

MATH 307 — Quiz #1

Instructions: You have 50 minutes to solve all three multipart problems. All answers should be exact – no decimal approximations. No calculators, notes, or other aids. Hand in only your solution booklet. There are 21 points available, plus 1 bonus point, but I'll record the grade out of 20.

1.

- (a) (2 points) Compute $|z|^2$, where $z = \frac{5i}{(1-i)(2-i)(3-i)}$.

Solution:

$$\bar{z}z = |z|^2 = \frac{|5i|^2}{|1-i|^2|2-i|^2|3-i|^2} = \frac{25}{2 \cdot 5 \cdot 10} = \frac{1}{4}$$

- (b) (2 points) Express $\text{Log} \left(\frac{1}{4} - \frac{1}{4}i \right)$ in the form $x + iy$. Here, Log denotes the **principal branch** of the logarithm.

Solution:

$$\begin{aligned} \text{Log} \left(\frac{1}{4} - \frac{1}{4}i \right) &= \log \left| \frac{1}{4} - \frac{1}{4}i \right| + i \text{Arg} \left(\frac{1}{4} - \frac{1}{4}i \right) \\ &= \log \sqrt{\frac{1}{4^2} + \frac{1}{4^2}} + i \left(-\frac{\pi}{4} \right) \\ &= -\frac{3}{2} \log 2 - i \frac{\pi}{4} \end{aligned}$$

- (c) (2 points) Express $z = (\sqrt{3} - i)^6$ in the form $re^{i\theta}$.

Solution: Notice that $\sqrt{3} - i = 2e^{-i\pi/6}$. Therefore,

$$z = (\sqrt{3} - i)^6 = (2e^{-i\pi/6})^6 = 2^6 e^{-6i\pi/6} = 64e^{-i\pi}.$$

- (d) (2 points) Let $w = e^{\bar{z}^2/2}$. Express $\text{Re}(w)$ and $\text{Im}(w)$ in terms of $x = \text{Re}(z)$ and $y = \text{Im}(z)$.

Solution: We compute:

$$\begin{aligned} e^{\bar{z}^2/2} &= e^{(x-iy)^2/2} \\ &= e^{(x^2-y^2-2xyi)/2} \\ &= e^{(x^2+y^2)/2} e^{-xyi} \\ &= e^{(x^2+y^2)/2} (\cos(-xy) + i \sin(-xy)) \\ &= e^{(x^2+y^2)/2} \cos(xy) - i e^{(x^2+y^2)/2} \sin(xy). \end{aligned}$$

Thus,

$$\operatorname{Re}(w) = e^{(x^2+y^2)/2} \cos(xy) \quad \text{and} \quad \operatorname{Im}(w) = e^{(x^2+y^2)/2} \sin(xy).$$

- (e) (2 points) Find all values of $(-8i)^{1/3}$. Express them in the form $x + iy$.

Solution:

$$\begin{aligned} (-8i)^{1/3} &= e^{\frac{1}{3} \log(-8i)} = e^{1/3(\log|-8i| + i \arg(-8i))} \\ &= e^{\frac{1}{3} \log 8 + \frac{i}{3}(-\frac{\pi}{2} + 2k\pi)} \\ &= e^2 e^{i(-\frac{\pi}{6} + \frac{k\pi}{3})} \\ &= e^2 \cos\left(-\frac{\pi}{6} + \frac{k\pi}{3}\right) + i e^2 \sin\left(-\frac{\pi}{6} + \frac{k\pi}{3}\right), \quad k \in \mathbb{Z}. \end{aligned}$$

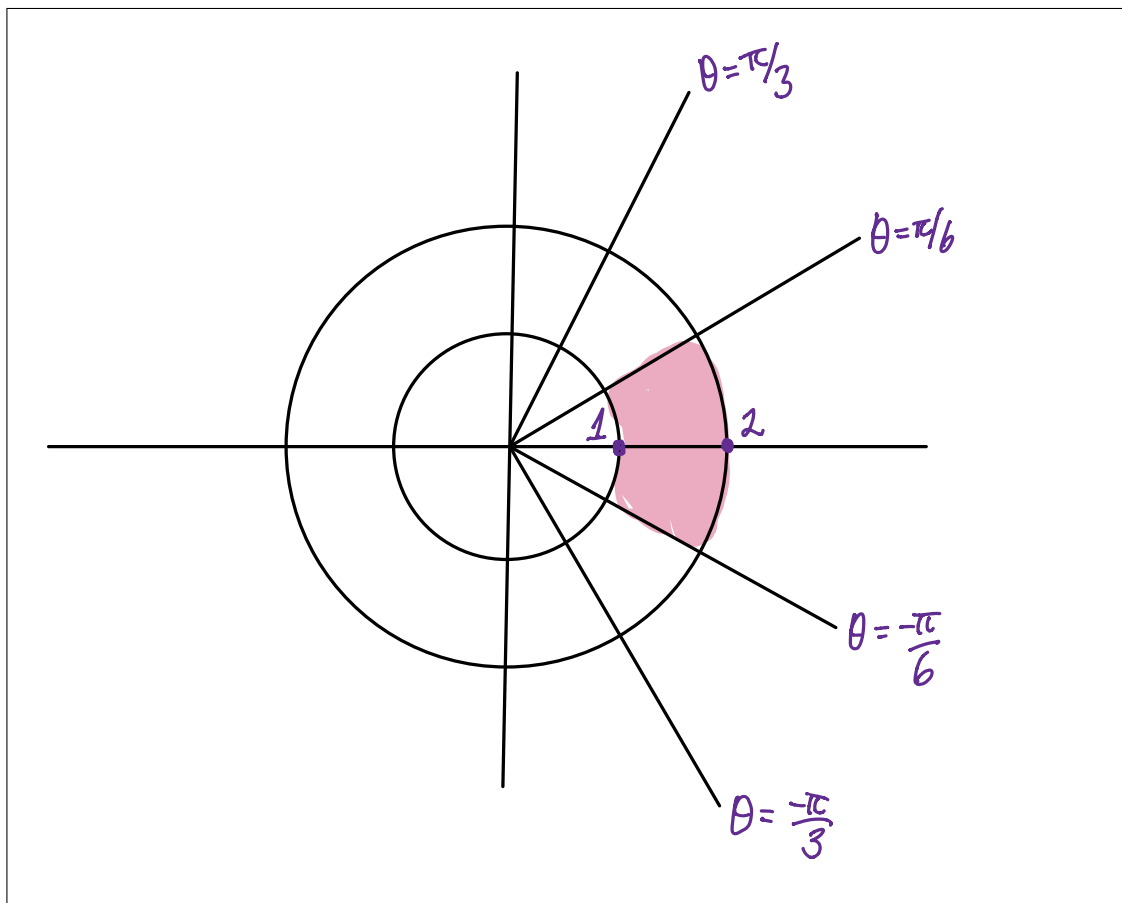
2. Sketch the sets in the complex plane, each on its own set of axes. Label points, radii, angles, etc. so that your meaning is unambiguous.

- (a) (2 points) $A = \{z : |z| < |z + 1|\}$

Solution: The set of points whose distance from 0 is less than their distance from -1 is the half-plane $\operatorname{Re} z > -\frac{1}{2}$.

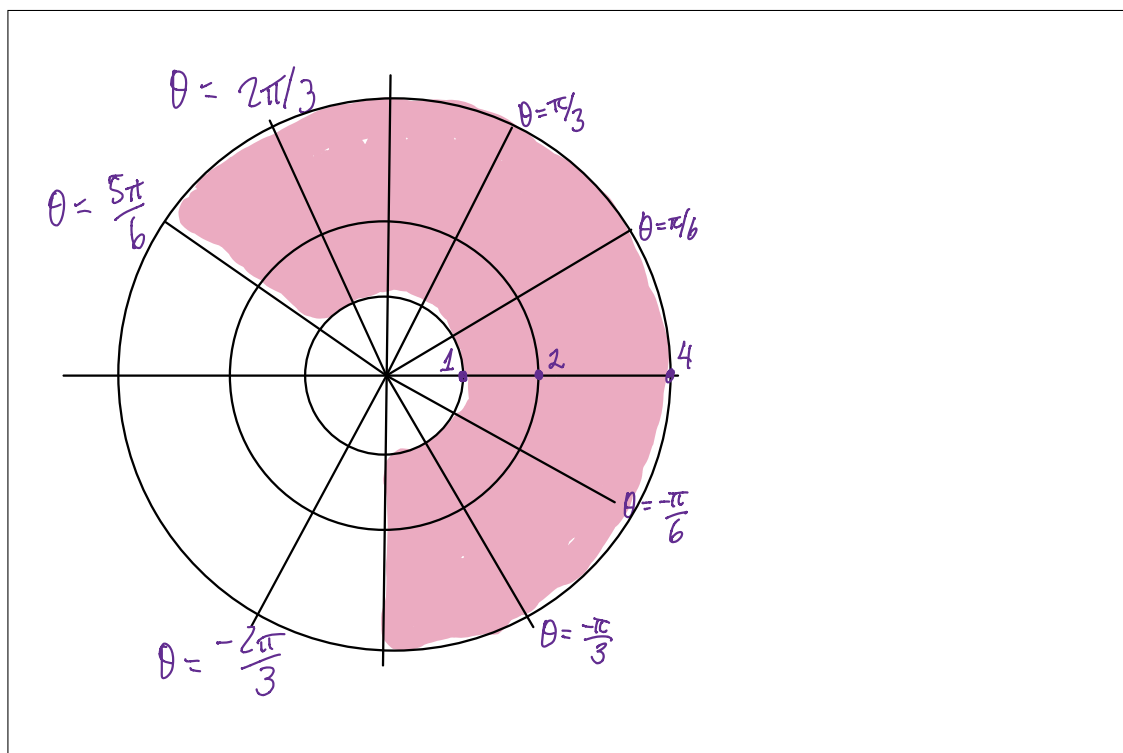
- (b) (2 points) $B = \left\{ z = r e^{i\theta} : 1 \leq r \leq 2 \text{ and } -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3} \right\}$

Solution:

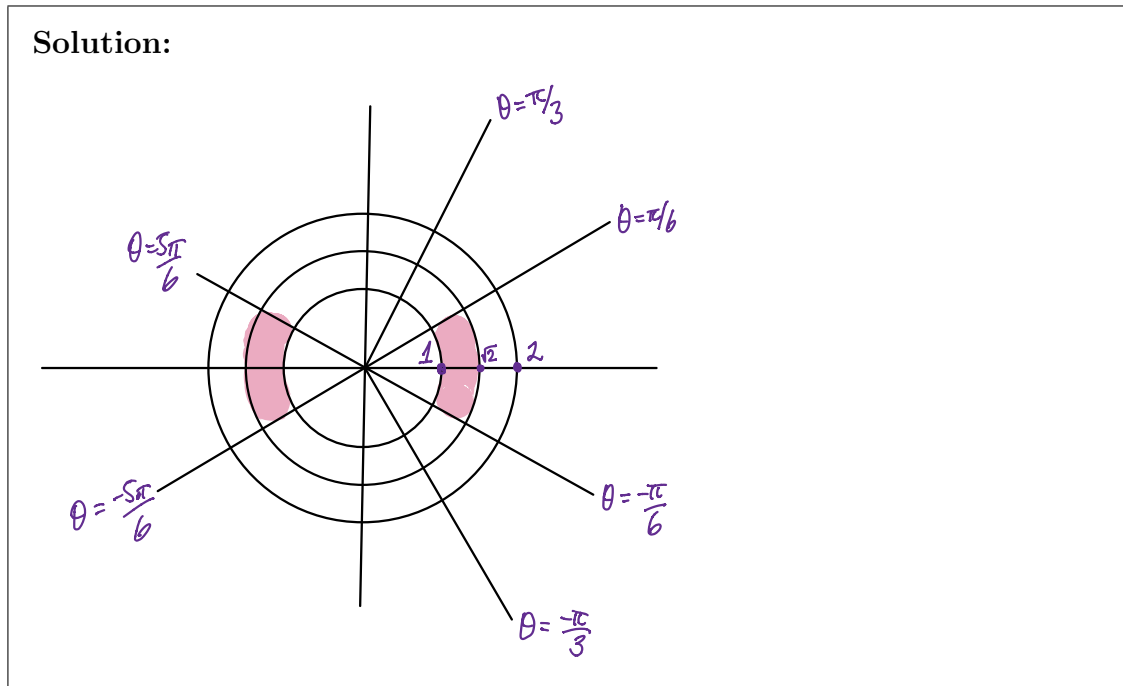


(c) (2 points) $C = \{e^{i\pi/6}z^2 : z \in B\}$

Solution:



(d) (1 point) ★★ BONUS ★★ $D = \{z : z^2 \in B\}$ ★★BONUS★★



3. (5 points) Let $\sqrt{\cdot}$ be the branch of the square root defined by

$$\sqrt{re^{i\theta}} = \sqrt{r} e^{i\theta/2} \quad \text{for} \quad \theta \in [\pi, 3\pi).$$

For which z does $\sqrt{z^2} = z$ hold?

Hint: Write $z = re^{i\psi}$ with $\psi \in [-\frac{\pi}{2}, 3\pi)$. When computing z^2 , consider the cases $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2})$ and $\psi \in [\frac{\pi}{2}, \frac{3\pi}{2})$ separately. In each case, identify a k such that $2\psi + 2\pi k$ belongs to $[\pi, 3\pi)$.

Solution: Suppose $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2})$. Then $2\psi \in [-\pi, \pi)$, so $\theta := \psi + 2\pi \in [\pi, 3\pi)$ and

$$\sqrt{z^2} = \sqrt{r^2 e^{i(2\psi)}} = \sqrt{r^2 e^{i(2\psi+2\pi)}} = re^{i(\psi+\pi)} = re^{i\psi} e^{i\pi} = -z.$$

If, on the other hand, $\psi \in [\frac{\pi}{2}, \frac{3\pi}{2})$, then $2\psi \in [\pi, 3\pi)$ and

$$\sqrt{z^2} = \sqrt{r^2 e^{i(2\psi)}} = re^{i(\psi+\pi)} = re^{i\psi} = z.$$

Thus, $\sqrt{z^2} = z$ when $\psi \in [\frac{\pi}{2}, \frac{3\pi}{2})$, i.e., when $\operatorname{Re}(z) < 0$ or when $\operatorname{Re}(z) = 0$ and $\operatorname{Im}(z) \geq 0$.