# MATH 307 — Worksheet #1

1. Express the following in the form $x + iy$:

   (a) $\dfrac{2i - 1}{5 + 6i}$

   (b) $(3 - 2i)^3$

2. Let $z = x + iy$. Express the following in the form $u(x, y) + iv(x, y)$.

   (a) $1 - z^2$

   (b) $\dfrac{1}{z^2}$

   (c) $z^3$

3. Verify the identities $\operatorname{Re}(iz) = -\operatorname{Im}(z)$ and $\operatorname{Im}(iz) = \operatorname{Re}(z)$.

4. For which $z$ does the identity $\operatorname{Re}(z^2) = \operatorname{Re}(z)^2$ hold?

5. Express $\dfrac{i^3(1 - i)}{2(1 + i\sqrt{3})}$ in the form $re^{i\theta}$ with $r > 0$ and $\theta \in [5\pi, 7\pi)$.

6. Let $a, b, c, d \in \mathbb{R}$ be such that $cd \neq 0$ and let $z \in \mathbb{C} \setminus \mathbb{R}$.

   (a) Express $\operatorname{Im} \dfrac{az + b}{cz + d}$ in terms of $\operatorname{Im} z$.

   (b) When is $\operatorname{Im} \dfrac{az + b}{cz + d}$ equal to 0?

7. Describe and sketch the set solution set.

   (a) $|z - i| = 2$

   (b) $|z + i| = |z - 1|$

   (c) $|z + 2i| + |z - 2i| = 6$

(d) $|z+3| - |z-3| = 4$

(e) $\operatorname{Im} z^2 = 4$

8. Solve the equation.

(a) $z^2 + 2z + (1-i) = 0$

(b) $z^2 + (2i-3)z + 5 - i = 0$

9. Given $x, y \in \mathbb{R}$, show that

$$a = \sqrt{\frac{x + \sqrt{x^2+y^2}}{2}} \quad b = \operatorname{sign}(y)\sqrt{\frac{-x + \sqrt{x^2+y^2}}{2}}.$$

is the unique solution to $(a+ib)^2 = x + iy$ with $a \geq 0$.

10. Prove the identities:

$$\cos z = \cosh(iz), \quad \cos(iz) = \cosh z, \quad \sin z = -i\sinh(iz), \quad \sin(iz) = i\sinh z$$

11. Prove the identities:

$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y,$$
$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$$

12. Prove the identity:

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

Deduce that

$$\lim_{y\to\infty} |\cos z| = \infty \quad \text{and} \quad \lim_{y\to\infty} |\sin z| = \infty$$

13. Solve the equation $|\cot z| = 1$.

14. Solve the equations:

(a) $e^{\bar{z}} = \overline{e^z}$

(b) $\cos(i\bar{z}) = \overline{\cos(iz)}$