P1) $\operatorname{sen} \pi z = \dfrac{e^{i\pi z} - e^{-i\pi z}}{2i}$.

$\operatorname{sen} \pi z = 0 \iff 0 = e^{i\pi z} - e^{-i\pi z}$

$\iff e^{-i\pi z} = e^{i\pi z} \iff 1 = e^{2i\pi z} \iff z = k \in \mathbb{Z}$.

$f(z) = \operatorname{sen} \pi z$. $f(k) = 0 \ \forall k \in \mathbb{Z}$.

$f'(z) = (\cos \pi z)\pi \Rightarrow f'(k) = \pi \cos k\pi = \begin{cases} \pi, & k \in 2\mathbb{Z} \\ -\pi, & k \in 2\mathbb{Z}+1 \end{cases}$

$\therefore f(z) = (z-k)\underbrace{f'(k)}_{\neq 0} + (z-k)^2 \dfrac{f'(k)}{2} + \dots$

$\therefore$ $z=k$ cero de orden 1 para f.

$z=k$ cero de orden 1 para $f \iff z=k$ polo de orden 1 para $1/f$.

$\dfrac{1}{f(z)} = \dfrac{1}{\operatorname{sen}\pi z} = \dfrac{a_{-1}}{z-k} + a_0 + a_1(z-k) + a_2(z-k)^2 + \dots \quad \forall z \in \underbrace{V_{z=k}}_{\text{vecindad}}$

$\dfrac{1}{2\pi i}\displaystyle\int_\gamma \dfrac{1}{f(z)}\,dz = a_{-1} \underbrace{\dfrac{1}{2\pi i}\displaystyle\int_\gamma \dfrac{1}{z-k}\,dz}_{1} \qquad \gamma = \partial B(k,\varepsilon)$

$\partial B(k,\varepsilon) \subseteq V_{z=k}$.

$a_{-1} = \operatorname{Res}(f, k) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \dfrac{1}{f(z)}\,dz$

Como $z=k$ polo de orden 1 para $1/f$.

$\Rightarrow \operatorname{Res}(f,k) = \lim\limits_{z\to k} (z-k)\, 1/f(z)$

$\dfrac{(z-k)}{f(z)} = 2i \dfrac{(z-k)}{e^{i\pi z} - e^{-i\pi z}}$

$e^{i\pi z} = \displaystyle\sum_{n=0}^{\infty} \dfrac{(i\pi z)^n}{n!} = 1 + \dfrac{i\pi z}{1} - \dfrac{\pi^2 z^2}{2!} - \dfrac{i\pi^3 z^3}{3!} + \dfrac{\pi^4 z^4}{4} - \dots$

$e^{-i\pi z} = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-i\pi z)^n}{n!} = 1 - \dfrac{i\pi z}{1} - \dfrac{\pi^2 z^2}{2!} + \dfrac{i\pi^3 z^3}{3!} + \dfrac{\pi^4 z^4}{4}$

$$e^{i\pi z} - e^{-i\pi z} = \sum_{n=0}^{\infty}\left(\frac{(i\pi z)^n}{n!} - \frac{(-i\pi z)^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{(i\pi z + i\pi z)}{n!}\Big((i\pi z)^{n-1} + (i\pi z)^{n-2}(-i\pi z) + (i\pi z)^{n-3}(-i\pi z)^2 + \dots + (i\pi z)(-i\pi z)^{n-2} + (-i\pi z)^{n-1}\Big)$$

$$a_{-1} = Res(f,k) = \lim_{z\to k} 2i \frac{z-k}{e^{i\pi z}-e^{-i\pi z}} = 2i \lim_{z\to k}\frac{z-k}{e^{i\pi z}-e^{-i\pi z}}.$$

$z = x\in\mathbb{R}$: $e^{i\pi x} - e^{-i\pi x} = (\cos(\pi x)+i\,sen(\pi x)) - (\cos(\pi x) - i\,sen(\pi x))$

$= 2i\,sen(\pi x)$

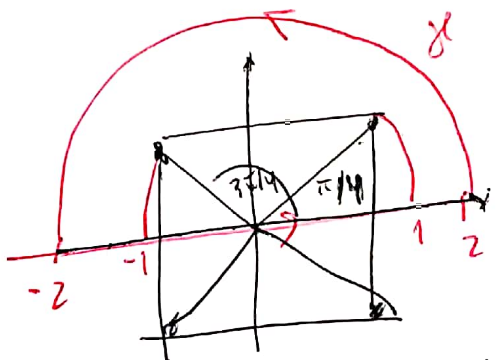
$$\lim_{x\to k}\frac{x-k}{2i\,sen\,\pi x} \underset{\text{L'hôpital.}}{=} \lim_{x\to k}\frac{1}{2i\pi\cos(\pi x)} = \frac{1}{2i\pi\cos(\pi k)}$$

$\therefore$ $Res(f,k) = \dfrac{1}{\pi\cos(\pi k)}$

P2| Evaluar $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^4}$

$f(z)=1+z^4$. $f(z)=0 \iff z^4=-1 \iff z = e^{\left(\frac{2\pi k+\pi}{4}\right)i}$

$k=0,1,2,3$



polos de $\frac{1}{f(z)}$ son $\{e^{i\pi/4}, e^{\frac{3\pi}{4}i}, e^{\frac{5\pi}{4}i}, e^{\frac{7\pi}{4}i}\}$

y polos simples.

$R\geq 2$. $\gamma = [-R,R]+\gamma_R$, $\gamma_R(t) = Re^{i\theta}$, $\theta\in[0,\pi]$

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{f(z)}\,dz = Res(1/f, e^{i\pi/4}) + Res(1/f, e^{3i\pi/4})$$

$$Res(1/f, e^{i\pi/4}) = \lim_{z\to e^{i\pi/4}} \frac{z-e^{i\pi/4}}{1+z^4}$$

$$= \frac{1}{(e^{3\pi i/4}-e^{i\pi/4})(e^{5\pi i/4}-e^{i\pi/4})(e^{7\pi i/4}-e^{i\pi/4})}$$

$$\operatorname{Res}(f, e^{3\pi i/4}) = \frac{1}{(e^{\pi i/4} - e^{3\pi i/4})(e^{5\pi i/4} - e^{3\pi i/4})(e^{7\pi i/4} - e^{3\pi i/4})}$$

$$\int_\gamma \frac{1}{f(z)}\,dz = \int_{-R}^{R} \frac{1}{1+x^4}\,dx + \int_0^{\pi} \frac{Rie^{i\theta}\,d\theta}{1+R^4e^{4i\theta}} \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx$$

obs: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = 2\int_0^{\infty} \frac{1}{1+x^4}\,dx$ etc. (sigue fácil)

———o———

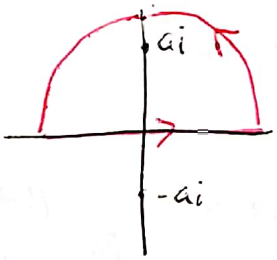
P3) Pd: $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \pi \frac{e^{-a}}{a}$ para $a>0$.

$\dfrac{\cos x}{x^2+a^2}$ es par. $f(z) = \dfrac{\cos(z)}{z^2+a^2}$, $f(z) = \infty \iff z \in \{ai, -ai\}$

$$\cos(z) = \frac{1}{2}(e^{iz}+e^{-iz}) = 0 \iff e^{iz} = -e^{-iz} \iff e^{2iz} = -1$$

$$\iff z \in \{(2k+1)\tfrac{\pi}{2}\}$$

$R \geq 2a$. $\gamma = [-R,R] + \gamma_R$, $\gamma_R(\theta) = Re^{i\theta}$, $\theta \in [0,\pi]$



$$\frac{1}{2\pi i}\int_\gamma \frac{\cos z}{z^2+a^2}\,dz = \operatorname{Res}(f, ai).$$

$$\operatorname{Res}(f, ai) = \lim_{z\to ai} (z-ai)\frac{\cos(z)}{(z+ai)(z-ai)} = \frac{\cos(ai)}{2ai}$$

$$\cos ai = \frac{1}{2}(e^{-a} - e^{+a}) = \frac{1}{2}(e^{-a} - e^{a})$$

$$\int_\gamma \frac{\cos z}{z^2+a^2}\,dz = \int_{-R}^{R} \frac{\cos x}{x^2+a^2}\,dx + \int_0^{\pi} \frac{\cos(Re^{i\theta})}{R^2e^{2i\theta}+a^2} Rie^{i\theta}\,d\theta$$

$$\left|\int_0^{\pi} \frac{\cos(Re^{i\theta})Rie^{i\theta}\,d\theta}{R^2e^{2i\theta}+a^2}\right| \leq \int_0^{\pi} \left|\frac{R\cos(Re^{i\theta})}{R^2e^{2i\theta}+a^2}\right| d\theta \leq \int_0^{\pi} \frac{R|\cos e^{i\theta}|}{R^2-a^2}\,d\theta \xrightarrow{R\to\infty} 0$$

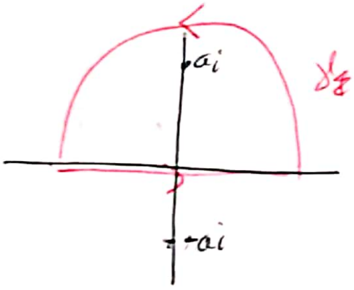
$$|\cos Re^{i\theta}| = \cancel{\tfrac{1}{2}|e^{R\theta}} = \frac{1}{2}\left| e^{(Re^{i\theta})i} + e^{-(Re^{i\theta})i} \right|$$

$$(Re^{i\theta})i = Ri(\cos\theta + i\,\mathrm{sen}\,\theta) = -R\,\mathrm{sen}\,\theta + Ri\cos\theta$$

$$|\cos Re^{i\theta}| \le \frac{1}{2}\left(\left|e^{(Re^{i\theta})i}\right| + \left|e^{-(Re^{i\theta})i}\right| \right)$$

$$= \frac{1}{2}\left(e^{-R\,\mathrm{sen}\,\theta} + e^{R\,\mathrm{sen}\,\theta} \right)$$

$$\cos x = \frac{e^{ix} - e^{-ix}}{2}, \qquad \frac{\cos x}{x^2+a^2} \longrightarrow \frac{e^{iz}}{z^2+a^2} = f(z)$$



$\gamma_R = \gamma_e + [-e, e]$

$R \ge 2a$

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{iz}}{z^2+a^2}\,dz = \mathrm{Res}(f(z), ai)$$

$$\mathrm{Res}(f, ai) = \lim_{z\to ai} (z-ai)\frac{e^{iz}}{(z+ai)(z-ai)} = \frac{e^{-a}}{2ai}$$

$$\int_\gamma f(z)\,dz = \int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}\,dx + \int_0^{\pi} \frac{e^{i(Re^{i\theta})}}{R^2e^{2i\theta}+a^2} Rie^{i\theta}\,d\theta$$

$$\left|\int_0^{\pi} \frac{e^{i(Re^{i\theta})}Rie^{i\theta}}{R^2e^{2i\theta}+a^2}\,d\theta\right| \le \int_0^{\pi} \left|\frac{e^{i(Re^{i\theta})}R}{R^2e^{2i\theta}+a^2}\right| d\theta \le \int_0^{\pi} \frac{Re^{-R\,\mathrm{sen}\,\theta}}{R^2-a^2}\,d\theta \xrightarrow{R\to\infty} 0$$

$$\int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}\,dx = \int_0^{R} \frac{e^{ix}}{x^2+a^2}\,dx + \int_{-R}^{0} \frac{e^{ix}}{x^2+a^2}\,dx$$

$$\int_{-R}^{0} \frac{e^{ix}}{x^2+a^2}\,dx = -\int_0^{-R} \frac{e^{ix}}{x^2+a^2}\,dx \underset{y=-x}{=} \int_0^{R} \frac{e^{-ix}}{x^2+a^2}\,dx$$

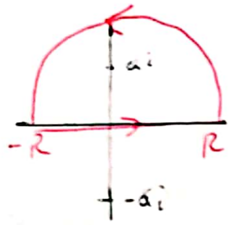
$$\therefore \int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}\,dx = 2\int_0^{R} \frac{\cos x}{x^2+a^2}\,dx$$

$$\frac{1}{2\pi i}\left(2\int_0^{\infty} \frac{\cos x}{x^2+a^2}\,dx\right) = \frac{e^{-a}}{2ai}$$

$$\Rightarrow \int_0^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{a}$$

P4| Pd: $\displaystyle\int_{-\infty}^{\infty} \frac{x \operatorname{sen} x}{x^2+a^2}\,dx = \pi e^{-a}$, $(a>0)$.

$$f(z) = \frac{z e^{iz}}{z^2+a^2} \quad , \quad \operatorname{sen} x = \frac{1}{2i}\left(e^{iz}-e^{-iz}\right)$$



$\gamma = \gamma_R + [-R,R]$

$R \geq 2a$

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \operatorname{Res}(f, ai)$$

$$\operatorname{Res}(f, ai) = \lim_{z\to ai} (z-ai)\frac{z e^{iz}}{z^2+a^2} = \frac{ai\, e^{-a}}{2ai} = \frac{e^{-a}}{2}$$

$$\int_\gamma f(z)\,dz = \int_{-R}^{R} \frac{x e^{ix}}{x^2+a^2}\,dx + \int_0^\pi \frac{Re^{i\theta} e^{i(Re^{i\theta})} Rie^{i\theta}}{R^2e^{2i\theta}+a^2}\,d\theta$$

$$\left|\int_0^\pi \frac{R^2ie^{2i\theta} e^{i(Re^{i\theta})}}{R^2e^{2i\theta}+a^2}\,d\theta\right| \leq \int_0^\pi \frac{R^2 e^{-R\operatorname{sen}\theta}}{R^2-a^2}\,d\theta = \int_0^\pi \frac{e^{-R\operatorname{sen}\theta}}{1-\left(\frac{a}{R}\right)^2}\,d\theta \xrightarrow{R\to\infty} 0$$

$$\int_{-R}^{R} \frac{x e^{ix}}{x^2+a^2}\,dx = \int_0^R \frac{x e^{ix}}{x^2+a^2}\,dx + \int_{-R}^{0} \frac{x e^{ix}}{x^2+a^2}\,dx$$

$$\int_{-R}^{0} \frac{x e^{ix}}{x^2+a^2}\,dx = -\int_0^{-R} \frac{x e^{ix}}{x^2+a^2}\,dx = -\int_0^R \frac{x e^{-ix}}{x^2+a^2}\,dx$$

$$\therefore \int_{-R}^{R} \frac{x e^{ix}}{x^2+a^2}\,dx = \int_0^R \frac{x\left(e^{ix}-e^{-ix}\right)}{x^2+a^2}\,dx = 2i\int_0^R \frac{x\operatorname{sen}x}{x^2+a^2}\,dx$$

$$\Rightarrow \int_0^R \frac{x\operatorname{sen}x}{x^2+a^2}\,dx = \frac{1}{2i}\int_{-R}^{R} \frac{x e^{ix}}{x^2+a^2}$$

$$\int_{-R}^{0} \frac{x\operatorname{sen}x}{x^2+a^2}\,dx = -\int_0^{-R} \frac{x\operatorname{sen}x}{x^2+a^2}\,dx = -\int_0^R \frac{-y\operatorname{sen}(-y)(-dy)}{y^2+a^2} = \int_0^R \frac{x\operatorname{sen}x\,dx}{x^2+a^2}$$

$$\Rightarrow \int_{-R}^{R} \frac{x\operatorname{sen}x}{x^2+a^2} = 2\int_0^R \frac{x\operatorname{sen}x}{x^2+a^2}$$

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$$\therefore \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\,dx = 2\int_{0}^{\infty} \frac{x \sin x}{x^2+a^2}\,dx = 2\cdot\frac{1}{2i}\left(\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+a^2}\,dx\right)$$

$$= \frac{1}{i}\,(2\pi i)\,\frac{e^{-a}}{2} = \pi e^{-a}.$$

P5| ~~Calcular~~ Pd: $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2}\,dx = \frac{\pi}{2}\,(1+2\pi|\xi|)\,e^{-2\pi|\xi|} \quad (\xi \in \mathbb{R})$

$$f(z) = \frac{e^{-2\pi i \xi z}}{(1+z^2)^2} \Rightarrow f(z) = \frac{g(z)}{(z-i)^2}, \quad g(z) = \frac{e^{-2\pi i z\xi}}{(z+i)^2}$$

$$g(z) = b_0 + b_1(z-i) + b_2(z-i)^2 +$$

$$\frac{g(z)}{(z-i)^2} = \frac{b_0}{(z-i)^2} + \frac{b_1}{(z-i)} + b_2 + b_3(z-i) + \dots$$

$$a_{-1} = b_1 = g'(i), \quad g'(z) = \frac{-2\pi i\xi e^{-2\pi i\xi z}(z+i)^2 - 2e^{-2\pi i z\xi}(z+i)}{(z+i)^4}$$

$$g'(z) = \frac{-2\pi i\xi e^{-2\pi i\xi z}(z+i) - 2e^{-2\pi i z\xi}}{(z+i)^3}$$

$$g'(i) = i\left(-2\pi i\xi e^{2\pi\xi}(2i) - 2e^{2\pi\xi}\right) = i\left(4\pi\xi e^{2\pi\xi} - 2e^{2\pi\xi}\right)$$

$$= ie^{2\pi\xi}(4\pi\xi - 1)$$

(Pendiente).

P9| Pd: $\int_0^1 \log(\operatorname{sen} \pi x)\, dx = -\log 2$ (Pendiente)

P13| f holomorfa en $D_r(z_0) \setminus \{z_0\}$, $|f(z)| \leq A|z-z_0|^{-1+\varepsilon}$ para algún $\varepsilon > 0$, $\forall z$ cerca de z_0. Muestre que $z = z_0$ es una singularidad removible para f.

dem $\lim_{z \to z_0} |(z-z_0) f(z)| = ~~\lim_{z\to z_0}~~$?

$$|(z-z_0) f(z)| \leq A|z-z_0|^{-1+\varepsilon} |z-z_0| = A|z-z_0|^{\varepsilon} \xrightarrow{z \to \infty} 0$$

$\therefore \lim_{z \to z_0} |(z-z_0) f(z)| = 0$

$\therefore \lim_{z \to z_0} (z-z_0) f(z) = 0$

$\therefore z = z_0$ singularidad removible para f.

P14| $f: \mathbb{C} \to \mathbb{C}$ entera e inyectiva $\Rightarrow f(z) = az+b$ $(a, b \in \mathbb{C}$, $a \neq 0)$.

dem $g(z) = f(1/z) \Rightarrow g: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ holomorfa.

sup: $\lim_{z \to 0} (z-0) g(z) = \lim_{z \to 0} z f(1/z) = 0$ ~~supon~~

$\Rightarrow$ ~~$\exists G: \mathbb{C} \to \mathbb{C}$ holo / $G(z) = g(z)$ $\forall z \neq 0$~~

~~$G(1/z) = f(z)$~~

$\Rightarrow z = 0$ sing. removible para g $\Rightarrow |g(z)| < \alpha$ $\forall z \in D(0,r) \setminus \{0\}$

$\Rightarrow |f(1/z)| \leq \alpha$ $\forall z \in D(0,r) \setminus \{0\}$

$\Rightarrow |f(z)| \leq \alpha$ $\forall z \in \{z / |z| \geq R\}$ $\Rightarrow f$ cte $(\Rightarrow \Leftarrow)$.

Sup $z=0$ polo de orden n para g

$$g(z) = \frac{h(z)}{z^n} \qquad h \text{ hol en vec de } z=0.$$

$$\Rightarrow f(1/z) = \frac{h(z)}{z^n} \Rightarrow f(z) = z^n h(1/z)$$

~~$z=0$ es polo para f de orden n.~~

$$\cancel{\infty = \lim_{z\to 0} |g(z)| = \lim_{z\to 0} |f(1/z)| = \lim_{z\to\infty} |f(z)|}$$

$$\cancel{\Rightarrow \lim_{z\to\infty} f(z) = \infty}$$

$$g(a) = g(b) \Leftrightarrow \frac{h(a)}{a^n} = \frac{h(b)}{b^n}$$

$$\Updownarrow$$

$$f(1/a) = f(1/b) \Rightarrow \frac{1}{a} = \frac{1}{b} \Rightarrow \boxed{a=b}$$

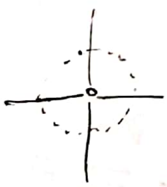
g inyectiva.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \forall z \in \mathbb{C}.$$

$$\Rightarrow g(z) = f(1/z) = \sum_{n=0}^{\infty} a_n z^{-n}$$

Si f es un polinomio $\Rightarrow f(z) = az+b$ ✓

Si f no es polinomio $\Rightarrow g(z)$ tiene singularidad esencial en $z=0$.

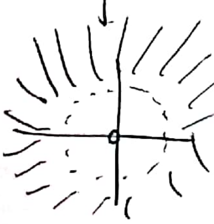


$g(D(0,r)\setminus\{0\})$ es denso en $\mathbb{C}$ $\forall r>0$

$\forall w \in \mathbb{C}$, $\forall \varepsilon>0$ $\exists z_0 \in D(0,r)\setminus\{0\}$ t.q.

$$|g(z)-w| < \varepsilon$$

$\frac{1}{z}$



Si $f(\overline{D(0,r)}) = A \Rightarrow f(z) \in A^c$ ($\rightarrow\leftarrow$) en $\{z \mid |z|>R\}$.

$z=0$ ~~es polo para g~~. no, sigue esencial para g

~~g también inyectiva en $\mathbb{C}\setminus\{0\}$~~. $\therefore f(z) = az+b$

P.15 | (a) f entera Tq $\sup_{|z|=R} |f(z)| \le AR^k + B \quad \forall R>0$, algun $k\in\mathbb{Z}^+$, $A,B>0$

$\Rightarrow$ f es polinomio $\deg f \le k$.

dem. $f^{(n)}(0) = \frac{n!}{2\pi i}\int_{|z|=R} \frac{f(z)}{z^{n+1}}\,dz = \frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(Re^{i\theta})}{R^{n+1}e^{i(n+1)\theta}}\, Rie^{i\theta}d\theta$

$\Rightarrow |f^{(n)}(0)| \le \frac{n!}{2\pi R^n}\int_0^{2\pi}|f(Re^{i\theta})|\,d\theta \le \frac{n!}{R^n}\sup_{\theta\in[0,2\pi]}|f(Re^{i\theta})|$

$\Rightarrow \sup_{|z|=R}|f(z)| \le AR^k+B \ge \frac{R^n}{n!}|f^{(n)}(0)|$

$\to |f^{(n)}(0)| \le \frac{n!(AR^k+B)}{R^n}$

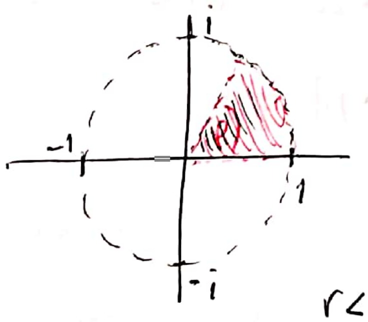
cuando $k > n$ $\quad |f^{(n)}(0)| \le \frac{n!(AR^k+B)}{R^n} \longrightarrow 0$

$\therefore\ |f^{(n)}(0)| = 0$.

$\therefore$ f polinomio de grado $\le k$.

(b) $f:\mathbb{D}\to\mathbb{C}$ holomorfa, acotada, converge a 0 uniformemente en $0<\arg z<\varphi$ cuando $|z|\to 1$, Por demo que $f=0$.

dem.



$\exists K>0,\ \forall z\in\mathbb{D} : |f(z)|\le K$

Sup que $f\neq$ cte. $\Rightarrow$

$r<1$

$f(z) = \frac{1}{2\pi i}\int_{|\xi|=r}\frac{f(\xi)}{\xi-z}\,d\xi = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(re^{i\theta})}{re^{i\theta}-z}\,rie^{i\theta}d\theta$

$f(z) = \frac{1}{2\pi i}\int_0^{\varphi}\frac{f(re^{i\theta})\,rie^{i\theta}}{re^{i\theta}-z}\,d\theta + \frac{1}{2\pi i}\int_{\varphi}^{2\pi}\frac{f(re^{i\theta})\,rie^{i\theta}}{re^{i\theta}-z}\,d\theta$

(pendiente)

(c) Sean $\{w_i\}_{i=1}^n$, $|w_i|=1$

Pd: $\exists \tilde{z} \in \mathbb{C}$, $|\tilde{z}|=1$ tq $|\tilde{z}-w_1||\tilde{z}-w_2|\dots|\tilde{z}-w_n| \geq 1$

Pd: $\exists w \in \mathbb{C}$, $|w|=1$ tq $|w-w_1||w-w_2|\dots|w-w_n| = 1$.

dem. $f(z) = \prod_{i=1}^n (z-w_i)$ f holomorfa en $\mathbb{C}$.

Sup que $|f(z)|<1$ $\forall |z|=1$ $\left(\sup_{z\in \overline{\mathbb{D}}\setminus\mathbb{D}} |f(z)| \leq 1 \right)$

f continua en $\overline{\mathbb{D}}$. $\Rightarrow$ $\sup_{z\in\mathbb{D}} |f(z)| \leq \sup_{z\in\overline{\mathbb{D}}\setminus\mathbb{D}} |f(z)|$

$\Rightarrow \sup_{z\in\overline{\mathbb{D}}^c} |f(z)| > 1$

$\exists K>0$: $|f(z)| \leq K$ $\Rightarrow$ $\exists \tilde{z} \in \overline{\mathbb{D}}\setminus\mathbb{D}$: $f(\tilde{z}) = K < 1$

$\Rightarrow$ $\exists z_0 \in \overline{\mathbb{D}}\setminus\mathbb{D}$ tq $\forall |z|=1$: $|f(z)| \leq |f(z_0)| < 1$

$\forall z \in \mathbb{D}$: $|f(z)| < |f(z_0)| < 1$

~~$\forall z \in \overline{\mathbb{D}}^c$: $|f(z)| \geq 1$~~

pero $f(0) = \prod w_i$ tq $|f(0)| = 1$ ($\Rightarrow\Leftarrow$)

$\boxed{\exists |z|=1 : |f(z)| \geq 1}$

~~Sup $|f(z)|>1$ $\forall |z|=1$ ... $|f(z)|<1$ $\forall |z|=1$~~

Sup que $\forall |z|=1$: $|f(z)|>1$. Como $f(w_i)=0$

$\Rightarrow$ $\exists w \in B(0,1)$: $|f(w)| = 1$.

(d) $f: \mathbb{C} \to \mathbb{C}$ entera. Pd: $|Re(f)| < \infty \Rightarrow f = cte$.

$f \neq cte \Rightarrow f$ ~~no alcanza máximo en $\mathbb{C}$~~.

$\Rightarrow g(z) = e^{f(z)}$ holomorfa no constante

$\Rightarrow g$ no alcanza maximo en $\mathbb{C}$

pero $|g(z)| = e^{Re(f)} < \infty$. $\Rightarrow$ Liouville: $g(z) = cte$.

$Re(f(z)) = \frac{1}{2}(f(z) + \overline{f(z)}) \Rightarrow |f(z) + \overline{f(z)}| = 2|Re(f(z))| < \infty$

$\Rightarrow |Re(f(z))| \leq |f(z) + \overline{f(z)}| \leq 2|f(z)|$

~~xxxxx~~ $f(z) f(\bar{z}) = |f(z)|^2 = Re^2(f) + Im^2(f) \leq K + Im^2(f(z))$

$|f(z)| = \sqrt{Re(f(z))^2 + Im(f(z))^2} \leq \sqrt{K + Im(f(z))^2}$

$f \neq cte \Rightarrow f$ no alcanza maximo en $\mathbb{C}$

$\forall r > 0, \exists w_r \in \overline{B(0,r)}$ tq $\forall z \in \overline{B(0,r)}: |f(z)| \leq |f(w_r)|$

$\exists \tilde{w}_r \in \overline{B(0,r)}^c : |f(w_r)| \not< |f(\tilde{w}_r)|$

$g: \mathbb{C}\setminus\{0\} \to \mathbb{C}$ holomorfa.

f no es polinomio ya que $|Re(f)| < \infty$ ✓ (tomar $z \in \mathbb{R}$)

g tiene singularidad esencial en $z = 0$.

$\forall w \in \mathbb{C}\ \exists z \in \mathbb{C}\setminus\{0\}$ tq $|g(z) - w| < \varepsilon$

$\exists \bar{z} \in \mathbb{C}\setminus\{0\}$ tq $|f(z) - w| < \varepsilon$ | Se puede construir $\{w_n\}$ tq $|Re(w_n)| \to \infty$.

$f: \mathbb{C} \to \mathbb{C}$ entera $\Rightarrow f$ no alcanza el máximo en $\mathbb{C}$

$\Rightarrow \exists (z_n)_{n\in\mathbb{N}}$ en $\mathbb{C}$ tq: $|f(z_n)| \leq |f(z_{n+1})| \quad \forall n$

$\Rightarrow g(z) = e^{f(z)}$ holomorfa no constante (no alcanza el máximo).

$\exists$

P13 $f: G=\{z \mid |z|>R\} \to \mathbb{C}$ tiene un polo en ∞

$\Rightarrow$ el orden del polo es el orden del ~~xxxx~~ polo de $F(z)=f(1/z)$

en $z=0$.

dem. $F(z)$ tiene polo en $z=0$ de orden m

$$F(z)=\frac{G(z)}{z^m}, \quad G(z) \text{ holomorfa en vec. de } z=0.$$

(a) $f:\mathbb{C}\to\mathbb{C}$ holomorfa con singularidad removible en ∞ $\Leftrightarrow$ f constante.

$(\Rightarrow)$ $F(z)=f(1/z)$ tiene singularidad removible en $z=0$

$F:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ $\Leftrightarrow$ $\lim\limits_{z\to 0} zF(z)=0$ $\lim\limits_{z\to\infty} z f(1/z)$

$$\Rightarrow \lim_{z\to 0} zF(z) = \lim_{z\to\infty}\frac{1}{z}F(1/z) = \lim_{z\to\infty} 1/z\, f(z)$$

F es acotada en vecindad de ~~xxxx~~. $V_{z=0}\setminus\{0\}$

$\Rightarrow$ f acotada en vecindad $\{z \mid |z|>R\}$ para algun $R>0$

$\therefore$ f acotada en $\mathbb{C}$

$\therefore$ Liouville $\Rightarrow$ f constante.

$(\Leftarrow)$ $f=\text{cte}$. $\lim\limits_{z\to 0} zF(z) = \lim\limits_{z\to\infty} 1/z\, f(z) = 0$

$\therefore$ F tiene singularidad removible en $z=0$

$\therefore$ f tiene singularidad removible en $z=\infty$.

(b) $f : \mathbb{C} \to \mathbb{C}$ holomorfa.

f tiene polo en ∞ de orden $m \Leftrightarrow f$ polinomio de grado m

dem. ($\Rightarrow$) $f(1/z)$ tiene polo en $z=0$ de grado m

$$f(1/z) = \frac{g(z)}{z^m} \quad , g(z) \text{ holomorfa en vec. de } z=0$$
$$g(0) \neq 0.$$

$$\Rightarrow f(z) = z^m g(1/z)$$

$$f(1/z) = \frac{a_{-m}}{z^m} + \frac{a_{-(m-1)}}{z^{m-1}} + \dots + \frac{a_{-1}}{z} + g(z) \qquad |z| < R$$

$$f(z) = a_{-m} z^m + a_{-(m-1)} z^{m-1} + \dots + a_{-1} z + g(1/z) \qquad |z| > R^{-1}$$

$$f(1/z) = \frac{g(z)}{z^m} \qquad g(z) \text{ holomorfa en } \mathbb{C}$$

$$f(z) = z^m g(1/z)$$

$g(1/z)$ $z \in \mathbb{C} \setminus \{0\}$ $\Rightarrow$ $G(z) = g(1/z)$ definida en $\hat{\mathbb{C}} \setminus \{0\}$
donde $G(\infty) \in \mathbb{C} \setminus \{0\}$.

$$G(z) = \begin{cases} g(z) , & |z| < R \\ g(1/z), & |z| > R^{-1} \end{cases}$$

$$f(z) = z^m g(1/z) \quad , \quad |z| < R : \quad g(z) = \sum_{n=0}^{\infty} a_n z^n$$

Sea $p_m(z) = a_{-m} z^m + a_{-(m-1)} z^{m-1} + \dots + a_{-1} z + a_0$

$\Rightarrow f(z) - p_m(z) : \mathbb{C} \to \mathbb{C}$ Tq $f(1/z) - p_m(1/z)$ no tiene polo en $z=0$. (tiene singularidad removible)

$\therefore F(z) = f(z) - p_m(z) : \mathbb{C} \to \mathbb{C}$ (entera)

y $\sup_{z \in \mathbb{C}} |F(z)| < \infty$

$f(1/z) - p_m(1/z)$ acotada en vec. de $z=0$.

$\therefore F$ cte (Liouville)

$\therefore$ $f(z) = p_m(z) + $ cte.

$(\Leftarrow)$ $f(z) = p_m(z)$ polinomio de grado m.

$f(1/z) = p_m(1/z)$. evidente que $\lim_{z\to\infty} p(z) = \infty$.

$\Rightarrow \lim_{z\to 0} f(1/z) = \infty$ $\therefore$ f tiene un polo en $z=\infty$.

$f(\cancel{1/z}) = p_m(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$

$f(1/z) = a_m \frac{1}{z^m} + a_{m-1} 1/z^{m-1} + \cdots + a_1 \frac{1}{z} + a_0$

$$f(1/z) = \frac{a_m + a_{m-1} z + \cdots + a_1 z^{m-1} + a_0 z^m}{z^m}$$

$f(1/z) = \frac{g(z)}{z^m}$, $g: \mathbb{C} \to \mathbb{C}$ holomorfa.

$g(0) = a_m \neq 0$ $\therefore$ f tiene polo en ∞ de orden m.

(c) Caracterizar las funciones racionales que tienen singularidad removible en ∞.

Dem. $f(z) = \frac{p(z)}{q(z)}$, $p(z), q(z) \in \mathbb{C}[x]$, $\text{mcd}(p,q) = 1$

$f(1/z) = \frac{p(1/z)}{q(1/z)}$ tiene singularidad removible en $z=\infty$.

$\Rightarrow \lim_{z\to 0} z \frac{p(1/z)}{q(1/z)} = \lim_{z\to\infty} \frac{1}{z} \frac{p(z)}{q(z)} = 0$ $\therefore$ $\deg zq(z) > \deg p(z)$

$\Rightarrow \deg q(z) \geq \deg p(z)$.

(d) Caracterice todas las funciones racionales que tienen un polo de orden m en ∞.

$$f(z) = \frac{p(z)}{q(z)} \quad , \quad p(z), q(z) \in \mathbb{C}[x] \quad , \quad \mathrm{mcd}(p(z), q(z)) = 1$$

$$f(1/z) = \frac{p(1/z)}{q(1/z)} \quad \text{tiene polo en } 0 \text{ de orden } m$$

$$f(1/z) = \frac{g(z)}{z^m} \quad , \quad g(0) \neq 0. \quad g \text{ holomorfa } \mathbb{C} \to \mathbb{C}$$

$$\frac{p(z)}{q(z)} \frac{1}{z^m} = g(1/z) \xrightarrow{z \to \infty} g(0) \neq 0$$

$\therefore$ $\deg p(z) = \deg q(z) + m$

$\therefore$ $\deg p(z) = \deg q(z) + m$

ó $f(z) = p_m(z) + \frac{\tilde{p}(z)}{q(z)}$ $\quad \deg p_m(z) = m$

Principio del argumento

P1| $f: G \xrightarrow{\text{región!!}} \mathbb{C}$ meromorfa ; $z_1, \dots, z_k$ ceros de f ; $p_1, \dots, p_l$ polos de f (~~contados con multiplicidad~~). $g: G \to \mathbb{C}$ analítica, γ curva cerrada simple en G tq $z_1, \dots, z_k, p_1, \dots, p_l \in A =$ parte interior de γ. Por demostrar

$$\frac{1}{2\pi i}\int_\gamma g\,\frac{f'}{f} = \sum_{i=1}^{k} n_i\, g(z_i) - \sum_{j=1}^{l} m_j\, g(p_j)$$

donde n_i, m_j son los ordenes de los ceros y polos z_i, p_j respectivamente.

dem.

$$f(z) = (z-z_1)^{n_1}\cdots(z-z_k)^{n_k}(z-p_1)^{-m_1}\cdots(z-p_l)^{-m_l}\varphi(z)$$

$\varphi(z)$ meromorfa sin ceros ni polos en A.

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{k}\frac{n_i}{z-z_i} - \sum_{j=1}^{l}\frac{m_j}{z-p_j} + \frac{\varphi'(z)}{\varphi(z)}$$

$$\frac{f'(z)}{f(z)}g(z) = \sum_{i=1}^{k}\frac{n_i}{z-z_i}g(z) - \sum_{j=1}^{l}\frac{m_j}{z-p_j}g(z) + \frac{\varphi'(z)}{\varphi(z)}g(z)$$

$$= \sum_{i=1}^{k}\left[\frac{n_i(g(z)-g(z_i))}{z-z_i} + \frac{n_i\, g(z_i)}{z-z_i}\right] - \sum_{j=1}^{l}\left[\frac{m_j(g(z)-g(p_j))}{z-p_j} + \frac{m_j\, g(p_j)}{z-p_j}\right] + \frac{\varphi'(z)}{\varphi(z)}g(z)$$

$$= \underbrace{\sum_{i=1}^{k}\frac{n_i(g(z)-g(z_i))}{z-z_i}}_{\text{holo en } G\setminus\{z_i\}} + \sum_{i=1}^{k}\frac{n_i\, g(z_i)}{z-z_i} - \underbrace{\sum_{j=1}^{l}\frac{m_j(g(z)-g(p_j))}{z-p_j}}_{\text{holo en } G-\{p_j\}} - \sum_{j=1}^{l}\frac{m_j\, g(p_j)}{z-p_j} + \frac{\varphi'(z)}{\varphi(z)}g(z)$$

$$\therefore \int_\gamma \frac{f'(z)}{f(z)} g(z)\,dz = \sum_{i=1}^{k} \int_\gamma \frac{n_i\, g(z_i)}{z-z_i}\,dz - \sum_{j=1}^{l} \int_\gamma \frac{m_j\, g(p_j)}{z-p_j}\,dz$$

$$= 2\pi i \left[\sum_{i=1}^{k} n_i\, g(z_i) - \sum_{j=1}^{l} m_j\, g(p_j) \right]$$

$$\therefore \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} g(z)\,dz = \sum_{i=1}^{k} n_i\, g(z_i) - \sum_{j=1}^{l} m_j\, g(p_j)$$

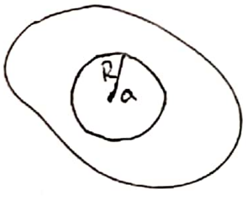
■

- Ocupar el resultado anterior para encontrar una inversa local para f holomorfa:

Prop. f holomorfa en una vecindad de $\overline{B}(a,R)$, f 1-1 en $B(a,R)$. Si $\Omega = f(B(a,R))$ y γ ~~es el~~ círculo $|z-a|=R$, entonces $\forall w \in \Omega$

$$f^{-1}(w) = \frac{1}{2\pi i} \int_\gamma \frac{z\, f'(z)}{f(z)-w}\,dz$$

dem. $\Omega \to$ podemos asumirla región. (nº de Lebesgue).



Sea $w \in \Omega$. $g(z) = f(z) - w$ (def en vec. de $\overline{B}(a,R)$) tiene ~~un solo~~ 1 cero (único) $\tilde{z}$ en $B(a,R)$.

$$g'(z) = f'(z) \Rightarrow \frac{g'(z)}{g(z)} = \frac{f'(z)}{f(z)-w}$$

Consideremos ahora $h(\zeta) \equiv \zeta$ (holomorfa)

$$\frac{1}{2\pi i} \int_\gamma h(z) \frac{f'(z)}{f(z)-w}\,dz = h(\tilde{z}) = \tilde{z}$$

, pero $f(\tilde{z}) - w = 0 \iff f(\tilde{z}) = w \iff \tilde{z} = f^{-1}(w)$

$$\therefore f^{-1}(w) = \frac{1}{2\pi i} \int_\gamma h(z) \frac{f'(z)}{f(z)-w}\,dz$$

Principio del argumento, Teo. de Rouché.

P2| f holomorfa en $\overline{B(0,1)}$, $|f(z)|<1$, $|z|=1$.
Encontrar nº de soluciones (contando multiplicidad) de $f(z)=z^n$ $(n\geq 1)$.

—o—

→ $g(z)=f(z)-z^n$ holomorfa en vec. de $\overline{B(0,1)}$

$f(z)=z^n \Rightarrow |f(z)|=|z|^n<1$ para $\boxed{z=1}$

∴ g no tiene ceros en $\partial B(0,1)$

$$|g(z)+z^n| = |f(z)-z^n+z^n| = |f(z)|<1=|z^n| \quad \forall z\in \overline{B(0,1)}$$

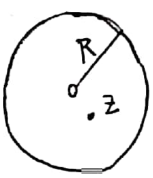
∴ Por teo de Rouché, z^n tiene 1 cero de mult. n en $z=0$ en $B(0,1)$

∴ $g(z)$ tiene 1 cero de mult. n en $B(0,1)$.

P3| f holo en $\overline{B(0,R)}$, $f(0)=0$, $f'(0)\neq 0$, $f(z)\neq 0$ $\forall 0<|z|\leq R$.
$\rho=\min(|f(z)|: |z|=R)>0$.

Def. $g: B(0,\rho)\to\mathbb{C}$, $g(w)=\frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)-w}\,dz$

γ es $|z|=R$. Pd que g es analítica y discutir propiedades de g.

dem.  → $f(z)\neq 0$. $\rho<|f(z)|$ $\forall z\in\overline{B(0,R)}$

$f(z)-w=0 \Leftrightarrow |f(z)|=|w|<\rho$ (⇒⇐).

∴ $f(z)-w\neq 0$ $\forall w\in B(0,\rho)$ $\forall z\in\gamma$.

→ g es continua en $B(0,\rho)$

$$g(w) = \frac{1}{2\pi i}\int_\gamma \frac{z\, f'(z)}{f(z)-w}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{Re^{i\theta} f'(Re^{i\theta})\, Rie^{i\theta}}{f(Re^{i\theta})-w}\,d\theta$$

$$= \int_0^{2\pi} F(\theta, w)\,d\theta \quad , \quad F(\theta,w) = \frac{Re^{i\theta} f'(Re^{i\theta})\, Re^{i\theta}}{f(Re^{i\theta})-w} \quad \text{continua}$$

$F(\theta,w)$ holomorfa en w

$\therefore$ g holomorfa.

f tiene sólo 1 cero ($z=0$) de orden 1 en $\overline{B(0,R)}$

~~$\forall w \in B(0,\rho)$: $\frac{1}{2\pi i}\int_\gamma \frac{z\, f'(z)}{f(z)-w}\,dz$~~

$g_w(z) = f(z) - w$, $|g_w(z) - f(z)| = |w| < \rho \leq |f(z)| \quad \forall |z| = R$

$\therefore$ ~~Teorema~~ $\therefore$ Teo de Rouché: $Z(g_w, B(0,R)) = Z(f, B(0,R)) = 1$

y $g_w'(z) = f'(z)$

$$\therefore \frac{1}{2\pi i}\int_\gamma z\,\frac{f'(z)}{f(z)-w}\,dz = 1\cdot(z_w) \qquad (f(z_w) = w)$$

$\therefore$ $g(w) = z_w$ donde z_w es cero de $f(z) - w$

P4/ (Conway) f meromorfa en $G \to \mathbb{C}$,

$\tilde{f} : G \to \mathbb{C}_\infty$

$$\tilde{f}(z) = \begin{cases} f(z), & z \in G \\ \infty, & z = \text{polo de } f \end{cases}$$

Pd: $\tilde{f}$ es continua.

Dem. Evidente que $\tilde{f}$ es continua en G (porque f es holo en G)

$\lim_{z \to p} |f(z)| = \infty$ (p polo de f)

$f(z) = \dfrac{g(z)}{(z-p)^n}$, g holomorfa en vec. de p.
g meromorfa en G.

$\forall z \neq p$: $\tilde{f}(z) = \dfrac{g(z)}{(z-p)^n}$, $\lim_{z \to p} \dfrac{g(z)}{(z-p)^n} = \infty$ S.P.G

$V \subseteq \mathbb{C}_\infty$ abierto tal que $\infty \in V$ $\Rightarrow$ $\exists r > 0$ tq $\{z \mid |z| > R\} \cup \{\infty\} \subseteq V$

$\tilde{f}^{-1}(\{z \mid |z| > R\} \cup \{\infty\}) \subset \tilde{f}^{-1}(V)$

$f^{-1}(\{z \mid |z| > R\}) \cup \{p_j\}_{j \in \mathbb{N}} \subseteq \tilde{f}^{-1}(V)$ ... falta.

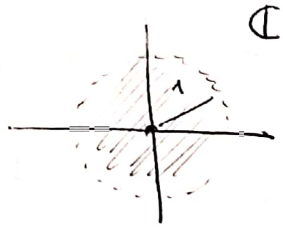
¿Qué significa $\lim_{z \to p} f(z) = \infty$? (Preguntar).

P4) $p(z) = 2z^7 - 12z^5 - 3z + 5$ $\quad A = B(0,1)$

Encontrar el número de soluciones de $p(z)$ en A

$p(z_0) = 0 \iff z_0^7 - 6z_0^5 - \frac{3}{2}z_0 + \frac{5}{2} = 0$

$\mathbb{C}$



$\frac{p(z)}{z^7} = 2 - \frac{12}{z^2} - \frac{3}{z^6} + \frac{5}{z^7}$

$p(z) = 2z^7 - 12z^5 - 3z + 5$

$q(z) = -12z^5 - 3z + 5$

$p(z) - q(z) = 2z^7$

$|p(z) - q(z)| = 2|z|^7 < 2$

$|q(z)| = |-12z^5 - 3z + 5| \geq \cancel{||12|z|^5 - 3|z|| - 5|}$

$\geq ||12z^5 - 3z| - 5|$

$\geq |12|z|^5 - |3z+5||$

$\forall z \in B(0,1)$

$\cdot\ ||3z| - 5| \leq |3z+5| \leq 3|z| + 5 \leq 5$

$= |3|z| - 5|$

$2 \leq$

$0 \leq 12|z|^5 \leq 12$

$\Rightarrow$

$\Rightarrow \quad -5 \leq -|3z+5| \leq -2$

$\Rightarrow \quad -5 \leq 12|z|^5 - |3z+5| \leq 10$

$\left.\begin{array}{l} 2 \leq |3z+5| \leq 5 \\ -12 \leq -12|z|^5 \leq 0 \end{array}\right\} \Rightarrow -10 \leq |3z+5| - 12|z|^5 \leq 5$

$\therefore \quad 5 \leq ||12|z|^5 - |3z+5|| \leq 10$

$\therefore \quad |p(z) - q(z)| < 2 \leq 5 \leq |12|z|^5 - |3z+5|| \leq |q(z)|$

$\therefore$ $p(z)$ y $q(z)$ tienen los mismos nº de raíces en $B(0,1)$

$p(z) = 2z^7 - 12z^5 - 3z + 5$

$q(z) = 2z^7 - 12z^5 - 3z$

$|p(z) - q(z)| = 5$

$|q(z)| = |2z^7 - 12z^5 - 3z| \geq \big| 3|z| - |2z^7 - 12z^5| \big|$

$= \big| 3|z| - |z|^5 |2z^2 - 2| \big|$

$|z| = 1$: $\quad |q(z)| \geqslant \big| 3 - |2z^2 - 2| \big|$

$\big| 2|z|^2 - 2 \big| \leq |2z^2 - 2| \leq 2|z|^2 + 2$

$0 \leq |2z^2 - 2| \leq 4$

$p(z) = 2z^7 - 12z^5 - 3z + 5$ | $|p(z) - q(z)| = 2|z|^7 < 2$

$q(z) = -12z^5 - 3z + 5$ | $|z| = 1$: $|p(z) - q(z)| = 2$

$|q(z)| = |-12z^5 - 3z + 5| \leq 12|z|^5 + 3|z| + 5 < 20$

$|-12z^5 - 3z + 5| \geqslant \big| |12z^5 + 3z| - 5 \big|$

$\big| 12|z|^5 - 3|z| \big| \leq |12z^5 + 3z| \leq 12|z|^5 + 3|z|$

$|z| = 1$: $\quad 9 \leq |12z^5 + 3z| \leq 15$

$\therefore |-12z^5 - 3z + 5| \geqslant 4 \qquad |z| = 1$

$\therefore |z| = 1$: $\quad |p(z) - q(z)| = 2 < 4 \leq |q(z)|$

$q(z) = -12z^5 - 3z + 5, \quad r(z) = -12z^5$

$|q(z) - r(z)| = |-3z + 5| \leq 3|z| + 5 < 8$

$|r(z)| = 12|z|^5 < 12$

$\therefore |z| = 1$: $\quad |r(z)| = 12$

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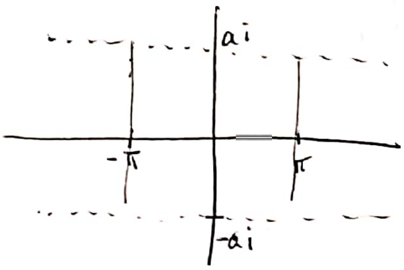
P2 | $a>0$, f holo en $\Omega := \{z\in\mathbb{C} \mid |\mathrm{Im}\, z|<a\}$

$f(z+2\pi)=f(z) \quad \forall z\in\Omega$

pd: $\exists (c_n)_{n\in\mathbb{Z}}$: $f(z)=\sum_{n\in\mathbb{Z}} c_n e^{inz}$

$$\frac{1}{1+\frac{1}{x^{2n}}} = \frac{x^{2n}}{x^{2n}+1}$$



$g(z)=e^{iz}$ $\longrightarrow$

$g(z)=e^{iz}$, $g(a+ib)=e^{i(a+ib)}=e^{-b}e^{ia}$ $\qquad \boxed{a+bi \mapsto e^{-b}e^{ia}}$

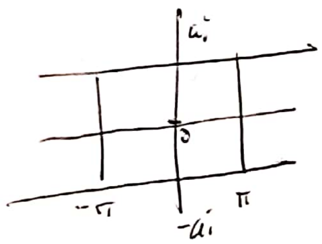
$g(Ri)=e^{i(Ri)}=e^{-R}$, $g(-ai)=e^{a}$, $g(ai)=e^{-a}$ $\qquad \boxed{a>0}$

no s.c

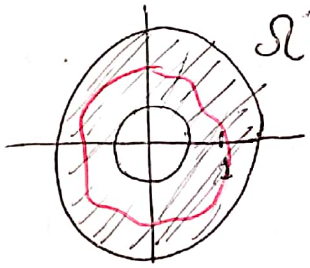
$\Omega \to 1$.

$-ai \mapsto e^{a}>1$

$ai \mapsto e^{-a}<1$

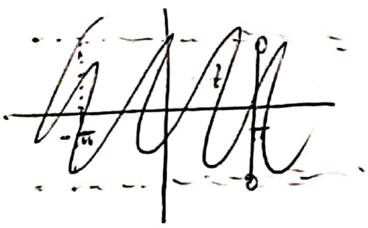


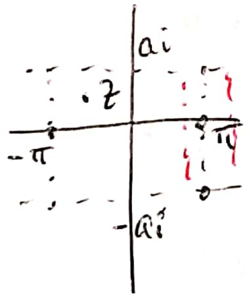
$f(z)=e^{iz}$ $\longrightarrow$



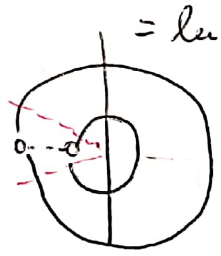
$\Gamma \xrightarrow{e^{iz}} \Omega$

$f(z)=g(e^{iz})$, g holo en Ω.





$\xrightarrow{e^{iz}}$



$\ln(e^{iz})$
$= \ln(e^{-b}e^{ia})$
$= -b+ai$

$h(z)=-i\ln(z)$

$h(e^{iz})=-i\ln(e^{iz})$
$= -i(iz)=z$

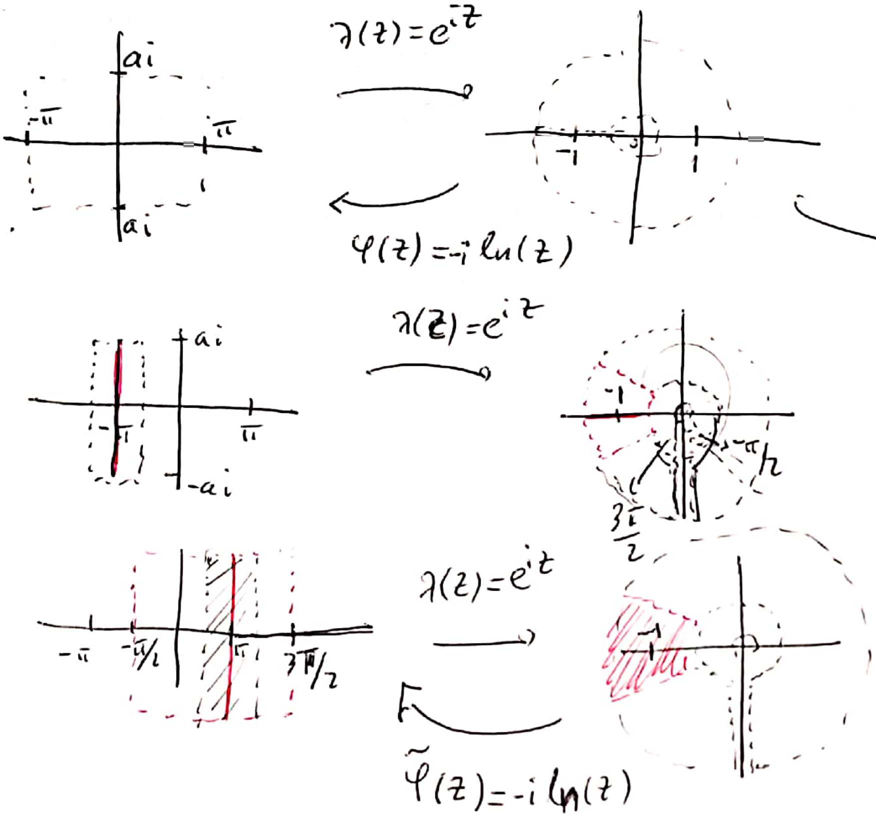
$f(z)=f(-i\ln(e^{iz}))$

$h(z)=-i\ln(z)$

$f(z)=(f\circ h)(e^{iz})$

$f(z)=f(\ln(e^{iz}))$
$=\underbrace{(f\circ \ln)}_{h}(e^{iz})$
$=h(e^{iz})$
$=\sum_{n\in\mathbb{Z}} c_n e^{inz}$

h holomorfa en Ω



$$\varphi(\lambda(z)) = i\ln(e^{iz}) = -i(iz) = z$$

$$f(z) = f(-i\ln(e^{iz})) = h(e^{iz})$$

$$0 \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

$$f(z) = \sum_{n\in\mathbb{Z}}{}' c_n e^{inz}$$

$$f(z) = \tilde{h}(e^{iz})$$

ok. $$f(z) = \sum_{n\in\mathbb{Z}}{}' \tilde{c}_n e^{inz}$$

Scanned with CamScanner

$$= re^{i\theta} \quad -\pi < \theta < \pi.$$

$$e^{\log(z)} = z = a+bi \quad , \quad \log(z) = \alpha + \beta i$$

$$e^{\log(z)} = e^{\alpha+\beta i} = e^{\alpha} e^{\beta i} = re^{i\theta} \quad / |\cdot|$$

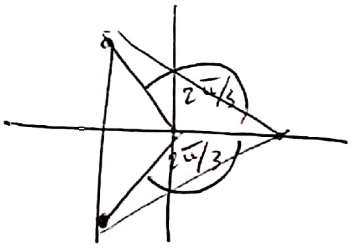
$$\Rightarrow e^{\alpha} = r \Rightarrow \alpha = \log(r)$$

$$\Rightarrow e^{\beta i} = e^{i\theta} \Rightarrow e^{(\beta-\theta)i} = 1 \Rightarrow \beta - \theta = 2\pi k$$

$$\theta = \beta + 2\pi k$$

$$\therefore \log(z) = \log(r) + (\theta + 2\pi k) i \qquad k \in \mathbb{Z}.$$

$$z_1 = z_2 = e^{2\pi i/3} \quad , \quad \log(z_1) = \log(z_2) = \frac{2\pi i}{3}$$



$$z_1 z_2 = e^{4\pi i/3} = e^{\pi i} e^{\pi i/3} = -e^{\pi i/3}$$

$$e^{2\pi i/3} e^{4\pi i/3} = e^{2\pi i} = 1$$

$$\therefore e^{4\pi i/3} = e^{-2\pi i/3}$$

$$w = a+bi \quad , \quad z = \alpha + bi$$

$$w = e^z \Leftrightarrow a+bi =$$

$$re^{i\theta} = e^z \Leftrightarrow$$

$$\theta \in [0, 2\pi)$$

$f : \Omega \to \mathbb{C}$ holo, Ω s.c. $\to \exists g : \Omega \to \mathbb{C}$: $f(z) = e^{g(z)}$

dem. $z_0 \xrightarrow{\gamma} z$ $\quad g(z) = \displaystyle\int_{\gamma_{z_0,z}} \frac{f'(w)}{f(w)} dw + c_0 \quad , \quad e^{c_0} = f(z_0)$

ok. g holomorfa. $\dfrac{d}{dz}\left(f(z) e^{-g(z)}\right) = 0$. ok.

$$g'(z) = \frac{f'(z)}{f(z)}$$

$|f(z) - g_{w_0}(z)| = |f(z) - (f(z) - w_0)|$

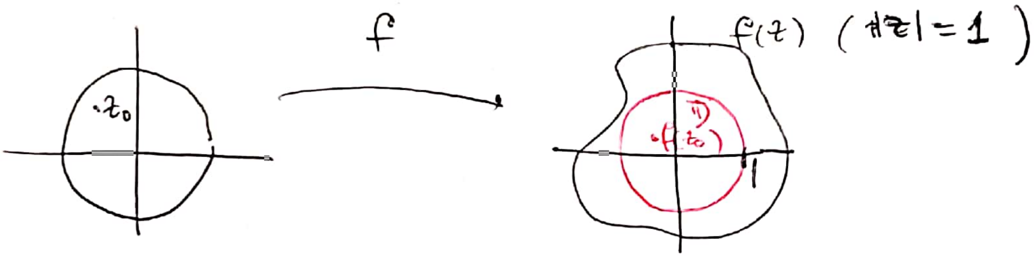
$= |w_0| < 1 = |f(z)| \quad \forall |z| = 1$

$\therefore$ Teo Rouché : $\exists$ raíz de $g_{w_0}(z)$ en $\mathbb{D}$.

$\therefore \exists \tilde{z} \in \mathbb{D}$ tq $f(\tilde{z}) = w_0$

$\therefore w_0 \in \mathbb{D} \Rightarrow w_0 \in f(\Omega)$.

(b) $w \in \mathbb{D}$, $g_w(z) = z - w$ holo en Ω.



$g_{\zeta_0}(z) = f(z) - \zeta_0$. Sea $f(z_0) = \zeta_0$

$w \in \mathbb{D}$: $|g_w(z) - g_{\zeta_0}(z)| = \cancel{|w + \zeta_0| \le |w| + |\zeta_0| < |w| + 1}$

$1 \le |f(z)| \le |f(z_1)| \quad \forall |z| = 1$ $\qquad \cancel{\le |w| + |f(z)|}$

$|f(z_0)| < 1 \le |f(z)| \le |f(z_1)|$

$|g_w(z) - g_{\zeta_0}(z)| = |f(z) - w - (f(z) - \zeta_0)| = |\zeta_0 - w| < 2$

$= |f(z) - g_{\zeta_0}(z) - w|$ | $|g_{\zeta_0}(z)| = |f(z) - \zeta_0| \ge ||f(z)| - |\zeta_0|| = |f(z)| - |\zeta_0| \ge 1 - |\zeta_0|$

$\sup_{z \in \mathbb{D}} |f(z)| \le \sup_{z \in \bar{\mathbb{D}} \setminus \mathbb{D}} |f(z)|$ | $\exists \zeta_0 \in \mathbb{D} \cap f(\Omega)$ $\Rightarrow \exists r > 0 : B(\zeta_0, r) \subset \mathbb{D} \cap f(\Omega)$

Podemos considerar $w \in \mathbb{D}$ tq $|\zeta_0 - w| < 1 - |\zeta_0|$ $(\le |g_{\zeta_0}(z)|)$

P16 f, g holomorfas en $\Omega \supseteq \overline{B(0,1)}$ → región

$f(0) = 0$, $f(z) \neq 0$ ~~[illegible]~~ $\forall z : 0 < |z| \leq 1$.

- $f_\varepsilon(z) = f(z) + \varepsilon g(z)$

Pd : Para ε suf pequeño $\Rightarrow \begin{cases} \text{(a) } f_\varepsilon(z) \text{ tiene cero único en } |z| \leq 1 \\ \text{(b) } z_\varepsilon \text{ cero de } f_\varepsilon, \ \varepsilon \mapsto z_\varepsilon \text{ es continua.} \end{cases}$

dem. ~~[illegible]~~ $|z| = 1$: $|f_\varepsilon(z)| = |f(z) + \varepsilon g(z)| \leq |f(z)| + |\varepsilon||g(z)|$

para $|\varepsilon| < 1$: $|f_\varepsilon(z)| = |f(z) + \varepsilon g(z)| \leq |f(z)| + |\varepsilon||g(z)|$

$< |f(z)| + |g(z)|$

$f_\varepsilon(\bar{z}) = 0 \Leftrightarrow f(\bar{z}) = \varepsilon g(\bar{z})$

$f_\varepsilon'(z) = f'(z) + \varepsilon g'(z)$

$|f_\varepsilon(z) - f(z)| = |\varepsilon||g(z)| \leq |\varepsilon| \alpha < \beta$ ε suf pequeño.

$\alpha = \sup_{|z|=1} |g(z)|$

~~$|f_\varepsilon(z)| = |f(z) + \varepsilon g(z)| \geq ||f(z)| - |\varepsilon||g(z)||$~~

donde $\beta = |f(z_0)|$ el mínimo de f en $\overline{B(0,1)}$

$\therefore$ $|f_\varepsilon(z) - f(z)| < |f(z)|$ $\forall |z| = 1$

- Teo Rouché : $f_\varepsilon(z)$ y $f(z)$ tiene la misma cantidad de ceros.

$\therefore$ $f_\varepsilon(z)$ tiene único cero en $B(0,1)$ para $\varepsilon < \beta/\alpha$ en $B(0,1)$

$\sigma : [0, \infty) \longrightarrow \mathbb{C}$
$\varepsilon \longmapsto z_\varepsilon$ $\Big\}$ (pendiente)

$|\varepsilon - \cancel{\varepsilon'}| < \delta$

P17| $f:\Omega\to\mathbb{C}$ holomorfa tq $\overline{B(0,1)}\in\Omega$; $f\neq cte$

(a) si $|f(z)|=1 \ \forall |z|=1 \Rightarrow \mathbb{D}\subset f(\Omega)$

(b) Si $|f(z)|\geq 1 \ \forall |z|=1 \ \wedge \ \exists z_0\in\mathbb{D} : |f(z_0)|<1$

$\Rightarrow \mathbb{D}\subset f(\Omega)$

—·—

dem. (a) Pd: $f(z)=0$ tiene una ~~raíz~~ z

$|f(z)-z| \leq |f(z)|+|z|$

Supongamos que $\forall z\in\mathbb{D} : f(z)\neq 0 \Rightarrow g=1/f : \overline{\mathbb{D}}\to\mathbb{C}$

~~holomorfa~~ continua. $g(\overline{\mathbb{D}})=\overline{\mathbb{D}}$ $|g(z)|=1$. $g:\mathbb{D}\to\mathbb{C}$ hdo.

g no alcanza el máximo en $\mathbb{D} \Rightarrow \sup_{z\in\mathbb{D}}|g(z)|\leq 1$

i.e $\forall z\in\mathbb{D} : |g(z)|\leq 1$ $\therefore$ $\forall z\in\mathbb{D} : |f(z)|\geq 1$. $(\Rightarrow\Leftarrow)$.

$\therefore \exists z\in\mathbb{D} : f(z)=0$.

$g_{w_0}:\Omega\to\mathbb{C}$, $g_{w_0}(z)=z-w_0$

$|f(z)-g_{w_0}(z)| \leq |f(z)|+|g_{w_0}(z)|$

Sup $\exists w_0\in\mathbb{D}$ tq $g(z)=f(z)-w_0$ no tiene solución en $\mathbb{D}$

$\Rightarrow h(z)=\dfrac{1}{g(z)}=\dfrac{1}{f(z)-w_0}$ holomorfa en $\mathbb{D}$. y continua en $\overline{\mathbb{D}}$.

$\forall |z|=1 : |h(z)| \leq \dfrac{1}{||f(z)|-w_0|} = \dfrac{1}{|1-|w_0||} = \dfrac{1}{1-|w_0|}$

$\Rightarrow$ ~~$\forall z\in\mathbb{D}$~~: $\sup_{|z|<1}|h(z)| \leq \dfrac{1}{1-|w_0|} \Rightarrow \forall z\in\mathbb{D} : |h(z)|< \dfrac{1}{1-|w_0|}$

$\Rightarrow \forall z\in\mathbb{D} : |g(z)|>1-|w_0|$

P18| PD: $f(z) = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(\zeta)}{\zeta - z}\, d\zeta$ usando homotopía de curvas.

dem.

$C: \gamma(t) = p + re^{it}, \quad t \in [0, 2\pi]$

$\tilde{\gamma}(t) = z + \tilde{r}e^{it}, \quad t \in [0, 2\pi]$

f es holomorfa en $z \Rightarrow \displaystyle\lim_{\zeta \to z} \dfrac{f(\zeta) - f(z)}{\zeta - z}$ existe

$F(\zeta) = \dfrac{f(\zeta) - f(z)}{\zeta - z}$ es acotada en vec de z.

$$\int_C F(\zeta)\, d\zeta = \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z)\int_C \frac{d\zeta}{\zeta - z}$$

$F(\zeta)$ tiene extensión holomorfa $\tilde{F}$ en vec de z

$\therefore \displaystyle\int_C F(\zeta)\, d\zeta = \int_C \tilde{F}(\zeta)\, d\zeta = 0.$

Además $\displaystyle\int_C \frac{d\zeta}{\zeta - z} = \int_{\tilde{\gamma}} \frac{d\zeta}{\zeta - z} = 2\pi i$

$\therefore f(z) = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(\zeta)}{\zeta - z}\, d\zeta.$

P19) Principio del módulo máximo para funciones armónicas

(a) u armónica en región Ω no constante $\Rightarrow$ u no tiene máximo (o mínimo) en Ω.

(b) Ω región con clausura compacta $\overline{\Omega}$. Si u armónica en Ω y continua en $\overline{\Omega}$, entonces

$$\sup_{z\in\Omega} |u(z)| \leq \sup_{z\in\overline{\Omega}\setminus\Omega} |u(z)| \qquad (\overline{\Omega}\setminus\Omega = \partial\Omega).$$

dem. Sup $\exists z_0 \in \Omega$ tq $\forall z\in\Omega$: $|u(z)| \leq |u(z_0)|$

$B(z_0, r) \subset \Omega$. $B(z_0,r)$ simplemente conexo, y $u|_{B(z_0,r)}$ armónica,

$\Rightarrow \exists$: $f: B(z_0,r) \to \mathbb{C}$ holomorfa con $Re(f) = u$

$u \neq cte \Rightarrow f \neq cte \Rightarrow f$ es holomorfa y abierta

$\therefore$ $Re\circ f$ abierta

$\therefore$ u abierta.

~~$|f(z)|^2 = Re(f(z))^2 + Im(f(z))^2$~~

pero $z_0 \in B(z_0,r) \Rightarrow \exists r'>0$ tq $B(f(z_0), r') \subset f(B(z_0,r))$

$\Rightarrow \exists w_1 \in B(f(z_0), r)$ tq

tq $|Re\, w_1| > |Re\, f(z_0)|$

$\therefore \exists z_1 \in B(z_0,r)$ tq $f(z_1) = w_1$, $|Re\, f(z_1)| > |Re\, f(z_0)|$

$\therefore$ $|u(z_1)| > |u(z_0)|$ $(\Rightarrow\Leftarrow)$

$\therefore$ f no es abierta

$\therefore$ f es constante

$\therefore$ u es constante.

(b) $u(z)$ es continua y no alcanza el máximo en Ω

Como $\overline{\Omega}$ es compacto, $\exists\, z_0 \in \overline{\Omega}$ tq $\sup\limits_{z\in\overline{\Omega}} |u(z)| = |u(z_0)|$

pero $\overline{\Omega} = \partial\Omega \,\dot{\cup}\, \Omega$ y $z_0 \notin \Omega$ $\therefore$ $z_0 \in \partial\Omega$

$\therefore$ $\forall z \in \Omega$: $|u(z)| < |u(z_0)| = \sup\limits_{z\in\overline{\Omega}} |u(z)| = \sup\limits_{z\in\partial\Omega} |u(z)|$

$\therefore$ $\sup\limits_{z\in\Omega} |u(z)| \leq \sup\limits_{z\in\partial\Omega} |u(z)|$

$\square$

P22) Probar que no hay función holomorfa f en el disco unitario $\mathbb{D}$ que se extiende continuamente a $\partial\mathbb{D}$ tal que $f(z) = 1/z$ para todo $z \in \partial\mathbb{D}$.