List of univariate probability distributions

Maximilian L. Greißl

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Contents

1 Discrete distributions

This list gives an overview of the most popular univariate probability distributions and their most important properties.
In the following, let K and X denote appropriately distributed discrete and continous random variables, respectively.

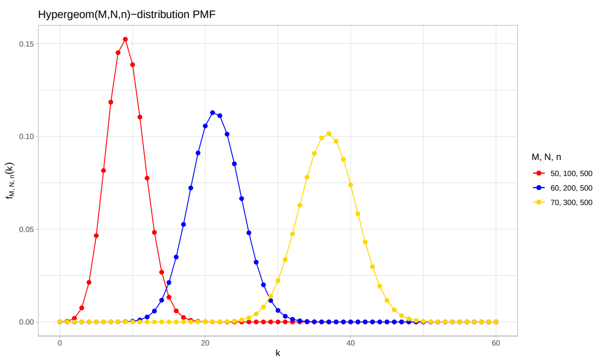
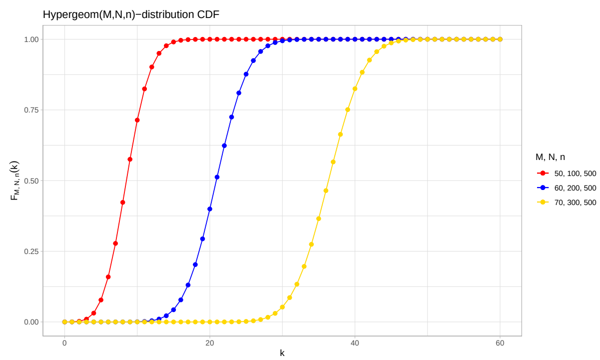
1.1 Discrete uniform(n) distribution

Mass function	Distribution function
$f(k) = \frac{1}{n}$	$P(\{K \leq k\}) = \frac{\lvert\{i : k_i \leq k\}\rvert}{n}$
Mean	**Variance**
$E[K] = \frac{1}{n}\sum_{i=0}^{n} k_i$	$\mathrm{Var}(K) = \frac{1}{n}\left(\sum_{i=0}^{n} k_i^2 - \frac{1}{n}\left(\sum_{i=0}^{n} k_i\right)^2\right)$

1.2 Bernoulli(p) distribution

Mass function		Distribution function
$f(k) = \begin{cases} p & \text{for } k = 1 \\ 1-p & \text{for } k = 0 \end{cases}$ $\mathrm{supp}\, f(k) = \{0, 1\}$		$P(\{K \leq k\}) = \begin{cases} 0 & \text{for } k < 0 \\ 1-p & \text{for } 0 \leq k < 1 \\ 1 & \text{for } k \geq 1 \end{cases}$
Mean	**Variance**	**Fisher Information**
$E[K] = p$	$\mathrm{Var}(K) = p(1-p)$	$\mathcal{I}(\theta) = \frac{1}{\theta(1-\theta)}$
Moment-generating function		**Characteristic function**
$M_X(t) = (1 - p + p\exp(t))$		$\varphi_X(t) = (1 - p + p\exp(it))$

1.3 Hypergeometric(M,N,n) distribution

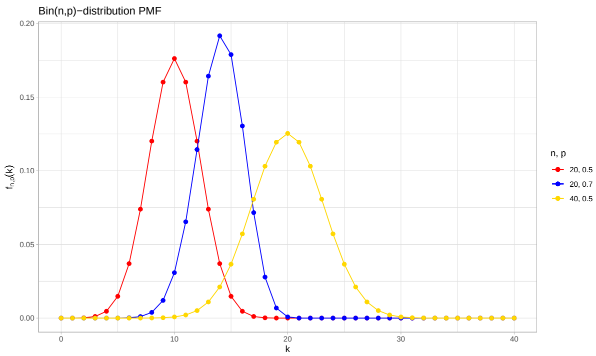
Mass function	Distribution function
$f\left(k\right)=\dfrac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$ 	$P\left(\left\{K\leq k\right\}\right)=\displaystyle\sum_{i=\max\{0,n-N\}}^{\lfloor k\rfloor}\frac{\binom{M}{i}\binom{N}{ni}}{\binom{M+N}{n}}$ 
Mean	**Variance**
$E\left[K\right]=n\dfrac{M}{N}$	$\mathrm{Var}\left(K\right)=n\dfrac{M}{N}\left(1-\dfrac{M}{N}\right)\dfrac{N-n}{N-1}$

1.4 Binomial(n,p) distribution

Mass function

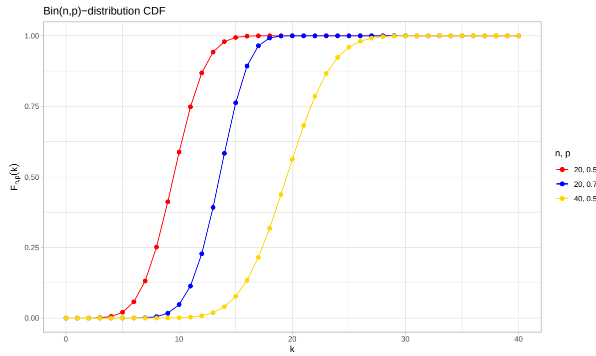
$$f(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\text{supp}\, f(k) = \{0, .., n\}$$



Distribution function

$$P(\{K \leq k\}) = \sum_{i=0}^{\lfloor k \rfloor} \binom{n}{i} p^i (1-p)^{n-i}$$



Mean

$$E[K] = np$$

Variance

$$\text{Var}(K) = np(1-p)$$

Fisher Information

$$\mathcal{I}(\theta) = \frac{n}{\theta(1-\theta)} \text{ for fixed } n$$

Moment-generating function

$$M_X(t) = (1 - p + p\exp(t))^n$$

Characteristic function

$$\varphi_X(t) = (1 - p + p\exp(it))^n$$

Maximum Likelihood Estimator

$$\hat{p} = \frac{1}{n} \sum_{i=1}^{n} k_i$$

- $Bin(m,p) + Bin(n,p) = Bin(m+n,p)$ [1]

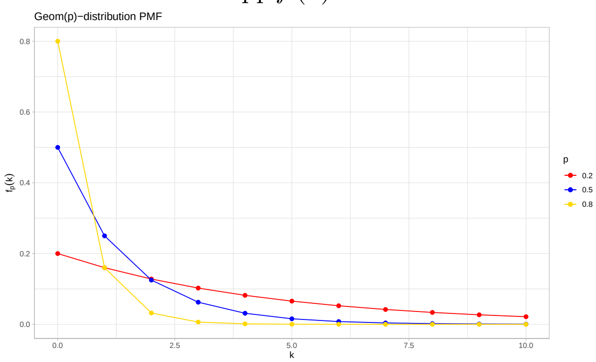
[1] Where e.g. $Bin(n,p)$ represents a random variable with $Bin(n,p)$-distribution, not the distribution itself.

1.5 Geometric(p) distribution

Mass function

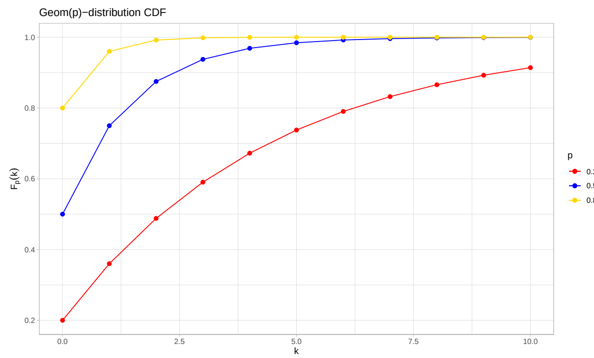
$$f(k) = p(1-p)^{k-1}$$

$$\operatorname{supp} f(k) = \mathbb{N}$$



Distribution function

$$P(\{K \leq k\}) = 1 - (1-p)^{\lfloor k \rfloor}$$



Mean

$$E[K] = \frac{1}{p}$$

Variance

$$\operatorname{Var}(K) = \frac{1}{p^2} - \frac{1}{p}$$

Fisher Information

$$\mathcal{I}(\theta) = \frac{1}{\theta^2} + \frac{1}{(1-\theta)^2 \theta}$$

Moment-generating function

$$M_X(t) = \frac{p \exp(t)}{1-(1-p)\exp(t)}$$

Characteristic function

$$\varphi_X(t) = \frac{p \exp(it)}{1-(1-p)\exp(it)}$$

Maximum Likelihood Estimator

$$\hat{p} = \frac{n}{\sum_{i=1}^{n} k_i}$$

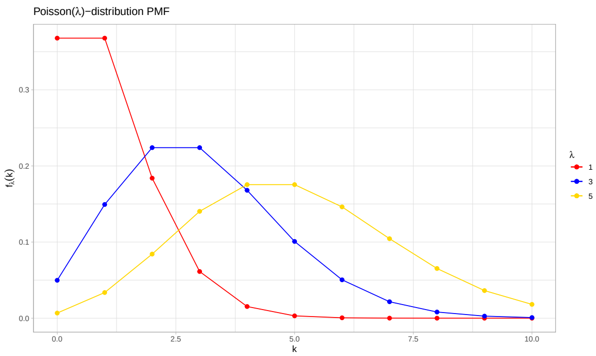
- $P(K \geq k+t \mid K \geq k) = P(K \geq t)$ (Memorylessness)

1.6 Poisson(λ) distribution

Mass function

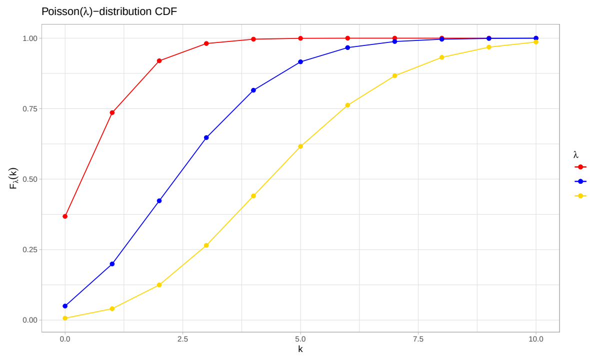
$$f(k) = \frac{\lambda^k}{k!} \exp(-\lambda)$$

$$\operatorname{supp} f(k) = \mathbb{N}_0$$



Distribution function

$$P(\{K \leq k\}) = \sum_{i=0}^{\lfloor k \rfloor} \frac{\lambda^i}{i!} \exp(-\lambda)$$



Mean

$$E[K] = \lambda$$

Variance

$$\operatorname{Var}(K) = \lambda$$

Fisher Information

$$\mathcal{I}(\theta) = \frac{1}{\theta}$$

Moment-generating function

$$M_X(t) = \exp(\lambda \exp(t) - 1)$$

Characteristic function

$$\varphi_X(t) = \exp(\lambda \exp(it) - 1)$$

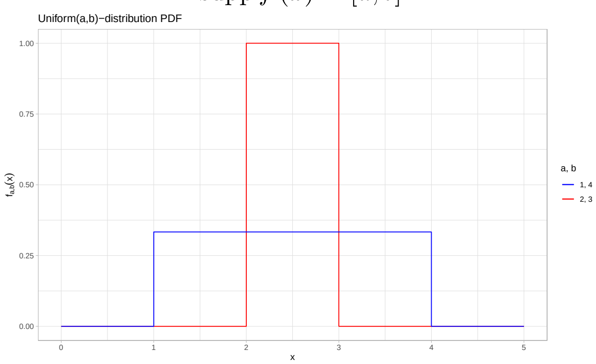
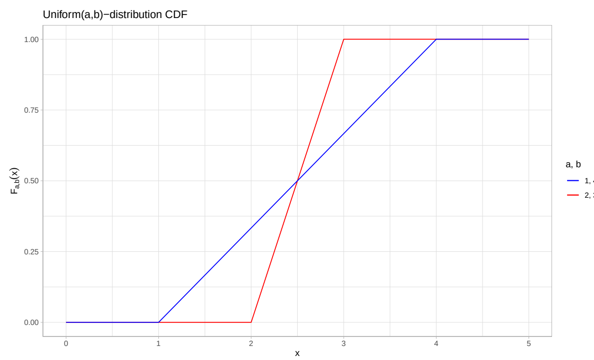
Maximum Likelihood Estimator

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} k_i$$

- $P(\alpha) + P(\beta) = P(\alpha + \beta)$
- $Bin(n, p) = P(np)$ as $n \to \infty$, $p \to 0$

2 Continous distributions

2.1 Continous uniform(a,b) distribution

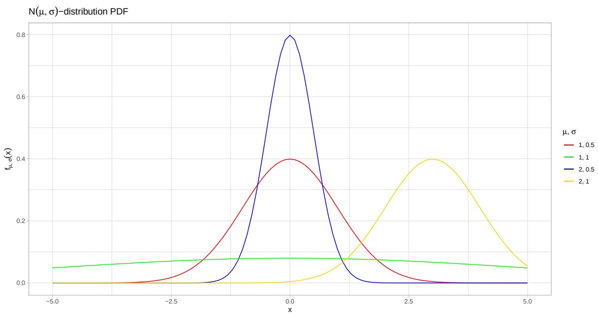
Density $f(x) = \begin{cases} \frac{1}{b-a} & \text{for } k \in [a,b] \\ 0 & \text{otherwise} \end{cases}$ $\operatorname{supp} f(x) = [a,b]$ 	Distribution function $F(x) = \begin{cases} 0 & \text{for } x \leq a \\ \frac{x-a}{b-a} & \text{for } a < x \leq b \\ 1 & \text{for } x > b \end{cases}$ 
Mean $E[X] = \frac{1}{2}(a+b)$	Variance $\operatorname{Var}(X) = \frac{1}{12}(b-a)^2$
Moment-generating function $M_X(t) = \frac{\exp(tb) - 1}{tb}$ for $a = 0$	Characteristic function $\varphi_X(t) = \frac{\exp(itb) - 1}{itb}$ for $a = 0$
Maximum Likelihood Estimator $\hat{b} = max\{x_1, ..., x_n\}$ for $a = 0$	

2.2 Normal(μ,σ^2) distribution

Density

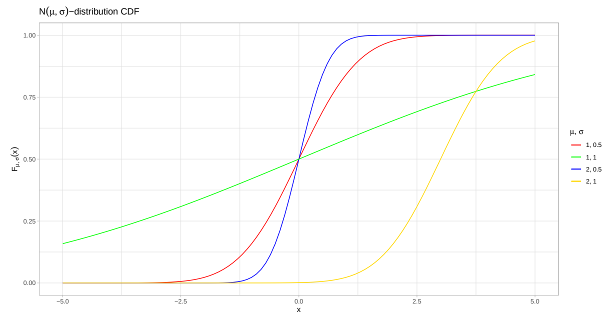
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\operatorname{supp} f(x) = \mathbb{R}$$



Distribution function

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) dt$$



Mean

$$E[X] = \mu$$

Variance

$$\operatorname{Var}(X) = \sigma^2$$

Fisher Information

$$\mathcal{I}(\mu, \sigma^2) = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix}$$

Moment-generating function

$$M_X(t) = \exp\left(t\mu + \frac{1}{2}\sigma_2 t^2\right)$$

Characteristic function

$$\varphi_X(t) = \exp\left(it\mu + \frac{1}{2}\sigma_2 t^2\right)$$

Order	Raw moment
2	$\mu^2 + \sigma_2$
3	$\mu_3 + 3\mu\sigma_2$
4	$\mu_4 + 6\mu_2\sigma_2 + 3\mu_4$
5	$\mu_5 + 10\mu_3\sigma_2 + 15\mu\sigma_4$

Maximum Likelihood Estimator

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x_n})^2$$

Confidence intervals with confidence level $(1-\alpha)$

$$\mu \in \left[\hat{\mu} - \tau_{n-1}\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{\hat{\sigma}^2}{n}}, \hat{\mu} + \tau_{n-1}\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{\hat{\sigma}^2}{n}}\right] \quad \sigma^2 \in \left[\frac{(n-1)\hat{\sigma}^2}{\chi^2_{n-1}\left(1-\frac{\alpha}{2}\right)}, \frac{(n-1)\hat{\sigma}^2}{\chi^2_{n-1}\left(\frac{\alpha}{2}\right)}\right]$$

- $N(\mu_1, \sigma_1^2) + N(\mu_2, \sigma_2^2) = N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

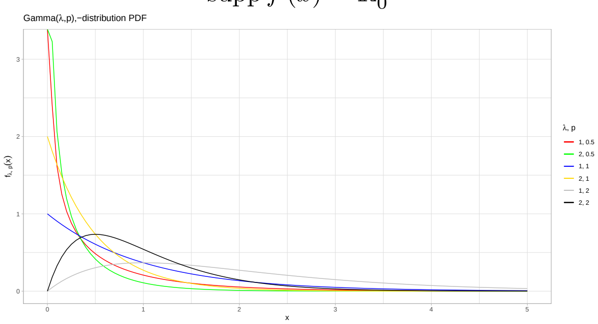
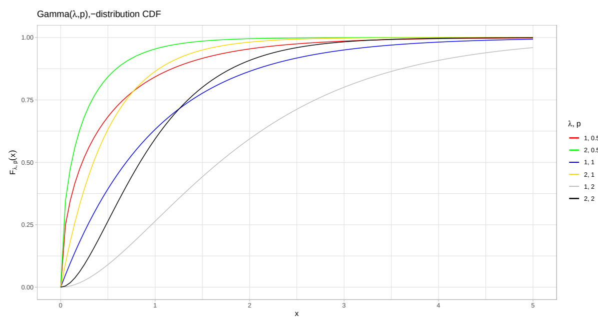
2.3 Gamma(λ,p) distribution

The Gamma-function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t)\, dt.$$

It has the following useful properties

- $\Gamma(x+1) = x\Gamma(x)\, \forall x > 0$,
- $\Gamma(n+1) = n! \forall n \in \mathbb{N}$,
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

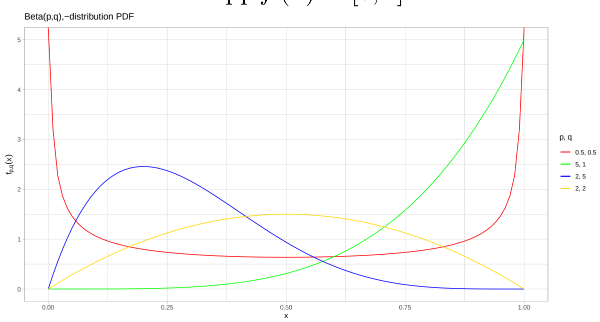
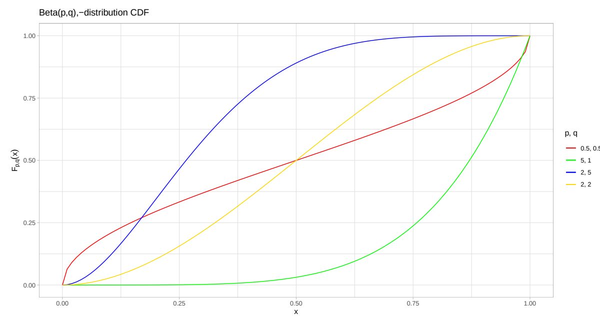
<table>
<tr><td colspan="3">Density
$$f(x) = \frac{\lambda^p}{\Gamma(p)} x^{p-1} \exp(-\lambda x)$$
$$\operatorname{supp} f(x) = \mathbb{R}_0^+$$

</td><td colspan="3">Distribution function
$$F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{\lambda^p}{\Gamma(p)} \int_0^x t^{p-1} \exp(-\lambda t)\, dt & \text{for } x > 0 \end{cases}$$

</td></tr>
<tr><td colspan="2">Mean
$$E[X] = \frac{p}{\lambda}$$</td><td colspan="2">Variance
$$\operatorname{Var}(X) = \frac{p}{\lambda^2}$$</td><td colspan="2">Fisher Information
$$\mathcal{I}(\lambda, p) = \begin{pmatrix} \frac{d^2}{dp^2} \log(\Gamma(p)) & -\frac{1}{\lambda} \\ -\frac{1}{\lambda} & \frac{p}{\lambda^2} \end{pmatrix}$$</td></tr>
<tr><td colspan="3">Moment-generating function
$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-p}$$</td><td colspan="3">Characteristic function
$$\varphi_X(t) = \left(1 - \frac{it}{\lambda}\right)^{-p}$$</td></tr>
<tr><td colspan="6">Maximum Likelihood Estimator
$$\hat{\lambda} = \frac{p}{\sum_{i=1}^n x_i}$$</td></tr>
</table>

- $\Gamma(\lambda, p) + \Gamma(\lambda, q) = \Gamma(\lambda, p + q)$
- $\Gamma(\lambda, 1) = Exp(\lambda)$

2.4 Beta(p,q) distribution

The Beta-function $B:(0,\infty)\times(0,\infty)\to\mathbb{R}^2$ is defined by

$$B\left(q,p\right)=\frac{\Gamma\left(q\right)\Gamma\left(p\right)}{\Gamma\left(q+p\right)}.$$

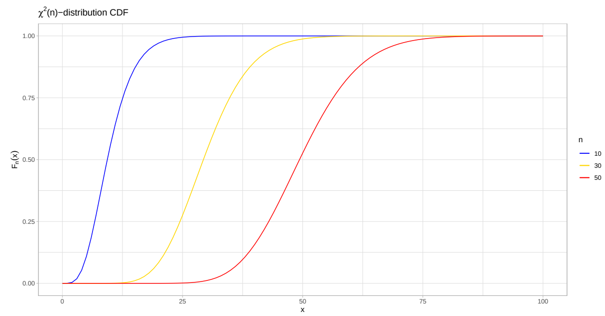
Density	Distribution function
$f\left(x\right)=\frac{1}{B\left(p,q\right)}x^{p-1}\left(1-x\right)^{q-1}$ $\operatorname{supp} f\left(x\right)=[0,1]$ 	$F\left(x\right)=\begin{cases}0 & \text{for } x<0\\ \frac{1}{B(p,q)}\int_0^x t^{p-1}\left(1-t\right)^{q-1} & \text{for } 0\leq x\leq 1\\ 1 & \text{for } x>1\end{cases}$ 
Mean	Variance
$E\left[X\right]=\frac{p}{p+q}$	$\operatorname{Var}\left(X\right)=\frac{pq}{\left(p+q+1\right)\left(p+q\right)^2}$
Fisher Information	
$\mathcal{I}\left(p,q\right)=\begin{pmatrix}\frac{\partial^2\log(\Gamma(p))}{\partial p^2}-\frac{\partial^2\log(\Gamma(p+q))}{\partial(p+q)^2} & \frac{\partial^2\log(\Gamma(q))}{\partial q^2}-\frac{\partial^2\log(\Gamma(p+q))}{\partial(p+q)^2}\\ \frac{\partial^2\log(\Gamma(q))}{\partial q^2}-\frac{\partial^2\log(\Gamma(p+q))}{\partial(p+q)^2} & -\frac{\partial^2\log(\Gamma(p+q))}{\partial(p+q)^2}\end{pmatrix}$	
Moment-generating function	Characteristic function
$M_X\left(t\right)=1+\sum_{i=1}^{\infty}\left(\prod_{j=0}^{i-1}\frac{p+j}{p+q+j}\right)\frac{t^i}{i!}$	—[2]

- $B\left(p,q\right)=\frac{\Gamma\left(\lambda,p\right)}{\Gamma\left(\lambda,p\right)+\Gamma\left(\lambda,q\right)}$

[2]The function has no closed-form expression.

2.5 Chi-square(n) distribution

<table>
<tr>
<td>Density
$$f\left(x\right)=\frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}x^{\frac{n}{2}-1}\exp\left(-\frac{x}{2}\right)$$
$$\operatorname{supp} f\left(x\right)=\mathbb{R}_0^+$$

</td>
<td>Distribution function
$$F\left(x\right)=\begin{cases}0 & \text{for } x\leq 0\\ 1-\frac{\gamma\left(\frac{n}{2},\frac{x}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} & \text{for } x>0\end{cases}$$

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</tr>
<tr>
<td>Mean
$$E\left[X\right]=n$$</td>
<td>Variance
$$\operatorname{Var}\left(X\right)=2n$$</td>
</tr>
<tr>
<td>Moment-generating function
$$M_X\left(t\right)=\left(1-2t\right)^{-\frac{n}{2}}$$</td>
<td>Characteristic function
$$\varphi_X\left(t\right)=\left(1-2it\right)^{-\frac{n}{2}}$$</td>
</tr>
</table>

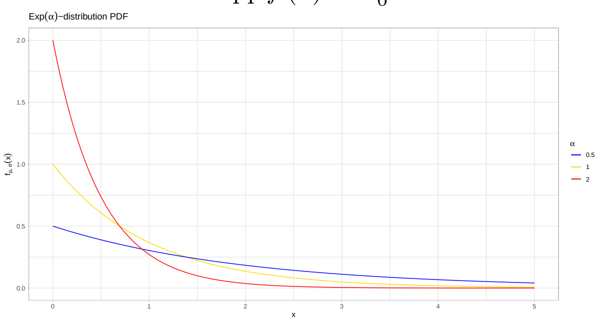
- $\chi^2\left(k\right)+\chi^2\left(l\right)=\chi^2\left(k+l\right)$
- $\chi^2\left(n\right)=N\left(n,2n\right)$ as $n\to\infty$
- $\chi^2\left(n\right)=\Gamma\left(\frac{1}{2},\frac{n}{2}\right)$

2.6 Exponential(α) distribution

Density

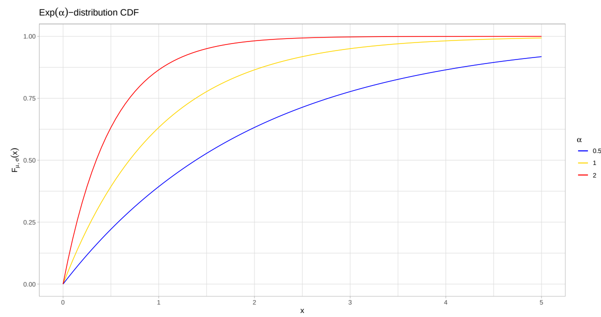
$$f(x) = \begin{cases} \alpha \exp(-\alpha x) & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

$$\operatorname{supp} f(x) = \mathbb{R}_0^+$$



Distribution function

$$F(x) = \begin{cases} 1 - \exp(-\alpha x) & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$



Mean

$$E[X] = \frac{1}{\alpha}$$

Variance

$$\operatorname{Var}(X) = \frac{1}{\alpha^2}$$

Fisher Information

$$\mathcal{I}(\theta) = \frac{1}{\theta^2}$$

Moment-generating function

$$M_X(t) = \frac{\alpha}{\alpha - t}$$

Characteristic function

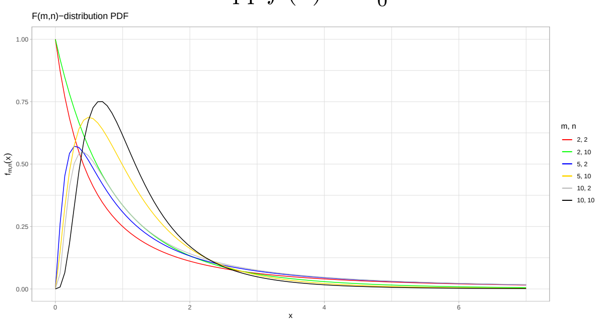
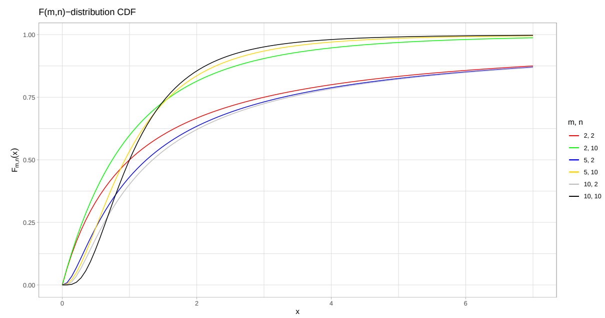
$$\varphi_X(t) = \frac{\alpha}{\alpha - it}$$

Maximum Likelihood Estimator

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} x_i}$$

- $P(X \geq x + t \mid X \geq x) = P(X \geq t)$ (Memorylessness)

2.7 Fisher(m,n) distribution

Density	Distribution function
$$f\left(x\right)=\frac{\Gamma\left(\frac{m+n}{2}\right)\left(\frac{m}{n}\right)^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}x^{\left(\frac{m}{2}-1\right)}\left(1+\frac{m}{n}x\right)^{-\frac{m+n}{2}}$$ $$\operatorname{supp} f\left(x\right)=\mathbb{R}_0^+$$ 	$-$ 
Mean	**Variance**
$$E\left[X\right]=\frac{n}{n-2}$$	$$\operatorname{Var}\left(X\right)=\frac{2n^2\left(m+n-2\right)}{m\left(n-2\right)^2\left(n-4\right)}$$

2.8 Student's(n) distribution

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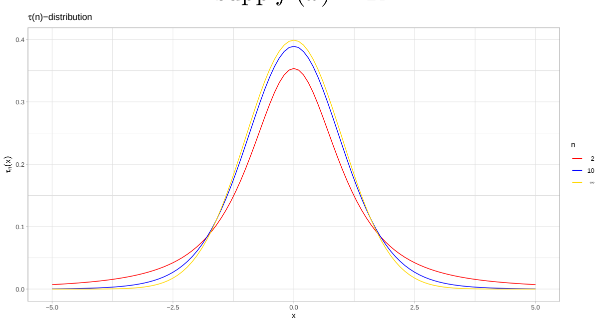
Density

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

$$\operatorname{supp} f(x) = \mathbb{R}$$



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Distribution function

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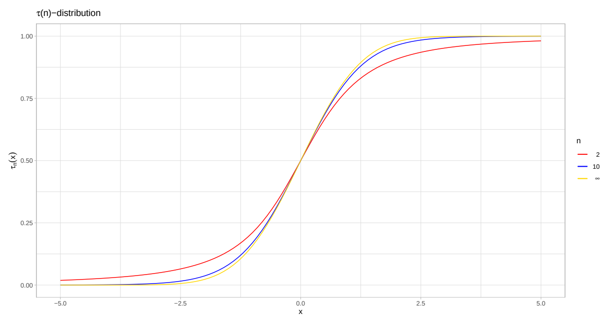

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Mean

$$E[X] = 0$$

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Variance

$$\operatorname{Var}(X) = \frac{n}{n-2}$$

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- $\tau(n) = N(0,1)$ as $n \to \infty$