Assignment PS1 - Q3.

Poisson distrⁿ : $p(y;\lambda) = \frac{e^{-\lambda}\lambda^{y}}{y!}$

(a) Exponential family:-

$$P(y;\eta) = b(y)\, e^{[\eta^T T(y) - a(\eta)]}$$

$$p(y;\eta) = b(y)\, e^{[\eta^T y - a(\eta)]}$$

Poisson : $p(y;\lambda) = \frac{\lambda^{y}}{y!} e^{-\lambda}$

So,

$$b = \frac{\lambda^{y}}{y!} e^{0y - \lambda}$$

$$\eta = 0$$

(X) $a(\eta) = \lambda$ [constant] (i.e. a independent of η)

$$T(y) = 0$$

~~$e(-\lambda + y\log\lambda)$~~

OR ✓ Trick

$$p(y;\lambda) = \exp\left(\log\left(\frac{e^{-\lambda}\lambda^{y}}{y!}\right)\right)$$

$$= \exp(-\lambda + y\log\lambda - \log y!)$$

$$= \frac{1}{y!} e^{(-\lambda + y\log\lambda)} = \frac{1}{y!} e^{[(\log\lambda)y - \lambda]}$$

So, $b(y) = 1/y!$

$T(y) = y$

$\eta = \log \lambda \quad \Rightarrow \quad \lambda = e^{\eta}$

$a(\eta) = \lambda \quad = e^{\eta}$

(b) <u>canonical response function</u>

$E[y; \eta] = \lambda = e^{\eta}$

<u>Design choice for GLM</u>

Ass (i) (1) $y|x;\theta \sim$ Exponential family (η)

Ass (ii) (2) $h(x) = E[y|x]$

Ass (iii) (3) $\eta = \theta^T x$ (design choice)

(c) <u>1 example</u> $l(\theta) = \log (p(y^{(i)} | x^{(i)}; \theta)$

$= \cancel{\log (p(y^{(i)}; \theta^T x^{(i)})}$

$= \cancel{\log p(y^{(i)}; \eta)}$

$= \log [b(y^{(i)}) \exp (\eta^T T(y^{(i)}) - a(\eta))$

$= \log [\frac{1}{y^{(i)}!} \exp ((\theta^T x)^T y^{(i)} - e^{\theta^T x^{(i)}})]$

$(\sum_{j=1} x_j^{(i)} \theta_j) y^{(i)}$ $\theta^T x^{(i)}, \Sigma \theta_j x_j^{(i)}$

$l(\theta) = -\log y^{(i)}! + x^T \theta y^{(i)} - e^{\theta^T x^{(i)}}$

from Ass (ii) & (iii) $h(x) = E[y|x] = \lambda = e^{\eta} = \underline{\underline{e^{\theta^T x}}}$.

$h_\theta(x^{(i)})$

$\frac{\partial l(\theta)}{\partial \theta_j} = 0 + x_j^{(i)} y^{(i)} - e^{\theta^T x^{(i)}} (x_j^{(i)})$

$\frac{\partial l(\theta)}{\partial \theta_j} = x_j^{(i)} (y^{(i)} - h_\theta(x^{(i)}))$

Generalise $\theta_j := \theta_j + \alpha \sum_{i=1} (y^{(i)} - h_\theta(x^{(i)})) x_j^{(i)}$