Question on subgradients When working with non-differentiable functions in optimisation we need to find their sub-gradient.

For example for the absolute value $f(x) = |x|$ the subgradient is

$$\partial f(x) = \begin{cases} sign(x) & \text{if } x \neq 0 \\ \{s : s \in [-1, 1]\} & \text{if } x = 0 \end{cases} \tag{0.1}$$

For an ℓ_2 norm $f(\mathbf{x}) = ||\mathbf{x}||_2 = \sqrt{\mathbf{x}^T\mathbf{x}}$ it is

$$\partial f(\mathbf{x}) = \begin{cases} \mathbf{x}/||\mathbf{x}||_2 & \text{if } \mathbf{x} \neq 0 \\ \{\mathbf{s} : ||\mathbf{s}||_2 \leq 1\} & \text{if } \mathbf{x} = 0 \end{cases} \tag{0.2}$$

For the generalised ℓ_2 norm $f(\mathbf{x}) = ||\mathbf{A}\mathbf{x}||_2 = \sqrt{\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x}}$ it is

$$\partial f(\mathbf{x}) = \begin{cases} \mathbf{A}^T\mathbf{A}\mathbf{x}/||\mathbf{A}\mathbf{x}||_2 & \text{if } \mathbf{x} \neq 0 \\ \{\mathbf{s} :??????\} & \text{if } \mathbf{x} = 0 \end{cases} \tag{0.3}$$

Any idea and how to get there?

Definition 1 *A vector* $\mathbf{v} \in \mathbb{R}^n$ *is a subgradient of (not necessarily convex) function* $f : \mathbb{R}^n \to \mathbb{R}$ *at point* $\mathbf{x} \in dom f$ *if for all* $\mathbf{z} \in dom f$

$$f(\mathbf{z}) - f(\mathbf{x}) \geq \mathbf{v}^T(\mathbf{z} - \mathbf{x}) \tag{0.4}$$

Note: If f is convex and differentiable at point $\mathbf{x}$ than the subgradient is equal to the gradient $\mathbf{v} = \nabla f(\mathbf{x})$.

Definition 2 *The set of all subgradients of function* f *at point* $\mathbf{x}$ *is called the subdifferential and denoted* $\partial f(\mathbf{x})$

$$\partial f(\mathbf{z}) = \{\mathbf{v} | f(\mathbf{z}) - f(\mathbf{x}) \geq \mathbf{v}^T(\mathbf{z} - \mathbf{x})\} \quad \textit{for all } \mathbf{z} \in dom f \tag{0.5}$$

Note: $\partial f(\mathbf{z})$ is a closed convex set (though may be empty)

Note: For a convex subdifferentiable function f the standard optimality condition for a minimum $f(\mathbf{x}^*) = \inf_x f(\mathbf{x}) \Leftrightarrow 0 = \nabla f(\mathbf{x})$ changes to $f(\mathbf{x}^*) = \inf_x f(\mathbf{x}) \Leftrightarrow 0 \in \partial f(\mathbf{x})$.

Yes, I know the formal definition of subgradients but that does not mean that I really know how to apply it so that it actually yields anything sensible.

Solving for x Once you're at it ... :-) Actually, in the end I need to figure out how to solve this equation for vector $\mathbf{x}$. Preferably in closed form (cause it shall simplify an algo. I have a lengthy descent-alternative already.)

$$\frac{\alpha\mathbf{K}\mathbf{x}}{||\mathbf{x}||_K} = \mathbf{v} - \mathbf{x}, \tag{0.6}$$

where $\mathbf{K}$ is positive definite matrix and $||\mathbf{x}||_K = \sqrt{\mathbf{x}^T\mathbf{K}\mathbf{x}}$