1 Math cheat-sheet

1.1 Vec operator

$$vec(\mathbf{AB}) = (\mathbf{I} \otimes \mathbf{A})\, vec(\mathbf{B}) = (\mathbf{B}' \otimes \mathbf{I})\, vec(\mathbf{A}) \tag{1.1}$$

$$vec(\mathbf{ABC}) = (\mathbf{C} \otimes \mathbf{A})\, vec(\mathbf{B}) \tag{1.2}$$

$$tr(\mathbf{ABC}) = vec(\mathbf{A}')'(\mathbf{I} \otimes \mathbf{B})vec(\mathbf{C}) = vec(\mathbf{A}')'(\mathbf{C}' \otimes \mathbf{I})vec(\mathbf{B}) \tag{1.3}$$

$$tr(\mathbf{ABCD}) = vec(\mathbf{A}')'(\mathbf{I} \otimes \mathbf{B})vec(\mathbf{CD}) = vec(\mathbf{A}')'(\mathbf{D}' \otimes \mathbf{B})vec(\mathbf{C}) \tag{1.4}$$

1.2 Matrix multiplications

For $(m \times p)$ matrix $\mathbf{A}$, $(p \times n)$ matrix $\mathbf{B}$ and $(m \times n)$ matrix $\mathbf{AB}$ we have

$$\mathbf{AB}_{ij} = \sum_k^p A_{ik} B_{ki} \tag{1.5}$$

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{1:} \\ \mathbf{A}_{2:} \\ \vdots \\ \mathbf{A}_{m:} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{:1} & \mathbf{B}_{:2} & \cdots & \mathbf{B}_{:n} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{1:}\mathbf{B} \\ \mathbf{A}_{2:}\mathbf{B} \\ \vdots \\ \mathbf{A}_{m:}\mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{AB}_{:1} & \mathbf{AB}_{:2} & \cdots & \mathbf{AB}_{:n} \end{bmatrix} \tag{1.6}$$

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{:1} & \mathbf{A}_{:2} & \cdots & \mathbf{A}_{:p} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{1:} \\ \mathbf{B}_{2:} \\ \vdots \\ \mathbf{B}_{p:} \end{bmatrix} = \sum_k^p \mathbf{A}_{:k} \otimes \mathbf{B}_{k:} = \sum_k^p \mathbf{A}_{:k}(\mathbf{B}_{k:})' \tag{1.7}$$

For matrix $\mathbf{A}$ and vectors $\mathbf{c}$ and $\mathbf{b}$

$$\sum_i^n c_i \mathbf{A}_{:i} = \mathbf{Ac} \qquad \sum_i^n c_i \mathbf{A}_{i:} = \mathbf{c}'\mathbf{A} \qquad \sum_i^n \sum_j^m c_i b_j \mathbf{A}_{ij} = \sum_i^n c_i \mathbf{A}_{i:}\mathbf{b} = \mathbf{c}'\mathbf{Ab} \tag{1.8}$$

$$\sum_i^n \sum_j^m c_j \mathbf{A}_{ij} \sum_k^m b_k \mathbf{A}_{ik} = \sum_i^n \mathbf{A}_{i:}\mathbf{c}\, \mathbf{A}_{i:}\mathbf{b} = \mathbf{c}'\mathbf{A}'\mathbf{Ab} \tag{1.9}$$

For matrix $\mathbf{A}$ and diagonal matrix $\mathbf{D}$

$$\mathbf{AD} = \begin{bmatrix} D_{11}\mathbf{A}_{:1} & D_{22}\mathbf{A}_{:2} & \cdots & D_{nn}\mathbf{A}_{:n} \end{bmatrix} \qquad \mathbf{DA} = \begin{bmatrix} D_{11}\mathbf{A}_{1:} \\ D_{22}\mathbf{A}_{2:} \\ \vdots \\ D_{nn}\mathbf{A}_{n:} \end{bmatrix} \tag{1.10}$$

1.3 Matrix traces

$$tr(\mathbf{A}) = \sum_i A_{ii} \qquad tr(\mathbf{AB}) = \sum_{ij} A_{ij} B_{ji} \tag{1.11}$$

$$tr(\mathbf{A}'\mathbf{BA}) = \sum_{ijk} A'_{ij} B_{jk} A_{ki} = \sum_{ij} A'_{ij} \mathbf{B}_{j:}\mathbf{A}_{:i} = \sum_i \mathbf{A}'_{i:}\mathbf{BA}_{:i} = \sum_i (\mathbf{A}_{:i})'\mathbf{BA}_{:i} \tag{1.12}$$

$$tr(\mathbf{ABA}') = \sum_{ijk} A_{ij} B_{jk} A'_{ki} = \sum_{ij} A_{ij} \mathbf{B}_{j:}\mathbf{A}'_{:i} = \sum_i \mathbf{A}_{i:}\mathbf{B}(\mathbf{A}_{i:})' \tag{1.13}$$

$$||\mathbf{A}||_F^2 = tr(\mathbf{A}'\mathbf{A}) = tr(\mathbf{A}\mathbf{A}') = \sum_i \sum_k A_{ik} A'_{ki} = \sum_i \langle \mathbf{A}_{i:}, \mathbf{A}_{i:} \rangle = \sum_i ||\mathbf{A}_{i:}||_2^2 = \sum_k ||\mathbf{A}_{:k}||_2^2 \tag{1.14}$$

$$\langle \mathbf{A}, \mathbf{B} \rangle_F = tr(\mathbf{A}'\mathbf{B}) = \sum_i \langle \mathbf{A}_{\mathbf{i}:}, \mathbf{B}_{\mathbf{i}:} \rangle = \sum_k \langle \mathbf{A}_{:\mathbf{k}}, \mathbf{B}_{:\mathbf{k}} \rangle \tag{1.15}$$

1.4 Inversion identities

$$(I+P)^{-1} = (I+P)^{-1}(I+P-P) = I - (I+P)^{-1}P \tag{1.16}$$

$$\begin{aligned} (I+PQ)^{-1}P &= P(I+QP)^{-1} \qquad (1.17) \\ (I+PQ)^{-1}P(I+QP) &= P \\ (I+PQ)^{-1}(P+PQP) &= P \\ (I+PQ)^{-1}(I+PQ)P &= P \end{aligned}$$

$$\begin{aligned} (A+kI)^{-1}A &= (I+kA^{-1})^{-1} \qquad (1.18) \\ (A+kI)^{-1}A\,(I+kA^{-1}) &= I \\ (A+kI)^{-1}\,(A+kI) &= I \end{aligned}$$