MATH 307 — Worksheet #3

1. Compute the derivatives:

   (a) $\dfrac{\partial}{\partial \bar{z}} |z|^2$

   Solution:
   $$\frac{\partial}{\partial \bar{z}} |z|^2 = \frac{\partial}{\partial \bar{z}} z\bar{z} = z$$

   (b) $\dfrac{\partial}{\partial \bar{z}} y$

   Solution:
   $$\frac{\partial}{\partial \bar{z}} y = \frac{\partial}{\partial \bar{z}} \frac{1}{2i}(z - \bar{z}) = -\frac{1}{2i} = \frac{i}{2}$$

   (c) $\dfrac{\partial}{\partial \bar{z}} \dfrac{1 - |z|}{1 + |z|}$

   Solution:
   $$\begin{aligned}
   \frac{\partial}{\partial \bar{z}} \frac{1 - |z|}{1 + |z|} &= \frac{\partial}{\partial \bar{z}} \frac{1 - (z\bar{z})^{1/2}}{1 + (z\bar{z})^{1/2}} \\
   &= \frac{\frac{1}{2}(z\bar{z})^{-1/2} z (1 + (z\bar{z})^{1/2}) - (1 - (z\bar{z})^{1/2}) \frac{1}{2}(z\bar{z})^{-1/2} z}{(1 + (z\bar{z})^{1/2})^2} \\
   &= \frac{\frac{1}{2}|z|^{-1} z (1 + |z| - (1 - |z|))}{(1 + |z|)^2} \\
   &= \frac{\frac{1}{2}|z|^{-1} z (2|z|)}{(1 + (z\bar{z})^{1/2})^2} \\
   &= \frac{z}{(1 + |z|)^2}
   \end{aligned}$$

2. Being uncomfortable with complex square roots, you're skeptical of the calculation
   $$\frac{\partial}{\partial \bar{z}} |z| = \frac{\partial}{\partial \bar{z}} \sqrt{z\bar{z}} = \frac{\partial}{\partial \bar{z}} \sqrt{z}\sqrt{\bar{z}} = \frac{\sqrt{z}}{2\sqrt{\bar{z}}} = \frac{\sqrt{z}}{2\sqrt{\bar{z}}} \cdot \frac{\sqrt{z}}{\sqrt{z}} = \frac{z}{2|z|}.$$
   Confirm its result by switching to Cartesian coordinates, i.e., evaluate
   $$\frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) \sqrt{x^2 + y^2}.$$

Solution:

$$\frac{1}{2}\left(\frac{\partial}{\partial x}\sqrt{x^2+y^2}+i\frac{\partial}{\partial y}\sqrt{x^2+y^2}\right)=\frac{1}{2}\left(\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}+i\frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}}\right)$$
$$=\frac{1}{2}\frac{x+iy}{\sqrt{x^2+y^2}}$$
$$=\frac{z}{2|z|}$$

3. Let f be a complex-valued function on an open subset U of $\mathbb{C}$ and let $z \in U$. Explain the difference between the statements "f is differentiable at z" and "f is analytic at z".

Solution: The statement "f is differentiable at z" means that

$$\lim_{w\to z}\frac{f(w)-f(z)}{w-z}$$

exists, the derivative being the limiting value.

The statement "f is analytic at z" means that there is an $\varepsilon > 0$ such that f is differentiable at w for all w with $|w-x| < \varepsilon$. In other words, it means that that f is differentiable on a *neighborhood* of z.

4. At which z is $f'(z)$ differentiable? analytic?

(a) $f(z) = x^3 + iy^3$

Solution: Write $u = x^3$, $v = y^3$. Then

$$u_x = 3x^2, \quad u_y = 0, \quad v_x = 0, \quad v_y = 3y^2.$$

The Cauchy-Riemann equations are satisfied only when $3x^2 = 3y^2$, i.e., when $x = \pm y$. In particular, f is not analytic at $z = x + iy$ when $x \neq \pm y$. On the other hand, since u and v have continuous partials on $\mathbb{C}$ and satisfy the Cauchy-Riemann equations along $y = \pm x$, if follows that f is differentiable along these lines. It's not analytic along these lines, however, because they have empty interior, i.e., they don't contain any disk.

(b) $f(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2 y - 3y)$

> **Solution:** Write $u = x^3$, $v = y^3$. Then
>
> $$u_x = 3x^2 + 3y^2 - 3, \quad u_y = 6xy, \quad v_x = 6xy, \quad v_y = 3y^2 + 3x^2 - 3.$$
>
> The Cauchy-Riemann equations are satisfied only when $3x^2 = 3y^2$, i.e., when $x = 0$ or $y = \frac{1}{2}$. In particular, f is not analytic at $z = x + iy$ when $x \neq 0$ and $y \neq 1/2$. On the other hand, since u and v have continuous partials on $\mathbb{C}$ and satisfy the Cauchy-Riemann equations along $x = 0$ and $y = \frac{1}{2}$, if follows that f is differentiable these lines. It's not analytic along these lines, however, because they have empty interion, i.e., they don't contain any disk.

(c) $f(z) = \dfrac{z^2 + \bar{z}^2}{2} + iz\bar{z}$

> **Solution:**
> $$\frac{\partial f}{\partial \bar{z}} = \bar{z} + iz = (x - y) + i(x + y) = 0$$
> if and only if $z = 0$. Thus, $f(z)$ is not analytic at $z \neq 0$. However, f is differentiable at $z = 0$ by the above equation together with the fact that the partials of $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are polynomials in x and y and, hence, continuous.

5. (∗) Find a $f = u + iv$ such that:

    1. u and v have continuous partials on $\mathbb{C}$,
    2. f is nowhere analytic,
    3. f is differentiable on the unit circle, $|z| = 1$.

6. Solve the equation $\dfrac{\partial u}{\partial \bar{z}} = 2x$.

> **Solution:** Use the formula $x = \frac{1}{2}z + \bar{z}$ to express the equation as
>
> $$\frac{\partial u}{\partial \bar{z}} = z + \bar{z}.$$
>
> Now integrate with respect to $\bar{z}$:
>
> $$u = z\bar{z} + \frac{1}{2}\bar{z}^2.$$

7. Show that $\dfrac{\partial^2 f}{\partial \bar{z}^2} = 0$ if and only if $f(z) = \bar{z}g(z) + h(z)$, where g and h are analytic.

Solution: Functions of the form $f(z) = g(z) + ih(z)$ are clearly solutions of the differential equation. Conversely, let f be a solution. We'll show it must have the required form. Since

$$\frac{\partial}{\partial \bar{z}} \frac{\partial f}{\partial \bar{z}} = 0,$$

$\partial f / \partial \bar{z}$ does not depend on $\bar{z}$, i.e.,

$$\frac{\partial f}{\partial \bar{z}} = g(z)$$

for some g. Moreover, g is analytic as

$$\frac{\partial g}{\partial \bar{z}} = \frac{\partial^2 f}{\partial \bar{z}^2} = 0.$$

Integrate this equation with respect to $\bar{z}$ to get

$$f(z) = \bar{z}g(z) + h(z)$$

for some h. (The function h is the "constant" of integration, which may depend on z as $\bar{z}$ is the variable of integration.) I claim that h is analytic. To see this, differentiate the equation $f(z) = \bar{z}g(z) + h(z)$ with respect to $\bar{z}$ and use the product rule:

$$\frac{\partial f}{\partial \bar{z}} = g(z) + \bar{z}\frac{\partial h}{\partial \bar{z}}$$

But $\partial f / \partial \bar{z} = g(z)$, so we get

$$\bar{z}\frac{\partial h}{\partial \bar{z}} = 0$$

Cancelling the $\bar{z}$, we conclude that $h(z)$ is analytic.

Remark: I've swept something under the rug, here. The above argument shows only that $\partial h / \partial \bar{z} = 0$ for $z \neq 0$. (You can't cancel zeros!) Can you figure out how to deduce the analyticity of $h(z)$ at $z = 0$?

8. Suppose $f(z)$ is analytic on an open set U. Show that $\overline{f(\bar{z})}$ is analytic on $\bar{U}$, where

$$\bar{U} = \{\bar{z} : z \in U\}.$$

9. Suppose v is a harmonic conjugate of u. Show that $-u$ is a harmonic conjugate of v.

10. Prove the identities

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}} \quad \text{and} \quad \overline{\frac{\partial f}{\partial \bar{z}}} = \frac{\partial \bar{f}}{\partial z}.$$

Style points if you deduce one from the other rather than arguing twice.

11. Which of the following identities are true? Prove or give a counterexample.

Solution: They're all false. See below for counterexamples.

(a) $\left|\int_\gamma f(z)dz\right| = \int_\gamma |f(z)|dz$

Solution: Let $f(z) = z^{-1}$. Then

$$\int_\gamma z^{-1}\, dz = 2\pi i,$$

as we've seen many times. On the other hand, $|z^{-1}| = 1$ on γ. Therefore,

$$\int_\gamma |z^{-1}|\, dz = \int_0^{2\pi} 1\, dz = 0.$$

(b) $\left|\int_\gamma f(z)dz\right| = \int_\gamma |f(z)||dz|$

Solution: Take $f(z) = 1$. Then

$$\int_\gamma 1\, dz = 0,$$

but

$$\int_\gamma |1||dz| = \text{length}(\gamma) = 2\pi.$$

Remark: Using Riemann sums and the triangle inequality, you can show that

$$\left|\int_\gamma f(z)dz\right| = \int_\gamma |f(z)||dz|$$

for all f. Is this inequality sharp?

(c) $\text{Re}\int_\gamma f(z)dz = \int_\gamma \text{Re}(f(z))\, dz$

Solution: Take $f(z) = z$. Then

$$\mathrm{Re} \int_\gamma z\, dz = \int_\gamma z\, dz = 0.$$

Noting that

$$\int_\gamma \bar{z}\, dz = \int_0^{2\pi} \overline{e^{i\theta}} i e^{i\theta}\, d\theta = i \int_0^{2\pi} 1\, d\theta = 2\pi i,$$

it follows that

$$\int_\gamma \mathrm{Re}(z)\, dz = \int_\gamma \frac{z+\bar{z}}{2}\, dz = \frac{1}{2}\int_\gamma z\, dz + \frac{1}{2}\int_\gamma \bar{z}\, dz = \frac{0+2\pi i}{2} = \pi i.$$

(d) $\mathrm{Im} \int_\gamma f(z) dz = \int_\gamma \mathrm{Im}(f(z))\, dz$

Solution: Take $f(z) = z$ as above. Then, as above,

$$\mathrm{Im} \int_\gamma z\, dz = \int_\gamma z\, dz = 0$$

and

$$\int_\gamma \mathrm{Im}(z)\, dz = \int_\gamma \frac{z-\bar{z}}{2i}\, dz = \frac{1}{2i}\int_\gamma z\, dz - \frac{1}{2i}\int_\gamma \bar{z}\, dz = \frac{0-2\pi i}{2i} = -\pi.$$

12. Compute the line integral. All curves are traversed counterclockwise.

(a) $\int_{|z|=1} \bar{z}^n\, dz$

Solution:

$$\begin{aligned}\int_{|z|=1} \bar{z}^n\, dz &= \int_0^{2\pi} (\overline{e^{i\theta}})^n i e^{i\theta}\, d\theta \\ &= i\int_0^{2\pi} e^{i(1-n)\theta}\, d\theta \\ &= \begin{cases} 2\pi i & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

(b) $\displaystyle\int_{|z|=1} z^m \bar{z}^n \, dz$

Solution:

$$\begin{aligned}\int_{|z|=1} z^m \bar{z}^n \, dz &= \int_0^{2\pi} e^{im\theta}(\overline{e^{i\theta}})^n ie^{i\theta} \, d\theta \\ &= i\int_0^{2\pi} e^{i(m-n+1)\theta} \, d\theta \\ &= \begin{cases} 2\pi i & \text{if } n = m+1, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

(c) $\displaystyle\int_\gamma x \, dz$, γ is the arc of the parabola $y = x^2$ from $(0,0)$ to $(2,2)$.

Solution: We take our parametrization to be $\gamma(t) = t + t^2 i$, $0 \leq t \leq 1$. Then

$$\begin{aligned}\int_{|z|=1} x \, dz &= \int_0^{2\pi} t(1+2it) \, dt \\ &= \int_0^{2\pi} t \, dt + 2i\int_0^{2\pi} t^2 \, dt \\ &= 2\pi^2 + \frac{16\pi^3 i}{3}\end{aligned}$$

(d) $\displaystyle\int_\gamma e^z \, dz$, $\gamma(t) = e^{it}$, $t \in [0, \pi]$.

Solution: By FTC4LI,

$$\int_\gamma e^z \, dz = e^{\gamma(1)} - e^{\gamma(0)} = e^{-1} - e.$$

13. Explain why

$$\int_\gamma \frac{dz}{z} = \int_\gamma i\frac{-y\,dx + x\,dy}{x^2+y^2}$$

for all closed curves γ not passing through 0.

Solution:

$$\begin{aligned}\frac{dz}{z} &= \frac{\bar{z}\,dz}{|z|^2}\\ &= \frac{(x-iy)(dx+i\,dy)}{x^2+y^2}\\ &= \frac{x\,dx+y\,dy}{x^2+y^2}+i\frac{-y\,dx+x\,dy}{x^2+y^2}\end{aligned}$$

Thus,

$$\int_\gamma \frac{dz}{z} = \int_\gamma i\frac{-y\,dx+x\,dy}{x^2+y^2}$$

for all closed curves γ not passing through 0 if and only if

$$\int_\gamma \frac{x\,dx+y\,dy}{x^2+y^2} = 0 \tag{$*$}$$

for all closed curves γ not passing through 0. Identity $(*)$ follows from

$$\frac{x\,dx+y\,dy}{x^2+y^2} = d\log\sqrt{x^2+y^2},$$

the fact that γ is closed, and the FTC4LI.

Quiz 2

14. For which z is $f(z) = |z|^2$ differentiable? analytic?

Solution: In Cartesian coordinates, $f(z) = u(x,y) + iv(x,y)$ with $u(x,y) = x^2+y^2$ and $v(x,y) = 0$. We compute partials:

$$u_x = 2x, \quad u_y = 2y, \quad v_x = 0, \quad v_y = 0$$

The Cauchy-Riemann equations are satisfied only at $z = 0$. In particular, $f(z)$ is neither differentiable nor analytic at z for $z \neq 0$. Since the partials of u and v are continuous at $z = 0$ in addition to the Cauchy-Riemann equations being satisfied there, $f(z)$ is differentiable at $z = 0$. It is not analytic at $z = 0$, however, because any disk around $z = 0$ contains points at which $f(z)$ is not differentiable.

15. Evaluate the line integrals over the curve γ given by $\gamma(t) = e^{it}$, $t \in [0, \frac{\pi}{2}]$.

(a) $\displaystyle\int_\gamma \frac{dz}{z-\frac{1}{2}}$, γ is the line segment from 1 to $\dfrac{i}{2}$.

Solution: Write $\operatorname{Log} z$ for the principal branch of the logarithm, so that $\operatorname{Log} z$ is analytic on $\mathbb{C}\backslash(-\infty,0)$. $\operatorname{Log}(z-\frac{1}{2})$ an antiderivative of $(z-\frac{1}{2})^{-1}$ on $\mathbb{C}\backslash(-\infty,\frac{1}{2})$, a region containing γ. Therefore, by FTC4LI,

$$\begin{aligned}
\int_\gamma \frac{dz}{z} &= \operatorname{Log}\left(\gamma\left(\frac{\pi}{2}\right)-\frac{1}{2}\right) - \operatorname{Log}\left(\gamma(0)-\frac{1}{2}\right) \\
&= \operatorname{Log}\left(\frac{i}{2}-\frac{1}{2}\right) - \operatorname{Log}\left(1-\frac{1}{2}\right) \\
&= \log\left|\frac{i}{2}-\frac{1}{2}\right| + i\arg\left(\frac{i}{2}-\frac{1}{2}\right) - \log\left|\frac{1}{2}\right| - i\arg\frac{1}{2} \\
&= \log\frac{1}{\sqrt{2}} + \frac{3\pi i}{4} - \log\frac{1}{2} \\
&= \frac{\log 2}{2} + \frac{3\pi i}{4}
\end{aligned}$$

(b) $\displaystyle\int_\gamma \frac{dz}{\bar{z}}$, $\quad \gamma(t)=e^{it}$, $\quad t\in[0,\frac{\pi}{2}]$

Solution:

$$\begin{aligned}
\int_\gamma \frac{dz}{\bar{z}} &= \int_0^{\frac{\pi}{2}} \frac{1}{\overline{e^{it}}}\, ie^{it}\,dt \\
&= i\int_0^{\frac{\pi}{2}} e^{2it}\,dt \\
&= \frac{i}{2i}e^{2it}\Big|_0^{\frac{\pi}{2}} \\
&= \frac{1}{2}\left(e^{i\pi}-1\right) \\
&= -1
\end{aligned}$$