$$\underline{PS-1 \ Q4}$$

(a)

$$p = b y) e^{ny-a(n)}$$

$$\frac{\partial p}{\partial n} = b e^{ny-a} \left[y - \frac{\partial a}{\partial n} \right] = p \left[y - \frac{\partial a}{\partial n} \right]$$

$$\Rightarrow \quad y p = \frac{\partial p}{\partial n} + p \frac{\partial a}{\partial n}$$

$$E[Y \mid x; \theta] = \int y p \, dy$$

$$= \int \frac{\partial p}{\partial n} + p \frac{\partial a}{\partial n} \, dy$$

$$= \int \frac{\partial p}{\partial n} \, dy + \int p \frac{\partial a}{\partial n} \, dy$$

using Hint → 0

$$= \left(\frac{\partial}{\partial n} \int p \, dy = 1 \right) + \frac{\partial a}{\partial n} \int p \, dy = 1$$

$$E[Y \mid x; \theta] = 0 + \frac{\partial a}{\partial n}$$

$$\boxed{E[y \mid x; \theta] = \frac{\partial a}{\partial n}}$$

(b)

$$\frac{\partial^2 p}{\partial n^2} = p \left[0 - \frac{\partial^2 a}{\partial n^2} \right] + \left[y - \frac{\partial a}{\partial n} \right] \frac{\partial p}{\partial n}$$

$$= p \left(-\frac{\partial^2 a}{\partial n^2} \right) + \left(y - \frac{\partial a}{\partial n} \right) (p) \left(y - \frac{\partial a}{\partial n} \right)$$

$$\frac{\partial^2 p}{\partial n^2} = p \left(-\frac{\partial^2 a}{\partial n^2} \right) + \left(y - \frac{\partial a}{\partial n} \right)^2 p \quad - \quad \textcircled{Eq1}$$

$$\text{Var}(y) = \int (y - E(y|x;\theta))^2 \, p \, dy$$

$$= \int \left(y - \frac{\partial a}{\partial \eta}\right)^2 p \, dy$$

$\underbrace{\qquad\qquad}_{\text{substitute form}}$ $\underline{E\phi I}$

$$= \int \frac{\partial^2 p}{\partial \eta^2} + p \frac{\partial^2 a}{\partial \eta^2} \, dy$$

$$= \int \frac{\partial^2 p}{\partial \eta^2} \, dy + \frac{\partial^2 a}{\partial \eta^2} \underbrace{\int p \, dy}_{1}$$

$$= \frac{\partial}{\partial \eta^2} \underbrace{\int p \, dy}^{0} + \frac{\partial^2 a}{\partial \eta^2}$$

$$\boxed{\text{Var}(y|x;\theta) = \frac{\partial^2 a}{\partial \eta^2}}$$

ⓒ $\ l(\theta) = $ Likelihood $= \prod p(y;\eta)$

log-likelihood $\quad \log \prod p(y^{(i)}, \eta)$

$- (\quad \|\quad) = l(\theta) = - \log \left(\prod_{i=1}^{m} p(y^{(i)}; \eta)\right)$

$$= -\log \left(\prod_{i=1}^{m} b(y) \; e^{\eta y - a(\eta)}\right)$$

$$= -\log .. \; b(y)! \; \{ + -(\eta y - a(\eta))$$

$$l(\theta) = a(\theta^T x) - \theta^T x y - \log b(y)$$

$$\nabla_\theta l(\theta) = x \frac{\partial a(\theta^T x)}{\partial \theta} - xy$$

Hessian $\nabla^2_\theta l(\theta) = \underbrace{x x^T}_{\text{square} > 0} \underbrace{\frac{\partial^2 a(\theta^T x)}{\partial \theta^2}}_{\text{Var} > 0}$ $\implies > 0$ $\implies$ PSD $\implies$ Convex