MATH 307 — Worksheet #4

1. Suppose $f(z)$ is analytic on an open set U. Show that $\overline{f(\bar{z})}$ is analytic on $\bar{U}$, where
$$\bar{U} = \{\bar{z} : z \in U\}.$$

2. Suppose v is a harmonic conjugate of u. Show that $-u$ is a harmonic conjugate of v.

3. Prove the identities
$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}} \quad \text{and} \quad \overline{\frac{\partial f}{\partial \bar{z}}} = \frac{\partial \bar{f}}{\partial z}.$$
Style points if you deduce one from the other rather than arguing twice.

4. Which of the following identities are true? Prove or give a counterexample.
   (a) $\left| \int_\gamma f(z) dz \right| = \int_\gamma |f(z)| dz$

   (b) $\left| \int_\gamma f(z) dz \right| = \int_\gamma |f(z)| |dz|$

   (c) $\operatorname{Re} \int_\gamma f(z) dz = \int_\gamma \operatorname{Re}(f(z))\, dz$

   (d) $\operatorname{Im} \int_\gamma f(z) dz = \int_\gamma \operatorname{Im}(f(z))\, dz$

5. Compute the line integral. All curves are traversed counterclockwise.
   (a) $\int_{|z|=1} \bar{z}^n \, dz$

   (b) $\int_{|z|=1} z^m \bar{z}^n \, dz$

   (c) $\int_\gamma x \, dz$, γ is the arc of the parabola $y = x^2$ from $(0,0)$ to $(2,2)$.

   (d) $\int_\gamma e^z \, dz$, $\gamma(t) = e^{it}$, $t \in [0, \pi]$.

6. Explain why
$$\int_\gamma \frac{dz}{z} = \int_\gamma i \frac{-y\, dx + x\, dy}{x^2 + y^2}$$
for all closed curves γ not passing through 0.