1 Kernel version of FVAR

For vector valued output $y \in \mathcal{Y} = \mathbb{R}^K$ the optimal solution for the minimisation of the regularised functional is

$$\mathbf{f}(\mathbf{x}) = \sum_i^n \mathbf{H}(\mathbf{x}_i, \mathbf{x})\, \mathbf{c}_i = \sum_i^n \sum_k^K c_{ik}\, \mathbf{H}(\mathbf{x}_i, \mathbf{x})_{(:,k)}, \tag{1}$$

where n is the number of instances, K is the number of tasks, $\mathbf{c}_i \in \mathcal{Y}$, $\mathbf{H}(.,.) \in \mathcal{Y} \times \mathcal{Y}$ and $\mathbf{H}(.,.)_{(:,k)}$ is the kth column of $\mathbf{H}$.

Hence, for task t we have

$$f(\mathbf{x}, t) = \sum_i^n \sum_k^K c_{ik}\, H\big((\mathbf{x}_i, k), (\mathbf{x}, t)\big), \tag{2}$$

where $H\big((.,.), (.,.)\big) \in \mathbb{R}$

Assuming H is separable we can write it as a product of input $K_X(.,.) \in \mathbb{R}$ and output $K_Y(.,.) \in \mathbb{R}$ kernels

$$H\big((\mathbf{x}_i, k), (\mathbf{x}, t)\big) = K_X(\mathbf{x}_i, \mathbf{x})\, K_Y(k, t), \tag{3}$$

or equivalently

$$\mathbf{H}\big(\mathbf{x}_i, \mathbf{x}\big) = K_X(\mathbf{x}_i, \mathbf{x})\, \mathbf{K}_Y, \tag{4}$$

where $\mathbf{K}_Y \in \mathcal{Y} \times \mathcal{Y}$ is the output kernel Gram matrix.

Since a linear combination of kernels is a valid kernel we can also assume that

$$\mathbf{H}\big(\mathbf{x}_i, \mathbf{x}\big) = \sum_s^S K_X^s(\mathbf{x}_i, \mathbf{x})\, \mathbf{K}_Y^s, \tag{5}$$

or equivalently

$$H\big((\mathbf{x}_i, k), (\mathbf{x}, t)\big) = \sum_s^S K_X^s(\mathbf{x}_i, \mathbf{x})\, K_Y^s(k, t), \tag{6}$$

Now, assuming $K_X^s(\mathbf{x}_i, \mathbf{x}) = \langle \mathbf{x}_i^s, \mathbf{x}^s \rangle$ and $K_Y^s(k, t) = \delta_{k,t} \alpha_{k,t,s}$ we get

$$\begin{aligned}
f(\mathbf{x}, t) &= \sum_i^n \sum_k^K c_{ik} \sum_s^S K_X^s(\mathbf{x}_i, \mathbf{x})\, K_Y^s(k, t) \\
&= \sum_i^n \sum_k^K c_{ik} \sum_s^S \langle \mathbf{x}_i^s, \mathbf{x}^s \rangle\, \delta_{k,t} \alpha_{k,t,s} \\
&= \sum_k^K \delta_{k,t} \sum_i^n \sum_s^S \alpha_{k,t,s} c_{ik} \langle \mathbf{x}_i^s, \mathbf{x}^s \rangle \\
&= \sum_i^n \sum_s^S \beta_{t,s} c_{it} \langle \mathbf{x}_i^s, \mathbf{x}^s \rangle \\
&= \sum_i^n c_{it} \sum_s^S \beta_{t,s} \langle \mathbf{x}_i^s, \mathbf{x}^s \rangle
\end{aligned}$$