Final Review MATH2001

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Contents

1 Half Range Expansion

There will be at least one question in the final exam about finding the Fourier Series using the half range expansion.

When we want to find the Fourier series expansion of a function, we may encounter the case that this function may defined on a finite interval. in order to find the Fourier series expansion of these kinds of functions, we may use a method called **Half Range Expansion**, which can convert a non-periodic function into a periodic function.

Suppose $f(x) = x^2$ (the dashed line in the figure below) is defined over $0 < x < \pi$, the interval $0 < x < \pi$ is called the a **Half Period**. In order to complete the full interval, we can extend it by adding the symmetry interval, $-\pi < x < 0$. Because the symmetry part is not specified, we can extend it as either an odd function or even function. If the function is odd after extension, we will need to find the b_n and use it as the coefficient of sine function. If the function after completion becomes an even function, we need to find a_n and a_0 and use them as the coefficient as cosine function. The first one is called **Sine Expansion**,since the Fourier Series in the result has only sine terms

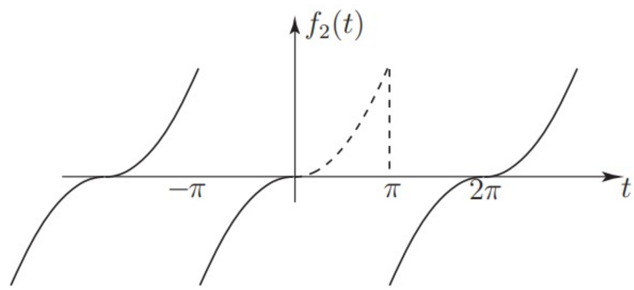

Figure 1: it is extended in odd function, thus is a **Sine Expansion**

and the other one is called **Cosine Expansion**

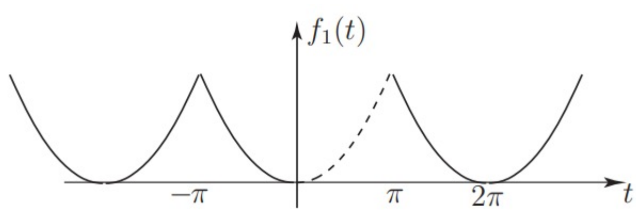

Figure 2: it is extended in even function, thus is a **Cosine Expansion**

Here is a table which may be helpful to remember the correspondence of sine and cosine expansion. The four columns are type of expansion, corresponding coefficient, corresponding integral, and the terms that will appear in the final, respectively.

odd	b_n	sine integral	sine
even	a_n	cosine integral	cosine

Note that after some calculation, the result always different in terms of the value of n, depending on its odd or even. we always introduce a parameter called m to examine the difference of n. Let $m = Ln$, where L is the half-period of the function. we then plug the m into the Fourier expression to find answers corresponding to different values n, and express the result as a piece-wise series.

Since the formula of finding Fourier Coefficients will be provided, there are no need to remember them.

2 Laplace Transformation and Laplace Inverse

The Laplace Transform part is estimated to have 3 questions in the final. Since the table of Laplace Transform will be given, there are no need to remember the corresponding transformation of different functions.

2.1 Important Properties of Laplace Transformation

2.1.1 Linearity

Laplace Transform is a **Linear Operator** (Same as the Fourier Series), which means that

$$L\{af(t) + bg(t)\} = L\{af(t)\} + L\{bg(t)\} = aL\{f(t)\} + bL\{g(t)\}1$$

2.1.2 Operation correspondence

Laplace Transform provides a tool to transform a function between the t-space and the s-space. Here are some operation correspondence between the two spaces.

The first one is the **Addition and Subtraction** of s, where the function is $f(t)$ in the t-space and $F(s)$ in the s-space

$$.\ F(s-a) = L\{e^{at}f(t)\}$$

Notice: There is **NO** minus sign on at.

The next correspondence is the **Differentiation of** $f(t)$. This operation is in the t-space. Notice that this correspondence is highly related to the part using Laplace Transform to solve differential equations, and is widely applied to find the Laplace Transform When The Function Is Given. The Given function means $f^{n-1}(0)$ can be calculated. We know that

$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f^{'}(0) - ... - f^{n-1}(0)$$

Hence,

$$s^n L\{f(t)\} = L\{f^n(t)\} + s^{n-1}f(0) + ... + f^{n-1}(0)$$

Therefore,

$$L\{f(t)\} = \frac{L\{f^n(t)\}+s^{n-1}f(0)+...+f^{n-1}(0)}{s^n}$$

If we want to find the Laplace Transform of a given function, then we can

- find the differentiation of that function
- find $f(0)$ to $f^n(0)$
- use the formula above to find $s^n L\{f(t)\}$
- divide it by s^n

Another differential transform is the **Differential of** $F(s)$, where $F(s)$ is the Laplace Transform of $f(t)$. This operation is in the s-space. Know that

$$F^{'}(s) = -L^{-1}\{tf(t)\}$$

Given that Laplace Transform is linear,

$$F^{'}(s) = L^{-1}\{-tf(t)\}$$

That's the operation correspondence of $F^{'}(s)$, which is to time a $-t$ in the t-space

Similar to the differential of $F(s)$, we may have an **Integration of** $F(s)$. This is also in the s-space and is to integrate the variable s from 0 to infinite. The expression is

$$\int_s^\infty f(\sigma)\, d\sigma = L^{-1}\{\tfrac{f(t)}{t}\}$$

Where σ is just a temporary variable.

The next correspondence is **Multiplication of** s^{-n}, where $n > 0$. This operation is in the s-space, and is highly used to find the inverse of Laplace Transform.

We want to find $L^{-1}\{s^n F(t)\}$, where the inverse Laplace Transform of $F(t)$ could be found. For the simplest case when $n = 1$ (which is s^{-1}) We may apply the working flow below,

- Firstly, find the $L^{-1}\{F(t)\}$. Denote the result as $f(t)$
- Then replace the variable t to τ, the $f(t)$ is converted into $f(\tau)$
- Integrate $f(\tau)$, from 0 to t. $\int_0^t f(\tau)\, dx$

The integration result corresponds to the multiplication of s^{-1} in the s-space

How about when $n \neq 1$? the result is to do the *iteration*. By using n times of integration to the function $f(t)$, which the L^{-1} can be found, we may find the value of $L^{-1}\{s^n f(t)\}$, when $n \neq 1$

Here's the end of this part. we look at five different operation correspondences in Laplace Transform. Only one action is in the t-space, called **Differentiation of** $f(t)$. Four are in the s-space, they are **Addition and Subtraction**, **Differentiation, Integration and Multiplication**. The next part will focus on methods selection, that's, under what circumstance should we use what method.

2.2 Find the Laplace Transformation And Its Inverse

This part will mainly focus on feature detection and method selection. The main focus will be finding the inverse of Laplace Transform, since it is more difficult and useful.

There are ten basic transformations on the table and the table will be provided in the final exam. When finding the Laplace Transform and its inverse, our goal is to convert a function with some other forms into these ten basic forms.

Fraction is the most frequent form that we may encounter. The general method is to use partial fractions. The goal for using partial fraction is to reduce the power of s. Some method like **Multiplication** introduced in the last part can also be applied. Here are some features that strongly indicate the method we will use.

The first one is **The Appearance of** e^{-as}. Knowing that

$$L\{u(t-a)f(t)\} = e^{-as}L\{f(t+a)\}$$

Notice: There **Is** a minus sign in front of as and there **Is** a $t+a$ term in the $f(t)$

And we also know that

$$L^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

Notice: There **Is** a $t-a$ term within the bracket of f

The $u(t-a)$ is the Step Unit Function where steps at $t=a$. Keep in mind that the term e^{-as} in the s-space is a strong indication that there will be a step unit function term involved in the original $f(t)$ But in real practice, we always write the answer in the piecewise form.

The second feature is the **Appearance of Piecewise Function**. This is a t-space feature since $f(t)$ is divided into different intervals and have different values over these intervals.

The solution to piecewise $L\{f(t)\}$ is to use unit step function. For example

$$f(t) = \begin{cases} 2 & 0 < t < 1 \\ t^2/2 & 1 < t < \pi/2 \\ \cos t & t > \pi/2 \end{cases}$$

Solution.

$$f(t) = 2(1-u(t-1)) + t^2/2 \cdot (u(t-1) - u(t-\pi/2))$$
$$+ \cos t u(t-\pi/2)$$

Figure 3: Example of $L\{f(t)\}$, where $f(t)$ is a piecewise function

Above is a screen shot of the lecture side. For an interval ranging from a to b, we use $u(t-a)-u(t-b)$ to represent that interval. By multiply the value of function as the coefficient in front of the $u(t-a)-u(t-b)$, we successfully convert a piecewise function into a continue function.

One more indication is the **Product of Two Functions**. Denote them as $f(t)$ and $g(t)$ respectively. This is an indication in the s-space, which $L^{-1}\{f(t)\}$ and $L^{-1}\{g(t)\}$ is easy to be found but finding L^{-1} of their product is difficult.

In order to find the inverse of Laplace Transform in this case, we need to use Convolution. I would like to omit this part here since convolution is the next part and is estimated to have at least one question in the final

Another indication is the **High Power In the Denominator With Signal Term**. It is a feature in the s-space and may due to the quotient rule in taking differentiation, especially when the power is an even number. Therefore, if you find a s-space fraction that cannot be transformed using table directly and has an even power number with a signal term in the denominator, then this method should be considered.

The next indication is the **Polynomial In the Triangular Function**. This is a feature in the t-space. Like $sin(3t-1)$. The polynomial term requires to expand the triangular function, since there are no correspond operation of addition and subtraction in the t-space to the s-space.

Until now, 5 different features are covered. Two are in the t-space, they are **Appearance of Piecewise Function**, **Polynomial in the Triangular function**. And rest of them are in the s-space. They are **Appearance of** e^{-as}, **High Power in Denominator with Signal Term** and the **Product of two functions**.

3 Convolution

Convolution is also an important part in the Laplace Transform. One question will relate to this point. The usage of convolution is to **Find the Inverse of Laplace Transform**, which is also related to the last part of this review.

3.1 Calculate the Convolution

The convolution is defined using an integral. It is a calculation within the t-space. Denote $h(t)$ as the convolution of two functions $f(t)$ and $g(t)$, then,

$$f * g = h(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Here the variable τ is just a temporary variable. Notice that we use $f(\tau) * g(t-\tau)$ because we are finding $f * g$. If we want to find $g * f$, then we should use $f(t-\tau) * g(t)$.

The Convolution has the following properties.

- $f * g = g * f$ They are commutative
- $f * 0 = 0$ The convolution with 0 is still 0
- $f * 1 \neq f$

3.2 Application of Convolution

3.2.1 To Find the Laplace Inverse

Finding the Laplace inverse is one of the applications of convolution

If we let $F(s) = L\{f(t)\}$, $G(s) = L\{g(t)\}$ and $H(s) = L\{h(t)\}$, then convolution can be expressed by

$$h(t) = f(t) * g(t)$$

Substitute t-space functions with its Laplace Transform and inverse, we get

$$L^{-1}\{H(s)\} = L^{-1}\{F(s)\} * L^{-1}\{G(s)\}$$

This expression means the Laplace Inverse of one function can be represented by the convolution of two Laplace Inverse of two functions in the s-space.

Another important property of the two functions is

$$H(s) = F(s)G(s)$$

Remember that, another feature in the s-space which may have strong indication about using the convolution is the **Production of Two Functions**. The Laplace Inverse of two functions here should be easy to be found. We may first rewrite the product of two functions to be $F(s)$ and $G(s)$, them we can apply the above formula to get the result

3.2.2 To Solve the Integration equation

Another usage for convolution is to solve integration equation. Here gives an example of how to solve this question

Ex. 5 Solve the integral equation

$$y(t) = t + \int_0^t y(\tau)\sin(t-\tau)d\tau$$

Figure 4: We will use convolution to solve this integral equation

The work flow for solving this kind of equation is pretty straight forward.

- Firstly, identify the convolution integral part, find out which two functions are doing convolution. In this example, they are y and $sin(t)$. We may denote this convolution as $y * sin(t)$
- Then the expression can be changed into

  $$y = t + y * sin(t)$$

- Apply Laplace transform to this expression

  $$Y = \frac{1}{s^2} + \frac{Y}{s^2+1}$$

  Notice that the convolution operator $*$ is disappeared because convolution of two Laplace inverse can be regraded as the product of two Laplace transform.
- Finally solve for Y and find its inverse.

Here's all the matters for convolution. It is also estimated that there will be at least on problem related to convolution. In fact, using convolution to solve for differential equation is pretty similar to the next part — using Laplace transform to solve differential equations. I think this will be the last part for the Laplace transform.

4 Using Laplace Transform to Solve Differential Equations

It seems that all important topics of Laplace transform have been discussed in earlier parts. This part is just the application of finding the Laplace transform and its inverse. It is also estimate that there will be at least one question in the final about using the Laplace transform to solve differential equation. The systems of differential equations will not be included in the final, according to the professor.

4.1 Find the Subsidiary Equation

The subsidiary equation is the expression of the differential equation in the s-space. It makes the equation more easier to compute. The bridge between differential equation in the t-space and subsidiary equation in the s-space is Laplace transform.

4.1.1 Laplace Transform for Differential Equation

In order to convert differential equation into subsidiary equation, we need to do the Laplace transform.

We know that:

$$L\{f^n\} = s^n L\{f\} - s^{n-1}f(0) - ... - f^{n-1}(0)$$

And if we replace f^n with y^n and $L\{f\}$ with Y, then

$$L\{y^n\} = s^n Y - s^{n-1}f(0) - ... - f^{n-1}(0)$$

Notice: the power of s is decrease from n to 0, while the "power" of f is increase from 0 to $n-1$

The second formula provides method of transforming y^n to Y. For a general differential equation, there are always some coefficients in front of different y^n. Therefore, remember to multiply these coefficients after the transformation.

For other parts in the differential equations, just apply the Laplace transform method in the Chapter 2 is okay.

4.1.2 Inverse Laplace Transform for Subsidiary Equation

After finding the subsidiary equation, we may apply inverse Laplace transform to find the solution. Firstly, the LHS of the equation should just has a Y with the coefficient 1. The inverse Laplace should be applied to the RHS, which means the RHS is a whole.

Since methods of finding the inverse Laplace Transformation has been introduced, please refer to Chapter 2.2 to find more information about this part.

5 Undetermined Coefficient and Variation of Parameters

This part is estimated to have one question in the final and is not included in the midterm test. According to the professor, the high order differential equation is also included.

Since these two methods are used for solving the Inhomogeneous equation, and because solving the homogeneous equation is the basic for solving inhomogeneous equation, therefore, this part will firstly focus on how to solve an homogeneous equation

5.1 Solving the homogeneous Equation

To solve an inhomogeneous equation, we first need to solve the homogeneous equation.

There are two kinds of homogeneous equation. We focus on the simplest case first.

5.1.1 The Most Common Case

The standard form of homogeneous equation is

$$y^{''} + ay^{'} + by = 0$$

We can write down its characteristic equation, by letting $y = e^{\lambda x}$ and make substitution

$$\lambda^2 + a\lambda + b = 0$$

Denote the solutions to be λ_1 and λ_2 ,when $\Delta > 0$; $-\frac{a}{2}$, when $\Delta = 0$; and $\lambda_{1,2} = \alpha \pm \omega i$, when $\Delta < 0$

According to the value of Δ, there are 3 different situations.

Col1	x_1	x_2
$\Delta > 0$	$c_1 e^{\lambda_1 x}$	$c_2 e^{\lambda_2 x}$
$\Delta = 0$	$c_1 e^{-\frac{a}{2}x}$	$c_2 x e^{-\frac{a}{2}x}$
$\Delta < 0$	$e^{\alpha x} A cos\omega x$	$e^{\alpha x} B sin\omega x$

Notice: The power of e is the (real) solution of characteristic equation. The term within the triangular expression is the imaging part of the complex solution

This gives the solution of homogeneous equation in the standard form

5.1.2 The Euler-Cauchy Equation

The standard form of Eular-Cauchy Equation is

$$x^2 y^{''} + axy^{'} + by = 0$$

If we let $y = x^m$, then the characteristic equation is

$$m^2 + (a-1)m + b = 0$$

Notice: The coefficient in front of m is $a - 1$.

Similar to the homogeneous equation, we can also write down its solution in a table. Denote the solution to be $m = m_1$ and $m = m_2$ when $\Delta > 0$; $m = \frac{1-a}{2}$ when $\Delta = 0$, and $m = \mu \pm vi$ when $\Delta < 0$

Col1	x_1	x_2
$\Delta > 0$	$c_1 x^{m_1}$	$c_2 x^{m_2}$
$\Delta = 0$	$c_1 x^{\frac{1-a}{2}}$	$c_2 x^{\frac{1-a}{2}} lnx$
$\Delta < 0$	$x^{\mu} Acos(vlnx)$	$x^{\mu} Bsin(vlnx)$

Comparing these two tables, we can find these features in the "conversion" from the first table to the second table

- The e^{sth} replaced by x^{sth}
- Inside the triangular function, x replaced by lnx, but the position of real and imaginary part keep the same

After obtained the two solutions of equation, just sum them up and that's the final answer for solving a homogeneous equation Here's the end of this part.

5.2 Solving the Inhomogeneous Equation

The standard form of inhomogeneous equation is

$$y^{''} + p(x)y^{'} + q(x)y = r(x)$$

And its solution is

$$y = y_h(x) + y_p(x)$$

Where $y_h(x)$ is the solution of corresponding homogeneous equation, and the $y_p(x)$ is a particular solution to the inhomogeneous equation.

In the last section we introduced the solution to the homogeneous equation. This section will mainly focus on how to find h_p, by using undetermined coefficient and variation of parameters

5.2.1 Undetermined Coefficients

The method of undetermined coefficients is to imitate the form of $r(x)$ in the RHS, with some coefficients to be determined, by building a function $y_p(x)$. If unknown coefficients in the $y_p(x)$ can be found after substitution, then the $y_p(x)$ is the **Particular Solution** to the equation.

The most important step here is to find the $y = y_p(x)$ which has the correct form. Here is a table for reference about how to choose the form of $y_p(x)$

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n $(n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ $k \sin \omega x$	$\Big\}\, K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ $ke^{\alpha x} \sin \omega x$	$\Big\}\, e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

Figure 5: This is the table in the slide

From the table, we may found that some basic "elements" are more or less the same in the $r(x)$ and $y_p(x)$. But notice that the coefficient for triangular function is pretty special: for one $sin\omega x$ or $cos\omega x$ function, there are 2 coefficients here.

In fact, there is another way that no need to remember the table. That's, if we find the form of $r(x)$ is the same to one of the homogeneous solutions, then we can just multiply an x in the $y_p(x)$ we assumed

5.2.2 Variation of Parameters

The method of **Variation of Parameters** is used to find the particular solution y_p. To use this method, we should calculate the **Wronskian** first.

In second order differentiation, the Wronskian is the determinant of the Wronski matrix. The notation of Wronskian is $W(y_1, y_2)$, where y_1, y_2 are two solutions of the corresponding homogeneous equation

$$W(y_1, y_2) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_2 y_1' \tag{1}$$

The Wronskian can also be used to find the linear independent of two solutions. If $W \neq =$, then we say the two solutions is linear independent. What's more, if $\frac{y_1}{y_2} \neq 0$, they are also independent to each other.

The difficulty of using the variation of parameter is doing the integration. The method itself is not so hard because we have the following formula

$$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

Figure 6: This image is from the slide

Just plug in numbers and do the integration is okay.

Here is the end of this two methods.

6 Integrating Factor

The Integrating Factor is to convert an **Non-Exact** differential equation into an **Exact** differential equation.

6.1 First Order ODE

The 1st-order OED has the following forms:

$$M(x,y)dx + N(x,y)dy = 0$$

Or,

$$y^{'} + p(x)y = q(x)$$

They are interchangeable.
In the first form, we say the ODE is **Exact** if

$$\frac{\vartheta M}{\vartheta y} = \frac{\vartheta N}{\vartheta x}$$

If this condition is not satisfied, this is a not exact equation. Then the integrating factor is needed.

6.2 Calculate the Integrating Factor

To find the integrating factor, we should find R first. R is exclusively contains only x or y. Here's the steps to find R

- Find the $\frac{\vartheta M}{\vartheta y}$ and $\frac{\vartheta N}{\vartheta x}$. Calculate their difference
- Divide the difference by M or N, to make it only has variable x or y.

Once R is found,

- Calculate $e^{\int Rdx}$ or $e^{\int Rdy}$

That's the integrating factor for this form of differential equation.
We denote this integrating factor to be F. The solution of this ODE problem can be obtained by

$$FM(x,y)dx + FQ(x,y) = 0$$

Now we should the integration with respect to dx or dy, let's integrate with respect to dx, and denote the original function as $G(x)$

$$\int FM(x,y)dx = G(x)$$

Then take the partial derivative of $G(x)$ with respect to y, we will get G_y. Compare the G_y to the $FQ(x,y)$

$$h(y) = G_y - FQ(x,y)$$

Integrate the $h(y)$ with respect to y to get $H(y)$, we can find the solution of this ODE

$$y = G(x) + H(x)$$

That's all about how to find solution y to this kind of 1st order ODE

For the other form of the 1st order ODE, the integrating factor I is:

$$I = e^{\int p(x)dx}$$

The expression of y can be calculated directly

$$y = \frac{1}{I}(\int q(x)Idx \ + C)$$

Where $q(x)$ is the RHS of the differential equation and C is a constant