{\pi}(100 \arg z) = \frac{100}{\pi}\tan^{-1}\left(\frac{y}{x}\right).$$

Thus, the solution we seek is

$$u_0(x,y) = u(f(x,y))$$

where $f(x,y) = z^2 = x^2 - y^2 + 2ixy = (x^2 - y^2, 2xy)$. Hence

$$u_0(x,y) = \frac{100}{\pi}\tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right)$$

is the desired answer. It is understood that $\tan^{-1}$ is taken in the interval $[0, \pi]$. Another form of the answer may be obtained as follows:

$$u_0(x,y) = u(z^2) = \frac{100}{\pi}\arg(z^2) = \frac{200}{\pi}\arg z = \frac{200}{\pi}\tan^{-1}\left(\frac{y}{x}\right).$$

The isotherms and flux lines are indicated in Figure 5.3.6. ♦

Example 5.3.2 *Let A be the upper half of the unit disk $|z| \leq 1$. Find the temperature inside if the circular portion is insulated; $T = 0$ for $x > 0$ and $T = 10$ for $x < 0$ on the real axis.*

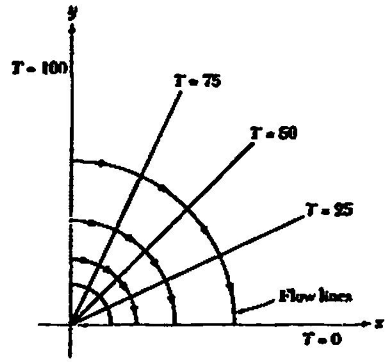


Figure 5.3.6: Isotherms and flux lines for Example 5.3.1.

Solution For this type of problem where there is a portion of the boundary where $\partial T/\partial n = 0$ (insulated), it is convenient to map the region to a half strip. This can be done for A by means of $\log z$ (using the principal branch); see Figure 5.3.7 (and Figure 5.2.11(viii)).

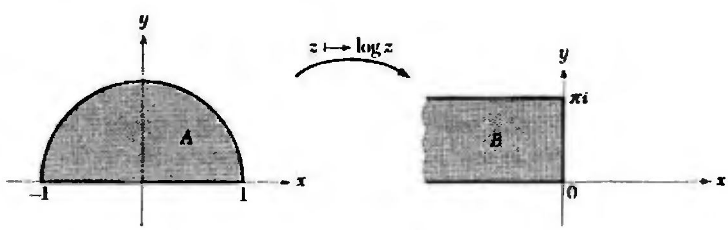


Figure 5.3.7: Mapping the semicircular region A to the half strip B in Example 5.3.2.

For the strip B we obtain, by inspection, the solution

$$T_0(x, y) = \frac{10y}{\pi}.$$

(Note that along the y axis $\partial T_0/\partial n = \partial T_0/\partial x = 0$.) Thus, our answer is

$$T(x, y) = T_0(\log(x + iy)) = \frac{10}{\pi}\tan^{-1}\left(\frac{y}{x}\right). \quad \blacklozenge$$

Electric Potential In physics we learn that if an electric potential ϕ is determined by static electric charges, ϕ must satisfy Laplace's equation (that is, be harmonic). The conjugate function Φ of ϕ is interpreted as follows: Lines along which Φ is constant are lines along which forces act on test charges and are called

flux lines. Tangent vectors to such lines are $-\operatorname{grad}\phi = \mathbf{E}$, called the ***electric field*** (see Figure 5.3.8). Thus *the flux lines and the equipotential lines (lines of constant ϕ) intersect orthogonally.*

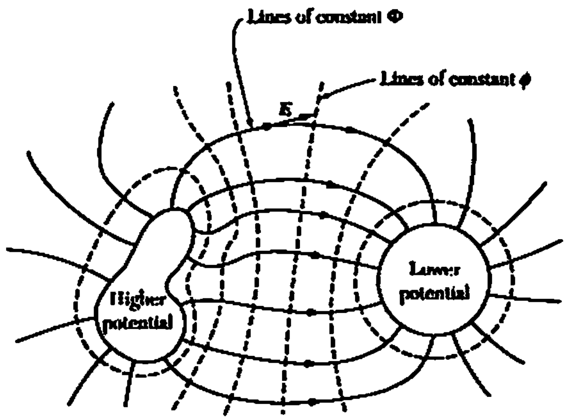


Figure 5.3.8: Electrical potential.

The Dirichlet problem arises in electrostatics when the boundary is maintained at a given potential (for example, by means of a battery or by grounding).

Example 5.3.3 *Consider the unit circle. The electric potential is maintained at $\phi = 0$ on the lower semicircle and at $\phi = 1$ on the upper semicircle. Find ϕ inside.*

Solution We can apply our general procedure for the Dirichlet problem by mapping the unit disk to the upper half plane. This may be done, for example, by the fractional linear transformation given in Figure 5.2.11(vi). See Figure 5.3.9.

With

$$u + iv = f(x+iy) = f(z) = \frac{1}{i}\frac{z+1}{z-1} = \frac{1}{i}\frac{(x+1)+iy}{(x-1)+iy},$$

we have

$$u = \frac{-2y}{(x-1)^2+y^2} \qquad \text{and} \qquad v = \frac{1-x^2-y^2}{(x-1)^2+y^2}.$$

As with temperature, it is physically reasonable that the potential be bounded. We can use the standard solution on the upper half plane:

$$\phi_0(u,v) = 0 + \frac{1}{\pi}(1-0)\tan^{-1}\left(\frac{v}{u}\right) = \frac{1}{\pi}\tan^{-1}\frac{v}{u}.$$

The solution on the unit disk is then

$$\phi(x,y) = \phi_0(f(x,y)) = \phi_0(u,v) = \frac{1}{\pi}\tan^{-1}\left(\frac{x^2+y^2-1}{2y}\right).$$

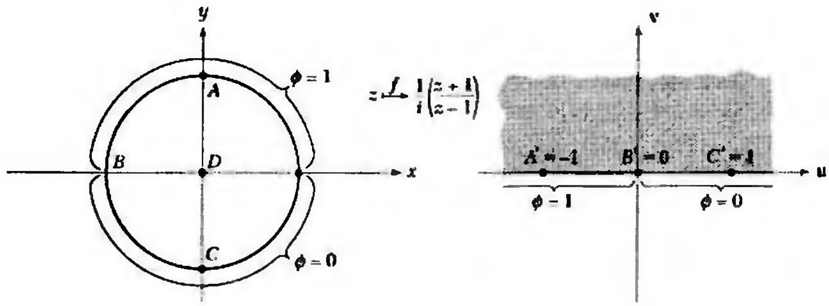


Figure 5.3.9: The conformal mapping used to solve a Dirichlet problem on the disk.

The values of the arctangent must be taken between 0 and π. The equipotential lines and flux lines are shown in Figure 5.3.10.

This example could also be solved by using Poisson's formula. The two answers would be equal, although this would not be obvious from their form. ♦

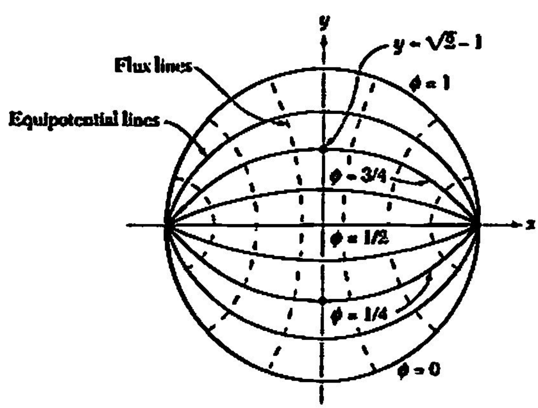


Figure 5.3.10: Equipotential and flux lines for the potential.

Example 5.3.4 *The harmonic function $\phi(z) = (Q/2\pi)\log|z - z_0| + K$ for a constant K, which is the real part of $(Q/2\pi)\log(z - z_0) + K$, represents the potential of a charge Q located at z_0. (This is because ϕ is a radial field such that if $\mathbf{E} = -\operatorname{grad}\phi$ is the electric force field, then the integral of $\mathbf{E}\cdot\mathbf{n}$ around a curve surrounding z_0 is Q by Gauss' Theorem).[5] The constant K may be adjusted to make any convenient*

[5] This is the potential for a charge in the *plane*. In space it corresponds to the potential produced by a charged line.

place such as infinity or some grounded object have potential* 0. *(This is reasonable since only the force* E *is observed, and it is not affected by changing* K.*) Sketch the equipotential lines for two charges of like or opposite signs.

Solution The potential of two charges is obtained by adding the respective potentials. Thus two charges $Q > 0$ located at z_1 and z_2 have the electrostatic potential $(Q/2\pi)\log(|z - z_1||z - z_2|)$; a charge $Q > 0$ at z_1 and a charge $-Q$ at z_2 have potential $(Q/2\pi)\log(|z - z_1|/|z - z_2|)$. The equipotential lines are sketched in Figure 5.3.11. The curves ϕ = constant in the drawing on the left are called lemniscates; in the drawing on the right they are called circles of Apollonius. The lines of force are the family of circles orthogonal to them that pass through the two points. Together they form the Steiner circles discussed in Exercise 35 of §5.2. ♦

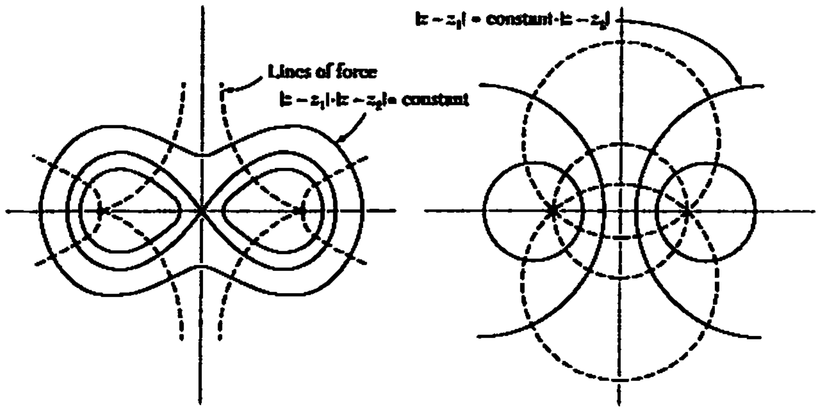


Figure 5.3.11: The field of like charges (left) and the field of opposite charges (right).

Example 5.3.5 ***Suppose a point charge of* +1 *is located at* $z_0 = \frac{1}{2}$ *and the unit circle is a grounded conductor maintained at potential* 0. *Find the potential at every point* $z \neq z_0$ *inside the unit circle.***

Solution 1. The function $f(z) = (2z - 1)/(2 - z)$ maps the unit disk D to itself taking $z_0 = \frac{1}{2}$ to 0. The function $u(z) = (1/2\pi)\log|z|$ is a solution on the image disk (point charge of +1 at 0 and 0 potential around the unit circle). Thus $\phi(z) = u(f(z)) = (1/2\pi)\log|(2z - 1)/(2 - z)|$ solves the original problem. (See Figure 5.3.12.)

Solution 2. We give a second solution that illustrates the method of reflection in a circle from §5.2 and an interesting idea from electrostatics called *image charges*. We need a field ϕ inside the unit disk D that has the unit circle C as an equipotential curve. The electric force lines must be a family of curves ending at z_0 that cross C

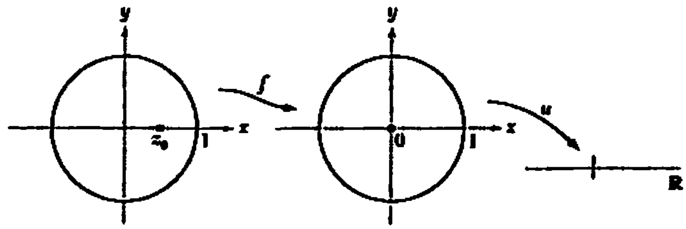


Figure 5.3.12: The conformal map f shifts the point charge from $z_0 = \frac{1}{2}$ to 0.

at right angles. This can be accomplished by placing an artificial "image charge" of -1 at the reflection $\bar{z}_0$ of z_0 in C. As in the last example, the electric force lines for such a pair of charges are the family of circles through z_0 and $\bar{z}_0$. We know from §5.2 that these cross C at right angles as desired. See Figure 5.3.13.

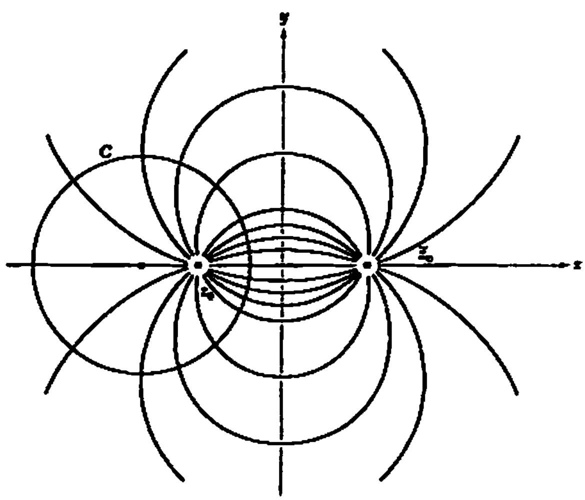


Figure 5.3.13: Reflection and image charge.

With charges of $+1$ at $z_0 = \frac{1}{2}$ and -1 at $\bar{z}_0 = 2$, we have

$$\begin{aligned}
\phi(z) &= \frac{1}{2\pi}\log|z - \frac{1}{2}| - \frac{1}{2\pi}\log|z-2| + K \\
&= \frac{1}{2\pi}\log\left|\frac{z-\frac{1}{2}}{z-2}\right| + K \\
&= \frac{1}{2\pi}\log\left|\frac{2z-1}{2-z}\right| - \frac{1}{2\pi}\log 2 + K.
\end{aligned}$$

Setting the constant K to $(1/2\pi)\log 2$ makes the potential 0 around C and gives the same answer as the first solution. ◆

Hydrodynamics If we have a (steady-state) incompressible, nonviscous fluid, we are interested in finding its velocity field, $\mathbf{V}(x, y)$. From vector analysis we know that "incompressible" means that the divergence $\operatorname{div} \mathbf{V} = 0$. (We say $\mathbf{V}$ is ***divergence free.***) We shall assume that $\mathbf{V}$ is also a ***potential flow*** and hence is circulation free; that is $\mathbf{V} = \operatorname{grad} \phi$ for some ϕ called the ***velocity potential.*** Thus, ϕ is harmonic because $\nabla^2 \phi = \operatorname{div} \operatorname{grad} \phi = \operatorname{div} \mathbf{V} = 0$. Thus when we solve for ϕ we can obtain $\mathbf{V}$ by taking $\mathbf{V} = \operatorname{grad} \phi$.

The conjugate ψ of the harmonic function ϕ (which will exist on any simply connected region) is called the ***stream function***, and the analytic function $F = \phi + i\psi$ is called the ***complex potential.*** Lines of constant ψ have $\mathbf{V}$ as their tangents (Why?), so *lines of constant ψ may be interpreted as the lines along which particles of fluid move*; hence the name stream function (see Figure 5.3.14).

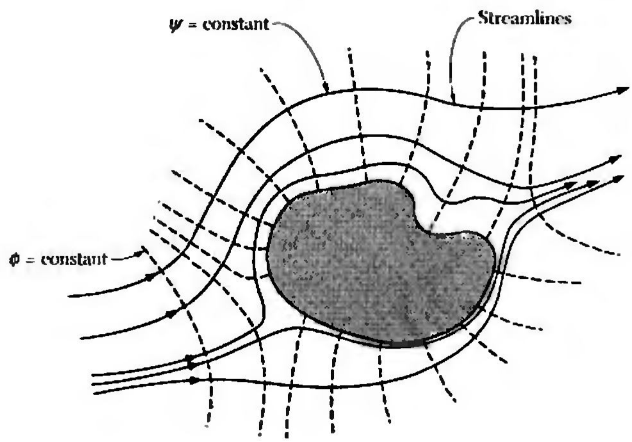


Figure 5.3.14: Fluid flow.

The natural boundary condition is that $\mathbf{V}$ should be parallel to the boundary. (The fluid flows parallel to the walls.) This means that $\partial\phi/\partial n = 0$, so we are led to the Neumann problem for ϕ.

Let us again consider the upper half plane. A physically acceptable motion is obtained by setting $\mathbf{V}(x, y) = \alpha = (\alpha, 0)$ or $\phi(x, y) = \alpha x = \operatorname{Re}(\alpha z)$, where α is real. The flow corresponding to V is parallel to the x axis, with velocity α. Notice that now ϕ is not bounded: thus the behavior at ∞ for fluids, for temperature, and for electric potential is different because of the different physical circumstances (see Figure 5.3.15).

Thus, to find the flow in a region we should map the region to the upper half plane and use the solution $\phi(x, y) = \alpha x$. We can specify α as the velocity at infinity. It should be clear that if f is the conformal map from the given region to the upper half plane, the required complex potential is given by $F(z) = \alpha f(z)$.

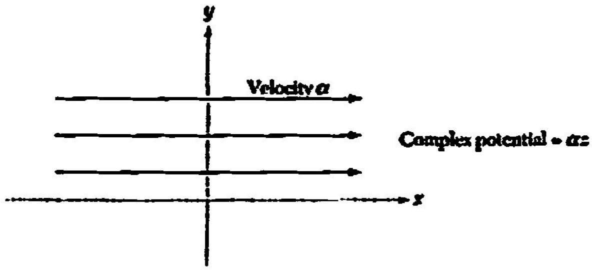


Figure 5.3.15: Flow in the upper half plane.

Example 5.3.6 *Find the flow around the upper half of the unit circle if the velocity is parallel to the x axis and is α at infinity.*

Solution We shall map the exterior of the given region to the upper half plane. Such a conformal map is $z \mapsto z + 1/z$ (Figure 5.3.16). Thus, $F_0(z) = \alpha z$ is the complex potential in the upper half plane, and so the required complex potential is

$$F(z) = \alpha\left(z + \frac{1}{z}\right).$$

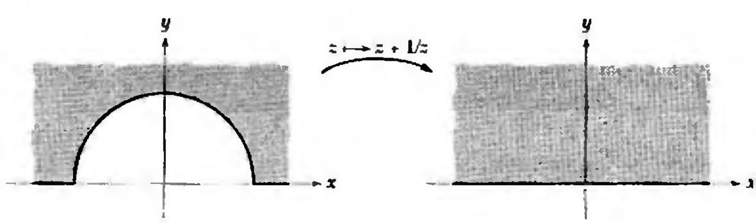


Figure 5.3.16: Effect of $z \mapsto z + 1/z$.

It is convenient to use polar coordinates r and θ to express ϕ and ψ. Then we get

$$\phi(r,\theta) = \alpha\left(r + \frac{1}{r}\right)\cos\theta \quad \text{and} \quad \psi(r,\theta) = \alpha\left(r - \frac{1}{r}\right)\sin\theta.$$

A few streamlines are shown in Figure 5.3.17.

Note: By slightly modifying the transformation $z \mapsto z + 1/z$ by the addition of appropriately chosen higher-order terms, the half circle can be replaced by something more closely resembling an airplane wing; these are called ***Joukowski transformations***. ♦

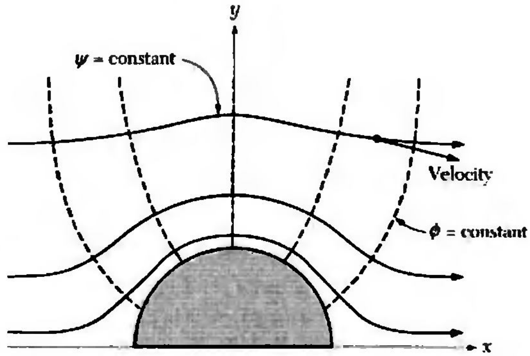


Figure 5.3.17: Streamlines for flow around a cylinder.

Exercises

1. Find a formula for determining the temperature in the region with the indicated boundary values shown in Figure 5.3.18 (left).

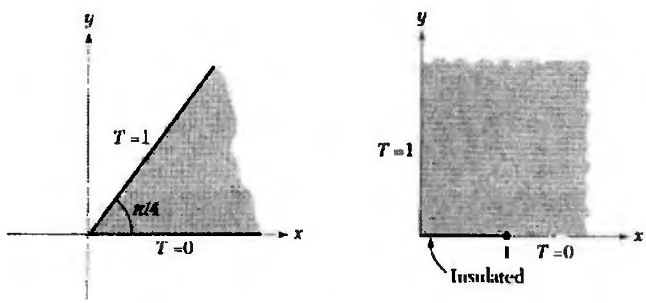


Figure 5.3.18: Find the temperature for these regions (Exercises 1 and 2).

2.* Find a formula for determining the temperature in the region illustrated in Figure 5.3.18 (right). *Hint:* Consider the map $z \mapsto \sin z$.

3. Find the electric potential in the region illustrated in Figure 5.3.19 (left). Sketch a few equipotential curves.

4. Find the electric potential in the region illustrated in Figure 5.3.19 (right).

5. Find the flow around a circular disk if the flow is at an angle θ to the x axis with velocity α at infinity, as in Figure 5.3.20 (left).

6. Suppose a point with charge $+1$ is located at $z_0 = (1+i)/\sqrt{2}$ and the positive real and imaginary axes (boundaries of the first quadrant $A = \{z \mid \operatorname{Re} z > 0$

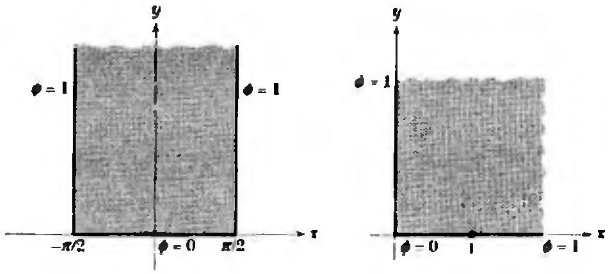


Figure 5.3.19: Find the electric potential for these regions (Exercises 3 and 4).

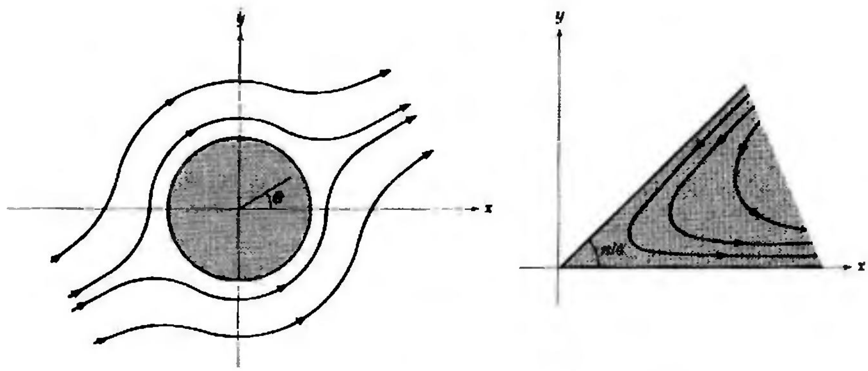


Figure 5.3.20: Flow around a disk (Exercise 5) and in a wedge (Exercise 7).

and $\operatorname{Im} z > 0\}$) are a grounded conductor maintained at potential 0. Find the potential at every point $z \neq z_0$ inside the region A.

7. Obtain a formula for determining the flow of a fluid in the region illustrated in Figure 5.3.20 (right). (The velocity at ∞ is α.)

Review Exercises for Chapter 5

1. Consider the map $z \mapsto z^3$. On what sets $A \subset \mathbb{C}$ is this map conformal (onto its image)?

2. Verify directly that the map $z \mapsto z^n$ preserves the orthogonality between rays from 0 and circles around 0.

3. Find a conformal map that takes the unit disk onto itself and maps $i/2$ to 0.

4. Find a conformal map of the unit disk onto itself with $f(\frac{1}{4}) = -\frac{1}{3}$ and $f'(\frac{1}{4}) > 0$.

5. Find a conformal map that takes the region

$$A = \{z \text{ such that } |z-1| > 1 \text{ and } |z-2| < 2\}$$

onto $B = \{z \mid 0 < \operatorname{Re} z < 1\}$.

6.• Let $z_1, z_2 \in \mathbb{C}$ and $a \in \mathbb{R}$, where $a > 0$. Show that

$$\left|\frac{z-z_1}{z-z_2}\right| = a$$

defines a circle and z_1, z_2 are inverse points in that circle (that is, they are collinear with the center z_0 and $|z_1 - z_0| \cdot |z_2 - z_0| = \rho^2$ where ρ is the radius of the circle).

7. Examine the image of the set $\{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0, 0 \leq \operatorname{Re} z \leq \pi/2\}$ under the map $z \mapsto \sin z$ by considering it to be the composition of the maps $z \mapsto e^{iz}, z \mapsto z - 1/z, z \mapsto z/2i$.

8. Let $f : A \to B$ be a conformal map, let γ be a curve in A, and let $\tilde{\gamma} = f \circ \gamma$. Show that

$$l(\tilde{\gamma}) = \int_a^b |f'(\gamma(t))| \cdot |\gamma'(t)| dt.$$

If f preserves the lengths of all curves, argue that $f(z) = e^{i\theta} z + a$ for some $a \in \mathbb{C}$ and for $\theta \in [0, 2\pi[$.

9. Find a conformal map that takes the set

$$A = \{z \text{ such that } |z - i| < 1\}$$

onto $B = \{z \text{ such that } |z-1| < 1\}$.

10. Show that the function $f(z) = (z-1)/(z+1)$ maps the region

$$A = \{z \text{ such that } |z| > 1 \text{ and } |z-1| < 2\}$$

one-to-one onto the set $B = \{z \mid 0 < \operatorname{Re} z < \frac{1}{2}\}$.

11. The region A in Exercise 10 is bounded by two circles, as is the region $\{z \mid 1 < |z| < 2\}$. Can a conformal map from this region to B be accomplished by a fractional linear transformation? If so, display the function. If not, why not?

12. Let $T(z) = (az+b)/(cz+d)$. Show that $T(T(z)) = z$ (that is, $T \circ T =$ identity) if and only if $a = -d$ or $T(z) = z$ for all z.

13. Is it possible to find a conformal map of the interior of the unit circle onto its exterior? Is $f(z) = 1/z$ such a map?

14. Find a conformal map of the quarter plane $A = \{z \mid \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0\}$ one-to-one onto the unit disk which takes $1+i$ to 0 with positive derivative at $1+i$.

15. Let F_1 and F_2 be conformal maps of the unit disk onto itself and let $F_1(z_0) = F_2(z_0) = 0$ for some fixed $z_0, |z_0| < 1$. Show that there is a $\theta \in [0, 2\pi[$ such that $F_1(z) = e^{i\theta}F_2(z)$.

16. Suppose f is a conformal map of the upper half plane one-to-one onto itself with $f(-1) = 0, f(0) = 2$, and $f(1) = 8$. Find $F(i)$.

17. Give a complete list of all conformal maps of the first quadrant $A = \{z \mid \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0\}$ onto itself. (Suggestion: See Exercise 18 in §5.2.)

18. Describe the region $A = \{z \text{ such that } |(z+3)/(z-1)| < 3\}$. *Hint*: $f(z) = (z+3)/(z-1)$ takes what points to the circle $|w| = 3$?

19. Find a conformal map that takes the region in Figure 5.R.1 to the upper half plane. Use this map to find the electric potential ϕ with the stated boundary conditions. *Hint*: Consider a branch of $z \mapsto \sqrt{z^2-1}$ after rotating the figure through 90°.

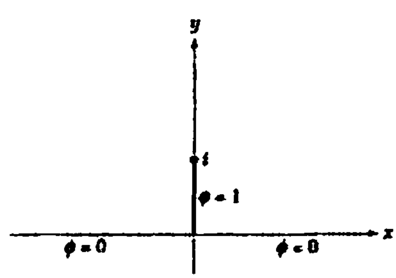
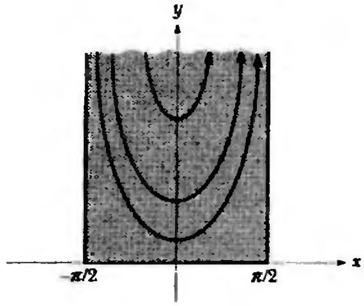


Figure 5.R.1: Boundary data for Exercise 19.

20.* Find the flow of a fluid in the region shown in Figure 5.R.2 (left).



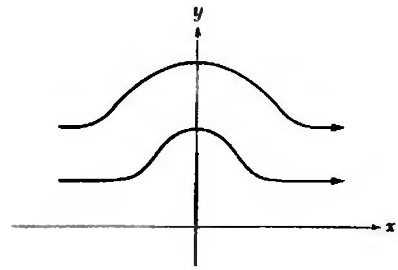


Figure 5.R.2: Regions for Exercise 20 (left) and Exercise 21 (right).

21. Use Exercise 19 to find the fluid flow over the obstacle in Figure 5.R.2 (right) and plot a few streamlines.

22. Let B be the open first quadrant, that is, $B = \{z \mid \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0\}$, and let $S = \{z \mid 0 < \operatorname{Im} z < \pi\}$.

    (a) Find an analytic function that maps B one-to-one onto S.

    (b) Find a function u harmonic on B and continuous on the closure of B except at $(0,0)$ that satisfies $u(x) = 0$ and $u(iy) = \pi$ for $y > 0$.

23. Find the electric potential in the region shown in Figure 5.R.3.

Figure 5.R.3: Boundary data for Exercise 23.

24. Suppose a point charge of $+1$ is placed at $z_0 = i$ in the upper half plane and the real axis is a grounded conductor maintained at constant potential 0. Find the potential at every point $z \neq i$ in the upper half plane.

25. Use the Schwarz-Christoffel Formula 5.2.11 to find a conformal map between the two regions shown in Figure 5.R.4 ($A = -1, B = 1, B' = 0$.)

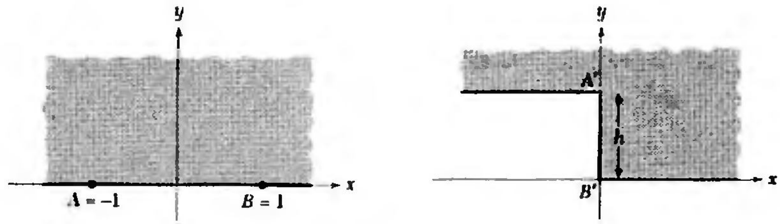


Figure 5.R.4: Regions for Exercises 25 and 26.

26. Use Exercise 25 to find the flow lines over the step in the bed of the deep channel shown on the right in Figure 5.R.4.

27. Find the temperature on the region illustrated in Figure 5.R.5 *Hint:* Use $z \mapsto \sin^{-1} z$.

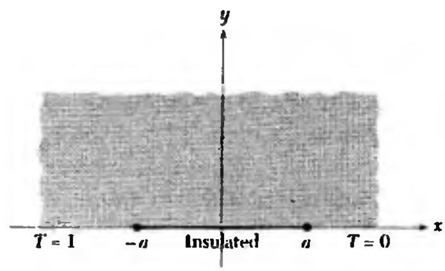


Figure 5.R.5: Boundary data for Exercise 27.

28. Let g_n be a sequence of analytic functions defined on a region A. Suppose that $\sum_{n=1}^{\infty} |g_n(z)|$ converges uniformly on A. Prove that $\sum_{n=1}^{\infty} |g_n'(z)|$ converges uniformly on closed disks in A.

29. Evaluate by residues:

$$\int_0^{\infty} \frac{\cos x}{x^2+3} dx.$$

30.• Let f be analytic on the set $\mathbb{C}\backslash\{0\}$. Suppose that $f(z) \to \infty$ as $z \to 0$ and $f(z) \to \infty$ as $z \to \infty$. Prove that f can be written in the form

$$f(z) = \frac{c_k}{z^k} + \ldots + \frac{c_1}{z} + c_0 + d_1 z + \ldots + d_l z^l$$

for constants c_i and d_j.

31. If $\sum_0^{\infty} a_n z^n$ has radius of convergence ρ, what is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^{2n}$? Of $\sum_{n=0}^{\infty} a_n^2 z^n$?

32. Find the Laurent expansion of $f(z) = z^4/(1-z^2)$ that is valid on the annulus $1 < |z| < \infty$.

Chapter 6

Further Development of the Theory

This chapter continues the development of the theory of analytic functions that was begun in Chapters 3 and 4. The main tools we will use in this development are Taylor series and the residue theorem.

The first topic in this chapter is analytic continuation, that is, the attempt to make the domain of an analytic function as large as possible. Further investigation in the theory of analytic continuation leads naturally to the concept of a Riemann surface, which is briefly discussed in §6.1. Additional properties of analytic functions are developed in subsequent sections. Some of these properties deal with such topics as counting zeros of an analytic function; others are generalizations of the Inverse Function Theorem 1.5.10.

6.1 Analytic Continuation and Elementary Riemann Surfaces

The first theorem in this section is called the Principle of Analytic Continuation, also referred to as the Identity Theorem. This theorem and its proof lead to a discussion of Riemann surfaces, which facilitates a more satisfactory treatment of what were previously referred to as "multiple-valued functions," such as $\log z$ and $\sqrt{z}$.

Analytic Continuation The basic idea is that if two analytic functions agree on a small portion of a (connected) region, then they agree on the whole region on which they are both analytic. This is stated precisely in the following theorem.

Theorem 6.1.1 (Principle of Analytic Continuation—Identity Theorem) *Let f and g be analytic in a region A. Suppose that there is a sequence $z_1, z_2, z_3, \ldots$ of distinct points of A converging to $z_0 \in A$, such that $f(z_n) = g(z_n)$ for all*

$n = 1, 2, 3, \ldots$. *Then $f = g$ on all of A* (see Figure 6.1.1). *The conclusion is valid, in particular, if $f = g$ on some neighborhood of some point in A.*

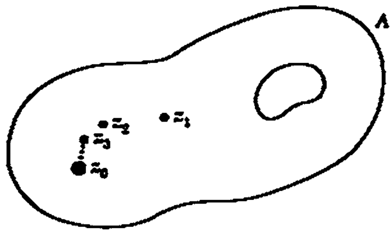


Figure 6.1.1: Identity Theorem: If $f = g$ at all the points $z_1, z_2, \ldots$, then $f(z) = g(z)$ for all $z \in A$.

Proof Let $h(z) = f(z) - g(z)$. Then h is analytic on A and $h(z_n) = 0$ for each n. We want to show that $h(z) = 0$ for all z in A. Corollary 3.2.10 on local isolation of zeros of a nonconstant function tells us that $h(z) = 0$ on some open disk around z_0. The connectedness of A will enable us to extend this conclusion to all of A. To this end, let $B = \{z \in A \mid h \text{ is } 0 \text{ on a neighborhood of } z\}$. Then B is a nonempty subset of A since $z_0 \in B$. If $w \in B$, then $h = 0$ on an open disk around w contained in A and hence on a neighborhood of each point of that disk. Thus, that disk is contained in B. Therefore, B is an open subset of A. On the other hand, suppose $w_1, w_2, w_3, \ldots$ are distinct points in B converging to a point z in A. Then $h(w_k) = 0$ for each k and, since A is open, z is interior to A. Corollary 3.2.10 applies again to show that h is 0 on a neighborhood of z. Thus $z \in B$. This shows that B is closed relative to A.

We have thus shown that the subset B is both open and closed relative to A and is not empty. Since A is connected, we must have $B = A$ (see Proposition 1.4.13). Thus, $h(z) = 0$ for all z in A as required. ∎

For example, this shows that there is exactly *one* analytic function on $\mathbb{C}$ that agrees with e^x on the x axis, namely, e^z, because the x axis contains a convergent sequence of distinct points (for example, $1/n$).

Notice that it is vital that A be connected. If A consisted of two disjoint disks, the function which were 0 on one of them and 1 on the other would agree with the zero function on one part of A but not on the other. Recall that for us, a *region* means an open connected subset of $\mathbb{C}$. Corollary 3.2.10 on the local isolation of zeros now extends to a global form on connected sets. We formulate this in the following corollary.

Corollary 6.1.2 *The zeros (or, more generally, points where a specified value w_0 is assumed) of a nonconstant analytic function are isolated in the following sense. If f is analytic and not constant in a region A and $f(z_0) = w_0$ for a point z_0 in A, then there is a number $\epsilon > 0$ such that $f(z)$ is not equal to w_0 for any z in the deleted neighborhood $\{z \mid 0 < |z - z_0| < \epsilon\}$.*

Proof If there were no such ϵ, then f would agree with the constant function defined by $h(z) = w_0$ at least on a sequence of points converging to z_0. But then it would agree with h everywhere on A by the Identity Theorem 6.1.1 and so be constant. ■

There can be a limit point of zeros on the boundary of the region of analyticity. (This is illustrated in Worked Example 6.1.11 with the function $\sin(1/z)$.) The Identity Theorem says that a nonconstant function cannot have a limit point of zeros in the *interior* of the region of analyticity.

Corollary 6.1.3 *Let $f : A \to \mathbf{C}$ and $g : B \to \mathbf{C}$ be analytic on regions A and B. Suppose that $A \cap B \neq \varnothing$ and $f = g$ on $A \cap B$. Define*

$$h(z) = \begin{cases} f(z) & \text{if } z \in A \\ g(z) & \text{if } z \in B \end{cases}.$$

*Then h is analytic on $A \cup B$ and is the only analytic function on $A \cup B$ equaling f on A (or g on B). We say that h is an **analytic continuation** of f (or g)* (see Figure 6.1.2).

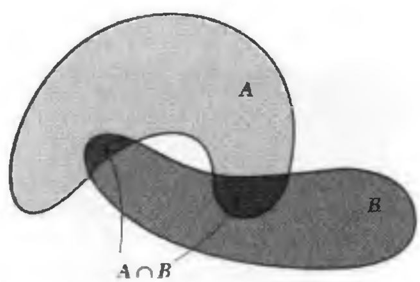


Figure 6.1.2: Analytic continuation.

Proof That h is analytic is obvious because f and g are. Uniqueness of h results from the Identity Theorem and from the facts that $A \cup B$ is a region and that $A \cap B$ is open. ■

Analytic continuation provides a method for increasing the domain of an analytic function. However, the following phenomenon can occur. Let f on A be continued to a region A_1 and let A_2 be as pictured in Figure 6.1.3.

If we continue f to be analytic on A_1, then continue this new function from A_1 to A_2, the result need not agree with the original function f on A. A specific example should clarify this point. Consider $\log z$, the principal branch $(-\pi < \arg z < \pi)$ on the region A consisting of the right half plane union the lower half plane. The log function may be continued uniquely to include A_1 = the upper half plane in

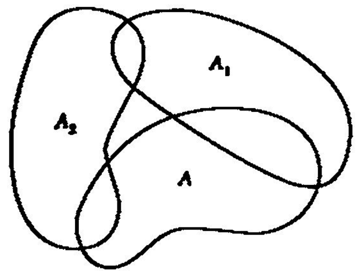


Figure 6.1.3: Continuation of a function from A to A_1 and from A_1 to A_2.

its domain. Similarly, we can continue the log again from the upper half plane so as to include A_2 = the left half plane in its domain by choosing the branch $0 < \arg z < 2\pi$. But these branches do not agree on the third quadrant; they differ by $2\pi i$ (see Figure 6.1.4). Therefore, in continuing a function we must be sure that the function on the extending region B agrees with the original on the whole intersection $A \cap B$ and not merely on part of it.

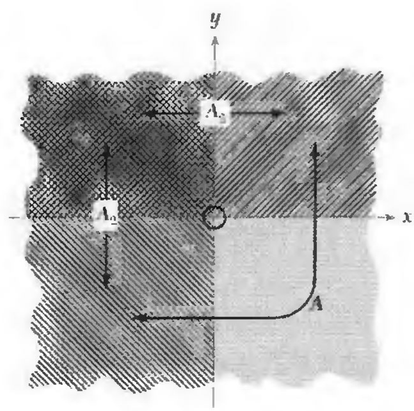


Figure 6.1.4: Continuing the log.

It is not always possible to extend an analytic function to a larger domain. The reader is asked in Exercise 5 to confirm that the power series $\sum_{n=0}^{\infty} z^{n!}$ converges to an analytic function $f(z)$ on the open unit disk but that this function cannot be analytically continued to *any* larger open set. The unit circle is called a *natural boundary* for this function. In the next two subsections we will examine techniques by which analytic continuation may sometimes be accomplished.

Schwarz Reflection Principle There is a special case of analytic continuation that can be dealt with directly as follows.

Theorem 6.1.4 (Schwarz Reflection Principle) *Let A be a region in the upper half plane whose boundary* $\mathrm{bd}(A)$ *intersects the real axis in an interval $]a,b[$ (or finite union of disjoint intervals). Let f be analytic on A and continuous on the set $A \cup]a,b[$. Let $A^* = \{z \mid \bar{z} \in A\}$, the reflection of A (see Figure 6.1.5), and define g on A^* by $g(z) = \overline{f(\bar{z})}$. Assume that f is real on $]a,b[$. Then g is analytic and is the unique analytic continuation of f to $A \cup]a,b[\cup A^*$.*

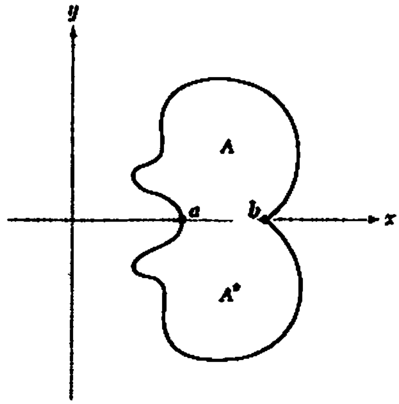


Figure 6.1.5: A^* is the reflection of A.

Proof Uniqueness is implied by the Identity Theorem since $A \cup]a,b[\cup A^*$ is connected and any such extension must agree with f on the segment $]a,b[$, which certainly contains a convergent sequence of points together with their limit. Note that $f = g$ on this segment since $\bar{z} = z = x$ there, and $f(x)$ is real, so $f(x) = \overline{f(x)} = \overline{f(\bar{x})} = g(x)$. The analyticity of g on A^* follows from the Cauchy-Riemann equations, as established in Worked Example 1.5.19. If h is defined on $A \cup]a,b[\cup A^*$ by $h(z) = f(z)$ on $A \cup]a,b[$ and $h(z) = g(z)$ on A^*, then h is analytic on $A \cup A^*$ and continuous across the mutual boundary $]a,b[$ since $f = g$ on the real axis. Analyticity on the whole set follows from Morera's Theorem 2.4.10, as was established in Worked Example 2.4.17. ■

This result is remarkable in that we required only that f be continuous and real on $]a,b[$. It followed *automatically* that f is analytic on $]a,b[$ when continued across the real axis. To help see that g (and thus h) is analytic on A^*, consider the map in three steps:

$$z \mapsto \bar{z}; \quad \bar{z} \mapsto f(\bar{z}); \quad f(\bar{z}) \mapsto \overline{f(\bar{z})}.$$

The middle map is conformal; the first and last are anticonformal in the sense that they reverse angles. Since angles are reversed twice, the net result is to preserve angles. The whole map is thus conformal.

A related reflection principle can be formulated using circles in place of the real axis and replacing complex conjugation by reflection in the circle. The Schwarz Reflection Principle 6.1.4 is a special case if lines are treated as circles of infinite radius, as in Chapter 5.

Theorem 6.1.5 (Schwarz Reflection Principle for a Circle) *Let A be a region in the interior or exterior of a circle C_1 (or on one side of a line) with part of its boundary an arc γ of C_1. Suppose f is analytic on A and continuous on $A \cup \gamma$ and $f(\gamma)$ is an arc Γ of another circle (or line) C_2. Let $\tilde{A} = \{z \mid \tilde{z} \in A\}$ be the reflection of A in C_1 and define g on $\tilde{A}$ by*

$$g(z) = \widetilde{f(\tilde{z})}$$

(the last $\tilde{}$ denotes reflection in C_2.) Then g is analytic and is the unique analytic continuation of f to $A \cup \gamma \cup \tilde{A}$.

Proof We assume A is interior to C_1 and $f(A)$ is interior to C_2. The other cases are similar. Let $T_i, i = 1, 2$, be fractional linear transformations taking C_i to the real axis and their interiors to the upper half plane. For w in $T_1(A)$, the function $h(w) = T_2(f(T_1^{-1}(w)))$ is analytic and by the Schwarz Reflection Principle 6.1.4, $\overline{h(\overline{w})}$ gives an analytic continuation to $T_1(A)^*$. Using the facts that fractional linear transformations preserve reflection in circles (Proposition 5.2.7) and that complex conjugation is reflection in the real axis, we find that

$$[f(\tilde{z})]^{\sim} = T_2^{-1}(\overline{h(\overline{T_1(z)})}),$$

so is an analytic continuation of f. (See Figure 6.1.6.) ■

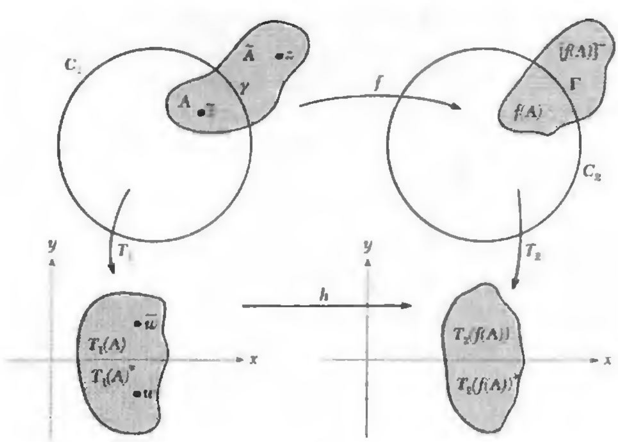


Figure 6.1.6: Analytic continuation by reflection.

An argument similar to that used to establish Worked Example 2.4.17 and the Schwarz Reflection Principle from Morera's Theorem is used to establish the following.

Theorem 6.1.6 (Analytic Continuation by Continuity) *Let A and B be disjoint simply connected regions whose boundaries intersect in a simple smooth curve γ. Let $C = A \cup (\text{interior } \gamma) \cup B$ (where* interior γ *means the image of γ without its endpoints) and suppose that*

(i) *Each point in interior γ has a neighborhood in C.*

(ii) *f is analytic in A and continuous on $A \cup \gamma$.*

(iii) *g is analytic in B and continuous on $B \cup \gamma$.*

(iv) *For $t \in \gamma$, we have $\lim_{z \to t, z \in A} f(z) = \lim_{z \to t, z \in B} g(z)$.*

Then there is a function h analytic on C that agrees with f on A and g on B.

Analytic Continuation by Power Series along Curves Suppose that f is analytic in a neighborhood U of z_0 and that γ is a curve joining z_0 to another point z' (as in Figure 6.1.7).

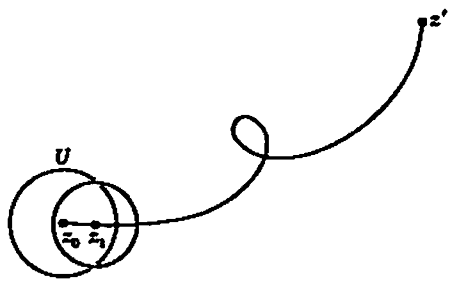


Figure 6.1.7: Continuation by power series.

If we want to continue f to z' we can proceed as follows. For z_1 on γ in U, consider the Taylor series of f expanded around z_1:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!}(z - z_1)^n.$$

This power series may have a radius of convergence such that the power series is analytic farther along γ than the portion of γ in U. The power series so obtained then defines an analytic continuation of f. We can continue this way along γ in hopes of reaching z', which will be possible if the successive radii of convergence do not shrink to 0 before we reach z'. If we succeed, we say f can be ***analytically***

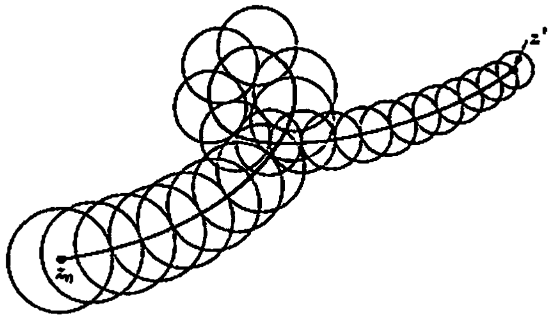


Figure 6.1.8: Continuation can lead to self-intersections.

continued along γ. However, we must be careful because the analytic continuation of f so defined might not be single-valued if γ intersects itself (as in Figure 6.1.8).

The coefficients of the power series around the new center z_1 can be computed in terms of those for the Taylor series of f around the original center z_0. (See Worked Example 6.1.13). If z' can be reached at all by this process, then it can be reached in a finite number of steps. This is essentially because of the Path-Covering Lemma 1.4.24 (see Exercise 7). Thus, the continuation at z' can be computed in terms of the original function.[1]

The example $\sum z^{n!}$ mentioned earlier shows that it can happen that there is no direction in which a power series can be continued. Fortunately this is not usually the case. However, there must always be at least one direction in which continuation is not possible.

Proposition 6.1.7 *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence $R < \infty$. Then there must be at least one point z_1 with $|z_0 - z_1| = R$ such that f cannot be analytically continued to any open set containing z_1.*

Proof Let $B = \{z \text{ such that } |z - z_0| < R\}$ and let C be its boundary circle $\{z \text{ such that } |z - z_0| = R\}$. We will show that if the assertion were false then f could be analytically continued to an open set A containing the closed disk $B \cup C$. If this were done, Worked Example 1.4.27 would show that A contains a larger disk $B_\epsilon = \{z \text{ such that } |z - z_0| < R + \epsilon\}$. (See Figure 6.1.9.) We would have continued f to a larger disk with the same center. This is not possible, since it implies a radius of convergence larger than R. (See Worked Example 6.1.12.)

To obtain A we proceed as follows. For each w on C there would be a neighborhood B_w of w and an analytic continuation f_w of f to $A_w = B \cup B_w$. (See Figure 6.1.10.)

[1] A discussion of how one carries out this computation, including its numerical aspects, may be found in P. Henrici, *Applied and Computational Complex Analysis* (New York: Wiley-Interscience, 1974), Chapter 3.

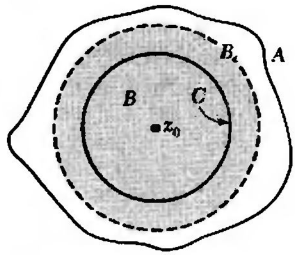


Figure 6.1.9: If A is an open set containing B and its boundary, then A contains a slightly larger disk.

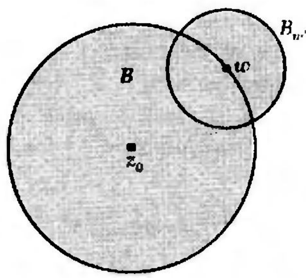


Figure 6.1.10: The set $A_w = B \cup B_w$.

Then $A =$ (union of all the A_w) would be an open set containing $B \cup C$. We try to define a continuation of f to A by setting $g(z) = f_w(z)$ for z in A_w. If this makes unambiguous sense it will certainly be analytic on A since f_w is analytic on A_w. For g to make sense we need to know that if z is in $A_{w_1} \cap A_{w_2}$, then $f_{w_1}(z) = f_{w_2}(z)$. But this is true. The two functions are both analytic on the region $A_{w_1} \cap A_{w_2}$ and they are both equal to f on the open set $B \subset A_{w_1} \cap A_{w_2}$. Therefore they must agree on the whole region, by the Identity Theorem. Thus the definition of g makes sense. It does not depend on which A_w we happen to select containing z. ■

Consequently, the radius of convergence of an analytic continuation is largely independent of the method used to obtain it. This agrees with what we saw in Chapter 3, namely that the radius of convergence is the distance to the nearest unavoidable singularity.

Proposition 6.1.8 *Suppose f is analytic on a neighborhood of a point z_0 of a region A and that f can be analytically continued along every curve joining z_0 to every other point z_1 of A. Then the radius of convergence of the Taylor series at z_1 for each such continuation to z_1 is the same and is at least as great as the distance from z_1 to the complement of A.*

Proof Suppose not. Then by extending the curve radially from z_1 to any point on the circle of convergence, the continuation could be analytically continued still further in every direction from z_1, contrary to Proposition 6.1.7. ■

This proposition does *not* claim that the continuations are all the same. They might not be, as the example of the logarithm shows. We may merely obtain local functions defined on disks but they need not agree on overlaps. This construction is one basic way in which multiple-valued functions arise. A point is called a ***branch point*** if analytic continuation around a closed curve surrounding it can produce a different value upon return to the starting point. The following result says that multiple-valued functions do not arise from continuation along curves in simply connected regions.

Proposition 6.1.9 (Monodromy Principle) *Let A be simply connected and let $z_0 \in A$. Let f be analytic in a neighborhood of z_0. Suppose that f can be analytically continued along any arc joining z_0 to another point $z \in A$. Then this continuation defines a (single-valued) analytic continuation of f on A.*

Proof We need to show that if z_1 is another point of A, then the process of continuation along a curve γ from z_0 to z_1 through A will always produce the same value at z_1 regardless of what curve is used. To this end, let γ_0 and γ_1 be two curves from z_0 to z_1 in A. Since A is simply connected, they are homotopic with fixed endpoints in A. That is, there is a continuous function $H : [0,1] \times [0,1] \to A$ from the unit square into A such that $H(0,t) = \gamma_0(t), H(1,t) = \gamma_1(t), H(s,0) = z_0$, and $H(s,1) = z_1$ for all s and t between 0 and 1, inclusive. The functions $\gamma_s(t) = H(s,t)$ are a family of curves from z_0 to z_1 in A deforming continuously from γ_0 to γ_1. See Figure 6.1.11.

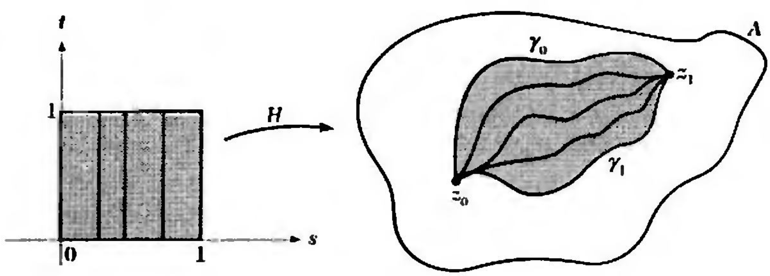


Figure 6.1.11: Homotopy between γ_0 and γ_1.

There is an analytic continuation f_s of f from z_0 to z_1 along each curve γ_s. We will show that $f_s(z_1)$ cannot change as s is shifted continuously from 0 to 1 and therefore that $f_0(z_1) = f_1(z_1)$. This is exactly what we need to establish the theorem.

The image of the square is a closed bounded subset of A. Thus by the Distance Lemma 1.4.21, it lies at a positive distance ρ from the complement of A. By Proposition 6.1.8, the radius of convergence always remains at least ρ as we analytically continue f along any of the curves γ_s. By the Path-Covering Lemma 1.4.24 the continuation along any γ_s may be completed to z_1 in a finite number of steps using disks of radius ρ. For each s this procedure produces an analytic continuation of f to a function f_s analytic on a "tube" A_s around γ_s, as in Figure 6.1.12.

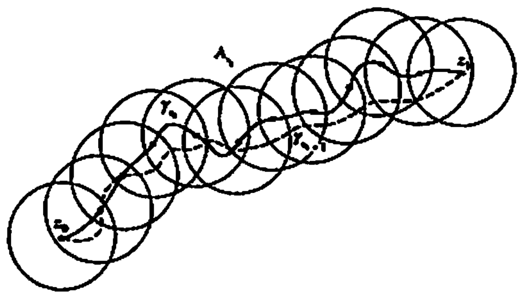


Figure 6.1.12: Each $\gamma_{s_{k+1}}$ is contained in A_{s_k}.

With a bit of care we can select a finite number of points $0 = s_0 < s_1 < s_2 < \ldots < s_N = 1$ and the values of t defining the centers of the disks making up the tubes A_s close enough together that γ_{s_k} is contained in the preceding tube $A_{s_{k-1}}$ and in the succeeding tube $A_{s_{k+1}}$. This is done using the uniform continuity of H, as in the proof of the Deformation Theorem 2.3.12; see, in particular, Figure 2.3.14. The functions f_{s_k} are each analytic on the region $A_{s_k} \cap A_{s_{k+1}}$ and agree on the open set $D(z_0;\rho) \subset A_{s_k} \cap A_{s_{k+1}}$, so they agree on the whole region by the Identity Theorem. In particular, $f_{s_k}(z_1) = f_{s_{k+1}}(z_1)$, so $f_0(z_1) = f_{s_1}(z_1) = f_{s_2}(z_1) = \ldots = f_{s_N}(z_1) = f_1(z_1)$. The continuation of f along γ_0 to z_1 agrees with that along γ_1 to z_1 at the point z_1. This is what we needed to show. ■

For nonsimply connected regions, we can get different values for the continuation of f when we traverse two different paths. This fact was already mentioned at the beginning of this section in connection with $\log z$. For example, in Figure 6.1.13, starting with log defined near 1 and continuing along γ_1, we get $\log(-1) = \pi i$, whereas along γ_2, we get $\log(-1) = -\pi i$. This is because the region $\mathbb{C}\setminus\{0\}$ is not simply connected.

Riemann Surfaces of the Log and Square Root Functions The phenomenon just described leads one to ask if there is a definition of log that does not introduce any artificial branch lines (which, after all, can be chosen arbitrarily). The answer is given by a brilliant idea of Georg Riemann in his doctoral thesis in 1851 that is briefly described here.

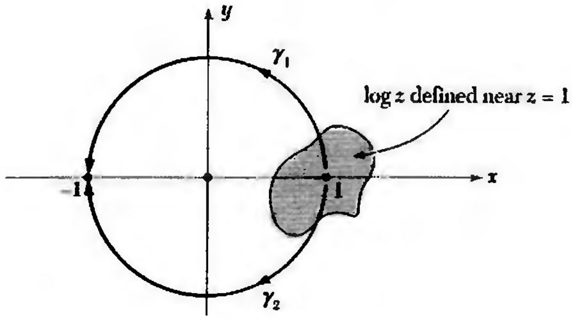


Figure 6.1.13: Continuation of $\log z$ along two different arcs from 1 to -1.

For the logarithm, if $\log z$ is to be single-valued, we should merely regard γ_1 and γ_2 in Figure 6.1.13 as ending up in different places. This can be pictured as in Figure 6.1.14.

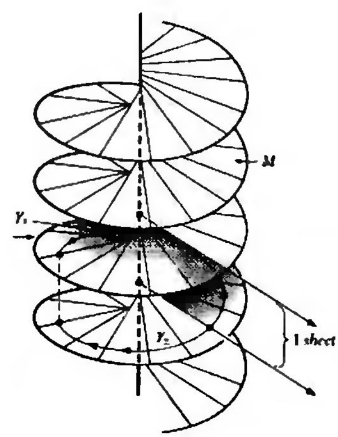


Figure 6.1.14: Riemann surface for $\log z$.

Only a core of the spiral staircase M, with axis over the origin, is shown —it should have infinite extent laterally. If we cut from 0 outward at any level and the one directly below it, we get a part of the surface called a *sheet*, the shaded portion in Figure 6.1.14. This shaded portion can be identified with the domain for a branch of log. Thus we have stacked up infinitely many copies of the complex plane $\mathbf{C}$ joined through 0 and glued together as shown in Figure 6.1.14. The arcs γ_1 and γ_2 now go to different points so we can assign different values of $\log z$ to each without ambiguity.

The main property of this surface that enables us to define $\log z = \log|z| + i \arg z$ as a single-valued function is that on this surface $\arg z$ is well defined, and the different sheets correspond to different intervals of length 2π in which $\arg z$ takes its values. Thus, we can take care of multiple-valued functions by introducing an enlarged domain on which the function becomes single-valued.

Let us briefly consider another example, the square root function: $z \mapsto \sqrt{z} = \sqrt{r}e^{i\theta/2}$. Here the situation is slightly different from that for the log function. If we go around the origin once, $\sqrt{z}$ takes on a different value, but if we go around twice (increase θ by 4π), we arrive back at the same value, so we want to be at the same point on the Riemann surface. The surface is illustrated in Figure 6.1.15.

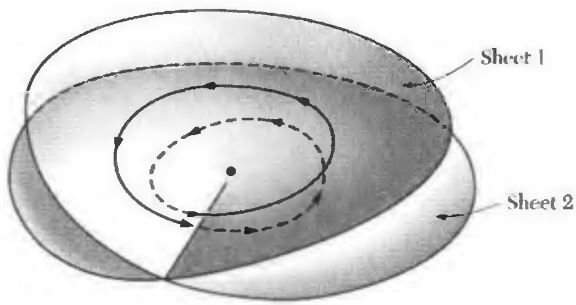


Figure 6.1.15: Riemann surface for $\sqrt{z}$.

Though the sheets in this figure appear to intersect, they are not supposed to. At fault is our attempt at visualization in $\mathbb{R}^3$. One can consider the Riemann surface to be in $\mathbb{R}^4$ or $\mathbb{C}^2$. Figure 6.1.15 is a picture of its "shadow" in $\mathbb{R}^3$. Here is another way to think about how the surface is related to analytic continuation. Let γ be the unit circle traveled twice counterclockwise by letting t change smoothly from 0 to 4π in $\gamma(t) = e^{it}$. Then $f(t) = e^{it/2}$ gives a smoothly changing square root for $\gamma(t)$. At the start, $\gamma(0) = 1$ and $f(0) = 1 =$ "$\sqrt{1}$."

As we make the first transit around the circle, $\gamma(t)$ successively hits points B, C, and D, namely the points i, -1 and $-i$, and the function $f(t)$ hits the corresponding points on the image circle. At $t = 2\pi$, the curve $\gamma(t)$ has returned to 1, but $f(t)$ has reached the other "square root," -1.

In the second transit around the circle, $\gamma(t)$ revisits the points it hit on the first circuit, while $f(t)$ goes through the other possible square roots in the lower half plane. At the end of the second circuit $\gamma(4\pi) = 1$, and $f(4\pi) = 1$ has returned to the original value. (See Figure 6.1.16.)

Riemann Surfaces of the Inverse Cosine Function For more complicated functions like $\cos^{-1}(z)$ the Riemann surface can be constructed as follows. On certain regions of $\mathbb{C}$, $\cos z$ is one-to-one and we define $\cos^{-1}(z)$ to be the inverse function. The period strips defined in §1.3 are examples of such regions for e^z and $\log z$. Such a region for $\cos z$ is shown in Figure 6.1.17.

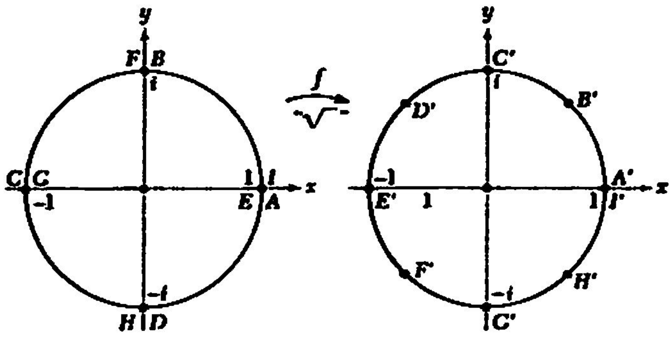


Figure 6.1.16: Tracking $\sqrt{z}$ as one traverses the unit circle.

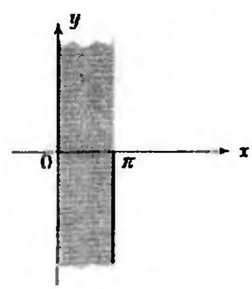


Figure 6.1.17: A region on which $\cos z$ is one-to-one.

The interior of each such strip is mapped conformally onto $\mathbb{C}$ minus the portions $]-\infty,-1]$ and $[1,\infty[$ of the real axis with half planes corresponding to half strips, as shown in Figure 6.1.18. Each of the deleted portions is the image of two different portions of the boundary of each strip. Each sheet of the Riemann surface is a copy of $\mathbb{C}$ slit along these portions of the real axis. The surface is then constructed by "gluing" the sheets together along these slits in such a way that half planes are joined in the same way as the corresponding preimage half strips.

A cross section of the surface over the circle $C = \{z \text{ such that } |z| = 2\}$ might be diagrammed somewhat as in Figure 6.1.19. The black dots in the diagram at the right indicate the places on the surface over 2 and -2 where the circle C crosses the slits along which the sheets are glued. To construct the model, one would roll the diagram at the right into a cylinder joining the top and bottom edges so that the labels on the sheets match. Then one would stand the cylinder over the circle C so that the rows of black dots are over 2 and -2. If we follow a suitably chosen curve winding around 1 and -1 passing sometimes between them and sometimes over the branch cuts, we may pass from any sheet to any other and obtain all possible values of $\cos^{-1} z$.

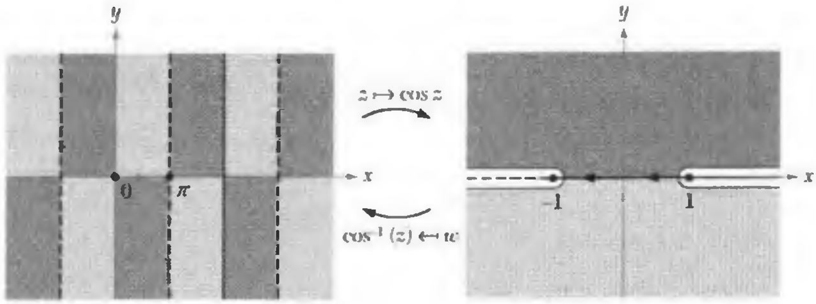


Figure 6.1.18: Construction of the Riemann surface for $\cos^{-1} z$.

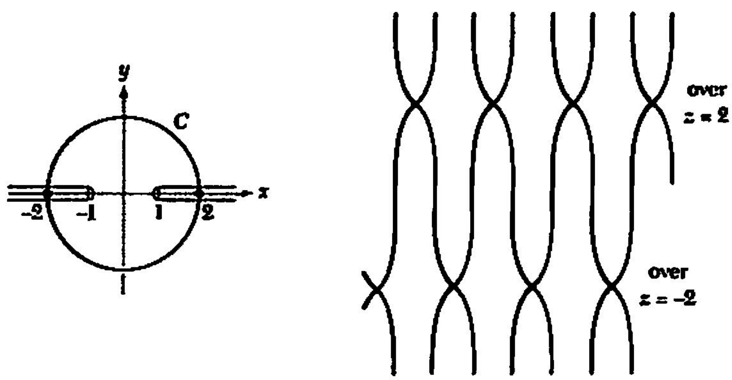


Figure 6.1.19: Cross section of Riemann surface for $\cos^{-1} z$ over the circle $|z| = 2$.

Worked Examples

Example 6.1.10 *Let f be an entire function equaling a polynomial on $[0, 1]$ on the real axis. Show that f is a polynomial.*

Solution Let $f(x) = a_0 + a_1x + \ldots + a_nx^n$ on $[0, 1]$. Then $f(z)$ and $a_0 + a_1z + \ldots + a_nz^n$ agree for $z \in [0, 1]$, and both are analytic on $\mathbb{C}$ (that is, both are entire). By the Identity Theorem 6.1.1, they are equal on all of $\mathbb{C}$, since $[0, 1]$ contains a convergent sequence of distinct points (for example, $z_n = 1/n$). This proves the assertion.

Example 6.1.11 *Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ have a radius of convergence $R > 0$.*

(a) *Is there always a sequence z_n with $|z_n - z_0| < R$ for $n = 1, 2, 3, \ldots$ and $|z_n - z_0| \to R$ such that $f(z_n) \to \infty$?*

(b) *Can f be continued analytically to a disk $|z - z_0| < R + \epsilon$ for some $\epsilon > 0$?*

Solution

(a) Such a sequence does not necessarily exist. Consider the series $\sum_{n=1}^{\infty} z^n/n^2$. By the ratio test, the radius of convergence is

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{n^2} = 1.$$

But for $|z| \le 1$ we have

$$\left|\sum_1^{\infty}\frac{z^n}{n^2}\right| \le \sum_1^{\infty}\left|\frac{z^n}{n^2}\right| = \sum_1^{\infty}\frac{|z|^n}{n^2} \le \sum_1^{\infty}\frac{1}{n^2} < \infty.$$

Thus $|f(z)|$ is bounded by $\sum_1^{\infty} 1/n^2$ on $\{z \text{ such that } |z| < 1\}$, so $f(z_n) \to \infty$ is impossible.

(b) No. Suppose that there is an analytic function g on $|z - z_0| < R + \epsilon$ with $g(z) = f(z)$ for $|z - z_0| < R$. Since f and g are analytic and agree on $|z-z_0| < R$, the Taylor series of g, $\sum_0^{\infty} a_n(z-z_0)^n$, is valid for $|z-z_0| < R+\epsilon$. Hence the radius of convergence of the given series is greater than R, which is impossible (since it equals R).

Example 6.1.12

(a) *Suppose f is given by the power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ valid for $|z-z_0| < R$. Show that if $|z_1 - z_0| < R$, then the Taylor series for f centered at z_1 is*

$$\sum_{k=0}^{\infty} b_k(z - z_1)^k, \quad \textit{where} \quad b_k = \sum_{m=0}^{\infty}\left[\frac{(k+m)!}{k!m!}a_{k+m}(z_1 - z_0)^m\right].$$

(b) *Work out the first few terms, starting with the principal branch of* $\log z$ *at* $z_0 = 1$ *and* $z_1 = (1+i)/2$.

Solution

(a) By taking the kth derivative of the series expansion for f about z_0, we find for $|z - z_0| < R$ that

$$\begin{aligned} f^{(k)}(z) &= \sum_{m=0}^{\infty}(k+m)(k+m-1)\dots(m+1)a_{k+m}(z-z_0)^m \\ &= \sum_{m=0}^{\infty}\frac{(k+m)!}{m!}a_{k+m}(z-z_0)^m. \end{aligned}$$

Thus the Taylor series of f around z_1 is

$$\sum_{k=0}^{\infty}\frac{1}{k!}f^{(k)}(z_1)(z-z_1)^k = \sum_{k=0}^{\infty}\left[\sum_{m=0}^{\infty}\frac{(k+m)!}{k!m!}a_{k+m}(z_1-z_0)^m\right](z-z_1)^k$$

This converges to $f(z)$ when $|z - z_1| < R - |z_1 - z_0|$ but may actually have a larger radius of convergence. If it does, then it gives an analytic continuation of f by power series.

(b) The principal branch of $\log(1 + w)$ for $|w| < 1$ has the expansion

$$\log(1 + w) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} w^n.$$

Setting $w = z - 1$ gives $\log z = \sum_{n=1}^{\infty} [(-1)^{n-1}/n](z-1)^n$, valid for $|z - 1| < 1$. Thus $a_0 = 0$ and $a_n = (-1)^{n-1}/n$ for $n > 0$. Since $z_1 - z_0 = (i - 1)/2$, we get

$$\begin{aligned}
b_0 &= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}\left(\frac{i-1}{2}\right)^m = \frac{1}{2}\log 2 + \frac{\pi i}{4} \\
b_1 &= \sum_{m=0}^{\infty} \frac{(m+1)!}{m!}\frac{(-1)^{m-1+1}}{m+1}\left(\frac{i-1}{2}\right)^m = \sum_{m=0}^{\infty}\left(\frac{1-i}{2}\right)^m \\
&= \frac{1}{1-(1-i)/2} = 1 - i
\end{aligned}$$

and

$$\begin{aligned}
b_2 &= \sum_{m=0}^{\infty} \frac{(m+2)!}{2!m!}\frac{(-1)^{m+1}}{m+2}\left(\frac{i-1}{2}\right)^m = \frac{1}{2}\sum_{m=0}^{\infty}(m+1)\left(\frac{1-i}{2}\right)^m \\
&= -\frac{1}{2}\frac{1}{\left[1-\left(\frac{1-i}{2}\right)\right]^2} = i.
\end{aligned}$$

Example 6.1.13 (Conformal Maps of Annuli)[2] *If* $0 < r < 1$, *and we define the annulus* $A_r = \{z \mid r < |z| < 1\}$, *let* $C_r = \{z$ such that $|z| = r\}$, *and let* $C_1 = \{s$ such that $|z| = 1\}$, *so that the closure of* A_r *is* $\mathrm{cl}(A_r) = C_r \cup A_r \cup C_1$. *Prove the following: Suppose* $0 < r < 1$ *and* $0 < R < 1$ *and that* f *is a one-to-one analytic map of* A_r *onto* A_R *that extends to a one-to-one continuous map of* $\mathrm{cl}(A_r)$ *onto* $\mathrm{cl}(A_R)$. *Then* $r = R$ *and* f *must be one of two types: either* (i) *a rotation, where there is a real constant* θ *such that* $f(z) = e^{i\theta}z$ *for all* z *in* A_r, *or* (ii) *a rotation and inversion, where there is a real constant* θ *such that* $f(z) = re^{i\theta}/z$ *for all* z *in* A_r.

Solution The function f must either map C_1 to C_1 and C_r to C_R or interchange the inner and outer circles. If the latter holds, then $f(r/z)$ is another map of A_r onto A_R that does not interchange them. Thus, we may assume that f takes C_1

[2]There is a rich literature about conformal maps of regions that are not simply connected. The subject is complicated because there is no theorem as broad in scope as the Riemann Mapping Theorem. In some sense the presence of more than one boundary component restricts the possible maps. This example shows how to use the Schwarz Reflection Principle 6.1.4 to study the situation for an annulus.

to C_1 and C_r to C_R continuously. The extended Schwarz Reflection Principle 6.1.5 shows how to continue f analytically to a map from the larger annulus A_{r^2} onto A_{R^2} so that the continuation is again continuous on the boundary and takes C_{r^2} to C_{R^2}. This process may be repeated indefinitely to extend f to an increasing sequence of annuli

$$A_r \subset A_{r^2} \subset A_{r^4} \subset \ldots$$

mapping respectively onto

$$A_R \subset A_{R^2} \subset A_{R^4} \subset \ldots .$$

(See Figure 6.1.20.) Each extension maps $C_{r^{2n}}$ to $C_{R^{2n}}$ and the annuli between them correspondingly.

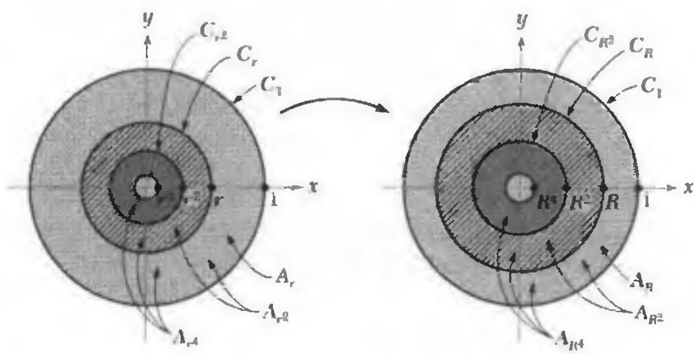


Figure 6.1.20: Conformal maps of annuli.

Since $R^{2n} \to 0$ as $n \to \infty$, we get $\lim_{z\to 0} f(z) = 0$. Thus $z = 0$ is a removable singularity and setting $f(0) = 0$ serves to complete the extension of f to an analytic function of the disk $D = \{z \text{ such that } |z| < 1\}$ to itself with $f(0) = 0$. The extended function satisfies the conditions of the Schwarz Lemma 2.5.7, so $|f(z)| \le |z|$ for all z. Since C_r goes to C_R, this forces $R \le r$. The process could just as well have been applied to f^{-1}, which takes A_R to A_r. This would give $r \le R$, so $r = R$. Finally, this shows $|f(z)| = |z|$ on each of the circles $C_{r^{2n}}$, so f must be a rotation by the Schwarz Lemma. ■

Exercises

1. (a) Let $f(z) = e^{1/z} - 1$. If $z_n = 1/2\pi n i$, then $z_n \to 0$ and $f(z_n) = 0$, yet f is not identically zero. Does this contradict the Identity Theorem 6.1.1? Why or why not?

   (b) Is the Identity Theorem true for harmonic functions?

2.* Let $h(x)$ be a function of a real variable $x \in \mathbb{R}$. Suppose that $h(x) = \sum_{n=0}^{\infty} a_n x^n$, which converges for x in some interval $]-\eta, \eta[$ around 0, where $\eta > 0$. Prove that h is the restriction of some analytic function defined in a neighborhood of 0.

3. Let f be analytic in a region A and let $z_1, z_2 \in A$. Let $f'(z_1) \neq 0$. Show that f is not constant on a neighborhood of z_2.

4. Let f be analytic and not identically zero on A. Show that if $f(z_0) = 0$, there is an integer k such that $f(z_0) = 0 = \ldots = f^{(k-1)}(z_0)$ and $f^{(k)}(z_0) \neq 0$.

5.* Prove the following result of Karl Weierstrass. Let $f(z) = \sum_{n=0}^{\infty} z^{n!}$. Then f cannot be analytically continued to *any* open set properly containing $A = \{z \text{ such that } |z| < 1\}$. *Hint*: First consider $z = re^{2\pi ip/q}$ where p and q are integers.

6. Formulate a Schwarz Reflection Principle for harmonic functions.

7.* Suppose that f can be continued analytically along a curve γ in the manner shown in Figure 6.1.21. Show that f can be continued by power series (in a finite number of steps).

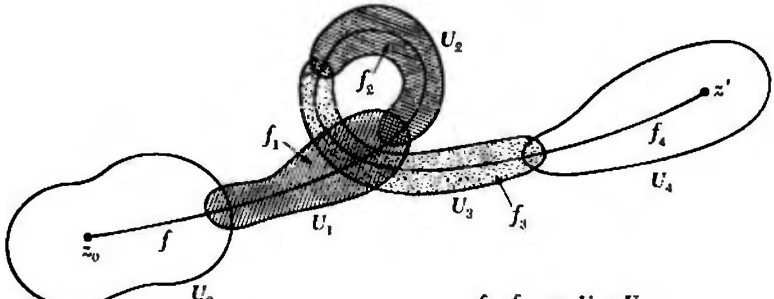


Figure 6.1.21: Analytic continuation of f along a curve from z_0 to z'.

8. Discuss the Riemann surface for $\sqrt{z^2 - 1}$.

9. Discuss the Riemann surface for $\sqrt[3]{z}$.

10. Discuss the relationship between Proposition 2.2.6 and the Monodromy Principle 6.1.9.

11. Consider the power series $\sum_0^{\infty}(-1)^n z^n$ defined in $|z| < 1$. To what domain in $\mathbb{C}$ can you analytically continue this function?

12. Show that if f is an analytic map of $\{z \mid r_1 < |z| < R_1\}$ one-to-one onto $\{z \mid r_2 < |z| < R_2\}$, which extends to a continuous map of $\{z \mid r_1 \leq |z| \leq R_1\}$

one-to-one onto $\{z \mid r_2 \le |z| \le R_2\}$, then $R_1/r_1 = R_2/r_2$. Give a description of all such functions.

13. Let A be a region, let $f : A \to \mathbb{C}$, and let $\gamma :]a, b[\to A$ be a smooth non-self-intersecting curve with $\gamma'(t) \neq 0$. Assume f is continuous on A and analytic on $A \backslash \gamma$.

    (a) Show that f is analytic.

    (b) Use (a) to prove the Schwarz Reflection Principle.

6.2 Rouché's Theorem and Principle of the Argument

In this section we develop some properties of analytic functions that are used to locate roots of equations within curves. The main tool will be the residue theorem.

Root and Pole Counting Formula The main results of this section will be those mentioned in the title. It is convenient to begin with a formula that counts the roots of an equation within a closed curve. A more intuitive version will be given as a corollary of the following precise version.

Theorem 6.2.1 (Root-Pole Counting Theorem) *Let f be analytic on a region A except for poles at $b_1, \dots, b_m$ and zeros at $a_1, \dots, a_n$, counted with their multiplicities (that is, if b_l is a pole of order k, then b_l is to be repeated k times in the list, and similarly for the zeros a_j). Let γ be a closed curve homotopic to a point in A and passing through none of the points a_j or b_l. Then*

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i \left[\sum_{j=1}^{n} I(\gamma; a_j) - \sum_{l=1}^{m} I(\gamma; b_l) \right].$$

This ***root-pole counting formula*** applies in particular to ***meromorphic functions***, that is, functions defined on $\mathbb{C}$ except for poles (see §3.3). There can be only a finite number of poles in any bounded region, since poles are isolated.

Proof First, it is clear that $f'(z)/f(z) = g(z)$ is analytic except at the points $a_1, \dots, a_n, b_1, \dots, b_m$. If f has a zero of order k at a_j, f' has a zero of order $k-1$, so $f'/f = g$ has a simple pole at a_j and the residue there is k. This is because we can write $f(z) = (z - a_j)^k \phi(z)$, as was shown in §3.2, where ϕ is analytic and $\phi(a_j) \neq 0$; therefore,

$$g(z) = \frac{k(z-a_j)^{k-1}\phi(z)}{(z-a_j)^k\phi(z)} + \frac{(z-a_j)^k\phi'(z)}{(z-a_j)^k\phi(z)} = \frac{k}{z-a_j} + \frac{\phi'(z)}{\phi(z)}.$$

Thus, the residue at a_j is clearly k. Similarly, if b_l is a pole of order k, we can write, near b_l,

$$f(z) = \frac{\phi(z)}{(z-b_l)^k},$$

where ϕ is analytic and $\phi(b_l) \neq 0$ (see Proposition 3.3.4(iv)). Proceeding as we did with the points a_j, we see that near b_l,

$$g(z) = \frac{-k}{z-b_l} + \frac{\phi'(z)}{\phi(z)},$$

so the residue is $-k$. By the residue theorem,

$$\int_\gamma g(z)dz = 2\pi i \left\{ \sum_j{}' [\operatorname{Res}(g;a_j)] I(\gamma;a_j) + \sum_l{}' [\operatorname{Res}(g;b_l)] I(\gamma;b_l) \right\},$$

where $\sum'$ means the sum over the distinct points. Since the residue equals the number of times a_j occurs and minus that number for the b_l, this expression becomes

$$2\pi i \left[\sum_{j=1}^{n} I(\gamma;a_j) - \sum_{l=1}^{n} I(\gamma;b_l) \right]. \quad \blacksquare$$

As its name implies, the root-pole counting formula may be used for counting zeros and poles.

Corollary 6.2.2 *Let γ be a simple closed curve:*

(i) *If f is analytic on an open set containing γ and its interior except for finitely many zeros and poles none of which lie on γ, then*

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i (Z_f - P_f),$$

where Z_f is the number of zeros of f inside γ and P_f the number of poles of f inside γ each counted with their multiplicities (orders).

(ii) ***Root Counting Formula*** *If f is analytic on an open set containing γ and its interior and $f(z)$ is never equal to w on γ, then*

$$\int_\gamma \frac{f'(z)}{f(z)-w} dz = 2\pi i N_w$$

where N_w is the number of roots of the equation $f(z) = w$ inside γ counted with their multiplicities as zeros of $f(z) - w$.

Proof Since γ is simple the index of γ with respect to a_j is 1 if a_j is inside γ and 0 if it is outside. Theorem 6.2.1 thus gives part (i). Part (ii) follows by applying part (i) to $g(z) = f(z) - w$. ∎

Principle of the Argument We now consider a useful consequence of the root-pole counting theorem. For a closed curve γ and z_0 not on γ, the change in argument of $z - z_0$ as z traverses γ is $2\pi \cdot I(\gamma; z_0)$. This is the intuitive basis on which the index was developed; it is written $\Delta_\gamma \arg(z - z_0) = 2\pi \cdot I(\gamma; z_0)$ (see Figure 6.2.1).

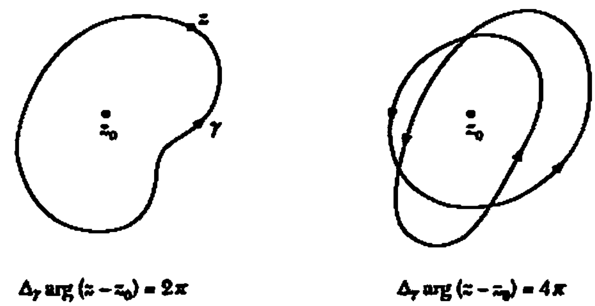


Figure 6.2.1: Change in the argument of $z - z_0$ when the two curves are traversed.

Next we want to define $\Delta_\gamma \arg f$, that is, the change in $\arg f(z)$ as z goes once around γ. Intuitively, and for practical computations, the meaning is clear; we merely compute $\arg f(\gamma(t))$ and let t run from a to b if $\gamma : [a, b] \to \mathbf{C}$, then look at the difference $\arg f(\gamma(b)) - \arg f(\gamma(a))$. We choose a branch of the argument such that $\arg f(\gamma(t))$ varies continuously with t. Equivalently, by changing variables, we can let $\tilde{\gamma} = f \circ \gamma$ and compute $\Delta_{\tilde{\gamma}} \arg z$. This leads to the following definition.

Definition 6.2.3 *Let f be analytic on a region A and let γ be a closed curve in A homotopic to a point and passing through no zero of f. We define*

$$\Delta_\gamma \arg f = 2\pi \cdot I(f \circ \gamma; 0).$$

(The index makes sense because 0 does not lie on $f \circ \gamma$.)

In examples, we can make use of our previous intuition about the index to compute $\Delta_\gamma \arg f$. The argument principle is as follows.

Theorem 6.2.4 (Principle of the Argument) *Let f be analytic on a region A except for poles at $b_1, \ldots, b_m$ and zeros at $a_1, \ldots a_n$ counted according to their multiplicity. Let γ be a closed curve homotopic to a point and passing through no a_j or b_l. Then*

$$\Delta_\gamma \arg f = 2\pi \left[\sum_{j=1}^{n} I(\gamma; a_j) - \sum_{l=1}^{m} I(\gamma; b_l) \right].$$

Proof By the Root-Pole Counting Theorem 6.2.1, it suffices to show that

$$i\Delta_\gamma \arg f = \int_\gamma \frac{f'(z)}{f(z)} dz$$

since f has no zeros or poles on γ. Indeed,

$$i\Delta_\gamma \arg f = 2\pi i \cdot I(f\circ\gamma; 0) = \int_{f\circ\gamma} \frac{dz}{z}$$

by the formula for the index (see §2.4). Letting $\gamma : [a, b] \to \mathbf{C}$, we have

$$\int_{f\circ\gamma} \frac{dz}{z} = \int_a^b \frac{\frac{d}{dt} f(\gamma(t))}{f(\gamma(t))} dt = \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)dt,$$

by the definition of the integral and the chain rule. The latter integral is equal to $\int_\gamma [f'(z)/f(z)]dz$ by definition. (If γ is only piecewise C^1, this holds only on each interval where γ' exists, and we get the result by addition.) ■

The Principle of the Argument 6.2.4 is usually applied in the case where γ is a *simple* closed curve. Then we may conclude that *the change in* $\arg f(z)$ *as we go once around* γ *(in a counterclockwise direction) is* $2\pi(Z_f - P_f)$ *where* Z_f *(or* P_f*) is the number of zeros (or poles) inside* γ *counted with their multiplicities.* It is somewhat surprising, a priori, that $Z_f - P_f$ and the argument change of f are even related.

This may sound familiar to the alert reader who remembers a trick from calculus called *logarithmic differentiation.* If γ is a small segment of curve short enough so that $f(\gamma)$ is a curve segment that lies in a half plane as in Figure 6.2.2, we can define a branch of logarithm with the branch cut leading away from that half plane by an appropriate choice of the reference angle for defining $\arg z$. Then along γ we have

$$\frac{d}{dz}[\log f(z)] = \frac{f'(z)}{f(z)},$$

so

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \Delta \log f(z) = \Delta \log |f(z)| + i\Delta \arg f(z).$$

For a closed curve γ we can do this along successive short parts of the curve using an appropriate choice of logarithm for each. When we return to the starting point, the contributions for $\Delta \log |f(z)|$ will all have canceled out, but not those for $\Delta \arg f(z)$, since we have kept changing determinations of argument.

Rouché's Theorem The argument principle can be used to prove a very useful theorem that has many applications, some of which will be given throughout the remainder of this chapter.

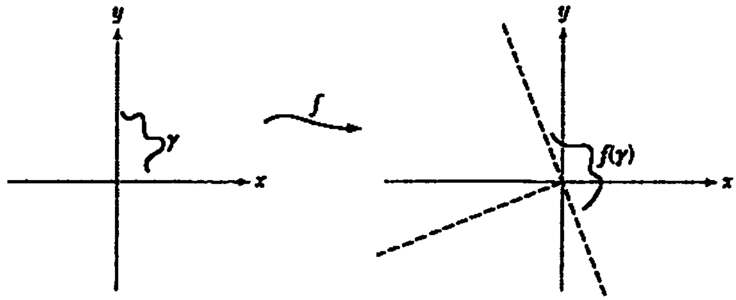


Figure 6.2.2: Logarithmic differentiation and the principle of the argument.

Theorem 6.2.5 (Rouché's Theorem) *Let f and g be analytic on a region A except for a finite number of zeros and poles in A. Let γ be a closed curve in A homotopic to a point and passing through no zero or pole of f or g. Suppose that on γ,*

$$|f(z) - g(z)| < |f(z)|.$$

Then (i) $\Delta_\gamma \arg f = \Delta_\gamma \arg g$ *and* (ii) $Z_f - P_f = Z_g - P_g$ *where Z_f is given by $Z_f = \sum_{j=1}^{n} I(\gamma; a_j)$, the a_j being the zeros of f counted with multiplicities, and with P_f, Z_g, P_g being defined similarly.*

Proof Since f and g have no zeros on γ, we can write our assumption as

$$\left|\frac{g(z)}{f(z)} - 1\right| < 1 \quad \text{on} \quad \gamma.$$

Thus, $g(z)/f(z) = h(z)$ maps γ into the unit disk centered at 1 (see Figure 6.2.3). We must have $I(h \circ \gamma; 0) = 0$, since $h \circ \gamma$ is homotopic to the point 1 in that disk (which does not contain 0). Thus,

$$\int_\gamma \frac{h'(z)}{h(z)} = i\Delta_\gamma \arg h = 2\pi i\, I(h \circ \gamma; 0) = 0.$$

We compute that

$$\frac{h'(z)}{h(z)} = \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}$$

and thus

$$\int_\gamma \frac{g'(z)}{g(z)} dz = \int_\gamma \frac{f'(z)}{f(z)} dz.$$

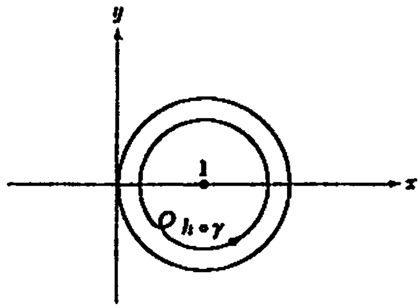


Figure 6.2.3: The image of γ under h.

Hence the result follows from the Root-Pole Counting Theorem 6.2.1 and the principle of the argument. ■

An important special case of Rouché's Theorem is the following. *Let γ be a simple closed curve and let f and g be analytic inside and on γ with γ passing through no zeros of f or g; suppose that $|f(z) - g(z)| < |f(z)|$ on γ. Then f and g have the same number of zeros inside γ.* Note that if $|f(z) - g(z)| < |f(z)|$ on γ, then γ automatically can pass through no zeros of f or g (Why?).

Rouché's Theorem can be used to locate the zeros of a polynomial. An illustration is given in Worked Example 6.2.12. Rouché's Theorem can also be used to give a simple proof of the Fundamental Theorem of Algebra, including the fact that an nth-degree polynomial has *exactly* n roots (see Exercise 9).

Hurwitz' Theorem One of the theoretical applications of Rouché's Theorem 6.2.5 is the following result of Hurwitz.

Theorem 6.2.6 (Hurwitz' Theorem) *Let f_n be a sequence of analytic functions on a region A converging uniformly on every closed disk in A to f. Assume that f is not identically zero, and let $z_0 \in A$. Then $f(z_0) = 0$ iff there is a sequence $z_n \to z_0$ and there is an integer N such that $f_n(z_n) = 0$ whenever $n \geq N$ (that is, a zero of f is a limit of zeros of the functions f_n).*

The theorem will follow from the next proposition.

Proposition 6.2.7 *Let f_n be a sequence of functions analytic on a region A that converge uniformly on every closed disk in A to f. Assume f is not identically* 0 *and that γ is a simple closed curve that together with its interior is contained in A and that passes through no zeros of f. Then there is an integer $N(\gamma)$ such that each f_n with $n \geq N(\gamma)$ has the same number of zeros inside γ as does f (counted according to multiplicity).*

Proof Since $|f|$ is continuous and never 0 on the compact set γ, it has a nonzero minimum m on γ; let us say $|f(z)| \geq m > 0$ for all z on γ. The curve is covered by a finite number of closed disks, so the convergence of f_n to f is uniform on γ. Accordingly, there is an integer $N(\gamma)$ such that $|f_n(z) - f(z)| < m \leq f(z)$ for all z on γ whenever $n \geq N(\gamma)$. Rouché's Theorem 6.2.5 applies, and we conclude that f_n and f have the same number of zeros inside γ, as desired. (Note that f is analytic on A by the Analytic Convergence Theorem 3.1.8.) ∎

Proof of Theorem 6.2.6 Again, f is analytic on A by the Analytic Convergence Theorem 3.1.8. Suppose $f(z_0) = 0$. Since f is not identically 0, the zeros are isolated by the Identity Theorem 6.1.1. There is a number $\delta > 0$ such that $f(z)$ is never 0 in the deleted neighborhood $\{z \mid 0 < |z - z_0| < \delta\}$. For each positive integer k, let γ_k be the circle $\{z \text{ such that } |z - z_0| = \delta/k\}$. Pick N_k as $N(\gamma_k)$ by Proposition 6.2.7. Then $n \geq N_k$ implies that f_n has at least one zero z_n inside γ_k. That is $f_n(z_n) = 0$. For $n \geq N_k$ we have $|z_n - z_0| < \delta/k$. This proves the theorem with $N = N_1$ (choose the z_n inside γ_k for $n \geq N_k$). ∎

We must assume that f is not identically zero. Consider, for example, the function $f_n(z) = e^z/n$, which approaches zero uniformly on closed disks (Why?) but for which f_n has no zeros.

Corollary 6.2.8 *Let f_n be a sequence of functions analytic on a region A that converge uniformly on closed disks in A to f. If each f_n is one-to-one on A and f is not constant, then f is one-to-one on A.*

Proof Suppose a and b are in A and $f(a) = f(b)$. We want to show that $a = b$. Consider $g_n(z) = f_n(z) - f_n(a)$ and $g(z) = f(z) - f(a)$. Then $g_n \to g$ uniformly on closed disks in A and $g(b) = 0$. Since g is not identically 0, Hurwitz' Theorem 6.2.6 says there is a sequence $z_n \to b$ with $g_n(z_n) = 0$. That is, $f_n(z_n) = f_n(a)$. But f_n is one-to-one, so $z_n = a$. Since $z_n \to b$, we must have $a = b$, as desired. ∎

It is possible for one-to-one functions to converge uniformly on closed disks to a constant function. For example, the functions $f_n(z) = z/n$ converge uniformly on the unit disk to the constant function $f(z) = 0$.

5 One-to-One Functions Analytic functions that are one-to-one find many useful applications. The term ***schlicht*** (simple) function is often used. We now relate one-to-one functions with the Inverse Mapping Theorem. Again Rouché's Theorem is the appropriate tool.

Proposition 6.2.9 *If $f : A \to \mathbf{C}$ is analytic and locally one-to-one, then $f'(z_0) \neq 0$ for all $z_0 \in A$. It follows from the Inverse Function Theorem that $f(A)$ is open and, if f is globally one-to-one, that f^{-1} is analytic from $f(A)$ to A.*

Proof Suppose that, on the contrary, for some point z_0 we have $f'(z_0) = 0$. Then $f(z) - f(z_0)$ has a zero of order $k \geq 2$ at z_0. Now f is not constant and thus the zeros of f' are isolated. Thus, there are a $\delta > 0$ and an $m > 0$ such that on the circle $|z - z_0| = \delta, |f(z) - f(z_0)| \geq m > 0$ and $f'(z) \neq 0$ for $0 < |z - z_0| \leq \delta$. For $0 < \eta < m$, we conclude that $f(z) - f(z_0) - \eta$ has k zeros inside $|z - z_0| = \delta$, by Rouché's Theorem 6.2.5. A zero cannot be a double zero, since $f'(z) \neq 0$ for $|z - z_0| \leq \delta, z \neq z_0$. Thus $f(z) = f(z_0) + \eta$ for two distinct points z and therefore is not one-to-one. This contradiction means that $f'(z_0) \neq 0$, as was to be shown. ■

Another basic property of one-to-one functions is the following.

Theorem 6.2.10 (One-to-One Theorem) *Let f be analytic on a region A and let γ be a closed curve homotopic to a point in A. Suppose that $I(\gamma; z) = 0$ or 1. Define the set $B = \{z \in A \mid I(\gamma; z) \neq 0\}$ (the "inside" of γ). If f is such that each point of $f(B)$ has index 1 with respect to the curve $\tilde{\gamma} = f \circ \gamma$, then f is one-to-one on B.*

Proof Consider, for $z_0 \in B$ and $w_0 = f(z_0)$,

$$N = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - w_0} dz.$$

By Corollary 6.2.2, N equals the number of times that $f(z) = w_0$ on B. We therefore must show that it equals 1. Letting $\tilde{\gamma} = f \circ \gamma$, we conclude, as in the principle of the argument, that

$$N = \frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{1}{z - w_0} dz,$$

which is the index of w_0 with respect to $\tilde{\gamma}$. Thus, $N = 1$ and therefore $f(z) = w_0$ has exactly one solution, $z = z_0$. This means that f is one-to-one. ■

The one-to-one theorem becomes more intuitive if we use the Jordan curve theorem. Let γ be a simple closed curve and let B be its interior. Suppose that the set $f(B)$ is bounded by the curve $\tilde{\gamma} = f \circ \gamma$. The hypothesis of the one-to-one theorem will be fulfilled if $\tilde{\gamma}$ is a simple closed curve (since this means that f should be one-to-one on γ). Therefore, the result may be rephrased as follows: *To see if an analytic function is one-to-one on a region, it is sufficient to check that it is one-to-one on the boundary.*

Worked Examples

Example 6.2.11 *Let n be a positive integer and let a be a real number satisfying $a > e$. Show that the equation $e^z = az^n$ has n solutions inside the unit circle.*

Solution Let $f(z) = e^z - az^n$ and let $g(z) = -az^n$. Notice that g has a zero of order n at the origin, so g has exactly n roots inside the unit circle $|z| = 1$. We will be done if we show that f and g have the same number of roots inside the unit circle. To do so we shall show that

$$|f(z) - g(z)| < |g(z)|$$

for $|z| = 1$. However,

$$|f(z) - g(z)| = |e^z| = e^x \leq e$$

since $|x| \leq 1$. Also, $|g(z)| = |az^n| = a > e$, so the result follows by Rouché's Theorem 6.2.5.

Example 6.2.12 *Use Rouché's Theorem to determine the quadrants in which the zeros of $z^4 + iz^2 + 2$ lie and the number of zeros that lie inside circles of varying radii.*

Solution Let $g(z) = z^4, f(z) = z^4 + iz^2 + 2$, and note that

$$|f(z) - g(z)| = |iz^2 + 2| \leq |z|^2 + 2$$

and that $|g(z)| = |z|^4$. Hence if $r = |z| > \sqrt{2}$, we have

$$|f(z) - g(z)| < |g(z)|.$$

Since g does not vanish on any circle of positive radius, the preceding inequality shows that f does not vanish on circles with radius $> \sqrt{2}$. Rouché's Theorem then shows that all four roots of f lie inside these circles, that is, inside the closed disk $|z| \leq \sqrt{2}$.

Next, let $h(z) = z^4 + 2iz^2 = z^2(z^2 + 2i)$. Clearly, h has a double root at 0 and two additional roots on the circle $|z| = \sqrt{2}$. Furthermore,

$$|f(z) - h(z)| = |-iz^2 + 2| = |z^2 + 2i| = \frac{|h(z)|}{|z|^2}.$$

For any choice of r with $1 < r < \sqrt{2}, h$ and hence f do not vanish on the circle $|z| = r$ and $|f(z) - h(z)| < |h(z)|$. Rouché's Theorem shows that f has precisely two zeros in $|z| < r$ for any of these values of r. Letting r approach 1 and $\sqrt{2}$, we see that f has two roots in the closed disk $|z| \geq 1$ and two on the circle $|z| = \sqrt{2}$.

Finally, let $k(z) = 2$. Then

$$|f(z) - k(z)| = |z^4 + iz^2| \leq |z|^4 + |z|^2 < 2 = |k(z)|$$

whenever $|z| < 1$. Arguing as before, for any r with $0 < r < 1, k$ and hence f do not vanish in $|z| < r$. Combining these three results we find that f has two zeros on $|z| = 1$ and two on $|z| = \sqrt{2}$.

Now we turn to an analysis of the quadrants in which the roots lie. For z either real or purely imaginary, $f(z) = z^4 + iz^2 + 2$ has a nonzero imaginary part unless $z = 0$. Thus f has no roots on the axes. Consider a large quarter circle as shown in Figure 6.2.4. We shall compute $\Delta_\gamma \arg(z^4 + iz^2 + 2)$ and use the Principle of the Argument 6.2.4. Along the x axis z is real and $f(z)$ lies in the first quadrant. Also $f(0) = 2$, and as $R \to \infty, \arg f(R) \to 0$ since

$$\arg f(R) = \arg R^4 \left(1 + \frac{i}{R^2} + \frac{2}{R^4}\right) = \arg\left(1 + \frac{i}{R^2} + \frac{2}{R^4}\right)$$

tends to 0 as $R \to \infty$. Since f takes its values in the first quadrant, we conclude that the change in the argument is zero as z moves from 0 to ∞. Along the curved portion of γ, z^4 clearly changes argument by $2\pi(= 4 \times \pi/2)$. As $R \to \infty, 2\pi$ is the limiting change in argument for $f(z)$ as well, as we see by writing

$$f(z) = z^4\left(1 + \frac{i}{z^2} + \frac{2}{z^4}\right).$$

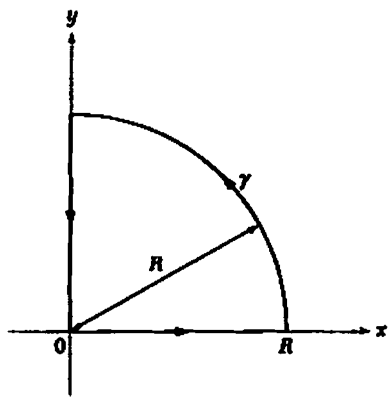


Figure 6.2.4: The curve γ used to locate the quadrants in which the zeros of the polynomial $z^4 + iz^2 + 2$ lie.

Similarly, coming down the imaginary axis there is, in the limit of large R, no change in the argument of f. (If $f(0)$ were not real, this device would still give the limiting behavior of the argument at infinity and the value at zero, so the change in the argument, at least up to multiples of 2π, can be inferred.) We conclude that the change in argument as we traverse γ is 2π. From the Principle of the Argument 6.2.4, there is exactly one zero in the first quadrant. By inspection, $f(z) = f(-z)$, so $-z$ is a root when z is. Thus there must be a root in each quadrant. Therefore, we must have one of the two possibilities shown in Figure 6.2.5. The methods used here do not enable us to tell which of these possibilities actually occurs without more detailed analysis. We can check this example by finding the roots directly using the quadratic formula twice; however, in other examples a direct computation may be impossible or impractical whereas the methods described here can nevertheless be used.

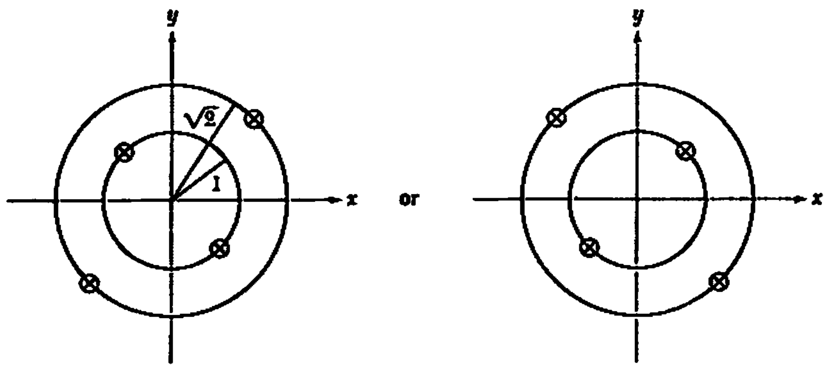


Figure 6.2.5: Locating the roots of the polynomial $z^4 + iz^2 + 2$.

Example 6.2.13 *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$. *Assume that* $a_0 = 0$ *and* $a_1 = 1$. *Prove that* f *is one-to-one on the unit disk* $\{z \text{ such that } |z| < 1\}$ *if* $\sum_{l=2}^{\infty} l|a_l| \le 1$.

Solution The series for f converges for $|z| < 1$ since, as a consequence of the assumed inequality $\sum_{l=2}^{\infty} l|a_l| \le 1$, we get $|a_n| \le 1$, and thus $|a_n z^n| \le |z|^n$; we know that $\sum |z|^n$ converges for $|z| < 1$. Thus f is analytic on $\{z \text{ such that } |z| < 1\}$.

Let $|z_0| < 1$. We want to show that $f(z) = f(z_0)$ has exactly one solution, z_0. Let $g(z) = z - z_0$, which has exactly one zero. If we set $h(z) = f(z) - f(z_0)$, then

$$h(z) - g(z) = \sum_{n=2}^{\infty} a_n z^n - \sum_{n=2}^{\infty} a_n z_0^n.$$

To estimate this we use the following trick. Let $\phi(z) = \sum_{n=2}^{\infty} a_n z^n$. Then

$$|\phi(z) - \phi(z_0)| \le |\max |\phi'(\zeta)|| \cdot |z - z_0|$$

where the maximum is over those ζ on the line joining z_0 to z (Why?). However.

$$|\phi'(\zeta)| = |\sum_{n=2}^{\infty} n a_n \zeta^{n-1}| < \sum_{n=2}^{\infty} n|a_n| \le 1,$$

since $|\zeta| < 1$. Hence

$$|h(z) - g(z)| = |\phi(z) - \phi(z_0)| < |z - z_0| = |g(z)|.$$

Thus, by Rouché's Theorem, $h(z) = f(z) - f(z_0)$ has exactly one solution, namely. $z = z_0$; this proves the assertion.

Example 6.2.14 *Find the largest disk centered at $z_0 = 1$ on which the function $f(z) = z^4$ is one-to-one.*

Solution This problem is intended to provide a warning against a common error. The derivative $f'(z) = 4z^3$ is 0 only at $z = 0$. In particular, $f'(z)$ is never 0 on the disk $D(1;1)$. However, we *cannot* conclude that f is one-to-one on this disk. In fact it is not. $f((1+i)/\sqrt{2}) = f((1-i)/\sqrt{2}) = -1$. If f is to be one-to-one near a point, the derivative must not be 0 at that point, and $f'(z) \neq 0$ is enough to guarantee that f is one-to-one in some neighborhood of z. But f' being never 0 on a large region is *not* enough to force f to be globally one-to-one on the whole region. In the present example $f(re^{i/4}) = f(re^{-i/4})$ for any r. Therefore the function will cease to be one-to-one as soon as the disk hits these 45° lines. This occurs for $D(1;R)$ when $R = 1/\sqrt{2}$. See Figure 6.2.6.

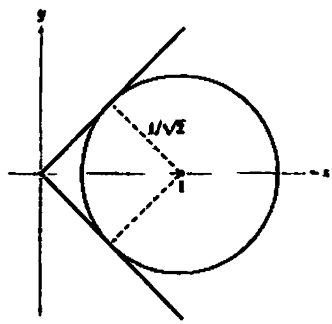


Figure 6.2.6: The function $f(z) = z^4$ is one-to-one on this disk.

Methods based on the one-to-one theorem (6.2.10) that involve looking at the boundary are usually more useful than examining the derivative. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are on the circle of radius R around 1 with $0 < R < \sqrt{2}$, then $-\pi/4 < \theta_1, \theta_2 < \pi/4$. $z_1^4 = z_2^4$ forces $r_1 = r_2$ and $e^{i4\theta_1} = e^{i4\theta_2}$, so $4(\theta_1 - \theta_2) = 2\pi n$. This cannot happen with θ_1 and θ_2 both between $-\pi/4$ and $\pi/4$ unless $\theta_1 = \theta_2$. But then $z_1 = z_2$. (We have actually shown that f is one-to-one on the open quarter plane $\{z \mid -\pi/4 < \arg z < \pi/4\}$.)

Exercises

1. How many zeros does $z^6 - 4z^5 + z^2 - 1$ have in the disk $\{z \text{ such that } |z| < 1\}$?

2. How many zeros does $z^4 - 5z + 1$ have in the annulus $\{z \mid 1 < |z| < 2\}$?

3. Show that there is exactly one point z in the right half plane $\{z \mid \operatorname{Re} z > 0\}$, at which $z + e^{-z} = 2$. *Hint*: Consider contours such as the one in Figure 6.2.7.

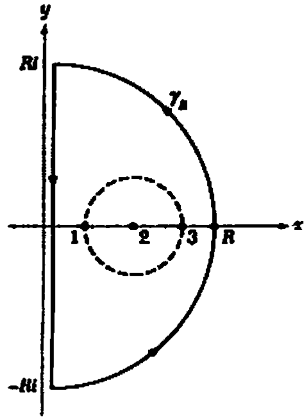


Figure 6.2.7: Contour for Exercise 3.

4. Show that if $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$, then there must be at least one point z with $|z| = 1$ and $|p(z)| \geq 1$. *Hint*: If $|p(z)| < 1$ everywhere on $\{|z| = 1\}$, how many zeros has $a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$?

5.• Let f be analytic inside and on the unit circle $|z| = 1$. Suppose that $0 < |f(z)| < 1$ if $|z| = 1$. Show that f has exactly one ***fixed point*** (defined to be a point z_0 such that $f(z_0) = z_0$) inside the unit circle.

6. Show that $e^z = 5z^3 - 1$ has three solutions in the disk $\{z \text{ such that } |z| < 1\}$. *Hint*: Think about Worked Example 6.2.11.

7. Show that the conclusion of Exercise 5 still holds if the assumption $0 < |f(z)| < 1$ is replaced by $0 < |f(z)| \leq 1$, allowing for the exception that the fixed point might be on the unit circle.

8. Let $g_n = \sum_{k=0}^{n} z^k/k!$. Let $D(0; R)$ be the disk of radius $R > 0$. Show that for n large enough, g_n has no zeros in $D(0; R)$.

9.• (*Fundamental Theorem of Algebra*) Use Rouché's Theorem 6.2.5 to prove that if $f(z) = a_0 + a_1 z + \ldots + a_n z^n, n \geq 1$, and $a_n \neq 0$, then f has exactly n roots counting multiplicity.

10. Supply the details of the following proof of Rouché's Theorem: Under the hypotheses of Theorem 6.2.5, the function $H(s,t) = sg(\gamma(t)) + (1-s)f(\gamma(t))$ is a closed curve homotopy between the curves $f \circ \gamma$ and $g \circ \gamma$ in $\mathbb{C}\backslash\{0\}$. It follows that $I(f \circ \gamma; 0) = I(g \circ \gamma; 0)$. The conclusion of Rouché's Theorem follows from this and the argument principle.

11.• Extend the Root-Pole Counting Theorem 6.2.1 to include the following result. If f is analytic on A except for zeros at $a_1, \ldots, a_n$ and poles at $b_1, \ldots, b_m$

(each repeated according to its multiplicity), if h is analytic on A, and if γ is a closed curve homotopic to a point in A, passing through none of $a_1, \ldots, a_n, b_1, \ldots, b_m$, then

$$\int_\gamma \frac{f'(z)}{f(z)} h(z)dz = 2\pi i \left[\sum_{i=1}^{n} h(a_i)I(\gamma; a_i) - \sum_{k=1}^{m} h(b_k)I(\gamma; b_k)\right].$$

12. Supply the details of the following proof (due to Caratheodory) of Rouché's Theorem: The function

$$F(\lambda) = \frac{1}{2\pi i}\int_\gamma \frac{\lambda g'(z) + (1-\lambda)f'(z)}{g(z) + (1-\lambda)f(z)} dz$$

is a continuous function of λ for $0 \le \lambda \le 1$. But its value is always an integer, so

$$Z_f - P_f = F(0) = F(1) = Z_g - P_g.$$

13. If $f(z)$ is a polynomial, use Exercise 11 to prove that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} z\, dz$$

is the sum of the zeros of f if the circle γ is large enough.

14. (a) Let $f : A \to B$ be analytic, one-to-one, and onto. Let $w \in B$ and let γ be a small circle centered at z_0 in A. Use Exercise 11 to prove that

$$f^{-1}(w) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)z}{f(z) - w} dz$$

for w sufficiently close to $f(z_0)$.

(b) Explain the meaning of

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - w} dz.$$

15. Let $f(z)$ be a polynomial of degree n. $n \ge 1$. Show that f maps $\mathbb{C}$ *onto* $\mathbb{C}$.

16. Suppose $g_n(z) = \sum_{k=0}^{n} 1/(k!z^k)$, and let $\epsilon > 0$. For large enough n, are all the zeros of g_n in the disk $D(0; \epsilon)$?

17. If $f(z)$ is analytic and has n zeros inside the simple closed curve γ, must it follow that $f'(z)$ has $n - 1$ zeros inside γ?

18.• Locate the zeros (as was done in Worked Example 6.2.12) for the polynomial $z^4 - z + 5 = 0$.

19. Find an $r > 0$ such that the polynomial $z^3 - 4z^2 + z - 4$ has exactly two roots inside the circle $|z| = r$.

20. Let f be analytic inside and on $|z| = R$ and let $f(0) \neq 0$. Let $M = \max |f(z)|$ on $|z| = R$. Show that the number of zeros of f inside $|z| = R/3$ does not exceed

$$\frac{1}{\log 2} \cdot \log \frac{M}{|f(0)|}.$$

Hint: Let $h(z) = f(z)/[(z - z_1) \cdot (z - z_n)]$ where z_i are the zeros of f inside $|z| = R/3$ and apply the Maximum Modulus Theorem 2.5.6 to h.

21.• Show that $z \mapsto z^2 + 3z$ is one-to-one on the set $\{z \text{ such that } |z| < 1\}$.

22. What is the largest disk around $z_0 = 0$ on which the function in Exercise 21 is one-to-one?

23.• Prove that the following statement is false: For every function f analytic on the annulus $\frac{1}{2} < |z| < \frac{3}{2}$, there is a polynomial p such that $|f(z) - p(z)| < \frac{1}{2}$ for $|z| = 1$.

24. Let f be analytic on $\mathbb{C}$ and let $|f(z)| \leq 5\sqrt{|z|}$ for all $|z| \geq 1$. Prove that f is constant.

6.3 Mapping Properties of Analytic Functions

Further local properties of analytic functions (that is, properties that depend only on the values of $f(z)$ for z in a neighborhood of a given point z_0) will be proved in this section. Alternative proofs will be given here of the Inverse Function Theorem 1.5.10, the Maximum Modulus Theorem 2.5.6, and the Open Mapping Theorem (stated formally for the first time in this section, but previously mentioned in Exercise 8, §1.5). We can prove these theorems and also obtain information concerning the behavior of a function near a point by using the root counting formula (see Corollary 6.2.2):

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - w} dz = \text{number of solutions of } f(z) = w \text{ inside } \gamma,$$

where the roots are counted with multiplicity.

Local Behavior of Analytic Functions If $f(z_0) = w_0$ with multiplicity k in the sense that $f(z) - w_0$ has a zero of order k at z_0, then we shall show that f is locally k-to-one near z_0. First consider the special example $f(z) = z^k$. This function has a zero of order k at $z_0 = 0$ (here $w_0 = 0$). For all w near $0, z^k = w$ has exactly k solutions near 0.

To see that this behavior is inherited by a more general function f for which $f(z_0) = w_0$ with multiplicity k, consider the power series expansion of f around z_0:

$$f(z) - w_0 = \sum_{n=k}^{\infty} a_n(z - z_0)^n.$$

For $|z - z_0|$ that are small enough, we might guess (correctly) that the behavior of the lowest-degree nonvanishing term, $a_k(z - z_0)^k$, will dominate.

Theorem 6.3.1 (Mapping Theorem: Informal Version) *Suppose f takes on the value w_0 at z_0 with multiplicity k. Then for all w sufficiently near w_0, the function f takes on the value w exactly k times near z_0 (counting multiplicities). For all w still nearer w_0, the k roots of $f(z) = w$ near z_0 are distinct.*

The more precise statement is the following.

Theorem 6.3.2 (Mapping Theorem) *Let f be analytic and not constant on a region A and let $z_0 \in A$. Suppose that $f(z) - w_0$ has a zero of order $k \geq 1$ at z_0. Then there is an $\eta > 0$ such that for any $\epsilon \in]0, \eta]$, there is a $\delta > 0$ such that if $|w - w_0| < \delta$, then $f(z) - w$ has exactly k roots (counted with their multiplicities) in the disk $|z - z_0| < \epsilon$* (see Figure 6.3.1).

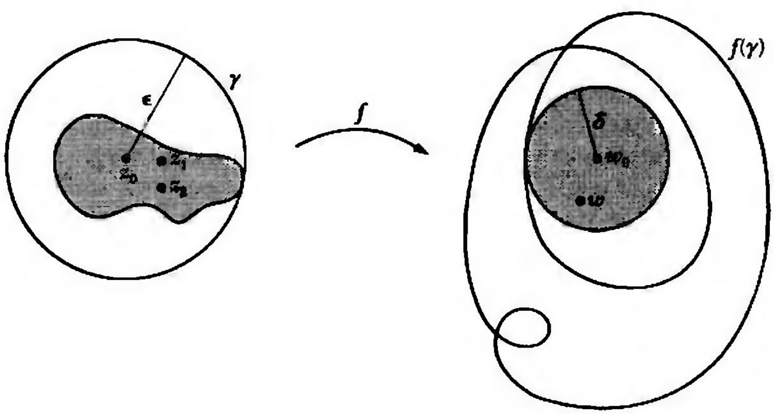


Figure 6.3.1: This function is two-to-one near z_0.

In fact, there is a $\lambda > 0$ (probably smaller than η) such that for any $\epsilon \in]0, \lambda]$, there is a $\delta > 0$ such that if $0 < |w - w_0| < \delta$, then $f(z) - w$ has exactly k distinct roots in the disk $0 < |z - z_0| < \epsilon$.

Proof Since f is not constant, the zeros of $f(z) - w_0$ are isolated. Thus there is an $\eta > 0$ such that for $|z - z_0| \leq \eta$, $f(z) - w_0$ has no zeros other than z_0. On the compact set $\{z \text{ such that } |z - z_0| = \epsilon\}$ (the circle γ in Figure 6.3.1), $f(z) - w_0$ is continuous and never zero. Hence there is a $\delta > 0$ such that $|f(z) - w_0| \geq \delta > 0$ for $|z - z_0| = \epsilon$. Thus if w satisfies $|w - w_0| < \delta$, then for $|z - z_0| = \epsilon$, the following hold:

(i) $f(z) - w_0 \neq 0$

(ii) $f(z) - w \neq 0$ (since $f(z) = w$ would mean that $|w - w_0| \geq \delta$)

(iii) $|(f(z) - w) - (f(z) - w_0)| = |w - w_0| < \delta \leq |f(z) - w_0|$

By Rouché's Theorem 6.2.5, $f(z) - w$ has the same number of zeros, counting multiplicities, as $f(z) - w_0$ inside the circle $|z - z_0| = \epsilon$. Thus we have proved the first part of the theorem. To prove the second part, notice that f' is not identically zero on A (because f is assumed to not be constant). The zeros of f' are thus isolated. Therefore, for some $\lambda \leq \eta$, neither $f(z) - w_0$ nor $f'(z)$ is zero in $|z - z_0| \leq \lambda$ except at z_0. Observe that $f(z) - w$ still has the same number of roots as $f(z) - w_0$ for any w near enough to w_0, but now the roots must be first-order, hence distinct, since f' is nonzero. ■

Open Mapping and Inverse Function Theorems The Mapping Theorem tells us that on some disk centered at z_0, f is exactly k-to-one. The theorem may not be directly helpful in finding the size of this disk (see the examples and exercises at the end of this section), but often knowledge of its existence can lead to interesting results.

A function $f : A \to \mathbb{C}$ is called ***open*** iff, for every open set $U \subset A, f(U)$ is open. By the definition of an open set, this statement is equivalent to: For every $\epsilon > 0$ sufficiently small, there is a $\delta > 0$ such that $|w - w_0| < \delta$ implies that there is a $z, |z - z_0| < \epsilon$ with $w = f(z)$. In other words, if f hits w_0, it hits every w sufficiently near w_0. Careful reading of the definition of open set and examination of Figure 6.3.1 show that the Mapping Theorem implies the next theorem:

Theorem 6.3.3 (Open Mapping Theorem) *Let $A \subset \mathbb{C}$ be open and $f : A \to \mathbb{C}$ be nonconstant and analytic. Then f is an open mapping; that is, the image of any open set under f is open.*

Using the Mapping Theorem 6.3.2, we can also get an alternative proof of the Inverse Function Theorem 1.5.10.

Theorem 6.3.4 (Inverse Function Theorem) *Let $f : A \to \mathbb{C}$ be analytic, let $z_0 \in A$, and suppose that $f'(z_0) \neq 0$. Then there is a neighborhood U of z_0 and a neighborhood V of $w_0 = f(z_0)$ such that $f : U \to V$ is one-to-one and onto and $f^{-1} : V \to U$ is analytic.*

Proof The function $f(z) - w_0$ has a simple zero at z_0 since $f'(z_0) \neq 0$. We can use Theorem 6.3.2 to find $\epsilon > 0$ and $\delta > 0$ such that each w with $|w - w_0| < \delta$ has exactly one preimage x with $|z - z_0| < \epsilon$. Let $V = \{w \text{ such that } |w - w_0| < \delta\}$ and let U be the inverse image of V under the map f restricted to $\{z$ such that

$|z-z_0| < \epsilon\}$ (the shaded region of Figure 6.3.1). By the Mapping Theorem, f maps U one-to-one onto V. Since f is continuous, U is a neighborhood of z_0. By the Open Mapping Theorem, $f = (f^{-1})^{-1}$ is an open map, so f^{-1} is continuous from V to U. To show that it is analytic, use

$$f^{-1}(w) = \frac{1}{2\pi i}\int_{|z-z_0|=\epsilon} \frac{f'(w)}{f(z)-w} z\, dz$$

(see Exercise 14 at the end of §6.2). This is analytic in w from Worked Example 2.4.15. ■

These ideas can be used as the basis for another proof of the Maximum Modulus Theorem (see §2.5), as follows.

Theorem 6.3.5 (Maximum Modulus Theorem) *Let f be analytic on a region (open connected set) A. If $|f|$ has a local maximum at $z_0 \in A$, then f is constant.*

Proof Suppose that f is not constant and that $z_0 \in A$. Since f is an open map, for $|w - f(z_0)|$ sufficiently small there is a z near z_0 with $w = f(z)$. Choose w with $|w| > |f(z_0)|$. Specifically, choose $w = (1 + \delta/2)f(z_0)$ if $f(z_0) \neq 0$ and $w = \delta/2$ if $f(z_0) = 0$ for δ small. Then it is clear that f does not have a relative maximum at z_0. ■

A similar proof shows that if $f(z_0) \neq 0$, then f has no minimum at z_0 unless f is constant. The Maximum Modulus Principle 2.5.6 follows, as in §2.5.

Worked Examples

Example 6.3.6 Prove the following: *If f is analytic near $z_0 \in A$ and if $f(z)-f(z_0)$ has a zero of order k at z_0, where $1 \leq k < \infty$, then there is an analytic function $h(z)$ such that $f(z) = f(z_0) + [h(z)]^k$ for z near z_0, and h is locally one-to-one.*

Solution Since $k < \infty$, f is not constant. Since $f(z) - f(z_0)$ has a zero of order k at z_0, we can write $f(z) - f(z_0) = (z - z_0)^k\phi(z)$, where $\phi(z_0) \neq 0$ and ϕ is analytic. For z near z_0, $\phi(z)$ lies in a small disk around $\phi(z_0)$ not containing 0, by continuity. On such a disk we can define $\sqrt[k]{\phi(z)}$ and let $h(z) = (z - z_0)\sqrt[k]{\phi(z)}$. Then $h'(z_0) \neq 0$, so by the Inverse Function Theorem, h is locally one-to-one. See Figure 6.3.2.

Example 6.3.7 *Determine the largest disk around $z_0 = 0$ on which the function $f(z) = 1 + z + z^2$ is one-to-one.*

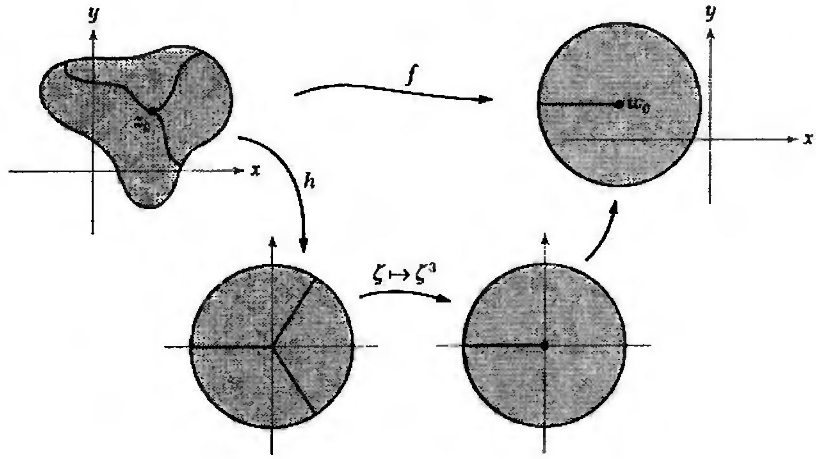


Figure 6.3.2: Worked Example 6.3.6 with $k = 3$.

Solution Since $f'(0) = 1, f(z) - 1$ has a simple zero at 0, and the Mapping Theorem (6.3.2) shows that f is one-to-one on some disk around $z_0 = 0$. Because $f(z) - 1 = z + z^2 = z(1 + z)$, which has roots at 0 and -1, we know that $f(z) - 1$ has only one root in the disk $\{z \text{ such that } |z| < 1\}$. This disk is the disk in the first part of the Mapping Theorem, but that does not guarantee that f is one-to-one on the disk; in fact, it is not. The Mapping Theorem shows only that f is one-to-one on the subregion of the disk shaded in Figure 6.3.1, the preimage of $\{w \text{ such that } |w - w_0| < \delta\}$. We can find out what causes this phenomenon by plotting the image of the unit circle. In this case $f(z) = 1 + z + z^2, z_0 = 0$, and $w_0 = 1$. Thus,

$$
\begin{aligned}
f(0) &= 1 \qquad & f\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) &= \left(1 + \frac{1}{\sqrt{2}}\right) - \left(1 + \frac{1}{\sqrt{2}}\right)i \\
f(1) &= 3 & & \\
f(i) &= i & f\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) &= \left(1 + \frac{1}{\sqrt{2}}\right) + \left(1 + \frac{1}{\sqrt{2}}\right)i \\
f(-1) &= 1 & & \\
f(-i) &= -i & f\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) &= \left(1 - \frac{1}{\sqrt{2}}\right) + \left(1 - \frac{1}{\sqrt{2}}\right)i \\
f(e^{2\pi i/3}) &= 0 & & \\
f(e^{4\pi i/3}) &= 0 & f\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) &= \left(1 - \frac{1}{\sqrt{2}}\right) - \left(1 - \frac{1}{\sqrt{2}}\right)i
\end{aligned}
$$

Plotting these points, we find that the image of the unit circle is as shown in Figure 6.3.3.

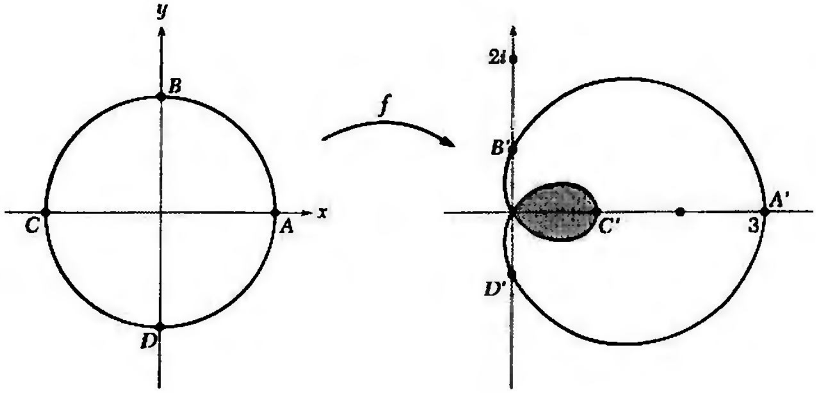


Figure 6.3.3: Image of the unit circle under $f(z) = 1 + z + z^2$.

The index of the image curve with respect to the small shaded region is 2. Therefore, each point here is hit twice by points in the unit disk; for example, $f'(-\frac{1}{2}) = 0$ and $f(-\frac{1}{2}) = \frac{3}{4}$. The Mapping Theorem shows that f is two-to-one on small neighborhoods of $-\frac{1}{2}$. Thus f will not be one-to-one on any disk containing a neighborhood of $-\frac{1}{2}$.

Consider the disk $D(0;r) = \{z \text{ such that } |z| < r\}$. The boundary curve is the circle $\gamma_r = \{z \text{ such that } |z| = r\}$. As r gets smaller, the troublemaking loop in the image curve shrinks. For some critical r_0 it disappears. For $r > r_0$, f is not one-to-one on γ_r. For $r < r_0$, f is one-to-one on γ_r. By the One-to-one Theorem (6.2.10), f is thus one-to-one on $D(0;r)$, and the desired disk is $D(0;r_0)$. To find r_0, suppose that $re^{i\theta}$ and $re^{i\psi}$ lie on γ_r and that $f(re^{i\theta}) = f(re^{i\psi})$. Then

$$1 + re^{i\theta} + r^2e^{i2\theta} = 1 + re^{i\psi} + r^2e^{i2\psi}.$$

Hence $e^{i\theta} + re^{i2\theta} = e^{i\psi} + re^{i2\psi}$, so

$$re^{i(\theta+\psi)}(e^{i(\theta-\psi)} - e^{i(\psi-\theta)}) = e^{i(\theta+\psi)/2}(e^{i(\psi-\theta)/2} - e^{i(\theta-\psi)/2}).$$

Thus,

$$re^{i(\theta+\psi)/2}\sin(\theta-\psi) = -\sin\left(\frac{\theta-\psi}{2}\right).$$

In other words,

$$2re^{i(\theta+\psi)/2}\sin\left(\frac{\theta-\psi}{2}\right)\cos\left(\frac{\theta-\psi}{2}\right) = -\sin\left(\frac{\theta-\psi}{2}\right).$$

One of two things must happen: either $\sin[(\theta-\psi)/2] = 0$, in which case $\theta-\psi = 2\pi n$ for some integer n, and thus $re^{i\theta} = re^{i\psi}$, or $\cos[(\theta-\psi)/2] = -(1/2r)e^{-i(\theta+\psi)/2}$.

If $r > \frac{1}{2}$, the latter can happen for $\psi = -\theta$; for example, at $r = 1$, it occurs at the points $e^{2\pi i/3}$ and $e^{4\pi i/3}$. If $r < \frac{1}{2}$, this same condition cannot hold, since $|\cos[(\theta-\psi)/2]| \leq 1$. If $r = \frac{1}{2}$, it can happen only for $\theta = \psi = \pi$. The critical radius is therefore $r_0 = \frac{1}{2}$. Hence f is one-to-one on the disk $D(0;\frac{1}{2}) = \{z \text{ such that } |z| < \frac{1}{2}\}$ but not on any larger open disk. ($D(0;\frac{1}{2})$ is the largest disk around $z_0 = 0$ on which $f'(z)$ is never zero. It is not generally true that this will also be the disk on which f is one-to-one (see Exercise 3)).

Exercises

1. Let $f(z) = z + z^2$. For each z_0 specified, find the largest disk centered at z_0 on which f is one-to-one:

   (a) $z_0 = 0$ (b) $z_0 = 1$

2.• What is the largest disk around $z_0 = 1$ on which $f(z) = e^z$ is one-to-one?

3. Let f be analytic on $D = \{z \text{ such that } |z - z_0| < r\}$. Let $f(z_0) = w_0$ and suppose that $f(z) - w_0$ has no roots in D other than z_0 and that $f'(z)$ is never zero in D. Show that it is not necessarily true that f is one-to-one on D. *Hint*: Consider z^3.

4. What is the largest disk centered at $z_0 = 1$ on which $f(z) = z^3$ is one-to-one? *Hint*: See Exercise 3.

5.• If f is analytic on A, $0 \in A$, and $f'(0) \neq 0$, then prove that near 0 we can write $f(z^n) = f(0) + [h(z)]^n$ for some analytic function h that is one-to-one near 0. *Hint*: Use Worked Example 6.3.6.

6. Let $u : A \to \mathbf{R}$ be harmonic and nonconstant on a region A. Prove that u is an open mapping.

7. Use Exercise 6 to prove the maximum and minimum principles for harmonic functions (see §2.5).

8. Let f be entire and have the property that if $B \subset \mathbf{C}$ is any bounded set, then $f^{-1}(B)$ is bounded (or perhaps empty). Show that for any $w \in \mathbf{C}$, there exists $z \in \mathbf{C}$ such that $f(z) = w$. *Hint*: Show that $f(\mathbf{C})$ is both open and closed and deduce that $f(\mathbf{C}) = \mathbf{C}$. Apply this result to polynomials to deduce yet another proof of the Fundamental Theorem of Algebra.

9. Show that the equation $z = e^{z-a}$, $a > 1$, has exactly one solution inside the unit circle.

10. Consider Worked Example 6.3.6 and take the case where $k = 4$. Visualize the local mapping in three steps as follows:

$$z \mapsto t = (z - z_0)\sqrt[4]{\phi(z)}; \qquad t \mapsto s = t^4; \qquad s \mapsto w = s + f(z_0).$$

    Sketch this mapping.

11. Suppose f is analytic in a region A containing the closed unit disk $D = \{z \text{ such that } |z| \leq 1\}$ and that $|f(z)| > 2$ whenever $|z| = 1$. If $f(0) = 1$, show that f has a zero in D.

12. Let $f(z) = \sum_0^\infty a_n z^n$ have a radius of convergence R. Suppose that $|a_1| \geq \sum_{n=2}^\infty n|a_n|r^{n-1}$ for some $0 < r \leq R$. Show that f is one-to-one on $\{z \text{ such that } |z| < r\}$ unless f is constant. Compare your method with that used to solve Worked Example 6.2.13.

Review Exercises for Chapter 6

1. Let f be analytic on $\{z \text{ such that } |z| < 1\}$ and let $f(1/n) = 0, n = 1, 2, \ldots$. What can be said about f?

2. Suppose that f and g are analytic on the disk $A = \{z \text{ such that } |z| < 2\}$ and that neither $f(z)$ nor $g(z)$ is ever 0 for $z \in A$. If
$$\frac{f'(1/n)}{f(1/n)} = \frac{g'(1/n)}{g(1/n)} \quad \text{for} \quad n = 1, 2, 3, 4 \ldots,$$
show there is a constant c such that $f(z) = cg(z)$ for all $z \in A$. *Hint*: Consider $(f/g)'(1/n)$.

3. Suppose that f is an entire function and that there is a bounded sequence of distinct real numbers $a_1, a_2, a_3, \ldots$ such that $f(a_k)$ is real for each k.
   (a) Show that $f(x)$ is real for all real x.
   (b) Suppose $a_1 > a_2 > a_3 > \ldots > 0$ and $\lim_{k\to\infty} a_k = 0$. Show that if $f(a_{2n+1}) = f(a_{2n})$ for all n, then f must be constant.

4. If f is analytic on the set $\{z \text{ such that } |z| < 1\}$ and $f(1 - 1/n) = 0, n = 1, 2, 3, \ldots$, does it follow that $f = 0$?

5.• Let f be analytic and bounded on $\{z \mid \operatorname{Im} z < 1\}$ and suppose that f is real on the real axis. Show that f is constant.

6. Let f be analytic and bounded on $|z + i| > \frac{1}{2}$ and real on $]-1, 1[$. Show that f is constant. *Hint*: Use the Schwarz Reflection Principle 6.1.4 from §6.1.

7. Let f be entire and suppose that for $z = x$ real, $f(x + 1) = f(x)$. Show that $f(z + 1) = f(z)$ for all $z \in \mathbf{C}$.

8.• Show that for $n > 2$, all the roots of $z^n - (z^2 + z + 1)/4 = 0$ lie inside the unit circle.

9. Suppose that f is analytic in $\mathbf{C}$ except for poles at $n \pm i, n = 0, \pm 1, \pm 2, \ldots$. What is the length of the longest interval $]x_0 - R, x_0 + R[$ in $\mathbb{R}$ on which $f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2/2 + \ldots + f^{(k)}(x_0)(x - x_0)^k/k! + \ldots$ converges?

10. Let $f : A \to B$ be analytic and onto; assume that $z_1, z_2 \in A, z_1 \neq z_2$, implies that $f(z_1) \neq f(z_2)$. Prove that f^{-1} is analytic.

11. Let f be a polynomial. Show that the integral of f'/f around every sufficiently large circle centered at the origin is $2\pi i$ times the degree of f.

12.• (a) Prove *Vitali's Convergence Theorem. Let f_n be analytic on a domain A such that*

   i. *For each closed disk B in A there is a constant M_B such that $|f_n(z)| \le M_B$ for all $z \in B$ and $n = 1, 2, 3, \ldots$.*
   ii. *There is a sequence of distinct points z_k of A converging to $z_0 \in A$ such that* $\lim_{n\to\infty} f_n(z_k)$ *exists for $k = 1, 2, \ldots$.*

   Then f_n converges uniformly on every closed disk in A; the limit is an analytic function. Hint: First take the case of a disk B with radius R and $z_k \to z_0 =$ the center of B. Use the Schwarz Lemma to show that $|f_n(z) - f_n(z_0)| \le 2M|z - z_0|/R$. Then show that

$$|f_n(z_0) - f_{n+p}(z_0)| \le \frac{4M|z - z_0|}{R} + |f_n(z) - f_{n+p}(z)|$$

   and deduce that $f_n(z_0)$ converges. Let

$$g_n(z) = \frac{f_n(z) - f_n(z_0)}{z - z_0}$$

   and conclude that $g_n(z_0)$ converges. Show that in general, if

$$f_n(z) = \sum_{k=0}^{\infty} a_{n,k}(z - z_0)^k,$$

   then $a_{n,k} \to a_k$ as $n \to \infty$. Deduce that $f_n(z)$ converges uniformly in $|z - z_0| < R - \epsilon$. Then use connectedness of A to deduce uniform convergence on any closed disk.

   (b) Show that if condition (i) is omitted, the conclusion is false. (Let $f_n(z) = z^n$.)

13. Let f be analytic on a region A and let γ be a closed curve in A homotopic to a point. Show that

$$\mathrm{Re}\left(\int_\gamma \frac{f'}{f}\right) = 0.$$

14. Let $f(z)$ be analytic on $\{z \mid 0 < |z| < 2\}$ and suppose that for $n = 0, 1, 2, \ldots$

$$\int_{|z|=1} z^n f(z)dz = 0.$$

   Show that f has a removable singularity at $z = 0$.

15. Let f be analytic and bounded on $A = \{z \text{ such that } |z| < 1\}$. Show that if f is one-to-one on $\{z \mid 0 < |z| < 1\}$, then f is one-to-one on A.

16.* Let $|f(z)| \leq 1$ when $|z| = 1$ and let $f(0) = \frac{1}{2}$ with f analytic. Prove that $|f(z)| \leq \frac{1}{2}(3|z|+1)$ for $|z| \leq \frac{1}{3}$ and $|f(z)| \leq 1$ for $\frac{1}{3} \leq |z| \leq 1$.

17. Let f and g be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Suppose that $f = g$ on the unit circle. Prove that $f = g$.

18. If $f(z)$ is analytic for $|z| < 1$ and if $|f(z)| \leq 1/(1-|z|)$, show that the coefficients of the expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ are subject to the inequality

$$|a_n| \leq (n+1)\left(1+\frac{1}{n}\right)^n < e(n+1).$$

19. Which of the following statements is/are true?

(a) The radius of convergence of $\sum_{n=0}^{\infty} 2^n z^{2n}$ is $1/\sqrt{2}$.

(b) An entire function that is constant on the unit circle is a constant.

(c) The residue of $1/[z^{10}(z-2)]$ at the origin is $-(2)^{-10}$.

(d) If f_n is a sequence of entire functions converging to a function f and if the convergence is uniform on the unit circle, then f is analytic in the open unit disk.

(e) $\displaystyle\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{2\pi}{a^2-1}$.

(f) For sufficiently large r, $\sin z$ maps the exterior of the disk of radius $r(\{z \text{ such that } |z| > r\})$ into any preassigned neighborhood of ∞.

(g) Let $f : \mathbb{C} \to \mathbb{C}$ be analytic in the open unit disk and let f have a nonremovable singularity at i. Then the radius of convergence of the Taylor series of f at 0 is 1.

(h) Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and nonconstant and let D be a domain in $\mathbb{C}$. Then f maps the boundary of D into the boundary of $f(D)$.

(i) Let f be analytic on $\{z \mid 0 < |z| < 1\}$ and suppose that $|f(z)| \leq \log(1/|z|)$. Then f has a removable singularity at 0.

(j) Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire and that f has exactly k zeros in the open unit disk but none on the unit circle. Then there exists an $\epsilon > 0$ such that any entire function g that satisfies $|f(z) - g(z)| < \epsilon$ for $|z| = 1$ must also have exactly k zeros in the open unit disk.

20. Prove the ***Phragmén-Lindelöf Theorem:***

(a) *Suppose that f is analytic in a domain that includes the strip*

$$G = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 1\}.$$

If $\lim_{z\to\infty, z\in G} f(z) = 0$ and if $|f(it)| \leq 1$ and $|f(1+it)| \leq 1$ for all real t, then $|f(z)| \leq 1$ for all $z \in G$.

(b) *If g is analytic in a domain containing G, if $\lim_{z\to\infty, z\in G} g(z) = 0$, and if $|g(it)| \le M$ and $|g(1+it)| \le N$ for all real t, then*

$$|g(z)| \le M^{1-\mathrm{Re}\, z} N^{\mathrm{Re}\, z}.$$

Hint: Apply the result of (a) to $f(z) = g(z)/M^{1-z}N^{z}$.

21. Is it correct to say that $1/\sqrt{z}$ has a pole at $z = 0$?

22.* Prove that for the principal value of the logarithm, $|\log z| \le r/(1-r)$ if $|1 - z| \le r < 1$.

23. (a) Let $f : \mathbf{C} \to \mathbf{C}$ be continuous on $\mathbf{C}$ and analytic on $\mathbf{C}\backslash\mathbf{R}$. Is f actually entire?

    (b) Let $f = \mathbf{C} \to \mathbf{C}$ be analytic on $\mathbf{C}\backslash\mathbf{R}$. Is f entire?

24. Let $P(z)$ be a polynomial. Prove that

$$\int_{|z|=1} P(z)d\bar{z} = -2\pi P'(0).$$

25. Find the radius of convergence of the series $\sum_0^\infty 2^n z^n$.

26. Show that $f(z) = (z^2+1)/(z^2-1)$ is one-to-one on $\{z \mid \mathrm{Im}\, z > 0\}$. Is it one-to-one on any larger set?

Chapter 7

Asymptotic Methods

This chapter gives an introduction to the theory of asymptotic methods, that is, to various approximation techniques for limits. The chapter begins with infinite products and the gamma function. These topics are of interest in their own right, and they provide motivation for the general study of asymptotic expansions, which is begun in §7.2. One of the main techniques used in this analysis, the method of steepest descent, and its variant, the method of stationary phase, are also considered and are applied to Stirling's formula and to Bessel functions in §7.3.

7.1 Infinite Products and the Gamma Function

To study the gamma function and subsequent topics, we first develop some basic properties of infinite products, which are somewhat analogous to the infinite sums considered in §3.1. For motivation, note that any polynomial $p(z)$ can be written in the form

$$p(z) = a_n(z-\alpha_1)\dots(z-\alpha_n) = a_n \prod_{j=1}^{n}(z-\alpha_j)$$

where $\alpha_1, \dots, \alpha_n$ are the roots of $p(z) = a_n z^n + \dots + a_1 z + a_0$ and $\prod$ stands for "take the product of" in the same way as $\sum$ stands for "take the sum of." It is natural to attempt to generalize this expression to entire functions, in which case the product becomes infinite.

Infinite Products Let $z_1, z_2, \dots$ be a sequence of complex numbers. We consider

$$\prod_{n=1}^{\infty}(1+z_n) = (1+z_1)(1+z_2)\dots.$$

We write $1+z_n$ because if the product is to converge, it is plausible that the general term should approach 1, that is, $z_n \to 0$. Roughly speaking, this is because if we take the logarithms of both sides, then for convergence of the resulting sum, the nth term, $\log(1+z_n)$ should go to zero (by the nth term test for series) and this means that z_n should go to zero.

Some technicalities are involved when $z_n = -1$ because the corresponding logarithms become infinite. We want to allow the product to be zero yet be able to impose some convergence condition. The following definition fits our needs.

Definition 7.1.1 *The product $\prod_{n=1}^{\infty}(1+z_n)$ is said to converge if only a finite number of z_n equal -1 and if the quantity*

$$\prod_{k=m}^{n}(1+z_k) = (1+z_m)\dots(1+z_n)$$

(where $z_k \neq -1$ for $k \geq m$), converges as $n \to \infty$, to a nonzero number. We set

$$\prod_{n=1}^{\infty}(1+z_n) = \lim_{n\to\infty}\prod_{k=1}^{n}(1+z_k).$$

(This product will be zero if some $z_k = -1$ and nonzero otherwise.)

For example, consider

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots$$

The nth ***partial product*** is

$$\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n-1}{n} = \frac{1}{n} \to 0.$$

Thus, the product does not converge, because we have demanded convergence to a *nonzero* number. In these circumstances, we would say that the product ***diverges to zero***. If we started at $n=1$, the product would still diverge. One reason for this terminology and convention is that the sequence of logarithms of the partial products diverges to $-\infty$.

By starting a given product beyond the point where some $z_n = -1$, we can assume that $z_n \neq -1$ for all n. Such an assumption imposes no real restrictions in the tests for convergence.

Theorem 7.1.2 (Convergence Theorem for Products)

(i) *If $\prod_{n=1}^{\infty}(1+z_n)$ converges, then $z_n \to 0$.*

(ii) *Suppose that $|z_n| < 1$ for all $n = 1, 2, \dots$ so that $z_n \neq -1$. Then $\prod_{n=1}^{\infty}(1+z_n)$ converges if and only if $\sum_{n=1}^{\infty}\log(1+z_n)$ converges. Here,* log *is the principal branch; $|z_n| < 1$ implies that* $\log(1+z_n)$ *is defined.*

(iii) *$\prod_{n=1}^{\infty}(1+|z_n|)$ converges iff $\sum_{n=1}^{\infty}|z_n|$ converges. We say that $\prod_{n=1}^{\infty}(1+z_n)$ converges absolutely in this case.*

(iv) *If $\prod_{n=1}^{\infty}(1+|z_n|)$ converges, then $\prod_{n=1}^{\infty}(1+z_n)$ converges.*

Criteria (iii) and (iv) are particularly important and are easy to apply. The proof of this theorem appears at the end of this section, but the plausibility of the theorem is discussed now. Criterion (i) was explained at the beginning of this section. To explain (ii), note that if we let $S_n = \sum_1^n \log(1+z_k)$ and $P_n = \prod_1^n(1+z_k)$, then $P_n = e^{S_n}$. That (ii) is plausible follows from this equation. Indeed, if $S_n \to S$, it is clear that $P_n \to e^S$. Once (ii) is shown, (iii) and (iv) follow. The following corollary is implicit in the preceding discussion.

Corollary 7.1.3 *If $|z_n| < 1$ and $\sum \log(1+z_n)$ converges to S, then $\prod(1+z_n)$ converges to e^S.*

This corollary is sometimes useful, but when it is applied to concrete problems, the sum of logarithms can be difficult to handle.

Let $f_n(z)$ be a sequence of functions defined on a set $B \subset \mathbf{C}$. How we should define the concept of the uniform convergence of $\prod_1^{\infty}(1+f_n)$ should be fairly clear. We do this next.

Definition 7.1.4 *The product*

$$\prod_{n=1}^{\infty}[1+f_n(z)]$$

is said to converge uniformly on B iff, for some m, $f_n(z) \neq -1$ for $n \geq m$ and all $z \in B$, if the sequence $P_n(z) = \prod_{k=m}^{n}[1+f_k(z)]$ converges uniformly on B to some $P(z)$, and if $P(z) \neq 0$ for all $z \in B$ (see §3.1 for the definition of uniform convergence of a sequence of functions).

The next result follows from the Analytic Convergence Theorem 3.1.8.

Theorem 7.1.5 (Analyticity of Infinite Products) *Suppose that $f_n(z)$ is a sequence of analytic functions on an open set A and that $\prod_{n=1}^{\infty}[1+f_n(z)]$ converges uniformly to $f(z)$ on every closed disk in A. Then $f(z)$ is analytic on A. Such uniform convergence holds if $|f_n(z)| < 1$ for $n \geq m$ and if either*

$$\sum_{n=m}^{\infty}\log[(1+f_n(z)] \quad \text{or} \quad \sum_{n=1}^{\infty}|f_n(z)|$$

converges uniformly (on closed disks in each case).

To check the validity of this statement, one must check that the proof of the convergence theorem for products works for uniform convergence; this is left as an exercise.

Canonical Products The following theorem is a special case of a result by Weierstrass that constructs the most general entire function with a given set of zeros. The special case described here is applicable to many examples, yet it illustrates the main ideas of the general case. (For a statement of the general case, see Exercises 10 and 14 at the end of this section.)

Theorem 7.1.6 (Theorem on Canonical Products) *Let $a_1, a_2, \ldots$ be a given sequence (possibly finite) of nonzero complex numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} < \infty.$$

If $g(z)$ is any entire function, then the function

$$f(z) = e^{g(z)} z^k \left[\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n} \right] \tag{7.1.1}$$

is entire. The product converges uniformly on closed disks, has zeros at $a_1, a_2, \ldots$, and has a zero of order k at $z = 0$, but it has no other zeros. Furthermore, if f is any entire function having these properties, it can be written in the same form. In particular, f is entire with no zeros if and only if f has the form $f(z) = e^{g(z)}$ for some entire function g. The product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

*is called a **canonical product**.*

The proof appears at the end of this section. The result is plausible if we note that the product vanishes exactly when z is equal to some a_n and that z^k has a zero of order k at 0. The finiteness condition on the a_n guarantees convergence of the product, as is shown in the proof (see Exercise 10 for more general conditions). Also, $e^{g(z)}$ vanishes nowhere, since $e^w \neq 0$ for all $w \in \mathbb{C}$. We note that the points $a_1, a_2, \ldots$ need not be distinct; each may be repeated finitely many times. If a_n is repeated l times, f will have a zero of order l at a_n.

The Theorem on Canonical Products 7.1.6 will be applied several times in the remainder of this section. For example, Worked Example 7.1.10 proves that

$$\sin \pi z = \pi z \prod_{n=-\infty, n\neq 0}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}. \tag{7.1.2}$$

Gamma Function The gamma function is a useful solution to an interpolation problem that has been studied since the 1700s. Here is the problem: *Find a continuous function of a real or complex variable that agrees with the factorial function at the integers.* The gamma function, $\Gamma(z)$ is one solution. It is analytic on $\mathbb{C}$ except for simple poles at $0, -1, -2, \ldots$, and $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$. The importance of this function was realized by Euler and Gauss as early as the eighteenth century.

Two equivalent definitions of the gamma function will be given: the first will be in terms of infinite products, the second in terms of an integral formula. These two formulas are due to Euler, with significant contributions made by Gauss and Legendre. The main facts that are included in the following discussion and in the end-of-section exercises are summarized in Table 7.1.1 at the end of this section.

For the first definition, we begin with an associated function that is defined by the canonical product

$$G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}. \tag{7.1.3}$$

By the Theorem on Canonical Products 7.1.6, this function is entire, with simple zeros at the negative integers $-1, -2, -3, \ldots$. This function satisfies the identity

$$zG(z)G(-z) = \frac{\sin \pi z}{\pi} \tag{7.1.4}$$

because of equation (7.1.2). Consider the function

$$H(z) = G(z-1), \tag{7.1.5}$$

which has zeros at $0, -1, -2, \ldots$. By the Theorem on Canonical Products, we can write

$$H(z) = e^{g(z)} z \prod_{1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} = z e^{g(z)} G(z) \tag{7.1.6}$$

for an entire function $g(z)$.

It will now be shown that $g(z)$ is constant. Using the convergence theorem for products, we get

$$\log H(z) = \log z + g(z) + \sum_{n=1}^{\infty} \left[\log\left(1 + \frac{z}{n}\right) - \frac{z}{n}\right].$$

Since the convergence is uniform on closed disks, we can differentiate term by term:

$$\frac{d}{dz} \log H(z) = \frac{1}{z} + g'(z) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right). \tag{7.1.7}$$

Similarly, by (7.1.3),

$$\begin{aligned}\frac{d}{dz}\log G(z-1) &= \sum_{n=1}^{\infty}\left(\frac{1}{z-1+n}-\frac{1}{n}\right)=\frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n+1}\right)\\ &= \frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right).\end{aligned}\tag{7.1.8}$$

Comparing (7.1.7) and (7.1.8) and using (7.1.5), we see that $g'(z)=0$, so $g(z)$ is constant. (Part (ii) of the convergence theorem for products is valid only for $|z|<1$, but this region of validity suffices since two entire functions that agree on $|z|<1$ are equal by Taylor's theorem or the identity theorem.)

The constant value $g(z)=\gamma$ is called ***Euler's constant.*** We can determine an expression for it as follows. By (7.1.3), (7.1.5), and (7.1.6), we get

$$G(z-1)=ze^{\gamma}G(z),\tag{7.1.9}$$

and therefore if we let $z=1$, then $G(0)=1=e^{\gamma}G(1)$. Thus, by (7.1.3),

$$e^{-\gamma}=\prod_{1}^{\infty}\left(1+\frac{1}{n}\right)e^{-1/n}=\prod_{1}^{\infty}\left(\frac{n+1}{n}\right)e^{-1/n}.$$

Noting that

$$\begin{aligned}\prod_{k=1}^{n}\left(\frac{k+1}{k}\right)e^{-1/k} &= \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{n+1}{n}e^{-1-1/2-1/3-\ldots-1/n}\\ &= (n+1)e^{-1-1/2-\ldots-1/n}\\ &= ne^{-1-1/2-\ldots-1/n}+e^{-1-1/2-\ldots-1/n},\end{aligned}$$

we get $e^{-\gamma}=\lim_{n\to\infty}ne^{-1-1/2-\ldots-1/n}$. Taking logs, we find that

$$\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\log n\right).\tag{7.1.10}$$

Our arguments show that this limit exists and is finite. Numerically, one finds from (7.1.10) that $\gamma\approx 0.57716\ldots$.

The ***gamma function*** is defined by

$$\Gamma(z)=[ze^{\gamma z}G(z)]^{-1}=\left[ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\right]^{-1}.\tag{7.1.11}$$

Since G is entire, with simple zeros at $-1,-2,\ldots$, we conclude that $\Gamma(z)$ is meromorphic, with simple poles at $0,-1,-2,\ldots$. From (7.1.9), $G(z-1)=ze^{\gamma}G(z)$, so

$$\Gamma(z+1)=z\Gamma(z)\quad\text{for}\quad z\neq 0,-1,-2,\ldots,\tag{7.1.12}$$

which is called the ***functional equation for the gamma function*** (see Exercise 7). Also, $\Gamma(1) = 1$, since $\Gamma(z) = [ze^{\gamma z}G(z)]^{-1}$ and $G(1) = e^{-\gamma}$ by our construction of γ. Thus, from (7.1.12) we see that $\Gamma(2) = 1 \cdot 1, \Gamma(3) = 2 \cdot 1, \Gamma(4) = 3 \cdot 2 \cdot 1$, and generally that, as earlier advertised,

$$\Gamma(n+1) = n!. \tag{7.1.13}$$

This formula leads to interesting approximations of $n!$, which are derived in §7.3 (Figure 7.1.1 shows a graph of $\Gamma(x)$ for x real.)

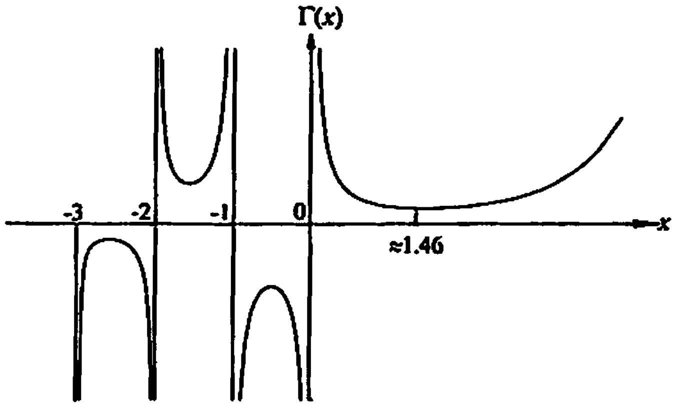


Figure 7.1.1: The graph of $\Gamma(x)$ for x real.

From the equation $zG(z)G(-z) = (\sin \pi z)/\pi$, we get

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{7.1.14}$$

We claim that $\Gamma(z) \neq 0$ for all $z, z \neq 0, -1, -2, \ldots$. Indeed, if $\Gamma(z) = 0$, we would have the contradiction $\pi = \Gamma(z)\Gamma(1-z)\sin \pi z = 0$ as long as $z \neq 0, \pm 1, \pm 2, \ldots$. (These are the points at which $\sin \pi z$ vanishes, so cross multiplication is invalid at those points.) We also know that $\Gamma(z) \neq 0$ if $z = 1, 2, 3 \ldots$, since $\Gamma(n+1) = n!, n = 0, 1, 2, \ldots$. This proves the claim.

If we let $z = 1/2$ in (7.1.14), we get $[\Gamma(1/2)]^2 = \pi$. But $\Gamma(1/2) > 0$. To see this, note that $\Gamma(z)$ is real for real positive z; we have shown that Γ has no zeros and that $\Gamma(n+1) = n! > 0$. Therefore, since $\Gamma(x)$ is continuous for $x \in]0, \infty[$ (because Γ is analytic), it follows from the intermediate value theorem that $\Gamma(x) > 0$ for all $x \in]0, \infty[$ (as in Figure 7.1.1). Thus, $\Gamma(1/2) = \sqrt{\pi}$ (rather than the other possibility, $-\sqrt{\pi}$).

Euler's Formula ***Euler's formula*** for the gamma function is

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}\right] = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)\ldots(z+n)}. \tag{7.1.15}$$

This formula is proven as follows. By definition,

$$\begin{aligned}\frac{1}{\Gamma(z)} &= z\left(\lim_{n\to\infty} e^{(1+1/2+\ldots+1/n-\log n)z}\right)\left[\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-z/k}\right]\\ &= z\lim_{n\to\infty}\left[e^{(1+1/2+\ldots+1/n-\log n)z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-z/k}\right]\\ &= z\lim_{n\to\infty}\left[n^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\right]\end{aligned}$$

since $n^{-z}=e^{-\log n \cdot z}$. Thus, we get

$$\begin{aligned}\frac{1}{\Gamma(z)} &= z\lim_{n\to\infty}\left[\prod_{k=1}^{n-1}\left(1+\frac{1}{k}\right)^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\right]\\ &= z\lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)^{z}\left[\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\left(1+\frac{1}{k}\right)^{-z}\right]\right\}.\end{aligned}$$

The first equality in (7.1.15) now follows. The student is asked to prove the second in Exercise 11.

Gauss' Formula Another important property of the gamma function is given in the *Gauss formula*: For any fixed positive integer $n\geq 2$,

$$\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\ldots\Gamma\left(z+\frac{n-1}{n}\right)=(2\pi)^{(n-1)/2}n^{(1/2)-nz}\Gamma(nz). \qquad (7.1.16)$$

To prove this formula we first note that we can write Euler's formula as

$$\begin{aligned}\Gamma(z) &= \lim_{m\to\infty}\frac{(m-1)!m^z m}{z(z+1)\ldots(z+m-1)(z+m)}=\lim_{m\to\infty}\frac{(m-1)!m^z}{z(z+1)\ldots(z+m-1)}\\ &= \lim_{m\to\infty}\frac{(mn-1)!(mn)^z}{z(z+1)\ldots(z+mn-1)}.\end{aligned}$$

(The first line follows since $m/(z+m)$ tends to 1 as $m\to\infty$. We define $f(z)$ as follows:

$$\begin{aligned}f(z) &= \frac{n^{nz}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\ldots\Gamma\left(z+\frac{n-1}{n}\right)}{n\Gamma(nz)}\\ &= \frac{n^{nz-1}\prod_{k=0}^{n-1}\lim_{m\to\infty}\frac{(m-1)!m^{z+k/n}}{\left(z+\frac{k}{n}\right)\left(z+\frac{k}{n}+1\right)\ldots\left(z+\frac{k}{n}+m-1\right)}}{\lim_{m\to\infty}\frac{(mn-1)!(mn)^{nz}}{nz(nz+1)\ldots(nz+nm-1)}}\\ &= \lim_{m\to\infty}\frac{[(m-1)!]^n m^{(n-1)/2}n^{mn-1}(nz)(nz+1)\ldots(nz+mn-1)}{(mn-1)!\prod_{k=0}^{n-1}[(nz+k)(nz+k+n)\ldots(nz+k+mn-n)]}\\ &= \lim_{m\to\infty}\frac{[(m-1)!]^n m^{(n-1)/2}n^{nm-1}}{(nm-1)!}.\end{aligned}$$

Thus, f is constant. Setting $z = 1/n$ gives

$$f(z) = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) > 0,$$

so

$$[f(z)]^2 = \frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n}}$$

using (7.1.14). From the fact that

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}} \quad \text{for} \quad n = 2, 3, \ldots$$

(see Exercise 28, §1.2), we get

$$[f(z)]^2 = \frac{(2\pi)^{n-1}}{n}.$$

Since $f(z) > 0$,

$$f(z) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}.$$

The Gauss formula therefore follows.

If we take the special case of (7.1.16), in which $n = 2$, we obtain the ***Legendre duplication formula:***

$$2^{2z-1}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z). \tag{7.1.17}$$

Residues of the Gamma Function We claim that *the residue of* $\Gamma(z)$ *at* $z = -m, m = 0, 1, 2, \ldots$ *is* $(-1)^m/m!$. Indeed, note that

$$(z+m)\Gamma(z) = (z+m)\frac{\Gamma(z+1)}{z} = (z+m)\frac{\Gamma(z+2)}{z(z+1)}.$$

More generally, we find that

$$(z+m)\Gamma(z) = \frac{\Gamma(z+m+1)}{z(z+1)\ldots(z+m-1)}.$$

Letting $z \to -m$, we get

$$\frac{\Gamma(1)}{-m(-m+1)\ldots(-1)} = \frac{(-1)^m}{m!}$$

as required.

Integral Formula for the Gamma Function There is an important expression for $\Gamma(z)$ as an integral. For $\operatorname{Re} z > 0$, we shall establish the following formula known as ***Euler's integral*** for $\Gamma(z)$:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt. \tag{7.1.18}$$

One might suspect that this expression can be evaluated by the methods of Chapter 4. Unfortunately, those methods are not applicable (the reader should contemplate why this is the case), so another method is needed to prove (7.1.18). To do this, we start by defining

$$F_n(z) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt.$$

We claim that

$$F_n(z) = \frac{n!n^z}{z(z+1)\dots(z+n)}. \tag{7.1.19}$$

By Euler's formula, we will then have proved that $F_n(z) \to \Gamma(z)$ as $n \to \infty$. To prove (7.1.19), we note that by changing variables and letting $t = ns$,

$$F_n(z) = n^z \int_0^1 (1-s)^n s^{z-1}ds.$$

Now integrate this expression successively by parts, the first step being

$$F_n(z) = n^z\left[\frac{1}{z}s^z(1-s)^n\,\Big|_0^1 + \frac{n}{z}\int_0^1 (1-s)^{n-1}s^z ds\right] = n^z\frac{n}{z}\int_0^1 (1-s)^{n-1}s^z ds.$$

Repeating this procedure, we integrate by parts n times and get

$$F_n(z) = n^z\frac{n\cdot(n-1)\dots 1}{z(z+1)\dots(z+n-1)}\int_0^1 s^{z+n-1}ds = \frac{n!n^z}{z(z+1)\dots(z+n)},$$

which establishes (7.1.19).

A theorem learned in calculus (usually in sections on exponential growth or compound interest) states that

$$\left(1-\frac{t}{n}\right)^n \to e^{-t} \quad \text{as} \quad n \to \infty. \tag{7.1.20}$$

If we let $n \to \infty$ in (7.1.19), the validity of (7.1.18) seems assured. However, such a conclusion is not so easily justified. To do this, we proceed as follows. From (7.1.15) and (7.1.19) we know that

$$\Gamma(z) = \lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt. \tag{7.1.21}$$

Let $f(z) = \int_0^\infty e^{-t}t^{z-1}dt$. This integral converges, since $|e^{-t}t^{z-1}| \le e^{-t}t^{\operatorname{Re} z-1}$ and $\operatorname{Re} z > 0$ (use the comparison test and compare this integral with $\int_1^\infty e^{-t}t^p dt$ and $\int_0^1 t^p dt$, where $p > -1$). We shall need to know "how fast" $[1-(t/n)]^n \to e^{-t}$. The following inequalities hold:

$$0 \le e^{-t} - \left(1 - \frac{t}{n}\right)^n \le \frac{t^2 e^{-t}}{n} \quad \text{for} \quad 0 \le t \le n. \tag{7.1.22}$$

(This follows from a calculus lemma whose proof is asked for in Exercise 15.)

From (7.1.21) and the definition of f we have

$$f(z) - \Gamma(z) = \lim_{n\to\infty} \left\{ \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}dt + \int_n^\infty e^{-t}t^{z-1}dt \right\}. \tag{7.1.23}$$

To show that this limit is zero, note that $\int_n^\infty e^{-t}t^{z-1}dt \to 0$ as $n \to \infty$. Indeed if $t > 1$, then $|e^{-t}t^{z-1}| \le e^{-t}t^m$ where m is an integer and $m \ge \operatorname{Re} z > 0$. But from calculus (or directly using integration by parts), we know that $\int_0^\infty e^{-t}t^m dt < \infty$, so $\int_n^\infty e^{-t}t^m dt \to 0$ as $n \to \infty$. It remains to be shown that

$$\int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}dt \to 0 \quad \text{as} \quad n \to \infty.$$

By inequality (7.1.22),

$$\left|\int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}dt\right| \le \int_0^n \frac{e^{-t}t^{\operatorname{Re} z+1}}{n}dt \le \frac{1}{n}\int_0^\infty e^{-t}t^{\operatorname{Re} z+1}dt,$$

which approaches zero as $n \to \infty$ because the integral converges. This completes the proof of (7.1.18); that is, for $\operatorname{Re} z > 0$,

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt.$$

In fact, if we examine that proof, we see that provided $0 < \epsilon < R, \epsilon \le |z| \le R$, and $(-\pi/2) + \delta \le \arg z \le (\pi/2) - \delta, \delta > 0$, the convergence is uniform in z (see Exercise 18).

Proof of Theorem 7.1.2: Convergence Theorem for Products

(i) We can assume that $z_n \ne -1$ for all n. Let $P_n = \prod_{k=1}^n (1 + z_k)$; therefore, by assumption, $P_n \to P$ for some $P \ne 0$. Thus, $P_n/P_{n-1} \to 1$ by the quotient theorem for limits. But $P_n/P_{n-1} = 1 + z_n$. Therefore, $z_n \to 0$.

(ii) Let $S_n = \sum_{k=1}^n \log(1 + z_k)$ and let $P_n = \prod_{k=1}^n (1 + z_k)$, so that $P_n = e^{S_n}$. It is clear that if S_n converges, then P_n also converges because e^z is continuous.

Conversely, suppose that $P_n \to P \ne 0$. To show that S_n converges, it suffices to show that for n sufficiently large, all S_n lie in a period strip (on which e^z has a continuous inverse).

We cannot write $\log P_n = \sum_{k=1}^{n} \log(1+z_k)$, because P_n could be on the negative real axis. Instead, for purposes of this proof, let us choose the branch of log such that P lies in its domain A. Now $P_n \to P$, so $P_n \in A$ if n is large and therefore we can write $S_n = \log P_n + k_n \cdot 2\pi i$ for an integer k_n. Subtracting this equation for $n+1$ from that for n gives

$$(k_{n+1} - k_n) \cdot 2\pi i = \log(1 + z_{n+1}) - (\log P_{n+1} - \log P_n).$$

Since the left side of the equation is purely imaginary,

$$(k_{n+1} - k_n) \cdot 2\pi i = i[\arg(1 + z_{n+1}) - \arg P_{n+1} + \arg P_n].$$

By (i), $z_{n+1} \to 0$. an so $\arg(1+z_{n+1}) \to 0$. Also, $\arg P_n \to \arg P$, and therefore $k_{n+1} - k_n \to 0$ as $n \to \infty$. Since the k_n's are integers, they must equal a fixed integer k for n large. Thus $S_n = \log P_n + k \cdot 2\pi i$, so, as $n \to \infty, S_n \to S = \log P + k \cdot 2\pi i$.

(iii) By (ii), it suffices to show that for $x_n \geq 0, \sum x_n$ converges iff $\sum \log(1+x_n)$ converges. To prepare for the proof, note that for $|z| < 1$,

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots, \quad \text{so} \quad \frac{\log(1+z)}{z} = 1 - \frac{z}{2} + \frac{z^2}{3} - \ldots.$$

which has a removable singularity at $z = 0$ and thus $\lim_{z\to 0}(\log(1+z))/z = 1$. Suppose that $\sum x_n$ converges. Since $x_n \to 0$, given $\epsilon > 0$, we have

$$0 \leq \log(1 + x_n) \leq (1+\epsilon)x_n$$

for sufficiently large n. By the comparison test, $\sum \log(1 + x_n)$ converges. If we use $(1-\epsilon)x_n \leq \log(1+x_n)$. we obtain the converse.

(iv) Suppose that $\prod(1 + |z_n|)$ converges. Then by (ii), $\sum \log(1 + |z_n|)$ converges. (We begin with terms such that the conditions in (ii) hold.) In fact, the argument in (iii) shows that $\sum \log(1 + z_n)$ converges absolutely and hence converges. Thus by (ii), $\prod(1 + z_n)$ converges. ■

Proof of Theorem 7.1.6: Canonical Products First we show that the function $\prod(1 - z/a_n)e^{z/a_n}$ is entire. For each $R > 0$, let $D_R = \{z \text{ such that } |z| \leq R\}$. the closed disk of radius R. Since $a_n \to \infty$, only a finite number of a_n's lie in D_R. say, $a_1, \ldots, a_{N-1}$. Therefore, for $z \in D_R$, only a finite number of terms $(1 - z/a_n)$ vanish. We will use the following lemma.

Lemma 7.1.7 *If* $1 + w = (1-a)e^a$ *and* $|a| < 1$, *then* $|w| \leq |a|^2/(1 - |a|)$.

Proof Writing e^a as a series, we get

$$(1-a)e^a = 1 - \frac{a^2}{2} - \ldots - \left(1 - \frac{1}{n}\right)\frac{a^n}{(n-1)!} - \ldots.$$

Thus, since $|a| < 1$,

$$\begin{aligned} |w| &= |(1-a)e^a - 1| \le \frac{|a|^2}{2} + \ldots + \frac{(n-1)}{n!}|a|^n + \ldots \\ &\le |a|^2 + |a|^3 + \ldots = \frac{|a|^2}{1-|a|}. \quad \blacktriangledown \end{aligned}$$

The next step in the proof is to show that the series

$$\sum_{n=1}^{\infty} w_n(z) = \sum_{n=1}^{\infty} \left[\left(1 - \frac{z}{a_n}\right) e^{z/a_n} - 1 \right]$$

converges uniformly and absolutely on $D_{R/2}$. This will show that

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

is entire (by Theorem 7.1.5).

Indeed, for $n \ge N$, $|z/a_n| < 1$ if $|z| \le R/2$, and so from the preceding lemma,

$$|w_n(z)| \le \frac{|z/a_n|^2}{1 - |z/a_n|} \le \frac{(R/2)^2}{1 - \frac{1}{2}} \cdot \frac{1}{|a_n|^2}$$

since $|z| \le R/2$ and $|a_n| \ge R$ for $n \ge N$. Thus,

$$|w_n(z)| \le \frac{R^2}{2} \cdot \frac{1}{|a_n|^2} = M_n.$$

By assumption, $\sum M_n$ converges, and so by the Weierstrass M test, $\sum w_n(z)$ converges uniformly and absolutely. Thus, the function f_1, defined by

$$f_1(z) = z^k \prod_{n=1}^{\infty} [1 - (z/a_n)] e^{z/a_n}$$

is entire. From the definition of the product it is clear that f_1 has exactly the required zeros. Thus, so does $e^g f_1$. If f has the given zeros, then f/f_1 will be entire and have no zeros (by Proposition 4.1.1). Therefore, we need only prove the following lemma.

Lemma 7.1.8 *Let $h(z)$ be entire with no zeros. Then there is an entire function $g(z)$ such that $h = e^g$.*

Proof This follows from Proposition 2.4.12 since the entire complex plane $\mathbb{C}$ is simply connected. ■

Table 7.1.1 The Gamma Function

Definition

$$\Gamma(z) = \frac{1}{ze^{\gamma z}\left[\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\right]},$$

where

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\log n\right) \approx 0.577.$$

Properties

1. Γ is meromorphic with simple poles at $0, -1, -2, \ldots$.
2. $\Gamma(z+1) = z\Gamma(z), z \neq 0, -1, -2, \ldots$.
3. $\Gamma(n+1) = n!, n = 0, 1, 2, \ldots$.
4. $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$.
5. $\Gamma(z) \neq 0$ for all z.
6. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}; \Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2n}\sqrt{\pi}$.
7. $\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right]$.
8. $\Gamma(z) = \lim_{n\to\infty}\frac{n!n^{z}}{z(z+1)\ldots(z+n)}$.
9. $\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\ldots\Gamma\left(z+\frac{n-1}{n}\right) = (2\pi)^{(n-1)/2}n^{(1/2)-nz}\Gamma(nz)$.
10. $2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2z)$.
11. The residue of Γ at $-m$ equals $(-1)^m/m!$.
12. (Euler's integral) $\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt$ for $\operatorname{Re} z > 0$. The convergence is uniform and absolute for $-\pi/2+\delta \leq \arg z \leq \pi/2-\delta, \delta > 0$, and for $\epsilon \leq |z| \leq R$, where $0 < \epsilon < R$.
13. $\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right) = \int_0^{\infty}\left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right)dt$.

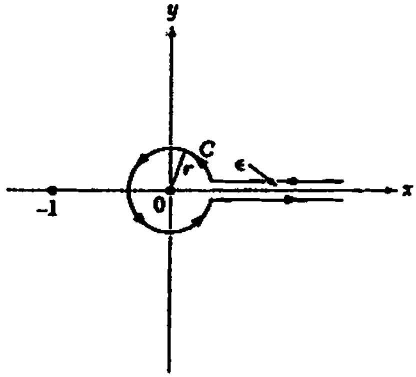


Figure 7.1.2: Contour for Hankel's formula.

14. (Hankel's formula) $\Gamma(z) = -\dfrac{1}{2i\sin \pi z}\displaystyle\int_C (-t)^{z-1}e^{-t}dt$. ($C$ as in Figure 7.1.2.)

15. $\dfrac{1}{\Gamma(z)} = \dfrac{i}{2\pi}\displaystyle\int_C (-t)^{-z}e^{-t}dt$. ($C$ as in Figure 7.1.2.)

16. $\Gamma(z+1) \approx \sqrt{2\pi}z^{z+1/2}e^{-z}$ for $|z|$ large, $\operatorname{Re} z > 0$. (This is Stirling's formula, to be proved in §7.3.)

Worked Examples

Example 7.1.9 *For what z does $(1+z)\prod_{n=1}^{\infty}(1+z^{2^n})$ converge absolutely? Show that the product equals $1/(1-z)$.*

Solution By Theorem 7.1.2(iii), we have absolute convergence iff $\sum_{n=1}^{\infty} z^{2^n}$ converges absolutely. This is the case for $|z| < 1$, since the radius of convergence of the series is 1. Thus, the product converges absolutely for $|z| < 1$.

Our product is $(1+z)(1+z^2)(1+z^4)(1+z^8)\ldots$. Notice that $(1+z)(1+z^2) = 1+z+z^2+z^3$, and

$$(1+z)(1+z^2)(1+z^4) = 1+z+z^2+z^3+\ldots+z^7.$$

Generally,

$$(1+z)\prod_{k=1}^{n}(1+z^{2^k}) = 1+z+z^2+\ldots+z^{2^{n+1}-1}.$$

This series converges to $1/(1-z)$ as $n \to \infty$ since it is the power series around $z=0$ for $1/(1-z)$.

Example 7.1.10 *Prove that*

$$\sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right).$$

Solution The zeros of $\sin z$ occur at 0 and $\pm n\pi$; let us define $a_1 = \pi, a_2 = -\pi, a_3 = 2\pi, a_4 = -2\pi, \ldots$. All the zeros are simple, and $\sum 1/|a_n|^2$ converges. Therefore, by the Theorem on Canonical Products 7.1.6, we can write

$$\begin{aligned}
\sin z &= e^{g(z)}z\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{z/a_n}\\
&= e^{g(z)}z\cdot\left[\left(1-\frac{z}{\pi}\right)e^{z/\pi}\right]\left[\left(1+\frac{z}{\pi}\right)e^{-z/\pi}\right]\\
&\quad\times\left[\left(1-\frac{z}{2\pi}\right)e^{z/2\pi}\right]\left[\left(1+\frac{z}{2\pi}\right)e^{-z/2\pi}\right]\ldots\\
&= e^{g(z)}z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)
\end{aligned}$$

(gathering the terms in pairs). It remains to be shown that $e^{g(z)} = 1$, which requires an unusual technique.

Let

$$P_n(z) = e^{g(z)}z\prod_{k=1}^{n}\left(1-\frac{z^2}{k^2\pi^2}\right),$$

so that $P_n(z)\to\sin z$ (uniformly on disks), and hence $P_n'(z)\to\cos z$. Thus,

$$\frac{P_n'(z)}{P_n(z)}\to\cot z\quad\text{for}\quad z\neq 0,\pm\pi,\pm2\pi,\ldots.$$

But

$$\begin{aligned}
\frac{P_n'(z)}{P_n(z)} &= \frac{d}{dz}\log P_n(z) = \frac{d}{dz}\left[g(z)+\log z+\sum_{k=1}^{n}\log\left(1-\frac{z^2}{k^2\pi^2}\right)\right]\\
&= g'(z)+\frac{1}{z}+\sum_{k=1}^{n}\left(\frac{2z}{z^2-k^2\pi^2}\right).
\end{aligned}$$

However, from §4.4,

$$\cot z = \frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2\pi^2}$$

for $z\neq n\pi$. Thus, $g'(z)=0$, and so $g(z)$ is a constant, say c. Therefore,

$$\frac{\sin z}{z} = e^c\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right).$$

Letting $z\to 0$, the left side approaches 1, while the right side approaches e^c (Why?). Thus $e^c = 1$ and we get the desired formula.

Exercises

1. Show that $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}$.

2. Show that $\prod_{n=2}^{\infty}\left(1-\frac{2}{n(n+1)}\right)=\frac{1}{3}$.

3.* Show that $\prod_{n=1}^{\infty}(1+z_n)$ converges absolutely if $\sum_{n=1}^{\infty}\log(1+z_n)$ converges absolutely.

4.* Complete the proof of the analyticity of infinite products (7.1.5).

5. Use Worked Example 7.1.10 to establish ***Wallis' formula***,

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots.$$

6. Show that $\prod_{n=2}^{\infty}\left(1-\frac{2}{n^3+1}\right)=\frac{2}{3}$.

7.* Prove formula 2 of Table 7.1.1.

8. Show that $\prod_{n=1}^{\infty}(1+z_n)$ converges (assuming that $z_n\neq-1$) iff, for any $\epsilon>0$, there is an N such that $n\geq N$ implies that

$$|(1+z_n)\ldots(1+z_{n+p})-1|<\epsilon \quad \text{for all} \quad p=0,1,2,\ldots.$$

Hint: Use the Cauchy Criterion 3.1.5 for sequences.

9.* Prove formula 4 of Table 7.1.1.

10. Let $a_1,a_2,\ldots$ be nonzero complex numbers and assume $\sum_{n=1}^{\infty}1/|a_n|^{1+h}$ converges, where $h\geq 0$ is a fixed integer. Show that the most general entire function having zeros at $a_1,a_2,\ldots$ and a zero of order k at 0 is

$$f(z)=e^{g(z)}z^k\prod_1^{\infty}\left[\left(1-\frac{z}{a_n}\right)e^{[z/a_n+(z/a_n)^2/2+\ldots+(z/a_n)^h/h]}\right],$$

where each of the points a_i may be repeated finitely often. *Hint*: Prove the following lemma: If $1+w=(1-a)e^{a+a^2/2+\ldots+a^h/h}$ for $|a|<1$, then $|w|\leq|a|^{h+1}/(1-|a|)$.

11. Prove formula 8 of Table 7.1.1.

12.* Using Euler's formula (formula 7 of Table 7.1.1), prove that $\Gamma(z+1)=z\Gamma(z)$.

13. Show that in the neighborhood $|z+m|<1$ for m a fixed positive integer,

$$\Gamma(z)-\frac{(-1)^m}{m!(z+m)}$$

is analytic (that is, has a removable singularity at $z=-m$).

14. (a) (**Weierstrass factorization theorem**) Let

$$E(z,h)=(1-z)e^{z+z^2/2+\dots+z^h/h}.$$

Show that the most general entire function having zeros at $a_1, a_2, \dots$, each repeated according to its multiplicity, where $a_n \to \infty$ and having a zero of order k at 0, is

$$f(z)=e^{g(z)}z^k\prod_{n=1}^{\infty}E\left(\frac{z}{a_n},n\right).$$

(b) Conclude that every meromorphic function is the quotient of two entire functions.

15.• Prove that, for $0\le t\le n$,

$$0\le e^{-t}-\left(1-\frac{t}{n}\right)^n\le\frac{t^2e^{-t}}{n}.$$

16. Prove that, for $\operatorname{Re} z\ge 0$,

$$\frac{\Gamma'(z)}{\Gamma(z)}=\int_0^{\infty}\left(\frac{e^{-t}}{t}-\frac{e^{-zt}}{1-e^{-t}}\right)dt.$$

Hint: If $\operatorname{Re} z\ge 0$, then $1/(z+n)=\int_0^{\infty}e^{-t(z+n)}dt$. Use $\gamma=\lim_{n\to\infty}(1+\frac{1}{2}+\dots+1/n-\log n)$ and Table 7.1.1, line 13.

17. Let γ be a circle of radius $\frac{1}{2}$ around $z_0=0$. Show that $\int_\gamma \Gamma(z)dz=2\pi i$.

18. Establish the uniform convergence in formula 12 of Table 7.1.1.

19.• Prove ***Hankel's formula*** (formula 14 of Table 7.1.1):

$$\Gamma(z)=\frac{-1}{2i\sin\pi z}\int_C(-t)^{-z}e^{-t}dt.$$

For what z is this formula valid? Using $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$, conclude that

$$\frac{1}{\Gamma(z)}=\frac{i}{2\pi}\int_C(-t)^{-z}e^{-t}dt.$$

20. Suppose one started by *defining* $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ for $\operatorname{Re} z > 0$.
    (a) Show that $\Gamma(z)$ is analytic on $\operatorname{Re} z > 0$ by showing that $\int_0^n t^{z-1}e^{-t}dt$ converges uniformly on closed disks as $n \to \infty$.
    (b) Show that $\Gamma(z+1) = z\Gamma(z), \operatorname{Re} z > 0$.
    (c) Use (b) and analytic continuation to prove that $\Gamma(z)$ can be extended to a meromorphic function having simple poles at $0, -1, -2, \ldots$. *Hint*: The procedure used is analogous to that used in proving the Schwarz Reflection Principle; see §6.1.

21.* Show that

$$\int_{-\infty}^{\infty} e^{-\gamma^2} d\gamma = \sqrt{\pi} \quad \text{and that} \quad \int_{-\infty}^{\infty} \gamma^2 e^{-\gamma^2} d\gamma = \frac{\sqrt{\pi}}{2}$$

by using the gamma function. *Hint*: Relate these equations using integration by parts and $\Gamma(1/2) = \sqrt{\pi}$.

7.2 Asymptotic Expansions and the Method of Steepest Descent

Asymptotic expansions provide a method of using the partial sums of a series to approximate the values of a function $f(z)$ for large z. A striking aspect is that the series itself might not converge to the function and might actually diverge. If we use only one term we say that we have an asymptotic approximation or asymptotic formula for f. Stirling's formula for the gamma function is such a formula. This result, proved in §7.3, states that

$$\Gamma(x) \sim e^{-x}x^{x-1/2}\sqrt{2\pi} \quad \text{for large } x.$$

The expression on the right side may be easier to handle than the Γ function itself and has important applications in fields such as probability and statistical mechanics. Another famous example is the prime number theorem, which asserts that if $\pi(x)$ is the number of primes less than or equal to the real number x, then

$$\pi(x) \sim \frac{x}{\log x}.$$

Exactly what such a formula means and in what sense it is an approximation will be developed in this section. The theory of asymptotic expansions considered in this section will be applied in the next, where Stirling's formula is proved and Bessel functions are studied.

There are methods for studying the asymptotic behavior of functions $f(z)$ other than those we shall develop. For example, if f satisfies a differential equation, then this equation frequently can be used to obtain an asymptotic formula. The reader who wishes to delve more deeply into these topics should consult the references listed in the Preface.

"Big Oh" and "Little oh" Notation Some notation is useful for keeping track of relationships in behavior between two functions. Suppose $f(z)$ and $g(z)$ are defined for z in some set A. We say $f(z)$ is $O(g(z))$ (pronounced "$f(z)$ is 'big oh' of $g(z)$") for z in A if there is a constant C such that $|f(z)| \leq C|g(z)|$ for all $z \in A$. We usually write $f(z) = O(g(z))$, although this is somewhat an abuse of notation since the object on the right is a statement of relationship and not a specific quantity to which $f(z)$ is equal. For example, $\sin x = O(x)$ for x in $\mathbb{R}$, since elementary calculus shows that $|\sin x| \leq |x|$ for all x. Note that $f(z) = O(1)$ just means that $f(z)$ is bounded.

More useful notation for us will be "little oh," which requires some sort of limiting behavior for its definition. Roughly speaking, the notation $f(z) = o(g(z))$ means that $f(z)/g(z)$ tends to 0 as $z \to z_0$ or $z \to \infty$, etc. (We say "roughly" only because $g(z)$ could vanish.) For example,

$$\begin{aligned}
&1 - \cos x = o(x) && \text{as} \quad x \to \infty \\
&\log x = o(x) && \text{as} \quad x \to \infty \\
&e^{-x} = o\left(\frac{1}{x^n}\right) && \text{as} \quad x \to \infty \text{ for any } n.
\end{aligned}$$

We will be concerned primarily with $z \to \infty$ in a sector $\alpha \leq \arg(z) \leq \beta$. For the remainder of this section, unless specified otherwise, the symbols will thus be defined as follows:

- $f(z) = O(g(z))$ *means* There are constants R and M such that whenever $|z| \geq R$ and $\alpha \leq \arg(z) \leq \beta$, $|f(z)| \leq M|g(z)|$.
- $f(z) = o(g(z))$ *means* For each $\epsilon > 0$, there is an R such that whenever $|z| \geq R$ and $\alpha \leq \arg(z) \leq \beta$, $|f(z)| \leq \epsilon|g(z)|$.

In some cases we will be interested only in behavior along the positive real axis and will then take $\alpha = \beta = 0$. Notice that if $f(z)$ is $O(1/z^{n+1})$, then it is $o(1/z^n)$, but the converse is not generally true.

Asymptotic Expansions Chapter 3 was concerned with representing a function by an infinite series that converges to the value of the function and it carefully avoided divergent series. Nonetheless, divergent series can sometimes be useful, though one must be very careful in their interpretation. We will see that it is possible to associate with a function an infinite series which may or may not converge, but whose partial sums can be made to yield good approximations to the value of the function.

Consider a series of the form

$$S = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

and let

$$S_n = a_0 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n}.$$

Thus, S_n is well defined for $z \neq 0$ but we do not demand that S converge. The correct way to say that S is asymptotic to a given function f is as follows.

Definition 7.2.1 *We say that $f \sim S$, or that f is **asymptotic to** S, or that S is an **asymptotic expansion** of f, if*

$$f - S_n = o\left(\frac{1}{z^n}\right)$$

for $\arg z$ *lying in a specified range* $[\alpha, \beta]$ (see Figure 7.2.1).

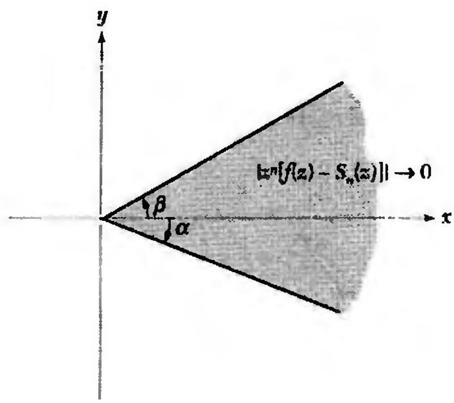


Figure 7.2.1: Asymptotic expansion.

Although S may be divergent, the partial sums may result in accurate approximations of f, the error being approximately $1/z^n$. This will be illustrated with an example in the following paragraphs.

If we allowed the full range $[-\pi, \pi]$ for $\arg z$, we might expect $a_0 + a_1/z + a_2/z^2 + \ldots$ to converge if $f(z)$ were analytic outside a large circle, because f has a convergent Laurent series of that form. However, f usually has poles $z_n \to \infty$ (such as $\Gamma(z)$, which has poles at $0, -1, -2, \ldots$), and therefore in many examples, we do not have a Laurent series that is valid on the exterior of any circle. If f has poles $z_n \to \infty$ in the sector $\arg z \in [\alpha, \beta]$ and $f \sim S$, then S cannot converge at any z_0. If it did, then S would converge uniformly for all $|z| > |z_0| + 1$. (See §3.3.) Definition 7.2.1 and the uniform convergence of S_n to S would say that for large enough $|z|$ in that sector, we have $|f(z) - S(z)| < 1$. But this cannot hold near the poles of f.

The following example should help to clarify the concept of asymptotic expansion.

Example 7.2.2 *Show that for x real and positive,*

$$\int_x^\infty t^{-1}e^{x-t}\,dt \sim \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \ldots.$$

Solution Define

$$f(x) = \int_x^\infty t^{-1} e^{x-t} dt.$$

(This is not the gamma function!) Integration by parts gives

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \ldots + \frac{(-1)^{n-1}(n-1)!}{x^n} + (-1)^n n! \int_x^\infty \frac{e^{x-t}}{t^{n+1}} dt.$$

We claim that

$$f(x) \sim S(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \ldots.$$

Note that the series *diverges*. Here the sector is $\alpha = \beta = 0$; that is, we are restricting z to the positive real axis.

Indeed, if

$$S_n = \frac{1}{x} - \frac{1}{x^2} + \ldots + \frac{(-1)^{n-1}(n-1)!}{x^n},$$

we have

$$\begin{aligned} |x^n[f(x) - S_n(x)]| &= x^n n! \int_x^\infty \frac{e^{x-t}}{t^{n+1}} dt = n! \int_x^\infty \left(\frac{x}{t}\right)^n \frac{e^{x-t}}{t} dt \\ &\le n! \int_x^\infty \frac{e^{x-t}}{t} dt \le \frac{n!}{x} \int_x^\infty e^{x-t} dt = \frac{n!}{x}, \end{aligned}$$

which approaches zero as $x \to \infty$. Thus $f(x) - S_n(x)$ is $o(1/x^n)$, so $f \sim S$ as required. Even though $n!$ grows quickly, we still have an accurate approximation because

$$|f(x) - S_n(x)| \le \frac{n!}{x^{n+1}} = O\left(\frac{1}{x^{n+1}}\right),$$

and if x is, say greater than n, then $n!/x^{n+1}$ is very small. ♦

The next proposition gives some basic properties of asymptotic expansions.

Proposition 7.2.3 (i) *If*

$$f(z) \sim S(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots,$$

then

$$f(z) - S_n(z) = O\left(\frac{1}{z^{n+1}}\right)$$

and conversely.

(ii) *If*

$$f \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

and

$$f \sim \tilde{a}_0 + \frac{\tilde{a}_1}{z} + \frac{\tilde{a}_2}{z^2} + \ldots,$$

then $a_i = \tilde{a}_i$. *(Asymptotic expansions are unique.)*

(iii) *If*

$$f \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

and

$$g \sim b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots,$$

both being valid in the same range of $\arg z$, *then in that range*

$$f + g \sim (a_0 + b_0) + \frac{(a_1 + b_1)}{z} + \frac{(a_2 + b_2)}{z_2} + \ldots$$

and

$$fg \sim c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots \quad \textit{where} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

(Asymptotic series may be added and multiplied.)

(iv) *Two different functions can have the same asymptotic expansion.*

(v) *Let* $\phi : [a, \infty[\to \mathbb{R}$ *be continuous and suppose that* $\phi(x) = o(1/x^n), n \geq 2$. *Then*

$$\int_x^{\infty} \phi(t)dt = o\left(\frac{1}{x^{n-1}}\right).$$

Proof

(i) Since $f \sim S$, we have, by definition, $f - S_{n+1} = o(1/z^{n+1})$. Therefore, we get

$$f - S_n = f - S_{n+1} + S_{n+1} - S_n = o\left(\frac{1}{z^{n+1}}\right) + \frac{a_{n+1}}{z^{n+1}} = O\left(\frac{1}{z^{n+1}}\right).$$

(ii) We shall show that $a_n = \tilde{a}_n$ by induction on n. First, by the definition of $f \sim S$, $f(z) - a_0 \to 0$ as $z \to \infty$, so $a_0 = \lim_{z\to\infty} f(z)$. Thus $a_0 = \tilde{a}_0$. Suppose we have proven that $a_0 = \tilde{a}_0, \ldots, a_n = \tilde{a}_n$. We shall show that $a_{n+1} = \tilde{a}_{n+1}$. Given $\epsilon > 0$, there is an R such that if $|z| \geq R$, we have

$$\left|z^{n+1}\left[f(z) - \left(a_0 + \frac{a_1}{z} + \ldots + \frac{a_{n+1}}{z^{n+1}}\right)\right]\right| < \epsilon$$

and

$$\left|z^{n+1}\left[f(z) - \left(\tilde{a}_0 + \frac{\tilde{a}_1}{z} + \ldots + \frac{\tilde{a}_{n+1}}{z^{n+1}}\right)\right]\right| < \epsilon.$$

Therefore, by the triangle inequality,

$$\begin{aligned}
&|a_{n+1} - \tilde{a}_{n+1}| \\
&= |z^{n+1}|\frac{|a_{n+1} - \tilde{a}_{n+1}|}{|z^{n+1}|} \\
&= |z^{n+1}|\left|\left[f(z) - \left(a_0 + \ldots + \frac{a_{n+1}}{z^{n+1}}\right)\right] - \left[f(z) - \left(\tilde{a}_0 + \ldots + \frac{\tilde{a}_{n+1}}{z^{n+1}}\right)\right]\right| \\
&< \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

Thus $|a_{n+1} - \tilde{a}_{n+1}| < 2\epsilon$ for any $\epsilon > 0$. Hence $a_{n+1} = \tilde{a}_{n+1}$.

(iii) Let $S_n(z) = a_0 + \ldots + a_n/z^n$ and $\tilde{S}_n(z) = b_0 + \ldots + b_n/z^n$. We must show that $f + g - (S_n + \tilde{S}_n) = o(1/z^n)$. To do this, write

$$f + g - (S_n + \tilde{S}_n) = (f - S_n) + (g - \tilde{S}_n) = o(1/z^n) + o(1/z^n) = o(1/z^n).$$

To establish the formula for the product, note that

$$c_0 + c_1/z + \ldots + c_n/z^n = S_n\tilde{S} + o(1/z^n),$$

since $S_n\tilde{S}_n = c_0 + c_1/z + \ldots + c_n/z^n$ plus higher-order terms. Thus,

$$fg - (c_0 + c_1 + \ldots + c_n/z^n) = fg - S_n\tilde{S}_n + o(1/z^n).$$

Now write $fg - S_n\tilde{S}_n = (f - S_n)g + S_n(g - \tilde{S}_n)$ and note that both terms are $o(1/z^n)$, since g and S_n are bounded as $z \to \infty$.

(iv) On $\mathbb{R}$, the function e^{-x} is $o(1/x^n)$ as $x \to \infty$ for any n. Thus if $f \sim a_0 + a_1/x + a_2/x^2 + \ldots$, then $f(x) + e^{-x} \sim a_0 + a_1/x + a_2/x^2 + \ldots$ as well.

(v) Since $\phi(t) = o(1/t^n)$, $\lim_{n\to\infty} t^n\phi(t) = 0$. Given $\epsilon > 0$, there is an $x_0 > 0$ such that $t > x_0$ implies $|t^n\phi(t)| < \epsilon$. Thus for $x > x_0$,

$$\left|\int_x^\infty \phi(t)dt\right| \leq \int_x^\infty \frac{\epsilon}{t^n}dt = \frac{\epsilon}{n-1}\cdot\frac{1}{x^{n-1}},$$

so for $x > x_0$,

$$\left|x^{n-1}\int_x^\infty \phi(t)dt\right| \leq \epsilon.$$

Therefore, $\lim_{x\to\infty} x^{n-1}\int_x^\infty \phi(t)dt = 0$, so $\int_x^\infty \phi(t)dt = o(1/x^{n-1})$. ■

Asymptotic Formulas and Asymptotic Equivalence If a function has an asymptotic series as just described, then the partial sums of that series can be used to obtain approximations to the function for large z. However, the applicability of this method is a bit restricted. If f has the asymptotic series

$$f(z) \sim S(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots,$$

then $f(z) - a_0 = o(1)$; that is, $\lim_{x\to\infty} f(x) = a_0$, so f has a finite limit at infinity in the specified sector. This is too restrictive, since we are commonly interested in functions that grow as x grows, such as the two examples mentioned at the beginning of this section, $\Gamma(z)$ and $\pi(z)$. To remedy this, we write

$$f(z) \sim g(z)\left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots\right)$$

to mean that

$$f(z) = g(z)\left[a_0 + \frac{a_1}{z} + \dots + \frac{a_n}{z^n} + o\left(\frac{1}{z^n}\right)\right].$$

In other words, if $g(z) \neq 0$, then

$$\frac{f(z)}{g(z)} \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots,$$

the hope being that $g(z)$ is an easier function to handle for large z than is f. Incorporating the factor a_0 into g, we have

$$f(z) \sim g(z)\left(1 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots\right).$$

The first term gives a function $g(z)$ with $f(z) = g(z)[1 + O(1/z)]$, or, slightly more generally, $f(z) = g(z)[1 + o(1)]$. In this case we say that f and g are asymptotically equivalent.

Definition 7.2.4 *Two functions $f(z)$ and $g(z)$ are **asymptotically equivalent** if $f(z) = g(z)[1 + o(1)]$. In this case we write $f(z) \sim g(z)$.*

Notice that if $g(z) \neq 0$, this says $f(z)/g(z) - 1 = o(1)$ so that $\lim_{z\to\infty}[f(z)/g(z)] = 1$ in the specified sector. The expression $g(z)$ is thought of as giving an asymptotic formula for $f(z)$. It is in this sense that Stirling's formula and the prime number theorem are to be interpreted.

The goal is to use $g(z)$ to approximate $f(z)$ for large z. However, the approximation need *not* be improving as $z \to \infty$ in the sense we have been using so far. That is, the absolute value of the error $\Delta f = g(z) - f(z)$ need not be shrinking. Instead it is the ***relative error*** or ***percentage error***, the error expressed as a fraction of the true value, which has to be shrinking. The relative error is

$$\frac{\Delta f}{f} = \frac{g(z) - f(z)}{f(z)} = \frac{g(z)}{f(z)} - 1,$$

and this goes to 0 as z goes to infinity in the specified sector since it is $o(1)$.

The following simple example should clarify the points made in the preceding paragraph. Let $f(z) = ze^z/(1+z)$. Then $f(z) \sim g(z) = e^z$. The asymptotic error incurred by using $g(z)$ to approximate $f(z)$ is $\Delta f = e^z - f(z) = e^z/(1+z)$. This error goes to infinity as z grows along the positive real axis. However, the relative error, the absolute error expressed as a fraction of the true value, is

$$\frac{\Delta f}{f} = \frac{e^z}{1+z} \cdot \frac{1+z}{ze^z} = \frac{1}{z},$$

which does go to 0 as z grows.

As we have noted, one function might have two different-looking asymptotic formulas even though the ratio of the two will tend to 1 as z grows in the specified sector.

Many of the functions one wishes to study either arise as or can be converted to integrals of the form

$$f(z) = \int_\gamma e^{zh(\xi)} g(\xi) d\xi.$$

The Γ function is of this form:

$$\Gamma(z) = \int_0^\infty e^{-t} t^z dt = \int_0^\infty e^{z \log t} e^{-t} dt.$$

We will pursue this idea in §7.3 to obtain Stirling's formula from a result at the end of this section. The plan is to find a point ξ_0 on the curve such that the factor $e^{zh(\xi)}$ is fairly large there but becomes small away from ξ_0 along the curve for large z. Then most of the contribution to the integral will come from the part of the curve near ξ_0 and we may be able to estimate it in terms of the behavior of h and g near ξ_0. First we turn our attention to some cases in which h is simple enough that we can obtain all terms of the series.

Laplace Transforms The Laplace transform is a construction very much like the Fourier transform we met earlier. Provided the integral makes sense, the Laplace transform of a function g defined on the positive real axis is

$$\tilde{g}(z) = \int_0^\infty e^{-zt} g(t) dt.$$

We will devote considerable attention to this construction and some of its applications in Chapter 8. Here we will see how asymptotic series might shed some light on the behavior of $\tilde{g}(z)$ for large z.

Proposition 7.2.5 *Suppose g is analytic in a region containing the positive real axis and is bounded on the positive real axis. Let the Taylor series for g centered at 0 be $\sum_{n=0}^\infty a_n z^n$ and let $\tilde{g}(z) = \int_0^\infty e^{-zt} g(t) dt$. Then*

$$\tilde{g}(z) \sim \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{2a_2}{z^3} + \ldots + \frac{n! a_n}{z^{n+1}} + \ldots$$

as $z \to \infty, \arg z = 0$.

Proof Let $h(z) = [g(z) - (a_0 + a_1 z + \ldots + a_{n-1}z^{n-1})]/z^n$. Then h is bounded on the positive real axis because first, g is bounded, second, the polynomial term in the numerator has degree less than n, and third, the limit as z tends to 0 is a_n. Thus there is a constant M such that

$$|g(t) - (a_0 + a_1 t + \ldots + a_{n-1}t^{n-1})| < Mt^n$$

for all $t \geq 0$ and so, for real z,

$$\left|\int_0^\infty e^{-zt}[g(t) - (a_0 + a_1 t + \ldots + a_{n-1}t^{n-1})]dt\right| \leq M \int_0^\infty e^{-zt}t^n dt.$$

Thus,

$$\left|\tilde{g}(z) - \sum_{k=0}^{n-1} a_k \int_0^\infty e^{-zt}t^k dt\right| \leq M \int_0^\infty e^{-zt}t^n dt.$$

Letting $x = zt$, we get

$$\left|\tilde{g}(z) - \sum_{k=0}^{n-1} a_k \frac{1}{z^{k+1}} \int_0^\infty e^{-x}x^k dx\right| \leq M \frac{1}{z^{n+1}} \int_0^\infty e^{-x}x^n dx,$$

$$\left|\tilde{g}(z) - \sum_{k=0}^{n-1} a_k \frac{\Gamma(k+1)}{z^{k+1}}\right| \leq M \frac{\Gamma(n+1)}{z^{n+1}} = \frac{Mn!}{z^{n+1}} = o\left(\frac{1}{z^n}\right),$$

which is what we wanted. ■

The assumption of analyticity for the function g fits nicely into the theme of this text and makes the simple proof just given possible. It is worth noting, and important for many applications, that the same result holds with different assumptions on g. Analyticity is not so essential as that g be infinitely differentiable.

Proposition 7.2.6 *Suppose g is infinitely differentiable on the positive real axis and that g and each of its derivatives are of exponential order. That is, there are constants A_n and B_n such that $|g^{(n)}(t)| \leq A_n e^{B_n t}$ for $t \geq 0$. Let $\tilde{g}(z) = \int_0^\infty e^{-zt}g(t)dt$. Then*

$$\tilde{g}(z) \sim \frac{g(0)}{z} + \frac{g'(0)}{z^2} + \frac{g''(0)}{z^3} + \ldots + \frac{g^{(n)}(0)}{z^{n+1}} + \ldots$$

as $z \to \infty, \arg z = 0$.

Proof Fix $n \geq 0$ and suppose $z > \max(B_0, B_1, \ldots, B_n)$. Then repeated integration by parts gives

$$\begin{aligned} \tilde{g}(z) &= -\sum_{k=0}^{n-1}\left(\lim_{T\to\infty}\left[\frac{e^{-zt}g^{(k)}(t)}{z^{k+1}}\Bigg|_{t=0}^{T}\right]\right) + \frac{1}{z^n}\int_0^\infty e^{-zt}g^{(n)}(t)dt \\ &= \sum_{k=0}^{n-1}\frac{g^{(k)}(0)}{z^{k+1}} + \frac{1}{z^n}\int_0^\infty e^{-zt}g^{(n)}(t)dt, \end{aligned}$$

since $|e^{-zt}g^{(k)}(T)| \le A_k e^{(B_k - z)T}$, and this last term goes to 0 as T grows. Therefore,

$$z^n|\tilde{g}(z) - S_n(z)| \le \int_0^\infty |e^{-zt}g^{(n)}(t)|dt \le \int_0^\infty A_n e^{(B_n - z)t}dt = \frac{A_n}{z - B_n},$$

and this goes to 0 as z grows, as we need. ■

The second result (7.2.6) applies to infinitely differentiable functions that are not analytic, as well as to functions such as polynomials, which are not bounded. The first (7.2.5) includes functions such as $g(t) = \sin e^{t^2}$ whose derivative is not of exponential order.

Watson's Theorem The argument used for Proposition 7.2.5 will also establish an expansion with a slightly more complicated function for h.

Theorem 7.2.7 (Watson's Theorem) *Let $g(z)$ be analytic and bounded on a domain containing the real axis. Set*

$$f(z) = \int_{-\infty}^{\infty} e^{-zy^2/2}g(y)dy$$

for z real. Then

$$f(z) \sim \frac{\sqrt{2\pi}}{\sqrt{z}}\left(a_0 + \frac{a_2}{z} + \frac{a_4 \cdot 1 \cdot 3}{z^2} + \frac{a_6 \cdot 1 \cdot 3 \cdot 5}{z^3} + \ldots\right)$$

as $z \to \infty, \arg z = 0$, where $g(z) = \sum_{n=0}^{\infty} a_n z^n$ near zero.

Proof We first observe that the function

$$h(z) = \frac{g(z) - (a_0 + a_1 z + \ldots + a_{2n-1}z^{2n-1})}{z^{2n}}$$

is bounded on the real axis since g is bounded and since $h(z) \to a_{2n}$ as $z \to 0$. Therefore, we obtain

$$\left|\int_{-\infty}^{\infty} e^{-zy^2/2}[g(y) - (a_0 + a_1 y + \ldots + a_{2n-1}y^{2n-1})]dy\right| \le M\int_{-\infty}^{\infty} e^{-zy^2/2}y^{2n}dy.$$

Now we use the fact that for $z > 0$,

$$\int_{-\infty}^{\infty} e^{-zy^2/2}y^{2k}dy = \sqrt{2\pi} \cdot 1 \cdot 3 \ldots \frac{2k-1}{z^{k+1/2}}$$

and

$$\int_{-\infty}^{\infty} e^{-zy^2/2}y^{2k+1}dy = 0$$

(see Exercise 7) to obtain

$$\left|\int_{-\infty}^{\infty} e^{-zy^2/2} g(y)dy - \left(\frac{a_0\sqrt{2\pi}}{z^{1/2}} + \frac{a_2\sqrt{2\pi}}{z^{1+1/2}} + \frac{a_4\sqrt{2\pi}\cdot 1\cdot 3}{z^{2+1/2}} + \dots \right.\right.$$
$$\left.\left. + \frac{a_{2n-2}\cdot\sqrt{2\pi}\cdot 1\cdot 3\dots(2n-3)}{z^{n-1+1/2}}\right)\right| \le M\sqrt{2\pi}\frac{1\cdot 3\dots(2n-1)}{z^{n+1/2}},$$

from which the theorem follows. ■

Method of Steepest Descent Finally we turn to situations in which h may be more complicated and in which more sophisticated techniques are needed. One of these is called the method of steepest descent or the saddle-point method. It was discovered by P. Debye around 1909 in connection with some high frequency approximations in optics.

We seek an expansion of the form

$$f(z) \sim g(z)\left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots\right)$$

for a particularly simple function $g(z)$, and we shall be mainly interested in obtaining the first term. The method described here works well if f has the special form $f(z) = \int_\gamma e^{zh(\xi)}d\xi$. We shall use contours $\gamma(t)$ defined for all $t \in \mathbb{R}$. We can integrate over such infinite contours in the same manner as we would integrate over ordinary ones, as long as we check convergence of the integrals.

Theorem 7.2.8 (Steepest Descent Theorem) *Let $\gamma :]-\infty,\infty[\to \mathbb{C}$ be a C^1 curve. (γ may also be defined only on a finite interval.) Let $\zeta_0 = \gamma(t_0)$ be a point on γ and let $h(\zeta)$ be a function continuous along γ and analytic at ζ_0. Make the following hypotheses: For $|z| \ge R$ and $\arg z$ fixed,*

(i) *$f(z) = \int_\gamma e^{zh(\zeta)}d\zeta$ converges absolutely.*

(ii) $h'(\zeta_0) = 0; h''(\zeta_0) \neq 0$.

(iii) *$\operatorname{Im}[zh(\zeta)]$ is constant for ζ on γ in some neighborhood of ζ_0.*

(iv) *$\operatorname{Re}[zh(\zeta)]$ has a strict maximum at ζ_0 along the entire curve γ.*

Then

$$f(z) \sim \frac{e^{zh(\zeta_0)}\sqrt{2\pi}}{\sqrt{z}\sqrt{-h''(\zeta_0)}}$$

as $z \to \infty, \arg z$ fixed. The sign of the square root is chosen such that

$$\sqrt{z}\sqrt{-h''(\zeta_0)}\cdot\gamma'(t_0) > 0.$$

The proof of this theorem is given in the Internet Supplement for this chapter.

Remarks

(i) To achieve conditions (i) to (iv) it may be necessary to deform γ by applying Cauchy's Theorem. A path γ verifying these conditions is called a ***path of steepest descent.***

(ii) The asymptotic expansion in the conclusion of the theorem depends only on $h(\zeta_0)$ and $h''(\zeta_0)$, not on the behavior of h elsewhere on γ (except, of course, that h must satisfy the hypotheses of the theorem). Higher-order derivatives would be used if further terms in the expansion were needed.

(iii) The origin of the term "steepest descent" can be traced to conditions (iii) and (iv) in the following way. Recall that $\operatorname{Im}[zh(\zeta)] = v(\zeta)$ and $\operatorname{Re}[zh(\zeta)] = u(\zeta)$ are harmonic conjugates, and recall the fact that v is constant on γ means that u is changing fastest in the direction of γ. Since ζ_0 is a maximum, $u(\zeta) = \operatorname{Re}[zh(\zeta)]$ is decreasing fastest when moving away from ζ_0 in the direction of γ. Hence the curve γ is the path of steepest descent. The term "saddle-point method" originated as follows. The function $u(\zeta) = \operatorname{Re}[zh(\zeta)]$ has a maximum on γ at ζ_0. But $h''(\zeta_0) \neq 0$ implies that u is not constant, so ζ_0 must be a saddle point of u since harmonic functions never have local maxima or minima (see Figure 7.2.2).

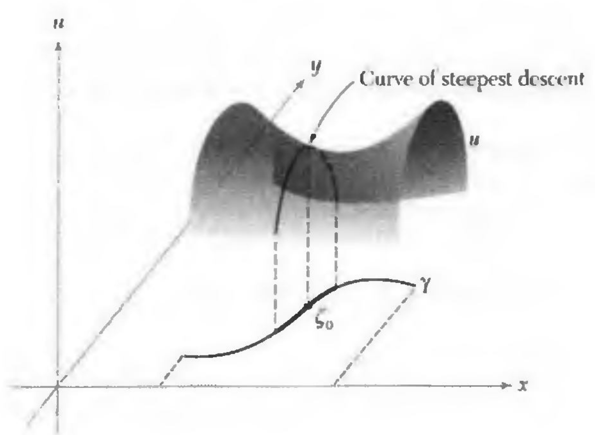


Figure 7.2.2: Saddle-point method.

(iv) Often the correct sign for the square root may be determined by examining the sign of the integral defining $f(z)$.

To obtain the higher-order terms in the expansion

$$f(z) \sim \frac{e^{zh(\zeta_0)}}{\sqrt{z}} \cdot \frac{\sqrt{2\pi}}{\sqrt{-h''(\zeta_0)}} \left(1 + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots\right),$$

one must be able to compute more terms in the series $\zeta = \zeta_0 + a_1 w + a_2 w^2 + \ldots$ in the preceding proof. In simple cases, these higher-order terms can be evaluated explicitly, as in Watson's Theorem 7.2.7. The details of the method of obtaining higher-order terms will not be given here, because such terms are needed only in refined calculations. The leading term given in the steepest descent theorem is the important one.

The applications of this theorem given in §7.3 deal primarily with the case in which z is real and positive. Clearly, in that case conditions (iii) and (iv) of the theorem can be written equivalently with or without the z.

A proof similar to the proof of the Steepest Descent Theorem shows the following (see Exercises 8 and 10).

Theorem 7.2.9 (Generalized Steepest Descent Theorem) *Let the conditions of the steepest descent theorem (7.2.8) hold but let f have the form*

$$f(z) = \int_\gamma e^{zh(\zeta)} g(\zeta) d\zeta$$

where $g(\zeta)$ is a bounded continuous function on γ with $g(\zeta_0) \neq 0$. Then

$$f(z) \sim \frac{e^{zh(\zeta_0)} \sqrt{2\pi} g(\zeta_0)}{\sqrt{z}\sqrt{-h''(\zeta_0)}}.$$

Method of Stationary Phase If the exponent in the integrand of Theorem 7.2.8 is purely imaginary, we can obtain a related result known as the ***method of stationary phase***. This method was developed in part by Lord Kelvin in 1887 and will be applied to the study of Bessel functions in §7.3.

Theorem 7.2.10 (Stationary Phase Theorem) *Let $[a, b]$ be a bounded interval on the real axis. Let $h(t)$ be analytic in a neighborhood of $[a, b]$ and be real for real t. Let $g(t)$ be a real- or complex-valued function on $[a, b]$ with continuous derivative. Suppose*

$$f(z) = \int_a^b e^{izh(t)} g(t) dt.$$

If $h'(t) = 0$ at exactly one point t_0 in $]a, b[$ and $h''(t_0) \neq 0$, then as $z \to \infty$ on the positive real axis, we have

$$f(z) \sim \frac{e^{izh(t_0)}\sqrt{2\pi}}{\sqrt{z}\sqrt{\pm h''(t_0)}} e^{\pm\pi i/4} g(t_0).$$

The plus signs are used if $h''(t_0) > 0$, and the minus signs are used if $h''(t_0) < 0$.

The asymptotic formula for f can also be written as

$$\lim_{z\to+\infty} \sqrt{z} e^{-izh(t_0)} \int_a^b e^{izh(t)} g(t) dt = \frac{\sqrt{2\pi} e^{\pm\pi i/4} g(t_0)}{\sqrt{\pm h''(t_0)}}.$$

We note that by breaking g into its real and imaginary parts, it is sufficient to prove the theorem for g real-valued. Also note that we did not require $g(t_0) \neq 0$.

The name "stationary phase" comes from the interpretation that the integrand is a complex quantity with amplitude (magnitude) $g(t)$ and phase angle $zh(t)$. The intuition behind the formula is that the main contribution to the integral should come from the neighborhood of t_0, where the phase angle is varying as slowly as possible. To see why, think of the integral in terms of its real and imaginary parts:

$$f(z) = \int_a^b g(t)\cos(zh(t))dt + i\int_a^b g(t)\sin(zh(t))dt.$$

If z is very large, then $zh(t)$ is changing rapidly in regions where $h'(t)$ is not zero. Thus, $\cos(zh(t))$ and $\sin(zh(t))$ are oscillating rapidly. Figure 7.2.3 illustrates this with $h(t) = t^2$ by the graphs of $\cos(10t^2)$ and $\cos(20t^2)$. If g is at all reasonable, the resulting oscillations of the integral should tend to cancel out *except* near the points where $h'(t) = 0$.

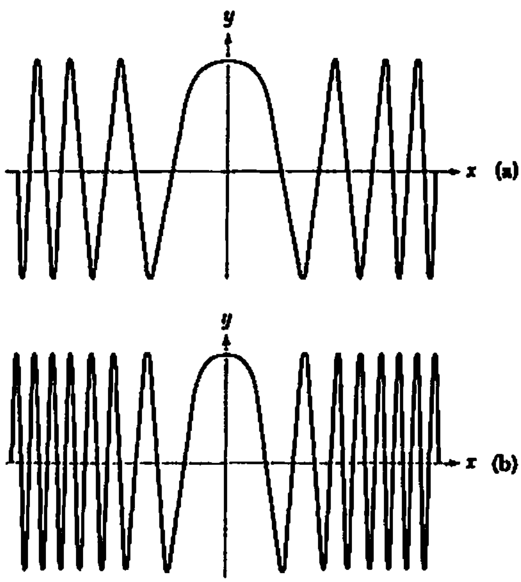


Figure 7.2.3: Graphs of (a) $y = \cos(10t^2)$ and (b) $y = \cos(20t^2)$.

The endpoints of the interval of integration might also be expected to contribute, but it turns out that this contribution is at worst proportional to $1/z$ and so will not interfere with the result we hope to prove: The integral behaves about like $1/\sqrt{z}$.

We should be able to estimate the integral for large z by using only a portion of the path near t_0. In this short interval we approximate $g(t)$ by the constant $g(t_0)$ and $h(t)$ by its second-order Taylor approximation $h(t_0) + [h''(t_0)/2](t - t_0)^2$

to obtain

$$f(z) \approx e^{izh(t_0)} g(t_0) \int e^{izh''(t_0)(t-t_0)^2/2} dt,$$

where the integral is over some short interval centered at t_0. Changing variables produces

$$f(z) \approx \frac{e^{izh(t_0)} g(t_0)\sqrt{2}}{\sqrt{z}\sqrt{h''(t_0)}} \int e^{ix^2}\, dx.$$

The integral is taken over an interval of the form $[-A\sqrt{z}, A\sqrt{z}]$. The real and imaginary parts of the integral

$$\int e^{ix^2}\, dx = \int \cos(x^2)\, dx + i \int \sin(x^2)\, dx$$

are called ***Fresnel integrals***. Residue methods may be used to show that each integral taken over the whole real line converges to $\sqrt{\pi/2}$. The details are given in the Internet Supplement for Chapter 4. Therefore, as z goes to infinity, the integral for f converges to $\sqrt{\pi/2}(1+i) = \sqrt{\pi}e^{\pi i/4}$. This leaves us with exactly the result we wish, with obvious modifications for the case in which $h''(t_0) < 0$.

We will see something of the applicability of this formula in §7.3 when we study the Bessel functions. Kelvin used it in 1891 to study the pattern of bow and stern waves from a moving ship. In any particular application the amplitude $g(t)$ is usually well behaved. But turning the intuitive derivation just given into a proof is a bit tricky. The first step requires that the function g be smooth enough so that when multiplied by the rapidly oscillating $\cos(zh(t))$ and $\sin(zh(t))$, it gives something for which the integral effectively cancels away from t_0 and for which any cancellation is not so effective near t_0. This may not happen if g itself has a lot of oscillation at very high frequencies. Continuity alone is not enough to prevent this, as can be seen from the following example.

Example 7.2.11 *Find a continuous function g for which the conclusion of the Stationary Phase Theorem 7.2.10 is false.*

Solution Let $\phi(t) = \sum_{k=1}^{\infty}(1/k^2)\cos(k^6 t)$. This series converges uniformly and absolutely for $t \in \mathbf{R}$ since the kth term is dominated by $1/k^2$. Thus, ϕ is continuous. Define $g(t)$ on the interval $I = [-\sqrt{2\pi}, \sqrt{2\pi}]$ by $g(t) = 2t\phi(t^2)$ when $t \geq 0$ and by $g(t) = 0$ when $t < 0$ and consider the integral

$$f(z) = \int_I e^{-izt^2} g(t)dt.$$

This fits the pattern of (7.2.10) with $h(t) = -t^2, t_0 = 0$, and $h''(0) < 0$. If n is a positive integer, we have

$$\begin{aligned} f(n) &= \int_I e^{-int^2} g(t)dt = \int_0^{\sqrt{2\pi}} e^{-int^2}\phi(t^2)2t\,dt \\ &= \int_0^{2\pi} e^{-inx}\phi(x)dx = \int_0^{2\pi} e^{-inx}\left[\sum_{k=1}^{\infty} \frac{1}{k^2}\cos(k^6 x)\right]dx \\ &= \sum_{k=1}^{\infty} \frac{1}{k^2}\int_0^{2\pi} e^{-inx}\cos(k^6 x)dx \\ &= \sum_{k=1}^{\infty} \frac{1}{k^2}\left[\int_0^{2\pi}\cos(nx)\cos(k^6 x)dx - i\int_0^{2\pi}\sin(nx)\cos(k^6 x)dx\right]. \end{aligned}$$

These integrals are all 0 except the first in the single case $k^6 = n$, and that one is π. Thus, $f(k^6) = \pi/k^2$. In other words, for positive integer z we have $f(z) = 0$ unless z is a sixth power, in which case $f(z) = \pi/\sqrt[3]{z}$. Thus, $\sqrt{z}f(z)$ does not remain bounded, so the conclusion of Theorem 7.2.10 cannot hold. ◆

The function g in this example is not smooth enough to make Theorem 7.2.10 work. It has too much influence from its high-frequency components, and some condition is needed to prevent this. The requirement of a continuous first derivative (in the sense of one real variable) specified in Theorem 7.2.10 is one such condition. It implies a property, called ***bounded variation***, which is phrased specifically in terms of the oscillations of g and which is important in the theory of integration. A few of the ideas about this property and a proof of Theorem 7.2.10 are given in the Internet Supplement to this chapter.

Worked Examples

Example 7.2.12 *Suppose that $f(z) = I(z) + J(z)$, that $I(z)/J(z) = O(1/z^M)$ for every positive integer M, and that*

$$J(z) \sim g(z)\left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots\right).$$

Show that

$$f(z) \sim g(z)\left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots\right).$$

Solution Since $I(z)/J(z) = O(1/z^M)$, we know that $z^M I(z)/J(z)$ stays bounded, and therefore $z^{M-1}I(z)/J(z) \to 0$. Thus $I(z)/J(z) = o(1/z^N)$ for every integer $N \geq 0$. In other words, there is a function $B_N(z)$ such that $z^N I(z) = B_N(z)J(z)$

and $B_N(z) \to 0$ as $z \to \infty$. Now compute

$$\begin{aligned}\left|z^N\left[\frac{f(z)}{g(z)} - S_N(z)\right]\right| &\le \left|\frac{z^N I(z)}{g(z)}\right| + \left|z^N\left[\frac{J(z)}{g(z)} - S_N(z)\right]\right| \\ &\le \left|B_N(z)\frac{J(z)}{g(z)}\right| + \left|z^N\left[\frac{J(z)}{g(z)} - S_N(z)\right]\right|.\end{aligned}$$

The first term goes to 0, since $B_N(z) \to 0$ and $J(z)/g(z) \to a_0$. The second term goes to 0, since $J(z) \sim g(z)(a_0 + a_1/z + a_2/z^2 + \dots)$. This completes the proof.

Example 7.2.13 *Let $h(\zeta) = \zeta^2$ and $\zeta_0 = 0$. Find a curve γ satisfying the hypotheses of the steepest descent theorem. (In other words, find a path of steepest descent.) Take $\arg z = 0$; that is, z is real, $z > 0$.*

Solution Let $h(\zeta) = u + iv$, so that if $\zeta = \xi + i\eta, u = \xi^2 - \eta^2$ and $v = 2\xi\eta$. The discussion following the steepest descent theorem indicated that the path of steepest descent is defined by $v =$ constant (since in our case z is real, $z > 0$). Thus the line of steepest descent through $\zeta_0 = 0$ is either $\xi = 0$ or $\eta = 0$. Since u must have a maximum at $\zeta_0 = 0$, the curve γ is defined by $\xi = 0$. ♦

Example 7.2.14 *Prove that*

$$f(z) = \int_{-\infty}^{\infty} e^{-zy^2/2}\cos y\, dy \sim \frac{\sqrt{2\pi}}{\sqrt{z}}\left(1 - \frac{1}{2z} + \frac{1}{2^2 2! z^2} - \dots\right)$$

as $z \to \infty$; $\arg z = 0$.

Solution We apply Watson's Theorem 7.2.7. Write $\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots$. Therefore, $a_0 = 1, a_2 = \frac{-1}{2!}, a_4 = \frac{1}{4!}, \dots$, and thus

$$f(z) \sim \sqrt{\frac{2\pi}{z}}\left(1 - \frac{1}{2z} + \frac{1\cdot 3}{4! z^2} - \dots\right) = \sqrt{\frac{2\pi}{z}}\left(1 - \frac{1}{2z} + \frac{1}{2^2 2! z^2} - \dots\right). \quad ♦$$

Exercises

1. Show that if $f(z) = O(h(z))$ and $g(z) = O(h(z))$ and a and b are constants, then $af(z) + bg(z) = O(h(z))$.

2. Show that asymptotic equivalence is an equivalence relation in the sense that the following three properties hold:

   (a) *Reflexive*: $f \sim f$.

   (b) *Symmetric*: If $f \sim g$, then $g \sim f$.

   (c) *Transitive*: If $f \sim g$ and $g \sim h$, then $f \sim h$.

3.• If $f(x) \sim a_2/x^2 + a_3/x^3 + \dots$ for $x \in [0, \infty[$, show that

$$g(x) = \int_x^\infty f(t)dt \sim \frac{a_2}{x} + \frac{a_3}{2x^2} + \frac{a_4}{3x^3} + \dots .$$

4. Let $f(x) = \int_x^\infty e^{-t}/t\,dt$. Use integration by parts to show

$$f(x) \sim e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{1 \cdot 2}{x^3} - \dots\right).$$

5.• Show that

$$f(x) = \int_0^\infty \frac{e^{-xt}}{1+t^2}dt \sim \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \dots .$$

6. Let $g(z)$ be analytic at z_0 and let $g'(z_0) = 0$ and $g''(z_0) \neq 0$, so that near $z_0, g(z) - g(z_0) = [w(z)]^2$ for w analytic, $w'(z_0) \neq 0$. Prove that there are exactly two perpendicular curves on which $\operatorname{Re} g$ (alternatively, $\operatorname{Im} g$) are constant through z_0. (Recall that Proposition 1.5.12 shows that if $f'(z_0) \neq 0, \operatorname{Re} f$ has exactly one level curve through z_0.) Show also that lines of constant $\operatorname{Re} g$ and $\operatorname{Im} g$ intersect at 45°.

7. (a) (See Exercise 21, §7.1.) Show that if $z > 0$, then for integers $k \geq 0$,

$$\int_{-\infty}^\infty e^{-zy^2/2} y^{2k} dy = \sqrt{2\pi}\frac{1 \cdot 3 \cdot 5 \dots (2k-1)}{z^{k+1/2}}$$

and

$$\int_{-\infty}^\infty e^{-zy^2/2} y^{2k+1} dy = 0.$$

(b) Show that for integers $m \geq 0$,

$$\int_{-\infty}^\infty e^{-y^2} y^{2m} dy = \sqrt{\pi}\frac{1 \cdot 3 \cdot 5 \dots (2m-1)}{2^m} = \frac{(2m)!\sqrt{\pi}}{m!2^{2m}}$$

and

$$\int_{-\infty}^\infty e^{-y^2} y^{2m+1} dy = 0.$$

8. Let $h_n(t) = \sqrt{n}e^{-nt^2}$. The area under the graph of $h_n(t)$ is $\sqrt{\pi}$ and for any $\epsilon > 0, h_n(t) \to 0$ uniformly outside $]-\epsilon, \epsilon[$. Such a sequence is called an *approximating δ sequence*. See Figure 7.2.4.

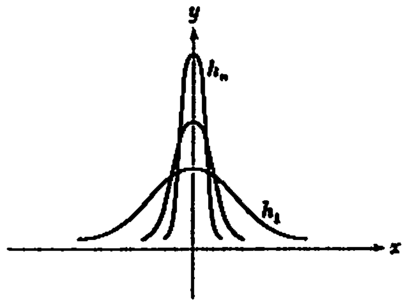


Figure 7.2.4: Approximating δ sequence.

(a) Show that if $g(t)$ is continuous and $0 < N < \infty$, then

$$\int_{-N}^{N} g(t)h_n(t)dt \to g(0)\sqrt{\pi}$$

as $n \to \infty$.

(b) Show that if $g(t)$ is continuous and bounded, then

$$\int_{-\infty}^{\infty} g(t)h_n(t)dt \to g(0)\sqrt{\pi}$$

as $n \to \infty$.

9. The expansion

$$\int_{x}^{\infty} t^{-1}e^{x-t}dt \sim \frac{1}{x} - \frac{1}{x^2} + \dots$$

was discussed in Example 7.2.2. Compute $S_4(10)$ and $S_5(10)$ numerically and find an upper bound for the respective errors. Discuss how the errors change in $S_n(x)$ as n and x increase. For example, for a given x, are errors reduced if we take n very large?

10.* Sketch the proof of the generalized steepest descent theorem (7.2.9) using Exercise 8 (you will need to read the relevant internet supplement).

11. Find an asymptotic expansion for

$$f(z) = \int_{-\infty}^{\infty} e^{-zy^2/2} \sin y^2 dy.$$

(Assume that $z \to \infty, z > 0$.)

12. Show that if $f(z) = O(\phi(z))$ and $g(z) = o(h(z))$, then $f(z)g(z) = o(\phi(z)h(z))$.

13. Find the path of steepest descent through $t_0 = 0$ if $h(t) = \cos t$. (Take z real, $z > 0$.)

14.• Prove that $\int_C e^{-z^2} dz = \sqrt{\pi}$, where C is the 45° line with equation $z = t + it$, where $-\infty < t < \infty$, by showing that

$$\int_C e^{-z^2} dz = \int_{-\infty}^{\infty} e^{-x^2} dx.$$

Hint: Show that $\int_{\gamma_x} e^{-\zeta^2} d\zeta \to 0$ as $x \to \infty$ where γ_x is the vertical line joining x to $x + ix$.

15. Repeat Exercise 13 but assume that z lies on the positive imaginary axis.

16. Show that the first term in Watson's Theorem may be obtained as a special case of the generalized steepest descent theorem if $g \geq 0$ on the real axis.

17. Find the asymptotic formula for f when the path found in Exercise 13 is used in the steepest descent theorem.

18. Use the steepest descent theorem to obtain the asymptotic formula for f using the path γ described in Worked Example 7.2.13.

19.• Find the asymptotic formula for f when the path γ found in Exercise 15 is used in the steepest descent theorem and $h(t) = t^2, t_0 = 0$ is chosen.

7.3 Stirling's Formula and Bessel Functions

In this section the method of steepest descent will be used to prove Stirling's formula for the gamma function $\Gamma(z)$. Some properties of Bessel functions $J_n(z)$, which are defined for $n = \ldots, -1, 0, 1, \ldots$, will also be developed and the method of stationary phase will be used to obtain an asymptotic formula for these functions.

Stirling's Formula We begin with the important asymptotic expansion for the gamma function.

Theorem 7.3.1 (Stirling's Formula)

$$\Gamma(z+1) \sim \sqrt{2\pi} z^{z+1/2} e^{-z}$$

as $z \to \infty$ on the positive real axis.

An extension of the proof given below shows that this result also holds for $-\pi/2 + \delta \leq \arg z \leq \pi/2 - \delta$ for any $\delta > 0$ (see Figure 7.3.1).

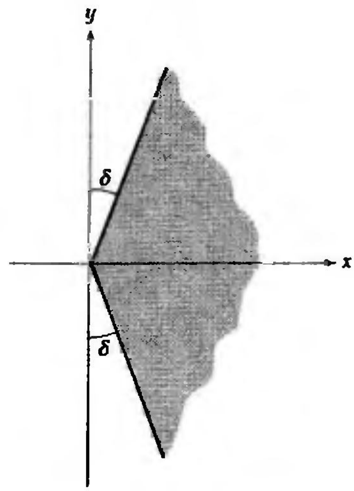


Figure 7.3.1: Region of validity for Stirling's formula.

Proof Recall from formula 12 in Table 7.1.1 that for $\operatorname{Re} z > 0$, we have Euler's integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

We are concerned with the case in which z is real and positive. We will first rewrite the integral so that the steepest descent theorem (7.2.8) applies. To do this, we make the change of variables $t = z\tau$ to get

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z dt = z^{z+1} \int_0^\infty e^{z(\log\tau - \tau)} d\tau.$$

Thus, $\Gamma(z+1)/z^{z+1}$ has the form

$$\int_\gamma e^{zh(\zeta)} d\zeta$$

where

$$h(\zeta) = \log\zeta - \zeta$$

and γ is the positive real axis, $[0, \infty[$. We must check hypotheses (i) to (iv) of the method of steepest descent (Theorem 7.2.8). Let $\zeta_0 = 1$. Clearly,

$$h(\zeta_0) = -1, h'(\zeta_0) = 0 \quad \text{and} \quad h''(\zeta_0) \neq 0.$$

Therefore, hypotheses (i) and (ii) of the method hold. Also, $h(\zeta)$ is real on γ, so (iii) is valid. To prove (iv), we know that $\operatorname{Re}[zh(t)] = xh(t)$ has a maximum iff $h(t)$ does. But $h(t)$ has a maximum of -1 at $\zeta_0 = 1$ on γ (see Figure 7.3.2).

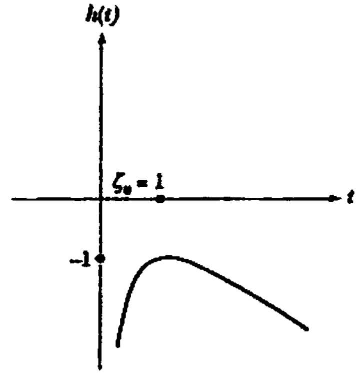


Figure 7.3.2: Graph of $h(t) = \log t - t$.

Thus (iv) holds. Therefore,

$$\frac{\Gamma(z+1)}{z^{z+1}} \sim \frac{e^{zh(\zeta_0)}}{\sqrt{z}\sqrt{-h''(\zeta_0)}} \cdot \sqrt{2\pi} = \frac{e^{-z}}{\sqrt{z}} \cdot \sqrt{2\pi}.$$

Hence $\Gamma(z+1) \sim z^{z+1/2}e^{-z}\sqrt{2\pi}$, as required. ■

If $z = re^{i\theta}$ were not real, the x axis would no longer be the path of steepest descent and the path of integration would have to be deformed into such a path using Cauchy's Theorem.

If one examines this method more carefully, one finds that the first few terms in the expansion are

$$\Gamma(z+1) \sim \sqrt{2\pi}z^{z+1/2}e^{-z}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots\right);$$

more precisely,

$$\Gamma(z+1) = \sqrt{2\pi}z^{z+1/2}e^{-z}\left[1 + \frac{1}{12z} + \frac{1}{288z^2} + O\left(\frac{1}{z^3}\right)\right].$$

When solving particular problems, the first term usually is the most important one.

Since $\Gamma(z+1) = z\Gamma(z)$, we obtain $\Gamma(x) \sim e^{-x}x^{x-1/2}(2\pi)^{1/2}$, mentioned earlier.

Bessel Functions The remainder of this section discusses some basic properties of Bessel functions and how the method of stationary phase can be applied to obtain an asymptotic formula. Bessel functions (the main properties of which are listed in Table 7.3.1 at the end of this section) arise naturally in solutions to certain partial differential equations, such as Laplace's equation, when these equations are expressed in terms of cylindrical coordinates. Bessel functions can be defined in several different ways. We will find the following definition convenient.

Definition 7.3.2 *Let $z \in \mathbf{C}$ be fixed and consider the function*

$$f(\zeta) = e^{z(\zeta - 1/\zeta)/2}.$$

Expand $f(\zeta)$ in a Laurent series around 0. The coefficient of ζ^n where n is positive or negative is denoted $J_n(z)$ and is called the ***Bessel function of order n****. We call $e^{z(\zeta-1/\zeta)/2}$ the* ***generating function****.*

The definition may be written as follows:

$$e^{z(\zeta-1/\zeta)/2} = \sum_{n=-\infty}^{\infty} J_n(z)\zeta^n.$$

From the formula for the coefficients of a Laurent expansion (see Theorem 3.3.1), we see that

$$J_n(z) = \frac{1}{2\pi i}\int_\gamma \zeta^{-n-1} e^{z(\zeta-1/\zeta)/2} d\zeta$$

where γ is any circle around 0. If we use the unit circle $\zeta = e^{i\theta}$ and write out the integral explicitly, we get

$$\begin{aligned} J_n(z) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-(n+1)i\theta} e^{iz\sin\theta} e^{i\theta} d\theta \\ &= \frac{1}{2\pi}\int_0^{\pi} e^{iz\sin\theta - ni\theta} d\theta + \frac{1}{2\pi}\int_0^{\pi} e^{-iz\sin\theta + ni\theta} d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta) d\theta, \end{aligned}$$

which we call the ***cosine representation*** for J_n. Although $J_n(z)$ will be defined for noninteger values of n later, this equation is valid only if n is an integer and it shows that $|J_n(z)| \le 1$ for z real. The graphs of $J_0(x)$ and $J_1(x)$ are shown in Figure 7.3.3.

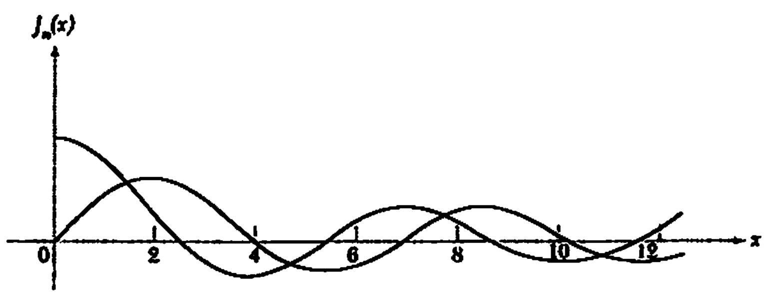


Figure 7.3.3: Bessel functions $J_0(x), J_1(x)$.

Next it will be shown that $J_n(z)$ is entire, and a method will be described for finding its power series. To carry out these tasks, it is convenient to change variables by $\zeta \mapsto 2\zeta/z$ and obtain, for each fixed z, the ***exponential representation***:

$$J_n(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^n \int_\gamma \zeta^{-n-1}\exp\left(\zeta - \frac{z^2}{4\zeta}\right) d\zeta.$$

Writing the exponential as a power series gives

$$\exp\left(\frac{-z^2}{4\zeta}\right) = 1 - \frac{z^2}{4\zeta} + \frac{z^4}{2\cdot(4\zeta)^2} - \cdots.$$

This series converges uniformly in ζ on γ (Why?), so we can integrate term by term (see Theorem 3.1.9) and obtain

$$J_n(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{z}{2}\right)^{n+2k}\int_\gamma \frac{e^\zeta}{\zeta^{n+k+1}}d\zeta.$$

If $n \geq 0$, the residue of e^ζ/ζ^{n+k+1} at $\zeta = 0$ is $1/(n+k)!$ (Why?), so

$$\begin{aligned} J_n(z) &= \sum_{k=0}^{\infty}\frac{(-1)^k(z)^{n+2k}}{2^{n+2k}k!(n+k)!} \\ &= \frac{z^n}{2^n n!}\left[1 - \frac{z^2}{2^2\cdot 1(n+1)} + \frac{z^4}{2^4\cdot 1\cdot 2(n+1)(n+2)} - \cdots\right], \end{aligned}$$

which is the ***power series representation*** for J_n. Thus, $J_n(z)$ is entire for $n \geq 0$ and has a zero of order n at $z = 0$.

Similarly, for $n \leq 0$, one finds that

$$J_n(z) = \sum_{k=0}^{\infty}\frac{(-1)^{k-n}z^{-n+2k}}{2^{-n+2k}(k-n)!k!}$$

(see Exercise 11). It follows that for $n \leq 0$,

$$J_n(z) = (-1)^n J_{-n}(z).$$

The relationship of Bessel functions to differential equations is as follows: $J_n(z)$ is a solution of ***Bessel's equation***:

$$\frac{d^2J_n}{dz^2} + \frac{1}{z}\frac{dJ_n}{dz} + \left(1 - \frac{n^2}{z^2}\right)J_n = 0.$$

This equation is obtained by differentiating the cosine or exponential representation for $J_n(z)$ and inserting the result into Bessel's equation (see Exercise 1). Note that both J_n and J_{-n} satisfy Bessel's equation.

If $n \geq 0$ but n is not an integer, we can still make sense of $J_n(z)$ in the power series representation by setting

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{n+2k}}{2^{n+2k} k! \Gamma(n+k+1)}.$$

Some basic identities can be obtained from the following relations for $z \neq 0$:

$$\frac{d}{dz}[z^{-n} J_n(z)] = -z^{-n} J_{n+1}(z).$$

This can be proven directly by differentiating the power series. The student is requested to establish such a proof in Exercise 4.

If we differentiate $z^{-n} J_n(z)$ in this identity, we obtain

$$\frac{d}{dz} J_n(z) = \frac{n}{z}[J_n(z)] - J_{n+1}(z).$$

Writing $-n$ for n, we get

$$\frac{d}{dz}[z^n J_{-n}(z)] = -z^n J_{-n+1}(z).$$

But $J_{-n}(z) = (-1)^n J_n(z)$, so

$$\frac{d}{dz}[z^n J_n(z)] = z^n J_{n-1}(z),$$

that is,

$$\frac{d}{dz} J_n(z) = J_{n-1}(z) - \frac{n}{z} J_n(z).$$

Combining these equations, we get

$$\frac{d}{dz} J_n(z) = \frac{1}{2}[J_{n-1}(z) - J_{n+1}(z)].$$

These are called the ***recurrence relations*** for Bessel functions. For example, if we know J_n and J_{n-1}, this relation determines J_{n+1}.

We conclude with the asymptotic formula for $J_n(z)$.

Theorem 7.3.3 (Asymptotic Formula for Bessel Functions) *The following formula holds for any integer n:*

$$J_n(z) \sim \sqrt{\frac{2}{\pi z}} \left[\cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]$$

as $z \to \infty$, z real and greater than zero. (This relation is also valid for $|\arg z| < \pi$.)

Proof We use the Stationary Phase Theorem 7.2.10 and the cosine representation

$$J_n(z) = \frac{1}{2\pi}\left(\int_0^{\pi} e^{iz\sin\theta - ni\theta}d\theta + \int_0^{\pi} e^{-iz\sin\theta+ni\theta}d\theta\right).$$

First, consider the function

$$f(z) = \int_0^{\pi} e^{iz\sin\theta - ni\theta}d\theta.$$

In the notation of the Stationary Phase Theorem, let $h(t) = \sin t$ and $g(t) = e^{-int}$. Clearly h is analytic and real for real t, and g is C^1. The interval $[a, b]$ is $[0, \pi]$, and $h'(t) = \cos t$ vanishes only at $t_0 = \pi/2$. At this point, $h''(t_0) = -\sin(\pi/2) = -1 < 0$. Thus, we use the minus sign in the asymptotic formula for f, giving

$$f(z) \sim \frac{e^{iz}\sqrt{2\pi}e^{-\pi i/4}}{\sqrt{z}} \cdot e^{-ni\pi/2} = \sqrt{\frac{2\pi}{z}}e^{i(z-n\pi/2-\pi/4)}.$$

Similarly, if we set $g(z) = \int_0^{\pi} e^{-iz\sin\theta+in\theta}d\theta$, we get

$$g(z) \sim \sqrt{\frac{2\pi}{z}}e^{-i(z-n\pi/2-\pi/4)}.$$

(A proof of this is requested in Exercise 9.) Adding the asymptotic expressions for f and g, we obtain

$$J_n(z) \sim \left(\frac{1}{2\pi}\right)\sqrt{\frac{2\pi}{z}} \cdot 2\cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right),$$

which is the result claimed. ■

Thus, for large x, $J_n(x)$ behaves like $\sqrt{2/\pi x}[\cos(x - \theta)]$, where θ is called the ***phase shift.***

Table 7.3.1 **Summary of Summary of Properties of Bessel Functions**

1. $J_n(z) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(n\theta - z\sin\theta)d\theta$, where n is an integer.

2. $|J_n(z)| \le 1$ for z real.

3. $J_n(z) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k z^{n+2k}}{2^{n+2k}k!(n+k)!}, n \ge 0.$

4. J_n is entire and has a zero of order n at $z = 0$; $J_0(0) = 1$.

5. $J_n(z) = (-1)^n J_{-n}(z)$.

6. Bessel's equation:

$$\frac{d^2J_n}{dz^2}+\frac{1}{z}\frac{dJ_n}{dz}+\left(1-\frac{n^2}{z^2}\right)J_n=0.$$

7. $\dfrac{d}{dz}[z^{-n}J_n(z)]=-z^{-n}J_{n+1}(z).$

8. $\dfrac{d}{dz}J_n(z)=\dfrac{n}{z}J_n(z)-J_{n+1}(z).$

9. $\dfrac{d}{dz}J_n(z)=\dfrac{J_{n-1}(z)-J_{n+1}(z)}{2}.$

10. $J_n(z)\sim\sqrt{\dfrac{2}{\pi z}}\cos\left(z-\dfrac{n\pi}{2}-\dfrac{\pi}{4}\right)$ as $z\to\infty$, z real and positive.

Exercises

1. Prove that $J_n(z)$ satisfies Bessel's equation (formula 6 of Table 7.3.1.).

2. Show that

$$\Gamma(z+1)=\sqrt{2\pi}z^{z+1/2}e^{-z}\left[1+O\left(\frac{1}{z}\right)\right]$$

and that

$$\Gamma(z+1)=\sqrt{2\pi}z^{z+1/2}e^{-z}\left[1+\frac{1}{12z}+O\left(\frac{1}{z^2}\right)\right]$$

as $z\to\infty$. (z is real and greater than zero.)

3. Prove that $J_0'(z)=-J_1(z)$ using the cosine representation.

4. Prove that $d[z^{-n}J_n(z)]/dz=-z^{-n}J_{n+1}(z)$ for all n.

5.* Prove that $J_2(z)=J_0''(z)-J_0'(z)/z$.

6. Use the recurrence relations for Bessel functions and Rolle's theorem from calculus to show that between two consecutive real positive zeros of $J_n(x)$, there is exactly one zero of $J_{n+1}(x)$. Show that $J_n(x)$ and $J_{n+1}(x)$ have no common roots.

7. Prove that $J_{1/2}(z)=\sqrt{2/\pi z}(\sin z)$, using the definition of $J_n(z)$ for nonintegral n.

8. Verify that the asymptotic expansion for $J_n(z)$ is consistent with Bessel's equation.

9.* Complete the proof that

$$J_n(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

by showing that

$$\int_0^{\pi} e^{-iz\sin\theta + ni\theta}\, d\theta \sim \sqrt{\frac{2\pi}{z}} e^{-i(z - n\pi/2 - \pi/4)}.$$

(The left side of the expression is called a ***Hankel function.***)

10. Verify that $\phi(x) = J_n(kx)$ is a solution of

$$\frac{d}{dx}(x\phi') + \left(k^2 x - \frac{n^2}{x}\right)\phi(x) = 0$$

with $\phi(0) = 0, \phi(a) = 0$, where ka is any of the zeros of J_n for $n \neq 0$.

11.* Establish the power series representation of J_n for $n \leq 0$.

Review Exercises for Chapter 7

1. Establish the convergence of and evaluate the infinite product

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n(n+2)}\right).$$

2. Establish the convergence of and evaluate the infinite product

$$\prod_{n=1}^{\infty} \left(\frac{n^2 + 3n + 2}{n^2 + 3n}\right).$$

3. Use Worked Example 7.1.10 to show that

$$\sqrt{2} = \left(\frac{3}{2}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{9}{10}\right)\left(\frac{11}{10}\right)\left(\frac{13}{14}\right)\cdots.$$

4. Use Worked Example 7.1.10 to show that

$$\sqrt{3} = 2\left(\frac{4}{5}\right)\left(\frac{8}{7}\right)\left(\frac{10}{11}\right)\left(\frac{14}{13}\right)\left(\frac{16}{17}\right)\left(\frac{20}{19}\right)\cdots.$$

5. On what region is each of the following absolutely convergent?

(a) $\prod_1^{\infty}(1 - z^n)$
(b) $\prod_1^{\infty}(1 - n^{-z})$

6.• Let $H_m = \sum_{k=1}^{\infty} 1/k^m$. Prove that

$$\log \Gamma(1+z) = -\gamma z + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} H_n z^n$$

for $|z| < 1$, where γ is Euler's constant.

7. Let $f(z) = \int_0^{\pi} e^{iz\sin t} \sin^2 t dt$. Use the method of stationary phase to find an asymptotic formula for $f(z)$ as $z \to \infty$, z real and positive.

8. Prove that

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + Q(z),$$

where $Q(z)$ is entire. In addition, show that

$$Q(z) = \int_1^{\infty} t^{z-1} e^{-t} dt.$$

9. Express the integral $\int_0^{\infty} e^{-t^3} dt$ in terms of the gamma function. *Hint:* Change to the variable $y = t^3$.

10. Show that $|d^k J_n(z)/dz^k| \leq 1, k = 0, 1, 2, \ldots$, for any n, z real.

11.• Prove that $\Gamma\left(\frac{1}{2} + iy\right) \to 0$ as $y \to \infty$.

12. Show that $\lim_{x\to\infty} J_n(x) = 0$.

13. Obtain an asymptotic expansion for $\int_{-\infty}^{\infty} e^{-zy^2/2} \cos y^2 dy$ (as $z \to \infty, z > 0$).

14. Prove that $x^n J_n(x) = \int_0^x t^n J_{n-1}(t) dt, n = 1, 2, \ldots$.

15.• Prove that $J_n(iy) \sim i^n e^y / \sqrt{2\pi y}$ (as $y \to \infty, y > 0$).

16.• In this exercise you are asked to develop some properties of the ***Legendre functions*** (see Review Exercise 34, Chapter 3). These functions are encountered in the study of differential equations (specifically Laplace's equation in three dimensions, which describes a wide range of physical phenomena) when spherical coordinates are used.[1]

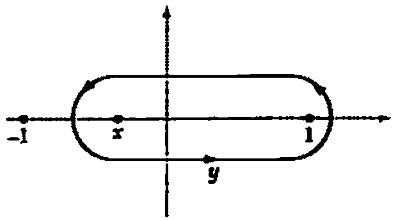
(a) For $-1 < x < 1$, set

$$P_n(x) = \frac{1}{2^{n+1}\pi i} \int_{\gamma} \frac{(t^2-1)^n}{(t-x)^{n+1}} dt$$

where γ is the contour as shown in the following figure. By differentiating under the integral sign, show that $P_n(x)$ solves ***Legendre's equation:***

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

[1]Consult, for example, G. F. D. Duff and D. Naylor, *Differential Equations of Applied Mathematics* (New York: Wiley, 1965).



(b) For an integer n, derive ***Rodrigues' formula:***

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

This formula provides an analytic extension $P_n(z)$.

(c) Show that

$$P_n(x) = \frac{1}{n!}\frac{d^n}{dt^n}\frac{1}{(t^2-2tx+1)^{1/2}}\bigg|_{t=0}$$

and deduce from Taylor's theorem that

$$\frac{1}{(1-2tz+t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(z)t^n.$$

(d) Develop recurrence relations for the coefficients of solutions of Legendre's equation. Use these relations to show that entire solutions must be of the form

$$\psi(x) = \sum_{k=0}^{\infty} a_k z^{2k} \quad \text{for } n \text{ even, and} \quad \rho(x) = \sum_{k=0}^{\infty} b_k x^{2k+1} \quad \text{for } n \text{ odd.}$$

Show that these series are actually polynomials, that is, that a_k and b_k vanish for large k.

(e) Using (c), show that $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$.

(f) Prove that $P_0(x) = 1, P_1(x) = x$, and $P_2(x) = (3x^2-1)/2$.

(g) Show that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0 & n \neq m \\ \frac{2}{(2n+1)} & n = m \end{cases}$$

Hint: Use (b) to prove the case where $n \neq m$; use (c) to prove the case where $n = m$.

17. Obtain the asymptotic formula $P_n(z) \sim [(2n)!2^n(n!)^2]z^n$ as $z \to \infty$, using part (b) of Exercise 16.

Chapter 8

Laplace Transform and Applications

This final chapter gives an introduction to the Laplace transform and some of its applications. §8.1 introduces two key properties that make the Laplace transform useful for differential equations: First, it behaves well with respect to differentiation, and second, a function can be recovered if its Laplace transform is known. The closely related Fourier transform also enjoys these properties. It was discussed in §4.3; see also the Internet Supplement for this chapter. §8.2 develops techniques for inverting Laplace transforms, while §8.3 considers some applications of Laplace transforms to ordinary differential equations.

8.1 Basic Properties of Laplace Transforms

The Laplace transform provides a powerful technique used in both pure and applied mathematics. For example, in control theory it has been an indispensable tool.[1] It is important, therefore, to have a good grasp of both its basic theory and its usefulness. Consider a (real- or complex-valued) function $f(t)$ defined on $[0, \infty[$. The ***Laplace transform*** of f is defined to be the function $\tilde{f}$ of a complex variable z given by

$$\tilde{f}(z) = \int_0^{\infty} e^{-zt} f(t)dt.$$

The Laplace transform $\tilde{f}$ is defined for those $z \in \mathbf{C}$ for which the integral converges. Other common notations for $\tilde{f}$ are $\mathcal{L}(f)$ or simply F.

For technical reasons, it will be convenient to impose a mild restriction on the functions we consider. We require that $f : [0, \infty[\to \mathbf{C}$ (or $\mathbf{R}$) be of ***exponential***

[1]See, for example, J. C. Willems and J. W. Polderman, *An Introduction to Mathematical Systems Theory and Control: A Behavioral Approach* (New York: Springer-Verlag, Texts in Applied Mathematics, 1997).

order. This means that there are constants $A > 0, B \in \mathbb{R}$, such that

$$|f(t)| \leq Ae^{tB}$$

for all $t \geq 0$. In other words, f should not grow too fast; for example, any polynomial satisfies this condition (Why?). *All functions considered in the remainder of this chapter will be assumed to be of exponential order*. It will also be assumed that on any finite interval $[0, a]$, f is bounded and integrable. (If, for example, we assume that f is piecewise continuous, this last condition will hold.)

Abscissa of Convergence The first important result in this chapter concerns the nature of the set on which $\bar{f}(z)$ is defined and is analytic.

Theorem 8.1.1 (Convergence Theorem for Laplace Transforms) *Assume tha $f : [0, \infty[\to \mathbb{C}$ (or $\mathbb{R}$) is of exponential order and let*

$$\bar{f}(z) = \int_0^\infty e^{-zt} f(t)dt.$$

There exists a unique number σ, $-\infty \leq \sigma < \infty$, such that this integral converges if $\operatorname{Re} z > \sigma$ *and diverges if* $\operatorname{Re} z < \sigma$. *Furthermore, $\bar{f}$ is analytic on the set*

$$A = \{z \mid \operatorname{Re} z > \sigma\}$$

and we have

$$\frac{d}{dz}\bar{f}(z) = -\int_0^\infty te^{-zt} f(t)dt$$

for $\operatorname{Re} z > \sigma$. *The number σ is called the **abscissa of convergence**, and if we define the number ρ by*

$$\rho = \inf\{B \in \mathbb{R} \mid \text{there exists an } A > 0 \text{ such that } |f(t)| \leq Ae^{Bt}\},$$

then $\sigma \leq \rho$.

The set $\{z \mid \operatorname{Re} z > \sigma\}$ is called the ***half-plane of convergence***. (If $\sigma = -\infty$, this set is all of $\mathbb{C}$.) See Figure 8.1.1. In general, it is difficult to tell whether $\bar{f}(z)$ will converge for z on the vertical line $\operatorname{Re} z = \sigma$. If there is any danger of confusion we can write $\sigma(f)$ for σ or $\rho(f)$ for ρ. A convenient way to compute $\sigma(f)$ is described in Worked Examples 8.1.12 and 8.1.13.

The proof of this theorem and more detailed convergence results are given at the end of this section. The basic idea is that if $\operatorname{Re} z > \rho$, then A and B may be selected with $\rho < B < \operatorname{Re} z$ and $|f(t)| < Ae^{Bt}$. The improper integral for $\bar{f}(z)$ converges by comparison with $\int_0^\infty Ae^{(B-\operatorname{Re} z)t}\, dt$.

The map $f \mapsto \bar{f}$ is linear in the sense that $(af + bg)\bar{} = a\bar{f} + b\bar{g}$, valid for $\operatorname{Re} z > \max[\sigma(f), \sigma(g)]$. It is also true that the map is one-to-one; that is, $\bar{f} = \bar{g}$ implies that $f = g$; in other words, a function $\phi(z)$ is the Laplace transform of at most one function.

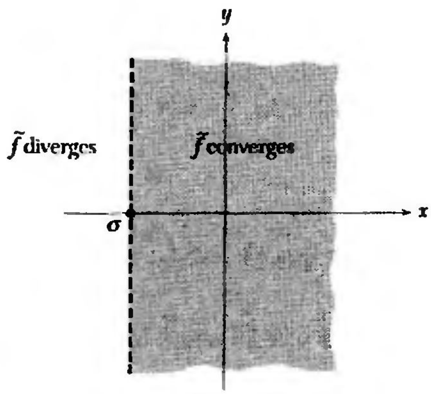


Figure 8.1.1: Half plane of convergence of the Laplace transform.

Theorem 8.1.2 (Laplace Transforms) *Suppose that the functions f and h are continuous and that $\tilde{f}(z) = \tilde{h}(z)$ for* $\operatorname{Re} z > \gamma_0$ *for some γ_0. Then $f(t) = h(t)$ for all $t \in [0, \infty[$.*

This theorem is not as simple as it seems. We do not have enough mathematical tools to give a complete proof, but the main ideas are given at the end of the section. Using ideas from integration theory, we could extend the result of the uniqueness theorem to discontinuous functions as well, but we would have to modify what we mean by "equality of functions." For example, if $f(t)$ is changed at a single value of t, then $\tilde{f}$ is unchanged.

The uniqueness theorem enables us to give a meaningful answer to the problem "Given $g(z)$, find $f(t)$ such that $\tilde{f} = g$," because it makes clear that there can be at most one such (continuous) f. We call f the ***inverse Laplace transform*** of g; methods for finding f when g is given are considered in §8.2.

Laplace Transforms of Derivatives The main utility of Laplace transforms is that they enable us to transform differential problems into algebraic problems. When the latter are solved, the answers to the original problems are obtained by using the inverse Laplace transform. The procedure is based on the following theorem.

Proposition 8.1.3 *Let $f(t)$ be continuous on $[0, \infty[$ and piecewise C^1, that is, piecewise continuously differentiable. Then for* $\operatorname{Re} z > \rho$ *(as defined in the convergence theorem* (8.1.1)*),*

$$\left(\frac{df}{dt}\right)^{\sim}(z) = z\tilde{f}(z) - f(0).$$

Proof By definition,

$$\left(\frac{df}{dt}\right)^{\sim}(z) = \int_0^\infty e^{-zt}\frac{df}{dt}(t)dt.$$

Integrating by parts, we get

$$\lim_{t_0\to\infty}\left(e^{-zt}f(t)\Big|_0^{t_0}\right) + \int_0^\infty ze^{-zt}f(t)dt.$$

By definition of $\rho, |e^{-Bt_0}\cdot f(t_0)| \le A$ for some $B < \operatorname{Re} z$. Thus, we get

$$|e^{-zt_0}\cdot f(t_0)| = |e^{-(z-B)t_0}||e^{-Bt_0}\cdot f(t_0)| \le e^{-(\operatorname{Re} z - B)t_0}A,$$

which approaches 0 as $t_0 \to \infty$. Therefore, we get $-f(0) + z\tilde{f}(z)$, as asserted. ■

While $(df/dt)^{\sim}(z)$ exists for $\operatorname{Re} z > \rho$, its abscissa of convergence might be smaller than ρ.

If we apply the preceding proposition to d^2f/dt^2, we obtain

$$\left(\frac{d^2f}{dt^2}\right)^{\sim}(z) = z^2\tilde{f}(z) - zf(0) - \frac{df}{dt}(0).$$

The formula for $d\tilde{f}/dz$ in the convergence theorem (8.1.1) is related to the formula

$$\tilde{g}(z) = d\tilde{f}(z)/dz, \quad \text{where} \quad g(t) = -tf(t).$$

In Exercise 19 the student is asked to prove the next proposition, which contains a similar formula for integrals.

Proposition 8.1.4 *Let* $g(t) = \int_0^t f(\tau)d\tau$. *Then for* $\operatorname{Re} z > \max[0, \rho(f)]$,

$$\tilde{g}(z) = \frac{\tilde{f}(z)}{z}.$$

Shifting Theorems Table 8.1.1 at the end of this section lists some formulas that are useful for computing $\tilde{f}(z)$. The proofs of these formulas are straightforward and are included in the exercises and examples. However, three of the formulas are sufficiently important to be given separate explanation, which is done in the following three theorems.

Theorem 8.1.5 (First Shifting Theorem) *Fix* $a \in \mathbb{C}$ *and let* $g(t) = e^{-at}f(t)$. *Then for* $\operatorname{Re} z > \sigma(f) - \operatorname{Re} a$, *we have*

$$\tilde{g}(z) = \tilde{f}(z + a).$$

Proof By definition,

$$\tilde{g}(z) = \int_0^\infty e^{-zt}e^{-at}f(t)dt = \int_0^\infty e^{-(z+a)t}f(t)dt = \tilde{f}(z+a),$$

which is valid if $\text{Re}(z+a) > \sigma$. ■

Theorem 8.1.6 (Second Shifting Theorem) *Let $H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t \geq 0$, which is called the **Heaviside**, or **unit step**, **function**. Also, let $a \geq 0$ and let $g(t) = f(t-a)H(t-a)$; that is, $g(t) = 0$ if $t < a$ while $g(t) = f(t-a)$ if $t \geq a$. (See Figure 8.1.2.) Then for* $\text{Re}\, z > \sigma$, *we have*

$$\tilde{g}(z) = e^{-az}\tilde{f}(z).$$

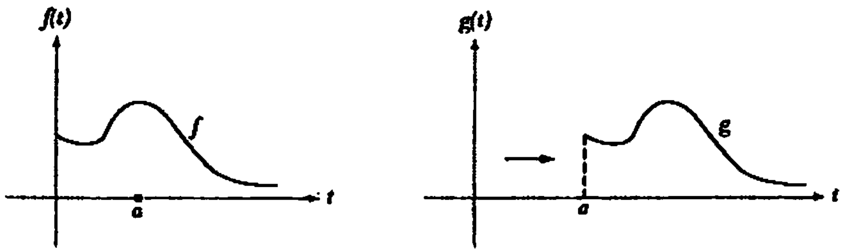


Figure 8.1.2: The function g in the second shifting theorem.

Proof By definition and because $g = 0$ for $0 \leq t < a$,

$$\tilde{g}(z) = \int_0^\infty e^{-zt}g(t)dt = \int_a^\infty e^{-zt}f(t-a)dt.$$

Letting $\tau = t - a$, we get

$$\tilde{g}(z) = \int_0^\infty e^{-z(\tau+a)}f(\tau)d\tau = e^{-za}\tilde{f}(z). \quad ■$$

From the second shifting theorem, we can deduce that if $a \geq 0$ and $g(t) = f(t)H(t-a)$, then $\tilde{g}(z) = e^{-az}\tilde{F}(z)$ where $F(t) = f(t+a), t \geq 0$ (see Figure 8.1.3).

Convolutions The ***convolution*** of two functions $f(t)$ and $g(t)$ is defined for $t \geq 0$ by

$$(f * g)(t) = \int_0^\infty f(t-\tau) \cdot g(\tau)d\tau$$

where we set $f(t) = 0$ if $t < 0$. Thus, the integration is really only from 0 to t. The convolution operation is related to Laplace transforms in the following way.

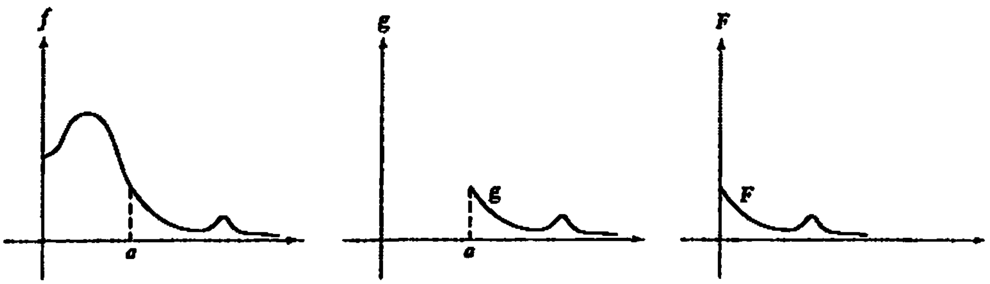


Figure 8.1.3: F is obtained from f by shifting and truncating.

Theorem 8.1.7 (Convolution Theorem) *The equalities $f * g = g * f$ and*

$$(f * g)\tilde{}(z) = \tilde{f}(z) \cdot \tilde{g}(z)$$

whenever $\operatorname{Re} z > \max[\rho(f), \rho(g)]$.

In brief, this theorem states that the *Laplace transform of a convolution of two functions is the product of their Laplace transforms.* It is precisely this property that makes the convolution an operation of interest to us.

Proof We have

$$\begin{aligned}(f * g)\tilde{}(z) &= \int_0^\infty e^{-zt}\left[\int_0^\infty f(t-\tau)\cdot g(\tau)d\tau\right]dt \\ &= \int_0^\infty \left[\int_0^\infty e^{-z\tau}e^{-z(t-\tau)}f(t-\tau)g(\tau)d\tau\right]dt.\end{aligned}$$

For $\operatorname{Re} z > \max[\rho(f), \rho(g)]$ the integrals for $\tilde{f}(z)$ and $\tilde{g}(z)$ converge absolutely, so we can interchange the order of integration[2] to obtain

$$\int_0^\infty e^{-z\tau}\left[\int_0^\infty e^{-z(t-\tau)}f(t-\tau)dt\right]g(\tau)d\tau.$$

Letting $s = t - \tau$ and remembering that $f(s) = 0$ if $s < 0$, we get

$$\int_0^\infty e^{-z\tau}\tilde{f}(z)g(\tau)d\tau = \tilde{f}(z)\cdot\tilde{g}(z). \quad \blacksquare$$

By changing variables, it is not difficult to verify that $f * g = g * f$, but such verification also follows from what we have done if f and g are continuous. We have

$$(f * g)\tilde{} = \tilde{f}\cdot\tilde{g} = \tilde{g}\cdot\tilde{f} = (g * f)\tilde{}.$$

Thus, $(f * g - g * f)\tilde{} = 0$, so by uniqueness theorem (8.1.2), $f * g - g * f = 0$.

[2]This is a theorem concerning integration theory from advanced calculus. See, for instance, J. Marsden and M. Hoffman, *Elementary Classical Analysis*, Second Edition (New York: W. H. Freeman and Company, 1993), Chapter 9.

Technical Proofs of Theorems To prove the convergence theorem (8.1.1), we shall use the following important result.

Lemma 8.1.8 *Suppose that $\tilde{f}(z) = \int_0^\infty e^{-zt} f(t)dt$ converges for $z = z_0$. Assume that $0 \leq \theta < \pi/2$ and define the set*

$$S_\theta = \{z \text{ such that } |\arg(z - z_0)| \leq \theta\}$$

(see Figure 8.1.4). *Then $\tilde{f}$ converges uniformly on S_θ.*

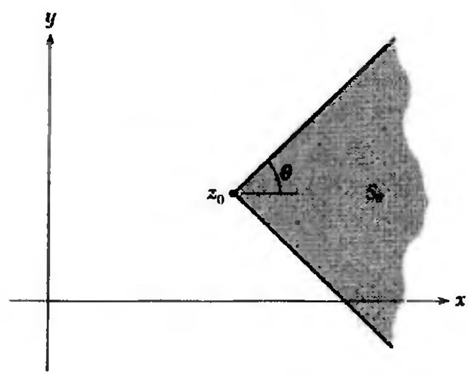


Figure 8.1.4: Sector of uniform convergence.

Proof Let

$$h(x) = \int_0^x e^{-z_0 t} f(t)dt - \int_0^\infty e^{-z_0 t} f(t)dt$$

so that $h \to 0$ as $x \to \infty$. We must show that for every $\epsilon > 0$, there is a t_0 such that $t_1, t_2 \geq t_0$ implies that

$$\left| \int_{t_1}^{t_2} e^{-zt} f(t)dt \right| < \epsilon$$

for all $z \in S_\theta$. It follows that $\int_0^x e^{-zt} f(t)dt$ converges uniformly on S_θ as $x \to \infty$, by the Cauchy Criterion. We will make use of the function $h(x)$ as follows. Write

$$\int_{t_1}^{t_2} e^{-zt} f(t)dt = \int_{t_1}^{t_2} e^{-(z-z_0)t}[e^{-z_0 t} f(t)]dt.$$

Integrating by parts, we get

$$e^{-(z-z_0)t_2} h(t_2) - e^{-(z-z_0)t_1} h(t_1) + (z - z_0) \int_{t_1}^{t_2} e^{-(z-z_0)t} h(t)dt.$$

Given $\epsilon > 0$, choose t_0 such that $|h(t)| < \epsilon/3$ and $|h(t)| < \epsilon' = \epsilon/(6\sec\theta)$ if $t \geq t_0$. Then for $t_2 > t_0$,

$$|e^{-(z-z_0)t_2}h(t_2)| \leq |h(t_2)| < \frac{\epsilon}{3},$$

since $|e^{-(z-z_0)t_2}| = e^{-(\mathrm{Re}\,z-\mathrm{Re}\,z_0)t_2} \leq 1$ because $\mathrm{Re}\,z > \mathrm{Re}\,z_0$. Similarly, for $t_1 > t_0$,

$$|e^{-(z-z_0)t_1}h(t_1)| < \frac{\epsilon}{3}.$$

We must still estimate the last term:

$$\left|(z-z_0)\int_{t_1}^{t_2} e^{-(z-z_0)t}h(t)dt\right| \leq |z-z_0|\epsilon' \int_{t_1}^{t_2} e^{-(x-x_0)t}dt,$$

where $x = \mathrm{Re}\,z$ and $x_0 = \mathrm{Re}\,z_0$. If $z = z_0$, this term is zero. If $z \neq z_0$, then $x \neq x_0$ (see the figure), and we get

$$\epsilon' \frac{|z-z_0|}{x-x_0}\left(e^{-(x-x_0)t_1} - e^{-(x-x_0)t_2}\right) < 2\epsilon' \frac{|z-z_0|}{x-x_0} \leq 2\epsilon' \sec\theta = \frac{\epsilon}{3}$$

(see Figure 8.1.5). Note that the restriction $0 \leq \theta < \pi/2$ is necessary for $\sec\theta = 1/\cos\theta$ to be finite.

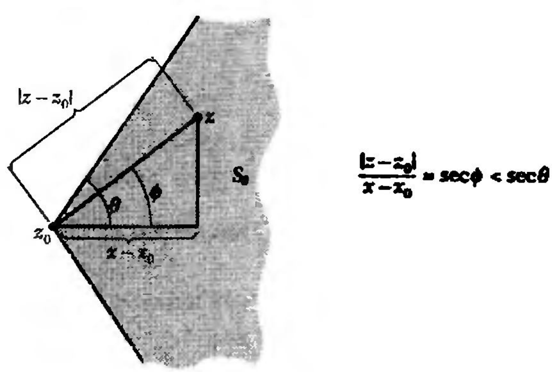


Figure 8.1.5: Some geometry in the region S_θ.

Combining the preceding inequalities, we get

$$\left|\int_{t_1}^{t_2} e^{-zt}f(t)dt\right| < \epsilon$$

if $t_1, t_2 \geq t_0$ for all $z \in S_\theta$, thus completing the proof of the lemma. ▼

Proof of the Convergence Theorem 8.1.1 Let

$$\sigma = \inf\left\{x \in \mathbf{R} \,\middle|\, \int_0^\infty e^{-x}f(t)dt \text{ converges}\right\},$$

where inf stands for "greatest lower bound." Note from Lemma 8.1.8 that if $\bar{f}(z_0)$ converges, then, for $\operatorname{Re} z > \operatorname{Re} z_0, \bar{f}(z)$ converges because z lies in some S_θ for z_0 (Why?).

Let $\operatorname{Re} z > \sigma$. By the definition of σ, there is an $x_0 < \operatorname{Re} z$ such that the integral $\int_0^\infty e^{-x_0 t}f(t)dt$ converges. Hence $\bar{f}(z)$ converges by Lemma 8.1.8. Conversely, assume $\operatorname{Re} z < \sigma$ and $\operatorname{Re} z < x < \sigma$. If $\bar{f}(z)$ converges, then so does $\bar{f}(x)$, and therefore $\sigma \leq x$ gives a contradiction. Thus $\bar{f}(z)$ does not converges if $\operatorname{Re} z < \sigma$.

We use the Analytic Convergence Theorem 3.1.8, to show that $\bar{f}$ is analytic on the set $\{z \mid \operatorname{Re} z > \sigma\}$. Let $g_n(z) = \int_0^n e^{-zt}f(t)dt$. Then $g_n(z) \to \bar{f}(z)$. By Worked Example 2.4.15, g_n is analytic with $g_n'(z) = -\int_0^n te^{-zt}f(t)dt$. We must show that $g_n \to \bar{f}$ uniformly on closed disks in $\{z \mid \operatorname{Re} z > \sigma\}$. But each disk lies in some S_θ relative to some z_0 with $\operatorname{Re} z_0 > \sigma$ (Figure 8.1.6).

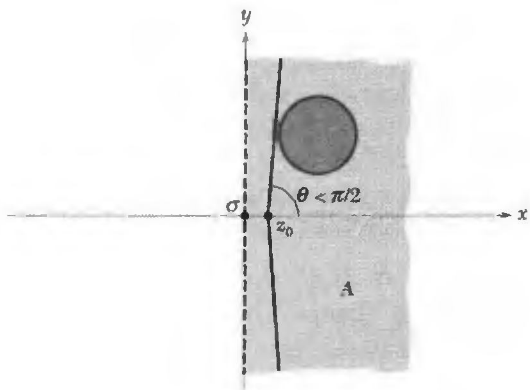


Figure 8.1.6: Each disk lies in S_θ for some $\theta, 0 \leq \theta < \pi/2$.

Thus, by the Analytic Convergence Theorem 3.1.8, $\bar{f}$ is analytic on the set $\{z \mid \operatorname{Re} z > \sigma\}$ and

$$(\bar{f})'(z) = -\int_0^\infty te^{-zt}f(t)dt.$$

It follows that this integral representation for the derivative of $\bar{f}$ converges for $\operatorname{Re} z > \sigma$, as do all the iterated derivatives.

It remains to be shown that $\sigma \leq \rho$. To prove this we need to show that $\sigma \leq B$ if $|f(t)| \leq Ae^{Bt}$. This will hold, by what we have proven, if $\bar{f}(z)$ converges for $\operatorname{Re} z > B$. Indeed, we show absolute convergence. Note that

$$|e^{-zt}f(t)| = |e^{-(z-B)t}e^{-Bt}f(t)| \leq e^{-(\operatorname{Re} z-B)t}A.$$

Since the integral $\int_0^\infty e^{-\alpha t}dt = 1/\alpha$ converges for $\alpha > 0$, it follows that the integral $\int_0^\infty e^{-zt}f(t)dt$ converges absolutely. ■

To prove that $\tilde{f} = \tilde{h}$ implies that $f = h$ for continuous functions f and h, it suffices, by considering $f - h$, to prove the following special case of Theorem 8.1.2.

Proposition 8.1.9 *Suppose that f is continuous and that for some real y_0, $\tilde{f}(z) = 0$ whenever $\operatorname{Re} z > y_0$. Then $f(t) = 0$ for all $t \in [0, \infty[$.*

The crucial lemma we use to prove this is the following.

Lemma 8.1.10 *Let f be continuous on $[0, 1]$ and suppose that $\int_0^1 t^n f(t)dt = 0$ for all $n = 0, 1, 2, \ldots$. Then $f = 0$.*

This assertion is reasonable since it follows that $\int_0^1 P(t)f(t)dt = 0$ for any polynomial P.

Proof The precise proof depends on the *Weierstrass approximation theorem*, which states that any continuous function is the uniform limit of polynomials.[3] By this theorem we get $\int_0^1 g(t)f(t)dt = 0$ for any continuous g. The result follows by taking $g(t) = \overline{f(t)}$ and applying the fact that if the integral of a nonnegative continuous function is zero, then the function is zero. ▼

Proof of Proposition 8.1.9 Suppose that

$$\tilde{f}(z) = \int_0^\infty e^{-zt}f(t)dt = 0$$

whenever $\operatorname{Re} z > \sigma$. Fix $x_0 > y_0$ real and let $s = e^{-t}$. By changing variables to express the integrals in terms of s and letting $z = x_0 + n$ for $n = 0, 1, 2, \ldots$, we get

$$0 = \int_0^\infty e^{-nt}e^{-x_0 t}f(t)dt = \int_1^0 s^n s^{x_0} f(-\log s)\left(-\frac{1}{s}\right) ds = \int_0^1 s^n h(s)ds = 0,$$

where $h(s) = s^{x_0 - 1}f(-\log s) = e^{-x_0 t + t}f(t)$. By the Lemma, h must be identically zero, and f must be also since the exponential function is never zero. ■

It is useful to note that $\tilde{f}(z) \to 0$ as $\operatorname{Re} z \to \infty$. This follows from the argument used to prove Theorem 8.1.1 (see Review Exercise 10).

[3]See, for example, J. Marsden and M. Hoffman, *Elementary Classical Analysis*, Second Edition (New York: W. H. Freeman and Company, 1993), Chapter 5.

Table 8.1.1 Some Common Laplace Transforms

Definition

$$\tilde{f}(z) = \int_0^\infty e^{-zt} f(t)dt$$

Properties

1. $\tilde{g}(z) = -\frac{d}{dz}\tilde{f}(z)$ where $g(t) = tf(t)$.
2. $(af + bg)\tilde{} = a\tilde{f} + b\tilde{g}$.
3. $\left(\frac{df}{dt}\right)\tilde{}(z) = z\tilde{f}(z) - f(0)$. (Assume that f is piecewise C^1.)
4. $\tilde{g}(z) = \frac{1}{z}\tilde{f}(z)$ where $g(t) = \int_0^t f(\tau)d\tau$.
5. $\tilde{g}(z) = \tilde{f}(z+a)$ where $g(t) = e^{-at}f(t)$.
6. $\tilde{g}(z) = e^{-az}\tilde{f}(z)$, where $a > 0$, and
$$g(t) = f(t-a) \quad \text{for} \quad t \geq a \quad \text{and} \quad 0 \quad \text{if} \quad t < a.$$
7. $\tilde{g}(z) = e^{-az}\tilde{F}(z)$, where $a \geq 0, F(t) = f(t+a)$, and
$$g(t) = f(t) \quad \text{if} \quad t \geq a \quad \text{and} \quad 0 \quad \text{if} \quad 0 \leq t < a.$$
8. $(f * g)\tilde{}(z) = \tilde{f}(z) \cdot \tilde{g}(z)$, where the ***convolution*** is defined by
$$(f * g)(t) = \int_0^\infty f(t-\tau)g(\tau)d\tau.$$
9. If $f(t) = e^{-at}$, then $\tilde{f}(z) = \frac{1}{z+a}$ and $\sigma(f) = -\operatorname{Re} a$.
10. For $f(t) = \cos at, \tilde{f}(z) = \frac{z}{z^2+a^2}$ and $\sigma(f) = |\operatorname{Im} a|$.
11. If $f(t) = \sin at, \tilde{f}(z) = \frac{a}{z^2+a^2}$ and $\sigma(f) = |\operatorname{Im} a|$.
12. If $f(t) = t^a, a > -1, \tilde{f}(z) = \frac{\Gamma(a+1)}{z^{a+1}}$ and $\sigma(f) = 0$.
13. If $f(t) = 1, \tilde{f}(z) = \frac{1}{z}$ and $\sigma(f) = 0$.

Worked Examples

Example 8.1.11 *Prove formula 9 in Table 8.1.1 and find $\sigma(f)$ in that case.*

Solution By definition,

$$\tilde{f}(z) = \int_0^\infty e^{-at}e^{-zt}dt = \int_0^\infty e^{-(a+z)t}dt = -\left.\frac{e^{-(a+z)t}}{a+z}\right|_0^\infty = \frac{1}{z+a}.$$

The evaluation at $t = \infty$ is justified by noting that $\lim_{t\to\infty} e^{-(a+z)t} = 0$ provided $\operatorname{Re}(a+z) > 0$, since $|e^{-(a+z)t}| = e^{-\operatorname{Re}(a+z)t} \to 0$ as $t \to \infty$. Thus, the formula is valid if $\operatorname{Re} z > -\operatorname{Re} a$.

Note that the formula for $\tilde{f}$ is valid only for $\operatorname{Re} z > -\operatorname{Re} a$, although $\tilde{f}$ coincides there with a function that is analytic except at $z = -a$. This situation is similar to that for the gamma function (see formula 12 of Table 7.1.1).

Finally, we show that for $f(t) = e^{-at}, \sigma(f) = -\operatorname{Re} a$. We have already shown that $\sigma(f) \le -\operatorname{Re} a$. But the integral diverges at $z = a$, so $\sigma(f) \ge -\operatorname{Re} a$, and thus $\sigma(f) = -\operatorname{Re} a$. If $a = 0$, this example specializes to formula 13 of Table 8.1.1.

Example 8.1.12 *Suppose that we have computed $\tilde{f}(z)$ and found it to converge for* $\operatorname{Re} z > \gamma$. *Suppose also that $\tilde{f}$ coincides with an analytic function that has a pole on the line* $\operatorname{Re} z = \gamma$. *Show that $\sigma(f) = \gamma$.*

Solution We know that $\sigma(f) \le \gamma$ by the basic property of σ in the convergence theorem. Also, since $\tilde{f}$ is analytic for $\operatorname{Re} z > \sigma$, there can be no poles in the region $\{z \mid \operatorname{Re} z > \sigma\}$. If $\sigma(f)$ were $< \gamma$, there would be a pole in this region. Hence $\sigma(f) = \gamma$ (see Figure 8.1.7).

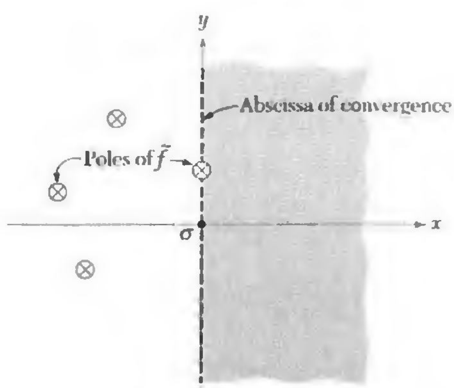


Figure 8.1.7: Location of poles of $\tilde{f}$.

Example 8.1.13 *Let $f(t) = \cosh t$. Compute $\tilde{f}$ and $\sigma(f)$.*

Solution $f(t) = \cosh t = (e^t + e^{-t})/2$. Thus, by formulas 2 and 9 of Table 8.1.1,

$$\tilde{f}(z) = \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1}\right) = \frac{z}{z^2-1}.$$

Here $\sigma(f) = 1$ by Worked Example 8.1.12; $\sigma(e^t) = 1$ and $\sigma(e^{-t}) = -1$, so $\sigma(f) \le 1$ but it cannot be < 1 since $\tilde{f}$ has a pole at $z = 1$.

Exercises

In Exercises 1 through 9, compute the Laplace transform of $f(t)$ and find the abscissa of convergence.

1. $f(t) = t^2 + 2$
2. $f(t) = \sinh t$
3. $f(t) = t + e^{-t} + \sin t$
4. • $f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ 1 & 1 < t < 2 \\ 0 & t \ge 2 \end{cases}$
5. $f(t) = (t+1)^n$, n a positive integer
6. $f(t) = \sin t$ if $0 \le t \le \pi$ and 0 if $t > \pi$
7. $f(t) = t \sin at$
8. $f(t) = t \sinh at$
9. $f(t) = t \cos at$
10. Use the shifting theorems to show the following:

    (a) If $f(t) = e^{-at}\cos bt$, then

    $$\tilde{f}(z) = \frac{z+a}{(z+a)^2 + b^2}.$$

    (b) If $f(t) = e^{-at}t^n$, then

    $$\tilde{f}(z) = \frac{\Gamma(n+1)}{(z+a)^{n+1}}.$$

    What is $\sigma(f)$ in each case?
11. Prove formula 10 of Table 8.1.1.
12. Prove formula 11 of Table 8.1.1.

13.* Prove formula 12 of Table 8.1.1.

14. Prove formula 13 of Table 8.1.1.

15.* Suppose that f is periodic with period p (that is, $f(t+p) = f(t)$ for all $t \geq 0$). Prove that

$$\tilde{f}(z) = \frac{\int_0^p e^{-zt} f(t)dt}{1 - e^{-pz}}$$

is valid if $\operatorname{Re} z > 0$. *Hint*: Write out $\tilde{f}(z)$ as an infinite sum.

16. Use Exercise 15 to prove that

$$\tilde{f}(z) = \frac{1}{z} \cdot \frac{1 - e^{-z}}{1 - e^{-2z}}$$

where $f(t)$ is the ***pulse function*** illustrated in Figure 8.1.8.

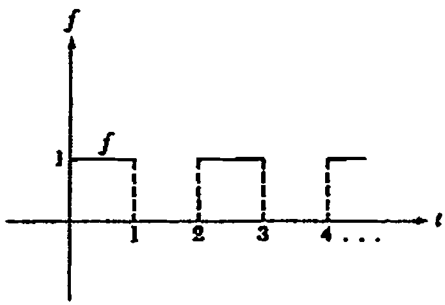


Figure 8.1.8: The unit pulse function.

17. Let $g(t) = \int_0^t e^{-s} \sin s\, ds$. Compute $\tilde{g}(z)$. Compute $\tilde{f}(z)$ if $f(t) = tg(t)$.

18. Let $f(t) = (\sin at)/t$. Show that $\tilde{f}(z) = \tan^{-1}(a/z)$.

19.* Prove Proposition 8.1.4. First establish that $\rho(g) \leq \max[0, \rho(f)]$.

20. Give a direct proof that $f * g = g * f$ (see the Convolution Theorem 8.1.7).

21.* Let $f(t) = e^{-e^t}, t \geq 0$. Show that $\sigma(f) = -\infty$.

22. Referring to the Convergence Theorem 8.1.1, show that, in general, $\sigma \neq \rho$. *Hint*: Consider $f(t) = e^t \sin e^t$ and show that $\sigma = 0, \rho = 1$.

8.2 Complex Inversion Formula

To be able to recover a function from its Laplace transform, it is important to be able to compute $f(t)$ when $\tilde{f}(z)$ is known. One technique for such a computation, using the *complex inversion formula*, will be established in this section. Using the formulas of Table 8.1.1 in reverse gives useful alternative techniques. (See Worked Examples 8.2.4 and 8.2.5.)

Main Inversion Formula The complex inversion formula, one of the key results for the Laplace transform, draws on many of the main points developed in the first four chapters of this book.

Theorem 8.2.1 *Suppose that $F(z)$ is analytic on $\mathbb{C}$ except for a finite number of isolated singularities and that for some real number σ, F is analytic on the half plane $\{z \mid \mathrm{Re}\, z > \sigma\}$. Suppose also that there are positive constants M, R, and β such that $|F(z)| \le M/|z|^\beta$ whenever $|z| \ge R$ (this is true, for example, if $F(z) = P(z)/Q(z)$ for polynomials P and Q with* $\deg(Q) \ge 1 + \deg(P)$*). For $t \ge 0$, let*

$$f(t) = \sum\{\text{residues of } e^{zt}F(z) \text{ at each of its singularities in } \mathbb{C}\}.$$

Then $\tilde{f}(z) = F(z)$ for $\mathrm{Re}\, z > \sigma$. *We call this the **complex inversion formula**.*

Proof Let $\alpha > \sigma$ and consider a large rectangle Γ with sides along the lines $\mathrm{Re}\, z = -x_1, \mathrm{Re}\, z = x_2, \mathrm{Im}\, z = y_2$, and $\mathrm{Im}\, z = -y_1$ selected large enough so that all the singularities of F are inside Γ and $|z| > R$ everywhere on Γ. Split Γ into a sum of two rectangular paths γ and $\tilde{\gamma}$ by a vertical line through $\mathrm{Re}\, z = \alpha$. (See Figure 8.2.1.)

The proof of the complex inversion formula could just as well be carried out using a large circle instead of the rectangle Γ. In fact, in the last paragraph of the proof, Γ is briefly deformed to such a circle. However, the rectangular path will be useful in Corollary 8.2.2, in which it plays a role like that of the rectangular path in the proof of Proposition 4.3.9 concerning the evaluation of Fourier transforms.

Since all singularities of F are inside γ, the definition of f gives

$$\int_\gamma e^{zt}F(z)dz = 2\pi i \sum\{\text{residues of } e^{zt}F(z)\} = 2\pi i f(t),$$

so

$$2\pi i \tilde{f}(z) = \lim_{r\to\infty}\int_0^r e^{-zt}\left[\int_\gamma e^{\zeta t}F(\zeta)d\zeta\right]dt = \lim_{r\to\infty}\int_\gamma\int_0^r e^{(\zeta-z)t}F(\zeta)dt\,d\zeta.$$

We may interchange the order of integration, because both integrals are over finite intervals. Therefore,

$$2\pi i \tilde{f}(z) = \lim_{r\to\infty}\int_\gamma \left(e^{(\zeta-z)r} - 1\right)\frac{F(\zeta)}{\zeta - z}d\zeta.$$

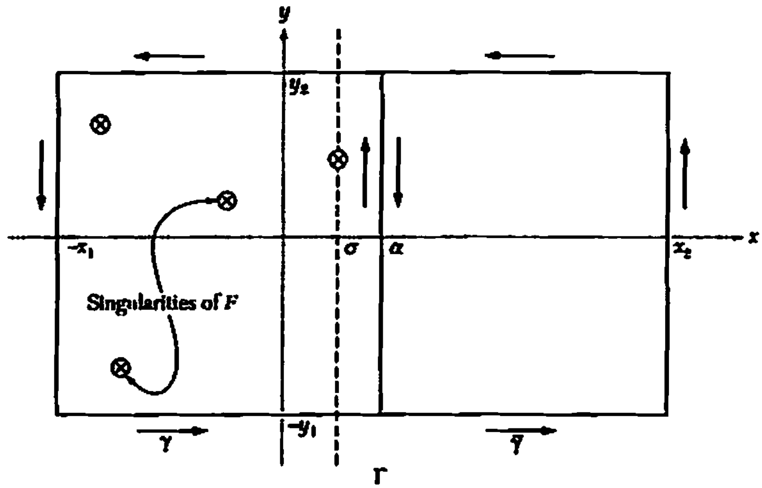


Figure 8.2.1: The large contour is the sum of the smaller two: $\Gamma = \gamma + \tilde{\gamma}$.

With z fixed in the half plane $\operatorname{Re} z > \alpha$, the term $e^{(\zeta - z)r}$ approaches 0 and the integrand converges uniformly to $-F(\zeta)/(\zeta - z)$ on γ. We obtain

$$\begin{aligned} 2\pi i \tilde{f}(z) &= -\int_{\gamma} \frac{F(\zeta)}{\zeta - z} d\zeta = \int_{\tilde{\gamma}} \frac{F(\zeta)}{\zeta - z} d\zeta - \int_{\Gamma} \frac{F(\zeta)}{\zeta - z} d\zeta \\ &= 2\pi i F(z) - \int_{\Gamma} \frac{F(\zeta)}{\zeta - z} d\zeta \end{aligned}$$

provided Γ is large enough so that z is inside $\tilde{\gamma}$. Finally,

$$\left| \int_{\Gamma} \frac{F(\zeta)}{\zeta - z} d\zeta \right| \le \int_{\Gamma} \frac{M}{|z|^{\beta} |\zeta - z|} |d\zeta| \le \frac{2\pi M \rho}{\rho^{\beta}(\rho - R)},$$

which is obtained by choosing Γ large enough so that it lies outside the circle $|\zeta| = \rho > R$ with all the singularities of $F(\zeta)/(\zeta - z)$ inside this circle, and then deforming Γ to this circle. This last expression goes to 0 as $\rho \to \infty$. Thus, letting Γ expand outward toward ∞, we obtain $\tilde{f}(z) = F(z)$. Since $\alpha > \sigma$ is arbitrary, the complex inversion formula holds for any z in the half plane $\operatorname{Re} z > \sigma$. ■

Corollary 8.2.2 *Let the conditions of the complex inversion formula hold. If $F(z)$ is analytic for $\operatorname{Re} z > \sigma$ and has a singularity on the line $\operatorname{Re} z = \sigma$, then* (i) *the abscissa of convergence of f is σ, and* (ii)

$$f(t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{zt} F(z) dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\alpha + iy)t} F(\alpha + iy) dy$$

for any constant $\alpha > \sigma$. The first integral is taken along the vertical line $\operatorname{Re} z = \alpha$ and converges as an improper Riemann integral; the second integral is used as alternative notation for the first.

Proof

(i) The complex inversion formula shows that $\sigma(f) \leq \sigma$ since $\tilde{f}(z)$ converges for $\operatorname{Re} z > \sigma$. If $\sigma(f)$ were $< \sigma$, then $\tilde{f}(z)$ would be analytic for $\operatorname{Re} z > \sigma(f)$ by the Convergence Theorem 8.1.1. But F has a singularity at a point z_0 on the line $\operatorname{Re} z = \sigma$, so there is a sequence of points $z_1, z_2, z_3, \ldots$ converging to z_0 with $F(z_n) \to \infty$. Since $\tilde{f}(z) = F(z)$ for $\operatorname{Re} z > \sigma$, and since both are analytic in a deleted neighborhood of z_0, they would be equal in that deleted neighborhood by the principle of analytic continuation. This would mean that $\tilde{f}(z_n) \to \infty$. But that is impossible, since $\tilde{f}(z)$ is analytic on $\operatorname{Re} z > \sigma(f)$. Thus, $\sigma(f) < \sigma$ is not possible, so $\sigma(f) = \sigma$.

(ii) From the complex inversion formula, $2\pi i f(t) = \int_\gamma e^{zt} F(z)dz$. This integral converges to the integral in the statement, exactly as in the proof of Proposition 4.3.9, as x_1, y_1, and $y_2 \to \infty$. Since y_1 and y_2 go independently to ∞, this establishes convergence of the improper integral. (The situation here is rotated by 90° from that of Proposition 4.3.9.) ■

In working examples, all conditions of the theorem must be checked. If they do not hold, these formulas for $f(t)$ may not be valid (see Example 8.2.5). The complex inversion formula is sometimes more convenient than Table 8.1.1 for computing inverse Laplace transforms since it is systematic and requires no guesswork as to which formula is appropriate. However, the table may be useful in cases in which hypotheses of the theorem do not apply or are inconvenient to check.

Heaviside Expansion Theorem Now we apply the complex inversion formula to the case in which $F(z) = P(z)/Q(z)$ where P and Q are polynomials. We give a simple case here.

Theorem 8.2.3 *Let $P(z)$ and $Q(z)$ be polynomials with* $\deg Q \geq \deg P + 1$. *Suppose that the zeros of Q are located at the points $z_1, \ldots, z_m$ and are simple zeros. Then the inverse Laplace transform of $F(z) = P(z)/Q(z)$ is given by the* ***Heaviside expansion formula:***

$$f(t) = \sum_{i=1}^{m} e^{z_i t} \frac{P(z_i)}{Q'(z_i)}.$$

Furthermore, $\sigma(f) = \max\{\operatorname{Re} z_i \mid i = 1, 2, \ldots, m\}$.

Proof Since $\deg Q \geq \deg P + 1$, the conditions of the complex inversion formula (8.2.1) are met (compare Proposition 4.3.9). Thus,

$$f(t) = \sum \left\{ \text{residues of } e^{zt} \frac{P(z)}{Q(z)} \right\}.$$

But the poles are all simple and so, by formula 4 of Table 4.1.1, we have

$$\operatorname{Res}\left(e^{zt}\frac{P(z)}{Q(z)}; z_i\right) = e^{z_i t}\frac{P(z_i)}{Q'(z_i)}.$$

The formula for $\sigma(f)$ is a consequence of Corollary 8.2.2. ■

Worked Examples

Example 8.2.4 *If $\tilde{f}(z) = 1/(z-3)$, find $f(t)$.*

Solution Refer to formula 9 of Table 8.1.1. Let $a = -3$; then we get $f(t) = e^{3t}$. Alternatively, we could get the same result by using the Heaviside expansion formula. In this example, $\sigma(f) = 3$.

Example 8.2.5 *If $\tilde{f}(z) = \log(z^2 + z)$, what is $f(t)$?*

Solution If f were such a function and $g(t) = tf(t)$, then by formula 1 of Table 8.1.1, we would have

$$\tilde{g}(z) = -\frac{d}{dz}\tilde{f}(z) = -\frac{d}{dz}\log(z^2 + z) = -\frac{2z+1}{z^2+z}.$$

To find $g(t)$ we could use partial fractions.

$$\tilde{g}(z) = -\frac{2z+1}{z^2+z} = -\frac{1}{z} - \frac{1}{z+1}.$$

Therefore $g(t) = -1 - e^{-t}$, and so

$$f(t) = -\frac{1}{t}(1 + e^{-t}).$$

Although this argument seems satisfactory, it is deceptive because *there is in fact no $f(t)$ whose Laplace transform is $\log(z^2 + z)$*. If there were, then this procedure would show that $f(t) = -(1 + e^{-t})/t$ is the only possibility. However, the integral

$$\int_0^\infty e^{-xt} f(t)\, dt$$

cannot converge for any real x because e^{-xt} is larger than $1/2$ near 0 and $|f(t)| \geq 1/t$. But $1/t$ is not integrable. Thus f does not exist in any sense we have studied. The argument above does not actually find such an f. It assumes that there is one and shows that there is only one possibility. But that one does not work. See also the remark at the end of §8.1.

Example 8.2.6 *Compute the inverse Laplace transform of*

$$F(z) = \frac{z}{(z+1)^2(z^2+3z-10)}.$$

Then compute $\sigma(f)$, the abscissa of convergence of f.

Solution In this case the hypotheses of the complex inversion formula clearly hold. Thus

$$f(t) = \sum \left\{ \text{residues of } \frac{e^{zt}z}{(z+1)^2(z^2+3z-10)} = \frac{e^{zt}z}{(z+1)^2(z+5)(z-2)} \right\}.$$

The poles are at $z = -1, z = -5$, and $z = 2$. The pole at -1 is double, whereas the others are simple. By formula 7 of Table 4.1.1, the residue at -1 is $g'(-1)$, where

$$g(z) = \frac{e^{zt}z}{z^2+3z-10}.$$

Thus, we obtain

$$\frac{-te^{-t}}{-12} + \frac{e^{-t}}{-12} - \frac{(-e^{-t})\cdot[2\cdot(-1)+3]}{144} = \frac{1}{12}\left(te^{-t} - e^{-t} + \frac{e^{-t}}{12}\right).$$

The residue at -5 is $e^{-5t}\cdot 5/16\cdot 7$; the residue at 2 is $e^{2t}\cdot 2/9\cdot 7$. Thus,

$$f(t) = \frac{1}{12}\left(te^{-t} - e^{-t} + \frac{e^{-t}}{12}\right) + \frac{5e^{-5t}}{16\cdot 7} + \frac{2e^{2t}}{63}.$$

By Corollary 8.2.2, $\sigma(f) = 2$.

Exercises

1. Compute the inverse Laplace transform of each of the following.

   (a) $F(z) = \dfrac{z}{z^2+1}$.

   (b) $F(z) = \dfrac{1}{(z+1)^2}$.

   (c) $F(z) = \dfrac{z^2}{z^3-1}$.

2. Check formulas 10 and 11 of Table 8.1.1 using Theorem 8.2.1.

3. Explain what is wrong with the following reasoning. Let $g(t) = 0$ on $[0,1[$ and be 1 on $[1,\infty)$. Then, by formulas 6 and 13 of Table 8.1.1, $\tilde{g}(z) = e^{-z}/z$. By the complex inversion formula, $g(t) = \text{Res}(e^{z(t-1)}/z; 0) = 1$. Therefore, $1 = 0$.

4.* Prove a Heaviside expansion formula for P/Q when Q has double zeros.

5. Compute the inverse Laplace transform of each of the following:

   (a) $\dfrac{z}{(z+1)(z+2)}$

   (b) $\sinh z$

Applying this again gives

$$\left(\frac{d^2y}{dt^2}\right)^{\bar{}}(z) = z^2\bar{y}(z) - zy(0) - y'(0) = z^2\bar{y}(z) - 1.$$

Therefore, our equation becomes $z^2\bar{y}(z) - 1 + 4z\bar{y}(z) + 3\bar{y}(z) = 0$, so

$$\bar{y}(z) = \frac{1}{z^2 + 4z + 3} = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\frac{1}{z+1} - \frac{1}{2}\frac{1}{z+3}.$$

By the inversion formula, the inverse Laplace transform of this function is

$$y(t) = \sum \left\{\text{residues of } \frac{e^{zt}}{(z+1)(z+3)} \text{ at } -1, -3\right\}.$$

Thus,

$$y(t) = \frac{e^{-t} - e^{-3t}}{2}.$$

(We could also apply line 9 of Table 8.1.1 to the partial fraction expansion.) This is the desired solution, as can be checked directly by substitution into the differential equation. ♦

Example 8.3.2 *Solve the equation $y'(t) - y(t) = H(t-1), t \geq 0, y(0) = 0$, where H is the Heaviside function.*

Solution Take the Laplace transforms of both sides of the equation. We get

$$z\bar{y}(z) - y(0) - \bar{y}(z) = e^{-z}/z.$$

Therefore, $\bar{y}(z) = e^{-z}/z(z-1)$. The inverse Laplace transform of $1/[z(z-1)]$ is $1 - e^{-t}$, so that of $e^{-z}/[z(z-1)]$ is, by formula 6 of Table 8.1.1,

$$y(t) = \begin{cases} 0 & 0 \leq t < 1 \\ -1 + e^{t-1} & t \geq 1 \end{cases}.$$

Note that the complex inversion formula does not apply as stated. This solution (see Figure 8.3.1) is not differentiable and thus cannot be considered a solution in the strict sense. However, it is a solution in a generalized sense, as previously explained. In Figure 8.3.1, the discontinuity in $H(t-1)$ causes the sudden jump in $y'(t)$. We say that $y(t)$ receives an "impulse" at $t = 1$. ♦

Example 8.3.3 *Find a particular solution of $y''(t) + 2y'(t) + 2y(t) = f(t)$.*

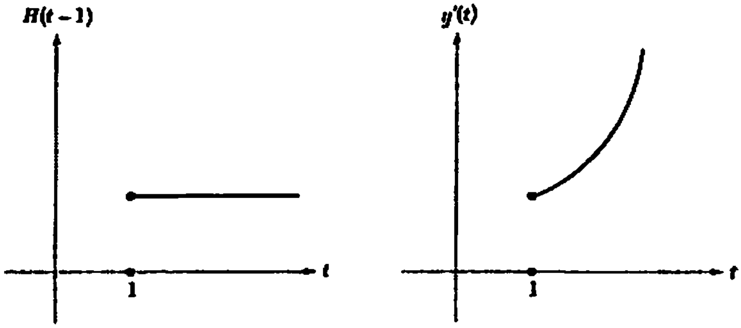


Figure 8.3.1: At $t = 1$, y receives an impulse.

Solution Let us find the solution with $y(0) = 0, y'(0) = 0$. Taking Laplace transforms,

$$z^2\tilde{y}(z) + 2z\tilde{y}(z) + 2\tilde{y}(z) = \tilde{f}(z),$$

so $\tilde{y}(z) = \tilde{f}(z)/(z^2 + 2z + 2)$. The inverse Laplace transform of $1/(z^2 + 2z + 2)$ is

$$g(t) = \frac{e^{z_1 t}}{2(z_1 + 1)} + \frac{e^{z_2 t}}{2(z_2 + 1)},$$

where z_1, z_2 are the two roots of $z^2 + 2z + 2$, namely, $-1 \pm i$. Simplifying, $g(t) = e^{-t}\sin t$. Thus, by formula 8 of Table 8.1.1,

$$y(t) = (g * f)(t) = \int_0^\infty f(t-\tau)g(\tau)d\tau = \int_0^\infty f(t-\tau)e^{-\tau}\sin\tau\, d\tau.$$

This is the particular solution we sought. ♦

Generally such particular solutions to differential equations of the form

$$a_n y^{(n)} + \ldots + a_1 y = f,$$

where $a_1, \ldots, a_n$ are constants, may be expressed in the form of a convolution. To obtain a solution with the values $y(0), y'(0), \ldots, y^{(n-1)}(0)$ prescribed, we can add a particular solution y_p satisfying

$$y_p(0) = 0, y_p'(0) = 0, \ldots, y_p^{(n-1)}(0) = 0$$

to a solution y_c of the homogeneous equation in which f is set equal to zero and with $y_c(0), y_c'(0), \ldots, y_c^{(n-1)}(0)$ prescribed. The sum $y_p + y_c$ is the solution sought. (These statements are easily checked.)

The method of Laplace transforms is a systematic method for handling constant coefficient differential equations. (Of course, these equations can be handled by other means as well.) If the coefficients are not constant, the method fails, because transformation of a product then involves a convolution, and then solving for $\tilde{y}(z)$ becomes difficult.

Exercises

Solve the differential equations in Exercises 1 through 6 using Laplace transforms.

1. $y'' - 4y = 0, y(0) = 2, y'(0) = 1$

2. $y'' + 6y - 7 = 0, y(0) = 1, y'(0) = 0$

3.• $y'' + 9y = H(t-1), y(0) = y'(0) = 0$

4. $y' + y = e^t, y(0) = 0$

5. $y' + y + \int_0^t y(\tau)d\tau = f(t)$ where $y(0) = 1$ and where $f(t) = 0$ for $0 \leq t < 1$ or $t \geq 2$ and $f(t) = 1$ if $1 \leq t < 2$.

6. $y'' + 9y = H(t), y(0) = y'(0) = 0$.

7. Solve the following systems of equations for $y_1(t), y_2(t)$ by using Laplace transforms.

   (a)
$$\begin{cases} y_1' + y_2 = 0 \\ y_2' + y_1 = 0 \end{cases} \quad \text{where } y_1(0) = 1, y_2(0) = 0.$$

   (b)
$$\begin{cases} y_1' + y_2' + y_1 = 0 \\ y_2' + y_1 = 3 \end{cases} \quad \text{where } y_1(0) = 0, y_2(0) = 0.$$

8. Solve: $y' + y = \cos t, y(0) = 1$.

9.• Solve: $y'' + y = t\sin t, y(0) = 0, y'(0) = 1$.

10. Study the solution of $y'' + \omega_0^2 y = \sin\omega t, y(0) = y'(0) = 0$, and examine the behavior of solutions for various ω, especially those near $\omega = \omega_0$. Interpret these solutions in terms of forced oscillations.

Review Exercises for Chapter 8

1. Compute the Laplace transform and the abscissa of convergence for $f(t) = H(t-1)\sin(t-1)$.

2.• Compute the Laplace transform and the abscissa of convergence for $f(t) = H(t-1) + 3e^{-(t+6)}$.

3. Compute the Laplace transform and the abscissa of convergence for
$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 1 & t > 1 \end{cases}.$$

4. Let $f(t)$ be a bounded function of t. Show that $\sigma(f) \leq 0$.

5. Compute the Laplace transform and the abscissa of convergence for
$$f(t) = \frac{e^t - 1}{t}.$$

6. If $f(t) = 0$ for $t < 0$, then $\hat{f}(y) = (1/\sqrt{2\pi})\tilde{f}(iy)$ is called the ***Fourier transform*** of f. Using Corollary 8.2.2, show that, under suitable conditions,
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(y) e^{ixy} \, dy.$$
(This result is called the ***Inversion Theorem for Fourier transforms.***)

7. Compute the inverse Laplace transform and the abscissa of convergence for
$$F(z) = \frac{e^{-z}}{z^2 + 1}.$$

8.• Compute the inverse Laplace transform and the abscissa of convergence for
$$F(z) = \frac{1}{(z+1)^2}.$$

9. Compute the inverse Laplace transform and the abscissa of convergence for
$$F(z) = \frac{z}{(z+1)^2} + \frac{e^{-z}}{z}.$$

10. (a) Let $\tilde{f}(z)$ be the Laplace transform of $f(t)$. Show that $\tilde{f}(z) \to 0$ as $\operatorname{Re} z \to \infty$.
    (b) Use (a) to show that, under suitable conditions, $z\tilde{f}(z) \to f(0)$ as $\operatorname{Re} z \to \infty$.
    (c) Can a nonzero polynomial be the Laplace transform of any $f(t)$?
    (d) Can a nonzero entire function F be the Laplace transform of a function $f(t)$?

11. Solve the following differential equations using Laplace transforms:
    (a) $y'' + 8y + 15 = 0, y(0) = 1, y'(0) = 0$
    (b) $y' + y = 3, y(0) = 0$

12.• Suppose that $f(t) \geq 0$ and is infinitely differentiable. Prove that $(-1)^k \tilde{f}^{(k)}(z) \geq 0, k = 0, 1, 2, \ldots,$ for $z \geq 0$. (The converse, called ***Bernstein's Theorem***, is also true but is more difficult to prove.)

13. Solve the following differential equations using Laplace transforms:
    (a) $y'' + y = H(t-1), y(0) = 0, y'(0) = 0$
    (b) $y'' + 2y' + y = 0, y(0) = 1, y'(0) = 1$

Answers to Odd-Numbered Exercises

1.1 Introduction to Complex Numbers

1. (a) $6+4i$ (b) $\frac{11}{17}+i\frac{10}{17}$ (c) $\frac{3}{2}-\frac{5}{2}i$ 3. $z=\pm(2-i)$

5. $\operatorname{Re}\left[\frac{1}{z^2}\right]=\frac{x^2-y^2}{(x^2+y^2)^2}$ $\operatorname{Im}\left[\frac{1}{z^2}\right]=-\frac{2xy}{(x^2+y^2)^2}$

$\operatorname{Re}\left[\frac{1}{3z+2}\right]=\frac{3x+2}{(3x+2)^2+9y^2}$ $\operatorname{Im}\left[\frac{1}{3z+2}\right]=\frac{-3y}{(3x+2)^2+9y^2}$

7. No; let $z=w=i$. 9. If $z=x+iy$, then $\operatorname{Re}(iz)=\operatorname{Re}(ix-y)=-y=-\operatorname{Im}(z)$, and $\operatorname{Im}(iz)=\operatorname{Im}(ix-y)=x=\operatorname{Re}(z)$.

11. The proof of the associative law for multiplication was outlined in the text. Here is how to show that addition is commutative: if $z=x+iy$ and $w=u+iv$, where x,y,u, and v are real, then z and w correspond to (x,y) and (u,v) respectively, and thus

$$\begin{aligned} z+w &= (x,y)+(u,v)=(x+u,y+v)\\ &= (u+x,v+y)=(u,v)+(x,y)=w+z. \end{aligned}$$

13. Show that $a=(x^2-y^2)/(x^2+y^2)$ and $b=-2xy/(x^2+y^2)$ and then $a^2+b^2=1$.

15. A complex number z can be written as $z=x+iy$ with x and y real in only one way, corresponding to the vector (x,y). The real numbers were to correspond to vectors of the form $(x,0)$, so $y=0$ and therefore $z=x=\operatorname{Re} z$.

17. (a) -4 (b) i

19. (a) $\sqrt{1+\sqrt{i}}=\pm\left(\frac{\sqrt{1+\sqrt{2}+\sqrt{4+2\sqrt{2}}}}{2^{3/4}}+i\frac{\sqrt{-1-\sqrt{2}+\sqrt{4+2\sqrt{2}}}}{2^{3/4}}\right)$

or $\sqrt{1+\sqrt{i}}=\pm\left(\frac{\sqrt{\sqrt{2}-1+\sqrt{4-2\sqrt{2}}}}{2^{3/4}}-i\frac{\sqrt{1-\sqrt{2}+\sqrt{4-2\sqrt{2}}}}{2^{3/4}}\right)$

(b) $\sqrt{1+i}=\pm\left(\sqrt{\frac{1+\sqrt{2}}{2}}+i\sqrt{\frac{\sqrt{2}-1}{2}}\right)$ (c) See Worked Example 1.1.6.

1.2 Properties of Complex Numbers

1. (a) $z=\sqrt[5]{2}\left(\cos\frac{2\pi k}{5}+i\sin\frac{2\pi k}{5}\right),\quad k=0,1,2,3,4$

(b) $z=\cos\left(\frac{3\pi}{8}+\frac{2\pi k}{4}\right)+i\sin\left(\frac{3\pi}{8}+\frac{2\pi k}{4}\right),\quad k=0,1,2,3$

3. $(3-8i)^4/(1-i)^{10}$
5. $\cos 5x = \cos^5 x - 10\cos^3 x \cdot \sin^2 x + 5\cos x \cdot \sin^4 x$
   $\sin 5x = \sin^5 x - 10\cos^2 x \cdot \sin^3 x + 5\cos^4 x \cdot \sin x$
7. $\sqrt{377/5}$ 9. Use the identity $(1-w)(1+w+w^2+\ldots+w^{n-1}) = 1-w^n$.
11. We use properties of the complex conjugate as follows:

$$\begin{aligned}|a-b|^2+|a+b|^2 &= (a-b)(\overline{a-b})+(a+b)(\overline{a+b})\\ &= (a-b)(\bar{a}-\bar{b})+(a+b)(\bar{a}+\bar{b})\\ &= |a|^2-a\bar{b}-b\bar{a}+|b|^2+|a|^2+a\bar{b}+b\bar{a}+|b|^2\\ &= 2(|a|^2+|b|^2)\end{aligned}$$

13. All the points must have the same argument and so lie on the same ray from the origin.
15. No; take $z=i$. In fact, $z^2=|z|^2$ if and only if z is real.
17. Each side is a positive real number whose square is $a^2(a')^2+a^2(b')^2+(a')^2b^2+b^2(b')^2$.
19. $|z-(8+5i)|=3$ 21. The real axis. 23. $1+|a|$
25. Using deMoivre's formula and $1+w+\ldots+w^n=(1-w^{n+1})/(1-w)$,

$$\begin{aligned}\sum_{k=0}^{n}\cos k\theta &= \mathrm{Re}\left[\sum_{k=0}^{n}(\cos\theta+i\sin\theta)^k\right]\\ &= \mathrm{Re}\,\frac{1-(\cos\theta+i\sin\theta)^{n+1}}{1-\cos\theta-i\sin\theta} = \mathrm{Re}\,\frac{1-\cos(n+1)\theta-i\sin(n+1)\theta}{1-\cos\theta-i\sin\theta}\\ &= \frac{1-\cos\theta-\cos(n+1)\theta+\cos(n+1)\theta\cos\theta+\sin(n+1)\theta\sin\theta}{2-2\cos\theta}\\ &= \frac{1}{2}+\frac{\cos n\theta-\cos(n+1)\theta}{2(1-\cos\theta)}\\ &= \frac{1}{2}+\frac{2\sin(\theta/2)\sin(n+\frac{1}{2})\theta}{2(1-\cos\theta)} = \frac{1}{2}+\frac{\sin(n+\frac{1}{2})\theta}{2\sin(\theta/2)}.\end{aligned}$$

27. (a) $(z_2-z_1)/(z_3-z_1)$ is real. (b) $\left(\dfrac{z_4-z_1}{z_4-z_2}\right)\cdot\left(\dfrac{z_3-z_2}{z_3-z_1}\right)$ is real.
29. Multiply by $1-w$ and use Exercise 9 to show that the sum is $-n/(1-w)$.

1.3 Some Elementary Functions

1. (a) $e^2(\cos 1+i\sin 1)$ (b) $\frac{1}{2}(\sin 1)\left(\frac{1}{e}+e\right)+i\frac{1}{2}(\cos 1)\cdot\left(e-\frac{1}{e}\right)$
3. (a) $z=\pm\left(\frac{\pi}{4}+2\pi n-i\frac{1}{2}\log 2\right)$
   (b) $z=\pm[2\pi n-i\log(4+\sqrt{15})]$, noting that $\log(4-\sqrt{15})=-\log(4+\sqrt{15})$.
5. (a) $\log 1 = 2\pi n i$ (b) $\log i = \pi i/2+2\pi n i$
7. (a) $e^{\pi/2}e^{-2\pi n}=e^{-2\pi(n-1/4)}$
   (b) $e^{(1/2)\log 2-2\pi n-\pi/4}\left[\cos\left(\frac{1}{2}\log 2+\frac{\pi}{4}\right)+i\sin\left(\frac{1}{2}\log 2+\frac{\pi}{4}\right)\right]$
9. $z=n\pi$ for any integer n
11. Since $|e^z|=e^{\mathrm{Re}\,z}$, $|e^z|$ goes to 0 along rays pointing into the left half plane. It is 1 along the imaginary axis, and it goes to $+\infty$ along rays into the right half plane.
13. (a) $e^{x^2-y^2}(\cos 2xy+i\sin 2xy)$ (b) $e^{-y}(\cos x+i\sin x)$
    (c) $e^{x/(x^2+y^2)}\left(\cos\frac{y}{x^2+y^2}-i\sin\frac{y}{x^2+y^2}\right)$

15. Using the definition of the sine and cosine of a complex variable gives

$$\sin(\pi/2 - z) = \frac{e^{i(\pi/2-z)} - e^{-i(\pi/2-z)}}{2i} = e^{\pi i/2}\frac{e^{-iz} - e^{iz}e^{-\pi i}}{2i}$$
$$= \frac{e^{-iz} + e^{iz}}{2} = \cos z.$$

The other two assertions follow in a similar way.

17. From Example 1.3.17, $\sin z = \sin x \cosh y + i \cos x \sinh y$, so

$$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$
$$= \sin^2(x)(1 + \sinh^2 y) + \cos^2 x \sinh^2 y$$
$$= \sin^2 x + \sinh^2 y \geq \sinh^2 y,$$

so $|\sin z| \geq |\sinh y|$. The other inequality follows in a similar manner.

19. No, not even for real a and b. Let $a = 2, b = -1$. Then $|a^b| = |2^{-1}| = \frac{1}{2}$, but $|a|^{|b|} = 2$.

21. If $|z| = 1$, then $z = e^{i\theta}$ for some θ, so $z + 1/z = e^{i\theta} + e^{-i\theta} = 2\cos\theta$. As θ varies from 0 to 2π, this covers the interval $[-2, 2]$ twice.

23. Since $|1/z| = 1/|z|$, the map interchanges the inside and outside of the unit circle. Circles of radius r are mapped to circles of radius $1/r$. If $z = re^{i\theta}$, then $1/z = (1/r)e^{-i\theta}$, so the ray defined by $\arg z = \theta$ is mapped to the ray with argument $-\theta$.

25. This holds iff $b \log a$ has its imaginary part in $[-\pi, \pi[$. Otherwise, the formula reads $\log a^b = b \log a + 2\pi i k$.

27. These are the nth roots of 1, since $(w^k)^n = [(e^{2\pi i/n})^k]^n = e^{2\pi k i} = 1$. They are all different, since $w^j = w^k$ implies $e^{2\pi i(k-j)/n} = 1$. By Proposition 1.3.2(vii), this forces $(k - j)/n$ to be an integer.

29. $\sin z = 0$ if and only if $e^{iz} = e^{-iz}$ or $e^{2iz} = 1$. By Proposition 1.3.2(vii), this happens exactly when $2iz = 2\pi n i$, or $z = n\pi$.

31. The maximum is $\cosh(2\pi) \approx 268$ attained at $z = 2\pi i, \pi + 2\pi i$, and $2\pi + 2\pi i$.

33. (a) $\approx 24 - i4.5$ (b) $\approx 1.17 - i(1.19) + 2\pi n i$ (c) $\approx 96.16 - i1644.43$

35. No, $\sin z$ is not one-to-one on $0 \leq \operatorname{Re} z < 2\pi$. For example, $\sin(0) = \sin(\pi) = 0$. If $\sin z = \sin w$, then

$$0 = \sin z - \sin w = 2 \sin\frac{z - w}{2} \cos\frac{z + w}{2}$$

and by Exercise 29 and a similar result for cosine, either $z - w = 2k\pi$ (for $k = 0, \pm 1, \pm 2, \dots$) or $z + w = n\pi$ (for $n = \pm 1, \pm 3, \pm 5, \dots$). Using Exercise 34 and this result, for each $z_0 \in \mathbb{C}$ there is *precisely one* w with $-\pi/2 \leq \operatorname{Re} w \leq \pi/2$ such that $\sin w = z_0$, provided, for example, that the portion of the boundary of this strip lying below the real axis is omitted. Taking this value of w defines a branch of $\sin^{-1} z_0$. The others are given by the above formulas for $z - w$ and $z + w$. The discussion for $\cos^{-1}$ is analogous.

1.4 Continuous Functions

1. Since $|w|^2 = (\operatorname{Re} w)^2 + (\operatorname{Im} w)^2$, all three assertions follow from the observation that if $a \geq 0$ and $b \geq 0$, then $a \leq \sqrt{a^2 + b^2} \leq a + b$.

3. Since f is continuous, there is a $\delta > 0$ such that $|z - z_0| < \delta$ implies that $|f(z) - f(z_0)| < |f(z_0)|/2$. Thus $f(z) \neq 0$, for if $f(z)$ were equal to 0, then $|f(z_0)|$ would be less than $|f(z_0)|/2$, which is absurd.

5. Let $\{a_1, a_2, \dots, a_n\}$ be a finite set of points and let z_0 be in its complement. Let $\delta_k = |z_0 - a_k|$ and let $\delta = \min\{\delta_1/2, \dots, \delta_n/2\}$. Then no a_k can lie in $D(z_0, \delta)$, since $\delta < |z_0 - a_k|$.

7. Let $\epsilon > 0$ and $\delta = \epsilon$. If $|z - z_0| < \delta$, then $|f(z) - f(z_0)| = |\bar{z} - \bar{z}_0| = |\overline{(z - z_0)}| = |z - z_0| < \epsilon$. Thus for any $z_0 \in \mathbb{C}$, $\lim_{z \to z_0} f(z) = f(z_0)$.

9. $\mathbb{C}\backslash\{2\pi ni \mid n \text{ is an integer}\}$ 11. $|z| < 1$

13. (a) Open, not closed (b) Neither open nor closed (c) Not open, closed

15. (a) Connected and compact (b) Compact, not connected
(c) Connected, not compact (d) Neither compact nor connected

17. $z \in \mathbb{C}\backslash f^{-1}(A)$ iff $z \notin f^{-1}(A)$ iff $f(z) \notin A$ iff $f(z) \in \mathbb{C}\backslash A$ iff $z \in f^{-1}(\mathbb{C}\backslash A)$

19. If the U_α's are open sets and $z \in \cup_\alpha U_\alpha$, then $z \in U_{\alpha_0}$ for some α_0. Since U_{α_0} is open, there is an $\epsilon > 0$ such that $D(z;\epsilon) \subset U_{\alpha_0} \subset \cup_\alpha U_\alpha$. This shows the union is open, since this argument holds for any such z.

21. $\cap_{n=1}^{\infty} D(0; 1/n) = \{0\}$, and this set is not open.

23. Let $R > 0$. We need to show that there is an N such that $|z^n/n| \geq R$ whenever $n \geq N$. A little algebra shows this is equivalent to $(nR)^{1/n} \leq |z|$. L'Hôpital's rule shows that $\lim_{n\to\infty}(nR)^{1/n} = 1$. (Take logarithms first and use L'Hôpital's rule to show that $\log(nR)^{1/n} \to 0$.) Since $|z| > 1$, we have the inequality we need for large enough n.

1.5 Basic Properties of Analytic Functions

1. (a) Analytic on all of $\mathbb{C}$. The derivative is $3(z+1)^2$.
(b) Analytic on $\mathbb{C}\backslash\{0\}$. The derivative is $1 - 1/z^2$.
(c) Analytic on $\mathbb{C}\backslash\{1\}$. The derivative is $-10[1/(z-1)]^{11}$.
(d) Analytic on $\mathbb{C}\backslash\{e^{2\pi i/3}, e^{4\pi i/3}, 1, i\sqrt{2}, -i\sqrt{2}\}$. The derivative is

$$-[1/(z^3-1)^2(z^2+2)^2]\cdot[(z^3-1)2z + 3z^2(z^2+2)].$$

3. (a) If $n \geq 0$, it is analytic everywhere. If $n < 0$, it is analytic everywhere except at 0. The derivative is nz^{n-1}. (b) Analytic on $\mathbb{C}\backslash\{0, i - i\}$. The derivative is

$$-2\frac{1}{(z+1/z)^3}\left(1 - \frac{1}{z^2}\right).$$

(c) Analytic except at the nth roots of 2, $\sqrt[n]{2}e^{2\pi ik/n}$. The derivative is

$$\frac{(1-n)z^n - 2}{(z^n-2)^2}.$$

5. (a) Locally, f rotates by $\theta = 0$ and multiplies lengths by 1.
(b) Locally, f rotates by an angle $\theta = 0$ and stretches lengths by a factor 3.
(c) Locally, f rotates by an angle π and stretches lengths by a factor 2.

7. $(f^{-1}\circ f)'(z) = (f^{-1})'(f(z))f'(z)$. But $(f^{-1}\circ f)(z) = z$, so $(f^{-1}\circ f)'(z) = 1$. Hence $(f^{-1})'(f(z))\cdot f'(z) = 1$.

9. $f(z) = z^2 + 3z + 2 = (x^2 - y^2 + 3x + 2) + i(2xy + 3y)$, so $\partial u/\partial x = 2x + 3 = \partial v/\partial y$ and $\partial u/\partial y = -2y = -\partial v/\partial x$.

11. Since $x = r\cos\theta$ and $y = r\sin\theta$, the chain rule gives

$$\frac{\partial u}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \quad\text{and}\quad \frac{\partial u}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}.$$

Solving for $\partial u/\partial x$ and $\partial u/\partial y$ gives

$$\frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \quad\text{and}\quad \frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}.$$

Similarly,

$$\frac{\partial v}{\partial x} = \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \quad\text{and}\quad \frac{\partial v}{\partial y} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta},$$

so the Cauchy-Riemann equations become

$$\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta}$$

and

$$\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} = -\cos\theta\frac{\partial v}{\partial r} + \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}.$$

Multiplying the first by $\cos\theta$ and the second by $\sin\theta$ and adding gives $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$. Similarly, $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

13. The definitions give

$$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right]$$
$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

Thus, the Cauchy-Riemann equations, $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$, are equivalent to saying that the complex quantity $\partial f/\partial \bar z$ is zero.

15. If $f = u + iv$, then $\partial u/\partial x = 0 = \partial u/\partial y$ since u is constant. By the Cauchy-Riemann equations, $\partial v/\partial y = \partial u/\partial x = 0$ and $\partial v/\partial x = -\partial u/\partial y = 0$. Thus, $f'(z) = \partial u/\partial x + i(\partial v/\partial x) = 0$ everywhere on A. Since A is connected, f is constant.

17. By the Cauchy-Riemann equations, $\partial u/\partial x = \partial v/\partial y$. Hence $2\partial v/\partial y = 0$, so $\partial u/\partial x$ and $\partial v/\partial y$ are identically 0 on A. Thus, u depends only on y and v depends only on x. But then $\partial u/\partial y$ can depend only on y and $\partial v/\partial x$ only on x. Since $\partial u/\partial y = -\partial v/\partial x$ for all x and y, $\partial u/\partial y$ and $-\partial v/\partial x$ equal the same real constant c. Thus $u = cy + d_1$ and $v = -cx + d_2$. Therefore, $f = u + iv = -ic(x + iy) + (d_1 + id_2)$.

19. (a) $\mathbb{C}\backslash\{1\}$ (b) Yes (c) x axis $\backslash\{1\}$, unit circle $\backslash\{1\}$ (d) $90°$

21. $\mathbb{C}\backslash\{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$

23. (a) u is the imaginary part of the analytic function $f(z) = z^2 + 3z + 1$, so u is harmonic on $\mathbb{C}$.
(b) Either check the second derivatives directly in Laplace's equation or notice that u is the real part of $f(z) = 1/(z-1)$ to see that u is harmonic on $\mathbb{C}\backslash\{1\}$.

25. Locally in B, $w = \mathrm{Re}\, g$, where g is analytic. Then $w \circ f = \mathrm{Re}(g \circ f)$. But $g \circ f$ is analytic, so $w \circ f$ is harmonic.

27. (a) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$ for all (x, y) (b) We need $\partial v/\partial y = \partial u/\partial x = e^x \cos y$; thus, $v(x,y) = e^x \sin y + g(x)$. Then $e^x \sin y + g'(x) = \partial v/\partial x = -\partial u/\partial y = e^x \sin y$. Thus $g'(x) = 0$, so g is constant. To obtain $v(0,0) = 0$, take $v(x,y) = e^x \sin y$.
(c) $e^z = e^x \cos y + ie^x \sin y$. By parts (a) and (b), the real and imaginary parts satisfy the conditions of the Cauchy-Riemann Theorem 1.5.8, so f is analytic.

29. (a) No. Counterexample: $u(x,y) = x^2 - y^2$ and $v(x,y) = x$ are harmonic, but $u(v(x,y), 0) = x^2$ is not harmonic. (b) No. Counterexample: $u(z) = v(z) = x$. Then $u(z) \cdot v(z) = x^2$ is not harmonic. (c) Yes

31. Write

$$U = \frac{\partial u}{\partial x} \quad \text{and} \quad V = -\frac{\partial u}{\partial y},$$

so $f = U + iV$. By assumption, U and V have continuous partial derivatives. By the assumption of continuous second partials for u and v, we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \text{that is,} \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x},$$

one of the Cauchy-Riemann equations for f. The other equation comes from

$$\frac{\partial U}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial V}{\partial y};$$

f is thus analytic by the Cauchy-Riemann Theorem.

1.6 Differentiation of the Elementary Functions

1. (a) Analytic for all z; the derivative is $2z+1$.
   (b) Analytic on $\mathbb{C}\backslash\{0\}$; the derivative is $-1/z^2$.
   (c) Analytic on $\mathbb{C}\backslash\{z=(2k+1)\pi/2 \mid k=0,\pm1,\pm2,\pm3,\ldots\}$; the derivative is $1/\cos^2 z$.
   (d) Analytic on $\mathbb{C}\backslash\{1\}$. The derivative is

$$\frac{2z^3-3z^2-1}{(z-1)^2}\exp\left(\frac{z^3+1}{z-1}\right).$$

3. (a) 1. (b) The limit does not exist. 5. No. 7. Yes.
9. (a) Analytic on $\mathbb{C}\backslash\{\pm1\}$. The derivative is $-(z^2+1)/(z^2-1)^2$.
   (b) Analytic on $\mathbb{C}\backslash\{0\}$. The derivative is $(1-1/z^2)e^{z+1/z}$.
11. The minimum is $1/e$, at $z=\pm i$.
13. The map $z\mapsto 2^{z^2}$ is a composition of entire functions and so is entire. The function $z^{2z}=e^{2z\log z}$ is analytic on the region of analyticity of the logarithm chosen.

Review Exercises for Chapter 1

1. (a) $e^i=\cos(1)+i\sin(1)$; (*b*) $\log(1+i)=\frac{1}{2}\log 2+\frac{\pi i}{4}+2\pi ni$, n an integer
   (c) $\sin i = i\frac{1}{2}\left(e-\frac{1}{e}\right)$; (*d*) $\exp[2\log(-1)]=1$
3. $e^{\pi i/16}1, e^{\pi i/16}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right), e^{\pi i/16}i, e^{\pi i/16}\left(\frac{-1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$,
   $e^{\pi i/16}(-1), e^{\pi i/16}\left(\frac{-1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right), e^{\pi i/16}(-i), e^{\pi i/16}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)$
5. $z=2\pi n\pm i\log(\sqrt{3}+\sqrt{2})$
7. (a) The real axis (b) A circle, centered at $(\frac{17}{8},0)$ of radius $\frac{3}{8}$.
9. (a) $(z^3+8)'=3z^2$ on all of $\mathbb{C}$. (b) $\left(\frac{1}{z^3+1}\right)'=\frac{-3z^2}{(z^3+1)^2}$ on $\mathbb{C}\backslash\{-1,e^{\pi i/3},e^{-\pi i/3}\}$.
   (c) $[\exp(z^4-1)]'=4z^3\exp(z^4-1)$ on all of $\mathbb{C}$. (d) $[\sin(\log z^2)]'=\frac{2}{z}\cos(\log z^2)$, on all $\mathbb{C}$ except the entire imaginary axis.
11. (a) Analytic on $\mathbb{C}\backslash\{0\}$; $(e^{1/z})'=-e^{1/z}/z^2$.
    (b) Analytic on $\mathbb{C}\backslash\{z=(\pi/2)+2\pi n \mid n=0,\pm1,\pm2,\pm3,\ldots\}$;
    $(1/(1-\sin z)^2)'=2(\cos z)/(1-\sin z)^3$.
    (c) $\mathbb{C}\backslash\{\pm ai\}$; $[e^{az}/(a^2+z^2)]'=[(a^2+z^2)ae^{az}-2ze^{az}]/(a^2+z^2)^2$.
13. (a) No. (b) Yes. (c) Yes. 15. If and only if f is constant. 17. $x=y=-1$
19. Note that $v(x,y)=0$ on A. Therefore, $\partial v/\partial x=\partial v/\partial y=0$, and the Cauchy-Riemann equations give $\partial u/\partial x=\partial u/\partial y=0$. Thus, f' is identically 0 on the connected set A. Hence, by Proposition 1.5.5, f is constant on A.
21. By hypothesis, $(d/dz)(f(z)-\log z)=0$. Now use Proposition 1.5.5.
23. $\lim\limits_{h\to0}\frac{(z_0-h)^n-z_0^n}{h}=f'(z_0)$ where $f(z)=z^n$.

25. Fix a branch of log, for example, the principal branch, which is analytic on $\mathbb{C}\setminus\{$nonpositive real axis$\}$. Write $z^z = e^{z\log z}$, which is analytic on the region of analyticity of logarithm chosen. The derivative is $z^z(1+\log z)$.

27. $z = 2e^{i\pi/2}, 2e^{7\pi i/6}, 2e^{11\pi i/6}$. 29. They are the real and imaginary parts of z^3.

31. (a) 0. (b) Differentiable at all points $z \in \mathbb{C}$.

33. (a) $\dfrac{\partial^2 u}{\partial x^2} = -6y, \dfrac{\partial^2 u}{\partial y^2} = 6y$, so $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. (b) $v(x,y) = x^3 - 3xy^2$.

35. $z = i, -1, -i, 1$.

37. $f(z) = f(2z) = f(4z) = \ldots = f(2^n z)$ for any z and positive integer n. Letting $w = 2^n z$, we get $f(w/2^n) = f(w)$ for all n. Letting $n \to \infty$ and using the continuity of f at 0 gives $f(0) = f(w)$. Since this can be done for any w in $\mathbb{C}$, f is constant.

2.1 Contour Integrals

1. (a) $2 + (i/2)$ (b) $\frac{1}{2}[\cos(2+2i) - \cos(2i)]$ (c) 0

3. The principal branch of the logarithm is a function that is analytic on an open set containing γ and whose derivative is $1/z$. Since γ is closed, the value of the integral is 0, by the Fundamental Theorem of Calculus for contour integrals, 2.1.7.

5. No; for example, let $f(z) = z, \gamma(t) = it$ for $t \in [0,1]$. Then

$$\int_\gamma \operatorname{Re} f = \int_0^1 0 \cdot i\,dt = 0.$$

However,

$$\operatorname{Re}\int_\gamma f = \operatorname{Re}\int_0^1 iti\,dt = \operatorname{Re}\left(-\frac{t^2}{2}\bigg|_0^1\right) = -\frac{1}{2}.$$

7. (a) $2\pi i$ (b) $-i/3$ 9. For $|z| = 1$ we have

$$\left|\frac{1}{2+z^2}\right| = \frac{1}{|2+z^2|} \le \frac{1}{2-|z|^2} = 1,$$

since $|z_1 + z_2| \ge |z_1| - |z_2|$. Hence

$$\left|\int_\gamma \frac{dz}{2+z^2}\right| \le 1 \cdot l(\gamma) = \pi.$$

11. (a) $\displaystyle\int_{|z|=1} \frac{dz}{z} = 2\pi i$; $\displaystyle\int_{|z|=1} \frac{dz}{|z|} = 0$; $\displaystyle\int_{|z|=1} \frac{|dz|}{z} = 0$; $\displaystyle\int_{|z|=1} \left|\frac{dz}{z}\right| = 2\pi$ (b) $-\dfrac{i}{3}$

13. 0 15. If $z = e^{i\theta}$ is on γ, then

$$\left|\frac{\sin z}{z^2}\right| = |\sin z| \le \frac{|e^{iz}| + |e^{-iz}|}{2} = \frac{e^{-\sin\theta} + e^{\sin\theta}}{2} \le e.$$

Since γ has length 2π, the estimate follows from Proposition 2.1.6.

2.2 Cauchy's Theorem—A First Look

1. (a) -6 (b) 0 (c) 0 (d) 0 0, by Cauchy's Theorem applied to $z = z_0 + re^{i\theta}$.

5. The integral is zero iff γ encircles neither or both roots $-(1/2) \pm (\sqrt{3}/2)i$ of $z^2 + z + 1 = 0$.

7. No; let $f(z) = z, \gamma(t) = e^{it}, t \in [0, 2\pi]$ (the unit circle). Then $\int_\gamma \operatorname{Re} f(z)dz = \pi i$, while $\int_\gamma \operatorname{Im} f(z)dz = -\pi$. 9. $-\dfrac{2}{3} - \dfrac{2}{3}i$ 11. $4\pi i$

2.3 A Closer Look at Cauchy's Theorem

1. If it were, then the circle $|z| = 1$ would be homotopic to a point in $\mathbb{C}\backslash\{0\}$ so that $\int_{|z|=1} dz/z = 0$. But $\int_{|z|=1} dz/z = 2\pi i$. This contradiction shows that $\mathbb{C}\backslash\{0\}$ is not simply connected.

3. Let $\gamma : [a,b] \to A$ be a closed curve in A. Define a homotopy $H : [a,b] \times [0,1] \to A$ by $H(t,s) = s\gamma(t) + (1-s)z_0$, which is clearly continuous and maps into A since A is starlike at z_0. Thus γ is homotopic by H to the constant map at z_0.

5. The set G contains the segment between each of its points and 0. Sample of part of a proof: Consider a point c for which $0 < \operatorname{Re} c < 1$ and $0 < \operatorname{Im} c < 3$. If $z = sc + (1-s)0 = sc$ with $0 \le s \le 1$ is on the segment between c and 0, then $\operatorname{Re} z = \operatorname{Re}(sc) = s\operatorname{Re} c$ and $\operatorname{Im} z = s\operatorname{Im} c$, so that $0 \le \operatorname{Re} z \le 1$ and $0 \le \operatorname{Im} z \le 3$. Thus $z \in G$. Points in other parts of G are handled similarly.

7. (a) $2\pi i$ (b) 0 (c) 0 (d) πi 9. (a) $2\pi i$ (b) $2\pi i$

2.4 Cauchy's Integral Formula

1. (a) $2\pi i$ (b) $2\pi i$

3. The Cauchy inequalities (2.4.7) show that $f^{(k)}(z)$ is identically 0 for $k > n$. The conclusion follows from Exercise 20 of §1.5.

5. (a) 0 (b) $-\pi i/3$

7. Using the Cauchy inequalities, $|f'(0)| \le 1/R$ for every $R < 1$. Hence $|f'(0)| \le 1$. This is the best possible bound, as is clear from the example $f(z) = z$.

9. Let $\tilde{\gamma}$ be the circle $\tilde{\gamma}(t) = z_1 + re^{it}, 0 \le t \le 2\pi, |z_0 - z_1| < r$. The curve γ is homotopic to $\tilde{\gamma}$ by $H(t,s) = s(z_1 + re^{it}) + (1-s)(z_0 + re^{it})$. Since z_1 is not in the image of the homotopy,
$$I(\gamma; z_1) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_1} = \frac{1}{2\pi i}\int_{\tilde{\gamma}} \frac{1}{z - z_1} = 1.$$

11. By Proposition 1.5.3, f is analytic on $A\backslash\{z_0\}$, so it is continuous there. Since
$$f(z_0) = F'(z_0) = \lim_{z\to z_0} \frac{F(z) - F(z_0)}{z - z_0} = \lim_{z\to z_0} f(z),$$
f is also continuous at z_0. By Corollary 2.4.11 to Morera's Theorem, f is analytic on A.

13. (a) 0 (b) 0 (c) 0 (d) $\pi i/2$ 15. $4\pi i$

17. $1/f$ is entire and $|1/f(z)| \le 1$ on $\mathbb{C}$, so $1/f$ (and hence f) is constant by Liouville's theorem.

19. (a) $\pi/2 + i(\pi/2)$ (b) 0

21. Start with Cauchy's Integral Formula 2.4.4 for $f(z_1), f(z_2)$ and $f'(z_0)$.

2.5 Maximum Modulus Theorem and Harmonic Functions

1. e 3. Let $A = \mathbb{C}\backslash\{0\}, f(z) = e^z$.

5. Note that $f - g$ is continuous on $\operatorname{cl}(A)$ and analytic on A. Also, $(f-g)(z) = 0$ for $z \in \operatorname{bd}(A)$, and so by the Maximum Modulus Theorem, $(f-g)(z) = 0$ for all $z \in A$. In other words, $f(z) = g(z)$ for all $z \in A$. Hence $f = g$ on all of $\operatorname{cl}(A) = A \cup \operatorname{bd}(A)$.

7. $|e^{z^2}|$ attains a maximum value of e at ± 1.

9. (a) $v(x,y) = -\cosh x \cos y$ on $\mathbb{C}$ (b) $v = \arctan(y/x)$ or $-\arctan(x/y)$ on $\mathbb{C}$ minus the negative real axis (c) $v(x,y) = e^x \sin y$ on $\mathbb{C}$
(Note that an arbitrary constant may be added to each.)

11. If $z = x + iy$, $\operatorname{Re} e^z = e^x \cos y$ and $\operatorname{Im} e^z = e^x \sin y$. The normal vector to the level curves of these functions is given by the gradient vectors

$$(e^x \cos y, -e^x \sin y) \quad \text{and} \quad (e^x \sin y, e^x \cos y).$$

Since these are orthogonal, the curves are also orthogonal.

13. Since f is analytic and nonconstant on A, z_0 cannot be a relative maximum. Thus, in every neighborhood of z_0, and, in particular, in the ϵ disk defined by $|z - z_0| < \epsilon$, there is a point z with $|f(z)| > |f(z_0)|$. If $f(z_0) \neq 0$, then by continuity, $f(z) \neq 0$ in some small disk D centered at z_0. Thus $1/f(z)$ is analytic on D. By the last argument, there is a ζ close to z_0 such that

$$\left|\frac{1}{f(\zeta)}\right| > \left|\frac{1}{f(z_0)}\right|.$$

Therefore, $|f(\zeta)| < |f(z_0)|$.

15. $|g(0)| = |0| = 0$, and so $g(0) = 0$. Also, $|g(z)| = |z| < 1$ for all z in the disk defined by $|z| < 1$. Thus, Schwarz's Lemma applies, and so $g(z) = cz$ with $|c| = 1$.

17. 0

Review Exercises for Chapter 2

1. (a) 0 (b) 0 (c) $2\pi i$ (d) $2\pi i \cos(1)$

3. For $r_1 > r_2 > 1$, γ_{r_1} is homotopic in $\{z \text{ such that } |z| > 1\}$ to γ_{r_2}, so the integrals are equal.

5. $-2/3$

7. If $z_1 \in A$, let γ be a path in A from z_0 to z_1. By the Distance Lemma 1.4.21, there is a $\delta > 0$ such that the set $B = \{z \mid \text{there is a point } w \text{ on } \gamma \text{ with } |z - w| < \delta\} \subset A$. The Maximum Modulus Theorem shows that f is constant on this bounded subregion and in particular $f(z_1) = f(z_0)$. Since z_1 was arbitrary, f is constant on A.

9. $v(x, y) = \dfrac{-y}{(x-1)^2 + y^2}$ on $\mathbb{C}\backslash\{1\}$ 11. Consider $\displaystyle\int_{|z|=1} \frac{e^z}{z^2} dz$ to obtain 2π.

13. i; $\log i = i(\pi/2) + 2\pi i n$; $\log(-i) = -i(\pi/2) + 2\pi i n$; $i^{\log(-i)} = e^{-\pi^2 k/2}$, where k is any odd integer.

15. By the Cauchy Integral Formulas, f' is analytic on A. Since f is nonzero in A, f'/f is analytic on A and the integral is 0 by the Cauchy Integral Theorem.

17. No; let γ be the unit circle. Then $\int_\gamma x\,dx + x\,dy = \pi$.

19. (a) No (b) Yes (c) Yes 21. $2e^{i\pi/6}, 2e^{i5\pi/6}$, and $2e^{i3\pi/2}$

23. By the Mean Value Property 2.5.9 for harmonic functions,

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})d\theta.$$

By Poisson's formula (2.5.13),

$$u(re^{i\phi}) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{u(Re^{i\theta})}{R^2 - 2Rr\cos(\phi - \theta) + r^2}d\theta.$$

Using these equalities, we get

$$\frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{u(Re^{i\theta})}{R^2 + 2Rr + r^2}d\theta \le u(z) \le \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{u(Re^{i\theta})}{R^2 - 2Rr + r^2}d\theta;$$

that is,

$$\frac{(R+r)(R-r)}{(R+r)(R+r)}\frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})d\theta \le u(z) \le \frac{(R-r)(R+r)}{(R-r)(R-r)}\frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})d\theta.$$

Therefore,

$$\frac{R - |z|}{R + |z|}u(0) \le u(z) \le \frac{R + |z|}{R - |z|}u(0).$$

3.1 Convergent Series of Analytic Functions

1. (a) Does not converge (b) 0.

3. The limit is 1 and the convergence is not uniform.

5. If $|z| \le r$, then $|z^n - 0| \le r^n$, so $|z^n - 0| < \epsilon$ whenever $n > (\log \epsilon)/(\log r)$. These n work for all z in D_r, so the convergence is uniform. The convergence is not uniform on $D(0;1)$. For example, if $z = r$, the required minimum n is $(\log \epsilon)/(\log r)$, which becomes arbitrarily large as r gets close to 1.

7. Neither of these series converges absolutely. However, both the real and imaginary parts of each are alternating series whose terms decrease in absolute value monotonically to zero and thus are convergent (by the alternating series test from calculus).

9. The sequence of partial sums converges uniformly; thus it converges to a continuous function and the assertion follows.

11. Let D be any closed disk in A. Then there exists an $r > 1$ such that $|z| > r$ for all $z \in D$. Hence $|1/z^n| < (1/r)^n$ for all $z \in D$. But $\sum_{n=1}^{\infty} (1/r)^n$ converges, since $1/r < 1$. Thus $\sum_{n=1}^{\infty} 1/z^n$ converges absolutely and uniformly on D. Since $1/z^n$ is analytic on A, the Analytic Convergence Theorem 3.1.8 shows that $\sum_{n=1}^{\infty} 1/z^n$ is analytic on A.

13. Let D be a closed disk in A and let δ be its distance from the boundary $\text{Im}\, z = \pm 1$. For $z = x + iy \in D$, prove that $|e^{-n} \sin(nz)| \le e^{-n\delta}$.

15. By Worked Example 3.1.15, $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ converges uniformly on closed disks in A and thus it is analytic on A with $\zeta'(z) = \sum_{n=1}^{\infty} (-\log n) n^{-z}$, which also converges uniformly on closed disks in A and thus is analytic. By induction, $\zeta^{(k)}(z) = \sum_{n=1}^{\infty} (-\log n)^k n^{-z}$ converges uniformly on closed disks in A and thus is analytic. Therefore, $(-1)^k \zeta^k(z) = \sum_{n=1}^{\infty} (\log n)^k n^{-z}$ is also analytic.

17. No; let $f_n(z) = \sum_{k=1}^{n} z^k/k^2$.

19. $\{z \mid |2z - 1| < 1\} = \{z \mid |z - \frac{1}{2}| < \frac{1}{2}\}$.

3.2 Power Series and Taylor's Theorem

1. (a) 1 (b) e (c) e (d) 1.

3. (a) $e^z = \sum_{n=0}^{\infty} (e/n!)(z-1)^n$, which converges everywhere.
(b) $1/z = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$; the series converges for $|z - 1| < 1$.

5. (a) $\frac{\sin z}{z} = \sin(1) + [\cos(1) - \sin(1)](z-1)$
$+ \left[\frac{\sin(1)}{2} - \cos(1)\right](z-1)^2 + \left[\frac{5}{6}\cos(1) - \frac{1}{2}\sin(1)\right](z-1)^3 + \ldots$
(b) $z^2 e^z = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} z^{n+2}$ (c) $e^z \sin z = z + z^2 + \dfrac{1}{3} z^3 - \dfrac{1}{30} z^5 + \ldots$

7. (a) $\displaystyle\sum_{n=0}^{\infty} z^{2n}/n!$ (b) $\displaystyle\sum_{n=0}^{\infty} [1 - 1/(2^{n+1})] z^n$

9. $\sqrt{z^2 - 1} = i - \dfrac{i}{2} z^2 - \dfrac{i}{8} z^4 + \ldots$

11. For $\sinh z$, the odd derivatives are 1 and the even derivatives are 0 at $z = 0$. Thus by Taylor's theorem,

$$\sinh z = \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n-1)!}.$$

The argument for $\cosh z$ is similar.

13. $$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1} \quad \text{and} \quad \frac{1}{(1-z)^3} = \frac{1}{2} \sum_{n=2}^{\infty} n(n-1) z^{n-2}.$$

15. If $|z| < R$, then $\sum a_n z^n$ converges absolutely; that is, $\sum |a_n||z|^n$ converges. However, $|(\text{Re}\, a_n)z^n| \le |a_n||z^n|$, so

$$\sum |(\text{Re}\, a_n)z^n| \le \sum |a_n||z^n|,$$

and hence $\sum(\text{Re}\, a_n)z^n$ converges. Since $\sum(\text{Re}\, a_n)z^n$ converges for any $|z| < R$, the radius of convergence must be $\ge R$.

17. The region for the first series is $A = \{z \text{ such that } |\text{Im}(z)| < \log 2\}$. The second series is not analytic anywhere. (Use $|e^{-ny} - e^{ny}|/2 \le |\sin nz| \le (e^{-ny} + e^{ny})/2$, where $y = \text{Im}\, z$).

19. (a) Suppose that the Taylor series for f, $\sum_{n=0}^{\infty} f^{(n)}(z_0)(z - z_0)^n/n!$, has radius of convergence R and converges to a function $g(z)$ on $D(z_0; R)$. Let R_0 be the radius of D. By Taylor's theorem (3.2.7), the restriction of g to D is equal to f and $R \ge R_0$. The function g is analytic and so continuous on $D(z_0; R)$. If R were $> R_0$, then g would be continuous and hence bounded on the compact closed disk cl $(D) = \{z \text{ such that } |z - z_0| \le R_0\}$. But g and f are the same on D, and f is not bounded on D, so R is not greater than R_0 and thus $R = R_0$.
(b) Branches of $\log(1 + z)$ may be defined with the plane cut along any ray from -1 to ∞. For the principal branch this is along the negative real axis. These determinations differ by an additive constant depending on the angle between the ray and the real axis. The series expansions around $z_0 = -2 + i$ differ only in the constant terms and so have the same radius of convergence, which we may call R. If we choose the ray leading from -1 directly away from z_0, then $D(z_0; \sqrt{2})$ lies in the region of analyticity, so $R \ge \sqrt{2}$. But $D = D(z_0; 1)$ is the largest disk centered at z_0 contained in the region of analyticity of the principal branch of $\log(1 + z)$, and $\sqrt{2} > 1$.

21. Suppose $|h(z)| \le M$ for z in B, and let $\epsilon > 0$. If $g(z) = \sum_i g_i(z)$, then uniform convergence given an N such that $|g(z) - \sum_{i=m+1}^{n} g_i(z)| < \epsilon/M$ whenever $n \ge N$ and $m \ge N$. Thus,

$$\left| h(z)g(z) - \sum_{i=m+1}^{n} h(z)g_i(z) \right| = |h(z)| \left| g(z) - \sum_{i=m+1}^{n} g_i(z) \right| < \epsilon.$$

23. $$\frac{d}{dz}\left[\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{n!}\right] = (2x - 2z)e^{2xz - z^2} = (2x - 2z)\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{n!}.$$

Therefore,

$$\sum_{n=1}^{\infty} H_n(x)\frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} 2xH_n(x)\frac{z^n}{n!} - \sum_{n=1}^{\infty} 2H_{n-1}(x)\frac{z^n}{(n-1)!}$$

$$H_1(x) + \sum_{n=1}^{\infty} H_{n+1}(x)\frac{z^n}{n!} = 2xH_0(x) + \sum_{n=1}^{\infty}[2xH_n(x) - 2nH_{n-1}(x)]z^n/n!.$$

Equating coefficients gives the desired results.

25. $$f(z) = 1 + 2z + \frac{5}{2}z^2 + \frac{5}{4}\sum_{n=3}^{\infty}\frac{2^n}{n!}z^n = \frac{5e^{2z} - 2z - 1}{4}.$$

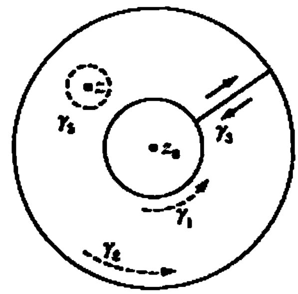
3.3 Laurent Series and Classification of Singularities

1. (a) $\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \dots, 0 < |z| < \infty$ (b) $\frac{1}{z} - 1 + z - z^2 + z^3 - z^4 + \dots, 0 < |z| < 1$
(c) $z - z^2 + z^3 - z^4 + z^5 - \dots, |z| < 1$ (d) $\frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{1}{3!}z + \frac{1}{4!}z^2 + \dots, 0 < |z| < \infty.$

3. $z/(z + 1) = 1/(1 + 1/z) = \sum_{n=0}^{\infty}(-1)^n(1/z)^n = \sum_{n=0}^{\infty}(-1)^n z^{-n}$ for $|z| > 1$.

5. Let γ_3 be a radial segment from γ_2 to γ_1 not passing through z. Let $\gamma_4 = \gamma_2 + \gamma_3 - \gamma_1 - \gamma_3$ as indicated in the Figure. Let γ_5 be a small circle about z lying between γ_1 and γ_2 and not crossing γ_3. Then γ_4 is clearly homotopic to γ_5 in the region of analyticity of $f(t)$ and of $f(t)/(t-z)$. Hence, by the Cauchy Integral Formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\gamma_5}\frac{f(t)}{t-z}dt = \frac{1}{2\pi i}\int_{\gamma_4}\frac{f(t)}{t-z}dt \\ &= \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(t)}{t-z}dt + \frac{1}{2\pi i}\int_{\gamma_3}\frac{f(t)}{t-z}dt - \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(t)}{t-z}dt - \frac{1}{2\pi i}\int_{\gamma_3}\frac{f(t)}{t-z}dt \\ &= \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(t)}{t-z}dt - \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(t)}{t-z}dt. \end{aligned}$$



7. Use the fact that there is an analytic function ϕ defined on a neighborhood of z_0 such that $\phi(z_0) \neq 0$ and $f(z) = \phi(z)/(z-z_0)^k$.

9. (a) No (b) No (c) Yes

11. $\dfrac{1}{e^z-1} = \dfrac{1}{z} - \dfrac{1}{2} + \dfrac{1}{12}z - \dfrac{1}{720}z^3 + \ldots$ 13. $\cot z = \dfrac{1}{z} - \dfrac{1}{3}z - \dfrac{1}{45}z^3 - \dfrac{2}{945}z^5 + \ldots$

15. Since $\sum_{n=1}^{\infty} b_n/(z-z_0)^n$ converges for $|z-z_0| = (R+r)/2 > R$,

$$b_n/(z-z_0)^n \to 0 \text{ for } |z-z_0| = (R+r)/2.$$

That is, $2^n|b_n|/(R+r)^n \to 0$ and so is bounded. Therefore, by the Abel-Weierstrass lemma, $\sum_{n=1}^{\infty} b_n(z-z_0)^n$ converges uniformly and absolutely on

$$A_\rho = \{z \text{ such that } |z-z_0| \le \rho\} \text{ if } \rho < 2/(R+r).$$

Taking $\rho = 1/r < 2/(R+r)$, $\sum_{n=1}^{\infty} b_n(z-z_0)^n$ converges uniformly and absolutely on $\{z \text{ such that } |z-z_0| \le 1/r\}$. In particular, $\sum_{n=1}^{\infty}|b_n|/r^n$ converges. If $z \in F_r$, then $|z-z_0| > r$, so

$$\left|\frac{b_n}{(z-z_0)^n}\right| < \frac{|b_n|}{r^n}.$$

Thus, $\sum_{n=1}^{\infty} b_n/(z-z_0)^n$ converges uniformly on F_r by the M test.

17. $\cos(1/z)$ has a zero of order 1 at

$$\frac{1}{z} = \frac{(2n+1)\pi}{2}; \qquad n = 0, \pm 1, \pm 2, \ldots,$$

that is, at

$$z = \frac{2}{(2n+1)\pi}.$$

Thus, $1/\cos(1/z)$ has simple poles at these points and $z = 0$ is not an isolated singularity.

19. (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) 0.

Review Exercises for Chapter 3

1. $\sum_{n=1}^{\infty}(-1)^{n-1}(z-1)^n/n$ 3. $\frac{1}{z^2}-\frac{1}{z}+1-z+z^2-z^3+\ldots$ 5. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)!}z^{4n}$

7. Suppose that $w \in \mathbb{C}, w \neq 0$. Solve $e^{1/z} = w$ for z as follows:
$$\frac{1}{z} = \log w = \log|w| + i(\arg w + 2\pi n); \quad z = \frac{1}{\log|w| + i(\arg w + 2\pi n)}.$$
Infinitely many of these solutions lie in any deleted neighborhood of the origin.

9. (a) $\frac{1}{2}$ (b) 1

11. The coefficients for either series are given by
$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^{n+1}}d\zeta$$
where γ is the circle of radius 2 centered at the origin.

13. The Schwarz Lemma applies to give $|f(z)| \le |z|$ for all $|z| < 1$. Also, $|f(z)| = |z|$ for any $|z| < 1$ implies that $f(z) = cz$ for a constant c with $|c| = 1$.

15. $-2\pi i/3$ 17. (a) $= \frac{1}{z} - z + z^3 - z^5 + z^7 - \ldots$ (b) $= \frac{1}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} - \frac{1}{z^9} + \frac{1}{z^{11}} - \ldots$

19. Yes. You could base your argument on the fact that a composition of analytic functions is analytic.

21. Since f is bounded near z_0, z_0 is neither a pole nor an essential singularity. 23. 2π

25. (a) Poles of order 1 at $z = 1$ and $z = 5$ (b) Removable singularity at $z = 0$
(c) Pole of order 1 at $z = 0$ (d) Pole of order 1 at $z = 1$.

27. $2\pi i$.

29. (a) $a_0 = -\frac{1}{2}, a_1 = 0, a_2 = (4-\pi^2)/8$ (b) The denominator has zeros of order 2 at the odd integers. The numerator has zeros of order 1 at ± 1, so the function has simple poles at ± 1 and poles of order 2 at all the other odd integers.
(c) The closest singularities to 0 are at ± 1. The radius of convergence is 1.

31. No: $\sin z$. 33. (a) 2π (b) $B_n = \frac{n!}{2\pi}\int_0^{2\pi}\frac{e^{-i(n-1)\theta}}{e^{e^{i\theta}}-1}d\theta$ 35. 1

37. $a_0 + z + z^2 + z^4 + z^8 + z^{16} + \ldots$, where $a_0 = f(0)$. Show uniqueness by showing that the coefficients are uniquely determined.

39. $|f(z)| = \lim_{n\to\infty}\left|\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}z^k\right|$ for all z. But
$$\left|\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}z^k\right| \le \sum_{k=0}^{n}\frac{|f^{(k)}(0)|}{k!}|z|^k \le \sum_{k=0}^{n}\frac{1}{k!}|z|^k \le \sum_{k=0}^{\infty}\frac{1}{k!}|z|^k = e^{|z|},$$
so the limit is no more than $e^{|z|}$.

4.1 Calculation of Residues

1. (a) 0 (b) 1 (c) -1 (d) $\frac{1}{6}$ (e) 0. 3. Let $f(z) = 1/z$. 5. The correct residue is 2.

7. (a) $\text{Res}(f;0) = \frac{1}{64}, \text{Res}(f;-4) = -\frac{1}{64}$ (b) $\text{Res}(f;-1) = 0$
(c) $\text{Res}(f;\sqrt[3]{3}) = 3^{-5/3}, \text{Res}(f;\sqrt[3]{3}e^{2\pi i/3}) = 3^{-5/3}e^{-4\pi i/3}, \text{Res}(f;\sqrt[3]{3}e^{4\pi i/3}) = 3^{-5/3}e^{-2\pi i/}$

9. $1/6$. 11. $\text{Res}(f_1f_2;z_0) = a_1\text{Res}(f_2;z_0) + a_2\text{Res}(f_1;z_0)$ where a_i is the constant term in the expansion of f_i.

13. (a) 0 (b) $-e/2$ (c) $\text{Res}(f;0) = 1, \text{Res}(f;1) = -1$ (d) $\text{Res}(f;0) = 1, \text{Res}(f;1) = -e/2$.

4.2 Residue Theorem

1. (a) 0 (b) 0 3. 0 5. $-12\pi i$ 7. (a) 0 (b) 0 9. (a) $2\pi i$ (b) $2\pi i$

11. $-\int_\gamma g\left(\frac{1}{z}\right)\frac{1}{z^2}dz = \int_{\tilde\gamma} g(w)dw$, where $\tilde\gamma$ is the curve $1/\gamma$.

13. . (a) -1 (b) $2\pi i$ 15. $-\pi i$

4.3 Evaluation of Definite Integrals

1. $\pi/\sqrt{3}$ 3. $\dfrac{\pi a}{(a^2-b^2)^{3/2}}$ 5. $\dfrac{\pi}{2\sqrt{2}}e^{-|m|/\sqrt{2}}\left(\cos\dfrac{|m|}{\sqrt{2}} + \sin\dfrac{|m|}{\sqrt{2}}\right)$

7. $\pi e^{-1}/2$ 9. $-\pi i/(a-1)^2$

11. The function is even and line 3 of Table 4.3.1 applies. 13. 0

15. $-\pi i/2$ (or $\pi i/2$ if a different branch is used). Construct $\sqrt{z^2-1}$ much as in Example 4.3.15, but make the branch cut for the factor $\sqrt{z-1}$ go from 1 to $-\infty$ and that for $\sqrt{z+1}$ from -1 to $-\infty$. Crossing the real axis at x with $|x|>1$ requires crossing either both cuts or neither. The product is analytic on $\mathbb{C}\backslash\{z \mid \operatorname{Im} z = 0 \text{ and } |\operatorname{Re} z| \le 1\}$, as in Example 4.3.15.

17. $\pi e^{-ab}/a$ 19. Use Exercise 18; $\operatorname{Res}((-z)^{a-1}f(z);1) = -(e^{\pi i})^{a-1}$.

21. After checking that all the integrals exist and the operations are justified, compute

$$\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right)\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dx\,dy$$
$$= \int_0^{\infty}\int_0^{2\pi} e^{-r^2}r\,d\theta\,dr = 2\pi\int_0^{\infty} e^{-r^2}r\,dr = -\pi e^{-r^2}\Big|_0^{\infty} = -\pi(0-1) = \pi,$$

so $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

23. $2\pi(97\sqrt{3}-168)/3$ 25. (a) $\sqrt{\pi i}$

4.4 Infinite Series and Partial-Fraction Expansions

1. As in the proof of Proposition 4.4.2,

$$\int_{C_N}\frac{\pi\cot\pi z}{z^4}dz \to 0 \text{ as } N\to\infty.$$

The residue at $n \ne 0$ is $1/n^4$. Compute the first few terms of the Laurent series $\cot z = 1/z - \frac{1}{3}z - \frac{1}{45}z^3 - \ldots$ to find that the residue at 0 is $-\pi^4/45$, so

$$\lim_{N\to\infty}\left[-\frac{\pi^4}{45} + \sum_{n=1}^{N}\frac{1}{n^4} + \sum_{n=-1}^{-N}\frac{1}{(-n)^4}\right] = \lim_{N\to\infty}\left(-\frac{\pi^4}{45} + 2\sum_{n=1}^{N}\frac{1}{n^4}\right),$$

and thus $\sum_1^\infty 1/n^4 = \pi^4/90$.

3. Apply the summation theorem 4.4.1 and Proposition 4.4.2,

$$\operatorname{Res}\left(\frac{\pi\cot\pi z}{z^2+a^2};\pm ai\right) = -\frac{\pi}{2a}\coth\pi a$$

(check this), so

$$\frac{\pi}{a}\coth\pi a = \sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} = \frac{1}{a^2} + 2\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a}\coth \pi a$$

and

$$\sum_{n=0}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{a^2} + \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{2a^2} + \frac{\pi}{2a}\coth \pi a.$$

5. $\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -$[sum of residues of $\pi \csc \pi z f(z)$ at the poles of f]. Here f should obey some conditions like: There exist an $R > 0$ and an $M > 0$ such that for $|z| > R$, $|f(z)| \le M/|z|^\alpha$ where $\alpha > 1$. If some of the poles of f should lie at the integers, the technique could still be used. After verifying that

$$\int_{C_N} \frac{\pi f(z)}{\sin \pi z} dz \to 0 \quad \text{as} \quad N \to \infty,$$

we would conclude that

$$-\sum_{n=-\infty}^{\infty} (-1)^n f(n)$$

(where the sum is over integers n which are *not* poles of f) is the sum of the residues of $\pi \csc \pi z\, f(z)$ at the poles of f.

7. Consider $f(z) = \cot z - 1/z$. Then $\lim_{z\to 0} f(z) = 0$ and f is analytic at 0. Check that it has simple poles at $z = n\pi$ for $n \ne 0$ with residue 1 at each. Let C_N be the square with corners at $(N+\frac{1}{2})\pi(\pm 1 \pm i)$. Along C_N we have $\cot z = -\cot(-z)$, and $-z$ is on C_N when z is. Thus it suffices to check $|\cot z|$ for any $y = \operatorname{Im} z \ge 0$. If $z = x + iy, y > 0$, then

$$|\cot z| = \left|\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}\right| = \left|\frac{e^{2ix-2y}+1}{e^{2ix-2y}-1}\right| \le \frac{2}{|e^{2ix-2y}-1|}$$

on the upper horizontal of the square $y = (n+\frac{1}{2})\pi > 1$, and so $|\cot z| \le 2/(1-e^{-2}) \le 4$. On the vertical sides, $x = \pm(N+\frac{1}{2})\pi$, and so $e^{2ix} = -1$, and

$$|\cot z| \le \left|\frac{2}{-e^{-2y}-1}\right| < 2.$$

In any case, $|f(z)| \le 4 + 2/\pi$ for z on C_N, and so with $R = (N+\frac{1}{2})\pi$, $M = 4 + 2/\pi$, and $S = 8$, the conditions of the partial-fraction theorem (4.4.5) are met and the data on poles and residues may be entered to give the desired formula.

9. An exact answer to this seemingly simple problem is not known. The sum is $\zeta(3)$ where ζ is the Riemann zeta function, important in analysis and number theory and a source of several famous open problems in mathematics. The method of the summation theorem (4.4.1) may be used for summing $\zeta(p) = \sum_1^\infty (1/n^p)$ for even p as in Proposition 4.4.3 and Exercise 1. One gets

$$\zeta(2m) = (-1)^{m+1}(2\pi)^{2m}\frac{B_{2m}}{2(2m)!}$$

where the B_{2m}'s are the Bernoulli numbers involved in the expansion of the cotangent function. (See also Review Exercise 33 of Chapter 3.) This method fails for odd p basically because $1/(-n)^p + 1/n^p = 0$, not $2/n^p$. An approximate value is $\zeta(3) \approx 1.2020569$, but until recently it was not even known if $\zeta(3)$ was irrational. This was shown in 1978 by R. Apery. (See *Mathematical Intelligencer* 1 (1979), 195–203.) Even irrationality is still unknown for $\zeta(p)$ for other odd values of p.

Review Exercises for Chapter 4

1. $2\pi/\sqrt{3}$ 3. $\pi/\sqrt{2}$ 5. $\frac{\pi}{2}\left[2-\frac{1}{e}-\cos(1)\right]$ 7. $\pi/\sqrt{5}$ 9. 2π 11. $2\pi i\sin(1)$

13. (a) $\frac{1}{2}+\frac{3}{4}z+\frac{7}{8}z^2+\ldots+\frac{2^{n+1}-1}{2^{n+1}}z^n+\ldots$ (b) $\frac{1}{z^2}+\frac{3}{z^3}+\frac{7}{z^4}+\frac{15}{z^5}+\ldots+\frac{2^n-1}{z^{n+1}}+\ldots$

15. $\pi/2$

17. (a) $\operatorname{Res}(f;0)=-1$. The other residues are at z with $z^2=2\pi ni, n=\pm1,\pm2,\ldots$, and are equal to $-1/2$. (b) $\operatorname{Res}(f;n\pi)=2\pi n\cos(n^2\pi^2)$ (c) $\cos(1)$

19. $\operatorname{Res}\left(\frac{f''}{f'};z_0\right)=k-1$ and $\operatorname{Res}\left(\frac{f''}{f};z_0\right)=\frac{2k}{k+1}\cdot\frac{f^{(k+1)}(z_0)}{f^{(k)}(z_0)}$ for $k\neq 0,1$

21. $\frac{1}{1+(z-1)}$ has been expanded incorrectly. The correct residue is -1. 23. $-\pi i$

25. (a) The radius of convergence is infinite.
(b) The radius of convergence is 1 (use the root test).
Note: To use the ratio test, the following facts are used:
(i) $\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n=e^z$. (ii) In the power series $\sum a_nz^n$, if the coefficients a_n tend to a nonzero finite limit, then the radius of convergence must be 1.

27. (a) $1+3z+6z^2+10z^3+\ldots+\frac{(n+1)(n+2)}{2}z^n+\ldots$
(b) $-\frac{1}{z^3}-\frac{3}{z^4}-\frac{6}{z^5}-\ldots-\frac{(n+1)(n+2)}{2}\frac{1}{z^{n+3}}-\ldots$
(c) $\frac{1}{8}+\frac{3}{16}(z+1)+\frac{6}{32}(z+1)^2+\ldots+\frac{(n+1)(n+2)}{2^{n+4}}(z+1)^n+\ldots$ (d) $\frac{-1}{(z-1)^3}$

29. (a) Use $\sin^3x=(3\sin x-\sin 3x)/4$. Use an argument like that for Cauchy principle value, checking directly that

$$\int_\gamma\frac{3e^{iz}-e^{3iz}}{z^3}\to-3\pi i\quad\text{as}\quad\rho\to0$$

where γ is a half circle in the upper half plane from $-\rho$ to ρ.
(b) Use line 5 of Table 4.3.1.

31. Use Exercise 5 of §4.4. $F(z)=(\pi\csc\pi z)/(z+a)^2$ has a pole at $z=-a$ with residue $-\pi^2\csc(\pi a)\cot(\pi a)$.

33. The last equality is wrong since the integral along the semicircle is omitted. We cannot conclude that the integral along the semicircle goes to 0 as $R\to\infty$ and thus must evaluate it more carefully.

35. It will suffice for f to be analytic in a region containing the real axis and the upper half plane and to be such that the integral of $f(z)/(z-x)$ along the upper semicircle of $|z|=R\to0$ as $R\to\infty$. These conditions will hold if $|f(z)|<M/R^\alpha$ for some $M>0$ and $\alpha>0$ for large enough R, and for z lying in the upper half plane. Use Exercise 34 for the last part.

37. $\frac{\pi}{2b}\tanh\frac{\omega\pi}{2b}$.

5.1 Basic Theory of Conformal Mappings

1. The first three quadrants.

3. (a) Everywhere except $z=0$ and $z=-2/3$. (b) Everywhere except $z=-1/5$.

5. $v(x,y)=1-2xy+\frac{x^2-y^2}{(x^2+y^2)^2}$

7. Let g and h be the functions guaranteed by the Riemann Mapping Theorem:

$$\begin{aligned} g : A \to D \quad &\text{with} \quad g(z_0) = 0 \quad \text{and} \quad g'(z_0) > 0 \\ h : B \to D \quad &\text{with} \quad h(w_0) = 0 \quad \text{and} \quad h'(w_0) > 0 \end{aligned}$$

Set $f(z) = h^{-1}(e^{i\theta}g(z))$ for $z \in A$ and check that f takes A one-to-one onto B, that $f(z_0) = h^{-1}(e^{i\theta}g(z_0)) = h^{-1}(0) = w_0$, and that $f'(z_0) = e^{i\theta}[g'(z_0)/h'(w_0)]$.

9. From $f^{-1}(f(z)) = z$ we get $(f^{-1})'(f(z))(f'(z)) = 1$. (Why is f^{-1} analytic?) It follows that $f'(z) \neq 0$, so f is conformal by the Conformal Mapping Theorem 5.1.1.

11. No for both parts. A function for (a) is a bounded entire function. Liouville's theorem says it would be constant and so certainly cannot map onto D. The inverse of a function for (b) is a function for (a).

13. $A \cup \operatorname{bd}(A)$ is closed. If A is bounded then $A \cup \operatorname{bd}(A)$ would also be bounded and so compact. If there were a continuous extension of f to this compact set its image would be compact and could not contain the unbounded set B.

5.2 Fractional Linear and Schwarz-Christoffel Transformations

1. (a) $\mathbb{R}\backslash\{1\}(f(\infty) = 1$ and $f(-1) = \infty)$
(b) The circle cutting the real axis at right angles at 3 and $\frac{1}{3}$ (center $\frac{5}{3}$, radius $\frac{4}{3}$)
(c) $\{\text{imaginary axis}\} \cup \{\infty\}$ $\quad (f(-1) = \infty)$ $\quad$ (d) $\{\text{unit circle}\}\backslash\{1\}$ $\quad (f(\infty) = 1)$

3. (a) $f(z) = (z+1)/(z-3)$ $\qquad$ (b) $f(z) = z - 2$

5. According to Figure 5.2.10(vi), $z \mapsto -i\left(\dfrac{z+1}{z-1}\right)$ takes the disk to the upper half plane with $0 \mapsto i$. The map $w \mapsto 2(w+1)$ takes the upper half plane to itself. Thus,

$$f(z) = 2\left[1 - i\left(\frac{z+1}{z-1}\right)\right]$$

does what we want.

7. $z \mapsto \dfrac{z - \frac{1}{2}}{1 - \frac{1}{2}z} = \dfrac{2z-1}{2-z}$ takes D to D and $\frac{1}{2}$ to 0. $w \mapsto \dfrac{3w-1}{3-w}$ takes D to D and $\frac{1}{3}$ to 0. Solving $\dfrac{3w-1}{3-w} = \dfrac{2z-1}{2-z}$ for w gives $w = \dfrac{5z-1}{5-z}$, so $f(z) = \dfrac{5z-1}{5-z}$ is the desired map.

9. $f(z) = e^{-3\pi i/4}\left(\dfrac{z-i}{z+i}\right)$

11. T is the composition of a translation, inversion in the unit circle, reflection in the real axis, a rotation, a magnification, and another translation.

13. $(2z-1)/(2-z)$. (This may be multiplied by $e^{i\theta}$ for any real constant θ.)

15. $e^{i\theta}\dfrac{z - Rz_0}{R - \bar{z}_0 z}$, where $|z_0| < 1$.

17. Suppose that T is such a map. Define W by

$$W(z) = T\left(\frac{1}{i} \cdot \frac{z+1}{z-1}\right).$$

W maps the unit disk conformally onto itself. Now use Proposition 5.2.2.

19. $\dfrac{z^8 - i}{z^8 + i}$

21. By Proposition 5.2.3, $T(\gamma_1)$ and $T(\gamma_2)$ are circles or straight lines, and since T is conformal, by Theorem 5.1.1, they intersect orthogonally.

23. $|\tilde{z} - z_0||z - z_0| = |\bar{\tilde{z}} - \bar{z}_0||z - z_0| = \left|\left(\frac{\bar{z}_0 z + R^2 - |z_0|^2}{z - z_0} - \bar{z}_0\right)(z - z_0)\right|$
$= |\bar{z}_0 z + R^2 - |z_0|^2 - \bar{z}_0 z + \bar{z}_0 z_0| = |R^2| = R^2$

25. $f(z) = \dfrac{1+e^z}{1-e^z}$

27. Use a branch of log defined on $\mathbb{C}\backslash\{\text{nonpositive real axis}\}$ by $\log(re^{i\theta}) = \log r + i\theta$, where $-\pi < \theta < \pi$.

29. By the Schwarz-Christoffel Formula 5.2.11, the image is a polygon with four sides, three of whose angles are $90°$. Hence the image is a rectangle.

31. No. The map

$$f(z) = \int_0^z (t-1)^{-\alpha_1} t^{-\alpha_2}\, dt$$

takes the upper half plane to a triangle with exterior angles $\pi\alpha_1, \pi\alpha_2, \pi\alpha_3$.

33. The boundary circles both go through 0 and $f(0) = \infty$. Since f takes $\mathbb{R}$ to $\mathbb{R}$, the circles go to lines through $f(2)$ and $f(4)$ that are orthogonal to $\mathbb{R}$. These are the lines $\operatorname{Re} z = 2$ and $\operatorname{Re} z = 1$. Check $f(3)$ to make sure the region is right.

35. $f(b) = 0$ and $f(d) = \infty$. Thus a circle through b and d must map to a circle through 0 and ∞, that is, a line through the origin. Since

$$|f(z)| = \left|a\frac{z-b}{z-d}\right| = \left|\frac{z-b}{z-d}\right| |a|,$$

the statement $\left|\frac{z-b}{z-d}\right| = \frac{r}{|a|}$ holds iff $|f(z)| = r$. This establishes (a) and (b).

The easiest way to obtain the orthogonality is to notice that the images under the map f are trivially orthogonal. Since the inverse of a fractional linear transformation is of the same form, hence conformal, the same must have been true of the preimage. (To confirm this directly a straightforward but lengthy calculation is required.)

5.3 Applications to Laplace's Equation, Heat Conduction, Electrostatics, and Hydrodynamics

1. $u(x,y) = \dfrac{1}{\pi}\arctan\dfrac{4x^3y - 4xy^3}{x^4 - 6x^2y^2 + y^4} = \dfrac{4}{\pi}\arctan\dfrac{y}{x}$

3. $\phi(x,y) = 1 - \dfrac{1}{\pi}\arctan\dfrac{\cos x \cdot \sinh y}{\sin x \cdot \cosh y - 1} + \dfrac{1}{\pi}\arctan\dfrac{\cos x \cdot \sinh y}{\sin x \cdot \cosh y + 1}$

5. $F(z) = |\alpha|\left(e^{-i\theta}z + \dfrac{1}{e^{-i\theta}z}\right)$. In polar coordinates,

$$\phi(r,\Theta) = |\alpha|\left[\left(r + \frac{1}{r}\right)\cos(\Theta - \theta)\right]; \qquad \psi(r,\Theta) = |\alpha|\left[\left(r - \frac{1}{r}\right)\sin(\Theta - \theta)\right].$$

7. In polar coordinates,

$$\phi(r,\theta) = \alpha r^4 \cos 4\theta \quad \text{and} \quad \psi(r,\theta) = \alpha r^4 \sin 4\theta.$$

In rectangular coordinates,

$$\phi(x,y) = (x^4 - 6x^2y^2 + y^4)\alpha \quad \text{and} \quad \psi(x,y) = (4x^3y - 4xy^3)\alpha.$$

Review Exercises for Chapter 5

1. Any region not containing zero.

3. $(2z - i)/(2 + iz)$. (This answer can be multiplied by $e^{i\theta}$ for any real constant θ.)

5. $f(z) = (2z-4)/z$ is one such. 7. The first quadrant. 9. $f(z) = z + 1 - i$.

11. No. The region $\{z \mid 1 < |z| < 2\}$ is not simply connected. The inverse would take the simply connected region B to it, which is impossible by Worked Example 5.1.7.

13. No 15. Use Proposition 5.2.2.

17. $F(z) = \sqrt{\dfrac{az^2+b}{cz^2+d}}$ where a, b, c, d are real and $ad > bc$.

19. $f(z) = \sqrt{z^2+1}$ where $\sqrt{\ }$ must be taken as the branch defined on $\mathbb{C}\backslash\{\text{positive real axis}\}$, which takes values in the upper half plane. The desired potential is $\phi(z) = \dfrac{1}{\pi}\arg\left(\dfrac{\sqrt{z^2+1}-1}{\sqrt{z^2+1}+1}\right)$.

21. The desired complex potential must be $F(z) = a\sqrt{z^2+1}$. The level curves $\psi(x,y) = K$, which are the streamlines, are described by

$$y^2 = \frac{K^2}{x^2+K^2} + K^2.$$

23. $\phi(x,y) = 1 - \dfrac{1}{\pi}\arctan\dfrac{e^x \sin y}{e^x\cos y - 1}$. The values of arctan are chosen between 0 and π.

25. $f(z) = \dfrac{h}{\pi}\left(\sqrt{z^2-1} + \cosh^{-1} z\right)$ 27. $T(x,y) = \dfrac{1}{2} - \dfrac{1}{\pi}\operatorname{Re}\left(\arcsin\dfrac{z}{a}\right)$.

29. $\dfrac{\pi e^{-\sqrt{3}}}{2\sqrt{3}}$ 31. $\sqrt{\rho}; \rho^2$.

6.1 Analytic Continuation and Elementary Riemann Surfaces

1. (a) No, it does not. An important condition of the identity theorem (6.1.1) is that the limit point z_0 must lie in A.
(b) No. Let A be the unit disk. Let $u_1(z) = \operatorname{Im} z$ and $u_2(z) = \operatorname{Im} e^z$. Both u_1 and u_2 are harmonic in A and are zero along the real axis, but they are not identically zero, nor do they agree with each other on A.

3. If f were constant on a neighborhood of z_2, it would be constant on all of A by the identity theorem. This would force $f'(z_1)$ to be 0, which is not true.

5. If $z = re^{2\pi ip/q}$, then $z^{n!} = r^{n!}$ whenever $n \geq q$. Any open set containing A contains a point $e^{2\pi ip/q} = z_0$. If f were analytically continued to include z_0 it would have to have a finite limit at z_0, but $\lim_{r\to 1} f(re^{2\pi ip/q}) = \infty$. Check this using the first observation.

7. The union of the sets U_k is an open set A containing γ. The Distance Lemma 1.4.21 gives a positive distance ρ from γ to the complement of A. Since the continuation is analytic on each U_k, the radius of convergence is at least ρ at each point along γ. The path covering lemma gives a finite chain of overlapping disks centered along γ, where each contains the center of the next so they can be used to implement the continuation by power series.

9. The situation is something like that for $\sqrt{z}$, except that now there are three sheets, each a copy of the plane cut along, say, the negative real axis. They are joined along these cuts so that following a path that winds once around zero carries one from the first sheet to the second. When the path winds once more around zero one is carried to the third sheet; when the path winds a third time around zero one is carried back to the first sheet.

11. $f(z) = 1/(1+z)$ extends it to $\mathbb{C}\backslash\{-1\}$.

13. Use Worked Example 2.4.16. (The implicit function theorem can be used to ensure that most small rectangles meet γ at most twice.)

6.2 Rouché's Theorem and Principle of the Argument

1. Five. Consider $g(z) = z^6 - 4z^5 + z^2 - 1$ and $f(z) = -4z^5$ and use Rouché's theorem.

3. For large enough R, the curve γ_R shown would include any finite number of possible solutions in the right half plane. Let $g(z) = z + e^{-z} - 2$ and $f(z) = z - 2$. Along γ_R, $|f(z) - g(z)| = e^{-\operatorname{Re} z} \leq 1 < |f(z)|$, and f has exactly one solution. Thus so does g.

5. Let $h(z) = f(z) - z$ and $g(z) = -z$. On the circle $|z| = 1$, $|h(z) - g(z)| = |f(z)| < 1 = |g(z)|$, so Rouché's theorem shows that h has one zero inside $\{z \text{ such that } |z| = 1\}$. A zero of h is a fixed point of f.

7. Let $r_n = 1 - 1/n$ and $f_n(z) = f(r_n z)$. Use Exercise 5 to get z_n with $f_n(z_n) = z_n$. (Use the Maximum Modulus Principle to obtain $|f(r_n z)| < 1$ if $|z| = 1$.) The z_n's are all in the closed disk $D = \{z \text{ such that } |z| \le 1\}$, so there is a subsequence converging to a point $z_0 \subset D$, say $z_{n_k} \to z_0$. Check the following: $r_{n_k} z_{n_k} \to z_0$, so $f(r_{n_k} z_{n_k}) \to f(z_0)$, but $f(r_{n_k} z_{n_k}) = z_{n_k} \to z_0$, so $f(z_0) = z_0$.

9. Let $g(z) = a_n z^n$, estimate $f(z) - g(z)$ along large circles as in the proof of the Fundamental Theorem of Algebra 2.4.9, and apply Rouché's theorem.

11. Use the method of Theorem 6.2.1 to compute the residue of $f'(z)h(z)/f(z)$, obtaining $kh(a_j)$ if f has a zero of order k at a_j and $-kh(b_l)$ if f has a pole of order k at b_l.

13. Apply Exercise 11 with $h(z) = z$. (The zeros are repeated in the sum according to their multiplicity.)

15. Apply the Fundamental Theorem of Algebra to the polynomial $f(z) - w$.

17. No. Let $f(z) = e^z - 1$. f has three zeros inside a circle of radius 3π, center 0, but $f'(z)$ has no zeros.

19. Any r such that $1 < r < 4$ will give the desired result. Rouché's theorem works with $r = 2$ and $g(z) = -4z^2$.

21. Suppose $e^{i\theta}$ and $e^{i\psi}$ are on the boundary circle. If $e^{i\phi} \ne e^{i\psi}$, then an equation $(e^{i\theta})^2 + 3(e^{i\theta}) = (e^{i\psi})^2 + 3(e^{i\psi})$ would become $(e^{i\theta})^2 - (e^{i\psi})^2 = 3(e^{i\theta} - e^{i\psi})$ or $(e^{i\theta} + e^{i\psi})(e^{i\theta} - e^{i\psi}) = 3(e^{i\theta} - e^{i\psi})$, requiring $e^{i\theta} + e^{i\psi} = 3$. This is not possible, since $e^{i\theta}$ and $e^{i\psi}$ both have absolute value 1. The function is one-to-one on the boundary circle, so on the whole region by the one-to-one theorem (6.2.10).

23. Consider $f(z) = 1/z$ and apply Rouché's theorem; you will get -1 "equal to" a nonnegative number.

6.3 Mapping Properties of Analytic Functions

1. (a) $\{z \text{ such that } |z| < \frac{1}{2}\}$ (b) $\{z \text{ such that } |z - 1| < \frac{3}{2}\}$

3. Let $f(z) = z^3, w_0 = z_0 = 1, r = 1$. The roots of $z^3 - 1$ lie at $1, e^{2\pi i/3}$, and $e^{4\pi i/3}$. Of these only one lies in $D = \{z \text{ such that } |z - 1| < 1\}$. $f'(z) = 3z^2$ and is 0 only at 0, which does not lie in D. However, $f(re^{\pi i/3}) = f(re^{-\pi i/3}) = -r^3$, and for small enough r, these points lie in D.

5. Use the chain rule to show that $g(z) = f(z^n) - f(0)$ has a zero of order n at $z_0 = 0$ and apply Worked Example 6.3.6.

7. Let u be harmonic and nonconstant on a region $A, z_0 \in A$, and let U be any open neighborhood of z_0 lying in A. By Exercise 6, u is an open mapping, so $u(U)$ is an open neighborhood of $u(z_0)$ in $\mathbb{R}$. This means that arbitrarily near z_0, u takes on values that are both larger and smaller than $u(z_0)$.

9. Let $f(z) = e^{z-a} - z$ and $g(z) = -z$. Then, for $z = x + iy$ on the unit circle,

$$|f(z) - g(z)| = |e^{x+iy-a}| = e^{x-a} < 1 = |g(z)|.$$

Rouché's theorem now applies.

11. $|f|$ has a minimum somewhere in D since it is continuous. It is not on the boundary, since $f(0) = 1 < 2$. The minimum is at an interior point z_0 of D. If $f(z)$ were never 0, then $1/f$ would be analytic with a local maximum at z_0. The Maximum Modulus Principle would say $1/f$ and so also f were constant. But it is not, since $f(0) = 1$ and $|f(1)| = 2$.

Review Exercises for Chapter 6

1. f is identically zero on $\{z$ such that $|z| < 1\}$.

3. (a) $g(z) = f(z) - \overline{f(\bar{z})}$ is entire and $g(a_k) = 0$ for all k. Since the a_k's are bounded, there is a subsequence convergent to some a_0. Thus $g = 0$ by the identity theorem, and so $\overline{f(\bar{z})} = f(z)$ for all z. Taking z real gives the result.
(b) By part (a), $f(x)$ is real for real x. Use the mean value theorem from calculus to obtain b_n with $a_{2n+1} \le b_n \le a_n$ and $f'(b_n) = 0$. Check that $b_n \to 0$ and so $f' = 0$ by the identity theorem. Conclude that f is constant.

5. Use the Schwarz Reflection Principle to define f on the upper half plane. The functions f and its reflection agree on the strip $\{z \mid 0 < \operatorname{Im} z < 1\}$ and together define a bounded entire function. Now use Liouville's theorem.

7. Use the identity theorem to show that $g(z) = f(z+1) - f(z)$ is identically equal to 0.

9. $\sqrt{5}$ 11. Use the root counting formula. 13. Use $\displaystyle\int_\gamma \frac{f'}{f} = 2\pi i \sum_{j=1}^{n} I(\gamma; a_j)$.

15. Let $0 < r < 1, D(0;r) = \{z$ such that $|z| < r\}, \gamma_r = \{z$ such that $|z| = r\}$. By assumption, f is one-to-one on γ_r, so $f(\gamma_r)$ is a simple closed curve. Since f is bounded on A, f must map $D(0;r)$ to the interior of γ_r. The one-to-one theorem (6.2.10) now shows that f is one-to-one on $D(0;r)$. Because this holds for any $r < 1, f$ is one-to-one on A.

17. Let $h(z) = f(z) - g(z)$ and use the maximum principle for harmonic functions.

19. (a) True (b) True (c) True (d) True (e) False
(f) False (g) True (h) False (i) True (j) True

21. No 23. (a) Yes (b) No 25. $\frac{1}{2}$

7.1 Infinite Products and the Gamma Function

1. The partial products are

$$\begin{aligned}\prod_{n=2}^{N}\left(1-\frac{1}{n^2}\right) &= \prod_{n=2}^{N}\frac{(n-1)(n+1)}{n^2}\\ &= \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdots\frac{(N-1)}{N}\cdot\frac{(N+1)}{N} = \frac{1}{2}\frac{N+1}{N},\end{aligned}$$

which converge to $\frac{1}{2}$ as $N \to \infty$.

3. Show that for small ϵ and large n,

$$0 < (1-\epsilon)|z_n| \le |\log(1+z_n)| \le (1+\epsilon)|z_n|$$

and use part (iii) of the convergence theorem for products (7.1.2).

5. Let $z = \frac{1}{2}$ in $\sin \pi z = \pi z \prod_{n=1}^{\infty}(1 - z^2/n^2)$ to obtain

$$1 = \frac{\pi}{2}\prod_{n=1}^{\infty}\left[1 - \frac{1}{(2n)^2}\right] = \frac{\pi}{2}\prod_{n=1}^{\infty}\frac{(2n-1)(2n+1)}{(2n)^2} = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{7}{8}\cdot\frac{9}{8}\cdots.$$

7. $G(z) = \prod_{n=1}^{\infty}(1 + z/n)e^{-z/n}$ has zeros at $-1, -2, -3, \ldots$, so $\Gamma(z) = [ze^{\gamma z}G(z)]^{-1}$ has poles at $0, -1, -2, \ldots$. We know that $G(z-1) = ze^{\gamma}G(z)$, so

$$\Gamma(z+1) = \frac{1}{(z+1)e^{\gamma z}e^{\gamma}G(z+1)} = \frac{1}{e^{\gamma z}G(z)} = \frac{z}{ze^{\gamma z}G(z)} = z\Gamma(z)$$

as long as we stay away from the poles.

9.

$$\begin{aligned} zG(z)G(-z) &= z\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n} \\ &= z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi} \end{aligned}$$

by Worked Example 7.1.10. Using Exercise 7,

$$\begin{aligned} \Gamma(z)\Gamma(1-z) &= -z\Gamma(z)\Gamma(-z) \\ &= \frac{-z}{ze^{\gamma z}G(z)(-z)e^{-\gamma z}G(-z)} = \frac{1}{zG(z)G(-z)} = \frac{\pi}{\sin \pi z}. \end{aligned}$$

11. Start with the third line of the proof of Euler's formula:

$$\begin{aligned} \frac{1}{\Gamma(z)} &= z\lim_{n\to\infty} n^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right) = \lim_{n\to\infty}\frac{z}{n^z}\prod_{k=1}^{n}\frac{k+z}{k} \\ &= \lim_{n\to\infty}\frac{z(z+1)(z+2)\ldots(z+n)}{n^z\cdot 1\cdot 2\cdot 3\ldots n} = \lim_{n\to\infty}\frac{z(z+1)\ldots(z+n)}{n!n^z}. \end{aligned}$$

13. Use the Laurent expansion and $\operatorname{Res}(\Gamma;-m) = (-1)^m/m!$.

15. One way is to proceed as follows. First, show that $1+y \le e^y \le (1-y)^{-1}$ for $0 \le y \le 1$ by using power series or calculus. Set $y = t/n$ to obtain $0 \le e^{-t} - (1-t/n)^n$ and conclude that $e^{-t} - (1-t/n)^n \le e^{-t}[1-(1-t^2/n^2)^n]$. Use the inequality $(1-k)^n \ge 1-nk$ for $0 \le k \le 1$ to get

$$1 - \left(1-\frac{t^2}{n^2}\right)^n \le \frac{t^2}{n}$$

for $0 \le t \le n$.

17. Γ has simple poles at $0,-1,-2,\ldots$ and is analytic elsewhere. Therefore,

$$\begin{aligned} \int_\gamma \Gamma(z)dz &= 2\pi i \operatorname{Res}(\Gamma;0) = 2\pi i \lim_{z\to 0} z\Gamma(z) \\ &= 2\pi i \lim_{z\to 0}\Gamma(z+1) = 2\pi i\Gamma(1) = 2\pi i. \end{aligned}$$

19. Let the radius of the circular part of C be $r < 1$. For $n > 1$ consider the functions $f_n(z) = \int_{C,|t|\le n}(-t)^{z-1}e^{-t}dt$.
(a) Use Worked Example 2.4.15 to show that $f_n(z)$ is entire.
(b) Estimate the part of the integral with $|t| > n$ to show that the improper integral converges and that the convergence of f_n to that integral is uniform on closed disks.
(c) Conclude that the Hankel integral is an entire function.
(d) Use Cauchy's Theorem to show that the value is independent of r and ϵ.
(e) Use $\arg(-t) = -\pi$ on the upper side of the real axis and π on the lower side to show that the straight-line portions combine to give $-2i\sin\pi z\int_r^\infty t^{z-1}e^{-t}dt$.
(f) The part along the circle goes to 0 as r goes to 0.
(g) Use Euler's integral for $\Gamma(z)$ to conclude that the formula holds for $\operatorname{Re} z > 0$.
(h) Use the identity theorem to conclude that the formulas agree everywhere both sides make sense; that is, at $z \ne 0,1,2,\ldots$.
(i) Use $\Gamma(z)\Gamma(1-z) = \pi/(\sin\pi z)$ to get the last assertion.

21. Do the first integral for positive y with the substitution $t = y^2$. For the second, integrate by parts with $u = y$ and $dv = ye^{-y^2}dy$.

7.2 Asymptotic Expansions and the Method of Steepest Descent

1. There are constants R_1, B_1, R_2, and B_2 such that $|f(z)/h(z)| < B_1$ whenever $|z| \geq R_1$ and $\alpha \leq \arg z \leq \beta$, and $|g(z)/h(z)| < B_2$ whenever $|z| \geq R_2$ and $\alpha \leq \arg z \leq \beta$. Therefore, if $|z| \geq R = \max(R_1, R_2)$ and $\alpha \leq \arg z \leq \beta$, then

$$|[af(z) + bg(z)]/h(z)| \leq a|f(z)/h(z)| + b|g(z)/h(z)| \leq aB_1 + bB_2.$$

Thus, $af(z) + bg(z) = O(h(z))$.

3. If $S_n(x) = \sum_2^n (a_k/x^k)$, then $f - S_n = o(1/x^n)$, so by Proposition 7.2.3(v), $\int_x^\infty f - \int_x^\infty S_n = o(1/x^{n-1})$. That is,

$$\int_x^\infty f - \sum_2^n [a_k/(k-1)x^{k-1}] = o(1/x^{n-1}),$$

or $\int_x^\infty f \sim (a_2/x) + (a_3/2x^2) + \ldots$, as desired.

5. Use the geometric series and apply Proposition 7.2.5.

7. For the even case of (a), integrate by parts repeatedly to reduce to the case $k = 0$. Then change variables by $zy^2/2 = t^2$ and use $\int_{-\infty}^\infty e^{-t^2} dt = \sqrt{\pi}$. In part (b) either do the same thing or put $z = 2$ in part (a). For the odd cases, the integrand is an odd function of y. Thus, if the integral converges, it must be to 0. Check that it converges.

9. For $S_4(10)$ the error is ≤ 0.00024, and for $S_5(10)$ it is 0.00012. For fixed x, the error term decreases as n increases until n becomes larger than x, at which point it begins to increase again. In fact, for fixed x, our bound on the error term goes to infinity with n. For fixed n, $\lim_{x\to\infty} n!/x^{n+1} = 0$, so the error goes to zero as x increases.

11. $f(z) \sim \dfrac{\sqrt{2\pi}}{\sqrt{z}} \left(\dfrac{1}{z} - \dfrac{1 \cdot 3 \cdot 5}{3!} \dfrac{1}{z^3} + \dfrac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{5!} \dfrac{1}{z^5} - \ldots \right).$

13. The path of steepest descent is the real axis.

15. $y = \log \dfrac{1 - \sin x}{\cos x}$ 17. $f(z) \sim e^z \sqrt{\dfrac{2\pi}{z}}$ 19. $f(z) \sim e^z(1 - i)\sqrt{\dfrac{\pi}{|z|}}$

7.3 Stirling's Formula and Bessel Functions

1. Differentiate any convenient formula for $J_n(z)$ twice and substitute it in the equation. For example:

$$J_n(z) = \frac{1}{2\pi i} \left(\frac{z}{2}\right)^n \int_\gamma t^{-n-1} e^{t-(z^2/4t)} dt.$$

3. From the text, $J_0(z) = (1/\pi) \int_0^\pi \cos(z \sin\theta) d\theta$, so

$$J_0'(z) = -(1/\pi) \int_0^\pi \sin(z \sin\theta) \sin\theta \, d\theta.$$

But

$$\begin{aligned} J_1(z) &= \frac{1}{\pi} \int_0^\pi \cos(\theta - z\sin\theta) d\theta \\ &= \frac{1}{\pi} \int_0^\pi [\cos\theta \cos(z\sin\theta) + \sin\theta \sin(z\sin\theta)] d\theta. \end{aligned}$$

Use symmetry in $\pi/2$ to show that the first term vanishes.

5. From line 8 of Table 7.3.1,

$$J_0'(z) = \frac{0}{z}J_0(z) - J_1(z) = -J_1(z),$$

so

$$J_0''(z) = -J_1'(z) = -\frac{1}{z}J_1(z) + J_2(z) = J_2(z) + \frac{1}{z}J_0'(z).$$

7. By definition and Legendre's duplication formula,

$$\begin{aligned} J_{1/2}(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{(1/2)+2k}}{2^{(1/2)+2k}\Gamma(\frac{1}{2}+k+1)k!} \\ &= \sqrt{\frac{2}{z}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\sqrt{\pi}\Gamma(2k+2)} = \sqrt{\frac{2}{\pi z}} \sin z. \end{aligned}$$

9. Use the Stationary Phase Theorem 7.2.10 with $\gamma = [0,\pi], h(\theta) = -\sin\theta, g(\theta) = e^{in\theta}, \theta_0 = \pi/2$, and $f(z) = \int_0^\pi e^{-iz\sin\theta + ni\theta}d\theta$. Clearly h is real on γ with a strict minimum at $\pi/2 = \theta_0$; $h'(\theta_0) = 0, h''(\theta_0) = 1$. All conditions of the theorem are met, so

$$\begin{aligned} f(z) &\sim \frac{e^{izh(\theta_0)}\sqrt{2\pi}e^{\pi i/4}}{\sqrt{z}\sqrt{h''(\theta_0)}} g(\theta_0) \\ &= \sqrt{\frac{2\pi}{z}} e^{-iz}e^{\pi i/4}e^{in\pi/2} = \sqrt{\frac{2\pi}{z}} e^{-i(z-n\pi/2-\pi/4)}. \end{aligned}$$

11. Let $n \le 0$ and $m = -n \ge 0$. The residue in the expansion now is $1/(k-m)!$, so

$$J_n(z) = \sum_{k=m}^{\infty} \frac{(-1)^k z^{n+2k}}{2^{n+2k}} \cdot \frac{1}{(k-m)!} \cdot \frac{1}{k!}.$$

Put $j = k - m$ and $k = j + m$ to get

$$\begin{aligned} J_n(z) &= \sum_{j=0}^{\infty} \frac{(-1)^{j+m} z^{n+2(j+m)}}{2^{n+2(j+m)}} \cdot \frac{1}{(j+m-m)!} \cdot \frac{1}{(j+m)!} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^{j-n} z^{m+2j}}{2^{m+2j}} \cdot \frac{1}{j!} \cdot \frac{1}{(j+m)!} = (-1)^n \sum_{j=0}^{\infty} \frac{(-1)^j z^{m+2j}}{2^{m+2j} j!(m+j)!} \\ &= (-1)^n J_m(z) = (-1)^n J_{-n}(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k-n} z^{-n+2k}}{2^{-n+2k} k!(k-n)!}. \end{aligned}$$

Review Exercises for Chapter 7

1. 2
3. Use $z = \pi/4$ and $z = \pi/2$ in Worked Example 7.1.10 to obtain

$$\begin{aligned} \sqrt{2} &= \frac{\prod_1^\infty (1 - 1/16n^2)}{\prod_1^\infty (1 - 1/4n^2)} = \frac{\lim_{N\to\infty} \prod_1^N (1 - 1/16n^2)}{\lim_{N\to\infty} \prod_1^N (1 - 1/4n^2)} \\ &= \lim_{N\to\infty} \frac{\prod_1^N (1 - 1/16n^2)}{\prod_1^N (1 - 1/4n^2)} = \lim_{N\to\infty} \prod_1^N \frac{(4n-1)(4n+1)}{(4n-2)(4n+2)} \\ &= \left(\frac{3}{2}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{9}{10}\right)\left(\frac{11}{10}\right)\left(\frac{13}{14}\right)\cdots. \end{aligned}$$

5. (a) $\{z \text{ such that } |z| < 1\}$ (b) $\{z \mid \operatorname{Re} z > 1\}$
7. $f(z) \sim e^{i(z-\pi/4)}\sqrt{2\pi}/\sqrt{z}$
9. $\Gamma(\frac{1}{3})/3$

11. First establish, for example from the product representation for $1/\Gamma$, that $\overline{\Gamma(z)} = \Gamma(\bar{z})$. With $z = \frac{1}{2} + iy$, this gives $|\Gamma(\frac{1}{2}+iy)|^2 = \Gamma(\frac{1}{2}+iy)\cdot\Gamma(\frac{1}{2}-iy) = \Gamma(\frac{1}{2}+iy)\cdot\Gamma(1-\frac{1}{2}+iy) = \pi/\sin(\pi(\frac{1}{2})+iy) = 2\pi/(e^{-\pi y}+e^{\pi y})$ which goes to 0 as $y\to\infty$.

13. $\sqrt{\frac{2\pi}{z}}\left(1-\frac{1\cdot 3}{2}\cdot\frac{1}{z^2}+\frac{1\cdot 3\cdot 5\cdot 7}{4!}\cdot\frac{1}{z^4}-\dots\right).$

15. Apply the generalized steepest descent theorem (7.2.9) after writing

$$\begin{aligned}J_n(iy) &= \frac{1}{2\pi}\left[\int_0^{2\pi} e^{-y\sin\theta}(\cos n\theta+1)d\theta - \int_0^{2\pi} e^{-y\sin\theta}d\theta\right]\\ &- \frac{i}{2\pi}\left[\int_0^{2\pi} e^{-y\sin\theta}(\sin n\theta+1)d\theta - \int_0^{2\pi} e^{-y\sin\theta}d\theta\right]\end{aligned}$$

and using $h(\theta) = -\sin\theta, \theta_0 = 3\pi/2$, and $\gamma = [0, 2\pi]$.

17. From part (b) of Exercise 16,

$$\begin{aligned}P_n(x) &= \frac{1}{2^n n!}\cdot(2n)(2n-1)\dots(n+1)\cdot x^n + \text{ (lower-order terms)}\\ &= \frac{(2n)!}{2^n(n!)^2}x^n + \text{ (lower-order terms).}\end{aligned}$$

8.1 Basic Properties of Laplace Transforms

1. $\tilde{f}(z) = \frac{2}{z^3}+\frac{2}{z}\quad \sigma(f)=0$ 3. $\tilde{f}(z) = \frac{1}{z^2}+\frac{1}{z+1}+\frac{1}{z^2+1}\quad \sigma(f)=0$

5. $\tilde{f}(z) = \frac{1}{z}+n\cdot\frac{1}{z^2}+n(n-1)\frac{1}{z^3}+\dots+n!\frac{1}{z^{n+1}}\quad \sigma(f)=0$

7. $\tilde{f}(z) = \frac{2az}{(z^2+a^2)^2}\quad \sigma(f) = |\operatorname{Im} a|$ 9. $\tilde{f}(z) = \frac{z^2-a^2}{(z^2+a^2)^2}\quad \sigma(f) = |\operatorname{Im}(z)a|$

11. $\tilde{f}(z) = \int_0^\infty e^{-zt}\cos at\,dt = \frac{1}{2}\int_0^\infty e^{-t(z-ia)}dt + \frac{1}{2}\int_0^\infty e^{-t(z+ia)}dt$
For $\operatorname{Re}(z-ia)>0$, the first integral converges to $1/2(z-ia)$. For $\operatorname{Re}(z+ia)>0$ the second converges to $1/2(z+ia)$. (See Worked Example 8.1.11.) Thus, for $\operatorname{Re} z > |\operatorname{Im} a|, f(z)$ converges to $\frac{1}{2}[1/(z-ia)+1/(z+ia)] = z/(z^2+a^2)$.

13. For z real and positive, put $u = zt$ to obtain $\tilde{f}(z) = \int_0^\infty e^{-u}u^{(a+1)-1}/z^{a+1}du$. Since $a > -1$, this converges to $\Gamma(a+1)/z^{a+1}$, so $f(z)$ converges on the positive real axis. Lemma 8.1.8 gives convergence on the open right half plane. The identity theorem shows that $f(z) = \Gamma(a+1)/z^{a+1}$ for $\operatorname{Re} z > 0$ and Worked Example 8.1.12 shows that $\sigma(f)=0$.

15. $\tilde{f}(z) = \int_0^\infty e^{-zt}f(t)dt = \sum_{n=0}^\infty \int_{np}^{(n+1)p} e^{-zt}f(t)dt$. In the nth integral put $u = t-np$. Then

$$\tilde{f}(z) = \sum_{n=0}^\infty \int_0^p e^{-z(u+np)}f(u+np)du = \left(\sum_{n=0}^\infty e^{-zpn}\right)\int_0^p e^{-zu}f(u)du,$$

and since $|e^{-zp}|<1$, this is $[\int_0^p e^{-zt}f(t)dt]/(1-e^{-zp})$.

17. $\tilde{g}(z) = \frac{1}{z}\cdot\frac{1}{(z+1)^2+1}\qquad \tilde{f}(z) = -\frac{3z^2+4z+2}{(z^3+2z^2+2z)^2}$

19. Suppose f is of exponential order ρ and let $g(T) = \int_0^T f(s)ds$. For $\epsilon>0$, there is an A with $|f(s)|\le Ae^{(\rho+\epsilon)s}$ for every $s\ge 0$. Then

$$|g(T)| \le A\int_0^T e^{(\rho+\epsilon)}ds = A(e^{(\rho+\epsilon)T}-1)/(\rho+\epsilon).$$

If $\rho \geq 0$, then $e^{(\rho+\epsilon)T} > 1$, so $|g(T)| \leq 2Ae^{(\rho+\epsilon)T}/(\rho+\epsilon)$ and thus $\rho(g) \leq 0$. In any case, $\rho(g) \leq \max[0, \rho(f)]$. If f is piecewise continuous, then g is continuous and piecewise C^1. For $\operatorname{Re} z \geq \max[0, \rho(f)]$, the above gives $\operatorname{Re} z > \rho(g)$ and Proposition 8.1.3 gives $(g')\tilde{}(z) = z g(z) - g(0)$. But $g'(t) = f(t)$, and $g(0) = 0$, so $\tilde{f}(z) = z\tilde{g}(z)$ and thus $\tilde{g}(z) = \tilde{f}(z)/z$.

21. If $B > 0$ and $t \geq 0$, then $-e^t \leq -1$ and $|f(t)| < e^{-1} < e^{Bt}$.
If $B < 0$, let $A = \max(1/e, e^{B-B\log(-B)})$, and let $g(t) = e^t + Bt + \log A$. Then $g(0) = 1 + \log A \geq 0$. Also, $g'(t) = 0$ only at $t = \log(-B)$ and
$$g(\log(-B)) = -B + B\log(-B) + \log A \geq 0,$$
so $-e^{-t} \leq Bt + \log A$ for $t \geq 0$ and $|f(t)| \leq Ae^{Bt}$. Thus $\rho(f) = -\infty$, and therefore $\sigma(f) = -\infty$.

8.2 Complex Inversion Formula

1. (a) $\cos t$ (b) $f(t) = te^{-t}$ (c) $f(t) = [e^t + 2e^{-t/2}\cos(\sqrt{3}t/2)]/3$
3. The conditions for the Complex Inversion Formula 8.2.1, do not hold. There are no constants M and R for which $|e^{-z}/z| < M/|z|$ whenever $|z| > R$.
5. (a) $f(t) = 2e^{-2t} - e^{-t}$.
(b) $\sinh z$ has no inverse Laplace transform. If $\tilde{f}(z) = \sinh z$, let $g(t) = tf(t)$ and $h(t) = tg(t)$. Then $\tilde{g}(z) = -\cosh z$ and $\tilde{h}(z) = \sinh z = \tilde{f}(z)$. This would force $f(t) = h(t) = t^2 f(t)$, so $\tilde{f}(z) = \sinh z = 0$, which is not true.
(c) $f(t) = t^2 e^{-t}/2$.
7. $g(t) = \int_0^t [\sin(t-s)]f(s)ds$ 9. $f(t) = (6te^{-3t} - e^{-3t} + 1)/9$

8.3 Application of Laplace Transforms to Ordinary Differential Equations

1. $y(t) = (5e^{2t} + 3e^{-2t})/4$ 3. $y(t) = \begin{cases} 0 & 0 \leq t < 1 \\ \frac{1}{9}[1 - \cos(3t-3)] & t \geq 1 \end{cases}$
5. $y(t) = g_1(t) + g_2(t) + g_3(t)$ where $g_1(t) = \frac{e^{-t/2}}{\sqrt{3}}\left(3\cos\frac{\sqrt{3}}{2}t - \sin\frac{\sqrt{3}}{2}t\right)$ and
$$g_2(t) = \begin{cases} 0 & 0 \leq t < 2 \\ -\frac{e^{-(t-2)/2}}{\sqrt{3}} 2\sin\frac{\sqrt{3}}{2}(t-2) & t \geq 2 \end{cases}$$
$$g_3(t) = \begin{cases} 0 & 0 \leq t < 1 \\ -\frac{e^{-(t-1)/2}}{\sqrt{3}} 2\sin\frac{\sqrt{3}}{2}(t-1) & t \geq 1 \end{cases}$$
7. (a) $y_1(t) = (e^t + e^{-t})/2 = \cosh t$; $y_2(t) = -(e^t - e^{-t})/2 = -\sinh t$
(b) $y_1(t) = -3t$, $y_2(t) = 3[(1+t)^2 - 1]/2$
9. $y(t) = [(t+4)\sin t - t^2\cos t]/4$

Review Exercises for Chapter 8

1. $\tilde{f}(z) = e^{-z}/(z^2+1)$, $\sigma(f) = 0$ 3. $\tilde{f}(z) = \dfrac{1-e^{-z}}{z^2}$
5. $\tilde{f}(z) = \log\left(\dfrac{z}{z-1}\right)$, $\sigma = 1$ 7. $f(t) = \begin{cases} 0 & t \leq 1 \\ \sin(t-1) & t > 1 \end{cases}$
9. $f(t) = \begin{cases} -te^{-t} + e^{-t} & t < 1 \\ -te^{-t} + e^{-t} + 1 & t > 1 \end{cases}$
11. (a) $y(t) = (-15 + 23\cos 2\sqrt{2}t)/8$ (b) $y(t) = 3(1 - e^{-t})$
13. (a) $y(t) = \begin{cases} 0 & 0 \leq t < 1 \\ 1 - \cos(t-1) & t \geq 1 \end{cases}$ (b) $y(t) = 2te^{-t} + e^{-t}$

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