17/03/2016

Bibliografía: Stein - Shakarchi. Complex analysis.

Ayudante: Nathan Aguirre.

— ∘ —

El cuerpo de los números complejos.

$$\mathbb{R}^2 = \{(x,y) ;\ x, y \in \mathbb{R}\}$$

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 y_1 - x_2 y_2,\ x_1 y_2 + x_2 y_1)$$

con neutros $(0,0)$ y $(1,0)$;

$$(x,y)^{-1} = \left(\frac{x}{x^2+y^2} , \frac{y}{x^2+y^2} \right)$$

$\{(x,0)\} \cong \mathbb{R}$. Identificamos $x = (x,0)$, $i = (0,1)$

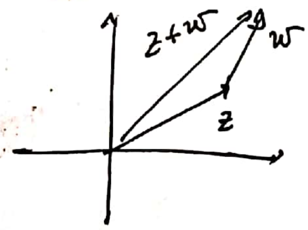
$$i^2 = -1$$

$$(x,y) = x(1,0) + y(0,1) = x + iy$$

Teo (Frobenius). Dado $n \in \mathbb{N}$, existe multiplicación en $\mathbb{R}^n$ que (junto con la suma usual) define estructura de cuerpo sii $n = 1$ ó $n = 2$.

Obs. Si en la def de cuerpo quitamos la propiedad $z_1 z_2 = z_2 z_1$, entonces $\exists$ producto en $\mathbb{R}^4$ ($\mathbb{R}^4 = \mathbb{H}$ = cuaterniones).

Visualización de los complejos:



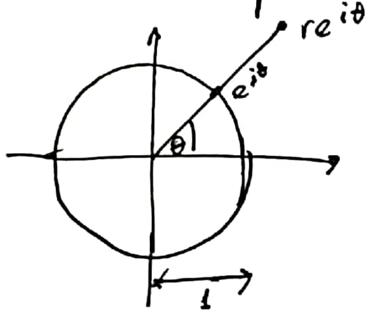
$$z = x + iy, \quad x, y \in \mathbb{R}$$

$$|z| := \sqrt{x^2 + y^2}$$

Si $\theta \in \mathbb{R}$, definimos $e^{i\theta} := \cos\theta + i \operatorname{sen}\theta$

$$|e^{i\theta}| = 1$$

En coordenadas polares: $z = re^{i\theta}$

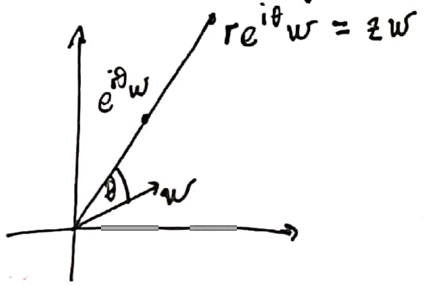


multiplicación en coordenadas polares:

$$\left\{\begin{array}{l} z_1 = r_1 e^{i\theta_1} \\ z_2 = r_2 e^{i\theta_2} \end{array}\right. \Rightarrow z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Consecuencia: Sea $z = re^{i\theta}$. La aplicación $\begin{array}{l} M_z : \mathbb{C} \longrightarrow \mathbb{C} \\ \qquad w \longmapsto zw \end{array}$

$$M_z = \begin{pmatrix} \text{rotación} \\ \text{de ángulo } \theta \end{pmatrix} \circ \begin{pmatrix} \text{homotecia} \\ \text{de factor } r \end{pmatrix}$$



En particular, M_z es:

- $\mathbb{R}$-lineal
- preserva la orientación de $\mathbb{R}^2$
- preserva ángulos (transformación conforme).

Para $z = x+iy$ definimos el conjugado por $\bar{z} = x - iy$.

Propiedad: $z\bar{z} = x^2+y^2 = |z|^2$; $z^{-1} = \dfrac{\bar{z}}{|z|^2}$

$\therefore |z| = \sqrt{z\bar{z}}$

Hecho: La conjugación $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$ es un automorfismo de $\mathbb{C}$, es decir, $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\,\bar{w}$.

Prop. Si $f : \mathbb{C} \to \mathbb{C}$ es automorfismo de $\mathbb{C}$ y además es continuo, entonces $f = id$ ó $f =$ conjugación.

dem. $f(0)=0$, $f(1)=1$

$f(-z)=-f(z)$

Se tiene fácilmente que $f|_{\mathbb{Q}} = id_{\mathbb{Q}}$. Por continuidad, $f|_{\mathbb{R}} = id_{\mathbb{R}}$.

$f(i) = ?$

$f(i^2) = f(i)^2 = f(-1) = -1 \Rightarrow f(i) \in \{i, -i\}$

$f(i)=i$: $f(x+iy) = f(x) + if(y) = x+iy$

$f(i)=-i$ $f(x+iy) = x-iy$

Para $z=x+iy$, $\begin{cases} Re(z) = x \\ Im(z) = y \end{cases}$, $\begin{cases} Re(z) = \dfrac{z+\bar{z}}{2} \\ Im(z) = \dfrac{z-\bar{z}}{2i} \end{cases}$

Producto interno : $\langle z, w \rangle = Re(z\bar{w})$

En efecto : $Re(z\bar{w}) = Re((x_1+iy_1)(x_2-iy_2))$

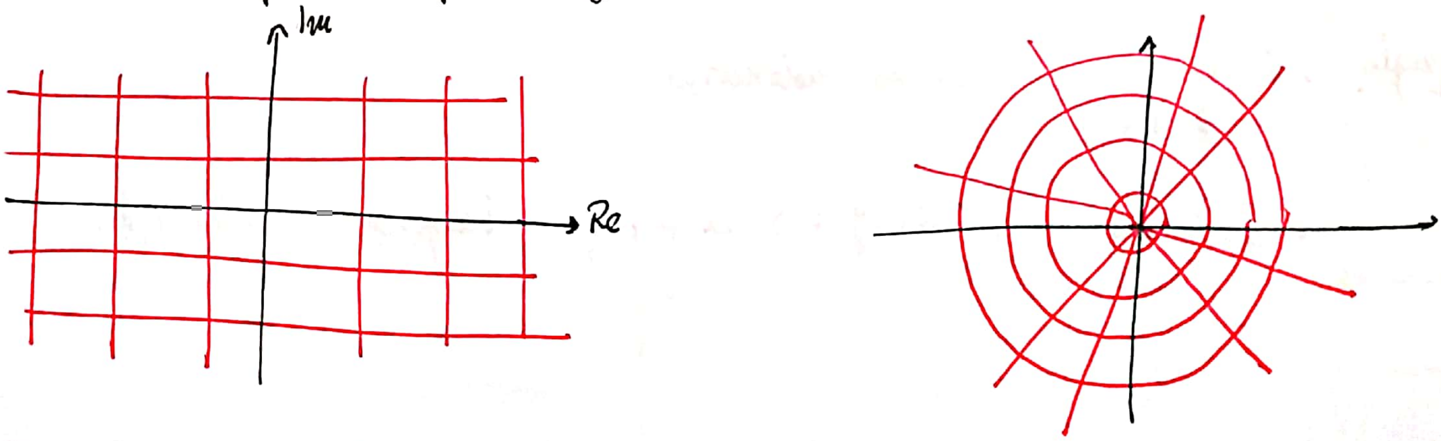
$= x_1x_2 + y_1y_2$

Función exponencial:

$z = x+iy$; $x, y \in \mathbb{R}$

$\exp(z) = e^z = e^x \underbrace{e^{iy}}_{(\cos(y)+i\,\mathrm{sen}(y))}$

tiene la siguiente propiedad geométrica:



obs. Otra definición es $\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Prop. $e^{z+w} = e^z e^w$

—→ —

Consideremos a $\mathbb{C}$ como espacio métrico mediante $d(z,w) = |z-w|$.

Sea $\Omega \subset \mathbb{C}$ abierto,

def. $f : \Omega \to \mathbb{C}$ es llamada holomorfa en el punto $z_0 \in \Omega$ si existe

$$f'(z_0) := \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

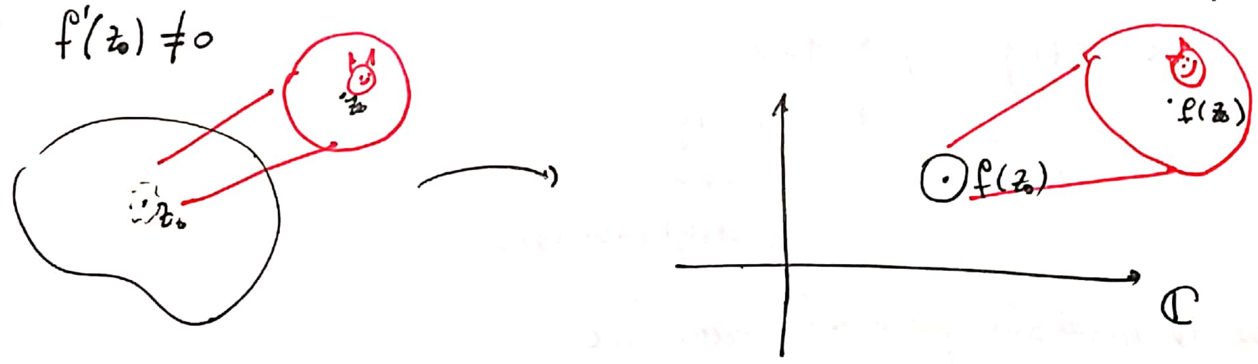
Si $\exists\, f'(z_0)$ para todos los $z_0 \in \Omega$, decimos que f es holomorfa en Ω.

O sea: f es holomorfa en $z_0 \in \Omega$ $\Leftrightarrow$ existen $a, b \in \mathbb{C}$ tq

$$f(z_0+h) = a + bh + o(h)$$

Notación. $o(\cdot)$ es una función $\varphi(h)$ cualquiera tq $\lim_{h \to 0} \frac{|\varphi(h)|}{|h|} = 0$

Si $f'(z_0) \neq 0$



Ejemplo. • $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$ no es holomorfa

• $f : \mathbb{C} \to \mathbb{C}$, $x+iy \mapsto 2x + i\frac{y}{2}$ tampoco es holomorfa.

Multiplicación compleja : $M_z : \mathbb{C} \to \mathbb{C}$

$w \mapsto zw$

$z = a + bi = (a, b)$

$w = x + yi = (x, y)$

$zw = (a+bi)(x+yi) = ax + ayi + bxi - by$

$= ax - by + (ay + bx)i$

$$M_z(w) = M_{\binom{a}{b}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

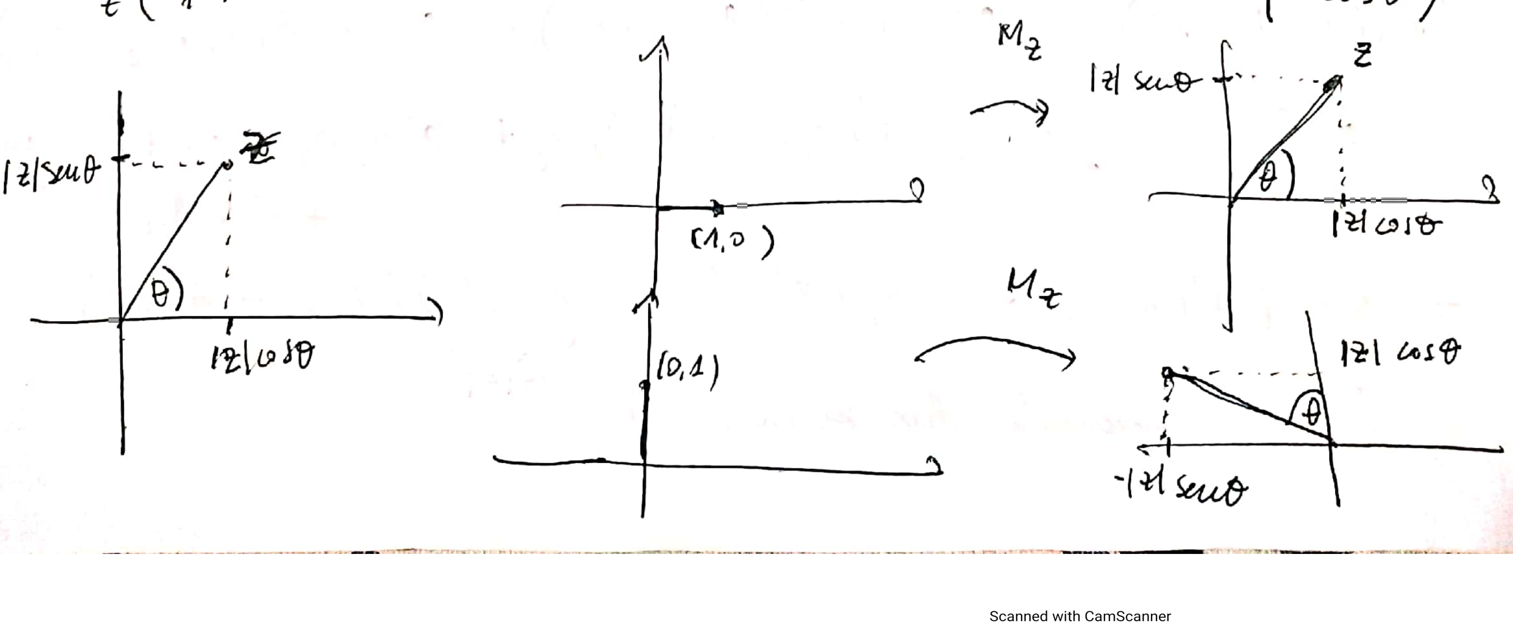
$$\begin{cases} a = |z| \cos\theta \\ b = |z| \operatorname{sen}\theta \end{cases} : \quad M_z(w) = \begin{pmatrix} |z|\cos\theta & -|z|\operatorname{sen}\theta \\ |z|\operatorname{sen}\theta & |z|\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$M_z(w) = |z| \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow M_z(w) = \begin{pmatrix} |z| & \\ & |z| \end{pmatrix} \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$M_z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} |z| & \\ & |z| \end{pmatrix} \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} |z|\cos\theta \\ |z|\operatorname{sen}\theta \end{pmatrix} = |z| \begin{pmatrix} \cos\theta \\ \operatorname{sen}\theta \end{pmatrix}$$

$$M_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} |z| & \\ & |z| \end{pmatrix} \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |z| \begin{pmatrix} -\operatorname{sen}\theta \\ \cos\theta \end{pmatrix}$$

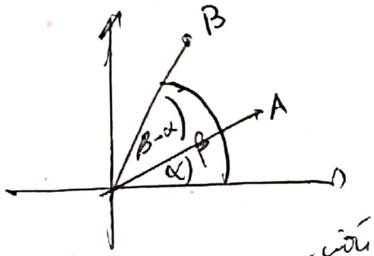


- La multiplicación compleja preserva ángulos y orientación.
- Averiguar qué significa que preserve orientación.

Preservar ángulos:

$$A = |A| e^{i\alpha} = (A_1, A_2)$$
$$B = |B| e^{i\beta} = (B_1, B_2)$$

$$M_z(A) = |z| e^{i\theta} |A| e^{i\alpha} = |z||A| e^{i(\theta+\alpha)}$$
$$M_z(B) = |z| e^{i\theta} |B| e^{i\beta} = |z||B| e^{i(\theta+\beta)}$$

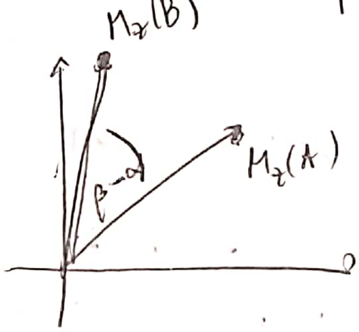


$\arg B = \beta$ | $\arg M_z(B) - \arg M_z(A)$

$\arg A = \alpha$ | $= \theta + \beta - (\theta + \alpha) = \beta - \alpha$

$= \arg B - \arg A$

no es transformación ortogonal (casi).



M_z Preservar ángulos: $\langle M_z(B), M_z(A) \rangle = \langle A, B \rangle$

$$M_z(A) = (a, b)(A_1, A_2) = (aA_1 - bA_2,\ aA_2 + bA_1)$$
$$M_z(B) = (a, b)(B_1, B_2) = (aB_1 - bB_2,\ aB_2 + bB_1)$$

$$\langle M_z(A), M_z(B) \rangle = (aA_1 - bA_2)(aB_1 - bB_2) + (aA_2 + bA_1)(aB_2 + bB_1)$$
$$= a^2 A_1 B_1 - \cancel{abA_1B_2} - \cancel{abA_2B_1} + b^2 A_2 B_2 + a^2 A_2 B_2 + \cancel{abA_2B_1} + \cancel{abA_1B_2} + b^2 A_1 B_1$$

$$= A_1 B_1 (a^2 + b^2) + A_2 B_2 (a^2 + b^2) = \underbrace{(a^2 + b^2)}_{=|z|} (A_1 B_1 + A_2 B_2)$$

(No necesariamente debe ser así...)

Funciones holomorfas:

- $f: \mathbb{C} \to \mathbb{C}$ $\quad z \mapsto \bar{z}$ no es holomorfa.

$r \in \mathbb{R}$ $(\mathbb{R} \times \{0\})$

$$\frac{f(z+r) - f(z)}{r} = \frac{\overline{z+r} - \bar{z}}{r} = \frac{\bar{z} + r - \bar{z}}{r} = 1 \xrightarrow{r \to 0} 1$$

$$\frac{f(z+ri) - f(z)}{ir} = \frac{\overline{z+ri} - \bar{z}}{r} = \frac{\bar{z} - ri - \bar{z}}{r} = -i \xrightarrow{ir \to 0} -i$$

$\therefore$ $f'(z)$ no existe para ningún $z \in \mathbb{C}$.

- $f: \mathbb{C} \to \mathbb{C}$, $\quad x+iy \mapsto 2x + i\frac{y}{2}$ no es holomorfa.

$r \in \mathbb{R}$:

$$\frac{f((x+iy)+(r+0i)) - f(x+iy)}{r+0i} = \frac{f((x+r)+iy) - f(x+iy)}{r+0i}$$

$$= \frac{2(x+r) + i\,y/2 - (2x + i\,y/2)}{r} = \frac{2r}{r} = 2 \xrightarrow{r \to 0} 2$$

$$\frac{f((x+iy)+(0+ir)) - f(x+iy)}{0+ir} = \frac{f(x+(y+r)i) - f(x+iy)}{0+ir}$$

$$= \frac{2x + i\frac{(y+r)}{2} - (2x + i\,y/2)}{ir} = \frac{\frac{ir}{2}}{ir} = \frac{1}{2} \xrightarrow{ir \to 0} \frac{1}{2}$$

$\therefore$ $f'(z)$ no existe para ningún punto de $\mathbb{C}$.

Evaluaciones

I1 : 25/Abril L

I2 : 8/Junio

Ex : 20/Junio.

09/03/2016

Prop. Sean $\Omega \subseteq \mathbb{C}$ abierto. $f, g : \Omega \to \mathbb{C}$ holomorfas en $z_0 \in \Omega$.
entonces:

i) $f+g$ es holomorfa en z_0 y $(f+g)'(z_0) = f'(z_0) + g'(z_0)$

ii) $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$

iii) si $g(z_0) \neq 0$:
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$$

Prop. (Regla de la cadena)

$f : \Omega_1 \to \mathbb{C}$, $g : \Omega_2 \to \mathbb{C}$

Si $\begin{cases} f \text{ es holomorfa en } z_0 \in \Omega_1 \\ g \text{ es holomorfa en } f(z_0) \in \Omega_2 \end{cases}$

entonces $g \circ f$ es holomorfa en z_0 y $(g \circ f)'(z_0) = g'(f(z_0))\, f'(z_0)$

- $\mathbb{C} = \mathbb{R}^2$

$f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ es llamada diferenciable en un punto $z_0 = (x_0, y_0)$ si existe $\exists$ transformación lineal
$$Df(z_0) : \mathbb{R}^2 \to \mathbb{R}^2$$
tal que $f(z_0 + h) = f(z_0) + Df(z_0)(h) + r(h)$

$r(h) = R(h)$, $\displaystyle\lim_{h \to 0} \frac{|R(h)|}{|h|} = 0$

diferenciabilidad en el sentido real.

De la definición de diferenciabilidad

$$f'(z_0) := \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

tenemos $f(z_0+h) = f(z_0) + \boxed{f'(z_0)\,h} + o(h)$

$$h \in \mathbb{C} = \mathbb{R}^2 \xmapsto{\mathbb{R}\text{-lineal}} f'(z_0)\,h \in \mathbb{C} = \mathbb{R}^2$$

$f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ diferenciable en $z_0 = (x_0, y_0)$

$(x,y) \mapsto (u(x,y), v(x,y))$

$$Df(z_0) = \begin{pmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{pmatrix}$$

Si f es holomorfa en z_0 y $f'(z_0) = a+bi$. Entonces

$$h = x+iy \longmapsto f'(z_0)\,h = (ax-by) + (ay+bx)$$

$= (x,y)$

$= \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad \begin{pmatrix} ax-by \\ bx+ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Teo. Sea $\Omega \subseteq \mathbb{C}$ abierto. Sea $f : \Omega \longrightarrow \mathbb{C}$ $\qquad u := Re(f)$, $v := Im(f)$

$$f(x+iy) = u(x,y) + i\,v(x,y) \quad (x, y \in \mathbb{R})$$

Entonces f es holomorfa en $z_0 = x_0 + iy_0 \in \Omega$ ssi

1) f es diferenciable (en el sentido real) en el punto z_0.

2) $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ en $z_0 = (x_0, y_0)$

Ecuaciones de Cauchy-Riemann.

$$Df(z_0)(i) = i \cdot Df(z_0)(1) \quad \text{(Linealidad compleja).}$$

$\downarrow$ producto complejo.

$$Df(z_0)\cdot i = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u_y \\ v_y \end{pmatrix}$$

$$i \cdot Df(z_0)(1) = i \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = i\begin{pmatrix} u_x \\ v_x \end{pmatrix}$$

$$= i(u_x + i v_x) = -v_x + i u_x = \begin{pmatrix} -v_x \\ u_x \end{pmatrix}$$

Ejemplo. $\exp : \mathbb{C} \to \mathbb{C}$, $z = x + iy$

$$\exp(z) = e^z = e^x(\cos(y) + i \,\text{sen}(y))$$

$$\begin{cases} u(x,y) = e^x \cos(y) \\ v(x,y) = e^x \,\text{sen}(y) \end{cases}$$

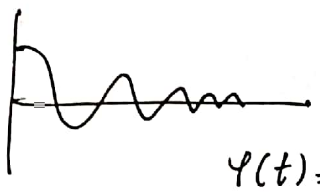
Satisfacen las ecuaciones de Cauchy-Riemann y por lo tanto exp es holomorfa.

Notar que $\exp'(0) = 1$

De lo cual tenemos lo siguiente:

$$\exp'(z) = \lim_{h\to 0} \frac{e^{z+h} - e^z}{h} = e^z \lim_{h \to 0} \frac{e^h - 1}{h} = e^z \cdot 1 = e^z$$

Oscilador armónico amortiguado



$$\varphi(t) = e^{at}\cos(bt) \qquad a<0$$

$$\varphi^{(100)}(t) = ??$$

$$\varphi(t) = Re\left(e^{(a+bi)t}\right)$$

$$\varphi^{(100)}(t) = Re\left(\left(\frac{d}{dt}\right)^{(100)} e^{(a+bi)t}\right) = Re\left((a+bi)^{100} e^{(a+bi)t}\right)$$

<u>Series de potencias</u>.

Sea $(a_n)_{n\in\{0,1,2,\dots\}}$ una sucesión en $\mathbb{C}$

$$\sum_{n=0}^{\infty} a_n := \lim_{N\to\infty} \sum_{k=0}^{N} a_k \quad \text{si el límite existe.}$$

En este caso decimos que la serie es convergente.

- No convergente = divergente.

Decimos que $\sum a_n$ es absolutamente convergente si $\sum |a_n|$ es convergente.

<u>Teo</u>. Si $\sum a_n$ es absolutamente convergente, entonces es convergente.

<u>dem</u>. Ejercicio.

Series del tipo $\sum_{n=0}^{\infty} a_n z^n$, o más generalmente, $\sum_{n=0}^{\infty} a_n (z-a)^n$ son llamadas <u>series de potencias</u>.

Dada una sucesión (a_n) en $\mathbb{C}$, definimos el radio de convergencia

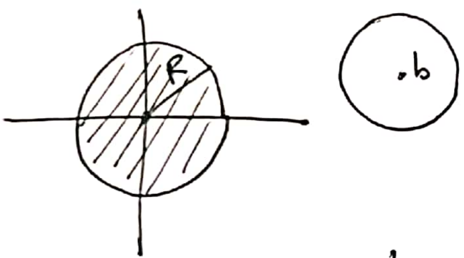
$$R := \left(\limsup_{n\to\infty} |a_n|^{1/n} \right)^{-1}, \qquad R \in [0,\infty]$$

Convenciones: $\begin{cases} 0^{-1} = \infty \\ \infty^{-1} = 0 \end{cases}$

Teo. (a_n) sucesión en $\mathbb{C}$, $z \in \mathbb{C}$. R = radio de convergencia.

(i) Si $|z| < R$, entonces $\sum_{n=0}^{\infty} a_n z^n$ es absolutamente convergente.

(ii) Si $|z| > R$ entonces la serie es divergente



$$\sum_{n=0}^{\infty} a_n (z-b)^n$$

$\{ z \in \mathbb{C} \,/\, |z| < R \}$

es llamado disco de convergencia

dem.

(i) Supongamos que $|z| < R$. Sea r tal que $|z| < r < R$. Entonces

$r^{-1} > R^{-1}$

$$r^{-1} > \limsup_{n\to\infty} |a_n|^{1/n}$$

$$\Rightarrow \exists n_0 \in \mathbb{N} \text{ tq } \forall n > n_0$$

$$r^{-1} > |a_n|^{1/n}$$

$$r^{-n} > |a_n|$$

$$|a_n z^n| = |a_n||z|^n$$
$$< \left(\frac{|z|}{r} \right)^n$$
$\in [0,1)$

$\Downarrow$

$\sum |a_n z^n|$ es convergente (comparación con la serie geométrica)

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Función holomorfa. Diferenciabilidad.

$$f \text{ holomorfa} \Leftrightarrow f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

definiendo $\sigma(h) = f(z_0+h) - f(z_0) - f'(z_0)h$

$$\Rightarrow \lim_{h\to 0} \frac{\sigma(h)}{h} = 0$$

$$\frac{\sigma(h)}{h} = \frac{f(z_0+h) - f(z_0) - f'(z_0)h}{h} = \frac{f(z_0+h) - f(z_0)}{h} - f'(z_0)$$

$$\Rightarrow f(z_0+h) = f(z_0) + f'(z_0)h + \sigma(h)$$

$T: \mathbb{C} \to \mathbb{C}$, $T(h) = f'(z_0)h$ es $\mathbb{R}$-lineal. ($\mathbb{C}$-lineal en ~~partic~~ general)
(visto en $T: \mathbb{R}^2 \to \mathbb{R}^2$)

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \Leftrightarrow \lim_{h\to 0} \frac{f(z_0+h) - f(z_0) - f'(z_0)h}{h} = 0$$

$$\Leftrightarrow \frac{f(z_0+h) - f(z_0) - f'(z_0)h}{h} = \frac{\sigma(h)}{h} \quad \text{tq} \quad \lim_{h\to 0} \frac{\sigma(h)}{h} = 0$$

Podemos considerar $\frac{\sigma(h)}{h} = \varphi(h)$

$\Rightarrow f$ diferenciable en $z = z_0 \in \mathbb{C}$ $\Leftrightarrow \exists a \in \mathbb{C} : f(z_0+h) - f(z_0) = ah + \varphi(h)h$ tq $\lim_{h\to 0} \varphi(h) = 0$.

· Ejercicio, ocupar lo anterior para demostrar regla de la cadena.

Pd: $(f\circ g)'(z_0) = f'(g(z_0))\, g'(z_0)$

$$\cancel{\frac{(f\circ g)(z_0+h) - (f\circ g)(z_0)}{h} = \frac{f(g(z_0+h)) - f(g(z_0))}{h}}$$

$$(f\circ g)(z_0+h) - (f\circ g)(z_0) = f(g(z_0+h)) - f(g(z_0))$$

$$= f\big(g(z_0) + g'(z_0)h + \varphi(h)h\big) - f(g(z_0))$$

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$$\frac{(f\circ g)(z_0+h)-(f\circ g)(z_0)}{h} = \frac{(f\circ g)(z_0+h)-(f\circ g)(z_0)}{g(z_0+h)-g(z_0)}\cdot\frac{g(z_0+h)-g(z_0)}{h}$$

~~f es diferencia~~

f es diferenciable en $g(z_0)$:

$$f(g(z_0)+h)-f(g(z_0)) = f'(g(z_0))h+\varphi(h)h$$

$$\cancel{\Rightarrow f(g(z_0)+h)g'(z_0) - f(g(z_0))g'(z_0) = f'(g(z_0))g'(z_0)h + g'(z_0)\varphi(h)h}$$

$$f(g(z_0+h))-f(g(z_0)) = f(g(z_0)+\underbrace{g'(z_0)h+\varepsilon(h)h}_{k_h})-f(g(z_0))$$

$\hookrightarrow =$

$$\Rightarrow f(g(z_0)+k_h)-f(g(z_0)) = f'(g(z_0))k_h+\varphi(k_h)k_h$$

$$= f'(g(z_0))\left[g'(z_0)h+\varepsilon(h)h\right]+\varphi(k_h)k_h$$

$$= f'(g(z_0))\left[g'(z_0)h+\varepsilon(h)h\right]+\varphi(g'(z_0)h+\varepsilon(h)h)\left[g'(z_0)h+\varepsilon(h)h\right]$$

$$= f'(g(z_0))g'(z_0)h+\Big[\underbrace{f'(g(z_0))\varepsilon(h)+\varphi(g'(z_0)h+\varepsilon(h)h)(g'(z_0)+\varepsilon(h))}_{\psi(h)}\Big]h$$

$$\lim_{h\to 0}\psi(h)=0.$$

$$\therefore (f\circ g)'(z_0) = f'(g(z_0))g'(z_0)$$

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Función exponencial.

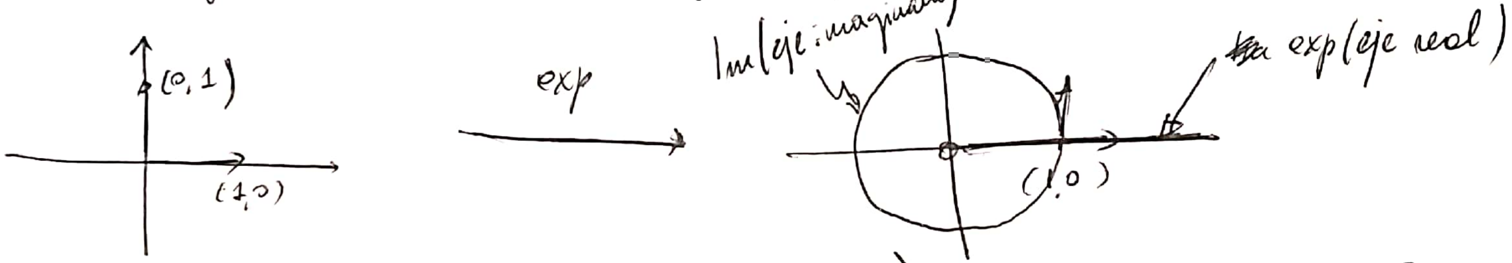
$$\exp : \mathbb{C} \to \mathbb{C}$$
$$z \mapsto \exp(z)$$

$z = x+iy$: $\exp(z) = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\,\text{sen}(y))$
$$= e^x \cos(y) + i\, e^x \text{sen}(y)$$

$$\begin{cases} u(x,y) = e^x \cos(y) \\ v(x,y) = e^x \text{sen}(y) \end{cases} \Rightarrow \begin{cases} u_x = e^x \cos(y) & v_x = e^x \text{sen}(y) \\ u_y = -e^x \text{sen}(y) & v_y = e^x \cos(y) \end{cases}$$

derivadas parciales continuas
$\Rightarrow$ exp diferenciable en el sentido real.

Cauchy - Riemann : $u_x = v_y$, $u_y = -v_x$ $\Rightarrow$ exp holomorfa.



$$(\exp'(x,y))_{\mathcal{C}} = \begin{pmatrix} e^x \cos(y) & -e^x \text{sen}(y) \\ e^x \text{sen}(y) & e^x \text{sen}(y) \end{pmatrix} \qquad \mathcal{C} = \{(1,0), (0,1)\}$$

$$\Rightarrow (\exp'(0))_{\mathcal{C}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \therefore \ \exp'(0) = 1$$

$z = a+bi$: $\exp'(0)\, z = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = z$ ∎

Shakarchi: Complex-valued functions as mappings.

Conexión:

Versión compleja — Versión $\mathbb{R}^2$

$$f: \Omega \to \mathbb{C} \qquad\qquad F: \Omega \to \mathbb{R}^2$$
$$z \mapsto f(z) \qquad\qquad (x,y) \mapsto (u(x,y), v(x,y))$$

$F: \Omega \to \mathbb{R}^2$ diferenciable en $P_0 = (x_0, y_0)$ ssi $\exists\, J: \mathbb{R}^2 \to \mathbb{R}^2$ lineal tq

$$\frac{|F(P_0+h) - F(P_0) - J(h)|}{|h|} \to 0 \quad \text{cuando } |h| \to 0,\ h \in \mathbb{R}^2$$

equiv: $F(P_0+h) - F(P_0) = J(h) + |h|\,\Psi(h)$, con $|\Psi(h)| \to 0$ cuando $|h| \to 0$

- J tiene representación matricial $J = J_F(x,y) = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}$.

Versión real $\mathbb{R}^2$.

- Versión $\mathbb{C}$.

f diferenciable en $z_0 \in \Omega$ ssi $\exists\ f'(z_0) = \lim_{h \to 0} \dfrac{f(z_0+h) - f(z_0)}{h}$.

- Supongamos que $h = h_1 + ih_2$, $h_2 = 0$:

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(z_0+h_1) - f(z_0)}{h_1}$$
$$= \lim_{h_1 \to 0} \frac{f(x_0+h_1+iy_0) - f(x_0+iy_0)}{h_1} = \frac{\partial f}{\partial x}$$

Sup. $h_1 = 0$:
$$f'(z_0) = \lim_{ih_2 \to 0} \frac{f(z_0+ih_2) - f(z_0)}{ih_2} = \lim_{ih_2 \to 0} \frac{f(x_0+(y_0+h_2)i) - f(x_0+iy_0)}{ih_2}$$
$$= \frac{1}{i}\frac{\partial f}{\partial y}$$

Por lo tanto: $\dfrac{\partial f}{\partial x} = \dfrac{1}{i}\dfrac{\partial f}{\partial y}$ | $f = u + iv$: $\dfrac{\partial f}{\partial x} = -i\dfrac{\partial f}{\partial y}$

$$\Rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

o.i. $\boxed{\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2mm] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}}$

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Trabajamos con los siguientes operadores diferenciales:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

- f holomorfa en $z_0 = (x_0, y_0)$, $f(z_0) = u(z_0) + iv(z_0)$.

$$\frac{\partial}{\partial z} f(z_0) = \frac{1}{2}\left(\frac{\partial}{\partial x} f(z_0) - i\frac{\partial}{\partial y} f(z_0)\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} u(z_0) + i\frac{\partial}{\partial x} v(z_0) - i\frac{\partial}{\partial y} u(z_0) + \frac{\partial}{\partial y} v(z_0)\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} u(z_0) + i\frac{\partial}{\partial x} v(z_0) + i\frac{\partial}{\partial x} v(z_0) + \frac{\partial}{\partial x} u(z_0)\right)$$

$$= \frac{\partial}{\partial x} u(z_0) + i\frac{\partial}{\partial x} v(z_0) = \frac{\partial}{\partial x} f(z_0) = f'(z_0)$$

$$\therefore \quad f'(z_0) = \frac{\partial}{\partial z} f(z_0)$$

$$\boxed{\begin{aligned}\frac{\partial}{\partial z} f(z_0) &= \frac{1}{2}\left(2\frac{\partial}{\partial x} u(z_0) - 2i\frac{\partial}{\partial y} v(z_0)\right)\\ &= 2\frac{\partial}{\partial z} u(z_0).\end{aligned}}$$

$$\frac{\partial}{\partial \bar{z}} f(z_0) = \frac{1}{2}\left(\frac{\partial}{\partial x} f(z_0) + i\frac{\partial}{\partial y} f(z_0)\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} u(z_0) + i\frac{\partial}{\partial x} v(z_0) + i\frac{\partial}{\partial y} u(z_0) - \frac{\partial}{\partial y} v(z_0)\right)$$

$$= \frac{1}{2}\left(\cancel{\frac{\partial}{\partial x} u(z_0)} + \cancel{i\frac{\partial}{\partial x} v(z_0)} - \cancel{i\frac{\partial}{\partial x} v(z_0)} - \cancel{\frac{\partial}{\partial x} u(z_0)}\right)$$

$$= 0$$

$$\therefore \quad \frac{\partial}{\partial \bar{z}} f(z_0) = 0.$$

$$\det J(x_0, y_0) = \det\begin{pmatrix} \frac{\partial}{\partial x} u(z_0) & \frac{\partial}{\partial y} u(z_0) \\ \frac{\partial}{\partial x} v(z_0) & \frac{\partial}{\partial y} v(z_0)\end{pmatrix} = \frac{\partial}{\partial x} u(z_0)\frac{\partial}{\partial y} v(z_0) - \frac{\partial}{\partial x} v(z_0)\frac{\partial}{\partial y} u(z_0)$$

$$= \frac{\partial}{\partial x} u(z_0)\frac{\partial}{\partial x} u(z_0) + \frac{\partial}{\partial y} u(z_0)\frac{\partial}{\partial y} u(z_0) = \left(\frac{\partial}{\partial x} u(z_0)\right)^2 + \left(\frac{\partial}{\partial y} u(z_0)\right)^2$$

$$= \left(\frac{\partial}{\partial x} u(z_0) + i\frac{\partial}{\partial y} u(z_0)\right)\left(\frac{\partial}{\partial x} u(z_0) - i\frac{\partial}{\partial y} u(z_0)\right) = \left(2\frac{\partial}{\partial \bar{z}} u(z_0)\right)\left(2\frac{\partial}{\partial z} u(z_0)\right) = \left|2\frac{\partial}{\partial z} u(z_0)\right|$$

$$\therefore \det J(x_0, y_0) = |f'(z_0)|^2 .$$

- f holomorfa en $z_0 \Rightarrow F(x,y) = f(z)$, F es diferenciable en el sentido real y $\det J_F(x_0, y_0) = |f'(z_0)|^2$.

Para demostrar que F es diferenciable basta observar que si ~~...~~ $H = (h_1, h_2)$ $h = h_1 + ih_2$:

$$J_F(x_0, y_0)(H) = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) = f'(z_0)h$$

$$J_F(x_0, y_0) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 \\ -\frac{\partial u}{\partial y}h_1 + \frac{\partial u}{\partial x}h_2 \end{pmatrix}.$$

$$\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) = \frac{\partial u}{\partial x}h_1 + i\frac{\partial u}{\partial x}h_2 - i\frac{\partial u}{\partial y}h_1 + \frac{\partial u}{\partial y}h_2$$

$$= \left(\frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2\right) + i\left(-\frac{\partial u}{\partial y}h_1 + \frac{\partial u}{\partial x}h_2\right)$$

$$\therefore J_F(x_0, y_0)(H) = f'(z_0)h$$

Th. (2.4) $f = u + iv$ definida en $\Omega \subseteq \mathbb{C}$. u, v continuamente diferenciable y satisfacen Cauchy-Riemann en Ω, entonces f es holomorfa en Ω y $f'(z) = \frac{\partial}{\partial z} f(z)$.

dem: u, v continuamente diferenciables, $h = (h_1, h_2)$

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\Psi_1(h)$$

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\Psi_2(h)$$

$\Psi_j(h) \to 0$, $h \to 0$.

$$f(z+h) - f(z) = u(z+h) + iv(z+h) - u(z) - iv(z)$$

$$= u(z+h) - u(z) + i\left(v(z+h) - v(z)\right)$$

$$= u(x+h_1, y+h_2) - u(x,y) + i\left(v(x+h_1, y+h_2) - v(x,y)\right)$$

$$\cancel{= \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + i\Big(\frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2}$$

$$= \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\psi_1(h) + i\left(\frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\psi_2(h)\right)$$

$$= \frac{\partial u}{\partial x}h_1 + i\frac{\partial v}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + i\frac{\partial v}{\partial y}h_2 + |h|\left(\psi_1(h) + i\psi_2(h)\right)$$

$$\cancel{= \frac{\partial f}{\partial x}h_1 + \frac{\partial f}{\partial y}h_2 + |h|\left(\psi_1 + i\psi_2(h)\right)}$$

$$\cancel{= \frac{\partial u}{\partial x}h_1 - i\frac{\partial u}{\partial y}h_2 + i\frac{\partial u}{\partial x}h_1 + i\frac{\partial u}{\partial x}h_2}$$

$$= \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + i\frac{\partial v}{\partial x}h_1 + i\frac{\partial v}{\partial y}h_2 + |h|\left(\psi_1(h) + i\psi_2(h)\right) \quad \Big| \quad \psi(h) := \psi_1(h) + i\psi_2(h)$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)h_1 + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)h_2 + |h|\left(\psi(h)\right)$$

$$= \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)h_1 + i\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)h_2 + \psi(h)|h| = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)h_1 + i\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)h_2 + \psi(h)|h|$$

$$= \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) + \psi(h)|h| = 2\left(\frac{\partial}{\partial z}u(z)\right)h + |h|\psi(h)$$

$$\left(\psi(h) \to 0,\ h \to 0\right)$$

$$\therefore\ f'(z) = 2\frac{\partial u(z)}{\partial z} = \frac{\partial f}{\partial z}(z).$$

Series de Potencias

Teo. $\sum a_n$ absolutamente convergente $\Rightarrow$ $\sum a_n$ convergente.

dem. Sea $S_N = \sum_{k=0}^{N} a_k$

$$|S_N - S_M| = \left| \sum_{k=0}^{N} a_k - \sum_{k=0}^{M} a_k \right| \leq \sum_{k=0}^{N} |a_k| - \sum_{k=0}^{M} |a_k| \xrightarrow{N,M\to\infty} 0$$

$\therefore$ (S_N) sucesión de Cauchy

$\therefore$ $\sum a_n$ convergente.

lim sup: $\limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \sup_{m\geq n} |a_m|^{1/m}$

$$R > \limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \sup_{n\geq m} |a_m|^{1/m}$$

$\Rightarrow \exists n_0 \in \mathbb{N}$ tq $\forall n \geq n_0$: $R > \sup_{m\geq n} |a_m|^{1/m} \geq |a_m|^{1/m}$ $\forall m \geq n$

$\therefore$ $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$: $R > |a_n|^{1/n}$

(Stein - Shakarchi). R radio de convergencia de $\sum_{n=0}^{\infty} a_n z^n$

$\Rightarrow$ $|z| < R$ converge absolutamente.

dem. Supongamos $R \neq 0, \infty$. $|z| < R \Rightarrow \frac{|z|}{R} < 1$

$L := \frac{1}{R}$ se tiene : $(L+\varepsilon)|z| = L|z| + \varepsilon|z| = \frac{|z|}{R} + \varepsilon|z| < \frac{|z|}{R} + \varepsilon R$, $\varepsilon > 0$.

Podemos considerar $\varepsilon > 0$ suf. pequeño tq : $\underbrace{(L+\varepsilon)}_{r}|z| < 1$

Como $L \leq L+\varepsilon \Rightarrow \limsup |a_n|^{1/n} \leq L + \varepsilon$

$\Rightarrow \exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$: $|a_n|^{1/n} \leq L + \varepsilon$

$\Rightarrow \quad |a_n||z|^n \leq (L+\varepsilon)^n |z^n| = r^n$

$$\therefore \quad \sum_0^\infty |a_n z^n| \leq \underbrace{\sum_0^\infty r^n}_{\text{serie geométrica convergente } (r<1)}$$

$\therefore \quad \sum_0^\infty a_n z^n$ absolutamente convergente en $|z|<R$.

Pd: $\sum_0^\infty a_n z^n$ diverge en $|z|>R$.

Estábamos estudiando las series de potencias $f(z) = \sum_{n=0}^{\infty} a_n (z-b)^n$.

Converge absolutamente en el disco $B(b, R)$, donde $R^{-1} = \limsup_{n\to\infty} |a_n|^{1/n}$

Además, la convergencia es uniforme en todo compacto $K \subset B(b,R)$

Ejemplo.

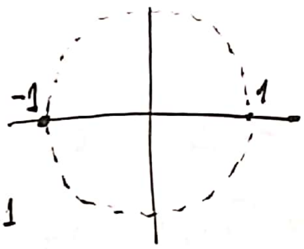
(1) $\sum n!\, z^n$ tiene radio de convergencia $R=0$.

$$\lim_{n\to\infty} (n!)^{1/n} = \infty.$$

(2) Para $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, $R = \infty$.

(3) Para $\sum_{n=0}^{\infty} z^n$, $R = 1$. $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ si $|z|<1$

(4) Para $\sum_{n=1}^{\infty} \frac{z^n}{n}$, $R=1$ $\Big|$ $\lim_{n\to\infty} n^{1/n} = 1 \iff \lim_{n\to\infty} \frac{\log n}{n} = 0$



$z=-1$

$\sum \frac{(-1)^n}{n}$ conv. condicionalmente $\Big|$ $z=1$ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge.

Teo. Las series de potencias se pueden derivar término a término en el disco de convergencia. O sea

Si $f(z) = \sum_{n=0}^{\infty} a_n (z-b)^n$, $z \in B(b,R)$, entonces $\exists f'(z) = \sum_{n=1}^{\infty} n a_n (z-b)^{n-1}$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} (z-b)^n$$

Además, el radio de convergencia de esta serie también es R.

dem. SPG: $b=0$. Para $z \in B(0,R)$ sea:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \underbrace{\sum_{n=0}^{N-1} a_n z^n}_{f_N(z)} + K_N(z)$$

$$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = g_N(z) + L_N(z)$$

Fijemos $z \in B(0,R)$

Fijemos r tal que $0<r<R$

Consideremos $h \neq 0$ tq $|z+h|<r$

$$\frac{f(z+h)-f(z)}{h} = \underbrace{\frac{f_N(z+h)-f_N(z)}{h}}_{\text{I}} + \underbrace{\frac{K_N(z+h)-K_N(z)}{h}}_{\text{II}}$$

Hecho general: $\sum a_n$, $\sum b_n$ absolutamente convergentes $\to \sum (a_n+b_n)$ es absolutamente convergente y $\sum (a_n+b_n) = \sum a_n + \sum b_n$.

$$|\text{II}| \leq \sum_{n=N}^{\infty} \frac{|a_n (z+h)^n - a_n z^n|}{|h|} = \sum_{n=N}^{\infty} |a_n| \left| \underbrace{\frac{(z+h)^n - z^n}{h}}_{\leq n r^{n-1}} \right| \leq \sum_{n=N}^{\infty} |a_n| n r^{n-1}$$

$$\frac{a^n-b^n}{a-b} = |a^{n-1} + a^{n-2}b + \dots + b^{n-1}|$$

$$\leq n(\max\{|a|,|b|\})^{n-1}$$

fijemos $N \in \mathbb{N}$ tal que

1) $\sum_{n=N}^{\infty} |a_n| n r^{n-1} < \varepsilon$

2) $|L_N(z)| < \varepsilon$

Entonces, si $|h|$ es suficientemente pequeño

$$\Bigg| \underbrace{\frac{f(z+h)-f(z)}{h}}_{I + II} - \underbrace{g(z)}_{g_N(z)+L_N(z)} \Bigg| \leq \underbrace{|I - g_N(z)|}_{<\varepsilon} + \underbrace{|II|}_{<\varepsilon} + \underbrace{|L_N(z)|}_{<\varepsilon} < 3\varepsilon \quad \blacksquare$$

$$I \xrightarrow[h\to 0]{} f_N'(z) = g_N(z)$$

Def. $\Omega \subset \mathbb{C}$ abierto. Una función $f: \Omega \to \mathbb{C}$ es llamada analítica en el punto $b \in \Omega$ si $\exists (a_n)$ tq la serie de potencias $\sum_{n=0}^{\infty} a_n (z-b)^n$ converge $\forall z \in V$ = alguna vecindad de b, y es $= f(z)$ $\forall z \in V \cap \Omega$

Def. $f: \Omega \to \mathbb{C}$ es llamada analítica si lo es en todo Ω.

Corolario del teorema: Si f es analítica en el punto b, entonces es holomorfa en una vecindad de b.

- Más adelante veremos la recíproca.

Hecho. $a_n = \frac{1}{n!} f^{(n)}(b)$ (Fórmula de Taylor) (Demostrar como ejercicio).

—o—

Series de Fourier. Sea $g: \mathbb{R} \to \mathbb{R}$ función C^2 2π-periódica

Entonces

$$g(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos(n\theta) + B_n \operatorname{sen} \quad)$$

convergencia absoluta y uniforme.

donde
$$A_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos(n\theta)\, d\theta$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \operatorname{sen}(n\theta)\, d\theta$$

Sea f analítica en 0

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad R > 1 \qquad a_n := \alpha_n + i\beta_n$$

$\alpha_n, \beta_n \in \mathbb{R} \quad \forall n.$

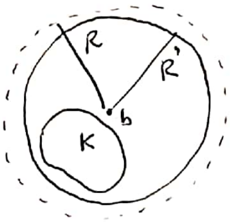
$$\begin{aligned} g(\theta) &= \mathrm{Re}\left(f(e^{i\theta})\right) \\ &= \mathrm{Re}\left(\sum_{n=0}^{\infty} a_n (e^{i\theta})^n\right) \\ &= \mathrm{Re}\left(\sum_{n=0}^{\infty} (\alpha_n + i\beta_n)(\cos n\theta + i \operatorname{sen} n\theta)\right) = \underbrace{\sum_{n=0}^{\infty} (\alpha_n \cos n\theta - \beta_n \operatorname{sen} n\theta)}_{\text{serie de Fourier}} \end{aligned}$$

obs. Los coeficientes A_n, B_n son únicos.

Series de Potencias.

$$f(z)=\sum_{n=0}^{\infty} a_n (z-b)^n \ , \quad R^{-1}=\limsup_{n\to\infty} |a_n|^{1/n}$$

Af. $f(z)=\sum_{n=0}^{\infty} a_n (z-b)^n$ uniforme en $K\subset B(b,R)$ compacto.



$\forall K\subseteq B(b,R)$ compacto, $\exists\, 0<R'<R$ tq $K\subset B(b,R')$

$0<R'<R$.

(s.p.g. ; $b=0$) $\quad |z|\leq R'$

$R'<R \Rightarrow R^{-1}<R'^{-1} \Rightarrow \limsup |a_n|^{1/n} < R'^{-1}$

$\Rightarrow \exists n_0\in\mathbb{N}$ tq $\forall n\geq n_0: \ |a_n|^{1/n} < R'^{-1} \ (\Leftrightarrow \ |a_n|<R'^{-n})$

$\Rightarrow |a_n||z|^n < R'^{-n}|z|^n$. $\qquad |z|^n\leq R'^n$

$|z|\leq R' \Rightarrow \dfrac{|z|}{R}\leq \dfrac{R'}{R}<1$

Debemos considerar $0<R'<\rho<R \quad \Rightarrow \ 0<R^{-1}<\rho^{-1}<R'^{-1}$

$\Rightarrow \limsup |a_n|^{1/n}<\rho^{-1} \rightarrow \exists n_0\in\mathbb{N},\ \forall n\geq n_0: \ |a_n|^{1/n}<\rho^{-1}$

$(\Rightarrow) \qquad$ " $\quad : \ |a_n|<\rho^{-n}$

para $|z|\leq R'$: $\ |a_n||z|^n<\left(\dfrac{R'}{\rho}\right)^n<1 \quad \forall n\geq n_0$

$\therefore$ $\qquad$ Y además $\displaystyle\sum_{n=1}^{\infty}\left(\frac{R}{\rho}\right)^n<\infty$

M-test Weierstrass : $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ uniformemente convergente en $|z|\leq R'$

$\therefore \displaystyle\sum_{n=0}^{\infty} a_n z^n$ converge uniformemente en el compacto K.

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(Conway - Shakarchi - De Silva).

- $f(z)=\sum_{n=0}^{\infty} a_n z^n$ con radio de convergencia $R \Rightarrow g(z)=\sum_{n=0}^{\infty} n a_n z^{n-1}$ es tal que f es holomorfa y $f'(z)=g(z)$. Radio de conv. de g $= R$.

— o —

$$f(z)=\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$$

$$f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \dots = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

R' = rad. de convergencia de $f'(z) \Rightarrow R'^{-1} = \limsup |(n+1) a_{n+1}|^{1/n}$

$\lim_{n\to\infty} (n+1)^{1/n} = 1$

Pd: $\limsup |a_{n+1}|^{1/n} = R$

$\to R^{-1} = \limsup |a_{n+1}|^{1/n}$ radio de convergencia de $\sum_{n=0}^{\infty} a_{n+1} z^n$

$$\sum_{n=0}^{\infty} a_{n+1} z^n = \frac{1}{z} \sum_{n=0}^{\infty} a_{n+1} z^{n+1} = \frac{1}{z}\left(\sum_{n=1}^{\infty} a_n z^n + a_0 - a_0\right) = \frac{1}{z}\sum_{n=0}^{\infty} a_n z^n - \frac{a_0}{z}$$

$$= \frac{1}{z}\left(\sum_{n=0}^{\infty} a_n z^n - a_0\right)$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n = z \sum_{n=0}^{\infty} a_{n+1} z^n + a_0$$

$|z|<R'$: $\sum_{n=0}^{\infty} |a_n z^n| = |z| \sum_{n=0}^{\infty} |a_{n+1} z^n| + |a_0| < \infty$ $\therefore R' \leq R$

$|z|<R$: $\sum_{n=0}^{\infty} |a_{n+1} z^n| = \frac{1}{|z|} \sum_{n=0}^{\infty} |a_n z^n| + \frac{|a_0|}{|z|} < \infty$ $\therefore R \leq R'$
$z \neq 0$

$\therefore R = R'$

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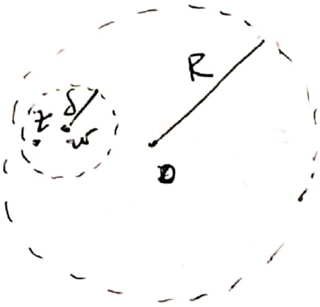
$$f(z)=\sum_{n=0}^{\infty} a_n z^n \quad , \quad g(z)=\sum_{n=1}^{\infty} n a_n z^{n-1}$$

Pd: f holomorfa y $f'(z)=g(z)$

Debemos estudiar $\dfrac{f(z)-f(w)}{z-w}-g(w)$

para cuando $z \to w$



$$\overline{B(w,\delta)} \subseteq B(0,R)$$

$$f(z)=\underbrace{\sum_{n=0}^{N} a_n z^n}_{S_N}+\underbrace{\sum_{n=N+1}^{\infty} a_n z^n}_{T_N}$$

$$\frac{f(z)-f(w)}{z-w}-g(w)=\frac{S_N(z)+T_N(z)-S_N(w)-T_N(w)}{z-w}-g(w)$$

$$=\frac{S_N(z)-S_N(w)}{z-w}+\frac{T_N(z)-T_N(w)}{z-w}-g(w)$$

$$=\frac{S_N(z)-S_N(w)}{z-w}-S_N'(w)+\frac{T_N(z)-T_N(w)}{z-w}+S_N'(w)-g(w)$$

$$\frac{T_N(z)-T_N(w)}{z-w}=\frac{\sum_{n=N+1}^{\infty} a_n z^n-\sum_{n=N+1}^{\infty} a_n w^n}{z-w}=\sum_{k=N+1}^{\infty} a_n\left(\frac{z^k-w^k}{z-w}\right)$$

$$\frac{z^k-w^k}{z-w}=z^{k-1}+z^{k-2}w^1+z^{k-3}w^2+\dots+z^2w^{k-3}+zw^{k-2}+w^{k-1}$$

$z\in \overline{B(w,\delta)}$. Ahora, fijado $w\in B(0,R)$, fijamos $r>0$ tq $0<r<R$ y $|w|<r$. Además $\overline{B(w,\delta)}\subset B(0,r)$

$$\therefore \left|\frac{z^k-w^k}{z-w}\right|=\left|z^{k-1}+z^{k-2}w+z^{k-3}w^2+\dots+z^2w^{k-3}+zw^{k-2}+w^{k-1}\right|$$

$$\leq |z^{k-1}|+|z^{k-2}w|+\dots+|zw^{k-2}|+|w^{k-1}|$$

$$\leq kr^{k-1}$$

$$\left|\frac{T_N(z)-T_N(w)}{z-w}\right| = \left|\sum_{k=N+1}^{\infty} a_k\left(\frac{z^k-w^k}{z-k}\right)\right| \leq \sum_{k=N+1}^{\infty} |a_k|\left|\frac{z^k-w^k}{z-w}\right|$$

$\leq \sum_{k=N+1}^{\infty} |a_k| k r^{k-1}$. Como $g(z)=\sum_{n=1}^{\infty} n a_n z^{n-1}$ tiene

radio de convergencia R y $r<R \Rightarrow \exists N_0 \in \mathbb{N}$ tq $\forall n \geq N_0$

$\sum_{k=N+1}^{\infty} |a_k| k r^{k-1} < \varepsilon$ (Previo fijar $\varepsilon>0$).

$S_N'(w)-g(w) = \sum_{k=1}^{N} k a_k w^{k-1} - \sum_{k=1}^{\infty} k a_n w^{k-1}$. Como $\lim_N S_N'(w)=g(w)$

Existe $N_1 \in \mathbb{N}$ tq $\forall n \geq N_1$: $|S_N'(w)-g(w)|<\varepsilon$

Tomamos $N_2 = \max\{N_1, N_2\}$ para estudiar $\left|\frac{S_N(z)-S_N(w)}{z-w} - S_N'(w)\right|$

Como $S_N(z)$ holomorfa y ~~$S_N'(w) = S_N'$~~ $\lim_{z\to w} \frac{S_N(z)-S_N(w)}{z-w} = S_N'(w)$

existe $\delta>0$ tal que $\forall z$, $0<|z-w|<\delta$

$$\left|\frac{S_N(z)-S_N(w)}{z-w} - S_N'(w)\right| < \varepsilon$$

$\therefore$ $\forall z$, $0<|z-w|<\delta$ se tiene que

$$\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| \leq \left|\frac{S_N(z)-S_N(w)}{z-w} - S_N'(w)\right| + \left|\frac{T_N(z)-T_N(w)}{z-w}\right|$$

$$+ |S_N'(w)-g(w)|$$

$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

$\therefore$ f holomorfa y $f'(z)=g(z)$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad , \quad g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = f'(z).$$

Inductivamente :

$$f''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$$

$$f'''(z) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n z^{n-3}$$

$\vdots$

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k) a_n z^{n-k}$$

$\Rightarrow \quad f(0) = a_0$

$f'(0) = a_1 = 1! a_1$

$f^{(2)}(0) = 2 \cdot 1 \, a_2 = 2! a_2$

$f^{(3)}(0) = 3 \cdot 2 \cdot 1 \, a_3 = 3! a_3$

$\Rightarrow \quad f^{(k)}(0) = k! a_k$

$$\boxed{\therefore \; a_k = \frac{f^{(k)}(0)}{k!}}$$