

**Problem 1**

Let maps  $T \in \mathcal{L}(U, V)$ ,  $S \in \mathcal{L}(V, W)$  be invertible, show that  $ST \in \mathcal{L}(U, W)$  is invertible and  $(ST)^{-1} = T^{-1}S^{-1}$ . Because  $T$  is invertible from  $U$  to  $V$ , we see that  $\dim U = \dim V$ . Because  $S$  is invertible from  $V$  to  $W$ , we also have  $\dim V = \dim W$ . Therefore,  $\dim V = \dim W$ . Because dimension is sufficient to show isomorphism, we see that  $ST$  is invertible because dimension is preserved. Now, we can write  $ST(ST)^{-1} = I$ . Now we must evaluate  $ST(ST)^{-1}$  to see if it gives the identity. Using composition of linear maps, we can rewrite this as  $ST(T^{-1}S^{-1})$ . Using linearity, we see that  $S(TT^{-1})S^{-1}$ . Because  $T$  is invertible, we have  $SIS^{-1}$ . Because  $I$  is the identity, we have  $SS^{-1}$ , which is equal to  $I$  because  $S$  is invertible. Therefore,  $(ST)^{-1} = T^{-1}S^{-1}$ .

**Problem 2**

Let  $V$  be finite-dimensional and  $S, T \in \mathcal{L}(V)$ . First assume that  $ST = I$ . Given vector  $v_1, v_2 \in V$  such that  $T(v_1) = v_2$ , we can multiply both sides by  $S$  to get  $ST(v_1) = S(v_2)$ . Now, we see that  $ST$  is defined as the identity, so we can rewrite this as  $v_1 = S(v_2)$ . Now, multiplying both sides by  $T$ , we see that  $T(v_1) = TS(v_2)$ . Substituting and using properties of linearity, we now have  $v_2 = (TS)v_2$ . Therefore,  $TS = I$ . Now assume that  $TS = I$ . Given any vectors  $v_1, v_2 \in V$  such that  $S(v_1) = v_2$ , we can write  $TS(v_1) = T(v_2)$ . Because  $TS$  is defined as the identity, we have  $v_1 = T(v_2)$ . Multiplying both sides by  $S$ , we get  $S(v_1) = ST(v_2)$ . Substituting and using properties of linearity, we see that  $v_2 = (ST)v_2$ . Therefore  $ST = I$ . Because this holds both ways,  $ST = I$  if and only if  $TS = I$ .

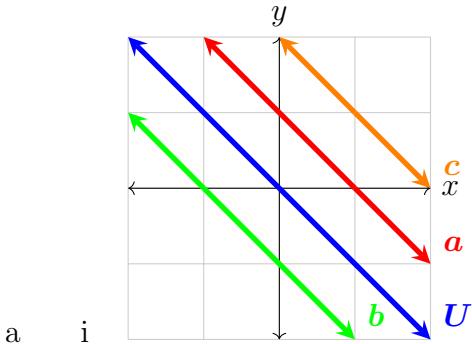
**Problem 3**

- a Given any field  $\mathbb{F}$ , show that  $\mathbb{F}^{n_1} \times \mathbb{F}^{n_2}$  is isomorphic to  $\mathbb{F}^{n_1+n_2}$ . According to Axler 3.76, we can find the dimension of the Cartesian product by summing each element's dimension. In this case, we get  $\dim \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} = \dim \mathbb{F}^{n_1} + \dim \mathbb{F}^{n_2} = n_1 + n_2$ . Because the dimension of  $\mathbb{F}^{n_1+n_2}$  is simply  $n_1 + n_2$ , we see that  $\mathbb{F}^{n_1} \times \mathbb{F}^{n_2}$  is isomorphic to  $\mathbb{F}^{n_1+n_2}$ .
- b Given a linear map  $T : U \times V \rightarrow W$ , show that  $T((u, v)) = T((u, 0)) + T((0, v))$ . To evaluate  $T((u, 0)) + T((0, v))$ , we must use properties of vector spaces. Adding these together, we get  $T((u, 0)) + T((0, v)) = T((u + 0), (v + 0))$ , which simplifies to  $T((u, 0)) + T((0, v)) = T((u, v))$
- c For all  $T \in \mathcal{L}(U \times V, W)$ , define  $T : U \times V \rightarrow W$  such that for all  $u \in U, v \in V$ ,  $T_1(u, v) = (u, 0)$  and  $T_2(u, v) = (0, v)$ . Now define map  $S : \mathcal{L}(U \times V, W) \rightarrow \mathcal{L}(U, W) \times \mathcal{L}(V, W)$  such that  $S(T) = (T_1 + T_2)$  (part b). To see if this is an isomorphism, we must construct an inverse map. We can define  $G : \mathcal{L}(U, W) \times \mathcal{L}(V, W) \rightarrow \mathcal{L}(U \times V, W)$

## Problem 4

Given nonempty subset  $A$  of  $V$ , show that it is an affine subset of  $V$  if and only if  $\lambda v + (1 - \lambda)w \in A$  for all  $v, w \in A$  and for all  $\lambda \in \mathbb{F}$ . Assume  $A$  is an affine subset of  $V$ , it can be expressed in the form  $A = a + U$  where  $a \in V$  and  $U$  is a subspace of  $V$ . Now, we can write  $v = a + u_1$  and  $w = a + u_2$  where  $u_1, u_2 \in U$ . Substituting this into the original expression, we have  $\lambda(a + u_1) + (1 - \lambda)(a + u_2)$ , which we can see is contained within the subset  $a + U$ , which we defined as  $A$ . Now assume that  $\lambda v + (1 - \lambda)w \in A$  for all  $v, w \in A$  and for all  $\lambda \in \mathbb{F}$ . We must show that  $A = a + U$  is an affine subset of  $V$ . To do this, we can show  $U = A - a$  is a subspace of  $V$ . If this is true, then  $A$  is an affine subset of  $V$  because it can be written in the form  $A = a + (A - a)$ . To show this, we can write  $A - a = \{x - a : x \in A\}$ . Following our assumption, we can substitute and see that for some  $x - a \in A - a$ , and  $\lambda \in \mathbb{F}$ , we have  $\lambda(x - a) + (1 - \lambda)(a) \in A$ . Simplifying, we get  $\lambda(x - a)$ , which shows that scalar multiplication is closed under  $U$ . Now we must show that  $U$  is closed under addition. Given any  $(x - a) + (y - a)$ , we can write this as  $2(\frac{x}{2} + \frac{y}{2} - a)$ . Because  $U$  is closed under scalar multiplication, we see that  $U$  is also closed under addition. Therefore,  $U$  is a subspace of  $V$ . As stated before, this shows that  $A$  is an affine subset.

## Problem 5



a      i       $a = U + (1, 0), b = U + (-1, 0), c = U + (2, 0)$

- ii Let  $W = (x, 0), x \in \mathbb{R}$ . Let  $\phi : V/U \rightarrow W$  be defined by  $\phi(v + U : v \in V) = (v, 0)$ . This map takes the vector that  $U$  is translated by and puts it on the x-axis. To show that this is isomorphic, we must show that the dimensions are equal. Using the rank-nullity theorem, we see that  $\dim(W) = \text{rank}(W) + \text{nullity}(W) = 1 + 0 = 1$ . We also see that  $\dim(V/U) = \dim(V) - \dim(U) = 2 - 1 = 1$ . Since the dimensions are equal, they are isomorphic. Hence, this map is an isomorphism.
- b Given that  $U$  is a subspace of  $V$  and  $v_1 + U, \dots, v_m + U$  is a basis of  $V/U$  and  $u_1, \dots, u_n$  is a basis of  $U$ . We must show that  $v_1, \dots, v_m, u_1, \dots, u_n$  is a basis of  $V$ , and that  $V/U$  is isomorphic to the subspace  $W = (v_1, \dots, v_m)$ . To start, we can write  $v + U$  as a linear combination in terms of constants  $a_1, \dots, a_m \in \mathbb{F}$ . Writing this out, we see

that  $v + U = a_1(v_1 + U) + \dots + a_m(v_m + U)$ . Distributing and rearranging, we have  $v = a_1v_1 + \dots + a_mv_m + (a_1 + \dots + a_m - 1)U$ . Because  $U$  is being multiplied by a constant, we can write it in terms of its basis, leaving us with  $v = a_1v_1 + \dots + a_mv_m + b_1u_1 + \dots + b_nu_n$ . Therefore, any vector  $v \in V$ , can be expressed as a linear combination of linear independent vectors  $v_1, \dots, v_m, u_1, \dots, u_n$ . Therefore, this list is a basis for  $V$ . Next, let  $\phi$  be a linear map from  $V/U$  to  $W$  where  $W = \text{span}(v_1, \dots, v_m)$ .

c

**Problem 6**