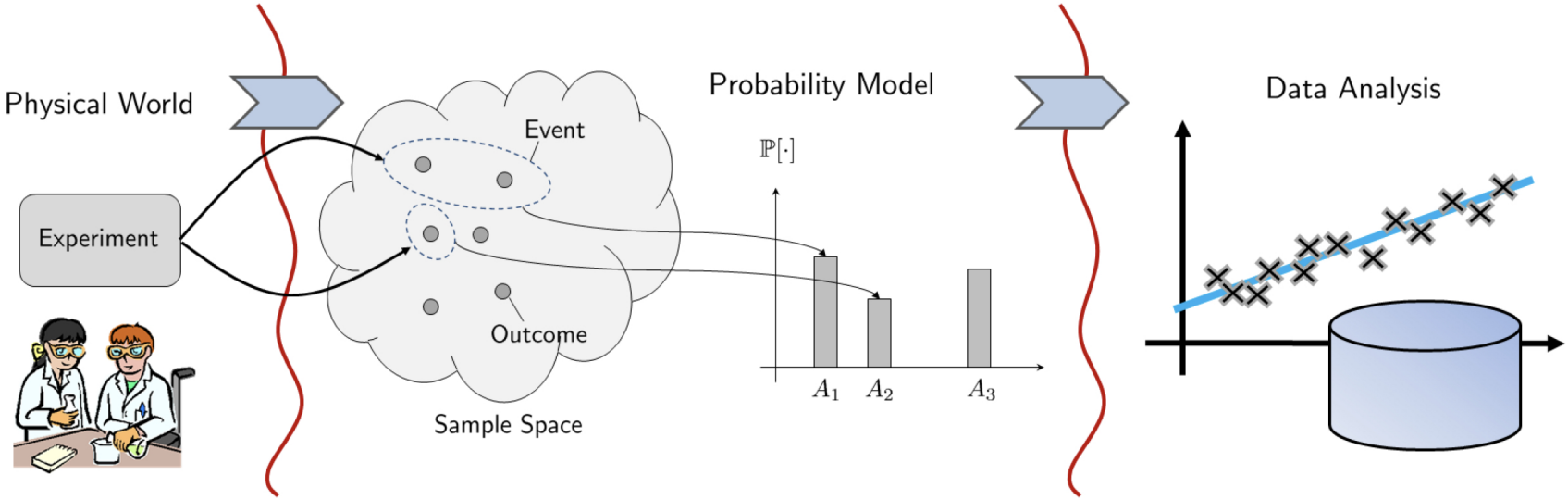
Probability & Statistics for DS & AI

Set Theory and Probability

Michele Guindani

Summer

- Probability: it is the tool that allows us to describe our world with uncertainty.
- We typically provide an approximate description of phenomena in the real world using the tools of probablity, and then we use probability to allow us to quantify the uncertainty in our description
- It's the base at the foundation of our statistics/data science models and algorithms



Probability and Set theory

- Ch.1 of the textbook provides mathematical backgrounds (skipped, will use what we need)
- We will not got into the phylosophical definition of probability, but we will present the axiomatic framework of Kolmogorov (1933)
- The notion of probability is related to that of an event, which mathematically can be described by the notion of set, a collection of elements

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- We will be interested in giving some sense/provide meaning to the following statement

$$P(A) = P(\{H\}) = \frac{1}{2}$$

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- $\Omega = \{\boxdot, \boxdot, \boxdot, \boxdot, \boxdot, \boxdot\} = \{1, 2, 3, 4, 5, 6\}$ ⇨ universal set (collection of all simple events when launching a die)
- Clearly, $B \subset \Omega$ is a complex event (formed by multiple simple elements) and $B = \{2, 4, 6\}$
- We will be interested in giving meaning to the following statement

$$P(B) = P(\{\text{even number}\}) = \frac{3}{6} = \frac{1}{2}$$

Set theory

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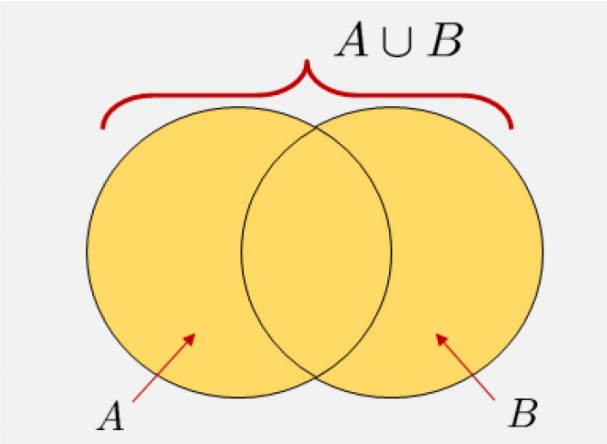
- In order to compute probabilities of complex events, it's often useful to be able to compute and represent an event through operations with sets
- We will recall 4 set operations:
  1. Union of sets
  2. Intersection of sets
  3. Complement of a set
  4. Difference between sets
- The textbook provides a more thorough discussion (invitation to read)

Union of two sets

- The union

$$A \cup B = \{\xi \mid \xi \in A \text{ or } \xi \in B\}$$

of two sets contains all elements in A or B.



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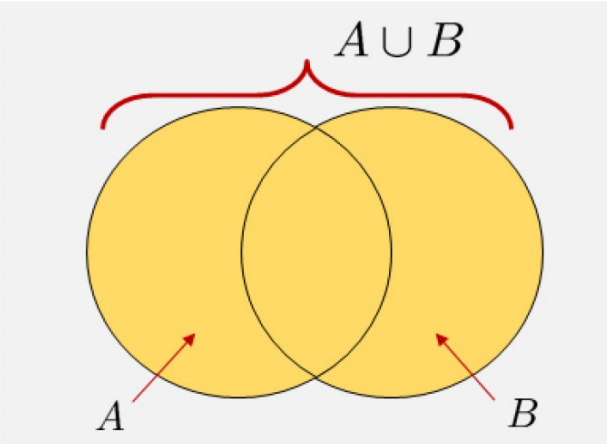

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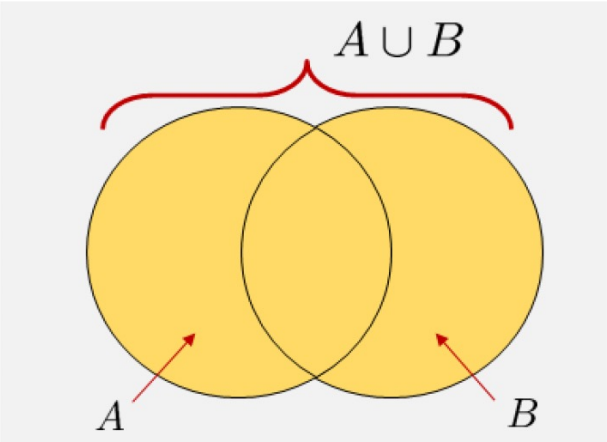

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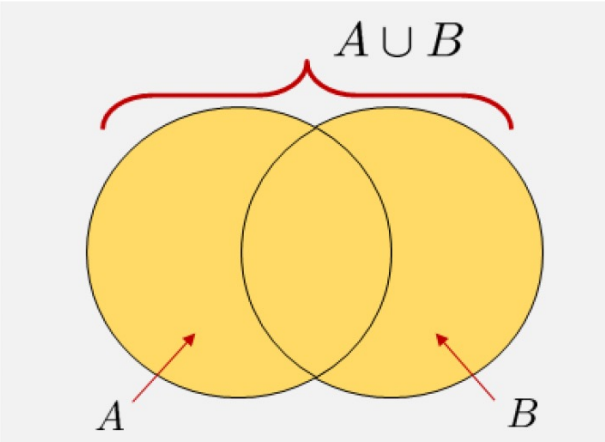

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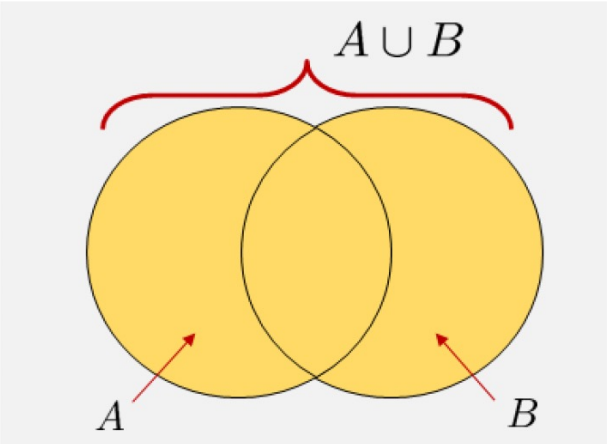

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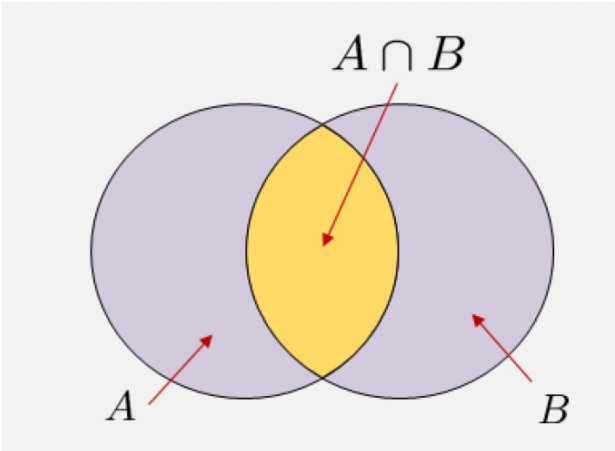

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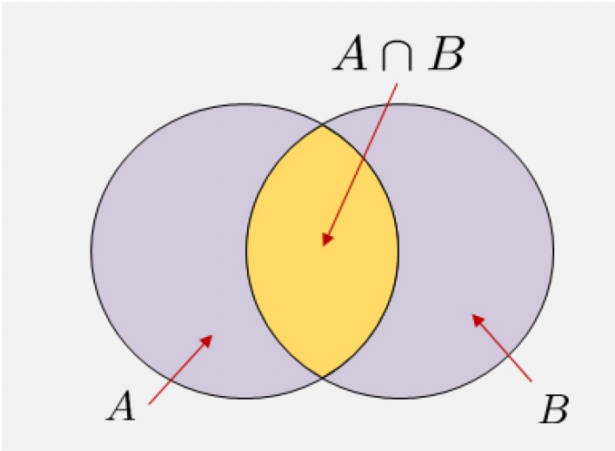

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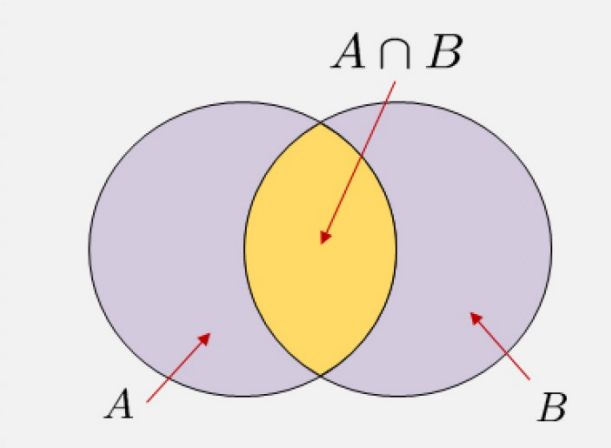

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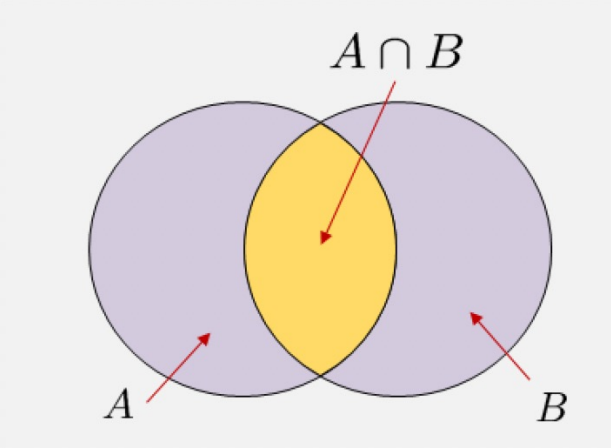

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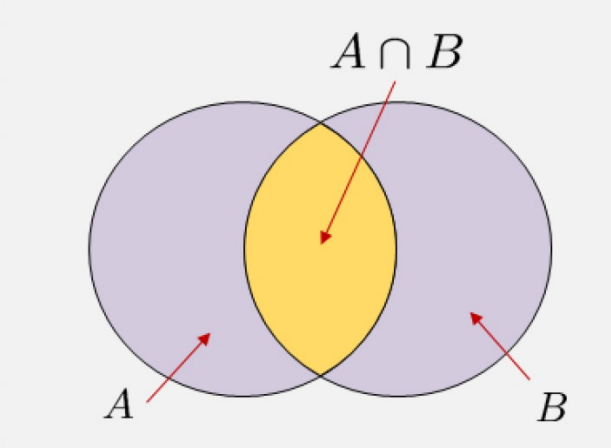

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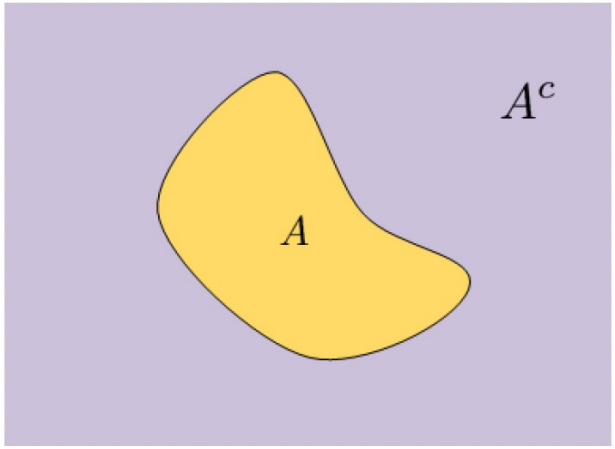
$$A^c = \{\xi \mid \xi \in \Omega \text{ and } \xi \notin A\}$$

contains all elements that are in Ω but not in A



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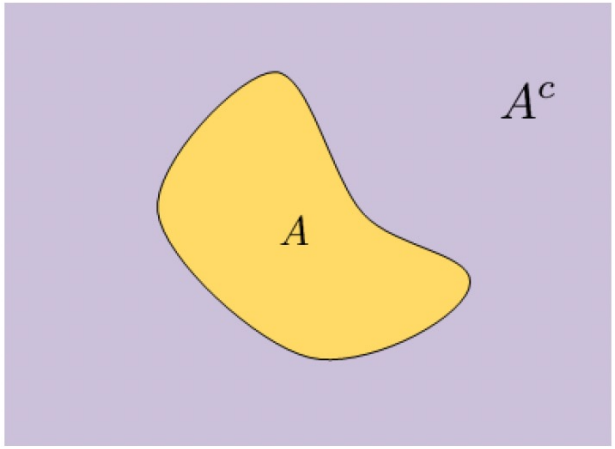

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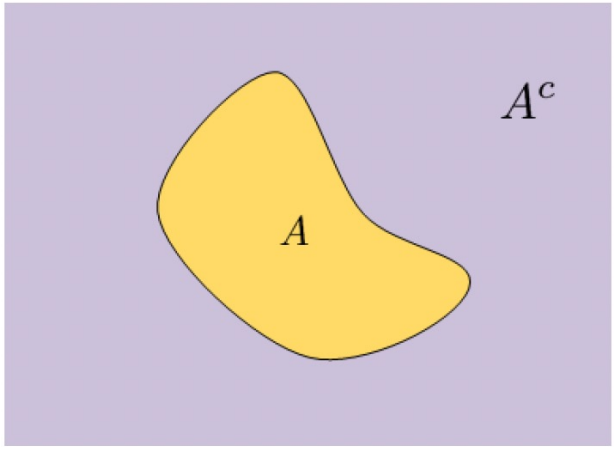

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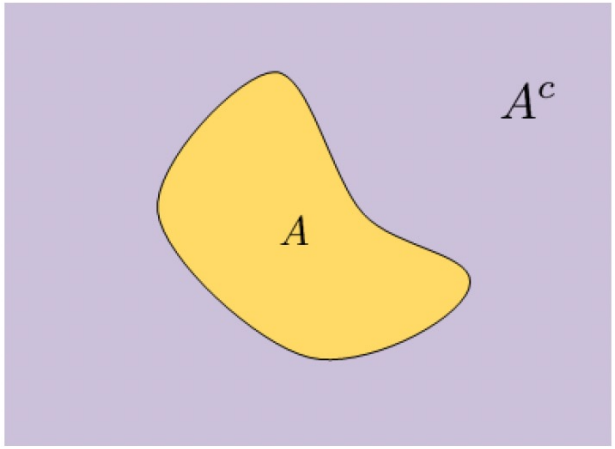

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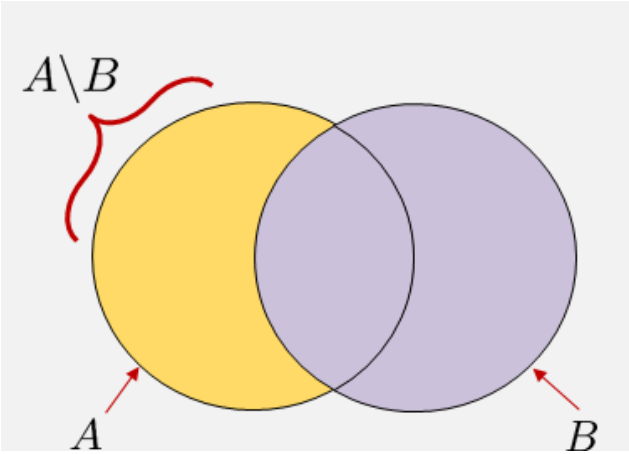

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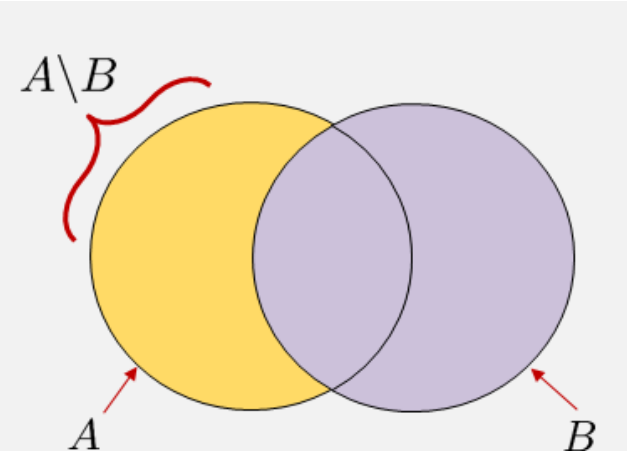

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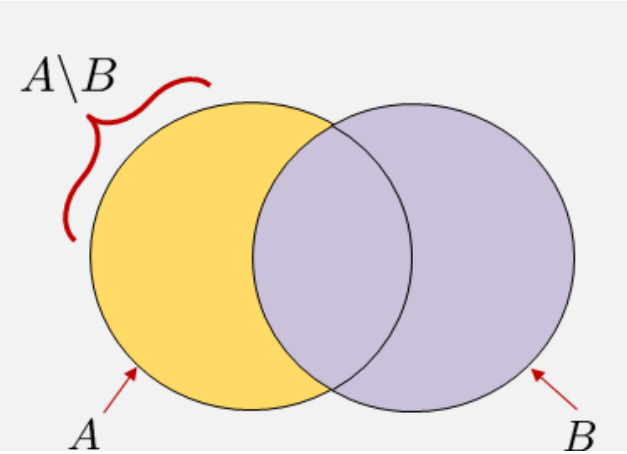

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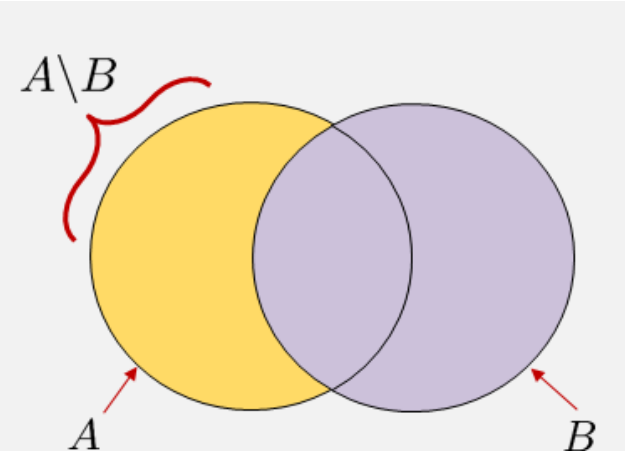

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- It can be shown that $A \setminus B = A \cap B^c$.

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Partition

A collection of sets $\{A_1, \ldots, A_n\}$ is a partition of the universal set Ω if

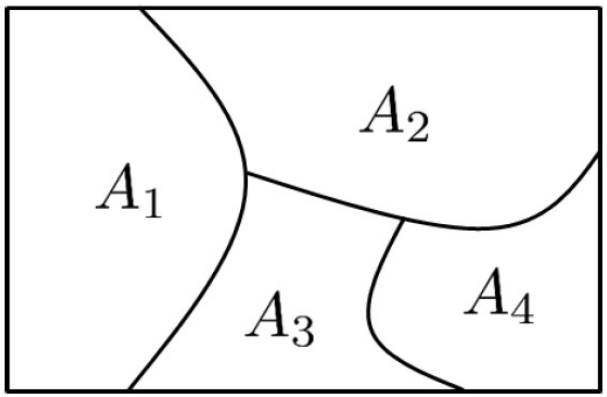
① the set don't overlap, i.e. $\{A_1, \ldots, A_n\}$ is dijoint:

$$A_i \cap A_j = \emptyset$$

② The union of $\{A_1, \ldots, A_n\}$ gives the universal set:

$$\bigcup_{i=1}^{n} A_i = \Omega$$

Example (Examples of Partition)



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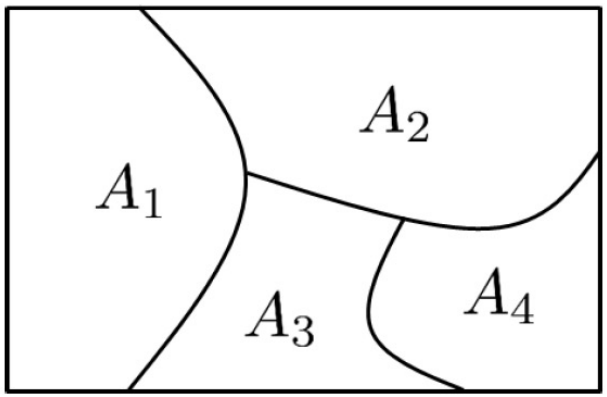

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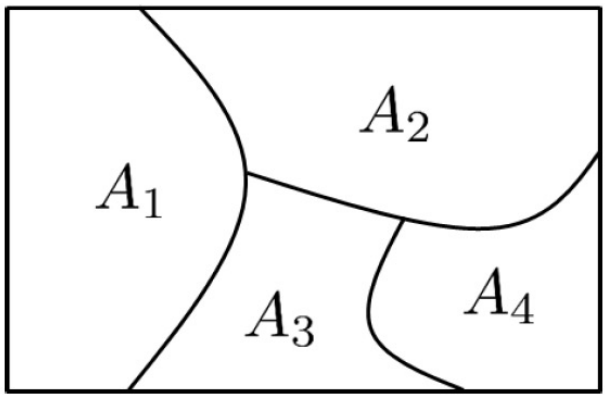

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- Sec. 2.1.9: look at other (simple) set operations

Probability & Statistics for DS & AI

Building the concept of Probability from a mathematical perspective

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Sample space

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Sample space Ω

A sample space Ω is the set of all possible outcomes from an experiment. We denote ξ as an element in Ω.

Example (Discrete outcomes)

- Coin flip: $\Omega = \{H, T\}$
- Throw a die: $\Omega = \{\boxdot, \boxdot, \boxdot, \boxdot, \boxdot, \boxdot\}$
- Numnber of points by Lebron James in a match: $\Omega : \{0, 1, 2, \ldots\}$
- Recovery from a disease $\Omega = \{\text{No, Yes}\} = \{0, 1\}$.

Example (Continuous Outcomes)

- Waiting time for a bus in Vashi: $\Omega = \{t \mid 0 \leq t \leq 30 \text{ minutes } \}$
- Winnings of golfers on the PCGA (Professionals' golfers association) tour $\Omega : \{x \mid x \geq 0\}$
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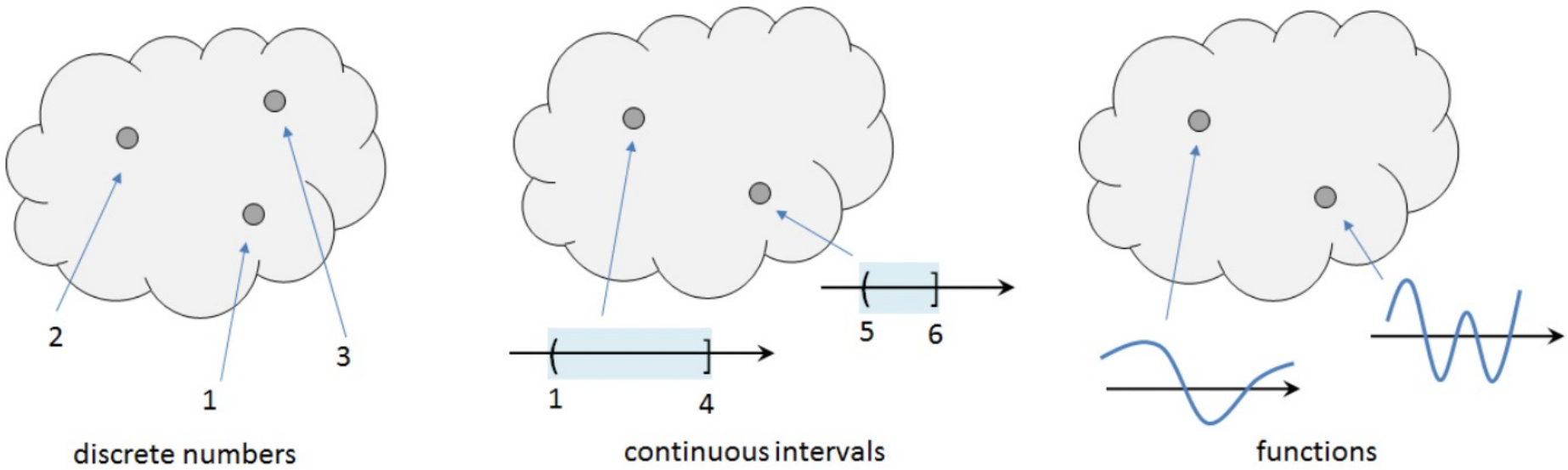
Elements in the sample space can be anything.



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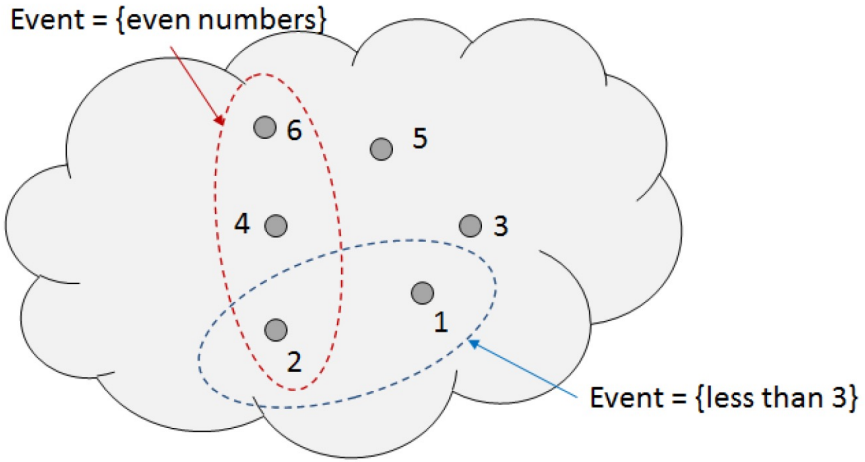
Die rolling

An event E is a subset in the sample space Ω. The set of all possible events is denoted as $\mathcal{F}$.

Example



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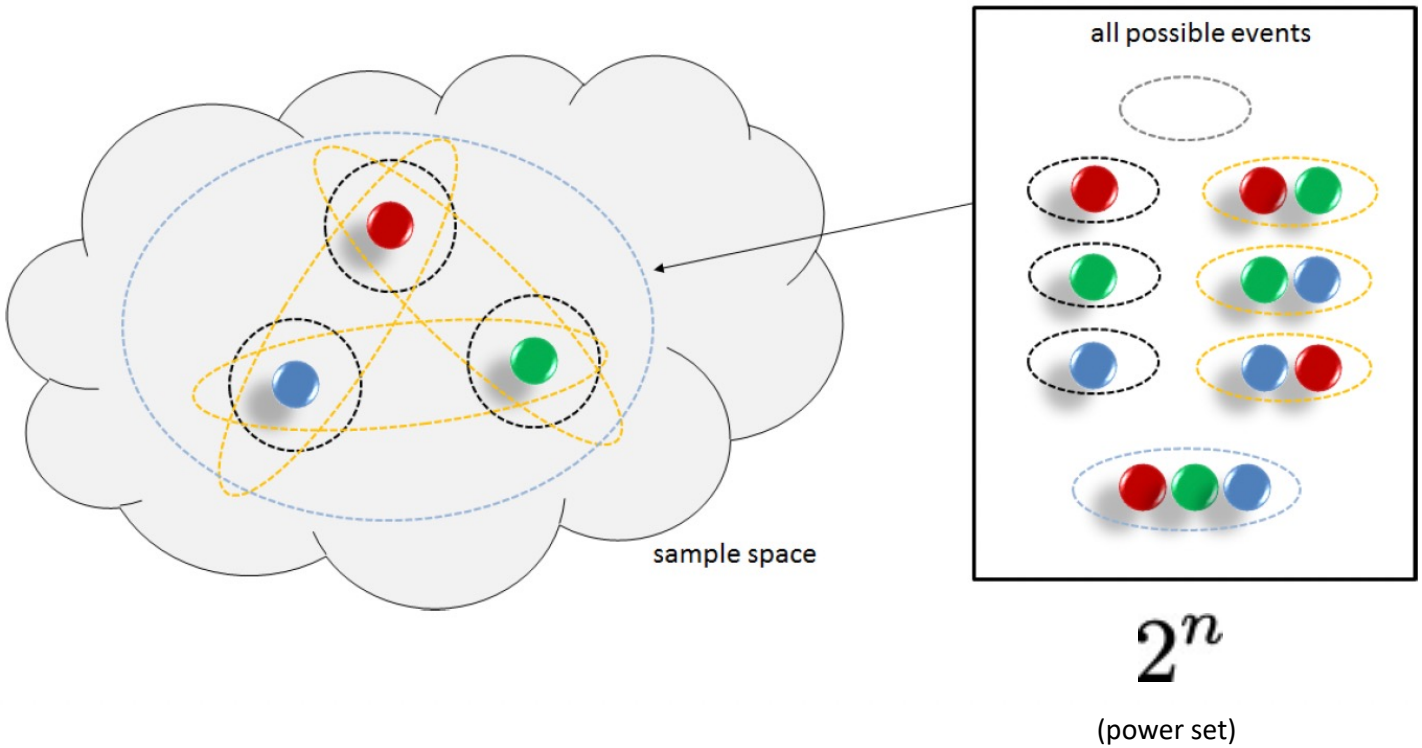

Probability law $\mathbb{P}$

Definition: Part 1

A probability law is a function $\mathbb{P}$ that associates to any event $E \in \mathcal{F}$ a real number in $[0, 1]$.

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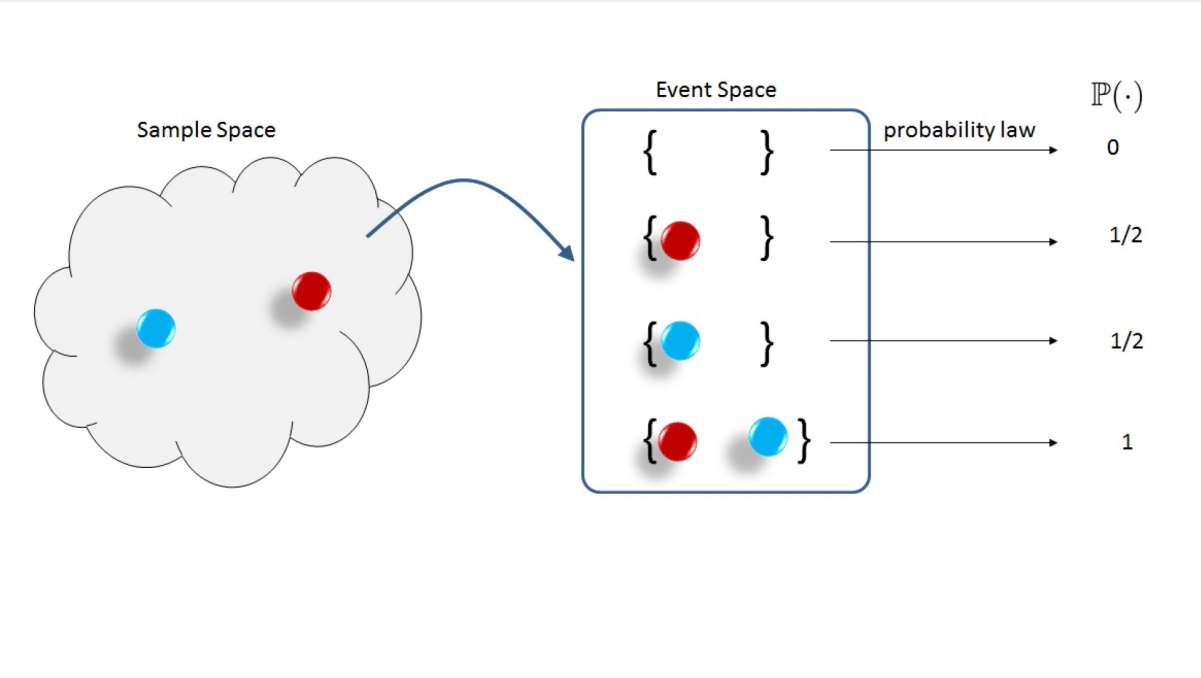

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2. Normalization: $\mathbb{P}[\Omega] = 1$
3. (Countable) Additivity: For any disjoint sets $\{A_1, A_2, \ldots\}$, it must be true that

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} \mathbb{P}\left[A_i\right]$$

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Why these three axioms?

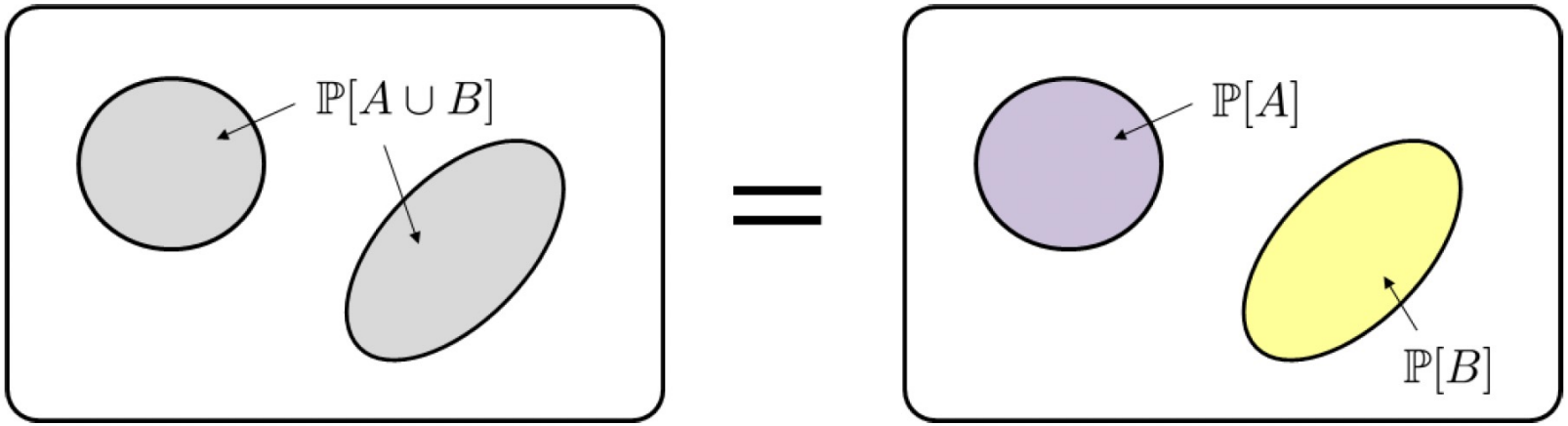
Axiom ① (Non-negativity) ensures that probability is never negative.

Axiom ② (Normalization) ensures that probability is never greater than 1.

Understanding additivity

If A and B are disjoint, then

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B],$$



Why these three axioms?

Axiom ③ (Additivity) allows us to add probabilities when two events do not overlap.

Example (Throwing a fair die)

- $\Omega = \{\boxdot, \boxdot, \boxdot, \boxdot, \boxdot, \boxdot\} = \{1, 2, 3, 4, 5, 6\}$
- Let's consider the probability of getting $\{\boxdot, \boxdot, \}$:

$$P(\{\boxdot, \boxdot, \}) = P(\{\boxdot\} \cup \{\boxdot\}) = P((\{\boxdot\}) + P(\{\boxdot\}) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

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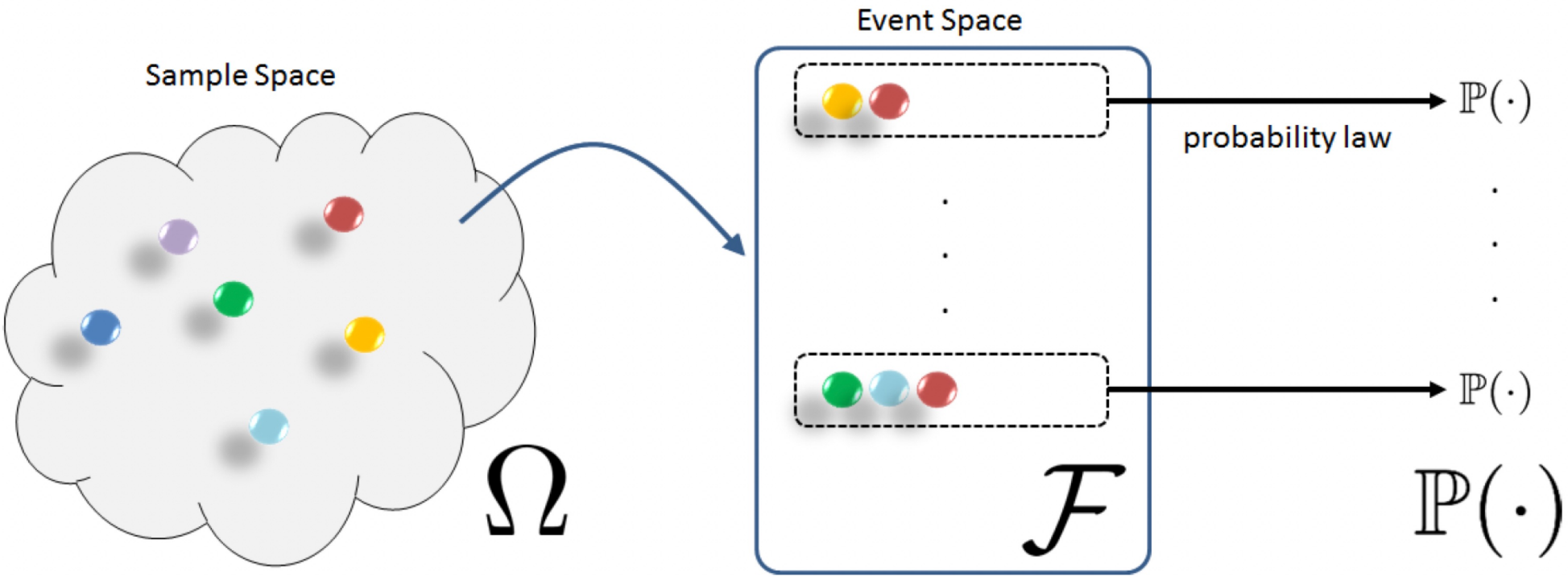
- Suppose that for a baseball player, $P(S) = 0.2$ and $P(D) = 0.05$. Then

$$P(S \text{ or } D) = P(S \cup D) = P(S) + P(D) = 0.25$$

Probability Space

A probability space consists of a triplet:

$$(\Omega, \mathcal{F}, \mathbb{P})$$



Properties of Probability

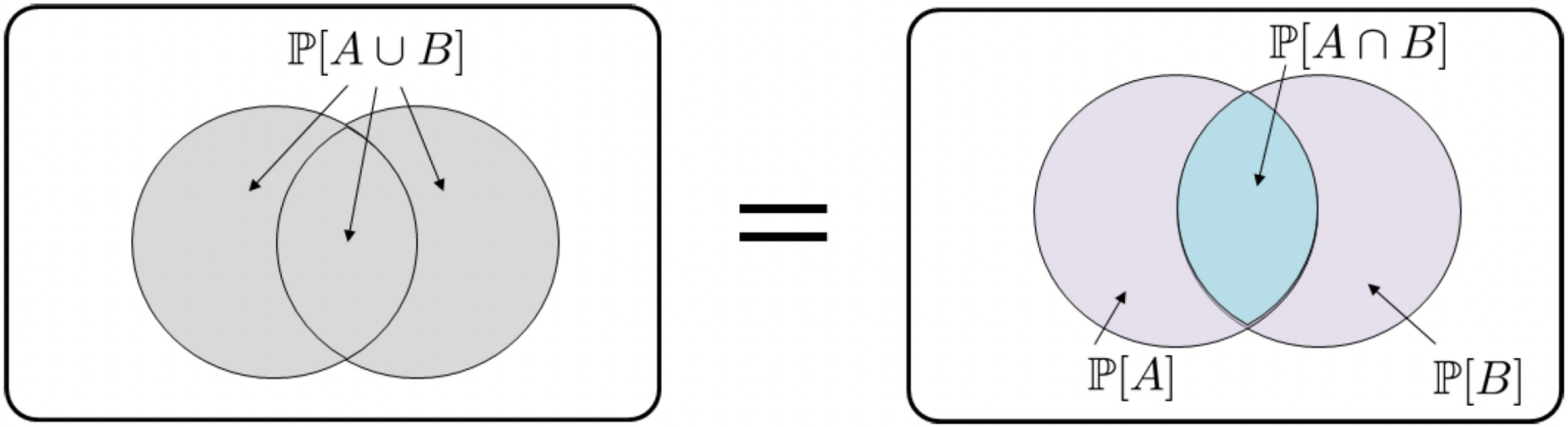
① $\mathbb{P}\left[A^c\right] = 1 - \mathbb{P}[A]$

② For any $A \subseteq \Omega, \mathbb{P}[A] \mid \leq 1$

③ $\mathbb{P}[\emptyset] = 0$

④ For any A and B

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$$



This statement is different from Axiom 3 because A and B are not necessarily disjoint.

Example (Fair coin tossing)

Let $\Omega = \{\boxdot, \boxdot, \boxdot, \boxdot, \boxdot, \boxdot\} = \{1, 2, 3, 4, 5, 6\}$ be the sample space of a fair die.

Let $A = \{\boxdot, \boxdot, \boxdot\}$ and $B = \{\boxdot, \boxdot, \boxdot\}$.

Then

$$P(A \cup B) = P(\{\boxdot, \boxdot, \boxdot, \boxdot, \boxdot\}) = \frac{5}{6}$$

but also

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(\{\boxdot, \boxdot, \boxdot\}) + P(\{\boxdot, \boxdot, \boxdot\}) - P(\{\boxdot\}) \\ &= \frac{3}{6} + \frac{3}{6} - \frac{1}{6} = \frac{5}{6} \end{aligned}$$

Properties of Probability

⑤ For any A and B

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⑥ If $A \subseteq B$, then

$$\mathbb{P}[A] \leq \mathbb{P}[B]$$

Example

$A = \{t \leq 5\}$, and $B = \{t \leq 10\}$, then $\mathbb{P}[A] \leq \mathbb{P}[B]$

Probability & Statistics for DS & AI

Conditional Probability

Michele Guindani

Summer

Conditional Probability

- In many practical data science problems, we are interested in the relationship between two or more events.
- For example, an event A may make B more likely or less likely to happen, and B may make C more or less likely to happen.
- A legitimate question in probability is then: If A has happened, what is the probability that B also happens?
- Of course, if A and B are correlated events, then knowing one event can tell us something about the other event.
- If the two events have no relationship (they are "independent"), knowing one event will not tell us anything about the other.

Conditional Probability

Consider two events A and B. Assume $\mathbb{P}[B] \neq 0$. The conditional probability of A given B is

$$\mathbb{P}[A \mid B] \stackrel{\text{def}}{=} \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

- Intuitively, the conditional probability of A given B is the probability that A happens when we know that B has already happened. Since B has already happened, the event that A has also happened is represented by $A \cap B$. However, since we are only interested in the relative probability of A with respect to B
- The difference between $\mathbb{P}[A \mid B]$ and $\mathbb{P}[A \cap B]$ is the denominator they carry:

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} \text{ and } \mathbb{P}[A \cap B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[\Omega]}$$

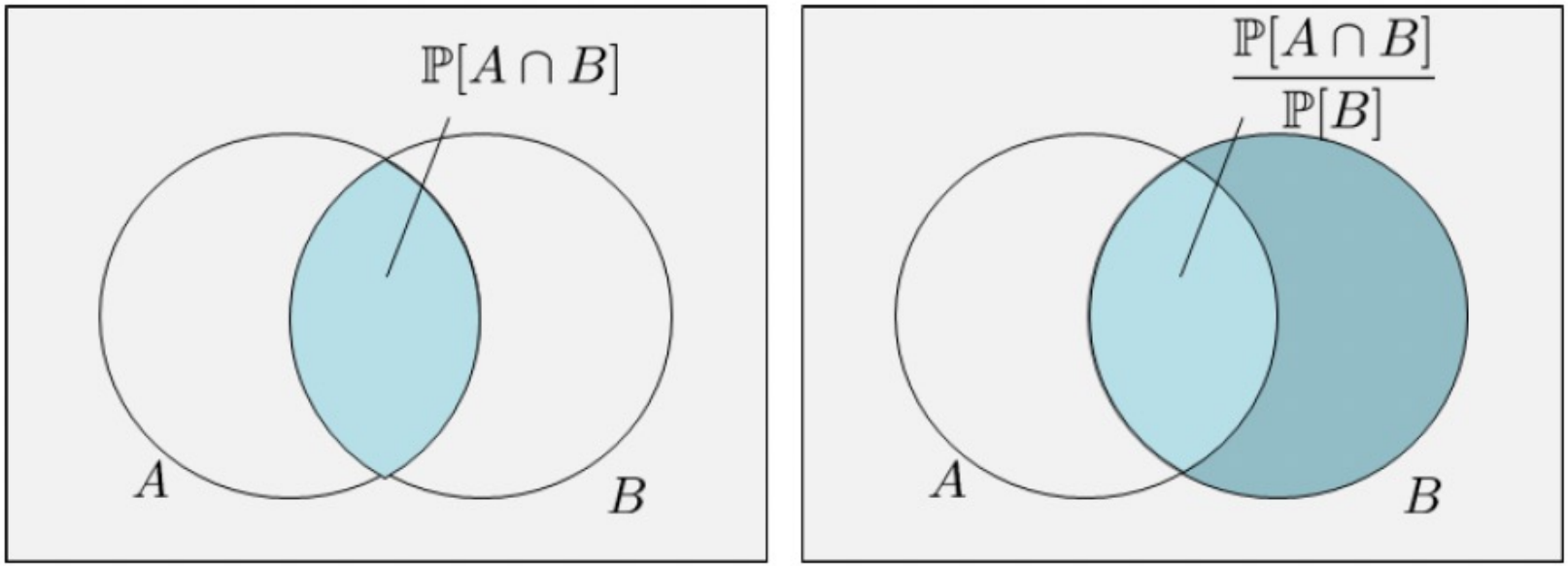

Figure: Illustration of conditional probability and its comparison with $\mathbb{P}[A \cap B]$.

Example (Throwing a die)

Consider throwing a die. Let

$$A = \{ \text{ getting a } 3\} \quad \text{and} \quad B = \{ \text{ getting an odd number } \}$$

The two probabilities are easy to compute:

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Then,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

And,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{6}}{\frac{1}{6}} = 1$$

Independence

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Throw a dice twice. Let

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- What is independence?
- One event does not affect the other event!
- Are A and B independent then?

Independence

Two events A and B are statistically independent if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$$

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- Why define independence in this way? Recall that $\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$.

  If A and B are independent, then $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$ and so

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}[A]\,\mathbb{P}[B]}{\mathbb{P}[B]} = \mathbb{P}[A]$$

- Intuitively, if the occurrence of B provides no additional information about the occurrence of A, then A and B are independent.

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Are A and B independent?
Let's compute

$$P(A) = \frac{1}{6} \qquad P(B) = \frac{1}{6}$$

We can also compute

$$P(A \cap B) = P(\text{ 1st dice is 3 and 2nd dice is 4 out of 36 combinations}) = \frac{1}{36}$$

Since

$$P(A \cap B) = P(A)\, P(B)$$

then the two events are statistically independent.

Example (Throwing a dice twice)

Let

$$A = \{1 \text{ st dice is } 1\} \quad \text{and } B = \{ \text{ sum is } 7\}$$

Are A and B independent?

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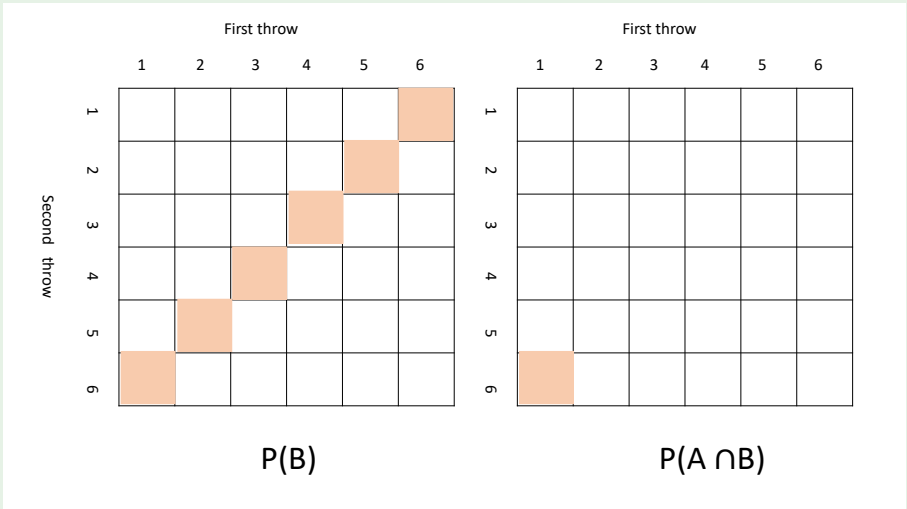
Let's compute

$$P(A) = \frac{1}{6} \quad P(B) = \frac{6}{36} = \frac{1}{6}$$

We can also check that

$$P(A \cap B) = \frac{1}{36}$$



So,

$$P(A \cap B) = P(A)\, P(B)$$

and the two events are independent.

Example (Throwing a dice twice)

Let

$$A = \{1 \text{ st dice is } 2\} \quad \text{and } B = \{ \text{ sum is } 8\}$$

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Are A and B independent?

Let's compute

$$P(A) = \frac{1}{6} \quad P(B) = \frac{5}{36}$$

We can also check that

$$P(A \cap B) = \frac{1}{36}$$

So,

$$P(A \cap B) \neq P(A)\, P(B)$$

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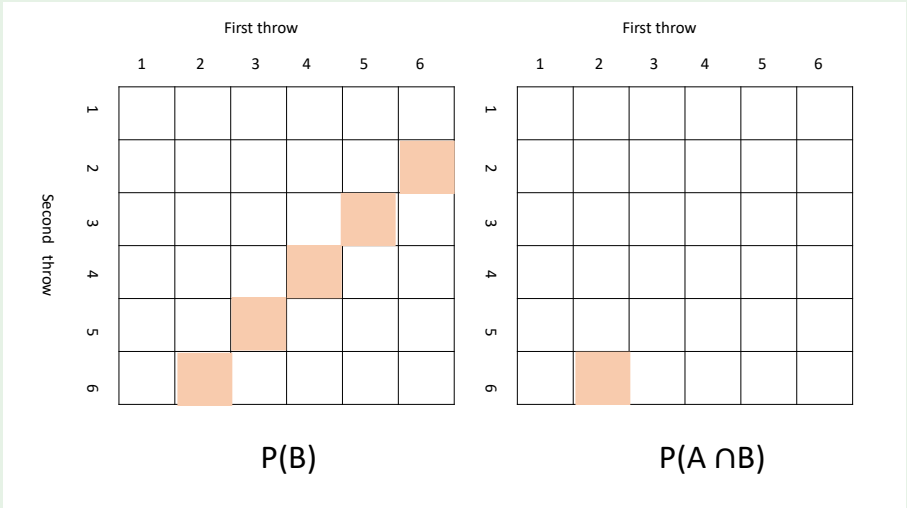

Example (Throwing a dice twice)

Let

$$A = \{1 \text{ st dice is } 3, 4, \text{ or } 5\} \quad \text{and } B = \{ \text{ sum is } 9\}$$

Are A and B independent?

Example (Throwing a dice twice)

Let

$$A = \{1 \text{ st dice is } 3, 4, \text{ or } 5\} \quad \text{and } B = \{ \text{ sum is } 9\}$$

Are A and B independent?

Let's compute

$$P(A) = \frac{3}{6} = \frac{1}{2} \quad P(B) = \frac{4}{36} = \frac{1}{9}$$

We can also check that

$$P(A \cap B) = \frac{3}{36} = \frac{1}{12}$$

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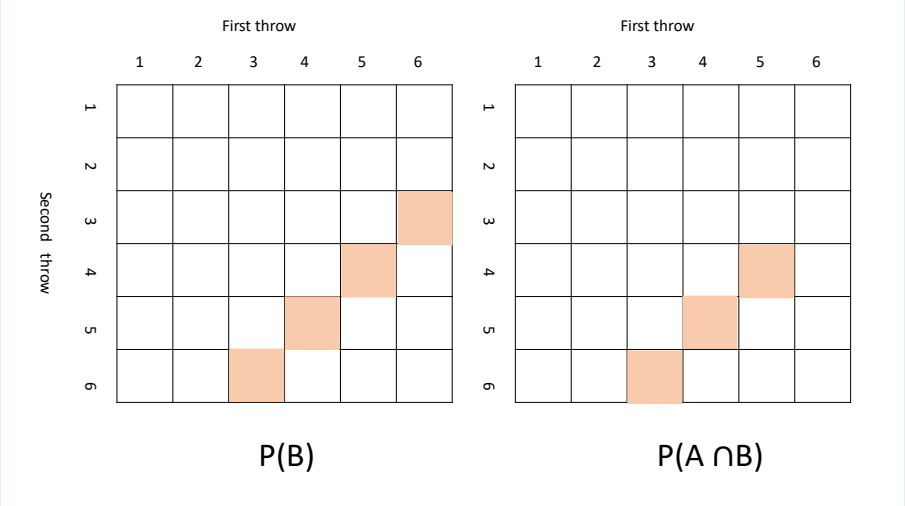

Bayes Theorem

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- Let $+$ indicate that someone tested *positive* for the disease in a screening test, and $-$ that they tested *negative*
- We are interested to know $P(D|+)$. But we may know:

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  Sensitivity of the test: how good is the test at identifying people who have the disease, $Pr(+|D) = 0.98$

  Specificity of the test: how good is the test at correctly identifying healthy individuals: $Pr(-|H) = 0.95$

Bayes Theorem

For any two events A and B such that $\mathbb{P}[A] > 0$ and $\mathbb{P}[B] > 0$,

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[B \mid A]\,\mathbb{P}[A]}{\mathbb{P}[B]}$$

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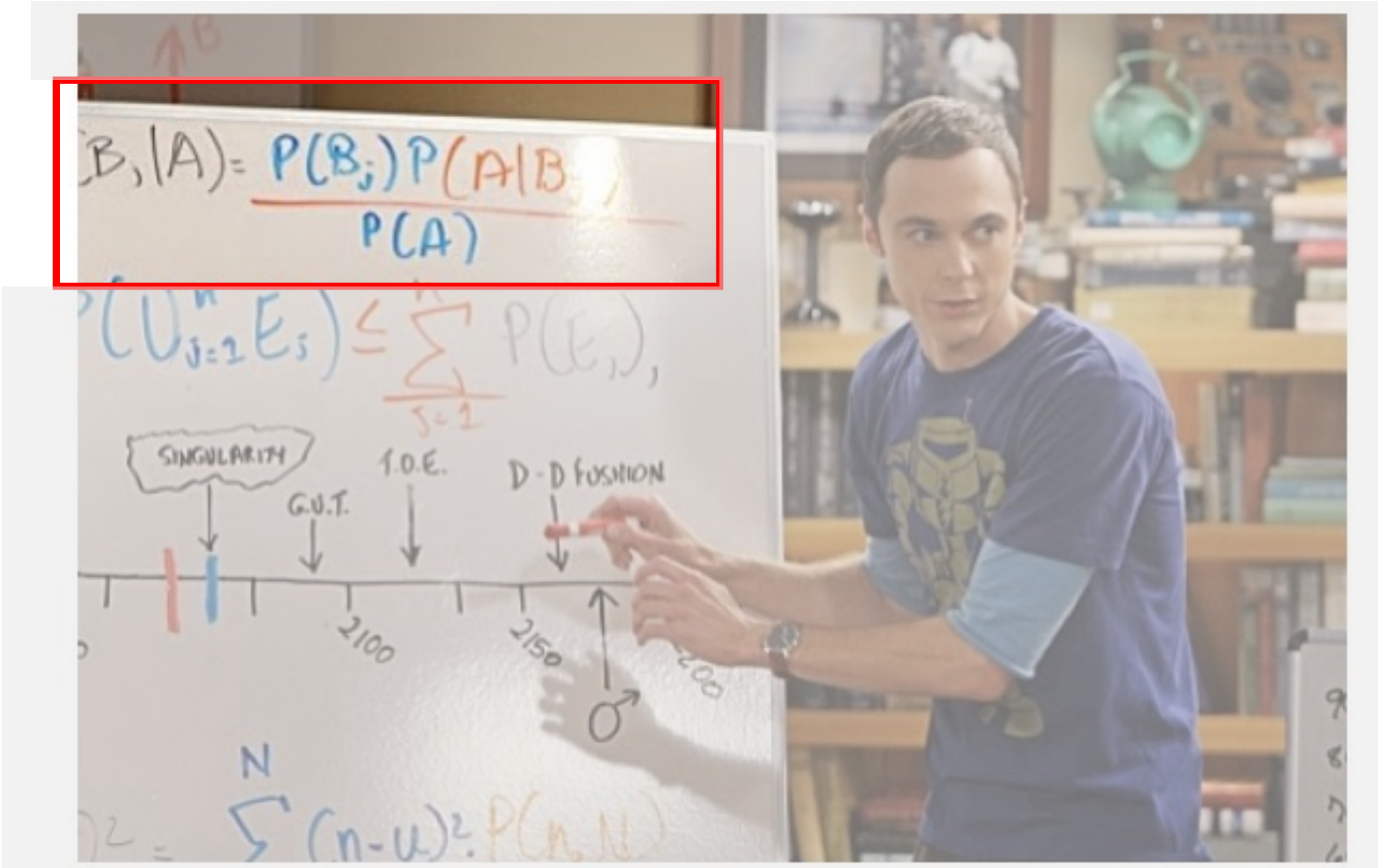
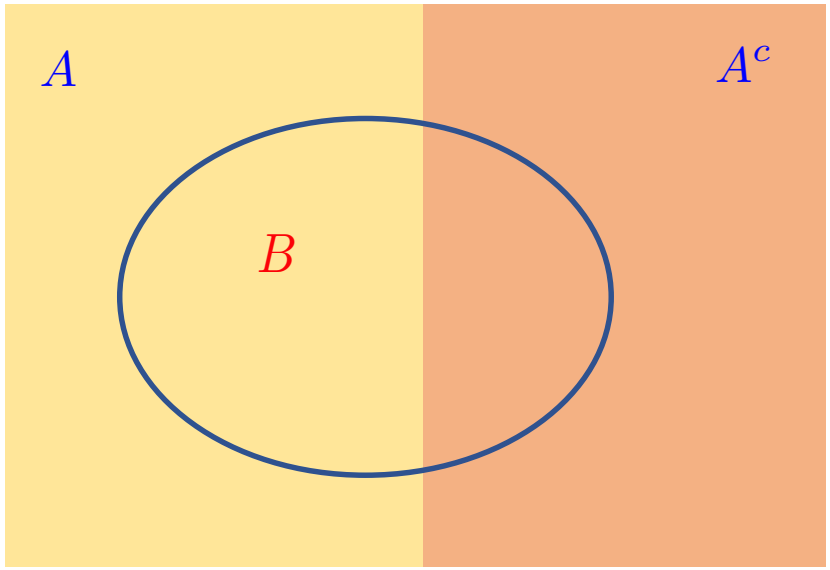
So, substitute in (*)

$$P(A \mid B) = \frac{P(B \mid A)\,P(A)}{P(B)}$$

Of course, this is a very important Theorem...



- If we don't know the denominator, i.e. the marginal probability $P(B)$, we can compute it using the law of total probability, in this way:



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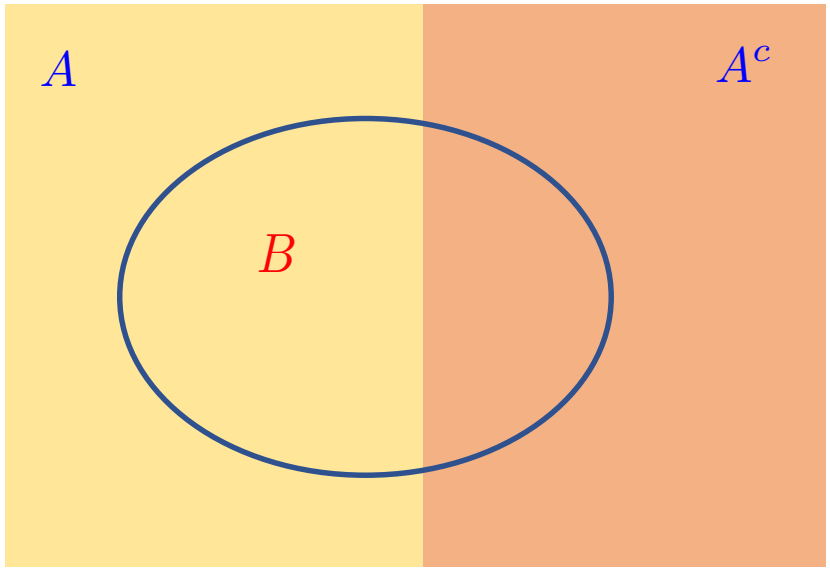

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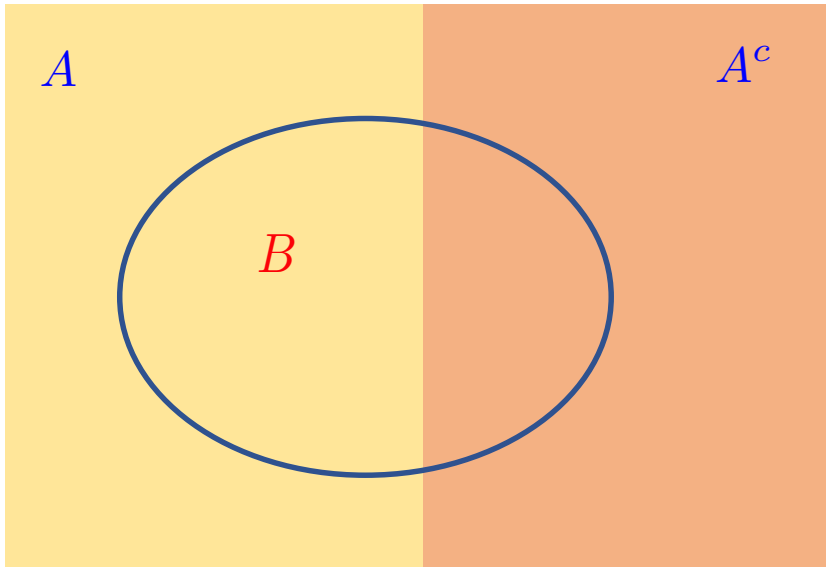

$$P(B) = P(B \cap A) + P(B \cap A^c) = P(B|A) \times P(A) + P(B|A^c) \times P(A^c)$$

Let's go back to the medical testing example

Example (Medical Testing)

- Recall:
- D indicates someone who has a disease and H a healthy subject
- $+$ indicates that someone tested *positive* for the disease in a screening test, and $-$ that they tested *negative*
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Let's go back to the medical testing example

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  Specificity of the test: how good is the test at correctly identifying healthy individuals: $Pr(-|H) = 0.95$

Example (Medical Testing)

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$$= \frac{Pr(+|D)Pr(D)}{Pr(+|D)Pr(D) + Pr(+|H)Pr(H)}$$

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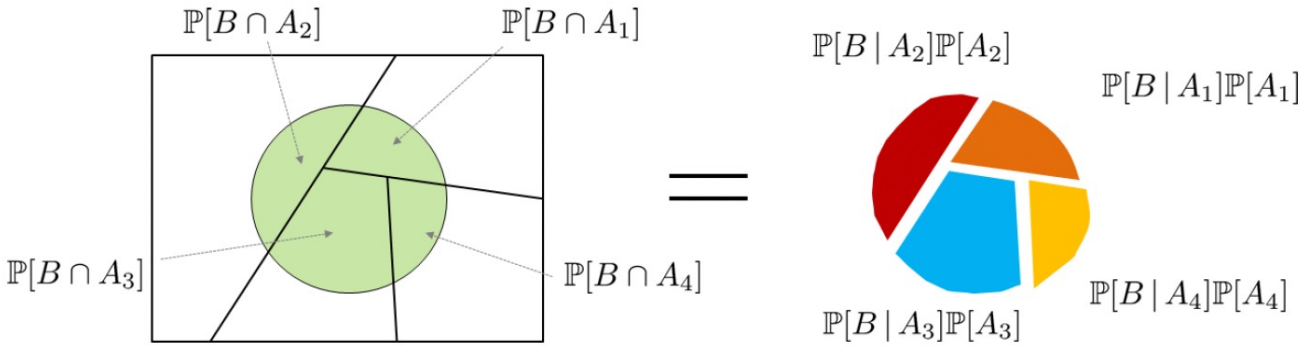
- Even after testing positive, there's still a 83.5% chance that the person is healthy!
- Notice the effect of the prevalence: if $Pr(D) = 0.5$, then $Pr(D|+) = 0.95$!!!

Law of total probability

- A more general version of the law of total probability is the following (on partition of a set B)



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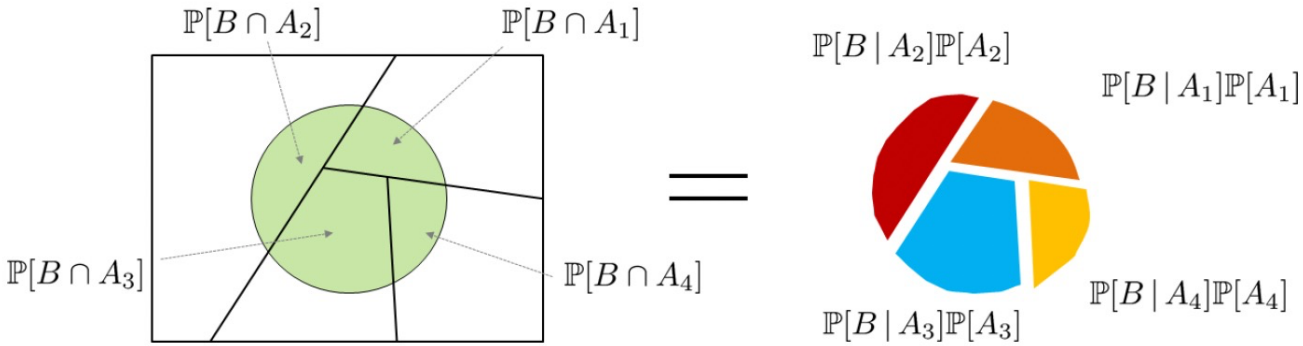

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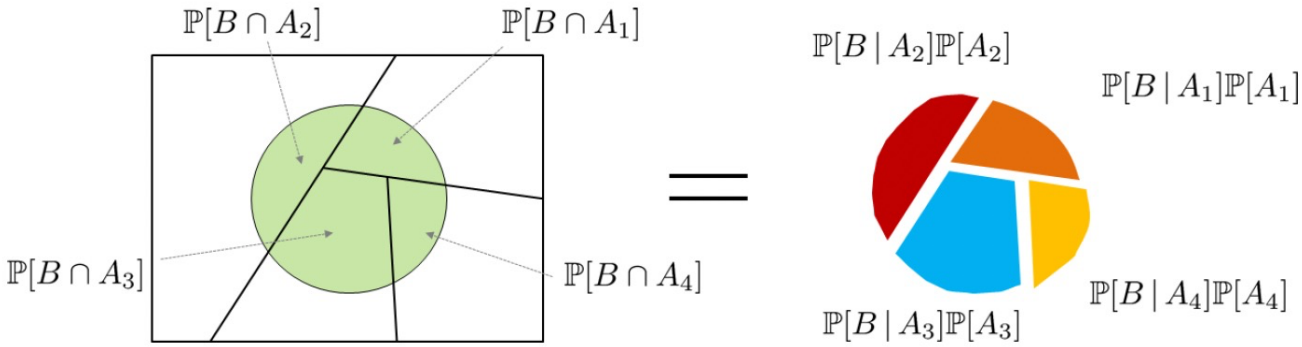

Theorem

Let $\{A_1, \ldots, A_n\}$ be a partition of Ω, i.e., $A_1, \ldots, A_n$ are disjoint and $\Omega = A_1 \cup \cdots \cup A_n$. Then, for any $B \subseteq \Omega$

$$\mathbb{P}[B] = \sum_{i=1}^{n} \mathbb{P}\left[B \mid A_i\right] \mathbb{P}\left[A_i\right]$$

Example (Tennis tournament)

Suppose your probability of winning the game is

- 0.3 against $\frac{1}{2}$ of the players (Event A).
- 0.4 against $\frac{1}{4}$ of the players (Event B).
- 0.5 against $\frac{1}{4}$ of the players (Event C).

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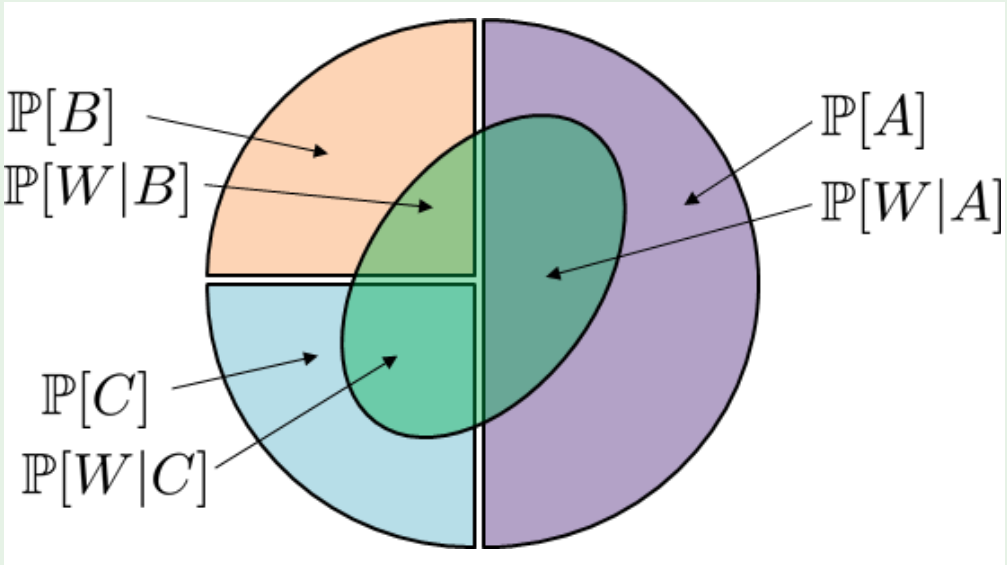
and we know that

$$\mathbb{P}[W \mid A] = 0.3, \quad \mathbb{P}[W \mid B] = 0.4, \quad \mathbb{P}[W \mid C] = 0.5$$

Example (Tennis tournament)

We can apply the law of total probability:



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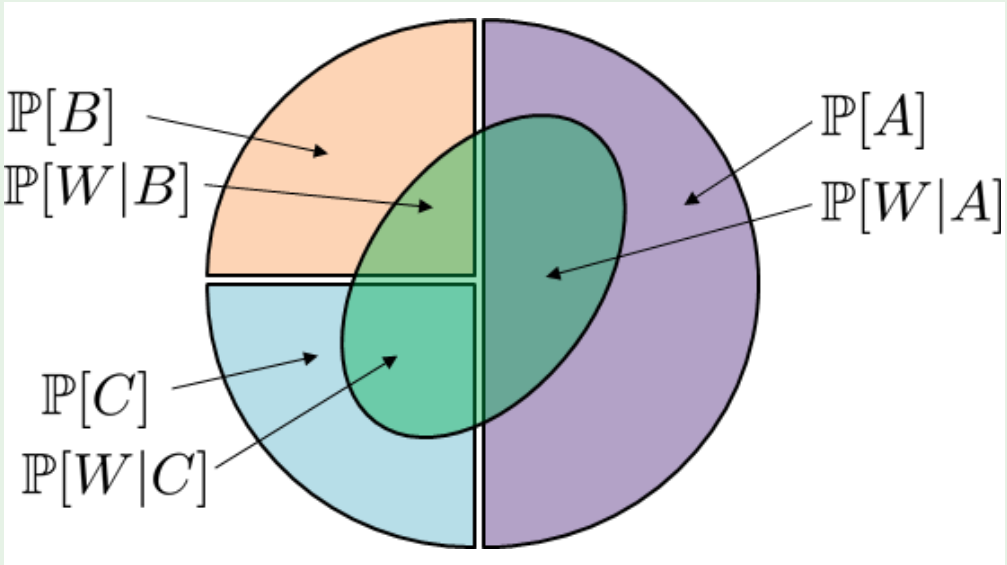


$$\begin{aligned}\mathbb{P}[W] &= \mathbb{P}[W \mid A]\mathbb{P}[A] + \mathbb{P}[W \mid B]\mathbb{P}[B] + \mathbb{P}[W \mid C]\mathbb{P}[C] \\ &= (0.3)(0.5) + (0.4)(0.25) + (0.5)(0.25) = 0.375\end{aligned}$$

Example (Tennis tournament - ctd)

② Supposing that you have won a match, what is the probability that you played against an A player?

Example (Tennis tournament - ctd)

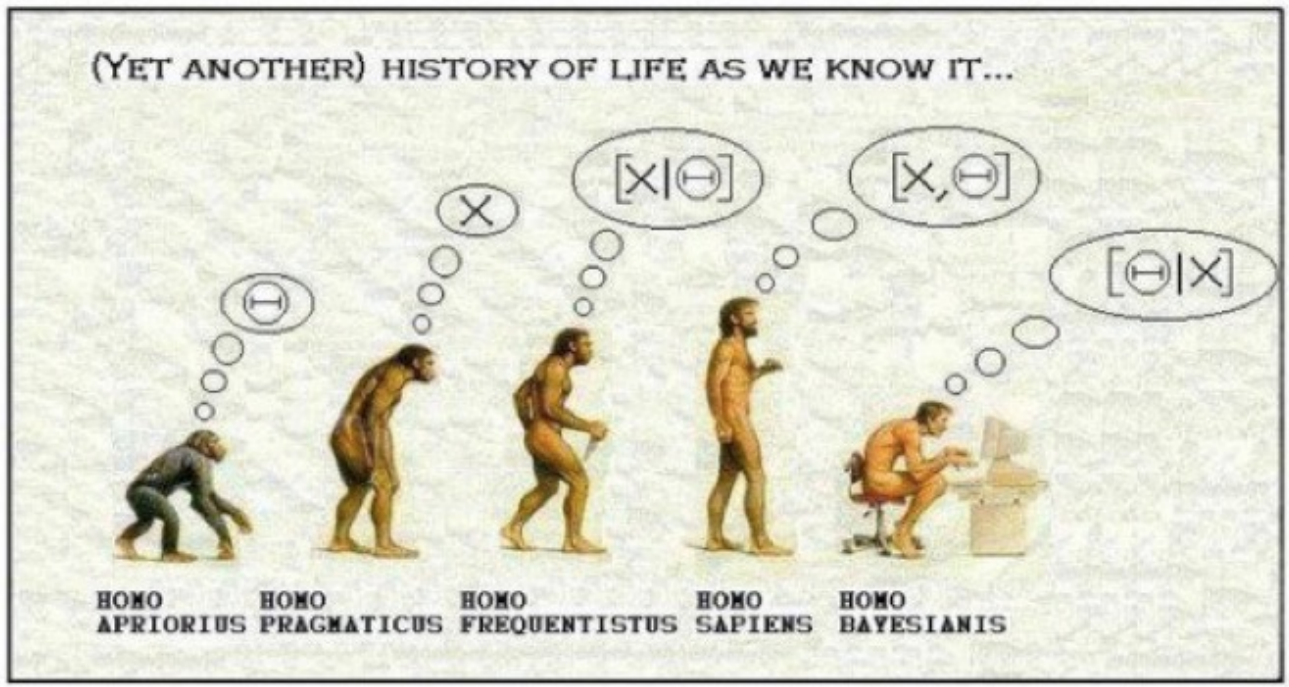
② Supposing that you have won a match, what is the probability that you played against an A player?
We can use the Bayes theorem to reply to this question. Given that you have won the match, the probability of A given W is

$$\mathbb{P}[A \mid W] = \frac{\mathbb{P}[W \mid A]\mathbb{P}[A]}{\mathbb{P}[W]} = \frac{(0.3)(0.5)}{0.375} = 0.4$$

Bayesian Analysis!

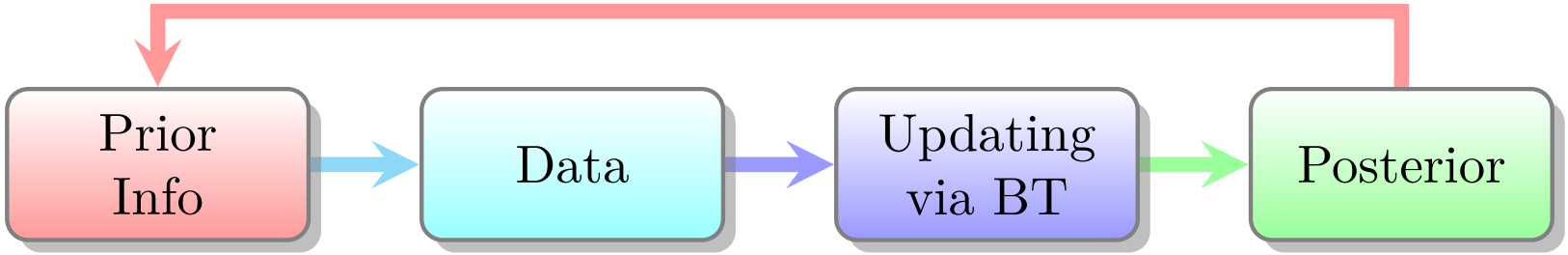
Bayesian Analysis!

- The Bayes Theorem is at the basis of a specific field of Statistics: Bayesian Analysis!



- The next evolution of Statistics! 🤪 😆

The tenets of Bayesian analysis



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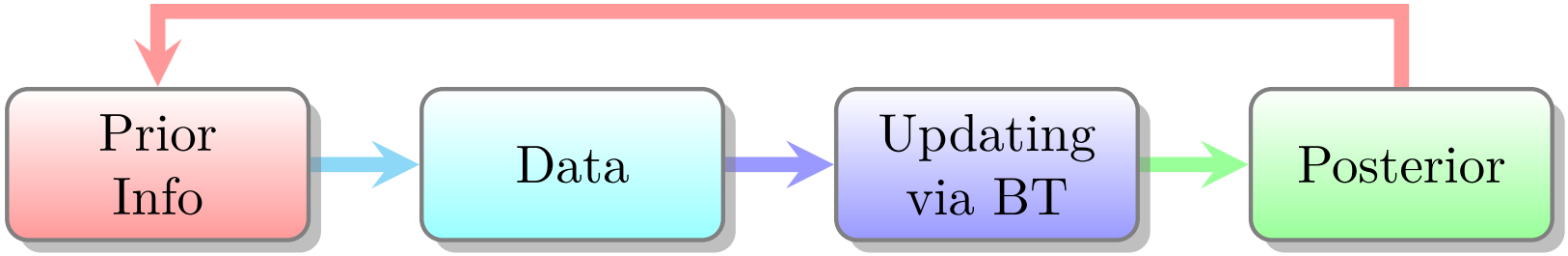

- Bayesian statistics starts by using (prior) probabilities to describe your current state of knowledge, say $P((\text{available}) \text{ info})$
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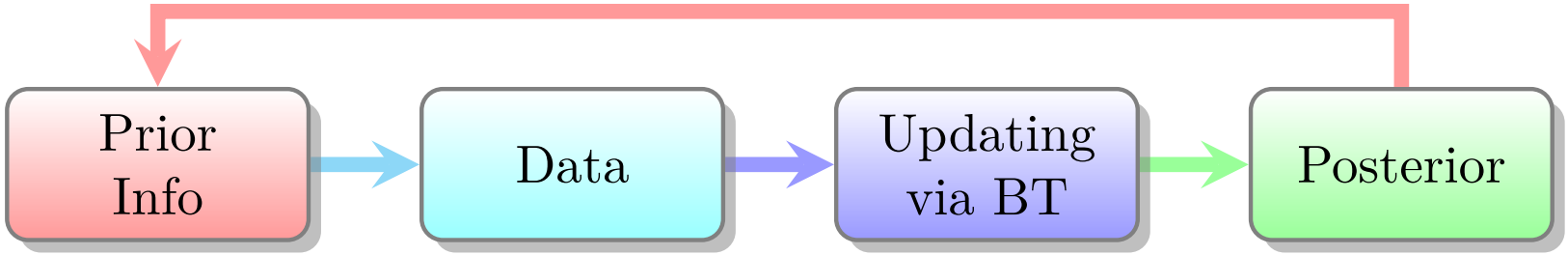

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- It then incorporates information through the collection of data, say $P(\text{data} \mid (\text{available}) \text{ info})$
- By combining the prior probabilities with the data, you can obtain new (posterior) probabilities to describe an updated state of knowledge:

$$P((\text{updated}) \text{ info}|\text{data}) = \frac{P(\text{data} \mid (\text{available}) \text{ info}) \times P((\text{available}) \text{ info})}{P(\text{data})}.$$