Problem 1

Let maps $T \in \mathcal{L}(U, V), S \in \mathcal{L}(V, W)$ be invertible, show that $ST \in \mathcal{L}(U, W)$ is invertible and $(ST)^{-1} = T^{-1}S^{-1}$. Because T is invertible from U to V, we see that $dimU = dimV$. Because S is invertible from V to W, we also have $dimV = dimW$. Therefore, $dimV = dimW$. Because dimension is sufficient to show isomorphism, we see that ST is invertible because dimension is preserved. Now, we can write $ST(ST)^{-1} = I$. Now we must evaluate $ST(ST)^{-1}$ to see if it gives the identity. Using composition of linear maps, we can rewrite this as $ST(T^{-1}S^{-1})$. Using linearity, we see that $S(TT^{-1})S^{-1}$. Because T is invertible, we have SIS^{-1}. Because I is the identity, we have SS^{-1}, which is equal to I because S is invertible. Therefore, $(ST)^{-1} = T^{-1}S^{-1}$.

Problem 2

Let V be finite-dimensional and $S, T \in \mathcal{L}(V)$. First assume that $ST = I$. Given vector $v_1, v_2 \in V$ such that $T(v_1) = v_2$, we can multiply both sides by S to get $ST(v_1) = S(v_2)$. Now, we see that ST is defined as the identity, so we can rewrite this as $v_1 = S(v_2)$. Now, multiplying both sides by T, we see that $T(v_1) = TS(v_2)$. substituting and using properties of linearity, we now have $v_2 = (TS)v_2$. Therefore, $TS = I$. Now assume that $TS = I$. Given any vectors $v_1, v_2 \in V$ such that $S(v_1) = v_2$, we can write $TS(v_1) = T(v_2)$. Because TS is defined as the identity, we have $v_1 = T(v_2)$. Multiplying both sides by S, we get $S(v_1) = ST(v_2)$. Substituting and using properties of linearity, we see that $v_2 = (ST)v_2$. Therefore $ST = I$. Because this holds both ways, $ST = I$ if and only if $TS = I$

Problem 3

a Given any field $\mathbb{F}$, show that $\mathbb{F}^{n_1} \times \mathbb{F}^{n_2}$ is isomorphic to $\mathbb{F}^{n_1+n_2}$. According to Axler 3.76, we can find the dimension of the Cartesian product by summing each element's dimension. In this case, we get $dim\mathbb{F}^{n_1} \times \mathbb{F}^{n_2} = dim\mathbb{F}^{n_1} + dim\mathbb{F}^{n_2} = n_1 + n_2$. Because the dimension of $\mathbb{F}^{n_1+n_2}$ is simply $n_1 + n_2$, we see that $\mathbb{F}^{n_1} \times \mathbb{F}^{n_2}$ is isomorphic to $\mathbb{F}^{n_1+n_2}$.

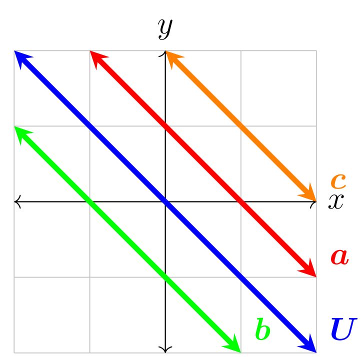
b Given a linear map $T : U \times V \to W$, show that $T((u, v)) = T((u, 0)) + T((0, v))$. To evaluate $T((u, 0)) + T((0, v))$, we must use properties of vector spaces. Adding these together, we get $T((u, 0)) + T((0, v)) = T((u + 0), (v + 0))$, which simplifies to $T((u, 0)) + T((0, v)) = T((u, v))$

c For all $T \in \mathcal{L}(U \times V, W)$, define $T : U \times V \to W$ such that for all $u \in U, v \in V, T_1(u, v) = (u, 0)$ and $T_2(u, v) = (0, v)$. Now define map $S : \mathcal{L}(U \times V, W) \to \mathcal{L}(U, W) \times \mathcal{L}(V, W)$ such that $S(T) = (T_1 + T_2)$ (part b). To see if this is an isomorphism, we must construct an inverse map. We can define $G : \mathcal{L}(U, W) \times \mathcal{L}(V, W) \to \mathcal{L}(U \times V, W)$

Problem 4

Given nonempty subset A of V, show that it is an affine subset of V if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and for all $\lambda \in \mathbb{F}$. Assume A is an affine subset of V, it can be expressed in the form $A = a + U$ where $a \in V$ and U is a subspace of V. Now, we can write $v = a + u_1$ and $w = a + u_2$ where $u_1, u_2 \in U$. Substituting this into the original expression, we have $\lambda(a + u_1) + (1 - \lambda)(a + u_2)$, which we can see is contained within the subset $a + U$, which we defined as A. Now assume that $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and for all $\lambda \in \mathbb{F}$. We must show that $A = a + U$ is an affine subset of V. To do this, we can show $U = A - a$ is a subspace of V. If this is true, then A is an affine subset of V because it can be written in the form $A = a + (A - a)$. To show this, we can write $A - a = \{x - a : x \in A\}$. Following our assumption, we can substitute and see that for some $x - a \in A - a$, and $\lambda \in \mathbb{F}$, we have $\lambda(x) + (1 - \lambda)(a) \in A$. Simplifying, we get $\lambda(x - a)$, which shows that scalar multuplication is closed under U. Now we must show that U is closed under addition. Given any $(x - a) + (y - a)$, we can write this as $2(\frac{x}{2} + \frac{y}{2} - a)$ Because U is closed under scalar multiplication, we see that U is also closed under addition. Therefore, U is a subspace of V. As stated before, this shows that A is an affine subset.

Problem 5

a i 

$a = U + (1, 0), b = U + (-1, 0), c = U + (2, 0)$

ii Let $W = (x, 0), x \in \mathbb{R}$. Let $\phi : V/U \to W$ be defined by $\phi(v + U : v \in V) = (v, 0)$. This map takes the vector that U is translated by and puts it on the x-axis. To show that this is isomorphic, we must show that the dimensions are equal. Using the rank-nullity theormen, we see that $dim(W) = rank(W) + nullity(W) = 1 + 0 = 1$. We also see that $dim(V/U) = dim(V) - dim(U) = 2 - 1 = 1$. Since the dimensions are equal, they are isomorphic. Hence, this map is an isomorphism.

b Given that U is a subspace of V and $v_1 + U, \ldots, v_m + U$ is a basis of V/U and $u_1, \ldots, u_n$ is a basis of U. We must show that that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V, and that V/U is isomorphic to the subspace $W = (v_1, \ldots, v_m)$. To start, we can write $v + U$ as a linear combination in terms of constants $a_1, ..., a_m \in \mathbb{F}$. Writing this out, we see that $v + U = a_1(v_1 + U) + ... + a_m(v_m + U)$. Distributing and rearranging, we have $v = a_1v_1 + ... + a_mv_m + (a_1 + ... + a_m - 1)U$ Because U is being multiplied by a constant, we can write it in terms of its basis, leaving us with $v = a_1v_1 + ... + a_mv_m + b_1u_1 + ... + b_nu_n$. Therefore, any vector $v \in V$, can be expressed as a linear combination of linear independent vectors $v_1, ..., v_m, u_1, ..., u_n$. Therefore, this list is a basis for V. Next, let ϕ be a linear map from V/U to W where $W = span(v_1, ..., v_m)$. Define $\phi((v_1+U)+...+(v_m+U)) = (v_1+...+v_m)$. To show that this is an isomorphism, we must show that the dimensions are equal. Using the rank-nullity theorem, we see that $dim(V/U) = dim(V) - dim(U)$. Because the basis of V/U is $v_1, ..., v_m$ merged with the basis of U, and the basis of W is $v_1, ..., v_m$, we see that $dim(V)$ is also equal to $dim(V) - dim(U)$. Therefore, they are isomorphic by the map ϕ.

c

Problem 6