MATH 307 — Worksheet #5

1. Compute the integral. All curves are oriented counterclockwise.

   (a) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{z\cos z}{(z+2i)^2}\mathrm{d}z$, where C is the unit circle

   Solution: The integrand is analytic on and inside the simple, closed curve C. Therefore,

   $$\frac{1}{2\pi i}\int_C \frac{z\cos z}{(z+2i)^2}\mathrm{d}z = 0.$$

   (b) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{z\cos z}{z-2i}\mathrm{d}z$, where C is the circle $|z-i| = 2$

   Solution: The function e^{2z} is analytic on and inside C. Since $2i$ lies inside C,

   $$\frac{1}{2\pi i}\int_C \frac{z\cos z}{z-2i}\mathrm{d}z = e^{2i},$$

   by Cauchy's integral formula.

   (c) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{2e^{2z}}{z^2+1}\mathrm{d}z$, where C is the square with vertices at 1, $1+2i$, $-1+2i$, and -1.

   Solution: Do a partial fraction decomposition:

   $$\frac{2}{z^2+1} = \frac{1}{(z-i)(z+i)} = -\frac{i}{z-i} + \frac{i}{z+i}$$

   Then

   $$\frac{1}{2\pi i}\int_C \frac{2e^{2z}}{z^2+1}\mathrm{d}z = -\frac{1}{2\pi i}\int_C \frac{ie^{2z}}{z-i}\mathrm{d}z + \frac{1}{2\pi i}\int_C \frac{ie^{2z}}{z+i}\mathrm{d}z$$

   By Cauchy's integral formula,

   $$\frac{1}{2\pi i}\int_C \frac{ie^{2z}}{z-i}\mathrm{d}z = ie^{2i}.$$

   Since $\dfrac{ie^{2z}}{z+i}$ is analytic on and inside C,

   $$\int_C \frac{ie^{2z}}{z+i}\mathrm{d}z = 0,$$

   by Cauchy's theorem. Therefore,

   $$\frac{1}{2\pi i}\int_C \frac{2e^{2z}}{z^2+1}\mathrm{d}z = -ie^{2i}.$$

(d) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{2ze^z}{z^2+1}\mathrm{d}z$, where C is the circle $|z|=2$

Solution: Arguing as in the previous problem,

$$\frac{1}{2\pi i}\int_C \frac{2ze^{2z}}{z^2+1}\mathrm{d}z = -\frac{1}{2\pi i}\int_C \frac{ize^{2z}}{z-i}\mathrm{d}z + \frac{1}{2\pi i}\int_C \frac{ize^{2z}}{z+i}\mathrm{d}z$$

By two applications of Cauchy's integral formula,

$$\frac{1}{2\pi i}\int_C \frac{2ze^{2z}}{z^2+1}\mathrm{d}z = -i^2e^{2i} + i^2e^{-2i} = e^{2i} - e^{-2i}.$$

(e) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{e^{3z}}{z^3}\mathrm{d}z$, where C is the unit circle

Solution: By Cauchy's integral formula for derivatives,

$$\frac{1}{2\pi i}\int_C \frac{e^{3z}}{z^3}\mathrm{d}z = \frac{1}{2!}\left.\frac{d^2}{dz^2}\right|_{z=0} e^{3z} = \frac{9}{2}.$$

(f) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{e^{3z}}{z^3-2z^2}\mathrm{d}z$, where C is the unit circle $|z|=2$

Solution: Do a partial fraction decomposition:

$$\frac{1}{z^2} = -\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z-1}$$

Therefore,

$$\frac{1}{2\pi i}\int_C \frac{e^{3z}}{z^3-2z^2}\mathrm{d}z = -\frac{1}{2\pi i}\int_C \frac{e^{3z}}{z}\mathrm{d}z - \frac{1}{2\pi i}\int_C \frac{e^{3z}}{z^2}\mathrm{d}z + \frac{1}{2\pi i}\int_C \frac{e^{3z}}{z-1}\mathrm{d}z.$$

By three applications of Cauchy's integral formula, one involving the derivative,

$$\frac{1}{2\pi i}\int_C \frac{e^{3z}}{z^3-2z^2}\mathrm{d}z = -e^{3(0)} - 3e^{3(0)} + e^{3(1)} = -4 + e^3.$$

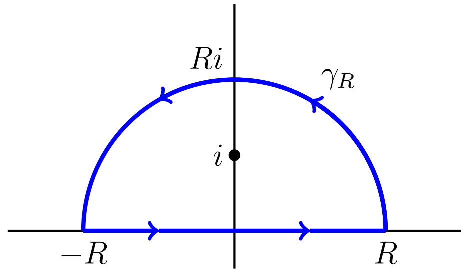
2. In this problem, we evaluate the real, improper integral

$$I := \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)^2}.$$

(a) Let $R > 0$. Use Cauchy's integral formula to compute

$$J_R := \int_{\gamma_R} \frac{\mathrm{d}z}{(z^2+1)^2},$$

where γ_R is the closed curve drawn in blue below.



Solution: Do a partial fraction decomposition:

$$\begin{aligned}\frac{1}{(z^2+1)^2} &= \frac{\mathrm{d}z}{(z-i)^2(z+i)^2} \\ &= \frac{1}{4}\left(\frac{i}{z+i} - \frac{1}{(z+i)^2} - \frac{i}{z-i} - \frac{1}{(z-i)^2}\right)\end{aligned}$$

Since $-i$ is outside γ_R,

$$\begin{aligned}J_R &= -\frac{1}{4}\left(i\int_{\gamma_R} \frac{\mathrm{d}z}{z-i} + \frac{\mathrm{d}z}{(z-i)^2}\right) \\ &= -\frac{1}{4}(i(2\pi i) + 0) \\ &= \frac{\pi}{2}\end{aligned}$$

By two applications of Cauchy's integral formula, one for the value of $f(z) = 1$ at $z = i$ and one for the derivative of $f(z) = 1$ at $z = i$.

(b) Let δ_R, be the semicircular portion of γ_R. Show that

$$|K_R| \leq \frac{\pi R}{(R^2-1)^2},$$

where

$$K_R := \int_{\delta_R} \frac{\mathrm{d}z}{(z^2+1)^2}.$$

Hint: Show that $|z^2+1| \geq R^2 - 1$ for z on γ_R, then use the ML-bound.

Solution: If z is on δ_R, then z^2 is on the circle of radius R^2 centered at 0 and $z^2 + 1$ lies outside the circle with radius $R^2 - 1$ centered at 0. Therefore, $|z^2 + 1| \geq R^2 - 1$ and

$$\frac{1}{|z^2+1|^2} \leq \frac{1}{(R^2-1)^2}.$$

Therefore, by the ML-bound,

$$\left| \int_{\delta_R} \frac{\mathrm{d}z}{(z^2+1)^2} \right| \leq \frac{\pi R}{(R^2-1)^2}.$$

(c) Briefly justify the identity

$$J_R = K_R + I_R, \qquad \text{where} \qquad I_R := \int_{-R}^{R} \frac{\mathrm{d}x}{(x^2+1)^2}.$$

Let $R \to \infty$ and evaluate I.

Solution: $J_R = K_R + I_R$ by path additivity of line integrals. Therefore,

$$\begin{aligned} I &= \lim_{R\to\infty} I_R \\ &= \lim_{R\to\infty} (J_R - K_R) \\ &= \frac{\pi}{2} - \lim_{R\to\infty} K_R \\ &= \frac{\pi}{2}, \end{aligned}$$

since $K_R \to 0$ as $R \to \infty$ by (b).