MATH 307 — Quiz #1

Question:	1	2	3	Total
Points:	10	6	5	20
Score:				

1.

(a) (2 points) Compute $|z|^2$, where $z = \dfrac{5i}{(1-i)(2-i)(3-i)}$.

Solution:

$$\bar{z}z = |z|^2 = \frac{|5i|^2}{|1-i|^2|2-i|^2|3-i|^2} = \frac{25}{2 \cdot 5 \cdot 10} = \frac{1}{4}$$

(b) (2 points) Express $\operatorname{Log}\left(\frac{1}{4} - \frac{1}{4}i\right)$ in the form $x + iy$. Here, Log denotes the principal branch of the logarithm.

(c) (2 points) Express $z = (\sqrt{3} - i)^6$ in the form $re^{i\theta}$.

Solution: Notice that $\sqrt{3} - i = 2e^{-i\pi/6}$. Therefore,

$$z = (\sqrt{3} - i)^6 = (2e^{-i\pi/6})^6 = 2^6 e^{-6i\pi/6} = 64e^{-i\pi}.$$

(d) (2 points) Let $w = e^{\bar{z}^2/2}$. Express $\operatorname{Re}(w)$ and $\operatorname{Im}(w)$ in terms of $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$.

Solution: We compute:

$$\begin{aligned}
e^{\bar{z}^2/2} &= e^{(x-iy)^2/2} \\
&= e^{(x^2-y^2-2xyi)/2} \\
&= e^{(x^2+y^2)/2} e^{-xyi} \\
&= e^{(x^2+y^2)/2} \big(\cos(-xy) + i\sin(-xy)\big) \\
&= e^{(x^2+y^2)/2} \cos(xy) - ie^{(x^2+y^2)/2} \sin(xy).
\end{aligned}$$

Thus,

$$\operatorname{Re}(w) = e^{(x^2+y^2)/2} \cos(xy) \quad \text{and} \quad \operatorname{Im}(w) = e^{(x^2+y^2)/2} \sin(xy).$$

(e) (2 points) Find all values of $(-8i)^{1/3}$. Express them in the form $x + iy$.

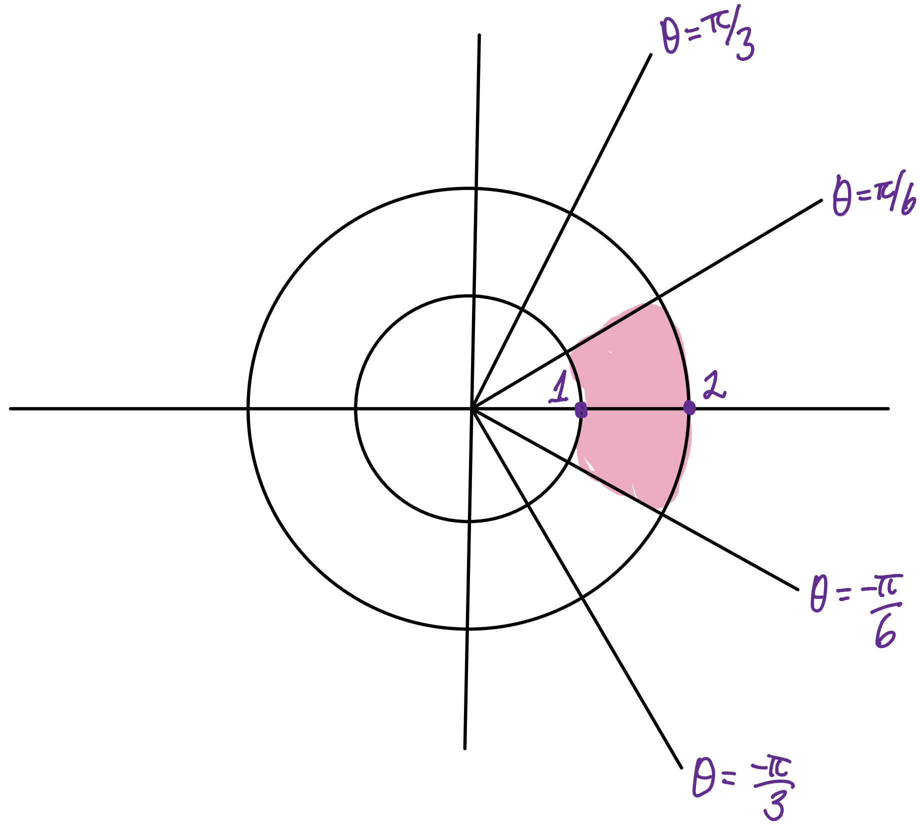
Solution:

$$\begin{aligned}(-8i)^{1/3} = e^{\frac{1}{3}\log(-8i)} &= e^{1/3(\log|-8i| + i\arg(-8i))} \\ &= e^{\frac{1}{3}\log 8 + \frac{i}{3}(-\frac{\pi}{2}+2k\pi)} \\ &= e^2 e^{i(-\frac{\pi}{6}+\frac{k\pi}{3})} \\ &= e^2\cos\left(-\frac{\pi}{6}+\frac{k\pi}{3}\right) + ie^2\sin\left(-\frac{\pi}{6}+\frac{k\pi}{3}\right), \quad k \in \mathbb{Z}.\end{aligned}$$

2. Sketch the sets in the complex plane, each on its own set of axes. Label radii, angles, etc. so that your meaning is unambiguous

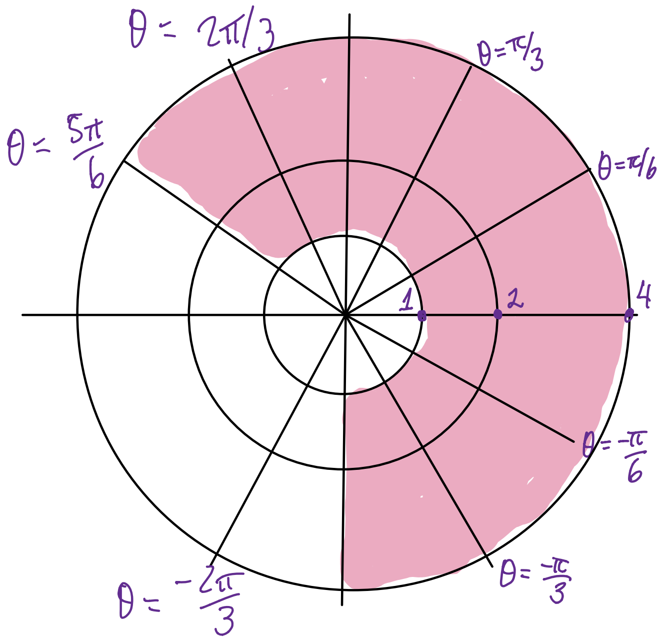
(a) (2 points) $A = \left\{z = re^{i\theta} : 1 \le r \le 2 \text{ and } -\frac{\pi}{3} \le \theta \le \frac{\pi}{3}\right\}$

Solution:



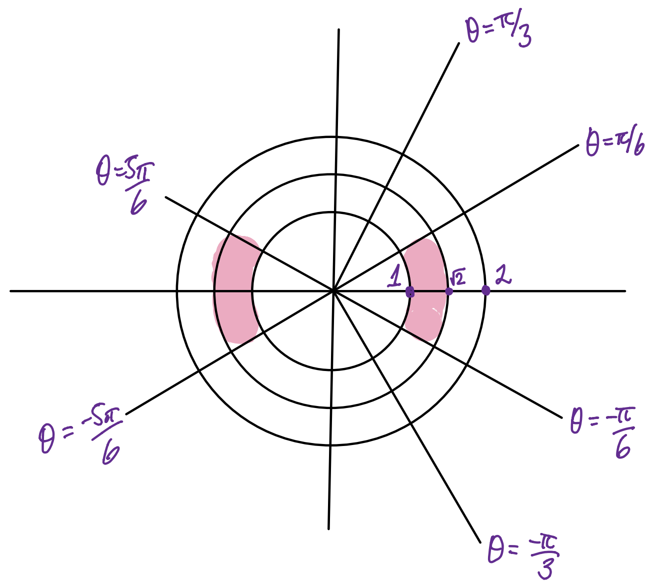
(b) (2 points) $B = \{e^{i\pi/6}z^2 : z \in A\}$

Solution:



(c) (2 points) $C = \{z : z^2 \in A\}$

Solution:



3. (5 points) Let $\sqrt{\cdot}$ be the branch of the square root defined by

$$\sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2} \qquad \text{for} \qquad \theta \in [\pi, 3\pi).$$

For which z does $\sqrt{z^2} = z$ hold?

Hint: Write $z = re^{i\psi}$ with $\psi \in [-\frac{\pi}{2}, 3\pi)$. When computing z^2, consider the cases $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2})$ and $\psi \in [\frac{\pi}{2}, \frac{3\pi}{2})$ separately. In each case, identify a k such that $2\psi + 2\pi k$ belongs to $[\pi, 3\pi)$.

Solution: Suppose $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2})$. Then $2\psi \in [-\pi, \pi)$, so $\theta := \psi + 2\pi \in [\pi, 3\pi)$ and

$$\sqrt{z^2} = \sqrt{r^2 e^{i(2\psi)}} = \sqrt{r^2 e^{i(2\psi+2\pi)}} = re^{i(\psi+\pi)} = re^{i\psi}e^{i\pi} = -z.$$

If, on the other handm, $\psi \in [\frac{\pi}{2}, \frac{3\pi}{2})$, then $2\psi \in [\pi, \frac{3\pi}{2})$ and

$$\sqrt{z^2} = \sqrt{r^2 e^{i(2\psi)}} = re^{i(\psi+\pi)} = re^{i\psi} = z.$$

Thus, $\sqrt{z^2} = z$ when $\psi \in [\frac{\pi}{2}, \frac{3\pi}{2})$, i.e., when $\mathrm{Re}(z) < 0$ or when $\mathrm{Re}(z) = 0$ and $\mathrm{Im}(z) \geq 0$.