CS 250 Final Review

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Contents

1 Exam 1 Review

Note: we won't have questions from this material, but we may need to use it for questions from later sections, so I'm circling back to it once finished with more relevent sections.

1.1 Section 1.1: Proof Primer Logic

Content coming soon

1.2 Section 1.2: Sets and Set Operators

Content coming soon

1.3 Section 1.4: Graphs and Trees

Content coming soon

1.4 Section 2.1: Functions, Definitions, and Examples

Content coming soon

1.5 Section 2.2: Composition of Functions

Content coming soon

1.6 Section 2.3: Properties and Applications

Content coming soon

2 Section 2.4: Countability

2.1 Cardinality Notation

Given a set "A", A's cardinality is denoted as |A |

The expression |A |= |B |indicates that there is bijection between A and B.

The expression |A |<= |B |indicates that there is injection from A to B

2.2 General Concepts

Sets "A" and "B" have the same cardinality IFF there is a one-to-one correspondance (i.e. a Bijection) from A to B.

IF a set is finite or has the same cardinality as **N**, then it is called **countable**, otherwise it is **uncountable**

In mathematics, the **cardinality** of a set is a measure of the number of elements of the set. The cardinality of the natural numbers **N** is denoted aleph-null

If the cardinality of some set "S" is equal to that of the set of natural numbers. then the set S is called **countably infinite**.

2.3 Cardinality Proof Example

Prove: that the set of positive rational numbers is countable.

$$E = \{n \mid n \bmod 2 = 0 \}$$

Let us define: f: **N** –> E as f:(n) = 2n.

- f maps N to E
- the defined function f is a one-to-one correspondence

Therefore: f is countable.

2.4 Cantor Diagonalization Proof Method

Prove: that the set of positive rational numbers is countable.

Q = { m/n — m, n $\in N$}

RETURNING TO THIS BIT LATER BECAUSE I HATE FORMATTING THIS PROOF

2.5 Cardinality of Integers Proof

The set of **Z** is countable, so it has the same cardinality as **N**.

Prove: there is a one-to-one correspondence from **N** to the Integers.

f(n) = n/2 if **N** is even and -(n-1)/2 if odd.

- f maps N to E
- the defined function f is a one-to-one correspondence

Therefore: the function f is countable, as f maps the set of integers.

2.6 Some Countability Results

1. Subsets and images of countable sets are also countable.
2. Set "S" is countable IFF |S |<= |**N** |

3. **N** x **N** is countable. This can be proved by using Cantor's Bijection to associate (x,y) with ((x+y)$^2 + 3x + y)/2$

2.7 Countability Fun Facts (wow so fun)

- The set **R** of real numbers is not countable.
- **Cantor's Result:** |A |< |power(A) |for any set A.
  - Power(**N**) is uncountable because |**N** |< |power(**N**)|.
  - The power set of **N** has the same cardinality as **R**.
- Most of the other sets that we have used thus far are countable:
  - **N** is a subset of **R**, but **R** is not a subset at **N**

- **Q** is a subset of **R**, but **R** is not a subset at **Q**
- **N**, **Z**, and **Q** are all the same, which is to say aleph-null
- Every finite set "A" of elements from N is a subset of N, so |A |< |N |

- The cardinality of the set of real numbers (**R**) js not the same as that of **N**

3 Section 3.1: Inductively-defined Sets

3.1 Inductive Definition Components

An inductively-defined set "S" has three primary components:

1. **Basis:** Specify one of more elements of S (having more than one of these is fine).

2. **Induction:** Specify one or more rules to construct elements of S from existing elements of S.

3. **Closure:** Specify that no other elements are in S (this step is always implicit)

The basis elements and the induction rules are called **constructors**.

3.1.1 Inductively-defined Set Example #1

Problem: Find an inductive definition for S = {3, 16, 29, 42...}
Solution:

1. **Basis:** 3 $\in$ S

2. **Induction:** If x $\in$ $3, then x + 13 \in$ 3. The constructors are 3, and the operation of adding 13.

3.1.2 Inductively-defined Set Example #2

Problem: Find an inductive definition for S = {3, 16, 29, 42, ...}
Solution: To simplify, we might try the method of Ḋivide and Conquer;̈ by writing S as the union of more familiar sets, like this:

S = {3, 4, 5, 8, 9, 12, 16, 17, 20, 24, 33, ...} $\cup \{4, 8, 12, 16, 20, 24, ...\}$

1. **Basis:** 3, 4 $\in$ S

2. **Induction:**

```
If x $\in$ S then:
 if x is odd:
     2x-1 $\in$ S
 else:
     x + 4 $\in$
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3.1.3 Example of Inductively Defined Sets of Strings

Problem: Find an Inductive definition for S = { λ, ac, aacc,aaaccc, ...} = / a$^n c^n | n \in$ **N**

Solution:

1. **Basis:** $\lambda \in$ S.

2. **Induction:** if x $\in$ S then axc $\in$ S.

NOTE: For strings, we start with a *middle* element, and inductive build outwards.

3.1.4 Example of Inductively Defined Sets of Lists

Problem: Describe the set S defined by:

1. **Basis:** $\langle 0 \rangle \in$ S

2. **Induction:** x $\in$ S implies cons(1, x) $\in$ S.

Solution: S = $\{\langle 0\rangle, \langle 1, 0\rangle, \langle 1, 1, 0\rangle, ...\}$

3.2 Infix Notation for Lists and Cons

Notation: cons(h,t) = h :: t. Associate to the right. This means that "x :: y :: z = x :: (y :: z)" is equivalent to cons(x, const(y, z)).

3.2.1 Infix notation Example #1:

Problem: Find an inductive definition for S = { $\langle\rangle, \langle a, b\rangle, \langle a, b, a, b\rangle, ...\}$
Solution:
Basis: $\langle\rangle \in$ S.
Induction: x $\in$ S implies a :: b :: x $\in$ S (or cons(a, cons(b, x))), if you prefer.

3.2.2 Infix Notation Example #2 (A more confusing case)

Problem: Find and inductive definition for S = { $\langle\rangle, \langle\langle\rangle\rangle, \langle\langle\langle\rangle\rangle\rangle, ...\}$
Solution:
Basis: $\langle\ \rangle \in$ S.
Induction: x $\in$ S implies x :: $\langle\rangle \in$ S.

3.3 Notation for Binary Trees

Let:

- t(L, x, R) be the tree with root x, left subtree L, and right subtree R
- $\langle\ \rangle$ denote the empty binary tree.

If T = t(L, x, R), then root(T) = x, left(T) = L, and right(T) = R.

3.3.1 Binary Tree Notation Example

Problem: Describe the set "S" defined inductively as follows:

1. **Basis:** $t(\langle\ \rangle, \cdot, \langle\ \rangle) \in S$.
2. **Induction:** $T \in S$ implies $t(T, \cdot, t(\langle\ \rangle, \cdot, \langle\ \rangle)) \in S$.