(1.1) Sea L/K una extensión algebraica. Probar que $L_{sep} \cap L_{t.i} = K$.
Aquí $L_{t.i}/K$ es la mayor subextensión totalmente inseparable.

-Demostración-

Evidente que $K \subset L_{sep} \cap L_{t.i}$.
Supongamos que $\alpha \in L_{sep} \cap L_{t.i}$. Como $\alpha \in L_{sep}$, $m_{\alpha,K}(x)$ tiene raíces distintas, y como $\alpha \in L_{t.i}$, $m_{\alpha,K}(x)$ tiene una sola raiz (α)

$$\therefore \quad m_{\alpha,K}(x) = x - \alpha$$
$$\therefore \quad \alpha \in K$$
$$\therefore \quad K = L_{sep} \cap L_{t.i}.$$

$\mathbb{F}_p(x_1, x_2)$

---

(3) $L = \mathbb{F}_p(x)$, $K = \mathbb{F}_p(f)$ algún $f \in \mathbb{F}_p[x]$

Dem: L_{sep}/K galoisiana $\Rightarrow$ L/K normal

$$\begin{tikzcd} L = \mathbb{F}_p(x) \arrow[d, dash, "t.i"] \arrow[dd, dash, bend right, "\cancel{\text{galoisiana}}\ t.i"'] \\ L_{sep} \arrow[d, dash, "\text{galois (normal)}"] \\ K = \mathbb{F}_p(f) \end{tikzcd}$$

$[L : L_{sep}] = p^t$ (algún t)

L/K normal $\Leftrightarrow$ $\forall \alpha \in L$: $m_{\alpha,K}(x)$ tiene raíces en L

L_{sep}/K galoisiana $\Rightarrow$ L_{sep}/K normal y separable

$\overset{a}{\cancel{h(x)}} \in K \Rightarrow \cancel{h}\ a = h(f(x))$, $f(x) \in \mathbb{F}_p[x]$

$[L:K] = \deg f \longrightarrow \boxed{L = K(\alpha_1, \dots, \alpha_n)}$, $n = \deg f$

$\alpha_{ij} \in L \setminus L_{sep}$

~~$\alpha \in L$ raiz de $m_{\alpha,K}(y) \in K[y] = \mathbb{F}_p(f)[y]$~~

$\alpha \in L$ raiz de $m_{\alpha,K}(T) \in K[T] = \mathbb{F}_p(f)[T]$

$$m_{\alpha,K}(T) = \sum_{i=0}^{n} a_i x^i, \quad a_i = \frac{g(f(x))}{h(f(x))}$$

$g(x), h(x) \in \mathbb{F}_p[x]$

(4) $L = \mathbb{F}_p(x)$, $K = \mathbb{F}_p(f)$ $(f \in \mathbb{F}_p[x])$

Pd: $L/L_{t.i.}$ is separable

$$\begin{tikzcd}
& L = K(\alpha_1, \dots, \alpha_m) \arrow[dl, dash, "t.i"'] \arrow[dr] & \\
L_{sep} \arrow[dr, dash, "sep"'] & & L_{t.i} \arrow[dl, dash, "t.i"] \\
& K &
\end{tikzcd}$$

, $\cancel{n \leq \deg f}$

$$[L : K] = \deg f$$

Queremos verificar $p(x)=x^4-2x^2+5$.

$$p(x)=x^4-2x^2+5=(x^2-1)^2+4=q(x^2-1)$$

donde $q(x)=x^2+4$. $q(x)$ irreducible pero no necesariamente implica $p(x)$ irreducible.

(c) $\mathbb{Q}(\sqrt{2},\sqrt{3})$

<u>Afirmación</u>. $\mathbb{Q}(\sqrt{2},\sqrt{3})=\mathbb{Q}(\sqrt{2}+\sqrt{3})$

Tenemos que $\sqrt{2},\sqrt{3}\in\mathbb{Q}(\sqrt{2},\sqrt{3})$, luego $\sqrt{2}+\sqrt{3}\in\mathbb{Q}(\sqrt{2},\sqrt{3})$

$$\therefore\ \mathbb{Q}(\sqrt{2}+\sqrt{3})\subseteq\mathbb{Q}(\sqrt{2},\sqrt{3})$$

. Ahora, $\sqrt{2}(\sqrt{2}+\sqrt{3})=2+\sqrt{6}\Rightarrow\sqrt{6}\in\mathbb{Q}(\sqrt{2},\sqrt{3})$

$$\underbrace{(\sqrt{2}+\sqrt{3})^2}_{\in\,\mathbb{Q}(\sqrt{2}+\sqrt{3})}=5+2\sqrt{6}\Rightarrow\sqrt{6}\in\mathbb{Q}(\sqrt{2}+\sqrt{3})$$

Entonces, $\sqrt{6}(\sqrt{2}+\sqrt{3})=\sqrt{12}+\sqrt{18}=2\sqrt{3}+3\sqrt{2}\in\mathbb{Q}(\sqrt{2}+\sqrt{3})$

$$(2\sqrt{3}+3\sqrt{2})-(2\sqrt{2}+2\sqrt{3})=\sqrt{2}\in\mathbb{Q}(\sqrt{2}+\sqrt{3})$$

$$(\sqrt{2}+\sqrt{3})-\sqrt{2}=\sqrt{3}\in\mathbb{Q}(\sqrt{2}+\sqrt{3})$$

$$\therefore\ \sqrt{2},\sqrt{3}\in\mathbb{Q}(\sqrt{2}+\sqrt{3})$$

$$\therefore\ \mathbb{Q}(\sqrt{2},\sqrt{3})\subseteq\mathbb{Q}(\sqrt{2}+\sqrt{3})$$

$$.\ \mathbb{Q}(\sqrt{2},\sqrt{3})=\mathbb{Q}(\sqrt{2}+\sqrt{3})$$

Sea $a=\sqrt{2}+\sqrt{3}$, $\quad a=\sqrt{2}+\sqrt{3}\Rightarrow a^2=5+2\sqrt{6}$

$$\Rightarrow a^2-5=2\sqrt{6}$$

$$\Rightarrow a^4+25-10a^2=24$$

$$\Rightarrow a^4-10a^2+1=0$$

a es raíz de $p(x)=x^4-10x^2+1$. $p(x)$ no posee factores lineales en $\mathbb{Q}[x]$, $\mathbb{Z}[x]$ ($p(-1),p(1)\neq0$).

Supongamos $x^4-10x^2+1=(x^2+ax+b)(x^2+cx+d)$. Por caso anterior

$$\begin{cases} a+c=0 \longrightarrow ad+cd=0 \\ ac+b+d=-10 \\ ad+bc=0 \\ bd=1 \end{cases}$$

Tenemos $\begin{matrix} ad+cd=0 \\ ad+bc=0 \end{matrix} \Rightarrow c(d-b)=0$

$d=b$: Tenemos $d=b=1 \Rightarrow ac=-12$ ~~$ac=12$~~

$\therefore$ ~~$x^4-10x^2+1=(x^2-12x+1)(x^2+12x+1)$~~

~~Comprobamos~~

~~$(x^2-12x+1)(x^2+12x+1)$~~

$\Rightarrow ac=-12,\quad a+c=0 \Rightarrow$

$$\Rightarrow \begin{cases} ac=-12 \\ a+c=0 \end{cases} \longrightarrow \begin{matrix} ac=-12 \\ ac+c^2=0 \end{matrix} \longrightarrow c^2=12$$

No existe $c\in\mathbb{Q}$ tal que $c^2=12$.

$d\neq b$: de $c(d-b)=0 \Rightarrow c=0 \Rightarrow a=0$. Luego

$$\begin{cases} b+d=-10 \longrightarrow b^2+db=-10b \\ db=1 \end{cases}$$

$\Rightarrow b^2+10b+db=b^2+10b+1$ no tiene raíces en $\mathbb{Q}$

$\therefore p(x)$ no se puede factorizar por polinomios de grado 2 ($p(x)$ irreducible)

$\therefore [\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}]=4$

(d) $\mathbb{Q}(\sqrt[4]{2}, i\sqrt[4]{2})$

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt[4]{2}, i\sqrt[4]{2}) \arrow[dl, dash] \arrow[dr] & \\
\mathbb{Q}(\sqrt[4]{2}) & & \mathbb{Q}(i\sqrt[4]{2}) \\
 & \mathbb{Q} \arrow[ul, dash, "4"] \arrow[ur, dash, "4"'] &
\end{tikzcd}$$

$\mathbb{Q}(\sqrt[4]{2}, i\sqrt[4]{2})=\mathbb{Q}(\sqrt[4]{2})(i\sqrt[4]{2})$. $p(x)=x^4+2\in\mathbb{Q}(\sqrt[4]{2})$

$p(x)$ no tiene raíces en $\mathbb{Q}(\sqrt[4]{2})$, luego no tiene factores lineales

$$x^4+2=(x^2+ax+b)(x^2+cx+d)$$

$$\begin{cases} a+c=0 \\ ac+b+d=0 \\ ad+bc=0 \\ bd=2 \end{cases}$$

o

$a=\sqrt[4]{2}+i\sqrt[4]{2}$, $a^2=(\sqrt[4]{2}+i\sqrt[4]{2})^2=\sqrt{2}-\sqrt{2}+2i\sqrt[4]{4}=\cancel{\sqrt{2}-\sqrt{2}}+2i\sqrt{2}$

$\Rightarrow a^2=2i\sqrt{2} \Rightarrow a^4=-4\cdot 2=-8 \Rightarrow a^4+8=0$

a es raíz de $p(x)=x^4+8$. $p(x)$ no tiene factores lineales.

factores cuadráticos: $x^4+8=(x^2+ax+b)(x^2+cx+d)$

$$\begin{cases} a+c=0 \\ ac+b+d=0 \\ ad+bc=0 \\ bd=8 \end{cases}$$

$\Rightarrow \begin{cases} a+c=0 \\ ad+bc=0 \end{cases} \Rightarrow \begin{cases} ad+cd=0 \\ ad+bc=0 \end{cases} \Rightarrow c(d-b)=0$

Como $bd=8 \rightarrow$ ~~$d\neq d$~~ $b\neq d \Rightarrow c=0 \Rightarrow a=0$

luego $\begin{cases} b+d=0 \\ bd=8 \end{cases} \Rightarrow \begin{cases} bd+d^2=0 \\ bd=8 \end{cases} \Rightarrow d^2=-8$

no existe $d\in\mathbb{Q}$ tq $d^2=-8$ $\therefore$ $p(x)$ irreducible.

Como $a^4=-8$; $-8(\sqrt[4]{2}+i\sqrt[4]{2})=\sqrt[4]{-16}+i\sqrt[4]{-16}=2i+i(2i)=-2+2i$

$\therefore i\in\mathbb{Q}(\sqrt[4]{2}+i\sqrt[4]{2})$

$i(\sqrt[4]{2}+i\sqrt[4]{2})=i\sqrt[4]{2}-\sqrt[4]{2}$; $-\sqrt[4]{2}+i\sqrt[4]{2}\in\mathbb{Q}(\sqrt[4]{2}+i\sqrt[4]{2})$

$a^3=(\sqrt[4]{2}+i\sqrt[4]{2})^3=(\sqrt[4]{2})^3+3(\sqrt[4]{2})^2(i\sqrt[4]{2})+3(\sqrt[4]{2})(i\sqrt[4]{2})^2+(i\sqrt[4]{2})^3$

$=(\sqrt[4]{2})^3+3i(\sqrt[4]{2})^3-3(\sqrt[4]{2})^3-i(\sqrt[4]{2})^3$

$=(\sqrt[4]{2})^3(1+3i-3-i)=(\sqrt[4]{2})^3(-2+2i) \Rightarrow (\sqrt[4]{2})^3\in\mathbb{Q}(\sqrt[4]{2}+i\sqrt[4]{2})$

$(\sqrt[4]{2})^3=(\sqrt[4]{2})^2(\sqrt[4]{2})=\sqrt{2}(\sqrt[4]{2})$. Como $\sqrt{2}\in\mathbb{Q}(\sqrt[4]{2}+i\sqrt[4]{2})$, $(\sqrt[4]{2})\in\mathbb{Q}(\sqrt[4]{2}+i\sqrt[4]{2})$

Análogamente, $i\sqrt[4]{2}\in\mathbb{Q}(\sqrt[4]{2}+i\sqrt[4]{2})$ $\therefore$ $\mathbb{Q}(\sqrt[4]{2}+i\sqrt[4]{2})=\mathbb{Q}(\sqrt[4]{2},i\sqrt[4]{2})$

En particular, $[\mathbb{Q}(\sqrt[4]{2}, i\sqrt[4]{2}) : \mathbb{Q}] = 8$.

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$$\psi : K \longrightarrow \Omega$$

$$\bar{\psi} : K[X] \longrightarrow \Omega[X]$$

$$f(x) = a_n x^n + \cdots + a_1 x + a_0 \in K[X]$$

$$\bar{\psi}(f)(X) = \psi(a_n) x^n + \cdots + \psi(a_1) x + \psi(a_0)$$

$$= a \prod_{i=1}^{n} (x - \lambda_i)$$

$$-1 = a^2 + 2c^2 + 4bd$$

$$0 = b^2 + 2d^2 + 2ab + 2ac + 4cd$$

$$0 = 2ab$$

$$0 = 2ad + 2bc$$

$a=0$:

$b=0$:

$c=0$:

$x^2 + \sqrt{2}$

$\mathbb{Q}(\sqrt[4]{2}, i\sqrt[4]{2})/\mathbb{Q}$ galoisiana

raíces de X^4-2 son $\sqrt[4]{2}, i\sqrt[4]{2}, -\sqrt[4]{2}, -i\sqrt[4]{2}$

$\sigma(i) \in \{i, -i\}$; $\forall \sigma \in Aut(K/\mathbb{Q})$

<u>Afirmación</u>. $|Gal(K/\mathbb{Q})| = 8$

$\sigma_1(\sqrt[4]{2}) = \sqrt[4]{2}$, $\sigma_1(i) = i$

$\sigma_1(i\sqrt[4]{2}) = \sigma_1(i)\sigma_1(\sqrt[4]{2}) = i\sqrt[4]{2}$

$$\sigma_1 : \begin{array}{l} \sqrt[4]{2} \longmapsto \sqrt[4]{2} \\ i\sqrt[4]{2} \longmapsto i\sqrt[4]{2} \end{array}$$

$\sigma_2(\sqrt[4]{2}) = i\sqrt[4]{2}$

$$\sigma_2 : \begin{cases} \sqrt[4]{2} \longmapsto i\sqrt[4]{2} \\ i\sqrt[4]{2} \longmapsto -\sqrt[4]{2} \end{cases}$$

$\sigma_3(\sqrt[4]{2}) = -\sqrt[4]{2}$

$$\sigma_3 : \begin{array}{l} \sqrt[4]{2} \longmapsto -\sqrt[4]{2} \\ i\sqrt[4]{2} \longmapsto \sqrt[4]{2} \end{array}$$

$\sigma_4(\sqrt[4]{2}) = -i\sqrt[4]{2}$

$$\sigma_4 : \begin{array}{l} \sqrt[4]{2} \longmapsto -i\sqrt[4]{2} \\ i\sqrt[4]{2} \longmapsto \sqrt[4]{2} \end{array}$$

$$\begin{tikzcd}[column sep=small]
 & \mathbb{Q}(\sqrt[4]{2}, i\sqrt[4]{2}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt[4]{2}) \arrow[dr, dash, "4"'] & & \mathbb{Q}(i\sqrt[4]{2}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Af $\sqrt[4]{2}\, i \notin \mathbb{Q}(\sqrt[4]{2})$

~~supongamos que~~ $\cancel{i = a + b\sqrt[4]{2} + c\sqrt[4]{2} + d(\sqrt[4]{2})^3}$

Supongamos que $i\sqrt[4]{2} \in \mathbb{Q}(\sqrt[4]{2})$

$$\bar{\sigma} : \mathbb{Q}(i\sqrt[4]{2}) \longrightarrow \mathbb{Q}(i\sqrt[4]{2})$$
$$\bar{\sigma}(i\sqrt[4]{2}) = -i\sqrt[4]{2}$$

Por teo. extensión de homomorf. existe $\sigma \in Gal(\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q})$

tq $\sigma(i\sqrt[4]{2}) = -i\sqrt[4]{2}$.

Consideremos $\tau(\sqrt[4]{2}) = -\sqrt[4]{2}$

$\tau((\sqrt[4]{2})^2) = \tau(\sqrt[4]{2})\tau(\sqrt[4]{2}) = (-\sqrt[4]{2})(-\sqrt[4]{2}) = \sqrt{2}$

$\tau((\sqrt[4]{2})^3) = \tau(\sqrt[4]{2})^3 = (-\sqrt[4]{2})^3 = -(\sqrt[4]{2})^3$

$\therefore\ \tau(a + b(\sqrt[4]{2}) + c(\sqrt[4]{2})^2 + d(\sqrt[4]{2})^3) = a - b\sqrt[4]{2} + c(\sqrt[4]{2})^2 - d(\sqrt[4]{2})^3$

$\tau(a + b\sqrt[4]{2} + c\sqrt{2} + d(\sqrt[4]{2})^3) = a - b\sqrt[4]{2} + c\sqrt{2} - d(\sqrt[4]{2})^3$

Sup $i\sqrt[4]{2} = a + b\sqrt[4]{2} + c\sqrt{2} + d(\sqrt[4]{2})^3$

$\tau(i\sqrt[4]{2}) = a - b\sqrt[4]{2} + c\sqrt{2} - d(\sqrt[4]{2})^3 = -a - b\sqrt[4]{2} - c\sqrt{2} - d(\sqrt[4]{2})^3 = -i\sqrt[4]{2}$

$\Rightarrow i\sqrt[4]{2} = -a + b\sqrt[4]{2} - c\sqrt{2} + d(\sqrt[4]{2})^3$

$\therefore\ i\sqrt[4]{2} = b\sqrt[4]{2} + d(\sqrt[4]{2})^3$

$i\sqrt[4]{2} = b\sqrt[4]{2} + d\sqrt{2}\sqrt[4]{2}$

$\Rightarrow i = b + d\sqrt{2} \Rightarrow -1 = b^2 + 2d^2 + 2\sqrt{2}bd \Rightarrow -1 = b^2 + 2d^2$ $(\Rightarrow\Leftarrow)$

$\sqrt{5} = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$

$\Big\downarrow \lambda$

$-\sqrt{5} = a - b\sqrt{2} + c\sqrt{3} - d\sqrt{6}$

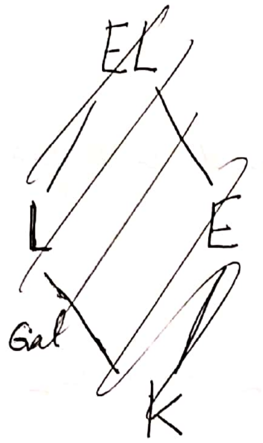
$\sqrt{5} = -a + b\sqrt{2} - c\sqrt{3} + d\sqrt{6}$

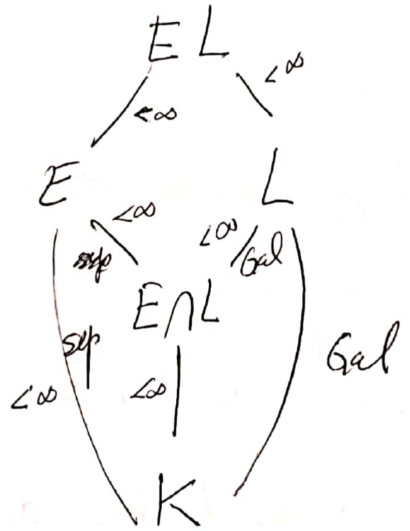
$\sqrt{5} = b\sqrt{2} + d\sqrt{6}$

L/K, E/K separables finitas

L/K Galoisiana

$\Rightarrow EL/E$ Galoisiana, $Gal(EL/E) \cong Gal(L/E\cap L)$



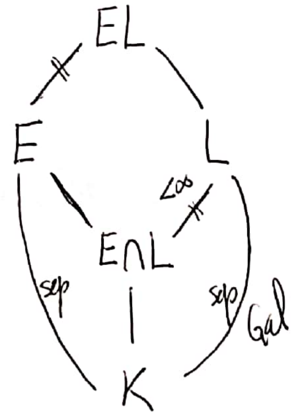


Resultado: Diagrama

Si K/F Galoisiana

$\Rightarrow KF'/F'$ Galoisiana

y $Gal(KF'/F') \cong Gal(K/K\cap F')$



Por demostrar que EL/E Galoisiana con $Gal(EL/E) \cong Gal(L/E\cap L)$

- Dem - L/K Galoisiana $\Rightarrow$ L cuerpo de descomposición de $f(x) \in K[x]$ separable

pero $K \hookrightarrow E$, $L \hookrightarrow EL$

$\Rightarrow$ EL cuerpo de desc. de $f(x) \in E[x]$ (separable)

$\therefore$ EL/E Galoisiana

Tomamos
$$\varphi : Gal(EL/E) \longrightarrow Gal(L/K)$$
$$\sigma \longmapsto \sigma|_K$$

Como $L \hookrightarrow EL$, entonces $\sigma \in Aut(EL) \Rightarrow \sigma \in Aut(L)$

y $K \hookrightarrow E \Rightarrow \left(\sigma|_E = id_E \Rightarrow \sigma|_K = id_K \right)$

$\therefore$ φ bien definida

$\therefore$ El resto es más o menos evidente.

$L = \mathbb{Q}(\sqrt{d})$

$\alpha = a + b\sqrt{d} \Rightarrow \frac{\alpha}{2} = \frac{a}{2} + \frac{b\sqrt{d}}{2}$

$(x - \frac{\alpha}{2})(x + \frac{\alpha}{2}) = x - ax + \frac{a^2 - b^2 d}{4}$

$t = a$ entero $\quad \frac{a^2 - b^2 d}{4}$ entero!

$\frac{a^2 - b^2 d}{4} = s \Rightarrow t^2 - b^2 d = 4s \rightarrow t^2 - 4s = b^2 d$

$b = \frac{p}{q} \Rightarrow t^2 - 4s = \frac{p^2}{q^2} d \Rightarrow \underset{r}{\overset{b \in \mathbb{Z}}{\parallel}}$

$\therefore \quad t - r^2 d = 4s$

ρ raíz de la unidad, $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}(\rho) = L$

- probar que $F/\mathbb{Q}$ galoisiana

$Gal(\mathbb{Q}(\rho)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*$ cíclico!

$F \longleftrightarrow H \leq (\mathbb{Z}/n\mathbb{Z})^* \cong (\mathbb{Z}/(n-1)\mathbb{Z})$

$\downarrow$

cíclico!

$\therefore$ ~~$\exists$ ..~~ $\sigma \in Gal(\mathbb{Q}(\rho)/\mathbb{Q})$; $H = \langle a \rangle$.

$Gal(L^H/\mathbb{Q})$ cíclico! $\Rightarrow$ normal $\Rightarrow F/\mathbb{Q}$ galois trivial: $\mathbb{Z}[\sqrt{-5}] \subset \mathbb{Q}(\sqrt{-5})_{ent}$

Sea $\beta \in \mathbb{Q}(\sqrt{-5})_{ent} \Rightarrow \beta$ entero sobre $\mathbb{Z}[\sqrt{-5}]$

$\Rightarrow f(\beta)=0$ para algún $f(x) \in \mathbb{Z}[\sqrt{-5}][x]$ (mónico)

$\mathrm{Quot}(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Q}(\sqrt{-5})$.

$$\beta^n + \underbrace{a_{n-1}}_{\in \mathbb{Z}[\sqrt{-5}]}\beta^{n-1} + \cdots + a_1\beta + a_0 = 0$$

)

$\cos(x)$, $\cos(\sqrt{2}x)$, $\cos(\sqrt{3}x)$ algeb. independientes

-Dem-. $a\cos(x) + b\cos(\sqrt{2}x) + c\cos(\sqrt{3}x) = 0 \quad \forall x$

$x=\frac{\pi}{2}$: $b\cos\left(\frac{\sqrt{2}}{2}\pi\right) + c\cos\left(\frac{\sqrt{3}}{2}\pi\right) = 0$

$x = \frac{\sqrt{2}}{2}\pi$: $a\cos\left(\frac{\sqrt{2}}{2}\pi\right) + (-b) + c\cos\left(\frac{\sqrt{6}}{2}\pi\right) = 0$

$x = \frac{\sqrt{3}}{2}\pi$: $a\cos\left(\frac{\sqrt{3}}{2}\pi\right) + b\cos\left(\frac{\sqrt{6}}{2}\pi\right) + c\cos\left(\frac{3}{2}\pi\right) = 0$

$x=0$: $a+b+c=0$ $\qquad \cos(\sqrt{2}\sqrt{6}\pi)$ ''

$x=2\pi$: $a + b\cos(2\sqrt{2}\pi) + c\cos(2\sqrt{3}\pi) = 0$

$x=3\pi$: $-a + b\cos(\sqrt{3}\sqrt{6}\pi) + c\cos(3\sqrt{3}\pi) = 0$

$x=\sqrt{2}\pi$: $a\cos(\sqrt{2}\pi) + b + c\cos(\sqrt{6}\pi) = 0$

$x=\sqrt{3}\pi$ $a\cos(\sqrt{3}\pi) + b\cos(\sqrt{6}\pi) + -c = 0$

[1] Probar que $\sqrt[5]{3+\sqrt[4]{\frac{1}{2}+\frac{\sqrt{5}}{2}}}$ es un entero algebraico.

-Demostración-

$$\alpha=\sqrt[5]{3+\sqrt[4]{\frac{1+\sqrt{5}}{2}}} \Rightarrow \alpha^5 = 3+\sqrt[4]{\frac{1+\sqrt{5}}{2}}$$

$$\Rightarrow \alpha^5-3=\sqrt[4]{\frac{1+\sqrt{5}}{2}}$$

$$\Rightarrow (\alpha^5-3)^4 = \frac{1+\sqrt{5}}{2} \Rightarrow 2(\alpha^5-3)^4-1=\sqrt{5}$$

$$\Rightarrow 4(\alpha^5-3)^8-4(\alpha^5-3)^4+1=5$$

$$\cancel{\Rightarrow 4(\alpha^5-3)^8-4(\alpha^5-3)^4=}$$

$$\Rightarrow 4(\alpha^5-3)^8-4(\alpha^5-3)^4-4=0$$

$$\Rightarrow (\alpha^5-3)^8-(\alpha^5-3)^4-1=0$$ (polinomio mónico de grado ~~8~~ 40 en $\mathbb{Z}[X]$)

$\therefore$ α es entero algebraico.

---

[2] Probar que $\mathbb{Z}[\sqrt{-5}]$ es normal.

-Dem-

$\mathbb{Z}[\sqrt{-5}]$ normal $\Leftrightarrow$ $\mathbb{Z}[\sqrt{-5}]$ integralmente cerrado en $\mathbb{Q}(\sqrt{-5})$.

$\Leftrightarrow \mathbb{Q}(\sqrt{-5})_{ent} = \mathbb{Z}[\sqrt{-5}]$

$\mathbb{Q}(\sqrt{-5})_{ent} = \{\beta\in\mathbb{Q}(\sqrt{-5}) \,/\, \beta$ entero algebraico sobre ~~$\mathbb{Z}$~~ $\mathbb{Z}[\sqrt{-5}]\}$

Como $-5\equiv 3\,(4) \Rightarrow \mathbb{Z}[\sqrt{-5}] = \mathcal{O}_{\mathbb{Q}(\sqrt{-5})}$ y ~~$\mathbb{Q}$~~ $\mathcal{O}_K$ es normal

$\therefore$ $\mathbb{Z}[\sqrt{-5}]$ normal.

$$L = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}) \longrightarrow (\mathbb{F}_2)^5 ?$$

$$[L : \mathbb{Q}] = 32$$

Calcular cuántos cuerpos F ~~$\mathbb{Q}$~~ cumplen $F \subseteq L$ y

$$[F : \mathbb{Q}] = 16$$

$L/\mathbb{Q}$ galoisiana!

$$\begin{array}{c} L \\ | \\ F \\ | \\ \mathbb{Q} \end{array}$$

$\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$
$\sqrt{2} \cdot \sqrt{5} = \sqrt{10}$
$\sqrt{2} \cdot \sqrt{7} = \sqrt{14}$
$\sqrt{2} \cdot \sqrt{11} = \sqrt{22}$
$\sqrt{3} \cdot \sqrt{5} = \sqrt{15}$
$\sqrt{3} \cdot \sqrt{7} = \sqrt{21}$
$\sqrt{3} \cdot \sqrt{11} = \sqrt{33}$
$\sqrt{5} \cdot \sqrt{7} = \sqrt{35}$
$\sqrt{5} \cdot \sqrt{11} = \sqrt{55}$
$\sqrt{7} \cdot \sqrt{11} = \sqrt{77}$

Recordatorio:

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt{2}, \sqrt{3}) \arrow[d, phantom, "\parallel"] & \\
 & \mathbb{Q}(\sqrt{2} + \sqrt{3}) \arrow[dl, dash, "Gal"'] \arrow[d, dash, "Gal"] \arrow[dr, dash, "Gal"] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr, dash, "Gal"'] & \mathbb{Q}(\sqrt{6}) \arrow[d, dash, "Gal"] & \mathbb{Q}(\sqrt{3}) \arrow[dl, dash, "Gal"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

$$[L : \mathbb{Q}] = [L : F][F : \mathbb{Q}]$$

$$\therefore [L : F] = 2$$

$$\therefore L = F(\sqrt{d})$$

$\sigma \in Gal(L/\mathbb{Q})$ $\quad \sigma(\sqrt{2}) = \sqrt{3}$

$$\sigma(\sqrt{2})\,\sigma(\sqrt{2}) = \sqrt{3} \cdot \sqrt{3} = 3$$

$$2 = \sigma(2) = 3 \quad \ast$$

$F/\mathbb{Q}$ Galoisiana $\Rightarrow |Gal(F/\mathbb{Q})| = [F : \mathbb{Q}] = 16$

$$\Updownarrow$$

$$Gal(L/F) \triangleleft Gal(L/\mathbb{Q})$$

pero como $[\, Gal(L/\mathbb{Q}) : Gal(F/\mathbb{Q}) \,] = 2$

$$\therefore Gal(F/\mathbb{Q}) \triangleleft Gal(L/\mathbb{Q})$$

$L/\mathbb{Q}$ extensión cuadrática

$\alpha \in L$ entero algebraico

α unidad $\Leftrightarrow$ α^{-1} entero algebraico

Pd: α unidad $\Rightarrow \alpha/2$ ¬entero

-Dem-. $L/\mathbb{Q}$ cuadrática $\Rightarrow L = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Q}$

sup $\alpha/2$ entero $\Rightarrow$ $a_0 + a_1 \alpha/2 + a_2(\alpha/2)^2 + \cdots + a_{n-1}(\alpha/2)^{n-1} + (\alpha/2)^n = 0$

donde $a_i \in \mathbb{Z}$

$\alpha = a + b\sqrt{d}$ entero $\Rightarrow$ $a - b\sqrt{d}$ entero

$$(x - a - b\sqrt{d})(x - a + b\sqrt{d}) = (x-a)^2 - b^2 d$$
$$= x^2 - 2ax + a^2 - b^2 d$$

$$\alpha^{-1} = \frac{1}{a+b\sqrt{d}} = \frac{1}{a+b\sqrt{d}} \cdot \frac{a-b\sqrt{d}}{a-b\sqrt{d}} = \frac{a-b\sqrt{d}}{a^2-b^2d} = \frac{a}{a^2-b^2d} - \frac{b}{a^2-b^2d}\sqrt{d}$$

entero $\Rightarrow$ $\overline{\alpha^{-1}}$ entero

$$\left(x - \frac{a}{a^2-b^2d} + \frac{b}{a^2-b^2d}\sqrt{d}\right)\left(x + \frac{a}{a^2-b^2d} - \frac{b}{a^2-b^2d}\sqrt{d}\right)$$

$$= x^2 - \frac{2a}{a^2-b^2d}x + \frac{a^2}{(a^2-b^2d)^2} - \frac{b^2d}{(a^2-b^2d)^2}$$

$$\alpha = a + b\sqrt{d} \Rightarrow \alpha/2 = \frac{a}{2} + \frac{b}{2}\sqrt{d}$$

$$\Rightarrow p(x) = x^2 - ax + \frac{a^2}{4} - \frac{b^2}{4}d = x^2 - ax + \frac{a^2-b^2d}{4}$$

$\underset{s}{\overset{a}{\shortparallel}}$ entero, $\frac{a^2-b^2d}{4}$ entero $\Rightarrow$ $s^2 - b^2d = 4t$

$$\frac{1+\sqrt{d}}{2} = \alpha \Rightarrow \alpha^{-1} = \frac{2}{1+\sqrt{d}} = \frac{2}{1+\sqrt{d}} \cdot \frac{1-\sqrt{d}}{1-\sqrt{d}} = \frac{2(1-\sqrt{d})}{1-d}$$

e^{x^2} algebraico sobre $\mathbb{C}(e^x)$

$$\cancel{\begin{array}{c}\mathbb{C}(e^x)\\ | \\ \mathbb{C}(e^{x^2})\end{array}}$$

$$\begin{tikzcd}
\mathbb{C}(e^{x^2}, e^x) \arrow[d, dash, "\text{algebraica ext.}"] \\
\mathbb{C}(e^x)
\end{tikzcd}$$

e^{x^2} trascendente sobre $\mathbb{C}(e^x)$

$p \in \mathbb{C}(e^x)$: $p(x) = a_0 + a_1 e^x + a_2 e^{2x} + \dots + a_n e^{nx}$

$p(x)\, e^{x^2} = a_0 e^{x^2} + a_1 e^x + a_2 e^{2x} + \dots + a_n e^{nx} = 0$

$$\cancel{a_0 + a_1 \frac{e^x}{e^{x^2}} + a_2 \frac{e^{2x}}{e^{x^2}} + \dots + a_n \frac{e^{nx}}{e^{x^2}} = 0}$$

cuando $x \to \infty$: $a_0 = 0$

$$a_0 e^{x^2} + a_1 e^{x+x^2} + a_2 e^{2x+x^2} + \dots + a_n e^{nx+x^2} = 0$$

L cuerpo, $D \subseteq B \subseteq L$. B entero/$D$

(i) Si $b \in L$ ent/B $\Rightarrow$ b ent/D

$b \in L$ ent/B $\Rightarrow$ $\cancel{b^n = \sum_{k=0}^{n-1} b_k b^k}$

$$\Rightarrow b^n = \sum_{k=0}^{n-1} c_k b^k \quad , \quad c_k \in B$$

Considerar $B' = B[c_0, \dots, c_{n-1}] \Rightarrow b$ ent/B'

$$B'[b] = B[c_0, \dots, c_{n-1}][b]$$

$$\alpha = \sum_{i=0}^{n-1} \beta_i' b^i \quad , \quad \beta_i' \in B'$$

$$\Rightarrow \quad B'[b] = B' + bB' + b^2 B' + \dots + b^{n-1} B'$$

Como B' finit/. generado como D-módulo $\Rightarrow$ $b^i B'$ f.g. D-módulo!

$\therefore$ $B'[b]$ f.g como D-módulo

b entero $/D$

//

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{1,3,5,7\} \cong \mathrm{Gal}(\mathbb{Q}(\eta)/\mathbb{Q})$$

$$\begin{tikzcd}
 & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \arrow[dl, dash, "2"'] \arrow[d, dash, "2"] \arrow[dr, dash, "2"] & \\
\langle 3 \rangle \arrow[dr, dash, "2"'] & \langle 5 \rangle \arrow[d, dash, "2"] & \langle 7 \rangle \arrow[dl, dash, "2"] \\
 & \langle 1 \rangle &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Q}(\eta) \arrow[d, dash] \\
F = \mathbb{Q}(\eta)^{\langle \sigma_3 \rangle} \arrow[d, dash] \\
\mathbb{Q}
\end{tikzcd}$$

$$\sigma_3(\eta) = \eta^3$$

$$\sigma_3(1) = 1$$

$$\sigma_3\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = e^{3i\pi/4} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$\sigma_3\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = e^{6i\pi/4} = e^{3i\pi/2} = -i$$

$$\sigma_3\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = e^{15i\pi/4} = e^{12i\pi/4} e^{3i\pi/4} = e^{3i\pi} e^{3i\pi/4} = -1\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$\sigma_3\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = e^{21i\pi/4} = e^{20i\pi/4} e^{i\pi/4} = -\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$\sigma_3(i) = -i$$

$$\sigma_3(-i) = (-i)^3 = -i^3 = i$$

$\eta = e^{i\pi/4}$, $\eta^2 = e^{i\pi/2} = \cos(\pi/2) + i\,\mathrm{sen}(\pi/2) = i$

$\eta^3 = e^{3i\pi/4}$

$\eta^4 = e^{i\pi} = -1$

$\eta^5 = e^{5i\pi/4}$

$\eta^6 = e^{3i\pi/2} = -i$

$\eta^7 = e^{7i\pi/4}$

$\eta^8 = e^{2i\pi} = 1$

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
s	0	1	2	3	4
c	4	3	2	1	0

$$\frac{\sqrt{\;\;}}{2}$$

$\mathrm{sen}(\pi/4) = \cos(\pi/4) = \frac{\sqrt{2}}{2}$

$$e^{3i\pi/4} = \cos\left(\frac{3\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$e^{5i\pi/4} = \cos\left(\frac{5\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$e^{7i\pi/4} = \cos\left(\frac{7\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$\sigma_3(\sqrt{2}) = (\sqrt{2})^3 = 2\sqrt{2}$; $\sigma_3(i\sqrt{2}) = \sigma_3(i)\,\sigma_3(\sqrt{2})$
$= -i\,2\sqrt{2}$

$\sigma_3(i) = i^3 = -i$

$\sigma_3(\eta) = \eta^3 \Rightarrow \sigma_3\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$\sigma_5(i) = i^5 = i$

$\sigma_5(\sqrt{2}) = \sqrt{2}(\sqrt{2})^4$
$= 4\sqrt{2}$

$\sigma_5\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$
$= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$\mathbb{Q}(\eta + \eta^7) = \mathbb{Q}(\sqrt{2})$

$\mathbb{Q}(\eta^5 + \eta^7) = \mathbb{Q}(-i\sqrt{2}) = \mathbb{Q}(i\sqrt{2}) = \mathbb{Q}(\sqrt{-2})$

$\mathbb{Q}(\eta^2) = \mathbb{Q}(i) = \mathbb{Q}(\sqrt{-1})$

$\alpha^2 = -2$

$\alpha^2 + 2 = 0$

$\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} - \frac{2}{4} + \frac{2}{4}\,i\,2 = i$

Scanned with CamScanner

Como $\mathbb{Q}(\eta)/\mathbb{Q}$ Galois, $[\mathbb{Q}(\eta):\mathbb{Q}] = |Gal(\mathbb{Q}(\eta)/\mathbb{Q})| = 4$

una base es ~~η,η^2,η~~ $1, \eta, \eta^2, \eta^3$ $(\eta, \eta^2, \eta^3, \eta^4)$

~~$\sigma_3(a+b\eta+c\eta^2+d\eta^3) = a+b\eta^3+c\eta^6+d\eta$~~

~~$\sigma_3(\lambda) = \lambda \Leftrightarrow b=d$~~

$\sigma_3(\underbrace{a\eta+b\eta^2+c\eta^3+d\eta^4}_{\lambda}) = a\eta^3+b\eta^6+c\eta+d\eta^4$

$\Rightarrow a\eta+b\eta^2+c\eta^3+d\eta^4 = a\eta^3+b\eta^6+c\eta+d\eta^4$

$\eta = e^{2\pi i/8} = e^{\pi i/4}$

base $1,\eta,\eta^2,\eta^3$: $\sigma_3(\underbrace{a+b\eta+c\eta^2+d\eta^3}_{\lambda}) = a+b\eta^3+c\eta^6+d\eta$

$\Rightarrow a+b\eta+c\eta^2+d\eta^3 = a+b\eta^3+c\eta^6+d\eta$
$= a+b\eta^3 - c\eta^2+d\eta$

$\Rightarrow \eta(b-d)+\eta^2(2c)+\eta^3(d-b)=0 \rightarrow$ $b=d$, $c=0$

$\lambda = a+b\eta+b\eta^3 = a+b(\eta+\eta^3)$ $\quad e^{i\pi/4}+e^{3i\pi/4}$

$x^8-1 = (x^4-1)(x^4+1) \rightarrow (x+1)(x^3-x^2+x-1)$
$= (x^2-1)(x^2+1)(x^2-i)(x^2+i)$
$= (x-1)(x+1)(x+i)(x-i)(x$
$\rightarrow (x-1)(x^3+x^2+x+1)$

$$Gal\left(\mathbb{Q}(\eta_n)/\mathbb{Q}\right) \cong \left(\mathbb{Z}/n\mathbb{Z}\right)^*$$

$n=8 : \left(\mathbb{Z}/8\mathbb{Z}\right)^* = \{1,3,5,7\}$ | $1^2 = 1$, $3^2 = 9 = 1$, $5^2 = 25 = 1$, $7^2 = 49 = 1$

$$\left(\mathbb{Z}/8\mathbb{Z}\right)^* = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} .$$

tenemos $\left(\mathbb{Z}/8\mathbb{Z}\right)^* \longrightarrow Gal\left(\mathbb{Q}(\eta_8)/\mathbb{Q}\right)$

$$a \longmapsto \sigma_a \quad ; \quad \sigma_a(\eta_8) = \eta_8^{\,a}$$

$$\therefore \; Gal\left(\mathbb{Q}(\eta_8)/\mathbb{Q}\right) = \{\sigma_1, \sigma_3, \sigma_5, \sigma_7\}$$

$$\begin{tikzcd}
 & \left(\mathbb{Z}/8\mathbb{Z}\right)^* \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle 3 \rangle \arrow[dr, dash] & \langle 5 \rangle \arrow[d, dash] & \langle 7 \rangle \arrow[dl, dash] \\
 & \langle 1 \rangle &
\end{tikzcd}$$

$\eta, \eta^2, \ldots, \eta^7$ base de $\mathbb{Q}(\eta)/\mathbb{Q}$

$$\sigma_3\left(a\eta + b\eta^2 + c\eta^3 + d\eta^4 + e\eta^5 + f\eta^6 + g\eta^7\right)$$

$$= a\eta^3 + b\eta^6 + c\eta + d\eta^4 + e\eta^7 + f\eta^2 + g\eta^5$$

$$\Rightarrow a\eta + b\eta^2 + c\eta^3 + d\eta^4 + e\eta^5 + f\eta^6 + g\eta^7 = a\eta^3 + b\eta^6 + c\eta + d\eta^4 + e\eta^7 + f\eta^2 + g\eta^5$$

$a = c$, $b = f$, $e = g$ | $\sigma_3(\lambda) = \lambda \Leftrightarrow \lambda = a(\eta+\eta^3) + b(\eta^2+\eta^6) + d\eta^4 + e(\eta^5+\eta^7)$

$a = b = d = e = -1$:

$$\lambda = -\eta - \eta^3 - \eta^2 - \eta^6 - \eta^4 - \eta^5 - \eta^7 = 1$$

Problema 2

$$\begin{tikzcd}[column sep=small] & FL & \\ F \arrow[ur, dash, "Gal"] & & L \arrow[ul, dash, "Gal"'] \\ & K \arrow[ul, dash, "Gal"] \arrow[ur, dash, "Gal"'] \arrow[uu, dash] & \end{tikzcd}$$

$\Rightarrow \begin{cases} F/K \text{ Gal} \Rightarrow F \text{ cuerpo de descomp. de } f_1(x) \in K[x] \text{ sep.} \\ L/K \text{ Gal} \Rightarrow L \text{ cuerpo de desc. de } f_2(x) \in K[x] \text{ sep.} \end{cases}$

(no poseen raíces en común).

FL cuerp. de descomp. de $f_1(x) f_2(x)$ (sep). $\quad \therefore FL/K$ Gal!

$$\varphi : Gal(FL/K) \longrightarrow Gal(F/K) \times Gal(L/K)$$
$$\sigma \longmapsto (\sigma|_F, \sigma|_L)$$

φ homom! $\quad \ker \varphi = \{\sigma \in Gal(FL/K) \;/\; (\sigma|_F, \sigma|_L) = (1_F, 1_L)\}$

$$\sigma|_F = 1_F,\ \sigma|_L = 1_L \Rightarrow \sigma|_{F\cap L} = 1_{F\cap L} \Rightarrow \sigma|_{FL} = 1_{FL} \quad \therefore \sigma = 1_{FL}$$

$$\varphi(Gal(LF/K)) = H = \{(\sigma|_F, \sigma|_L) \;/\; \sigma \in Gal(FL/K)\}$$

$$(\sigma|_F)|_K = \sigma|_K = (\sigma|_L)|_K$$

Afi $\quad |H| = |Gal(F/K)| \cdot |Gal(L/K)|$

Si $\quad (\tau, \xi) \in Gal(F/K) \times Gal(L/K) \Rightarrow \begin{cases} \tau \in Gal(F/K) \\ \xi \in Gal(L/K) \end{cases}$

$\Rightarrow \forall \tau \in Gal(F/K)$ hay $|Gal(L/K)|$ elementos $\xi \in Gal(L/K)$ tal que $\xi|_K = \tau|_K$

$$|H| = |Gal(LF/K)| = [LF:K] = [L:K][F:K]$$
$$= |Gal(L/K)|\,|Gal(F/K)| \qquad \therefore Gal(LF/K) \cong Gal(L/K) \times Gal(F/K)$$
$$\| $$
$$|Gal(L/K) \times Gal(F/K)|$$

$$\begin{array}{c}\mathbb{F}_2(x)\\ |\\ \mathbb{F}_2\end{array} \Rightarrow \mathbb{F}_2(x)/\mathbb{F}_2 \text{ extensión trascendente} \Rightarrow x \text{ trascendente}/\mathbb{F}_2$$

$$\Rightarrow \{x\} \text{ alg. ind}/\mathbb{F}_2 .$$

$\{x\}$ genera algebraicamente $\mathbb{F}_2(x)/\mathbb{F}_2 \Rightarrow \deg tr(\mathbb{F}_2(x)/\mathbb{F}_2) = 1$

$$\begin{array}{ccc} & & \alpha \in \mathbb{F}_2(x)\\ & & |\\ & & F\\ \mathbb{F}_2(\alpha) & \overset{c}{—} & |\\ | & & \\ \mathbb{F}_2 & & \mathbb{F}_2\end{array}$$ no algebraica!?

$F/\mathbb{F}_2$ algebraica $\Rightarrow \alpha \in F\setminus\mathbb{F}_2$ algebraica$/\mathbb{F}_2$

$$\Rightarrow \alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_1\alpha + a_0 = 0$$

$$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{F}_2[x].$$

$[F:\mathbb{F}_2] < \infty$, $|F| < \infty \Rightarrow F = \mathbb{F}_2(\alpha)$

Como x no algebraico$/\mathbb{F}_2$ $\nexists\, a \in \mathbb{F}_2(x)$ algebraico$/\mathbb{F}_2$.

$f(x) \in \mathbb{F}_2(x) \Rightarrow f(x)^2 = f(x^2)$

$$\begin{array}{ccc} \mathbb{F}_2(x) & & \\ | & & \\ K & \rightarrow & \mathbb{F}_2(f_i)\\ | & \text{trascendente} & \\ \mathbb{F}_2 & & \end{array}$$

alg. ind

alg ind

base de trascendencia $\{x_1, x_2, \dots, x_n\}$ ó $\{f_1, f_2, \dots, f_n\}$

$f_i \in \mathbb{F}_2(x)$

$\Rightarrow$ $F/\mathbb{F}_2(f_1, \dots, f_n)$ algebraica

Af. x trasc$/\mathbb{F}_2 \Rightarrow \forall g(x) \in \mathbb{F}_2(x)\setminus\mathbb{F}_2$: $g(x)$ trasc$/\mathbb{F}_2$

$$=\infty\left[\begin{array}{c}\mathbb{F}_2(x)\\ |<\infty\\ \mathbb{F}_2(g(x))\\ |=\infty\\ \mathbb{F}_2\end{array}\right. \quad \therefore g(x) \text{ trasc}/\mathbb{F}_2$$

$$\begin{array}{c}\mathbb{F}_2(x)\\ |\\ K\\ \neq |\\ \mathbb{F}_2\end{array} \quad \exists\, g(x) \in K\setminus\mathbb{F}_2(x) \Rightarrow \begin{array}{c}\mathbb{F}_2(x)\\ |\\ K\\ |\\ \mathbb{F}_2(g(x))\\ |\\ \mathbb{F}_2\end{array}$$

n° $<\infty$ cuerpos

$<\infty$. pero, $\mathbb{F}_2(x) = \mathbb{F}_2(g(x))(x)$

$\therefore$

$p_1(x), \dots, p_r(x)$ polinomios cuadráticos!

$\Rightarrow e^{p_1(x)}, \dots, e^{p_r(x)}$ linealmente independientes

-Dem-
$$a_1 e^{p_1(x)} + \dots + a_r e^{p_r(x)} = 0$$

con $p_1(x)$ el mayor, para $|x|$ suf grande

$$\Rightarrow a_1 + a_2 \frac{e^{p_2(x)}}{e^{p_1(x)}} + a_3 \frac{e^{p_3(x)}}{e^{p_1(x)}} + a_4 \frac{e^{p_4(x)}}{e^{p_4(x)}} + \dots + a_r \frac{e^{p_r(x)}}{e^{p_1(x)}} = 0$$

$$\Rightarrow a_1 + a_2 e^{p_2(x)-p_1(x)} + a_3 e^{p_3(x)-p_1(x)} + a_4 e^{p_4(x)-p_1(x)} + \dots + a_r e^{p_r(x)-p_1(x)} = 0$$

$$a_1 + a_2 e^{p_1(x)\left(\frac{p_2(x)}{p_1(x)} - 1\right)} + \dots + a_r e^{p_1(x)\left(\frac{p_r(x)}{p_1(x)} - 1\right)} = 0$$

$x \to \infty$, $p_1(x) \to \infty$ / $a_1 = 0$

$\therefore a_i = 0$

(b) e^{x^2} trascendente sobre ~~$\mathbb{C}$~~ $\mathbb{C}(e^x)$

$\mathbb{C}(x)$
|
$\mathbb{C}(x^2)$

$\mathbb{C}(x) \ni e^{x^2}$
|
$\mathbb{C}(e^x)$
|
$\mathbb{C}$

~~$e^x e^x \leq 0$~~

e^x algebraico sobre $\mathbb{C}(e^x)$

x trascendente

$e^{x^2} = (e^x)^x$ e^x alg sobre $\mathbb{C}(e^x)$

x trascendente sobre $\mathbb{C}(e^x)$

$p(e^x)x = 0 \;/ \frac{d}{dt}$

$\Rightarrow p'(e^x)e^x x + p(e^x) = 0$

x^2 es algebraico sobre $\mathbb{C}(x^2)$ (sobre $\mathbb{C}(x)$)

supongamos e^{x^2} algebraico sobre $\mathbb{C}(e^x)$

$x e^{x^2}$ algebraico sobre

$\Rightarrow e^{x^2}$

$\cos(x)$, $\cos(\sqrt{2}x)$, $\cos(\sqrt{3}x)$ alg. independientes $\forall x$

$$a\cos(x)+b\cos(\sqrt{2}x)+c\cos(\sqrt{3}x)=0 \quad \forall x$$

$x=0$: $\quad a+b+c=0$

$x=\pi$: $\quad -a+b\cos(\sqrt{2}\pi)+c\cos(\sqrt{3}\pi)=0$

$x=\sqrt{2}\pi$: $\quad a\cos(\sqrt{2}\pi)+b+c\cos(\sqrt{6}\pi)=0$

$$\Rightarrow (-b-c)\cos(x)+b\cos(\sqrt{2}x)+c\cos(\sqrt{3}x)=0$$

$$b\left(\cos(\sqrt{2}x)-\cos(x)\right)+c\left(\cos(\sqrt{3}x)-\cos(x)\right)=0$$

$\cos(\sqrt{2}x)=\cos(x)$

$x=\sqrt{2}\pi$: $\quad b\left(1-\cos(\sqrt{2}\pi)\right)+c\left(\cos(\sqrt{6}\pi)-\cos(\sqrt{2}\pi)\right)=0$

$x=\sqrt{3}\pi$: $\quad b\left(\cos(\sqrt{6}\pi)-\cos(\sqrt{3}\pi)\right)+c\left(-1-\cos(\sqrt{3}\pi)\right)=0$

$x=\pi$: $\quad b\left(\cos(\sqrt{2}\pi)-1\right)+c\left(\cos(\sqrt{3}\pi)-1\right)=0$

$$\Rightarrow b\left(1-\cos(\sqrt{2}\pi)\right)=c\left(\cos(\sqrt{3}\pi)-1\right)$$

$$\therefore c\left(\cos(\sqrt{3}\pi)-1\right)+c\left(\cos(\sqrt{6}\pi)-\cos(\sqrt{2}\pi)\right)=0$$

$$\Rightarrow c\left(\underbrace{\cos(\sqrt{3}\pi)+\cos(\sqrt{6}\pi)-\cos(\sqrt{2}\pi)-1}_{\neq 0}\right)=0$$

$$\cos(\sqrt{3}\pi)+\cos(\sqrt{6}\pi)=1+\cos(\sqrt{2}\pi)$$

$f(x)=\cos(\sqrt{3}x)+\cos(\sqrt{6}x)-\cos(\sqrt{2}x)-1$

$f'(x)=-\sqrt{3}\,\mathrm{sen}(\sqrt{3}x)-\sqrt{6}\,\mathrm{sen}(\sqrt{6}x)+\sqrt{2}\,\mathrm{sen}(\sqrt{2}x)$

~~$f(x)=0$~~ $\quad x=n\pi$, $n\in\mathbb{Z}$: $f'(x)=0$ máximos o mínimos

$\therefore c=0$

$\therefore b=0$

$\therefore a=0$

$\therefore$ $\cos(x)$, $\cos(\sqrt{2}x)$, $\cos(\sqrt{3}x)$ alg. independientes!

Tenemos

$$\begin{tikzcd}
& \mathbb{Q}(\sqrt{2},\sqrt{3}) \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr, dash, "2"'] & & \mathbb{Q}(\sqrt{3}) \arrow[dl, dash, "2"] \\
& \mathbb{Q} &
\end{tikzcd}$$

Afirmación. $\mathbb{Q}(\sqrt{2},\sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$

Evidente $\mathbb{Q}(\sqrt{2},\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2}+\sqrt{3})$

Ahora, $x_0 = \sqrt{2}+\sqrt{3} \Rightarrow x_0^2 = 2+3+2\sqrt{6}$

$$\begin{aligned}
&\Rightarrow x_0^2 - 5 = 2\sqrt{6} \\
&\Rightarrow x_0^4 + 25 - 10x_0^2 = 24 \\
&\Rightarrow x_0^4 - 10x_0^2 + 1 = 0
\end{aligned}$$

$\therefore$ $\sqrt{2}+\sqrt{3}$ es raíz de $p(x) = x^4 - 10x^2 + 1$

Afirmación. ~~p(x)=x^4+~~ $p(x) = x^4 - 10x^2 + 1 \in \mathbb{Q}[x]$ es irreducible en $\mathbb{Q}$

Si $\exists\, q(x) \in \mathbb{Q}[x]$ irreducible tq $q(x_0) = 0$ ~~...~~ tal que $\partial q < \partial p$

$\Rightarrow \partial q = 2 \Rightarrow q(x) = ax^2 + bx + c$, $a,b,c \in \mathbb{Q}$

Luego $0 = q(x_0) = q(\sqrt{2}+\sqrt{3}) = a(\sqrt{2}+\sqrt{3})^2 + b(\sqrt{2}+\sqrt{3}) + c$

$$= a(5+2\sqrt{6}) + b(\sqrt{2}+\sqrt{3}) + c$$

$$\Rightarrow 5a + 2a\sqrt{6} + b\sqrt{2} + b\sqrt{3} + c = 0$$

$$\Rightarrow \begin{cases} 5a + c = 0 \\ b\sqrt{2} = 0 \\ b\sqrt{3} = 0 \\ 2a\sqrt{6} = 0 \end{cases}$$

$\therefore$ $a,b,c = 0$ $(\Rightarrow\Leftarrow)$

$\therefore$ $p(x)$ es irreducible.

Calcular $[\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5}):\mathbb{Q}]$

Tenemos $\sqrt{2}+\sqrt{3}+\sqrt{5} \in \mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5})=K$.

$\sqrt{2}\sqrt{3}\sqrt{5} = \sqrt{30} \in K$

$$(\sqrt{2}+\sqrt{3}+\sqrt{5})^2 = 10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15} \quad , \quad \therefore \sqrt{6},\sqrt{10},\sqrt{15} \in K$$

Por otro lado, $\sqrt{6}\sqrt{2}=\sqrt{12}=2\sqrt{3}$

$\sqrt{6}\sqrt{3}=\sqrt{18}=3\sqrt{2}$ $\qquad \therefore \sqrt{2},\sqrt{3},\sqrt{5} \in \mathbb{Q}(\sqrt{2}+\sqrt{3}+\sqrt{5})$

$\sqrt{10}=\sqrt{2}\sqrt{5}$

Afirmación. $\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5}) = \mathbb{Q}(\sqrt{2}+\sqrt{3}+\sqrt{5})$

$a=\sqrt{2}+\sqrt{3}+\sqrt{5}$, $\qquad a^2 = 10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}$

$\Rightarrow \quad a^2-10 = 2(\sqrt{6}+$

(3) Probar que $\mathbb{C}$ es la única extensión cuadrática de $\mathbb{R}$.

Supongamos que $L/\mathbb{R}$ es una extensión cuadrática. Como $ch(\mathbb{R})=0$, por (1), existe $a \in L \setminus \mathbb{R}$ t.q. $a^2 \in \mathbb{R}$. Por propiedad de completitud de $\mathbb{R}$, $a^2 = -r$, donde $r>0$. Luego

$$a^2 = -r \Rightarrow \frac{a^2}{r} = -1 \rightarrow \left(\frac{a}{\sqrt{r}}\right)^2 = -1$$

$\alpha = \frac{a}{\sqrt{r}} \in L \setminus \mathbb{R}$ Luego existe $\alpha \in L$ tal que $\alpha^2 = -1$. Se sigue que $\alpha \in L \setminus \mathbb{R}$ es solución de $p(x) = x^2+1 \in \mathbb{R}[x]$.

$$\therefore L \cong \mathbb{R}[x]/(p(x)) = \mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$$

$$\therefore L \cong \mathbb{C}.$$

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(5) Si K es un cuerpo de característica 2, probar que existen matrices $A \in M_2(K)$ que satisfacen $A^2 = I$ pero que no son diagonalizables.

– Demostración –

Basta ver que $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ satisface $A^2 = I$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1+1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

pero A no es diagonalizable:

$$A = PDP^{-1} \text{ (} D \text{ diagonal)} \Rightarrow A^2 = PD^2P^{-1} = I$$

$$\Rightarrow D^2 = I \Rightarrow D = I \Rightarrow A^2 = I \;(\Rightarrow\Leftarrow)$$

$$[L:\mathbb{R}]=2 \quad , c(\mathbb{R})=0$$

$\exists a \in L$; $x^2-a^2 \in \mathbb{R}[x]$ ~~es~~ irreducible.

Se tiene que $\mathbb{R}(a)=\mathbb{R}[x]/(x^2-a^2)$

$$x^2-a^2 = a^2\left(\frac{x^2}{a^2}-1\right) = a^2\left(\left(\frac{x}{a}\right)^2-1\right) = a^2\left(\left(\frac{1}{a^2}\right)x^2-1\right)$$

Tomando $y=\frac{x}{a}$, $x^2-a^2=a^2(y^2-1)$

$$y^2-1 \in \mathbb{R}[y] = \mathbb{R}\left[\frac{x}{a}\right] = \mathbb{R}[x]$$

$$L=\{a+b\alpha \mid \alpha^2 \in \mathbb{R}\}$$

$f: \mathbb{C} \to L$, $f(a+bi)=a+b\alpha$ homomorfismo.

$$f((a+bi)+(c+di)) = f((a+c)+(b+d)i)$$
$$= (a+c)+(b+d)\alpha = (a+b\alpha)+(c+d\alpha) =$$
$$= f(a+bi)+f(c+di)$$

$$f((a+bi)(c+di)) = f(ac+adi+bci-bd) = f((ac-bd)+(ad+bc)i)$$
$$= (ac-bd)+(ad+bc)\alpha$$

$$(\tilde{a}+\tilde{b}\alpha)(\tilde{c}+\tilde{d}\alpha) = \tilde{a}\tilde{c}+\tilde{a}\tilde{d}\alpha+\tilde{b}\tilde{c}\alpha+\tilde{b}\tilde{d}\alpha^2$$
$$= (\tilde{a}\tilde{c}+\tilde{b}\tilde{d}\alpha^2)+(\tilde{a}\tilde{d}+\tilde{b}\tilde{c})\alpha$$

$$f(i)=\alpha \Rightarrow f(i^2)=\alpha^2=-1 \quad \therefore \quad \alpha^2=-1$$

$\alpha^2=a>0$, si $\alpha^2 \in \mathbb{R} \Rightarrow \alpha^2<0. \Rightarrow -\alpha^2>0$

$\alpha \notin \mathbb{R}$

$$-\alpha^2=\alpha^2(-1) \Rightarrow -\alpha^2=r \Rightarrow -\left(\frac{\alpha}{\sqrt{r}}\right)^2=1 = \left(\frac{\alpha}{\sqrt{r}}\right)^2=-1$$

$$\Leftrightarrow \frac{\alpha^2}{r}=-1 \Leftrightarrow \alpha^2=-r$$

$$\therefore \quad L=\{a+b\alpha \mid \alpha^2=-r \ , a,b\in\mathbb{R}\}$$
$$= \{a+b\alpha \mid \left(\frac{\alpha}{\sqrt{r}}\right)^2=-1 \ , a,b\in\mathbb{R}\}$$
$$= \{a+b\tfrac{-r}{-r}\alpha \mid \left(\frac{\alpha}{\sqrt{r}}\right)^2=-1 \ , a,b\in\mathbb{R}\}$$

$\{a+ \ldots \}$

(1) Sea L/K una extensión cuadrática. Probar que si la característica de K no es 2, entonces existe $a \in L$ con $a \notin K$ tal que $a^2 \in K$.

Demostración

Si $[L:K]=2$, entonces el polinomio irreducible $m(x) \in K[X]$ ~~puede ser~~ es de la forma $m(x)=x^2+\alpha x+\beta$, con $\beta \neq 0$.

Ahora se sabe que las raíces de $m(x)$ son de la forma $x = \dfrac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$

Por otro lado, si $x_1 \in L$ es raíz de $m(x)$, entonces

$$m(x)=(x-x_1)(x-x_2)$$

donde $-x_1-x_2=\alpha$, $x_1x_2=\beta$. Se sigue de esto que $x_2 \notin K$ $(x_2 \in L)$

Por lo tanto, $\cancel{\alpha^2-4\alpha\beta=(x_1+x_2)^2+4(x_1+x_2)x_1x_2}$

$$\Delta=\alpha^2-4\beta=(x_1+x_2)^2-4x_1x_2 = x_1^2-2x_1x_2+x_2^2=(x_1-x_2)^2$$

~~Supongamos~~ $\cancel{x_1=\dfrac{-\alpha+\sqrt{\Delta}}{2} \Rightarrow \sqrt{\Delta}=2x_1+\alpha}$

$\cancel{x_2=\dfrac{-\alpha-\sqrt{\Delta}}{2} \Rightarrow -\sqrt{\Delta}=2x_2+\alpha \Rightarrow \sqrt{\Delta}=-2x_2-\alpha}$

$$\therefore (x_1-x_2)^2=\alpha^2-4\beta \in K$$

Tomando $\tilde{y}=x_1-x_2$, tenemos que $\tilde{y} \notin K$, pero $\tilde{y}^2 \in K$

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(2) Probar que $\mathbb{C}$ no tiene extensiones cuadráticas

Demostración

Supongamos que $L/\mathbb{C}$ es una extensión de grado 2. Como $ch(\mathbb{C})=0$, entonces por (1), existe $a \in L$, $a \notin \mathbb{C}$ tq $a^2 \in \mathbb{C}$; pero esto es equivalente a que $x^2-a^2 \in \mathbb{C}[X]$ es irreducible $(\Rightarrow\Leftarrow)$
($\mathbb{C}$ algebraicamente cerrado)

( ) Consideremos $L=\mathbb{Q}(\sqrt[8]{2},\rho)$, $\rho=\frac{1+i}{\sqrt{2}}$. L cuerpo de descomposición de $p(x)=x^8-2\in\mathbb{Q}[x]$.

$$\begin{tikzcd} & L \arrow[dl, dash, "2"'] \arrow[dr, dash, "8"] & \\ \mathbb{Q}(\sqrt[8]{2}) \arrow[dr, dash, "8"'] & & \mathbb{Q}(i) \arrow[dl, dash, "2"] \\ & \mathbb{Q} & \end{tikzcd}$$

Evidentemente se cumple que $\mathbb{Q}(\sqrt[8]{2},\rho)=\mathbb{Q}(\sqrt[8]{2},i)$ (Notar que $\sqrt{2}\in L$)

Por teorema de extensión de homomorfismos existen

$\sigma:\begin{matrix} i\mapsto -i \\ \sqrt[8]{2}\mapsto\sqrt[8]{2}\end{matrix}$, $\tau:\begin{matrix} i\mapsto i \\ \sqrt[8]{2}\mapsto\rho\sqrt[8]{2}\end{matrix}$, donde $\sigma^2=\tau^8=id$

$\tau^r\sigma(i)=\tau^r(-i)=-i$; $\tau\sigma^r(i)=\tau(i^r)=i^r$

$\tau^r\sigma(\sqrt[8]{2})=\tau^r(\sqrt[8]{2})=\rho^r\sqrt[8]{2}$; $\tau\sigma^r(\sqrt[8]{2})=\tau(\sqrt[8]{2})=\rho\sqrt[8]{2}$

$\sigma\tau^r(i)=\sigma(i)=-i$

$\sigma\tau^r(\sqrt[8]{2})=\sigma(\rho^r\sqrt[8]{2})=\rho^{-r}\sqrt[8]{2}$

$\left(\rho=\frac{1+i}{\sqrt{2}}\Rightarrow\sigma(\rho)=\frac{1-i}{\sqrt{2}}\right.$, pero $\frac{1+i}{\sqrt{2}}\cdot\frac{1-i}{\sqrt{2}}=\frac{1+1}{2}=1\therefore\left.\sigma(\rho)=\rho^{-1}\right)$

En particular, $\tau^7\sigma=\sigma\tau^{-1}$ $\therefore\sigma^{-1}\tau\sigma=\tau^{-1}$

$\therefore\ Gal(L/\mathbb{Q})=\langle\tau,\sigma\ /\ \sigma^2=\tau^8=id,\ \sigma^{-1}\tau\sigma=\tau^{-1}\rangle$

$\therefore\ Gal(L/\mathbb{Q})\cong D_{16}$

( ) $p(x)=x^5-2$

Raíces de $p(x)$ son $\sqrt[5]{2},\sqrt[5]{2}\eta,\sqrt[5]{2}\eta^2,\sqrt[5]{2}\eta^3,\sqrt[5]{2}\eta^4$; $\eta=e^{2\pi i/5}$

$\eta=\cos\left(\frac{2\pi}{5}\right)+i\,sen\left(\frac{2\pi}{5}\right)$ ... . Sea L cuerpo de descomp. de $p(x)$. $L=\mathbb{Q}(\sqrt[5]{2},\eta)$

$$\begin{tikzcd} & L \arrow[dl, dash, "5"'] \arrow[dr, dash, "4"] & \\ \mathbb{Q}(\eta) \arrow[dr, dash, "4"'] & & \mathbb{Q}(\sqrt[5]{2}) \arrow[dl, dash, "5"] \\ & \mathbb{Q} & \end{tikzcd}$$

Por teorema de extensión de homomorfismos, existen

$$\begin{array}{c|c} \sigma & \tau \\ \eta \mapsto \eta^2 & \eta \mapsto \eta \\ \sqrt[5]{2} \mapsto \sqrt[5]{2} & \sqrt[5]{2} \mapsto \eta\sqrt[5]{2} \end{array}$$

Ejercicio 3. Determine el grupo de Galois de los siguientes polinomios (sobre $\mathbb{Q}$)

(1) $x^3 - x^2 - 4$,

Las posibles raíces racionales de $x^3 - x^2 - 4$ son:
$\pm 1, \pm 2, \pm 4$. Como $2^3 - 2^2 - 4 = 8 - 4 - 4 = 0$

$$\therefore \; x-2 \mid x^3 - x^2 - 4$$

Se tiene que $x^3 - x^2 - 4 = (x-2)(x^2 + x + 4)$

Como $x^2 + x + 4$ ired $/_{\mathbb{Q}}$, $Gal(x^3 - x^2 - 4) \cong C_2$

(2) $x^3 - 2x + 4$,

Posibles raíces racionales son: $\pm 1, \pm 2, \pm 4$.

Como $(-2)^3 - 2(-2) + 4 = -8 + 4 + 4 = 0$

$$\therefore \; x+2 \mid x^3 - 2x + 4$$

Además, $x^3 - 2x + 4 = (x+2)(x^2 - 2x + 2)$. Como $x^2 - 2x + 2$ es irreducible $/_{\mathbb{Q}}$, $Gal(x^3 - 2x + 4) \cong C_2$

(3) $x^3 - x + 1$,

Posibles raíces de $x^3 - x + 1$ son ± 1

$$\therefore \; x^3 - x + 1 \text{ ired } /_{\mathbb{Q}}.$$

Así, si L es cdd de $x^3 - x + 1$, $L = \mathbb{Q}(\sqrt{D}, \theta)$, donde D es discriminante y θ una raíz de $x^3 - x + 1$.

$$D = -4(-1)^3 - 27(1)^2 = 4 - 27 = -23$$

Como $\sqrt{D} \in \mathbb{Q}$, $Gal(x^3 - x + 1) \cong S_3$.

(4) x^3+x^2-2x-1,

Posibles raíces racionales son: ± 1.

$1^3+1^2-2\cdot 1-1 = 1+1-2-1 = -1 \neq 0$

$(-1)^3+(-1)^2-2(-1)-1 = -1+1+2-1 = 1 \neq 0$

$\therefore\ x^3+x^2-2x-1$ irred $/\mathbb{Q}$.

discriminante:

$D = a^2b^2-4b^3-4a^3c-27c^2+18abc$, donde

$a=1,\ b=-2,\ c=-1$

$D = 1\cdot 4-4(-8)-4(1)(-1)-27(1)+18(1)(-2)(-1)$

$= 4+32+4-27+36$

$= 76-27$

$= 49$

Como $\sqrt{D}\in\mathbb{Q} \Rightarrow Gal(x^3+x^2-2x-1)\cong A_3$.

(5) x^4-25,

x^4-25 irreducible sobre $\mathbb{Q}$, ya que tiene raíces $\sqrt{5}, -\sqrt{5}, i\sqrt{5}, -i\sqrt{5}$.

obs. Debemos calcular el resolvente cúbico y el discriminante.

1°) Discriminante:

$x^4+ax^3+bx^2+cx+d$, donde $\begin{cases} a=0\\ b=0\\ c=0\\ d=-25\end{cases}$

$D = 256(-25)^3 = 2^7\cdot(-5^2)^3 = -2^7\cdot 5^6$

$\therefore \sqrt{D}\notin\mathbb{Q}$.

$\sqrt{D} = \sqrt{-2^7\cdot 5^6} = 5^3\sqrt{-2^7} = 5^3\cdot 2^3\sqrt{2}\,i = 10^3\sqrt{2}\,i$

Ahora calcular el resolvente cúbico.

$$\cancel{g(x)} \quad h(x) = x^3 - 2px^2 + (p^2 - 4r)x + q^2$$

donde $p = \frac{1}{8}(-3a^2 + 8b)$.

$q = \frac{1}{8}(a^3 - 4ab + 8c)$

$r = \frac{1}{256}(-3a^4 + 16a^2 b - 64ac + 256d)$

Se tiene $p = 0$, $q = 0$, $r = \frac{1}{256} \cdot 256(-25) = -25$

$$\therefore h(x) = x^3 + 100x$$

$h(x) = x(x^2 + 100)$ ($x^2 + 100$ irred $/\mathbb{Q}$)

• Estudiar si $g(y)$ es irreducible o no sobre $\mathbb{Q}(10\sqrt[3]{2}i)$
$= \mathbb{Q}(\sqrt{2}i)$

$g(y) = y^4 + py^2 + qy + r$

$$\therefore g(y) = y^4 - 25 = (y^2 + 5)(y^2 - 5)$$

$$\therefore G = C$$

(6) $x^4 + 3x^3 - 3x - 2$,
$a = 3$, $b = 0$, $c = -3$, $d = -2$

$D = -4(3)^3(-3)^3 - 27(3)^4(-2)^2 - 192 \cdot 3(-3) \cdot 4 - 6 \cdot 9 \cdot 9(-2)$
$+ 256(-8) - 27(-3)^4$

$D = 4 \cdot 27^2 - 27 \cdot 81 \cdot 4 + 192 \cdot 4 \cdot 9 + 12 \cdot 81 - 256 \cdot 8 - 27 \cdot 81$
$= 2916 - 8748 + 6912 + 972 - 2048 - 2187$
$= -2183$ $\qquad \therefore \sqrt{D} \notin \mathbb{Q}$

Resolvente cúbico:

$$h(x) = x^3 - 2px^2 + (p^2 - 4r)x + q^2 ,$$

$$q = \frac{1}{8}\left(27 + 8(-3)\right) = \frac{1}{8}(27 - 24) = \frac{3}{8}$$

$$p = \frac{1}{8}(-3 \cdot 9) = -\frac{27}{8}$$

$$r = \frac{1}{256}\left(-3 \cdot 81 + \cancel{16} - 64 \cdot 3(-3) + 256(-2)\right)$$

$$= \frac{1}{256}\left(-243 + 576 - 512\right) = -\frac{179}{256}$$

$$h(x) = x^3 + \frac{27}{4}x^2 + \left(\frac{729}{64} + \frac{179}{64}\right)x + \frac{9}{64}$$

$$h(x) = x^3 + \frac{27}{4}x^2 + \frac{908}{64}x + \frac{9}{64}$$

Buscamos raíces de $h(x)$,

$$h(x) = 0 \iff 64x^3 + 432x^2 + 908x + 9 = 0$$

...

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Ejercicio 4. (Fórmula de Newton).

Sea $p(x)$ polinomio mónico de grado n con raíces $\alpha_1, \dots, \alpha_n$. Sean s_i las funciones simétricas elementales de grado i en las raíces y defina $s_i = 0$ para $i > n$. Sea $p_i = \alpha_1^i + \dots + \alpha_n^i$ $(i \geq 0)$. Pruebe las fórmulas:

$$p_1 - s_1 = 0$$

$$p_2 - s_1 p_1 + 2 s_2 = 0$$

$$p_3 - s_1 p_2 + s_2 p_1 - 3 s_3 = 0$$

$$\vdots$$

$$p_i \; s_1 p_{i-1} + s_2 p_{i-2} - \dots + (-1)^{i-1} s_{i-1} p_1 + (-1)^i i s_i = 0$$

dem.

$$p_1 - s_1 = (\alpha_1 + \dots + \alpha_n) - (\alpha_1 + \dots + \alpha_n) = 0$$

$$p_2 - s_1 p_1 + 2 s_2 = (\alpha_1^2 + \dots + \alpha_n^2) - (\alpha_1 + \dots + \alpha_n)(\alpha_1 + \dots + \alpha_n) + 2(\alpha_1\alpha_2 + \dots + \alpha_1\alpha_n + \alpha_2\alpha_3 + \dots + \alpha_2\alpha_n + \dots + \alpha_{n-1}\alpha_n)$$

$$= (\alpha_1^2 + \dots + \alpha_n^2) - \left(\alpha_1^2 + \dots + \alpha_n^2 + 2(\alpha_1\alpha_2 + \dots + \alpha_1\alpha_n + \alpha_2\alpha_3 + \dots + \alpha_2\alpha_n + \dots + \alpha_{n-1}\alpha_n)\right)$$
$$+ 2(\alpha_1\alpha_2 + \dots + \alpha_1\alpha_n + \alpha_2\alpha_3 + \dots + \alpha_2\alpha_n + \dots + \alpha_{n-1}\alpha_n)$$

$= 0$ (evidente).