MATH 307 — Worksheet #4

1. Suppose $f(z)$ is analytic on an open set U. Show that $\overline{f(\bar{z})}$ is analytic on $\bar{U}$, where

$$\bar{U} = \{\bar{z} : z \in U\}.$$

> **Solution:** Let $u \in U$. Let's show that $g(z) := \overline{f(\bar{z})}$. Let $v \in \bar{U}$. Then
>
> $$\begin{aligned}
\lim_{w \to v} \frac{g(w) - g(v)}{w - v} &= \lim_{z \to u} \frac{g(\bar{z}) - g(\bar{u})}{\bar{z} - \bar{u}} && (u := \bar{v},\ z := \bar{w}) \\
&= \lim_{z \to u} \frac{\overline{f(z)} - \overline{f(u)}}{\bar{z} - \bar{u}} \\
&= \lim_{z \to u} \overline{\frac{f(z) - f(u)}{z - u}} \\
&= \overline{\lim_{z \to u} \frac{f(z) - f(u)}{z - u}} && \text{(continuity of conjugation)} \\
&= \overline{f'(u)} \\
&= \overline{f'(\bar{v})}
\end{aligned}$$
>
> Thus, g is differentiable at v and
>
> $$g'(v) = \overline{f'(\bar{v})}.$$

2. Suppose v is a harmonic conjugate of u. Show that $-u$ is a harmonic conjugate of v.

> **Solution:**
>
> Let u be harmonic and let v be a harmonic conjugate of u. Then $f := u + iv$ is analytic. But then $if = -v + iu$ is analytic, too. Therefore, u is a harmonic conjugate of $-v$.

3. Prove the identities

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}} \quad \text{and} \quad \overline{\frac{\partial f}{\partial \bar{z}}} = \frac{\partial \bar{f}}{\partial z}.$$

Style points if you deduce one from the other rather than arguing twice.

Solution:

$$\begin{aligned}
\frac{\partial \bar{f}}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\bar{f} \\
&= \overline{\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f} \\
&= \overline{\frac{\partial f}{\partial z}}
\end{aligned}$$

This proves the first identity.

Letting $\bar{f}$ play the role of f in the first identity gives

$$\overline{\frac{\partial \bar{f}}{\partial z}} = \frac{\partial f}{\partial \bar{z}}$$

Conjugating this gives

$$\frac{\partial \bar{f}}{\partial z} = \overline{\frac{\partial f}{\partial \bar{z}}},$$

which is the second identity.

4. Which of the following identities are true? Prove or give a counterexample.

Solution: They're all false. See below for counterexamples.

(a) $\left|\int_\gamma f(z)dz\right| = \int_\gamma |f(z)|dz$

Solution: Let $f(z) = z^{-1}$. Then

$$\int_\gamma z^{-1}\,dz = 2\pi i,$$

as we've seen many times. On the other hand, $|z^{-1}| = 1$ on γ. Therefore,

$$\int_\gamma |z^{-1}|\,dz = \int_0^{2\pi} 1\,dz = 0.$$

(b) $\left|\int_\gamma f(z)dz\right| = \int_\gamma |f(z)||dz|$

Solution: Take $f(z) = 1$. Then

$$\int_\gamma 1\, dz = 0,$$

but

$$\int_\gamma |1||dz| = \text{length}(\gamma) = 2\pi.$$

Remark: Using Riemann sums and the triangle inequality, you can show that

$$\left|\int_\gamma f(z)dz\right| = \int_\gamma |f(z)||dz|$$

for all f. Is this inequality sharp?

(c) $\text{Re} \int_\gamma f(z)dz = \int_\gamma \text{Re}(f(z))\, dz$

Solution: Take $f(z) = z$. Then

$$\text{Re} \int_\gamma z\, dz = \int_\gamma z\, dz = 0.$$

Noting that

$$\int_\gamma \bar{z}\, dz = \int_0^{2\pi} \overline{e^{i\theta}} ie^{i\theta}\, d\theta = i\int_0^{2\pi} 1\, d\theta = 2\pi i,$$

it follows that

$$\int_\gamma \text{Re}(z)\, dz = \int_\gamma \frac{z+\bar{z}}{2}\, dz = \frac{1}{2}\int_\gamma z\, dz + \frac{1}{2}\int_\gamma \bar{z}\, dz = \frac{0+2\pi i}{2} = \pi i.$$

(d) $\text{Im} \int_\gamma f(z)dz = \int_\gamma \text{Im}(f(z))\, dz$

Solution: Take $f(z) = z$ as above. Then, as above,

$$\text{Im} \int_\gamma z\, dz = \int_\gamma z\, dz = 0$$

and

$$\int_\gamma \text{Im}(z)\, dz = \int_\gamma \frac{z-\bar{z}}{2i}\, dz = \frac{1}{2i}\int_\gamma z\, dz - \frac{1}{2i}\int_\gamma \bar{z}\, dz = \frac{0-2\pi i}{2i} = -\pi.$$

5. Compute the line integral. All curves are traversed counterclockwise.

(a) $\displaystyle\int_{|z|=1} \bar{z}^n \, dz$

Solution:

$$\begin{aligned}
\int_{|z|=1} \bar{z}^n \, dz &= \int_0^{2\pi} (\overline{e^{i\theta}})^n i e^{i\theta} \, d\theta \\
&= i \int_0^{2\pi} e^{i(1-n)\theta} \, d\theta \\
&= \begin{cases} 2\pi i & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

(b) $\displaystyle\int_{|z|=1} z^m \bar{z}^n \, dz$

Solution:

$$\begin{aligned}
\int_{|z|=1} z^m \bar{z}^n \, dz &= \int_0^{2\pi} e^{im\theta} (\overline{e^{i\theta}})^n i e^{i\theta} \, d\theta \\
&= i \int_0^{2\pi} e^{i(m-n+1)\theta} \, d\theta \\
&= \begin{cases} 2\pi i & \text{if } n = m+1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

(c) $\displaystyle\int_\gamma x \, dz$, γ is the arc of the parabola $y = x^2$ from $(0,0)$ to $(2,2)$.

Solution: We take our parametrization to be $\gamma(t) = t + t^2 i$, $0 \le t \le 1$. Then

$$\begin{aligned}
\int_{|z|=1} x \, dz &= \int_0^{2\pi} t(1 + 2it) \, dt \\
&= \int_0^{2\pi} t \, dt + 2i \int_0^{2\pi} t^2 \, dt \\
&= 2\pi^2 + \frac{16\pi^3 i}{3}
\end{aligned}$$

(d) $\displaystyle\int_\gamma e^z\,dz$, $\gamma(t) = e^{it}$, $t \in [0, \pi]$.

Solution: By FTC4LI,

$$\int_\gamma e^z\,dz = e^{\gamma(1)} - e^{\gamma(0)} = e^{-1} - e.$$

6. Explain why

$$\int_\gamma \frac{dz}{z} = \int_\gamma i\frac{-y\,dx + x\,dy}{x^2+y^2}$$

for all closed curves γ not passing through 0.

Solution:

$$\begin{aligned}\frac{dz}{z} &= \frac{\bar{z}\,dz}{|z|^2}\\ &= \frac{(x-iy)(dx+i\,dy)}{x^2+y^2}\\ &= \frac{x\,dx+y\,dy}{x^2+y^2} + i\frac{-y\,dx+x\,dy}{x^2+y^2}\end{aligned}$$

Thus,

$$\int_\gamma \frac{dz}{z} = \int_\gamma i\frac{-y\,dx + x\,dy}{x^2+y^2}$$

for all closed curves γ not passing through 0 if and only if

$$\int_\gamma \frac{x\,dx+y\,dy}{x^2+y^2} = 0 \tag{$*$}$$

for all closed curves γ not passing through 0. Identity $(*)$ follows from

$$\frac{x\,dx+y\,dy}{x^2+y^2} = d\log\sqrt{x^2+y^2},$$

the fact that γ is closed, and the FTC4LI.