Problem 1

Let maps $T \in \mathcal{L}(U, V), S \in \mathcal{L}(V, W)$ be invertible, show that $ST \in \mathcal{L}(U, W)$ is invertible and $(ST)^{-1} = T^{-1}S^{-1}$. Because T is invertible from U to V, we see that $dimU = dimV$. Because S is invertible from V to W, we also have $dimV = dimW$. Therefore, $dimV = dimW$. Because dimension is sufficient to show isomorphism, we see that ST is invertible because dimension is preserved. Now, we can write $ST(ST)^{-1} = I$. Now we must evaluate $ST(ST)^{-1}$ to see if it gives the identity. Using composition of linear maps, we can rewrite this as $ST(T^{-1}S^{-1})$. Using linearity, we see that $S(TT^{-1})S^{-1}$. Because T is invertible, we have SIS^{-1}. Because I is the identity, we have SS^{-1}, which is equal to I because S is invertible. Therefore, $(ST)^{-1} = T^{-1}S^{-1}$.

Problem 2

Let V be finite-dimensional and $S, T \in \mathcal{L}(V)$. First assume that $ST = I$. Given vector $v_1, v_2 \in V$ such that $T(v_1) = v_2$, we can multiply both sides by S to get $ST(v_1) = S(v_2)$. Now, we see that ST is defined as the identity, so we can rewrite this as $v_1 = S(v_2)$. Now, multiplying both sides by T, we see that $T(v_1) = TS(v_2)$. substituting and using properties of linearity, we now have $v_2 = (TS)v_2$. Therefore, $TS = I$. Now assume that $TS = I$. Given any vectors $v_1, v_2 \in V$ such that $S(v_1) = v_2$, we can write $TS(v_1) = T(v_2)$. Because TS is defined as the identity, we have $v_1 = T(v_2)$. Multiplying both sides by S, we get $S(v_1) = ST(v_2)$. Substituting and using properties of linearity, we see that $v_2 = (ST)v_2$. Therefore $ST = I$. Because this holds both ways, $ST = I$ if and only if $TS = I$

Problem 3

a Given any field $\mathbb{F}$, show that $\mathbb{F}^{n_1} \times \mathbb{F}^{n_2}$ is isomorphic to $\mathbb{F}^{n_1+n_2}$. According to Axler 3.76, we can find the dimension of the Cartesian product by summing each element's dimension. In this case, we get $dim\mathbb{F}^{n_1} \times \mathbb{F}^{n_2} = dim\mathbb{F}^{n_1} + dim\mathbb{F}^{n_2} = n_1 + n_2$. Because the dimension of $\mathbb{F}^{n_1+n_2}$ is simply $n_1 + n_2$, we see that $\mathbb{F}^{n_1} \times \mathbb{F}^{n_2}$ is isomorphic to $\mathbb{F}^{n_1+n_2}$.

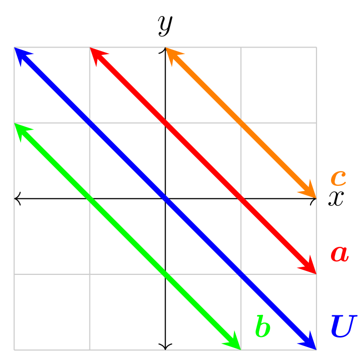
b Given a linear map $T : U \times V \to W$, show that $T((u, v)) = T((u, 0)) + T((0, v))$. To evaluate $T((u, 0)) + T((0, v))$, we must use properties of vector spaces. Adding these together, we get $T((u, 0)) + T((0, v)) = T((u + 0), (v + 0))$, which simplifies to $T((u, 0)) + T((0, v)) = T((u, v))$

c For all $T \in \mathcal{L}(U \times V, W)$, define $T_1 : U \times V \to W$ and $T_2 : U \times V \to W$ such that for all $u \in U, v \in V, T_1(u, v) = (u, 0)$ and $T_2(u, v) = (0, v)$. Now define map $S : \mathcal{L}(U \times V, W) \to \mathcal{L}(U, W) \times \mathcal{L}(V, W)$ such that $S(T_1, T_2) = (T_1 + T_2)$ (part b). To see if these are isomorphic, we must see if the dimensions are equal (axler 3.59). Because the dimension of $\mathcal{L}(U \times V, W)$ is equal to $(dim(U) + dim(V)) \times dim(W)$ and the dimension of $\mathcal{L}(U, W) \times \mathcal{L}(V, W)$ is equal to $dim(U) \times dim(W) + dim(V) \times dim(W)$. Because these two are equivilant, the dimensions are equal, and therefore it is isomporphic.

Problem 4

Given nonempty subset A of V, show that it is an affine subset of V if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and for all $\lambda \in \mathbb{F}$. Assume A is an affine subset of V, it can be expressed in the form $A = a + U$ where $a \in V$ and U is a subspace of V. Now, we can write $v = a + u_1$ and $w = a + u_2$ where $u_1, u_2 \in U$. Substituting this into the original expression, we have $\lambda(a + u_1) + (1 - \lambda)(a + u_2)$, which we can see is contained within the subset $a + U$, which we defined as A (Assuming that $\mathbb{F}$ is over $\mathbb{R}$ or $\mathbb{C}$, as Axler does). Now assume that $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and for all $\lambda \in \mathbb{F}$. We must show that $A = a + U$ is an affine subset of V. To do this, we can show $U = A - a$ is a subspace of V. If this is true, then A is an affine subset of V because it can be written in the form $A = a + (A - a)$. To show this, we can write $A - a = \{x - a : x \in A\}$. Following our assumption, we can substitute and see that for some $x - a \in A - a$, and $\lambda \in \mathbb{F}$, we have $\lambda(x) + (1 - \lambda)(a) \in A$. Simplifying, we get $\lambda(x - a)$, which shows that scalar multuplication is closed under U. Now we must show that U is closed under addition. Given any $(x - a) + (y - a)$, we can write this as $2(\frac{x}{2} + \frac{y}{2} - a)$ Because U is closed under scalar multiplication, we see that U is also closed under addition. Therefore, U is a subspace of V. As stated before, this shows that A is an affine subset.

Problem 5

a i 

$a = U + (1, 0), b = U + (-1, 0), c = U + (2, 0)$

ii Let $W = (x, 0), x \in \mathbb{R}$. Let $\phi : V/U \to W$ be defined by $\phi(v + U : v \in V) = v$. This map takes the vector that U is translated by and puts it on the x-axis. To show that this is isomorphic, we must show that the dimensions are equal. Using the rank-nullity theormen, we see that $dim(W) = rank(W) + nullity(W) = 1 + 0 = 1$. We also see that $dim(V/U) = dim(V) - dim(U) = 2 - 1 = 1$. Since the dimensions are equal, they are isomorphic. Now, to show that this map is an isomorphism, we must show that it is bijective. First, to show surjectivity, we see that $range(W)$ is equal to the codomain of ϕ, which is $\mathbb{R}$. Because the only element that maps to zero is zero, $null(W) = 0$, and therefore $W\phi$ is injective. Because the dimensions are equal, and it is both injective and surjective, ϕ is an isomorhpism.

b Given that U is a subspace of V and $v_1 + U, \ldots, v_m + U$ is a basis of V/U and $u_1, \ldots, u_n$ is a basis of U. We must show that that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V, and that V/U is isomorphic to the subspace $W = (v_1, \ldots, v_m)$. To start, we can write $v + U$ as a linear combination in terms of constants $a_1, ..., a_m \in \mathbb{F}$. Writing this out, we see that $v + U = a_1(v_1 + U) + ... + a_m(v_m + U)$. Distributing and rearranging, we have $v = a_1v_1 + ... + a_mv_m + (a_1 + ... + a_m - 1)U$ Because U is being multiplied by a constant, we can write it in terms of its basis, leaving us with $v = a_1v_1 + ... + a_mv_m + b_1u_1 + ... + b_nu_n$. Therefore, any vector $v \in V$, can be expressed as a linear combination of linear independent vectors $v_1, ..., v_m, u_1, ..., u_n$. Therefore, this list is a basis for V. Next, let ϕ be a linear map from V/U to W where $W = span(v_1, ..., v_m)$. Define $\phi((v_1+U)+...+(v_m+U)) = (v_1+...+v_m)$. To show that this is an isomorphism, we must show that the dimensions are equal. Using the rank-nullity theorem, we see that $dim(V/U) = dim(V) - dim(U)$. Because the basis of V/U is $v_1, ..., v_m$ merged with the basis of U, and the basis of W is $v_1, ..., v_m$, we see that $dim(V)$ is also equal to $dim(V) - dim(U)$. Therefore, they are isomorphic by the map ϕ.

c Let W be any subspace of V such that $V = U \oplus W$. We must show that $V/U \simeq W$. First, let $u_1, ..., u_n$ be a basis of U and $v_1 + U, ..., v_n + U$ be a basis of V/U. Because $U \oplus W = V$ is a direct sum, we can represent the basis of W as $v_1 + U - U + ... + v_n + U - U$, or simply $v_1, ..., v_n$. Now, using the proof from part b, we see that a map $\phi : V/U \to W$ where $W = span(v_1, ..., v_n)$ is an isomprphism between V/U and W because it is both injective and surjective.

Problem 6

a Given basis $v_1 = e_1$, $v_2 = e_1 + e_2$, and $v_3 = e_1 + e_2 + e_3$ for $\mathbb{R}^3$. Write $\varphi_1 = v_1^*, \varphi_2 = v_2^*$ and $\varphi_3 = v_3^*$ for the corresponding dual basis of $Ł(\mathbb{R}^3, \mathbb{R})$. For map φ_1, we see that v_1^* is sent to e_1, which can also be represented as $1e_1 + 0e_2 + 0e_3$, giving the matrix [1 0 0]. For φ_2, v_2^* is sent to $1e_1 + 1e_2 + 0e_3$, and e_3 to zero. Therefore, the matrix is [1 1 0]. For $_3$, we see that v_3^* is sent to $1e_1 + 1e_2 + 1e_3$, so each standard basis is added together, giving the matrix [1 1 1].

b Given $\varphi \colon \mathbb{R}^3 \to \mathbb{R}$ given by $\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 2x - y + z$, we must write this in terms of basis vectors. To start, we see that the output of φ can be written as $2e_1 - e_2 + e_3$, or as the matrix [2 -1 1]. Now, we must write this matrix as the coefficients of a linear combination of $\varphi_1, \varphi_2, \varphi_3$. This results in a matrix of [3 -2 1].

Problem 7

Given $v_1, \ldots, v_n$ as a basis of V and let $\varphi_1 = v_1^*, \ldots, \varphi_n = v_n^*$ denote the corresponding dual basis of V^*. Suppose $\psi \in V^*$. We must show that $\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n.$. To show this, we must show that ϕ can be written as a linear combination $\phi = c_1 v_1^* + ... + c_n v_n^*$, because $v_1^*, ..., v_1^*$ is the dual basis for V^*, and $\psi / in V^*$. Because φ is an element of the dual of V, it maps a $v \in V$ to a scalar $\mathbb{F}$. This means that every $\varphi(v_1), ..., \varphi(v_n)$ can be denoted by some constant $c_1, ..., c_n$. Plugging this in, we get $\psi = c_1\varphi_1 + ... + c_n\varphi_n$. Because $\varphi_j = v_j^*$, we can substitute and get $\psi = c_1 v_1^* + ... + c_n v_n^*$, proving our proposition.

Problem 8

Given finite-dimensional vector spave V

a Given U, W are subsets of V with $U \subset W$, we must show that $W^0 \subset U^0$. To start, we can assume that $\varphi \in W^0$, so $\varphi(w) = 0 \ \forall w \in W$. Because U is contained within W, we see that $\varphi(u) = 0 \ \forall u \in W$. Because this holds for any chosen $\varphi \in W$, we can see that $W^0 \subset U^0$.

b Given U, W are subsets of V with $W^0 \subset U^0$, we must show that $U \subset W$. Let $w_1, ..., w_n$ be a basis for W. Assuming that U is not contained within W, we can extend this to a basis for V by making it $w_1, ..., w_n, v, v_1, ..., v_m$ where all $v_k \in U$. Let $\varphi \in V^*$ be defined by $\varphi(w_j) = 0$, $\varphi(v) = 1$, and $\varphi(v_i) = 0$. Here, we see that $\varphi \in W^0$ but $\varphi \notin U^0$. Therefore, this is a contradiction, and hence U must be contained within W.