$\mathcal{C} : \cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$

$\mathcal{C}' : \cdots \to C'_{n+1} \to C'_n \to C'_{n-1} \to \cdots$

$\mathcal{C}'' : \cdots \to C''_{n+1} \to C''_n \to C''_{n-1} \to \cdots$

$0 \to \mathcal{C} \to \mathcal{C}' \to \mathcal{C}'' \to 0$ . sucesión exacta sse

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r] \arrow[d] & C'_n \arrow[r] \arrow[d] & C'_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C''_{n+1} \arrow[r] \arrow[d] & C''_n \arrow[r] \arrow[d] & C''_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$\boxed{1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}}$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "1_{C_{n+1}}"] \arrow[dr, phantom, "\circlearrowright"] & C_n \arrow[r] \arrow[d, "1_{C_n}"] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"'] & C_n \arrow[r] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$1 : C_n \to C_n$

$$\boxed{H_n(\mathcal{C}) = Z_n(\mathcal{C}) / B_n(\mathcal{C})}$$

$$= \ker(d_n : C_n \to C_{n-1}) / \operatorname{Im}(d_{n+1} : C_{n+1} \to C_n)$$

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d_{n+1}"] \arrow[dr, "d_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "1"] & C_{n-1} \\
 & C_n \arrow[ur, "d_n"'] &
\end{tikzcd}$$

$\mathcal{C}$ complejo de cadena $\Rightarrow H_p(\mathcal{C}) = \ker(d_p : C_p \to C_{p-1}) / \operatorname{Im}(d_{p+1} : C_{p+1} \to C_p)$

$f : \mathcal{C} \to \mathcal{C}'$

$$\begin{array}{ccccc} C_{p+1} & \longrightarrow & C_p & \longrightarrow & C_{p-1} \\ \downarrow & & \downarrow f_p & & \downarrow \\ C'_{p+1} & \longrightarrow & C'_p & \longrightarrow & C'_{p-1} \end{array}$$

$(f_*)_p : H_p(\mathcal{C}) \to H_p(\mathcal{C}')$

$a + d_{p+1}(C_{p+1}) \mapsto f_p(a) + d'_{p+1}(C'_{p+1})$

$(f_*)_p$ bien definido:

$a + d_{p+1}(C_{p+1}) = b + d_{p+1}(C_{p+1}) \Leftrightarrow a - b \in d_{p+1}(C_{p+1})$

$a - b = d_{p+1}(e)$, $e \in C_{p+1}$

$f_p(a-b) = f_p d_{p+1}(e) = d'_{p+1} f_{p+1}(e)$,

$f_p(a-b) = d'_{p+1}(f_{p+1}(e))$, $f_{p+1}(e) \in C'_{p+1}$

$f_p(a-b) \in d'_{p+1}(C'_{p+1})$

$f_p(a) - f_p(b) \in d'_{p+1}(C'_{p+1}) \Leftrightarrow f_p(a) + d'_{p+1}(C'_{p+1}) = f_p(b) + d'_{p+1}(C'_{p+1})$

Para $f = 1 \Rightarrow (1_*)_p(a + d_{p+1}(C_{p+1})) = a + d_{p+1}(C_{p+1})$

$\therefore \cancel{f}\ (1_*)_p = 1_{H_p(\mathcal{C})}$

~~f~~ y g $\quad f : \mathcal{C} \to \mathcal{C}'$, $g : \mathcal{C}' \to \mathcal{C}'' \Rightarrow g \circ f : \mathcal{C} \to \mathcal{C}''$

$((g\circ f)_*)_p : H_p(\mathcal{C}) \to H_p(\mathcal{C}'')$

$a + d_{p+1}(C_{p+1}) \mapsto (g \circ f)_p(a) + d''_{p+1}(C''_{p+1})$

$(g_*)_p \circ (f_*)_p (a + d_{p+1}(C_{p+1})) = (g_*)_p(f_p(a) + d'_{p+1}(C'_{p+1}))$

$= g_p(f_p(a)) + d''_{p+1}(C''_{p+1}) = (g_p \circ f_p)(a) + d''_{p+1}(C''_{p+1})$

$0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0$ suc. exacta complejos de cadena

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
C': & \cdots \arrow[r] & C'_{p+1} \arrow[r, "d'_{p+1}"] \arrow[d, "i_{p+1}"] & C'_p \arrow[r, "d'_p"] \arrow[d, "i_p"] & C'_{p-1} \arrow[r] \arrow[d, "i_{p-1}"] & \exists!\, y \text{ ya que } i_{p-1} \text{ mono} \\
C: & \cdots \arrow[r] & C_{p+1} \arrow[r, "d_{p+1}"] \arrow[d, "j_{p+1}"] & C_p \arrow[r, "d_p"] \arrow[d, "j_p"] & C_{p-1} \arrow[r] \arrow[d, "j_{p-1}"] & \cdots \\
C'': & \cdots \arrow[r] & C''_{p+1} \arrow[r, "d''_{p+1}"'] \arrow[d] & C''_p \arrow[r, "d''_p"'] \arrow[d] & C''_{p-1} \arrow[r] \arrow[d] & \cdots \\
 & & 0 & 0 & 0 &
\end{tikzcd}
$$

suc. exacta

$\exists! y$ $\to$ mono

$\exists x''$ $\quad x' \in \ker(d_{p-1})$ $\quad \ker(d_{p-1}) = \operatorname{Im}(i_{p-1})$

$\to$ epi

$x \longmapsto 0$

$j_{p-1} \circ h = 1_{C_{p-1}}$

$$H_p(C') \xrightarrow{(i_*)_p} H_p(C) \xrightarrow{(j_*)_p} H_p(C'') \xrightarrow{\partial_p} H_{p-1}(C') \xrightarrow{(i_*)_p} H_{p-1}(C) \xrightarrow{(j_*)_p} H_{p-1}(C'') \to \cdots$$

$[x] \in \ker(d''_p) / \operatorname{Im}(d''_{p+1})$ $\qquad \ker(d'_{p-1}) / \operatorname{Im}(d'_p) \quad [y]$

$$[x] \longmapsto [y]$$

bien definida?

$[x] = [\bar{x}] \in H_p(C'') \iff x - \bar{x} \in \operatorname{Im}(d''_{p+1}) \iff x - \bar{x} = d''_{p+1}(e),\ e \in C''_{p+1} = j_{p+1}(C_{p+1})$

$x - \bar{x} = d''_{p+1}(e) \iff x = d''_{p+1}(e) + \bar{x}$

$x - \bar{x} \in d''_{p+1}(C''_{p+1}) = d''_{p+1}(j_{p+1} C_{p+1}) = j_p d_{p+1}(C_{p+1})$
$= j_p(d_{p+1}(C_{p+1})) \subset j_p(C_p) = C''_p$

$x - \bar{x} = j_p d_{p+1}(e), \quad e \in C_{p+1}$

$d''_p(x) = d''_p(d''_{p+1}(e)) + d''_p(\bar{x})$
$= d''_p(\bar{x}) \Rightarrow d''_p(x - \bar{x}) = 0$

$\partial_p : C_p(X) \to C_{p-1}(X)$ $\qquad \mathbb{Z}(S_{p-1}X)$ "

$\bar{\partial}_p : C_p(X) \to C_{p-1}(X,A) = C_{p-1}(X) / C_{p-1}(A)$ $\qquad \sigma : \Delta_{p-1} \xrightarrow[\text{cont}]{} X$

$\bar{\partial}_p(\sigma) = \partial_p(\sigma) + C_{p-1}(A)$ $\qquad$ bien definido, homomorfismo

$= \sum_{i=0}^{p} (-1)^i \sigma \circ d_i + C_{p-1}(A)$

$f : G \to H \;,\; A \trianglelefteq H \Rightarrow \bar{f} : G \to H/A$ $\qquad$ homo

$g \longmapsto f(g)A$

$g = \tilde{g} \Rightarrow f(g) = f(\tilde{g}) \Rightarrow f(g)A = f(\tilde{g})A \Rightarrow \bar{f}(g) = \bar{f}(\tilde{g})$

$\bar{f}(g\tilde{g}) = f(g\tilde{g})A = f(g)f(\tilde{g})A = (f(g)A)(f(\tilde{g})A)$

$\partial_p(\sigma) \in C_{p-1}(A) \Leftrightarrow \sum_{i=0}^{p} (-1)^i \sigma \circ d_i \in C_{p-1}(A)$

$\Leftrightarrow \sum_{i=0}^{p} (-1)^i \sigma \circ d_i = \sum_{r=1}^{m} k_r \sigma_r \;,\; \sigma_r : \Delta_{p-1} \xrightarrow{\text{cont}} A$

$\sigma \circ d_i : \Delta_{p-1} \to X$ cont. $\quad | \quad C_p(A) \subseteq \ker(\bar{\partial}_p)$

¡Importante!

$f : G \to H$ homomorfismo

$A \trianglelefteq G$ sup: $\exists \bar{f} : G/A \to H$ tq

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & H \\ G/A \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

$\bar{f} \neq \bar{\bar{f}} : G/A \to H$ tq $\bar{\bar{f}} \circ p = f \circ p$ $\quad$ epi

$\Rightarrow \bar{f} = \bar{\bar{f}}$

$a \in A \;,\; f(a) = \bar{f} \circ p(a) = \bar{f}(aA) = \bar{f}(A) = e \Rightarrow a \in \ker(f)$

Sup $A \subset \ker(f)$: $[g] = [h] \Rightarrow gh^{-1} \in A \Rightarrow gh^{-1} \in \ker f \Rightarrow f(gh^{-1}) = 1$

$\Rightarrow f(g) = f(h)$

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$0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0$ suc. exacta corta

$\Leftrightarrow$

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to & & C'_{p+1} & \to & C'_p & \to & C'_{p-1} \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to & & C_{p+1} & \to & C_p & \to & C_{p-1} \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to & & C''_{p+1} & \to & C''_p & \to & C''_{p-1} \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

$\Downarrow$

$\exists$: $H_n(C) \xrightarrow{(j_*)_n} H_{n+1}(C'') \xrightarrow{\partial_{n+1}} H_n(C') \xrightarrow{(i_*)_n} H_n(C) \xrightarrow{(j_*)_n} H_n(C'') \xrightarrow{\partial_n} H_{n-1}(C') \xrightarrow{(i_*)_n} H_{n-1}(C) \xrightarrow{j_*} \cdots$

$\partial_n$: morfismo conector

---

$f: X \to Y$ continua. $C_n X \xrightarrow{(f_\#)_n} C_n(Y)$ morfismo

$\sigma: \Delta_n \to X \longmapsto f \circ \sigma : \Delta_n \to Y$

$$\begin{array}{ccccccc}
\to \cdots \to C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow & \cdots \\
\downarrow (f_\#)_{n+1} & \circlearrowright & \downarrow (f_\#)_n & \circlearrowright & \downarrow (f_\#)_{n-1} & & \\
\cdots \to C_{n+1}(Y) & \xrightarrow[\partial'_{n+1}]{} & C_n(Y) & \xrightarrow[\partial'_n]{} & C_{n-1}(Y) & \longrightarrow & \cdots
\end{array}$$

$$(f_\#)_n \circ \partial_{n+1}(\sigma) = (f_\#)_n(\partial_{n+1}(\sigma)) = (f_\#)_n\left(\sum_{i=0}^{n+1}(-1)^i \sigma \circ d_i\right) = \sum_{i=0}^{n+1}(-1)^i (f_\#)_n(\sigma \circ d_i)$$

$$= \sum_{i=0}^{n+1}(-1)^i (f \circ \sigma \circ d_i) = \sum_{i=0}^{n+1}(-1)^i (f \circ \sigma) \circ d_i$$

$f \circ \sigma \circ d_i : \Delta_n \to Y$

$\therefore f \circ \sigma : \Delta_{n+1} \to Y$

$d_i : \Delta_{n+1} \to \Delta_n$

$(f_\#)_{n+1}(\sigma)$ ok

$\therefore\ f_\# : C(X) \to C(Y)$ morfismo de complejos de cadena.

$\Rightarrow \exists\ (f_\#)_n : H_n(C(X)) \to H_n(C(Y))$ $\left( \text{i.e.: } (f_\#)_n : H_n(X) \to H_n(Y) \right)$

Sea ahora $f: X \to Y$ tq $A \subset X$, $B \subset Y$

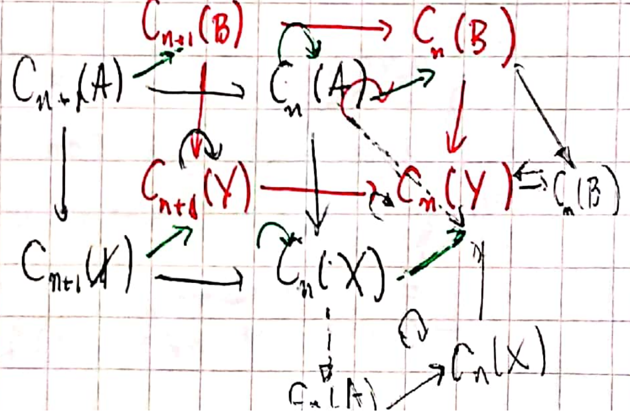
$0 \to C(A) \xrightarrow{i} C(X) \xrightarrow{j} C(X,A) \to 0$ (suc. exacta)

$0 \to C(B) \to C(Y) \to C(Y,B) \to 0$ (suc. exacta).

Pd:
$$\begin{array}{ccccccccc}
0 \to & C(A) & \to & C(X) & \to & C(X,A) & \to 0 \\
 & \downarrow f_\# & \circlearrowright & \downarrow f_\# & \circlearrowright & \downarrow f_\# & \\
0 \to & C(B) & \to & C(Y) & \to & C(Y,B) & \to 0
\end{array}$$

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \cdots \to C_{n+1}(B) \to C_n(B) \to C_{n-1}(B) \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & C_{n+1}(A) & \to & C_n(A) & \to & C_{n-1}(A) \to \cdots & \\
 & \downarrow & \circlearrowright & \downarrow & \circlearrowright & \downarrow & \cdots \to C_{n+1}(Y) \to C_n(Y) \to C_{n-1}(Y) \to \cdots \\
\cdots \to & C_{n+1}(X) & \to & C_n(X) & \to & C_{n-1}(A) \to \cdots & \\
 & \downarrow & \circlearrowright & \downarrow & \circlearrowright & \downarrow & \cdots \to C_{n+1}(Y,B) \to C_n(Y,B) \to C_{n-1}(Y,B) \to \cdots \\
\cdots \to & C_{n+1}(X,A) & \to & C_n(X,A) & \to & C_{n-1}(X,A) \to \cdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

pendiente



## Consecuencias teo. Hahn-Banach

Fuente (Rudin, pag 107)

$X$ espacio vectorial normado, $M \leq X$, $x_0 \in X$

(1) $x_0 \in \overline{M} \iff \nexists f: X \to \mathbb{C}$ l.a. $f(x)=0 \ \forall x \in M$, $f(x_0) \neq 0$

dem. $(\Rightarrow)$ $x_0 \in \overline{M}$, $f: X \to \mathbb{C}$ lineal acotado tq $f(x)=0 \ \forall x \in M$

$x_0 = \lim_{j \in \mathbb{N}} x_j$, $x_j \in M \Rightarrow f(x_0) = \lim_j f(x_j) = 0$.

$(\Leftarrow)$ Contrapositiva,

Sup $x_0 \notin \overline{M} \iff \exists \delta > 0 : \|x - x_0\| \geq \delta \quad \forall x \in M$

$M' = \langle x_0, M \rangle \leq X$ | def: $f: M' \to \mathbb{C}$

$f(x + \lambda x_0) := \lambda \quad \forall x \in M, \ \forall \lambda \in \mathbb{C}$

$x + \lambda x_0 = y + \sigma x_0 \Rightarrow \underbrace{x - y}_{\in M} + (\lambda - \sigma) x_0 = 0$

Si $\lambda \neq \sigma \Rightarrow x_0 = \frac{1}{\lambda - \sigma}(x - y) \in M$ $(\to\leftarrow)$ $\boxed{\therefore \lambda = \sigma}$ importante

$\therefore f(x + \lambda x_0) = \lambda$ bien definida, lineal.

¿f acotada? $|f(x + \lambda x_0)| \leq |\lambda|$

$\|x + \lambda x_0\| \underset{\lambda \neq 0}{=} |\lambda| \underbrace{\|\lambda^{-1} x + x_0\|}_{\text{condición } x_0 \notin \overline{M}} \geq |\lambda| \delta \Rightarrow |\lambda| \leq \|x + \lambda x_0\| \frac{1}{\delta}$

$\therefore |f(x + \lambda x_0)| \leq \frac{1}{\delta} \|x + \lambda x_0\| \quad \forall x + \lambda x_0 \in \langle M, x_0 \rangle$

$\therefore \|f\| \leq 1/\delta$ es también denúvame $f(x) = 0 \ \forall x \in M$, $f(x_0) = 1$

aplicar Hahn-Banach

(2) Ejemplo Rudin pag 108

$X$ espacio vectorial, $x_0 \in X \setminus \{0\} \Rightarrow \exists f: X \to \mathbb{C}$ ~~f~~ lineal acotado

tq $\|f\| = 1$, $f(x_0) = \|x_0\|$.

Sea $M = \langle x_0 \rangle \Rightarrow$ def: $f: M \to \mathbb{C}$ $f(\lambda x_0) = \lambda \|x_0\|$

$f$ bien definida y lineal: ~~$\lambda x_0 = \sigma x_0 \Rightarrow \lambda = \sigma \Rightarrow f(\lambda x_0) = f(\sigma x_0)$~~

etc

$f(x_0) = \|x_0\|$

$|f(\lambda x_0)| = |\lambda| \|x_0\| = \|\lambda x_0\| \Rightarrow |f(z)| = \|z\|$ $\forall z \in M$.

seminorma

$\exists F: X \to \mathbb{C}$ lineal tq $|F(z)| = \|z\|$

$\therefore \|F\| = 1$, ~~$F(z) = \|z\|$~~

$F(x_0) = f(x_0) = \|x_0\|$

(3) Aplicación de Hahn-Banach.

Pd: Af. $X \hookrightarrow X^{**}$

$T: X \to X^{**}$

$x \mapsto T_x: X^* \to \mathbb{C}$

$\Lambda \mapsto \langle \Lambda, x \rangle$

$T(x) = T(y) \Rightarrow \forall \Lambda: \langle \Lambda, x \rangle = \langle \Lambda, y \rangle$

$\Rightarrow \forall \Lambda: \langle \Lambda, x-y \rangle = 0$

$0 = |\langle \Lambda, x-y \rangle| \leq \|\Lambda\| \|x-y\|$

Sup $x \neq y \Rightarrow \sup \dfrac{|\langle \Lambda, x-y \rangle|}{x-y} \leq \|\Lambda\|$

$\Rightarrow |\langle \Lambda, x-y \rangle| = 0$

$\Rightarrow \dfrac{|\langle \Lambda, x-y \rangle|}{\|x-y\|} = 0$

$\therefore \|\Lambda\| = 0$ $\boxed{\Rightarrow \Leftarrow}$

$\boxed{\therefore x = y}$

$T$ es una isometría, ya que $\forall x \in X$ : $\|T(x)\| = \|x\|$

---

Estudiar $(C_c^\infty(\Omega), \rho)^*$

$\Lambda : C_c^\infty(\Omega) \to \mathbb{R}$ lineal, $\underbrace{|\langle \Lambda, \varphi \rangle| \leq c\, \rho(\varphi)}_{\text{continuo}}$ algún $c > 0$

$Im(D) = S \xrightarrow{D^{-1}} C_c^\infty(\Omega) \xrightarrow{\Lambda} \mathbb{R} \Rightarrow \Lambda \circ D^{-1} : S \to \mathbb{R}$ $\quad$ $C_c^\infty(\Omega, \mathbb{R}^n)$

$\forall F \in S \Rightarrow F = D\varphi$, $\varphi \in C_c^\infty(\Omega)$

$|\langle \Lambda \circ D^{-1}, F \rangle| = |\langle \Lambda \circ D^{-1}, D\varphi \rangle| = |\langle \Lambda, \varphi \rangle| \leq c\, \rho(\varphi)$

$= c \|D\varphi\|_{L^p} = c \|F\|$

$\therefore$ $\Lambda \circ D^{-1}$ lineal continuo.

Hahn-Banach: $\exists\, \Upsilon : \cancel{C_c^\infty(\Omega, \mathbb{R}^n)}\ L^p(\Omega, \mathbb{R}^n) \to \mathbb{R}$ tq $\langle \Upsilon, F \rangle = \langle \Lambda \circ D^{-1}, F \rangle$
lineal continuo $\quad \forall F \in S$

$\Leftrightarrow \langle \Upsilon, D\varphi \rangle = \langle \Lambda, \varphi \rangle \quad \forall \varphi \in C_c^\infty(\Omega)$

Además $\|\Upsilon\|_{(L^p)^*} = \|\Lambda\|$

$\langle \Upsilon, D\varphi \rangle = \langle \Lambda, \varphi \rangle \Rightarrow |\langle \Upsilon, D\varphi \rangle| = |\langle \Lambda, \varphi \rangle|$

$$\Rightarrow \frac{|\langle \Lambda, \varphi \rangle|}{\rho(\varphi)} = \frac{|\langle \Upsilon, D\varphi \rangle|}{\rho(\varphi)} = \frac{|\langle \Upsilon, D\varphi \rangle|}{\|D\varphi\|_{L^p}} \quad \forall \varphi.$$

$\Rightarrow \|\Lambda\| = \|\Upsilon\|_{(L^p)^*}$

## Aplicación de Hahn Banach.

$X^*$ separa puntos

$X^* = \{ f : X \to \mathbb{C} \ / \ f$ lineal continuo $\}$. Sean $x, y \in X$, $x \neq y$

$\Rightarrow z = x - y \neq 0 \in X \Rightarrow W = \langle z \rangle \supsetneq \{0\}$, $W \leq X$

H.B $\exists f : X \to \mathbb{C}$ lineal continuo tq $f(z) = \|z\| \neq 0$

$$\therefore f(x-y) = f(x) - f(y) \neq 0$$

$$\therefore f(x) \neq f(y).$$

---

- Formas geométricas

(Aforo de K). $K \subset X$, $\text{int}(K) = K$, $0 \in K$, $K$ convexo

$\rho : X \to \mathbb{R}$, $\rho(x) = \inf\{ \alpha > 0 \ / \ \frac{1}{\alpha} x \in K \}$

Pd: $\rho(\lambda x) = \lambda \rho(x) \quad \forall x \in X \quad \forall \lambda > 0$

- $\rho(\lambda x) = \inf\{ \alpha > 0 \ / \ \frac{1}{\alpha} \lambda x \in K \}$

$\Rightarrow \rho(\lambda x) < \alpha \quad \forall \alpha > 0$ tq $\frac{1}{\alpha} \lambda x \in K$

Sea $\alpha > 0$ tq $\frac{1}{\alpha} \lambda x = \frac{1}{\alpha/\lambda} x \in K \Rightarrow \rho(x) < \alpha/\lambda \Rightarrow \lambda \rho(x) < \alpha$

$\Rightarrow \lambda \rho(x)$ cota inferior de $A = \{ \alpha > 0 \ / \ \frac{1}{\alpha} \lambda x \in K \}$

$\Rightarrow \lambda \rho(x) \leq \rho(\lambda x)$

## Formas geométricas

**Af.** $K$ convexo, $K \neq \emptyset$, $c \in K \Rightarrow K-c$ convexo, $K-c \neq \emptyset$

**dem.** $c \in K \neq \emptyset \Rightarrow 0 \in K-c = \{x-c / x \in K\} \neq \emptyset$

$a, b \in K-c \Rightarrow \begin{array}{l} a = \alpha - c \\ b = \beta - c \end{array}, \quad \alpha, \beta \in K$

$\forall t \in [0,1]$, $z_t = (1-t)a + tb = (1-t)(\alpha - c) + t(\beta - c)$

$= \alpha - c - t\alpha + \cancel{tc} + t\beta - \cancel{tc}$

$= \underbrace{(1-t)\alpha + t\beta}_{\in K} - c \Rightarrow z_t \in K - c \ \forall t$

---

## Espacios de Hilbert

$H$ espacio de Hilbert $\Leftrightarrow$ $H$ espacio con producto interno completo.

**Prop.**
1) $x \mapsto \langle x, y \rangle$, $y \mapsto \langle x, y \rangle$, $x \mapsto \|x\|$ funciones continuas
2) $x \in H$, $M \leq H$, $x^\perp$, $M^\perp \leq H$ cerrados
3) Igualdad del paralelógramo:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

**Af.** $E \subseteq H$ convexo, cerrado, $E \neq \emptyset \Rightarrow \exists! \, x_0 \in E$; $\forall x \in E$: $\|x_0\| \leq \|x\|$

**dem.** $\delta = \inf\{\|x\| / x \in E\}$, $0 \leq \delta \leq \infty$

$E \neq \emptyset \Rightarrow \exists z \in E \Rightarrow 0 \leq \delta \leq \|z\| < \infty \Rightarrow 0 \leq \delta < \infty$

Identidad del paralelógramo: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

$\Rightarrow \|\frac{1}{2}x + \frac{1}{2}y\|^2 + \|\frac{1}{2}x - \frac{1}{2}y\|^2 = 2\|\frac{1}{2}x\|^2 + 2\|\frac{1}{2}y\|^2$

$\|\frac{x+y}{2}\|^2 + \frac{1}{4}\|x-y\|^2 = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2$

$E$ convexo ; $x, y \in E \Rightarrow \frac{1}{2}x + \frac{1}{2}y = \frac{x+y}{2} \in E$

$$\Rightarrow \frac{1}{4}\|x-y\|^2 = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \left\| \frac{x+y}{2} \right\|^2$$

$$\Rightarrow \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 4\left\| \frac{x+y}{2} \right\|^2$$

$$\frac{x+y}{2} \in E \Rightarrow \delta \leq \left\| \frac{x+y}{2} \right\| \Rightarrow \delta^2 \leq \left\| \frac{x+y}{2} \right\|^2$$

$$\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 4\left\| \frac{x+y}{2} \right\|^2 \leq 2\|x\|^2 + 2\|y\|^2 - 4\delta^2$$

$$(*) \quad \therefore \|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 - 4\delta^2 \quad \forall x, y \in E$$

$\exists (x_n)_{n\in\mathbb{N}}$ sucesión en $E$ tq: $\|x_n\| \to \delta$ (por propiedad del ínfimo)

Ocupando $(*)$: $\|x_n - x_m\|^2 \leq 2\|x_n\|^2 + 2\|x_m\|^2 - 4\delta^2$

$$2\|x_n\|^2 \xrightarrow{n\to\infty} 2\delta^2, \quad 2\|x_m\|^2 \xrightarrow{m\to\infty} 2\delta^2$$

$\therefore n, m \to \infty$ : $\|x_n - x_m\|^2 \to 0$ ( $(x_n)_n$ de Cauchy )

$H$ espacio de Hilbert: $\exists x_0 \in H$, $x_n \to x_0$ ,

$x_n \to x_0 \Rightarrow \|x_n\| \to \|x_0\|$ ( $\|\cdot\|$ continua )

$$\therefore \lim \|x_n\| = \|x_0\| = \delta$$

$$\therefore \|x_0\| = \delta$$

$E$ cerrado, $x_0 \in E$ ✓ ( $\|x_0\| \leq \|x\| \ \forall x \in E$ )

Unicidad: $\|x\| = \|y\| = \delta$

$\Rightarrow \|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 - 4\delta^2 = 0$

$\Rightarrow \|x-y\|^2 = 0$

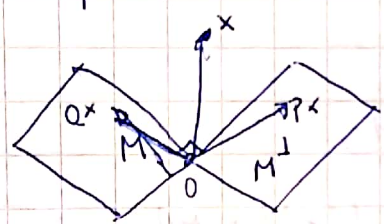
$\Rightarrow \|x-y\| = 0 \Leftrightarrow \boxed{x=y}$

---

Espacios de Hilbert - Proyecciones ortogonales.

Obs: Para la existencia de la descomposición, se necesita que $M$ sea cerrado.

$H$ espacio de Hilbert. $M \leq H$ cerrado




$\forall x \in H$, $x = Px + Qx$ escritura única,

$Px \in M$, $Qx \in M^\perp$

$P: H \to M$, $Q: H \to M^\perp$ proyecciones ortogonales.

- $Px \in M$ punto más cercano a $x$
- $Qx \in M^\perp$ punto más cercano a $x$

$\|x\|^2 = \|Px\|^2 + \|Qx\|^2$ "teo. de Pitágoras"

$P, Q$ lineales.

Corolario: $M \neq H \Rightarrow \exists\, y \in H,\ y \neq 0$ tq $y \perp M$

$y \perp M \Leftrightarrow \langle x, y \rangle = 0 \ \forall x \in M$

dem: $M \neq H \Rightarrow \exists x \in H \setminus M$ tq $x = Px + Qx$, $x \neq Px \in M$

$x \neq 0$

$\Rightarrow Qx = x - Px \in M^\perp$. Tomando $y = Qx$, se tiene que $\langle y, z \rangle = 0 \quad \forall z \in M$

$\therefore y \perp M$.

~~Obs: No es necesario que M sea cerrado.~~

· Espacios de Hilbert (Caracterización de los funcionales lineales continuos)

$\|z\| = 1 \iff \sqrt{(z,z)}^{1/2} = 1 \iff (z,z) = 1$ $\quad\Big|\quad$ $\underbrace{u = (Lx)z - (Lx)z}_{M}, \quad z \in M^{\perp}$

$L$ funcional lineal continuo

$$Lx = (Lx)(z,z) = ((Lx)z, z) = ((u + (Lz)x), z)$$

$$= \cancel{(u,z)}_{0} + ((Lz)x, z) = (Lz)(x,z)$$

$$\therefore \quad Lx = (Lx)(z,z) = (Lz)(x,z) \quad \forall x \in H$$

cambiando $x$ por ~~$\left(\frac{\alpha}{Lz}\right)x$, $x \in H$~~

$$\cancel{Lx = (Lz)\left(\left(\frac{\alpha}{Lz}\right)x, z\right) = (x,z) \quad \forall x \in H}.$$

$z = y/\alpha, \quad \alpha = \overline{Lz} : \quad Lx = (Lz)(x,z) = (Lz)(x, y/\alpha)$

$$= \frac{Lz}{\bar{\alpha}}(x,y) = \cancel{\frac{Lz}{Lz}}^{1}(x,y)$$

$$\boxed{\therefore \ Lx = (x,y)}$$

Unicidad: Supongamos $Lx = (x,y') = (x,y) \quad \forall x \in H$

$$\Rightarrow (x, y'-y) = 0 \ \forall x \Rightarrow y - y' = 0 \Rightarrow \boxed{y = y'}$$

Espacios de Hilbert (Poincaré)

$\Omega = (0,1)$, $\varphi \in C_c^\infty(\Omega) \Rightarrow \forall x \in \Omega : \varphi(x) = \varphi(a) + \int_a^x \varphi'(t)dt$

sop $\varphi \subseteq \Omega \Rightarrow \varphi(a) = 0$. $\therefore \varphi(x) = \int_a^x \varphi'(t)dt$.

$$|\varphi(x)| = \left| \int_a^x \varphi'(t)dt \right| \leq \int_a^x |\varphi'(t)|dt \leq \int_\Omega |\varphi'(t)|dt = \|\varphi'\|_{L^1} = \|D\varphi\|_{L^1}$$

$$\therefore \|\varphi\|_{L^\infty} \leq \|D\varphi\|_{L^1}$$

$$\|\varphi\|_{L^\infty} \leq \|D\varphi\|_{L^1} = \int_\Omega |\varphi'| \underset{\text{Hölder}}{\leq} \|D\varphi\|_{L^2} \|\chi_\Omega\|_{L^2} = \sqrt{b-a}\, \|D\varphi\|_{L^2}$$

Hölder → Hölder siempre es importante.

• Desigualdad de Poincaré:

$\Omega = \mathbb{R}^n$, $n > 1$, $\varphi \in C_c^\infty(\Omega)$ : $\|\varphi\|_{L^2} \leq C \|D\varphi\|_{L^2}$

—∘—

Falta averiguar: $\begin{cases} \text{Derivadas distribucionales} \\ H_0^1(\Omega) = \{ \varphi \in L^2(\Omega) \mid D\varphi \in L^2(\Omega),\ \varphi|_{\partial\Omega} = 0 \} \end{cases}$

$H_0^1(\Omega)$ es un espacio de Hilbert con el producto interno $(u,v)_{H_0^1} = \int Du \cdot Dv$

$$\|u\|_{H_0^1}^2 = (u,u)_{H_0^1} = \int Du \cdot Du = \int |Du|^2$$

—∘—

Ecuación de Poisson (forma débil)

tengo: $\begin{cases} -\Delta u = f, \ f \in H_0^1(\Omega) \\ u = 0 \text{ en } \partial\Omega \end{cases}$ tiene solución única $u$

$f \in (H_0^1)^*$ : $\begin{cases} -\Delta u = f \\ u = 0 \text{ en } \partial\Omega \end{cases}$ $\Leftrightarrow$ $\forall v \in H_0^1$ : $\langle f, v\rangle = -\int (\Delta u)\, v$

$\|$

$\{ H_0^1 \to \mathbb{C} \ /\ \text{lineales continuos} \}$

Por Riesz : $\langle f, v \rangle = (u, v) = \int Du \cdot Dv$

$\therefore \quad -\int \Delta u \cdot v = \int Du \cdot Dv$ forma débil de la ec. de Poisson.

---

Fourier.

$f \in L^2(T)$ $\to$ $f : \mathbb{R} \to \mathbb{C}$ $2\pi$-periódica $L^2$-integrable.

$\overline{C_c(T)}^{\|\cdot\|_{L^2}} = L^2(T) \xRightarrow{\forall \varepsilon > 0} \exists g \in C_c(T) : \|f - g\|_{L^2} < \varepsilon/2$

Stone - Weierstrass : $\overline{P(T, \mathbb{C})}^{\|\cdot\|_\infty} = C_c(T, \mathbb{C})$

$\Rightarrow \exists h \in P(T, \mathbb{C}) : \| h - g \|_\infty < \varepsilon/2$

$\therefore \quad \| h - f \|_{L^2} \le \| h - g \|_{L^2} + \| g - f \|_{L^2}$

$\| h - g \|_{L^2} \le \| h - g \|_\infty \sqrt{2\pi}$

$\therefore \quad \| h - f \|_{L^2} \le \frac{\varepsilon}{2}\sqrt{2\pi} + \varepsilon/2 \qquad \forall \varepsilon > 0$

$\therefore \quad \overline{P(T, \mathbb{C})}^{\|\cdot\|_{L^2}} = L^2(T, \mathbb{C})$.

Fourier $\rightarrow$ demostrar la desigualdad de Bessel : $\| S_N \|^2 \leq \| f \| \quad \forall N$

$f \in L^2(T, \mathbb{C})$ : $\quad S_N = \sum_{n=-N}^{N}{}' \hat{f}(n)\, u_n \quad , \quad u_n = e^{int}$

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-int}\, dt = (f, u_n)_{L^2}$$

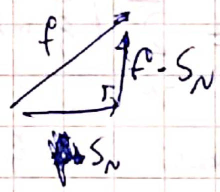
• Demostramos que $(S_N, S_N)_{L^2} = \sum_{n=-N}^{N}{}' |\hat{f}(n)|^2 \, , \; (S_N, f)_{L^2} = (S_N, S_N)_{L^2}$

$$\| S_N \|^2 = (S_N, S_N) = (S_N, f) \leq \| S_N \| \| f \|$$

$$(S_N, f) = (S_N, S_N) \Leftrightarrow (S_N, f - S_N) = 0$$

$$\Rightarrow 0 = |(S_N, f - S_N)| \leq \| S_N \| \| f - S_N \|$$

$(S_N, f - S_N) = 0 \Rightarrow S_N \perp f - S_N \Rightarrow$




$$\| f \| = \| f - S_N + S_N \| = \| f - S_N \| + \| S_N \| \Rightarrow \| S_N \| \leq \| f \|$$

$$\| S_N \| \leq \| f \| \; \forall n \Rightarrow \| S_N \|^2 = \sum_{n=-N}^{N}{}' |\hat{f}(n)|^2 \leq \| f \| \quad \forall N$$

$$\Rightarrow \sum_{n \in \mathbb{Z}}{}' |\hat{f}(n)|^2 \leq \| f \| \quad \forall N$$

$S_N = S_N(f)$

$\| \hat{f} \|^2 \lim_{N \to \infty} \| S_N(f) \|^2$

Bessel : $\overline{\sum_{n \in \mathbb{Z}}}{}' |\hat{f}(n)|^2 \leq \| f \|_{L^2} , \quad f \in L^2(T) \Rightarrow L^2(T) \longrightarrow \ell^2(\mathbb{Z})$

$$f \longmapsto \hat{f} = (\dots, \hat{f}(-2), \hat{f}(-1), \hat{f}(0), \hat{f}(1), \hat{f}(2), \dots)$$

$$\xi \in P(T,\mathbb{C}) \Rightarrow \quad \xi = \sum_{n=-N}^{N}{}' \lambda_n u_n = \sum_{n=-N}^{N}{}' \lambda_n e^{int}$$

$$(\xi, u_m) = \left( \sum_{-N}^{N}{}' \lambda_n e^{int}, e^{imt} \right) = \sum_{-N}^{N}{}' \lambda_n (e^{int}, e^{imt})$$

$m \notin \{-N, \dots, N\}$ : $(\xi, u_m) = 0$

$m \in \{-N, \dots, N\}$ : $(\xi, u_m) = \lambda_m$ , $\underline{n=m}$.

$$= \hat{\xi}(n)$$

$\therefore$ $\cancel{(\xi, u_m) = \sum_{-N}^{N}{}' \hat{\xi}(n) e^{int}}$ $\quad \xi = \sum_{-N}^{N}{}' \hat{\xi}(n) e^{int}$

$$\| \xi \|_{L^2}^2 = (\xi, \xi) = \sum_{-N}^{N}{}' \sum_{-N}^{N} \hat{\xi}(n) \overline{\hat{\xi}}(m) (e^{int}, e^{imt})$$

$$= \sum_{-N}^{N}{}' | \hat{\xi}(n) |^2 = \| \hat{\xi} \|_{\ell^2}^2$$

Por lo tanto: $\| \xi \|_{L^2}^2 = \| \hat{\xi} \|_{\ell^2}^2 \quad \forall \xi \in P(T,\mathbb{C})$

— o —

$\xi = \sum_{n=-N}^{N} \lambda_n u_n$ , $f \in L^2(T,\mathbb{C})$

$$(f, \xi) = \left( f, \sum_{n=-N}^{N} \lambda_n u_n \right) = \sum_{n=-N}^{N}{}' \overline{\lambda}_n (f, u_n) = \sum_{n=-N}^{N}{}' \hat{f}(n) \overline{\lambda}_n$$

∘ <u>Fourier</u>: La correspondencia $\Lambda : L^2(T,\mathbb{C}) \to \ell^2(\mathbb{Z})$ es una isometría.

$$\Lambda : L^2(T,\mathbb{C}) \to \ell^2(\mathbb{Z})$$
$$f \longmapsto \hat{f}$$

$$\|\xi\|^2 = (\xi,\xi) = \left(\sum_{-N}^{N}{}' \lambda_n u_n , \sum_{-N}^{N}{}' \lambda_n u_n\right), \quad \xi = \sum_{-N}^{N} \lambda_n u_n$$

$$(\xi, u_n) = \hat{\xi}(n) = \left(\sum_{-N}^{N}{}' \lambda_m u_m \, u_n\right) = \sum_{-N}^{N}{}' \lambda_m (u_m u_n) = \lambda_n$$

$$\Rightarrow (\xi,\xi) = \left(\sum_{-N}^{N}{}' \hat{\xi}(n) u_n , \sum_{-N}^{N}{}' \hat{\xi}(m) u_m\right) = \sum_{-N}^{N} |\hat{\xi}(n)|^2$$
$$= \|\xi\|_{\ell^2}$$

para $m > |N|$: $(\xi, u_m) = \hat{\xi}(m) = 0 \Rightarrow \xi = \sum_{n\in\mathbb{Z}}{}' \hat{\xi}(n) u_n = \lim_{N\to\infty} S_N(\xi)$

$$\Rightarrow \|\xi\|_{L^2(T)} = \sum_{n\in\mathbb{Z}}{}' |\hat{\xi}(n)|^2 = \|\hat{\xi}\|_{\ell^2} \qquad \text{ok.}$$

Ahora, $\forall f \in L^2(T)$, $\exists (\xi_j) \in P(T,\mathbb{C})$ tq $f = \lim_{j\in\mathbb{N}} \xi_j$

~~[illegible]~~

Se tiene que $\|\hat{f}\| = \lim_{N\to\infty} \|S_N(f)\|$ y Bessel $\|S_N(f)\| \le \|f\| \ \forall N$

$$\therefore \|\hat{f}\|_{\ell^2} \le \|f\|_{L^2} \quad \forall f \in L^2(T)$$

$$\|\widehat{f - \xi_j}\|_{\ell^2} \le \|f - \xi_j\|_{L^2} \xrightarrow{j\to\infty} 0$$

$\Lambda : L^2 \to \ell^2$ lineal

$$\widehat{f+g}(n) = (f+g, u_n) = (f, u_n) + (g, u_n) = \hat{f}(n) + \hat{g}(n)$$

$$\therefore \widehat{f+g} = \hat{f} + \hat{g}$$

$$\therefore \|\widehat{f - \xi_j}\|_{\ell^2} = \|\hat{f} - \hat{\xi}_j\| \xrightarrow{j\to\infty} 0$$

$$\| \qquad \|f - \xi_j\|_{L^2}$$

$$\therefore \|f\|_{L^2} = \lim_{j\to\infty} \|\xi_j\|_{L^2} = \lim_{j\to\infty} \|\hat{\xi}_j\|_{\ell^2} = \|\hat{f}\|_{\ell^2}$$

$$\therefore \wedge : L^2(T, \mathbb{C}) \longrightarrow \ell^2(\mathbb{Z}) \text{ isometría}$$

<u>Conclusión</u>: Primero, importante $\|f\|_{L^2} = \|\hat{f}\|_{\ell^2}$

$$f \in L^2 \Rightarrow S_N(f) = \sum_{n=-N}^{N} \hat{f}(n) u_n \in L^2$$

$$\|f - S_N(f)\|_{L^2} = \|\hat{f} - \widehat{S_N(f)}\|_{\ell^2} \qquad \Big| \ \widehat{S_N(f)} = \sum_{-N}^{N} \hat{f}(n)\hat{u}_n$$

$$= \sum_{n=-\infty}^{\infty} |\hat{f}(n) - \widehat{S_N(f)}(n)|^2 \qquad \hat{u}_n(m) = (u_n, u_m) = \delta_{nm}$$

$$\cancel{\widehat{S_N(f)} = \sum_{-N}^{N} \hat{f}(n)\delta_{nm}}$$

$$= \sum_{n=-\infty}^{\infty} \left|\hat{f}(n) - \sum_{j=-N}^{N} f(j)\hat{u}_j(n)\right|^2$$

$$= \sum_{|n|>N} \left|\hat{f}(n) - \sum_{j=-N}^{N} f(j)\hat{u}_j(n)\right|^2 + \sum_{n=-N}^{N} \left|\hat{f}(n) - \cancel{\sum_{j=-N}^{N} f(j) u_j(n)}\right|^2$$

$$= \sum_{|n|>N} |\hat{f}(n)|^2 \xrightarrow[N\to\infty]{} 0 \qquad \therefore S_N(f) \xrightarrow{N\to\infty} f \text{ en } L^2$$

$$\therefore \sum_{n\in\mathbb{Z}} \hat{f}(n) u_n = f$$

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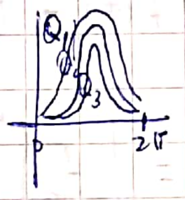
Fourier (Rudin habla sobre series de Fourier).

$Q_k \in P$, $k \in \mathbb{N}$ tq $\begin{cases} Q_k(t) \geq 0 \quad \forall t \in \mathbb{R} \\ \frac{1}{2\pi}\int_0^{2\pi} Q_k(t)\,dt = 1 \\ \forall \delta > 0,\ \delta < 2\pi : \ Q_k \xrightarrow{c.u} 0 \text{ en } [0,\delta] \cup [2\pi-\delta, 2\pi] \end{cases}$

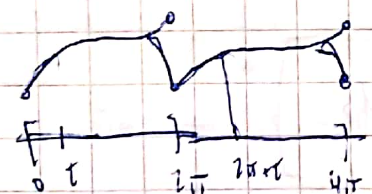



$$\forall f \in C(T) : \ P_k(t) = \frac{1}{2\pi}\int_0^{2\pi} f(t-s)Q_k(s)\,ds \qquad k \in \mathbb{N}.$$

$$t - s := u \Rightarrow t - u = s \ \begin{cases} s = 0 \to u = t \\ s = 2\pi \to u = t - 2\pi \end{cases} \quad du = -ds$$

$$P_k(t) = \frac{1}{2\pi}\int_t^{t-2\pi} f(u)Q_k(t-u)(-du) = \frac{1}{2\pi}\int_{t-2\pi}^{t} f(u)Q_k(t-u)\,du$$

$$= \frac{1}{2\pi}\int_t^{t+2\pi} f(u-2\pi)Q_k(t-u+2\pi)\,du = \frac{1}{2\pi}\int_t^{t+2\pi} f(u)Q_k(t-u)\,du$$




$$\int_t^{2\pi+t} = \int_0^t + \int_t^{2\pi} + \int_{2\pi}^{t+2\pi} - \int_0^t$$

$$\Rightarrow \frac{1}{2\pi}\int_t^{t+2\pi} f(u)Q_k(t-u)\,du = \frac{1}{2\pi}\int_0^{2\pi} f(u)Q_k(t-u)\,du$$

$$\therefore \frac{1}{2\pi}\int_0^{2\pi} f(t-s)Q_k(s)\,ds = \frac{1}{2\pi}\int_0^{2\pi} f(s)Q_k(t-s)\,ds$$

(*) $f \in C(\mathbb{T})$, $P_k(t) = \frac{1}{2\pi}\int_0^{2\pi} f(s)Q_k(t-s)\,ds$

$$|f(t) - P_k(t)| = \left|\frac{1}{2\pi}\int_0^{2\pi} Q_k f(t)\,ds - \frac{1}{2\pi}\int_0^{2\pi} f(s)Q_k(t-s)\,ds\right|$$

$$= \left|\frac{1}{2\pi}\int_0^{2\pi} Q_k(s) f(t)\,ds - \frac{1}{2\pi}\int_0^{2\pi} f(t-s)Q_k(s)\,ds\right|$$

$$= \left|\frac{1}{2\pi}\int_0^{2\pi} Q_k(s)\big(f(t) - f(t-s)\big)\,ds\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |Q_k(s)|\,|f(t) - f(t-s)|\,ds$$

$$\leq \frac{1}{2\pi} \underbrace{\sup_{s\in[0,2\pi]}\{|f(t) - f(t-s)|\}}_{=\beta} \int_0^{2\pi} Q_k(s)\,ds$$

Dado $\varepsilon > 0$, $\exists\, \delta > 0$ tq $\int_\delta^{2\pi-\delta} Q_k(s)\,ds < \varepsilon$

$\exists\, k_0 \in \mathbb{N}$ tq $\forall k \geq k_0$ $Q_k \rightrightarrows 0$ en $[0,\delta] \cup [2\pi-\delta, 2\pi]$ $\to$ $Q_k(s) < \varepsilon$

$\downarrow$ no depende de $t$.

$\therefore$ $|f(t) - P_k(t)| \leq$

$$\leq \frac{1}{2\pi}\beta\left[\int_\delta^{2\pi-\delta} Q_k(s)\,ds + \int_0^\delta Q_k(s)\,ds + \int_{2\pi-\delta}^{2\pi} Q_k(s)\,ds\right]$$

$$\leq \frac{1}{2\pi}\beta\left[\varepsilon + \varepsilon\delta + \varepsilon\delta\right] = \frac{1}{2\pi}\beta\left[\varepsilon + 2\varepsilon\delta\right]$$

$\therefore$ $P_k$ converge uniformemente a $f$

- Fourier ( Riesz - Fischer thm)

$$c \in \ell^2(\mathbb{Z}) \Rightarrow \sum_{n=-\infty}^{\infty}{}' |c_n|^2 < \infty$$

polinomio trigonométrico.

$$\text{Pd: } \exists f \in L^2(\mathbb{T}, \mathbb{C}) \text{ tq } \forall n : \quad c_n = \hat{f}(n)$$

$$\xi := \sum_{n=-\infty}^{\infty}{}' c_n u_n \quad , \quad \xi = \lim_{N\to\infty} \xi_N \quad , \quad \xi_N = \sum_{-N}^{N}{}' c_n u_n$$

$$\|\xi_N\|^2 = (\xi_N, \xi_N) = \sum_{-N}^{N}{}' \sum_{-N}^{N}{}' c_n \bar{c}_m (u_n, u_m) = \sum_{-N}^{N}{}' |c_n|^2 < \sum_{-\infty}^{\infty}{}' |c_n|^2 < \infty$$

$$\therefore \|\xi_N\|^2 < \|c\|^2_{\ell^2} < \infty \quad \forall N$$

$$\therefore \xi_N \xrightarrow{N\to\infty} \xi := \sum_{-\infty}^{\infty}{}' c_n u_n \in L^2$$

~~$\therefore (\|\xi_N\|)_N$ es creciente y acotada $\therefore \|\xi_N\|$ convergente~~

$$\text{Como } (\xi, u_m) = c_m = \hat{\xi}(m)$$

Obs. Hay Bessel encubierto en la demostración.

$$\xi_N = \sum_{-N}^{N}{}' \hat{\xi}_N(n) u_n$$

$$\hat{\xi}_N(n) = (\xi, u_n)$$

- Riesz - Fischer thm: $L^2$ es completo.

* $(\xi_N)_{N\in\mathbb{N}}$ es de Cauchy en $L^2$,

$$\|\xi_N - \xi_M\|^2_{L^2} = \Big\|\sum_{-N}^{N}{}' c_n u_n - \sum_{-M}^{M}{}' c_n u_m\Big\|^2 = \Big\|\sum_{|n| > \min\{N,M\}}^{\max\{N,M\}} c_n u_n\Big\|^2$$

$$\leq \sum_{\min\{N,M\} < |n| < \max\{N,M\}} \|c_n u_n\| .$$

$$\|\xi_N - \xi_M\|^2_{L^2} = \|\hat{\xi}_N - \hat{\xi}_M\|^2_{\ell^2} = \sum_{-\infty}^{\infty}{}' |\hat{\xi}_N(n) - \hat{\xi}_M(n)|^2$$

---

$$c \in \ell^2(\mathbb{Z}) \quad \|c\|^2_{\ell^2} = \sum_{-\infty}^{\infty}{}' |c_n|^2$$

tomar $\xi_N := \sum_{-N}^{N} c_n u_n$, donde $\underline{c_n = \hat{\xi}_N(n)}$

porque $(\xi_N, u_m) = \sum_{-N}^{N} c_n (u_n, u_m) = c_n$

$\xi_N = \sum_{-N}^{N}{}' \hat{\xi}_N(n)\, u_n$ es un polinomio Trigonométrico

$$\|\xi_N - \xi_M\|^2_{L^2} = \left( \cancel{\sum_{-N}^{N} \hat{\xi}_N(n) u_n, \sum_{-M}^{M} \hat{\xi}_M(n) u_n} \right)$$

$$= \cancel{\sum_{-M}^{N} \sum_{-M}^{M} \hat{\xi}_N(n) \hat{\xi}_M(m) (u_n, u_m)}$$

$$= \cancel{\sum \xi}$$

Efectivamente $(\xi_N)_N$ es suc. de Cauchy en $L^2(T)$.

$$= \left( \sum_{-N}^{N} \hat{\xi}_N(n) u_n - \sum_{-M}^{M} \hat{\xi}_M(n) u_n, \sum_{-N}^{N} \hat{\xi}_N(n) u_n - \sum_{-M}^{M} \hat{\xi}_M(n) u_n \right)$$

$$= \cancel{\left( \sum_{-N' = \min\{-N,-M\}}^{N' = \max\{N,M\}} \hat{\xi}_{N'}(n) u_n, \sum_{-N' = \min\{-N,-M\}}^{N' = \max\{N,M\}} \hat{\xi}_{N'}(n) u_n \right)} \neq$$

$$= \left( \sum_{\min\{N,M\} < |n| \leq \max\{N,M\}} \hat{\xi}(n) u_n, \sum_{\min\{N,M\} < |n| \leq \max\{N,M\}}{}' \hat{\xi}\, c_n u_n \right) = \sum_{\min\{N,M\} < |n| \leq \max\{M,N\}} |c_n|^2 \xrightarrow{N,M \to \infty} 0$$

Fourier (identidad de Parseval)

~~Fourier (identidad de Parseval)~~

Pd: $\displaystyle\sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)} = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt.$

$$(f,g)_{L^2} = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt$$

$$|(f,g)|_{L^2} \le \|f\|_{L^2}\|g\|_{L^2} = \|\hat{f}\|_{\ell^2}\|\hat{g}\|_{\ell^2}$$

$f = \sum_{-\infty}^{\infty}\hat{f}(n)u_n$ , $g = \sum_{-\infty}^{\infty}\hat{g}(n)u_n$ $\longrightarrow$ Resultado mega importante

$$(f,g) \overset{?}{=} \left(\sum_{-\infty}^{\infty}\hat{f}(n)u_n , \sum_{-\infty}^{\infty}\hat{g}(n)u_n\right) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\hat{f}(n)\overline{\hat{g}(n)}\,(u_n,u_m)$$

$$\overset{?}{=} \sum_{-\infty}^{\infty}\hat{f}(n)\overline{\hat{g}(n)}$$

$$\left(\sum_{-N}^{N}\hat{f}(n)u_n , \sum_{-N}^{N}\hat{g}(n)u_n\right) = \sum_{-N}^{N}\sum_{-N}^{N}\hat{f}(n)\overline{\hat{g}(m)}\,(u_n,u_m) = \sum_{-N}^{N}\hat{f}(n)\overline{\hat{g}(n)}$$

$$|(\cdot,\cdot)| \underset{C.S.}{\le} \left\|\sum_{-N}^{N}\hat{f}(n)u_n\right\|\left\|\sum_{-N}^{N}\hat{g}(n)u_n\right\| = \|S_N(f)\|\,\|S_N(g)\|$$

$$\underset{\text{Bessel}}{\le} \|f\|_{L^2}\|g\|_{L^2}$$

Se cumple! $\displaystyle\sum_{n\in\mathbb{Z}}\hat{f}(n)\overline{\hat{g}(n)} = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt$

$$|(S_N(f),S_N(g)) - (f,g)| = |(S_N(f),S_N(g)) - (f,g) + (f,S_N(g)) - (f,S_N(g))|$$

$$\le |(S_N(f),S_N(g)) - (f,S_N(g))| + |(f,S_N(g)) - (f,g)|$$

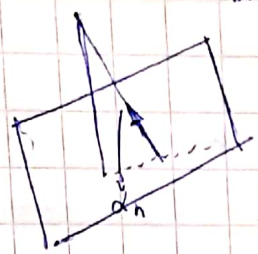
$$= |(S_N(f)-f, S_N(g))| + |(f, S_N(g)-g)|$$

$$\le \|S_N(f)-f\|\,\|S_N(g)\| + \|f\|\,\|S_N(g)-g\| \xrightarrow{N\to\infty} 0$$

Análisis II, 16 Nov. $\boxed{2015}$

Lema. $T \in L(X,X)$ compacto. $\forall \lambda \neq 0$, $R(T_\lambda)$ es cerrado.

Lema previo. $X$ normado, $M < X$ cerrado, $M \neq X$. $\forall \varepsilon > 0$, $\exists x_\varepsilon \in X$,
$\|x_\varepsilon\| = 1$, $dist(x_\varepsilon, M) \geq 1-\varepsilon$



dem Sea $z_0 \in X \setminus M$, sea $(\alpha_n)_n$ en $M$ tq $\|z - \alpha_n\| \to dist(z,M)$

Sea $x_n = \dfrac{z-\alpha_n}{\|z-\alpha_n\|}$ $\qquad$ $x_n - y = \dfrac{z-\alpha_n - y\|z-\alpha_n\|}{\|z-\alpha_n\|}$

$\Rightarrow z - \alpha_n = x_n \|$ $\qquad$ $\Rightarrow \inf\limits_{y \in M} \|x_n - y\| \geq \dfrac{dist(z,M)}{\|z-\alpha_n\|} \geq 1-\varepsilon$ algún $n$

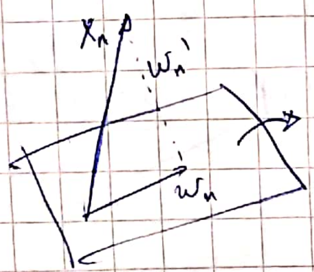
$dist(x_n, M) \geq 1-\varepsilon$

dem. del lema: Supo $T_\lambda x_n \to y$

Pd: $y = T_\lambda x$ para algún $x$.

Como $\underbrace{T_\lambda = \lambda I - T}_{\text{perturbación compacta de la identidad.}}$ $\Rightarrow$ $I = \dfrac{T + T_\lambda}{\lambda}$, $\quad x_n = \dfrac{T x_n + T_\lambda x_n}{\lambda}$ $\forall n$.

$\Rightarrow$ si los $Tx_n$ tienen subsucesión convergente. $\exists (n_k)$ tal que $x_{n_k} \to x$ para algún $x$.




$T_\lambda x_n = T_\lambda w_n + T_\lambda w_n' = T_\lambda w_n'$

Supongamos que $\alpha_n := dist(x_n, \ker T_\lambda) \xrightarrow{n\to\infty} \infty$

$$T_\lambda x = \lim T_\lambda x_n = 0 . \quad \text{Sea } w_n \in \ker T_\lambda \text{ tq}$$

$$\alpha_n \leq \| x_n - w_n \| \leq \left(1 + \frac{1}{n}\right) \alpha_n$$

Sea $z_n = \dfrac{x_n - w_n}{\|x_n - w_n\|}$ $\qquad T_\lambda z_n = \dfrac{T_\lambda x_n}{\|x_n - w_n\|} \longrightarrow 0$

$T$ compacto, $\|z_n\| = 1 \Rightarrow \exists n_k : z_{n_k} \to w_0$ algún $w_0$

$$\left( z_{n_k} = \frac{T z_{n_k} - T_\lambda z_{n_k}}{\lambda} \right)$$

$$T_\lambda w_0 = \lim T_\lambda z_n = 0 \quad \Rightarrow w_0 \in \ker T_\lambda$$

$$\Rightarrow \| x_n - (w_n - w_0 \| x_n - w_n \|) \| \geq \alpha_n \geq \frac{\|x_n - w_n\|}{1 + \frac{1}{n}}$$

$$\frac{1}{1 + \frac{1}{n}} \leq \left\| \frac{x_n - w_n}{\|x_n - w_n\|} - w_0 \right\| \xrightarrow[\text{def de } z_{n_k}]{} 0$$

$(\rightarrow\leftarrow)$

Se concluye que $\exists (w_n) \subset \ker(T_\lambda)$ tq $\sup_n \| x_n - w_n \| < \infty$

$$\therefore x_n - w_n = \frac{1}{\lambda}\Big( \underbrace{T(x_n - w_n)}_{T \text{ compacto}} + \underbrace{T_\lambda (x_n - w_n)}_{= T_\lambda x_n} \Big)$$

tiene subsucesión convergente $x_{n_k} - w_{n_k} \longrightarrow x'$

$$\Rightarrow y = \lim T_\lambda x_n = \lim T_\lambda (x_{n_k} - w_{n_k}) = T_\lambda x'$$

$$\Rightarrow y \in R(T_\lambda) \ \blacksquare$$

( En dimensión finita )

$$\sqrt{|\lambda_1|^2+|\lambda_2|^2} \leq \| \lambda_1 \alpha_1 + \lambda_2 \alpha_2 \| \leq |\lambda_1| \|\alpha_1\| + |\lambda_2| \|\alpha_2\|$$

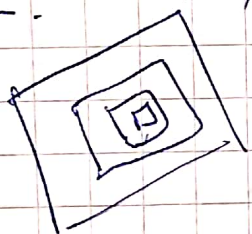
$$\leq \underbrace{\sqrt{|\lambda_1|^2+|\lambda_2|^2}}_{\mathbb{R}^n} \sqrt{\|\alpha_1\|^2+\|\alpha_2\|^2}$$

Teo. $T \in L(X,X)$, $\lambda \neq 0$ no es un valor propio ($T_\lambda$ es 1-1)

$\Rightarrow T_\lambda(X) = X$

y $T_\lambda^{-1}$ es continua.

dem. $X_0 = X$, $X_1 = T_\lambda(X)$, $X_2 = T_\lambda(X_1)$, ...



dem : $X_{n+1} \subset X_n$

$n=0$ ✓

$y \in X_{n+2} \Rightarrow y = T_\lambda x$, ~~$x\alpha$~~ $x \in X_{n+1} \subset X_n$

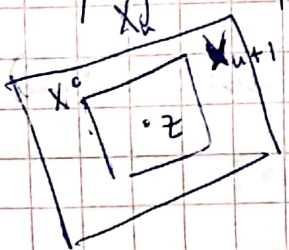
$\in T_\lambda(X_n) = X_{n+1}$

dem que $X_{n+1} \neq X_n$

Estamos suponiendo que $T_\lambda$ no es sobre

$\Rightarrow X_1 = T_\lambda(X) \neq X_0$.

Supongamos que $X_{n+1} \neq X_n \Rightarrow \exists x \in X_n \setminus X_{n-1}$. ?d: $T_\lambda x \in X_{n+1} \setminus X_{n+2}$




Supongamos que $T_\lambda x \in X_{n+2} = T_\lambda(X_{n+1})$

$\Rightarrow \exists z \in X_{n+1}$ tq $T_\lambda x = T_\lambda z \Rightarrow x = z$ ($T_\lambda$ 1-1)

pero $x \notin X_{n+1}$ ($\Rightarrow\Leftarrow$)

Por lema de las proyecciones casi ortogonales

$$\exists (y_n) : \| y_n \| = 1 \qquad y_n \in X_n \ \forall n$$
$$\operatorname{dist}(y_n, X_{n+1}) \geq 1/2$$

Sea $n > m$, $T_\lambda = \lambda I - T \Rightarrow T = \lambda I - T_\lambda$

$$\| T y_n - T y_m \| = \| - y_n - \underbrace{(T_\lambda y_n - \lambda y_m - T_\lambda y_m)}_{\in X_{n+1}} \| \geq \lambda/2$$

$(\rightarrow \leftarrow)$

Def. $T \in L(X,X)$, resolvente $\rho(T) = \{ \lambda \in \mathbb{C}$ tq

i) $R(T_\lambda)$ es denso en $X$

ii) $T_\lambda$ es 1-1

iii) $T_\lambda^{-1} : R(T_\lambda) \to X$ es continua

def. espectro de $T$, $\sigma(T) = \mathbb{C} \setminus \rho(T)$

$$T \text{ compacto} \Rightarrow \sigma(T) = \{ \text{valores propios} \}$$

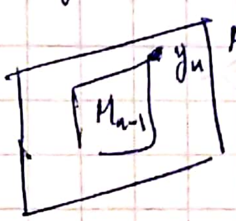
Teo. $T$ compacto $\Rightarrow$ (i) $\sigma(T)$ no tiene puntos de acumulación $\lambda \neq 0$.

(ii) $\forall \lambda \in \sigma(T)$ dim $\ker T_\lambda < \infty$

dem. Sup. que $T x_n = \lambda_n x_n$, $\lambda_n \to \lambda \neq 0$ $x_n$ l.i.

$M_n = \langle x_1, x_2, \ldots, x_n \rangle$. Sea $(y_n)$ $\| y_n \| = 1$, $\operatorname{dist}(y_n, M_{n-1}) \geq 1/2$

$y_n \in M_n$




$$y_n = \sum_{j=1}^{n} \beta_j x_j ,$$

$$T_{\lambda_n} y_n = \sum_{j=1}^{n-1} \beta_j (\lambda_n - \lambda_j) x_j \in M_{n-1}$$

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$n>m \quad Ty_n - Ty_m = \lambda_n y_n - \underbrace{\underbrace{T_{\lambda_n} y_n}_{\in M_{n-1}} - \underbrace{Ty_m}_{\in M_{n-1}}}$

$$\| Ty_n - Ty_m \| \geq \lambda_n/2 . \quad \boxed{}$$

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Algebra II, 16 Noviembre 2015.

Sea $G$ un grupo y $(V, \rho)$ representación ~~finita~~ de dimensión finita sobre un cuerpo $K$. Se define el caracter de $\rho$ como $\chi_\rho : G \to K$

$$\chi_\rho(g) = \text{traza}(\chi_\rho(g))$$

Ejemplos. (1) $U_n(K)$ el grupo unipotente

$$U_n = \{ A \in M_n(K) \mid A = I + N, \ N \text{ estrictamente triangular superior} \}$$

$$U_3 = \left\{ \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \middle/ a,b,c \in K \right\} , \quad i : U_n(K) \hookrightarrow GL_n(K)$$
$$A \longmapsto A$$

$\chi_i(A) = \text{traza}(A) = n \quad , \quad \forall A \in U_n(K)$

Si $\text{char}(K) = p$ y $p \mid n \Rightarrow \chi_i(A) = 0 \quad \forall A \in U_n(K)$

2) $V = KG$ $\quad \rho$ representación regular. $G$ grupo finito.

$\forall v \quad v = \sum_{g\in G} \alpha_g \delta_g \quad , \quad \rho(h) v = \sum_{g \in G} \alpha_g \delta_{hg}$

$\chi_\rho(h) = \text{traza}(\rho(h))$

$$[\rho(h)]_B = \begin{pmatrix} 0 & * & * & * \\ * & 0 & & \\ * & & \ddots & \\ & & & 0 \end{pmatrix} , \quad B = (\delta_g)_{g \in B}$$

Como $h \neq 1_G \Rightarrow hg \neq g \quad \forall g \in G$

$$\chi_g(h) = \begin{cases} |G| & h = 1_G \\ 0 & h \neq 1_G \end{cases}$$

la representación cuenta la cantidad de puntos fijos de ciertas acción, ¿cuál?

Propiedades de $\chi_\varrho$ (ejercicio).

$\chi_\varrho(1_G) = \dim(V, \varrho)$

$\chi_{\varrho \oplus \eta} = \chi_\varrho + \chi_\eta$

$\chi_{\varrho \otimes \eta} = \chi_\varrho \chi_\eta$

$\chi_{\varrho^*}(g) = \chi_\varrho(g^{-1}) \quad \forall g \in G.$

$(V, \varrho) \cong (W, \eta) \Rightarrow \chi_\varrho = \chi_\eta$

Teo. Sea $K$ algebraicamente cerrado y sean $(V, \varrho)$, $(W, \eta)$ dos representaciones irreducibles de un grupo $G$ de dimensión finita sobre $K$,

$$(V, \varrho) \cong (W, \eta) \iff \chi_\varrho = \chi_\eta$$

obs. $(V, \varrho) \cong (W, \eta) \Rightarrow \chi_\varrho = \chi_\eta$ siempre se cumple (averiguar).

dem. $\mathcal{L}(V, W) = \mathrm{Hom}_K(V, W) := \{ f : V \to W \,/\, f \ K\text{-lineal} \}$.

$\mathrm{Hom}_G(V, W) = \{ f : V \to W \,/\, f \text{ función de entrelazamiento} \}$.

Lema (de Schur). Sean $(V, \varrho)$, $(W, \eta)$ dos representaciones de un grupo $G$.

1) $(V, \varrho) \not\cong (W, \eta) \Rightarrow \mathrm{Hom}_G(V, W) = \{0\}$.

2) $\dim(V), \dim(W) < \infty \Rightarrow \dim(\mathrm{Hom}_G(V, W)) = 1 \vee 0$, siempre que $K$ sea algebraicamente cerrado.

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dem: Sea $\Phi \in Hom_G(V,W)$. Supongamos que $\Phi \neq 0$

$$(Ker(\Phi), \rho|_{Ker(\Phi)}) \leq (V,\rho) \Rightarrow ker(\Phi)=\{0\} \Rightarrow \Phi \text{ inyectiva}$$

Por otro lado, $(Im(\Phi), \lambda|_{Im(\Phi)}) \leq (W,\lambda) \Rightarrow Im(\Phi)=W \Rightarrow \Phi$ epiyectiva

$\therefore \Phi$ es isomorfismo ; $(V,\rho) \cong (W,\lambda)$

2) Supongamos $\exists \Phi$, $\Phi \in END_G(V) \ (= Hom_G(V,V))$. $\exists \xi \in K$ valor propio de $\Phi$ (existe pues $K$ es algebraicamente cerrado)

$$\Phi - \xi\, Id \in END_G(V), \quad ker(\Phi - \xi\, Id) \neq \{0\}$$

$$\therefore \Phi - \xi\, Id = 0$$

Ahora sea $\Phi \in Hom_G(V,W)$, $\Phi \neq \{0\}$. Sea $\Psi \in Hom_G(V,W)$, $\Psi \neq 0$

$$\Psi^{-1} \circ \Phi \in END_G(V)$$

$$\Psi^{-1} \circ \Phi = \xi\, Id \Rightarrow \Phi = \xi \Psi \quad \xi \neq 0.$$

Para el caso $\Psi = 0$, $\Psi = 0 \circ \Phi$.

Sea $(V,\rho)$ una representación de $G$. Definimos

$$(V_L, \rho_L) = (\mathcal{L}(V), \rho_L), \quad \rho_L : G \to \mathcal{L}(V)$$
$$g \mapsto \rho_L(g) : V \to V$$
$$T \mapsto \rho(g) \circ T$$

Teo: Si $(V,\rho)$ es de dimensión finita. Entonces

$$(V_L, \rho_L) \cong \bigoplus_{i=1}^{n} (V,\rho)$$

dem: Sea $w=(w_1,\dots,w_n)$ base ordenada de $V$. $\forall v \in \bigoplus_{i=1}^{n} V$, $v = \sum_{i=1}^{n} v_i = (v_1,\dots,v_n)$

$$\phi : \bigoplus_{i=1}^{n} V \longrightarrow V_L$$

$$v \longmapsto \phi(v) : V \longrightarrow V$$
$$w_i \mapsto v_i$$

$\phi$ es lineal

$$\phi(v) = 0 \ , \quad v = \sum_{i=1}^{n} v_i \qquad \phi(v) w_i = 0 \quad \forall i$$
$$\Rightarrow v_i = 0 \quad \forall i$$
$$\Rightarrow v = 0$$

$\therefore$ $\phi$ es monomorfismo.

Af. $\phi$ biyectiva.

$$\bigoplus_{i=1}^{n} (V, \rho) = \left( \bigoplus_{i=1}^{n} V, \bigoplus_{i=1}^{n} \rho \right)$$

$$\phi \circ \bigoplus_{i=1}^{n} \rho(g) v = \phi(\rho(g) v_1 + \dots + \rho(g) v_n)$$
$$\left(\text{con } v = \sum_{i=1}^{n} v_i\right)$$

$$\phi(\rho(g) v_1 + \dots + \rho(g) v_n)(w_j) = \rho(g) v_j \ ,$$
$$\rho(g)(\phi(v) w_j) = \rho(g) \circ \phi(v) w_j = \rho_L(g)(\phi(v)) w_j$$

$$\therefore \ \phi \circ \bigoplus_{i=1}^{n} \rho(g) v = \rho_L(g) \circ \phi(v)$$
$$\forall v \in \bigoplus_{i=1}^{n} V$$

Ayudantía Algebra II
16/11/2015

J.P. Serre - "Linear Representations of finite Groups".
A - Mcdonald - "Intr. al álg. conmutativa".

Representaciones, ejemplos.

1) $G = \langle \alpha \rangle \quad |G| = \infty \longrightarrow G = \mathbb{Z}$

$$G \xrightarrow{\varphi} GL_n(K) \qquad \varphi \text{ fiel} \Leftrightarrow |A| = \infty$$
$$\alpha \mapsto A$$

Ej. $K = \mathbb{Q}$, $n = 2$ $\quad \alpha \mapsto \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} = A$

$\forall n$, $A^n = \begin{bmatrix} 1 & n \\ & 1 \end{bmatrix}$

¿ E irreducible $\Leftrightarrow$ $0 \subsetneq V \subsetneq \mathbb{Q}^n$ tal que $V$ es invariante bajo $G$.

$G \circlearrowright K^n$
$G \circlearrowright \mathbb{Q}^2$

Una forma es conociendo los valores y vectores propios de la matriz

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ & 1-\lambda \end{vmatrix} = 0 \rightarrow \lambda = 1, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad y = 0$$

$$V = \langle (x,y) \in \mathbb{Q}^2 / \ y = 0 \rangle$$

$\therefore \varphi$ no es irreducible.

• 2do caso $|G| < \infty$ $G$ cíclico $\Rightarrow G = \mathbb{Z}/(n)$

$$G \to GL_r(\mathbb{C})$$

$$\alpha \mapsto \begin{bmatrix} \lambda & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix} \quad \text{donde } \lambda = e^{2\pi i/n}$$

otro ejemplo: $G \to GL_n(\mathbb{Q})$

(de la misma situación) $\alpha \mapsto \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & 0 & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = A \to$ matriz de orden $n$.

• Ejemplo (Representación de grupos Diedrales).

$$D_n = \langle r, s \mid r^n = s^2 = 1, \quad sr = r^{-1}s \rangle$$

$$R = \begin{pmatrix} \omega & \\ & \omega^{-1} \end{pmatrix}, \quad S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad \text{donde } \omega = e^{2\pi i/n}$$

$\varphi: \begin{cases} r \mapsto R \\ s \mapsto S \end{cases}$ , $S^2 = Id$, $R^n = \begin{pmatrix} \omega^n & \\ & \omega^{-n} \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = Id.$

representación

$$SR = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}, \quad R^{-1}S = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}$$

tenemos $D_n \to GL_2(\mathbb{Q}(\omega))$

Subespacios invariantes: $S \to \langle (1,-1) \rangle$ ——— linealmente independientes.

$R \to \langle (1,0) \rangle$ y $\langle (0,1) \rangle$

$\to \varphi$ no tiene subespacios invariantes propios no triviales

Representaciones inducidas: $G, H$ grupos $G \times H$ producto directo

$$\varphi: G \to GL_{n_1}(K)$$
$$\psi: H \to GL_{n_2}(K)$$
representaciones

tenemos la representación (en matriz) $\begin{bmatrix} X_1 & \\ & X_2 \end{bmatrix}$

$$\begin{array}{ccc} G & \xrightarrow{\varphi_1} & GL_{n_1}(K) \\ & \searrow^{\varphi_2} & GL_{n_2}(K) \end{array}$$

$$G \circlearrowleft K^{n_1} \otimes K^{n_2}$$

$$\varphi_1 \otimes \varphi_2 (g) = \varphi_1(g) \circ \varphi_2(g)$$

$$G \circlearrowleft V \otimes V = S^2(V) \oplus \Lambda^2(V)$$

$\dim V = n \Rightarrow \dim \frac{n(n+1)}{2}$ $\qquad$ $\dim \frac{n(n-1)}{2}$

$S^2(V)$, $\Lambda^2(V)$ son invariantes bajo $G$.

Otro ejemplo: $A_4^* = \langle \underbrace{(12)(34)}_{m_1}, \underbrace{(123)}_{m_2} \rangle \leq S_4$

$A_4 \to GL_4(\mathbb{Q})$, $\mathbb{Q}^4 = \langle e_1, e_2, e_3, e_4 \rangle$

$m_1 \mapsto M_1 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 0 & \\ & & & 1 \\ & & 1 & 0 \end{pmatrix}$, $\quad m_2 \mapsto M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$(12)(34)(123) = (243)$

$$\cancel{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$Q_8 = \langle i, j \mid i^4 = 1,\ i^2 = j^2 = (ij)^2 \rangle \quad , \quad |Q_8| = 8$$

$Q_8 \longrightarrow GL_2(\mathbb{C})$ (representación compleja)

$$i \longmapsto \begin{pmatrix} i & \\ & -i \end{pmatrix}$$

$$j \longmapsto \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

$G \geq H$ , $|G| < \infty$ , $H \xrightarrow{\varphi} GL(V)$ representación.

Sea $\{g_1 = e, g_2, \dots, g_r\} \subset G$ conjunto de representantes de $G/H$ , $r = [G:H]$.

$$W := V \oplus g_2 V \oplus \dots \oplus g_r V$$

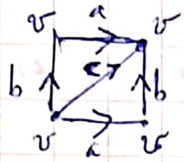
$g = g_k h$ , $h \in H$

$$g \cdot (v_1 + g_2 v_2 + \dots + g_r v_r) = g_k h v_1 + g_c h v_2 + \dots + g_k h v_r$$

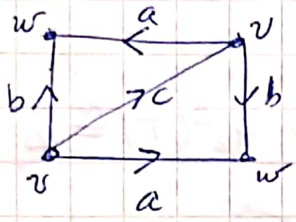
$\bar{\varphi} : G \to GL(V \times \dots \times V)$ es la representación inducida.

16/11/2015

Tenemos definidos los $\Delta$-símplices.

Ej.  $= T^2 =$ 

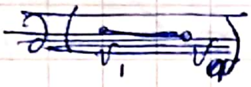


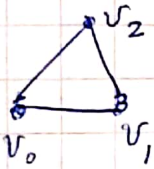
$\mathbb{R}P^2 =$ 



Luego (1) Se define el grupo abeliano libre de los "n-símplices"

$\Delta_n X =$ n-cadenas

(2)  $\partial(v_0 \longrightarrow v_1) = v_1 - v_0$

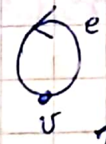
$\partial\Big($  $\Big) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$



$\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$ , $\partial_n(\sigma_\alpha) = \sum (-1)^i \sigma_\alpha \Big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$

de la tarea antigua $\partial\partial = 0$.

Y así, $H_n^\Delta(X) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$

Ejemplo. $S^1 =$ 

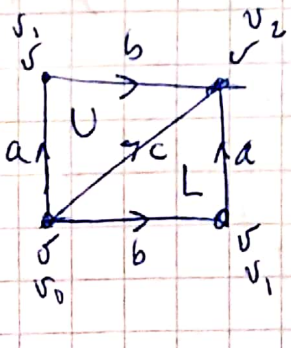
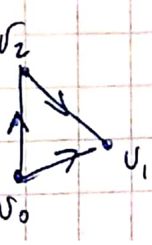


$\Delta_0(S^1) = \mathbb{Z} = \langle v \rangle$

$\Delta_1(S^1) = \mathbb{Z} = \langle e \rangle$

$$0 \to \underset{\mathbb{Z}}{\Delta_1(S^1)} \xrightarrow{\partial} \underset{\mathbb{Z}}{\Delta_0(S^1)} \to 0$$

$\partial(e) = 0$ $\quad H_1^\Delta(S^1) = \mathbb{Z}$ , $H_0^\Delta(S^1) = \mathbb{Z}$.

Ejemplo  





$\Delta_0(T) =$ [crossed out] $= \langle v \rangle \cong \mathbb{Z}$

$\Delta_1(T) = \langle a, b, c \rangle \cong \mathbb{Z}^3$

$\Delta_2(T) = \langle U, L \rangle \cong \mathbb{Z}^2$

$\partial_1 = 0 \Rightarrow H_0^\Delta(T) = \mathbb{Z}$

$\partial_2(U) = a+b-c = \partial_2(L)$ y como $\{a, b, a+b-c\}$ es base de $\Delta_1(T)$

$\Rightarrow H_1^\Delta(T) \cong \mathbb{Z}^2$

$0 \to \ker \to \mathbb{Z}^2 \to \mathbb{Z} \to 0 \quad \to H_2^\Delta(T) \cong \mathbb{Z}$

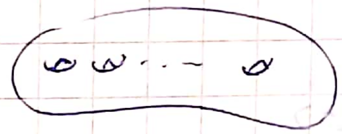
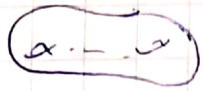
$\ker = \mathbb{Z}$

teo. El morfismo inducido $\Delta_n X \to \Delta_n X$ para un $\Delta$-símplice $X$, genera isomorfismo en homología, i.e,

$$H_n^\Delta(X) \simeq H_n(X), \ \forall n.$$

Aplicación Sea $X$ un $\Delta$-símplice finito

$$\Rightarrow e(X) = \sum_{n=0}^{\infty} (-1)^n \text{rango}(\Delta_n X) = \sum_{n=0}^{\infty} (-1)^n \text{rango}(H_n X))$$

$e($  $) = 2-2g$, $g = \text{rango}(H_1($  $))$

$g$-hoyos

El lema que se usa: $0 \to A \to B \to C \to 0$ suc. exacta de grupos abelianos y $g \Rightarrow \text{rango}(B) = \text{rango}(C) + \text{rango}\, A)$

dem. $0 \to \Delta_k(X) \to \Delta_{k-1}(X) \xrightarrow{d_{k-1}} \cdots$

$\Rightarrow 0 \to Z_n$

$$\Rightarrow \quad 0 \longrightarrow \underset{\ker \partial_n}{\underset{\shortparallel}{Z_n}} \longrightarrow \Delta_n X \longrightarrow \underset{\operatorname{Im} \partial_n}{\underset{\shortparallel}{B_{n-1}}} \longrightarrow .$$

$\Rightarrow$ ~~rango~~

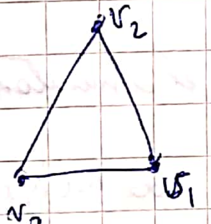
$\Rightarrow$ rango $(\Delta_n) =$ ~~rango~~ rango $(B_n)$ + rango $(B_{n-1})$ + rango $(H_n)$ $\square$

Notar que $f: I \to X$ camino es un 1-símplice singular y si es un loop $\Rightarrow \partial f = f(1) - f(0) = 0 \Rightarrow$ genera una función de

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "h"] \arrow[dr] & H_1(X) \\
& \pi_1(X, x_0)_{abel} := \pi_1(X, x_0) / [\pi_1 : \pi_1] \arrow[u, dashed, "\cong"']
\end{tikzcd}$$

(1) Sea $f: I \to X$, $f(t) \equiv x_0 \Rightarrow \partial(x_0) = 0$.

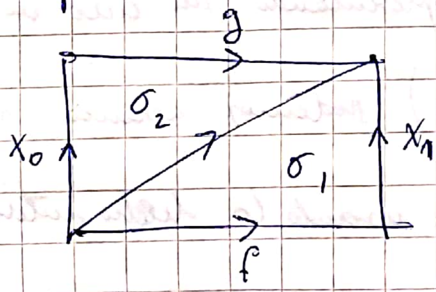
$[x_0] \in H_1(X)$, queremos $[x_0] = 0$ en $H_1(X)$




Tomar $\Delta_2 \xrightarrow{F} X \quad F(p) = x_0$

$$\partial(F) = [x_0] - [x_0] + [x_0] = [x_0]$$

(2) Si $f \underset{}{\overset{\text{caminos}}{\simeq}} g$ en $X$, tenemos $F: I \times I \longrightarrow X$ entre $f$ y $g$




$I \xrightarrow{F} X$

$$\therefore \partial(\sigma_1 - \sigma_2) = f - g + x_0 + x_1$$