17/03/2016

Bibliografía: Stein - Shakarchi. Complex analysis.
Ayudante: Nathan Aguirre.

— o —

El cuerpo de los números complejos.

$\mathbb{R}^2 = \{(x,y) ; x,y \in \mathbb{R}\}$

$$(x_1,x_2)+(y_1,y_2) = (x_1+y_1, x_2+y_2)$$
$$(x_1,x_2)\cdot(y_1,y_2) = (x_1y_1 - x_2y_2,\ x_1y_2 + x_2y_1)$$

con neutros $(0,0)$ y $(1,0)$;

$$(x,y)^{-1} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

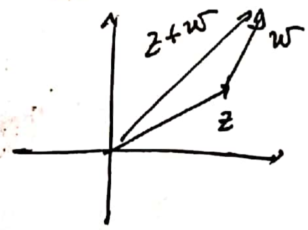
$\{(x,0)\} \cong \mathbb{R}$. Identificamos $x=(x,0)$, $i=(0,1)$

$$i^2=-1$$
$$(x,y) = x(1,0)+y(0,1) = x+iy$$

Teo (Frobenius). Dado $n \in \mathbb{N}$, existe multiplicación en $\mathbb{R}^n$ que (junto con la suma usual) define estructura de cuerpo ssi $n=1$ o $n=2$.

Obs. Si en la def de cuerpo quitamos la propiedad $z_1z_2 = z_2z_1$, entonces $\exists$ producto en $\mathbb{R}^4$ ($\mathbb{R}^4 = \mathbb{H}$ = cuaterniones).

Visualización de los complejos:



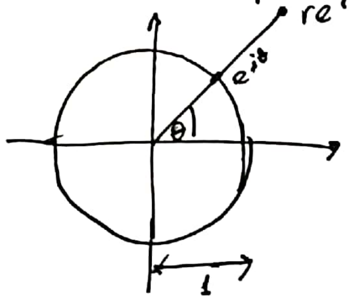
$z = x+iy$, $x,y \in \mathbb{R}$

$|z| := \sqrt{x^2+y^2}$

Si $\theta \in \mathbb{R}$, definimos $e^{i\theta} := \cos\theta + i \operatorname{sen}\theta$

$$|e^{i\theta}| = 1$$

En coordenadas polares: $z = re^{i\theta}$

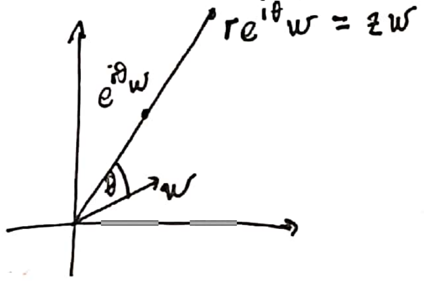


multiplicación en coordenadas polares:

$$\begin{cases} z_1 = r_1 e^{i\theta_1} \\ z_2 = r_2 e^{i\theta_2} \end{cases} \Rightarrow z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Consecuencia: Sea $z = re^{i\theta}$. La aplicación $M_z : \mathbb{C} \longrightarrow \mathbb{C}$, $w \longmapsto zw$

$$M_z = \begin{pmatrix} \text{rotación} \\ \text{de ángulo } \theta \end{pmatrix} \circ \begin{pmatrix} \text{homotecia} \\ \text{de factor } r \end{pmatrix}$$



En particular, M_z es:

- $\mathbb{R}$-lineal
- preserva la orientación de $\mathbb{R}^2$
- preserva ángulos (transformación conforme).

Para $z = x+iy$ definimos el conjugado por $\bar{z} = x - iy$.

Propiedad: $z\bar{z} = x^2+y^2 = |z|^2$; $z^{-1} = \dfrac{\bar{z}}{|z|^2}$

$\therefore |z| = \sqrt{z\bar{z}}$

Hecho: La conjugación $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$ es un automorfismo de $\mathbb{C}$, es decir, $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\,\bar{w}$.

Prop. Si $f : \mathbb{C} \to \mathbb{C}$ es automorfismo de $\mathbb{C}$ y además es continuo, entonces $f = id$ ó $f =$ conjugación.

dem. $f(0)=0$, $f(1)=1$

$f(-z) = -f(z)$

Se tiene fácilmente que $f|_{\mathbb{Q}} = id_{\mathbb{Q}}$. Por continuidad, $f|_{\mathbb{R}} = id_{\mathbb{R}}$.

$f(i) = ?$

$f(i^2) = f(i)^2 = f(-1) = -1 \Rightarrow f(i) \in \{i, -i\}$

$f(i) = i$: $f(x+iy) = f(x) + i f(y) = x+iy$

$f(i) = -i$ $f(x+iy) = x - iy$

Para $z = x+iy$, $\begin{cases} Re(z) = x \\ Im(z) = y \end{cases}$, $\begin{cases} Re(z) = \dfrac{z+\bar{z}}{2} \\ Im(z) = \dfrac{z-\bar{z}}{2i} \end{cases}$

Producto interno : $\langle z, w \rangle = Re(z\bar{w})$

En efecto : $Re(z\bar{w}) = Re((x_1+iy_1)(x_2-iy_2))$

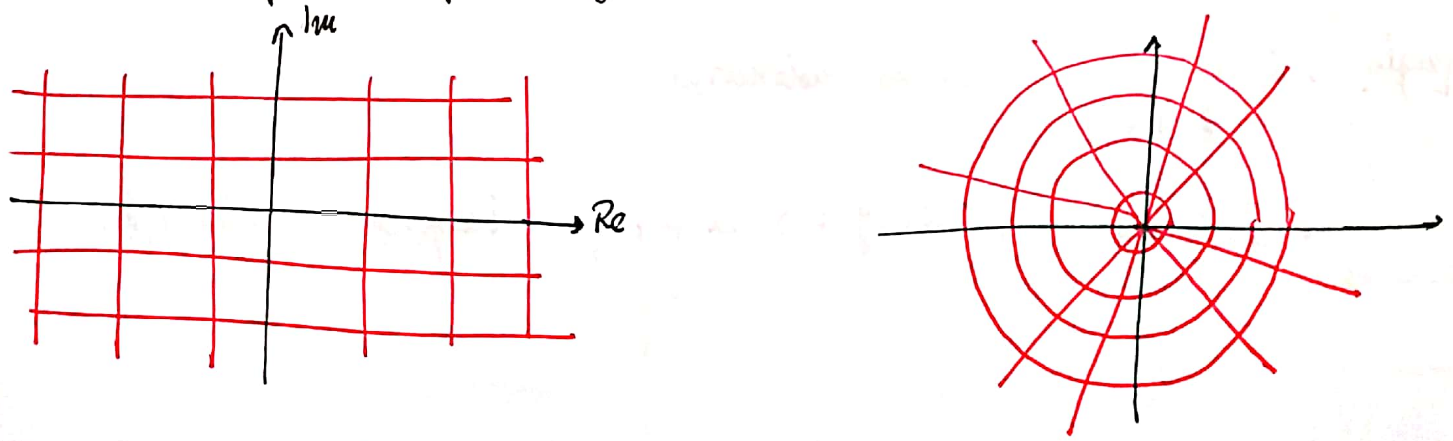
$= x_1x_2 + y_1y_2$

Función exponencial:

$z = x+iy$; $x, y \in \mathbb{R}$

$exp(z) = e^z = e^x \underbrace{e^{iy}}_{(\cos(y) + i\,sen(y))}$

tiene la siguiente propiedad geométrica:



obs. otra definición es $\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Prop. $e^{z+w} = e^z e^w$

—— ——

Consideremos a $\mathbb{C}$ como espacio métrico mediante $d(z,w) = |z-w|$.

Sea $\Omega \subset \mathbb{C}$ abierto,

def. $f : \Omega \to \mathbb{C}$ es llamada holomorfa en el punto $z_0 \in \Omega$ si existe

$$f'(z_0) := \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

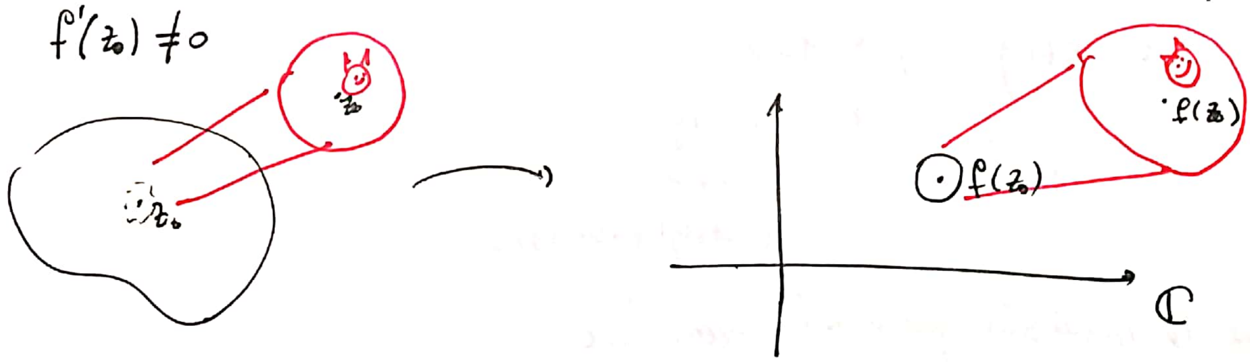
Si $\exists\, f'(z_0)$ para todos los $z_0 \in \Omega$, decimos que f es holomorfa en Ω.

O sea: f es holomorfa en $z_0 \in \Omega \iff$ existen $a, b \in \mathbb{C}$ tq

$$f(z_0+h) = a + bh + o(h)$$

Notación. $o(\cdot)$ es una función $\varphi(h)$ cualquiera tq $\lim_{h \to 0} \frac{|\varphi(h)|}{|h|} = 0$

Si $f'(z_0) \neq 0$



Ejemplo. $\cdot\ f : \mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$ no es holomorfa

$\cdot\ f : \mathbb{C} \to \mathbb{C}$, $x+iy \mapsto 2x + i\frac{y}{2}$ tampoco es holomorfa.

Multiplicación compleja: $\quad M_z : \mathbb{C} \to \mathbb{C}$

$\qquad\qquad\qquad\qquad\quad w \mapsto zw$

$z = a + bi = (a, b)$

$w = x + yi = (x, y)$

$$zw = (a+bi)(x+yi) = ax + ayi + bxi - by$$

$$= ax - by + (ay + bx)i$$

$$M_z(w) = M_{\binom{a}{b}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

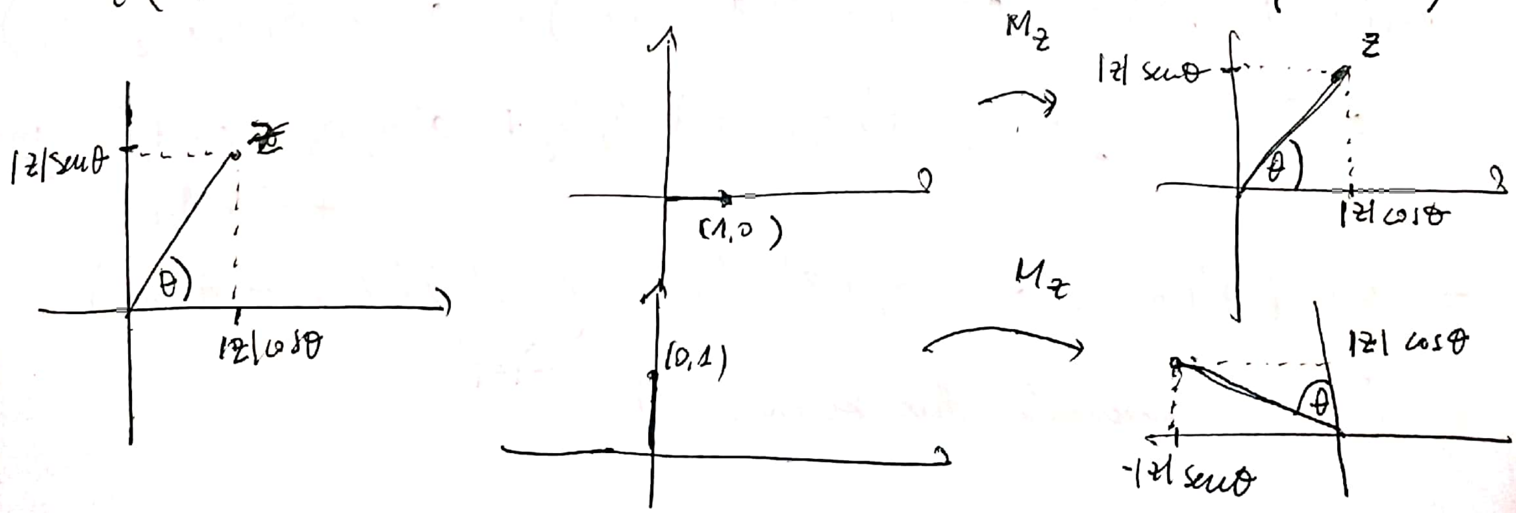
$$\begin{cases} a = |z| \cos\theta \\ b = |z| \operatorname{sen}\theta \end{cases} : \quad M_z(w) = \begin{pmatrix} |z|\cos\theta & -|z|\operatorname{sen}\theta \\ |z|\operatorname{sen}\theta & |z|\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$M_z(w) = |z| \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow M_z(w) = \begin{pmatrix} |z| & \\ & |z| \end{pmatrix} \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$M_z\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} |z| & \\ & |z| \end{pmatrix} \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} |z|\cos\theta \\ |z|\operatorname{sen}\theta \end{pmatrix} = |z| \begin{pmatrix} \cos\theta \\ \operatorname{sen}\theta \end{pmatrix}$$

$$M_z\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} |z| & \\ & |z| \end{pmatrix} \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |z| \begin{pmatrix} -\operatorname{sen}\theta \\ \cos\theta \end{pmatrix}$$

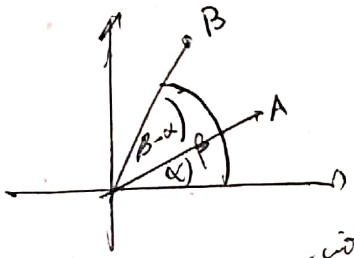


• La multiplicación compleja preserva ángulos y orientación.
• Averiguar qué significa que preserve orientación.

Preservar ángulos :

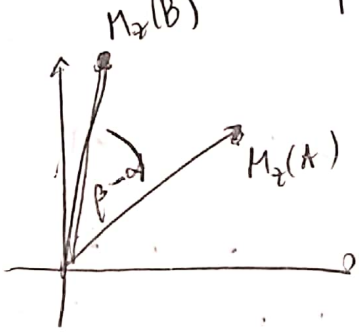
$$A = |A| e^{i\alpha} = (A_1, A_2)$$
$$B = |B| e^{i\beta} = (B_1, B_2)$$

$$M_z(A) = |z| e^{i\theta} |A| e^{i\alpha} = |z||A| e^{i(\theta+\alpha)}$$
$$M_z(B) = |z| e^{i\theta} |B| e^{i\beta} = |z||B| e^{i(\theta+\beta)}$$



$\arg B = \beta$
$\arg A = \alpha$

$\arg M_z(B) - \arg M_z(A)$
$= \theta + \beta - (\theta + \alpha) = \beta - \alpha$
$= \arg B - \arg A$



no es transformación ortogonal (casi).

M_z Preservar ángulos : $\langle M_z(B), M_z(A) \rangle = \langle A, B \rangle$

$$M_z(A) = (a, b)(A_1, A_2) = (aA_1 - bA_2,\ aA_2 + bA_1)$$
$$M_z(B) = (a, b)(B_1, B_2) = (aB_1 - bB_2,\ aB_2 + bB_1)$$

$$\langle M_z(A), M_z(B) \rangle = (aA_1 - bA_2)(aB_1 - bB_2) + (aA_2 + bA_1)(aB_2 + bB_1)$$
$$= a^2 A_1 B_1 - \cancel{abA_1 B_2} - \cancel{abA_2 B_1} + b^2 A_2 B_2 + a^2 A_2 B_2 + \cancel{abA_2 B_1} + \cancel{abA_1 B_2} + b^2 A_1 B_1$$
$$= A_1 B_1 (a^2 + b^2) + A_2 B_2 (a^2 + b^2) = \underbrace{(a^2 + b^2)}_{=|z|}(A_1 B_1 + A_2 B_2)$$

(No necesariamente debe ser así...)

Funciones holomorfas:

- $f: \mathbb{C} \to \mathbb{C}$ $\quad z \mapsto \bar{z}$ no es holomorfa.

$r \in \mathbb{R}$ $(\mathbb{R} \times \{0\})$

$$\frac{f(z+r) - f(z)}{r} = \frac{\overline{z+r} - \bar{z}}{r} = \frac{\bar{z} + r - \bar{z}}{r} = 1 \xrightarrow{r \to 0} 1$$

$$\frac{f(z+ri) - f(z)}{ir} = \frac{\overline{z+ri} - \bar{z}}{r} = \frac{\bar{z} - ri - \bar{z}}{r} = -i \xrightarrow{ir \to 0} -i$$

$\therefore$ $f'(z)$ no existe para ningún $z \in \mathbb{C}$.

- $f: \mathbb{C} \to \mathbb{C}$, $\quad x+iy \mapsto 2x + i\frac{y}{2}$ no es holomorfa.

$r \in \mathbb{R}$: $\quad \dfrac{f((x+iy)+(r+0i)) - f(x+iy)}{r+0i} = \dfrac{f((x+r)+iy) - f(x+iy)}{r+0i}$

$$= \frac{2(x+r) + i\,y/2 - (2x + i\,y/2)}{r} = \frac{2r}{r} = 2 \xrightarrow{r \to 0} 2$$

$$\frac{f((x+iy)+(0+ir)) - f(x+iy)}{0+ir} = \frac{f(x+(y+r)i) - f(x+iy)}{0+ir}$$

$$= \frac{2x + i\dfrac{(y+r)}{2} - \left(2x + i\,y/2\right)}{ir} = \frac{\frac{ir}{2}}{ir} = \frac{1}{2} \xrightarrow{ir \to 0} \frac{1}{2}$$

$\therefore$ $f'(z)$ no existe para ningún punto de $\mathbb{C}$.

Evaluaciones

I1 : 25/Abril

I2 : 8/Junio

Ex : 20/Junio.

09/03/2016

Prop. Sean $\Omega \subseteq \mathbb{C}$ abierto. $f, g : \Omega \to \mathbb{C}$ holomorfas en $z_0 \in \Omega$.
entonces:

i) $f+g$ es holomorfa en z_0 y $(f+g)'(z_0) = f'(z_0) + g'(z_0)$

ii) $(fg)'(z_0) = f'(z_0) g(z_0) + f(z_0) g'(z_0)$

iii) si $g(z_0) \neq 0$:

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$$

Prop. (Regla de la cadena)

$f : \Omega_1 \to \mathbb{C}$, $g : \Omega_2 \to \mathbb{C}$

Si $\begin{cases} f \text{ es holomorfa en } z_0 \in \Omega_1 \\ g \text{ es holomorfa en } f(z_0) \in \Omega_2 \end{cases}$

entonces $g \circ f$ es holomorfa en z_0 y $(g \circ f)'(z_0) = g'(f(z_0)) f'(z_0)$

- $\mathbb{C} = \mathbb{R}^2$

$f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ es llamada diferenciable en un punto $z_0 = (x_0, y_0)$ si existe $\exists$ transformación lineal

$$\left.\begin{array}{l} Df(z_0) : \mathbb{R}^2 \to \mathbb{R}^2 \\ \text{tal que } f(z_0 + h) = f(z_0) + Df(z_0)(h) + r(h) \\ r(h) = R(h) \quad , \quad \lim_{h \to 0} \dfrac{|R(h)|}{|h|} = 0 \end{array}\right\} \text{diferenciabilidad en el sentido real.}$$

De la definición de diferenciabilidad

$$f'(z_0) := \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

tenemos $f(z_0+h) = f(z_0) + \boxed{f'(z_0)\,h} + o(h)$

$$h \in \mathbb{C} = \mathbb{R}^2 \xmapsto{\mathbb{R}\text{-lineal}} f'(z_0)\,h \in \mathbb{C} = \mathbb{R}^2$$

$f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ diferenciable en $z_0 = (x_0, y_0)$

$(x,y) \mapsto (u(x,y), v(x,y))$

$$Df(z_0) = \begin{pmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{pmatrix}$$

Si f es holomorfa en z_0 y $f'(z_0) = a+bi$. Entonces

$h = x+iy \longmapsto f'(z_0)\,h = (ax-by) + (ay+bx)$

$= (x,y)$

$= \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad \begin{pmatrix} ax-by \\ bx+ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Teo. Sea $\Omega \subseteq \mathbb{C}$ abierto. Sea $f : \Omega \longrightarrow \mathbb{C}$ $\qquad u := Re(f)$, $v := Im(f)$

$$f(x+iy) = u(x,y) + i\,v(x,y) \quad (x,y \in \mathbb{R})$$

Entonces f es holomorfa en $z_0 = x_0 + iy_0 \in \Omega$ ssi

1) f es diferenciable (en el sentido real) en el punto z_0.

2) $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ en $z_0 = (x_0, y_0)$

Ecuaciones de Cauchy-Riemann.

$$Df(z_0)(i) = i \cdot Df(z_0)(1) \quad \text{(Linealidad compleja).}$$

$\downarrow$
producto complejo.

$$Df(z_0) \cdot i = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u_y \\ v_y \end{pmatrix}$$

$$i \cdot Df(z_0)(1) = i \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = i \begin{pmatrix} u_x \\ v_x \end{pmatrix}$$

$$= i(u_x + i v_x) = -v_x + i u_x = \begin{pmatrix} -v_x \\ u_x \end{pmatrix}$$

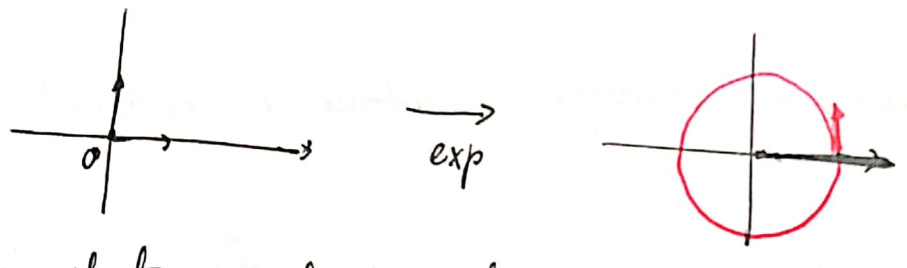
<u>Ejemplo</u>. $\exp : \mathbb{C} \to \mathbb{C}$, $z = x + iy$

$$\exp(z) = e^z = e^x(\cos(y) + i\,\mathrm{sen}(y))$$

$$\begin{cases} u(x,y) = e^x \cos(y) \\ v(x,y) = e^x \,\mathrm{sen}(y) \end{cases}$$

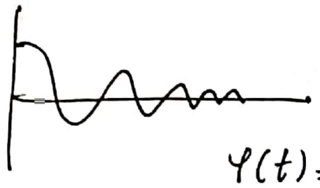
Satisfacen las ecuaciones de Cauchy-Riemann y por lo tanto exp es holomorfa.

Notar que $\exp'(0) = 1$



De lo cual tenemos lo siguiente:

$$\exp'(z) = \lim_{h \to 0} \frac{e^{z+h} - e^z}{h} = e^z \lim_{h \to 0} \frac{e^h - 1}{h} = e^z \cdot 1 = e^z$$



Oscilador armónico amortiguado

$$\varphi(t) = e^{at}\cos(bt) \qquad a<0$$

$$\varphi^{(100)}(t) = ??$$

$$\varphi(t) = \mathrm{Re}\left(e^{(a+bi)t}\right)$$

$$\varphi^{(100)}(t) = \mathrm{Re}\left(\left(\frac{d}{dt}\right)^{(100)} e^{(a+bi)t}\right) = \mathrm{Re}\left((a+bi)^{100} e^{(a+bi)t}\right)$$

<u>Series de potencias</u>.

Sea $(a_n)_{n\in\{0,1,2,\dots\}}$ una sucesión en $\mathbb{C}$

$$\sum_{n=0}^{\infty} a_n := \lim_{N\to\infty} \sum_{k=0}^{N} a_k \quad \text{si el límite existe.}$$

En este caso decimos que la serie es convergente.

- No convergente = divergente.

Decimos que $\sum a_n$ es absolutamente convergente si $\sum |a_n|$ es convergente.

<u>Teo</u>. Si $\sum a_n$ es absolutamente convergente, entonces es convergente.

<u>dem</u>: Ejercicio.

Series del tipo $\sum_{n=0}^{\infty} a_n z^n$, o más generalmente, $\sum_{n=0}^{\infty} a_n (z-a)^n$ son llamadas <u>series de potencias</u>.

Dada una sucesión (a_n) en $\mathbb{C}$, definimos el radio de convergencia

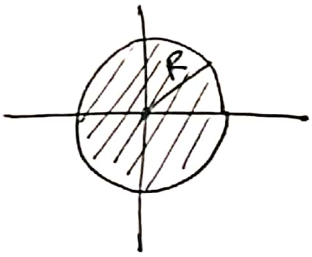
$$R := \left(\limsup_{n\to\infty} |a_n|^{1/n} \right)^{-1}, \qquad R \in [0,\infty]$$

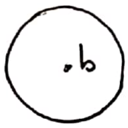
Convenciones: $\begin{cases} 0^{-1} = \infty \\ \infty^{-1} = 0 \end{cases}$

<u>Teo</u>. (a_n) sucesión en $\mathbb{C}$, $z \in \mathbb{C}$. R = radio de convergencia.

(i) Si $|z| < R$, entonces $\sum_{n=0}^{\infty} a_n z^n$ es absolutamente convergente.

(ii) Si $|z| > R$ entonces la serie es divergente





$$\sum_{n=0}^{\infty} a_n (z-b)^n$$

$\{ z \in \mathbb{C} \,/\, |z| < R \}$

es llamado disco de convergencia

<u>dem</u>.

(i) Supongamos que $|z| < R$. Sea r tal que $|z| < r < R$. Entonces

$r^{-1} > R^{-1}$

$$r^{-1} > \limsup_{n\to\infty} |a_n|^{1/n}$$

$$\Rightarrow \exists n_0 \in \mathbb{N} \text{ tq } \forall n > n_0$$

$$r^{-1} > |a_n|^{1/n}$$

$$r^{-n} > |a_n|$$

$$|a_n z^n| = |a_n||z|^n < \left(\frac{|z|}{r}\right)^n$$

$\in [0,1)$

$\Downarrow$

$\sum |a_n z^n|$ es convergente (comparación con la serie geométrica)

Función holomorfa. Diferenciabilidad.

f holomorfa $\Leftrightarrow$ $f'(z_0) = \lim\limits_{h\to 0} \dfrac{f(z_0+h) - f(z_0)}{h}$

definiendo $\sigma(h) = f(z_0+h) - f(z_0) - f'(z_0)h$

$$\Rightarrow \lim_{h\to 0} \frac{\sigma(h)}{h} = 0$$

$$\frac{\sigma(h)}{h} = \frac{f(z_0+h) - f(z_0) - f'(z_0)h}{h} = \frac{f(z_0+h) - f(z_0)}{h} - f'(z_0)$$

$$\Rightarrow f(z_0+h) = f(z_0) + f'(z_0)h + \sigma(h)$$

$T: \mathbb{C} \to \mathbb{C}$, $T(h) = f'(z_0)h$ es $\mathbb{R}$-lineal. ($\mathbb{C}$-lineal en ~~particular~~ general)
(visto en $T: \mathbb{R}^2 \to \mathbb{R}^2$)

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \Leftrightarrow \lim_{h\to 0} \frac{f(z_0+h) - f(z_0) - f'(z_0)h}{h} = 0$$

$$\Leftrightarrow \frac{f(z_0+h) - f(z_0) - f'(z_0)h}{h} = \frac{\sigma(h)}{h} \quad \text{tq} \quad \lim_{h\to 0} \frac{\sigma(h)}{h} = 0$$

Podemos considerar $\dfrac{\sigma(h)}{h} = \varphi(h)$

$\Rightarrow$ f diferenciable en $z = z_0 \in \mathbb{C}$ $\Leftrightarrow$ $\exists a \in \mathbb{C}$: $f(z_0+h) - f(z_0) = ah + \varphi(h)h$ tq $\lim\limits_{h\to 0} \varphi(h) = 0$.

· Ejercicio, ocupar lo anterior para demostrar regla de la cadena.

Pd: $(f\circ g)'(z_0) = f'(g(z_0))\, g'(z_0)$

$$\cancel{\frac{(f\circ g)(z_0+h) - (f\circ g)(z_0)}{h} = \frac{f(g(z_0+h)) - f(g(z_0))}{h}}$$

$$(f\circ g)(z_0+h) - (f\circ g)(z_0) = f(g(z_0+h)) - f(g(z_0))$$
$$= f\big(g(z_0) + g'(z_0)h + \varphi(h)h\big) - f(g(z_0))$$

$$\frac{(f\circ g)(z_0+h)-(f\circ g)(z_0)}{h} = \frac{(f\circ g)(z_0+h)-(f\circ g)(z_0)}{g(z_0+h)-g(z_0)}\cdot\frac{g(z_0+h)-g(z_0)}{h}$$

~~f es diferencia~~

f es diferenciable en $g(z_0)$:

$$f(g(z)+h)-f(g(z_0)) = f'(g(z_0))h+\varphi(h)h$$

$$\cancel{\Rightarrow f(g(z)+h)g'(z) - f(g(z_0))g'(z_0) = f'(g(z_0))g'(z_0)h+g'(z_0)\varphi(h)h}$$

$$f(g(z_0+h))-f(g(z_0)) = f\big(g(z_0)+\underbrace{g'(z_0)h+\varepsilon(h)h}_{k_h}\big)-f(g(z_0))$$

$\hookrightarrow =$

$$\Rightarrow f(g(z)+k_h)-f(g(z_0)) = f'(g(z_0))k_h+\varphi(k_h)k_h$$

$$= f'(g(z_0))\left[g'(z_0)h+\varepsilon(h)h\right]+\varphi(k_h)k_h$$

$$= f'(g(z_0))\left[g'(z_0)h+\varepsilon(h)h\right]+\varphi(g'(z_0)h+\varepsilon(h)h)\left[g'(z_0)h+\varepsilon(h)h\right]$$

$$= f'(g(z_0))g'(z_0)h+\Big[\underbrace{f'(g(z_0))\varepsilon(h)+\varphi(g'(z_0)h+\varepsilon(h)h)(g'(z_0)+\varepsilon(h))}_{\psi(h)}\Big]h$$

$$\lim_{h\to 0}\psi(h)=0.$$

$$\therefore\ (f\circ g)'(z_0) = f'(g(z_0))g'(z_0)$$

Función exponencial.

$$\exp : \mathbb{C} \to \mathbb{C}$$
$$z \mapsto \exp(z)$$

$z = x+iy$: $\exp(z) = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\,\mathrm{sen}(y))$
$= e^x\cos(y) + i\,e^x\,\mathrm{sen}(y)$

$$\begin{cases} u(x,y) = e^x\cos(y) \\ v(x,y) = e^x\,\mathrm{sen}(y) \end{cases} \Rightarrow \begin{cases} u_x = e^x\cos(y) & v_x = e^x\,\mathrm{sen}(y) \\ u_y = -e^x\,\mathrm{sen}(y) & v_y = e^x\cos(y) \end{cases}$$

derivadas parciales continuas
$\Rightarrow$ exp diferenciable en el sentido real.

Cauchy - Riemann : $u_x = v_y$, $u_y = -v_x$ $\Rightarrow$ exp holomorfa.

$$(\exp'(x,y))_{\mathcal{C}} = \begin{pmatrix} e^x\cos(y) & -e^x\,\mathrm{sen}(y) \\ e^x\,\mathrm{sen}(y) & e^x\,\mathrm{sen}(y) \end{pmatrix} \qquad \mathcal{C} = \{(1,0),(0,1)\}$$

$$\Rightarrow (\exp'(0))_{\mathcal{C}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \therefore \exp'(0) = 1$$

$z = a+bi$: $\exp'(0)\,z = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = z$ ∎

Shakarchi: Complex-valued functions as mappings.

Conexión:

	Versión compleja	Versión $\mathbb{R}^2$
	$f: \Omega \to \mathbb{C}$	$F: \Omega \to \mathbb{R}^2$
	$z \mapsto f(z)$	$(x,y) \mapsto (u(x,y), v(x,y))$

$F: \Omega \to \mathbb{R}^2$ diferenciable en $P_0 = (x_0, y_0)$ ssi $\exists\, J: \mathbb{R}^2 \to \mathbb{R}^2$ lineal tq

$$\frac{|F(P_0+h) - F(P_0) - J(h)|}{|h|} \to 0 \quad \text{cuando } |h| \to 0,\ h \in \mathbb{R}^2$$

equiv: $F(P_0+h) - F(P_0) = J(h) + |h|\,\psi(h)$, con $|\psi(h)| \to 0$ cuando $|h| \to 0$

- J tiene representación matricial $J = J_F(x,y) = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}$.

versión real $\mathbb{R}^2$.

- Versión $\mathbb{C}$.

f diferenciable en $z_0 \in \Omega$ ssi $\exists\ f'(z_0) = \lim_{h\to 0} \dfrac{f(z_0+h) - f(z_0)}{h}$.

- Supongamos que $h = h_1 + ih_2$, $h_2 = 0$:

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(z_0+h_1) - f(z_0)}{h_1}$$

$$= \lim_{h_1\to 0} \frac{f(x_0+h_1+iy_0) - f(x_0+iy_0)}{h_1} = \frac{\partial f}{\partial x}$$

Sup. $h_1 = 0$: $f'(z_0) = \lim_{ih_2 \to 0} \dfrac{f(z_0+ih_2) - f(z_0)}{ih_2} = \lim_{ih_2\to 0} \dfrac{f(x_0+(y_0+h_2)i) - f(x_0+iy_0)}{ih_2}$

$$= \frac{1}{i}\frac{\partial f}{\partial y}$$

Por lo tanto: $\dfrac{\partial f}{\partial x} = \dfrac{1}{i}\dfrac{\partial f}{\partial y}$ | $f = u + iv$: $\dfrac{\partial f}{\partial x} = -i\dfrac{\partial f}{\partial y}$

$$\Rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\text{ó } \boxed{\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2mm] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}}$$

Trabajamos con los siguientes operadores diferenciales:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

• f holomorfa en $z_0=(x_0,y_0)$, $f(z_0)=u(z_0)+iv(z_0)$.

$$\frac{\partial}{\partial z}f(z_0) = \frac{1}{2}\left(\frac{\partial}{\partial x}f(z_0) - i\frac{\partial}{\partial y}f(z_0)\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x}u(z_0) + i\frac{\partial}{\partial x}v(z_0) - i\frac{\partial}{\partial y}u(z_0) + \frac{\partial}{\partial y}v(z_0)\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x}u(z_0) + i\frac{\partial}{\partial x}v(z_0) + i\frac{\partial}{\partial x}v(z_0) + \frac{\partial}{\partial x}u(z_0)\right)$$

$$= \cancel{\tfrac{1}{2}} = \frac{\partial}{\partial x}u(z_0) + i\frac{\partial}{\partial x}v(z_0) = \frac{\partial}{\partial x}f(z_0) = f'(z_0)$$

$$\therefore \quad f'(z_0) = \frac{\partial}{\partial z}f(z_0)$$

$$\boxed{\begin{aligned}\frac{\partial}{\partial z}f(z_0) &= \frac{1}{2}\left(2\frac{\partial}{\partial x}u(z_0) - 2i\frac{\partial}{\partial y}v(z_0)\right)\\ &= 2\frac{\partial}{\partial z}u(z_0).\end{aligned}}$$

$$\frac{\partial}{\partial \bar z}f(z_0) = \frac{1}{2}\left(\frac{\partial}{\partial x}f(z_0) + i\frac{\partial}{\partial y}f(z_0)\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x}u(z_0) + i\frac{\partial}{\partial x}v(z_0) + i\frac{\partial}{\partial y}u(z_0) - \frac{\partial}{\partial y}v(z_0)\right)$$

$$= \frac{1}{2}\left(\cancel{\frac{\partial}{\partial x}u(z_0)} + \cancel{i\frac{\partial}{\partial x}v(z_0)} - \cancel{i\frac{\partial}{\partial x}v(z_0)} - \cancel{\frac{\partial}{\partial x}u(z_0)}\right)$$

$$= 0$$

$$\therefore \quad \frac{\partial}{\partial \bar z}f(z_0) = 0.$$

$$\det J(x_0,y_0) = \det\begin{pmatrix}\frac{\partial}{\partial x}u(z_0) & \frac{\partial}{\partial y}u(z_0)\\ \frac{\partial}{\partial x}v(z_0) & \frac{\partial}{\partial y}v(z_0)\end{pmatrix} = \frac{\partial}{\partial x}u(z_0)\frac{\partial}{\partial y}v(z_0) - \frac{\partial}{\partial x}v(z_0)\frac{\partial}{\partial y}u(z_0)$$

$$= \frac{\partial}{\partial x}u(z_0)\frac{\partial}{\partial x}u(z_0) + \frac{\partial}{\partial y}u(z_0)\frac{\partial}{\partial y}u(z_0) = \left(\frac{\partial}{\partial x}u(z_0)\right)^2 + \left(\frac{\partial}{\partial y}u(z_0)\right)^2$$

$$= \left(\frac{\partial}{\partial x}u(z_0) + i\frac{\partial}{\partial y}u(z_0)\right)\left(\frac{\partial}{\partial x}u(z_0) - i\frac{\partial}{\partial y}u(z_0)\right) = \left(2\frac{\partial}{\partial \bar z}u(z_0)\right)\left(2\frac{\partial}{\partial z}u(z_0)\right) = \left|2\frac{\partial}{\partial z}u(z_0)\right|^2$$

$\therefore \; \det J(x_0, y_0) = |f'(z_0)|^2$. ⊡

- f holomorfa en $z_0 \Rightarrow F(x,y) = f(z)$, F es diferenciable en el sentido real y $\det J_F(x_0, y_0) = |f'(z_0)|^2$.

Para demostrar que F es diferenciable basta observar que si ~~...~~ $H = (h_1, h_2)$
$h = h_1 + i h_2$:

$$J_F(x_0, y_0)(H) = \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}\right)(h_1 + i h_2) = f'(z_0) h$$

$$J_F(x_0, y_0) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 \\ -\frac{\partial u}{\partial y} h_1 + \frac{\partial u}{\partial x} h_2 \end{pmatrix}.$$

$$\left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}\right)(h_1 + i h_2) = \frac{\partial u}{\partial x} h_1 + i \frac{\partial u}{\partial x} h_2 - i \frac{\partial u}{\partial y} h_1 + \frac{\partial u}{\partial y} h_2$$

$$= \left(\frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2\right) + i\left(-\frac{\partial u}{\partial y} h_1 + \frac{\partial u}{\partial x} h_2\right)$$

$$\therefore \; J_F(x_0, y_0)(H) = f'(z_0) h$$

Th. (2.4) $f = u + iv$ definida en $\Omega \subseteq \mathbb{C}$. u, v continuamente diferenciable y satisfacen Cauchy-Riemann en Ω, entonces f es holomorfa en Ω y $f'(z) = \frac{\partial}{\partial z} f(z)$.

dem: u, v continuamente diferenciables, $h = (h_1, h_2)$

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + |h| \Psi_1(h)$$

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 + |h| \Psi_2(h)$$

$\Psi_j(h) \to 0$, $h \to 0$.

$$f(z+h) - f(z) = u(z+h) + iv(z+h) - u(z) - iv(z)$$

$$= u(z+h) - u(z) + i\big(v(z+h) - v(z)\big)$$

$$= u(x+h_1, y+h_2) - u(x,y) + i\big(v(x+h_1, y+h_2) - v(x,y)\big)$$

$$= \cancel{\frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + i\Big(\frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2}$$

$$= \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\psi_1(h) + i\Big(\frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\psi_2(h)\Big)$$

$$= \frac{\partial u}{\partial x}h_1 + i\frac{\partial v}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + i\frac{\partial v}{\partial y}h_2 + |h|\big(\psi_1(h) + i\psi_2(h)\big)$$

$$= \cancel{\frac{\partial f}{\partial x}h_1 + \frac{\partial f}{\partial y}h_2 + |h|(\psi_1 + i\psi_2(h))}$$

$$= \cancel{i\frac{\partial u}{\partial x}h_1 - i\frac{\partial u}{\partial y}h_2 + \frac{\partial u}{\partial x}h_1 + i\frac{\partial u}{\partial x}h_2}$$

$$= \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + i\frac{\partial v}{\partial x}h_1 + i\frac{\partial v}{\partial y}h_2 + |h|\big(\psi_1(h) + i\psi_2(h)\big) \qquad \Big|\ \psi(h) := \psi_1(h) + i\psi_2(h)$$

$$= \Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big)h_1 + \Big(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\Big)h_2 + |h|\big(\psi(h)\big)$$

$$= \Big(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\Big)h_1 + i\Big(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\Big)h_2 + \psi(h)|h| = \Big(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\Big)h_1 + i\Big(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\Big)h_2 + \psi(h)|h|$$

$$= \Big(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\Big)(h_1 + ih_2) + \psi(h)|h| = 2\Big(\frac{\partial}{\partial z}u(z)\Big)h + |h|\psi(h)$$

$$\big(\psi(h) \to 0,\ h \to 0\big)$$

$$\therefore\ f'(z) = 2\frac{\partial u(z)}{\partial z} = \frac{\partial f}{\partial z}(z).$$

Series de Potencias

Teo. $\sum a_n$ absolutamente convergente $\Rightarrow \sum a_n$ convergente.

dem. Sea $S_N = \sum_{k=0}^{N} a_k$

$$|S_N - S_M| = \left|\sum_{k=0}^{N} a_k - \sum_{k=0}^{M} a_k\right| \leq \sum_{k=0}^{N} |a_k| - \sum_{k=0}^{M} |a_k| \xrightarrow{N,M\to\infty} 0$$

$\therefore (S_N)$ sucesión de Cauchy

$\therefore \sum a_n$ convergente.

lim sup: $\limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \sup_{m\geq n} |a_m|^{1/m}$

$$R > \limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \sup_{n\geq m} |a_m|^{1/m}$$

$\Rightarrow \exists n_0 \in \mathbb{N}$ tq $\forall n \geq n_0$: $R > \sup_{m\geq n} |a_m|^{1/m} \geq |a_m|^{1/m} \; \forall m \geq n$

$\therefore \exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$: $R > |a_n|^{1/n}$

(Stein - Shakarchi). R radio de convergencia de $\sum_{n=0}^{\infty} a_n z^n$

$\Rightarrow |z| < R$ converge absolutamente.

dem. Supongamos $R \neq 0, \infty$. $|z| < R \Rightarrow \frac{|z|}{R} < 1$

$L := \frac{1}{R}$ se tiene : $(L+\varepsilon)|z| = L|z| + \varepsilon|z| = \frac{|z|}{R} + \varepsilon|z| < \frac{|z|}{R} + \varepsilon R$
$\varepsilon > 0$.

Podemos considerar $\varepsilon > 0$ suf. pequeño tq : $\underbrace{(L+\varepsilon)}_{r}|z| < 1$

Como $L \leq L+\varepsilon \Rightarrow \limsup |a_n|^{1/n} \leq L+\varepsilon$

$\Rightarrow \exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$: $|a_n|^{1/n} \leq L+\varepsilon$

$$\Rightarrow \quad |a_n||z|^n \leq (L+\varepsilon)^n |z^n| = r^n$$

$$\therefore \quad \sum_0^\infty |a_n z^n| \leq \underbrace{\sum_0^\infty r^n}$$

serie geométrica convergente ($r<1$)

$\therefore$ $\sum_0^\infty a_n z^n$ absolutamente convergente en $|z|<R$.

Pd: $\sum_0^\infty a_n z^n$ diverge en $|z|>R$.

Scanned with CamScanner

Estábamos estudiando las series de potencias $f(z) = \sum_{n=0}^{\infty} a_n (z-b)^n$.

Converge absolutamente en el disco $B(b, R)$, donde $R^{-1} = \limsup_{n\to\infty} |a_n|^{1/n}$

Además, la convergencia es uniforme en todo compacto $K \subset B(b, R)$

Ejemplo.

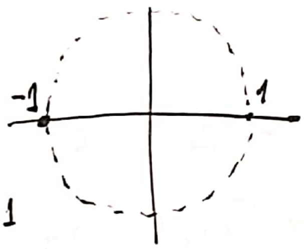
(1) $\sum n!\, z^n$ tiene radio de convergencia $R = 0$.

$$\lim_{n\to\infty} (n!)^{1/n} = \infty .$$

(2) Para $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, $R = \infty$.

(3) Para $\sum_{n=0}^{\infty} z^n$, $R = 1$. $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ si $|z| < 1$

(4) Para $\sum_{n=1}^{\infty} \frac{z^n}{n}$, $R = 1$ | $\lim_{n\to\infty} n^{1/n} = 1 \iff \lim_{n\to\infty} \frac{\log n}{n} = 0$



$z = -1$

$\sum \frac{(-1)^n}{n}$ conv. condicionalmente | $z = 1$ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge.

Teo. Las series de potencias se pueden derivar término a término en el disco de convergencia. O sea

Si $f(z) = \sum_{n=0}^{\infty} a_n (z-b)^n$, $z \in B(b, R)$, entonces $\exists f'(z) = \sum_{n=1}^{\infty} n a_n (z-b)^{n-1}$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} (z-b)^n$$

Además, el radio de convergencia de esta serie también es R.

dem. SPG: $b=0$. Para $z \in B(0,R)$ sea:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \underbrace{\sum_{n=0} a_n z^n}_{f_N(z)} + K_N(z)$$

$$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = g_N(z) + L_N(z)$$

Fijemos $z \in B(0,R)$

Fijemos r tal que $0<r<R$

Consideremos $h \neq 0$ tq $|z+h|<r$

$$\frac{f(z+h)-f(z)}{h} = \underbrace{\frac{f_N(z+h)-f_N(z)}{h}}_{\mathrm{I}} + \underbrace{\frac{K_N(z+h)-K_N(z)}{h}}_{\mathrm{II}}$$

Hecho general: $\sum a_n$, $\sum b_n$ absolutamente convergentes $\to \sum (a_n+b_n)$ es absolutamente convergente y $\sum (a_n+b_n) = \sum a_n + \sum b_n$.

$$|\mathrm{II}| \leq \sum_{n=N}^{\infty} \frac{|a_n (z+h)^n - a_n z^n|}{|h|} = \sum_{n=N}^{\infty} |a_n| \left| \frac{(z+h)^n - z^n}{h} \right| \leq \sum_{n=N}^{\infty} |a_n| n r^{n-1}$$

$$\leq n r^{n-1}$$

$$\frac{a^n - b^n}{a-b} = |a^{n-1} + a^{n-2} b + \dots + b^{n-1}|$$

$$\leq n (\max\{|a|,|b|\})^{n-1}$$

fijemos $N \in \mathbb{N}$ tal que

1) $\sum_{n=N}^{\infty} |a_n| n r^{n-1} < \varepsilon$

2) $|L_N(z)| < \varepsilon$

Entonces, si $|h|$ es suficientemente pequeño

$$\Bigg|\underbrace{\frac{f(z+h)-f(z)}{h}}_{I+II} - \underbrace{g(z)}_{g_N(z)+L_N(z)}\Bigg| \leq \underbrace{|I - g_N(z)|}_{<\varepsilon} + \underbrace{|II|}_{<\varepsilon} + \underbrace{|L_N(z)|}_{<\varepsilon} < 3\varepsilon$$

$$I \xrightarrow[h\to 0]{} f_N'(z) = g_N(z)$$

Def. $\Omega \subset \mathbb{C}$ abierto. Una función $f: \Omega \to \mathbb{C}$ es llamada analítica en el punto $b \in \Omega$ si $\exists (a_n)$ tq la serie de potencias $\sum_{n=0}^{\infty} a_n (z-b)^n$ converge $\forall z \in V$ = alguna vecindad de b, y es $= f(z)$ $\forall z \in V \cap \Omega$

Def. $f: \Omega \to \mathbb{C}$ es llamada analítica si lo es en todo Ω.

Corolario del teorema: Si f es analítica en el punto b, entonces es holomorfa en una vecindad de b.

- Más adelante veremos la recíproca.

Hecho. $a_n = \frac{1}{n!} f^{(n)}(b)$ (Fórmula de Taylor) (Demostrar como ejercicio).

—o—

Series de Fourier. Sea $g: \mathbb{R} \to \mathbb{R}$ función C^2 2π-periódica

Entonces

$$g(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos(n\theta) + B_n \operatorname{sen} \quad)$$

convergencia absoluta y uniforme.

donde $A_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos(n\theta)\, d\theta$

$B_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \operatorname{sen}(n\theta)\, d\theta$

Sea f analítica en 0

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad R > 1 \qquad a_n := \alpha_n + i\beta_n$$

$\alpha_n, \beta_n \in \mathbb{R} \quad \forall n.$

$$g(\theta) = \operatorname{Re}\left(f(e^{i\theta})\right)$$

$$= \operatorname{Re}\left(\sum_{n=0}^{\infty} a_n (e^{i\theta})^n\right)$$

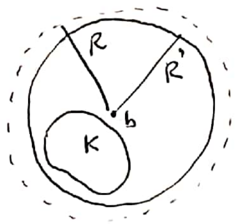
$$= \operatorname{Re}\left(\sum_{n=0}^{\infty} (\alpha_n + i\beta_n)(\cos n\theta + i \operatorname{sen} n\theta)\right) = \underbrace{\sum_{n=0}^{\infty} (\alpha_n \cos n\theta - \beta_n \operatorname{sen} n\theta)}_{\text{serie de Fourier}}$$

obs. Los coeficientes A_n, B_n son únicos.

Series de Potencias.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-b)^n, \quad R^{-1} = \limsup_{n\to\infty} |a_n|^{1/n}$$

Af. $f(z) = \sum_{n=0}^{\infty}{}' a_n (z-b)^n$ uniforme en $K \subset B(b,R)$ compacto.



$\forall K \subseteq B(b,R)$ compacto, $\exists\, 0 < R' < R$ tq $K \subset B(b,R')$

$0 < R' < R$.

(s.p.g. ; $b=0$) $\quad |z| \leq R'$

$R' < R \Rightarrow R^{-1} < R'^{-1} \Rightarrow \limsup |a_n|^{1/n} < R'^{-1}$

$\Rightarrow \exists n_0 \in \mathbb{N}$ tq $\forall n \geq n_0 : |a_n|^{1/n} < R'^{-1} \; (\Leftrightarrow |a_n| < R'^{-n})$

$\Rightarrow |a_n||z|^n < R'^{-n}|z|^n$. $\quad\Big|\; |z|^n \leq R'^n$

$|z| \leq R' \Rightarrow \frac{|z|}{R} \leq \frac{R'}{R} < 1$

Debemos considerar $0 < R' < \rho < R \Rightarrow 0 < R^{-1} < \rho^{-1} < R'^{-1}$

$\Rightarrow \limsup |a_n|^{1/n} < \rho^{-1} \to \exists n_0 \in \mathbb{N}, \forall n \geq n_0 : |a_n|^{1/n} < \rho^{-1}$

$(\Rightarrow) \quad$ " $: |a_n| < \rho^{-n}$

para $|z| \leq R'$: $|a_n||z|^n < \left(\frac{R'}{\rho}\right)^n < 1 \quad \forall n \geq n_0$

$\therefore$ y además $\sum_{n=1}^{\infty}{}' \left(\frac{R}{\rho}\right)^n < \infty$

M-test Weierstrass: $\sum_{n=0}^{\infty} a_n z^n$ uniformemente convergente en $|z| \leq R'$

$\therefore \sum_{n=0}^{\infty}{}' a_n z^n$ converge uniformemente en el compacto K.

(Conway - Shakarchi - De Silva).

- $f(z) = \sum_{n=0}^{\infty} a_n z^n$ con radio de convergencia $R \Rightarrow g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ es tal que f es holomorfa y $f'(z) = g(z)$. Radio de conv. de g $= R$.

— o —

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$$

$$f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \dots = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

R' = rad. de convergencia de $f'(z) \Rightarrow R'^{-1} = \limsup |(n+1) a_{n+1}|^{1/n}$

$\lim_{n\to\infty} (n+1)^{1/n} = 1$

Pd: $\limsup |a_{n+1}|^{1/n} = R$

$\to R^{-1} = \limsup |a_{n+1}|^{1/n}$ radio de convergencia de $\sum_{n=0}^{\infty} a_{n+1} z^n$

$$\sum_{n=0}^{\infty} a_{n+1} z^n = \frac{1}{z} \sum_{n=0}^{\infty} a_{n+1} z^{n+1} = \frac{1}{z}\left(\sum_{n=1}^{\infty} a_n z^n + a_0 - a_0\right) = \frac{1}{z} \sum_{n=0}^{\infty} a_n z^n - \frac{a_0}{z}$$

$$= \frac{1}{z}\left(\sum_{n=0}^{\infty} a_n z^n - a_0\right)$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n = z \sum_{n=0}^{\infty} a_{n+1} z^n + a_0$$

$|z| < R'$: $\sum_{n=0}^{\infty} |a_n z^n| = |z| \sum_{n=0}^{\infty} |a_{n+1} z^n| + |a_0| < \infty$ $\therefore R' \leq R$

$|z| < R$: $\sum_{n=0}^{\infty} |a_{n+1} z^n| = \frac{1}{|z|} \sum_{n=0}^{\infty} |a_n z^n| + \frac{|a_0|}{|z|} < \infty$ $\therefore R \leq R'$

$z \neq 0$

$$\therefore R = R'$$

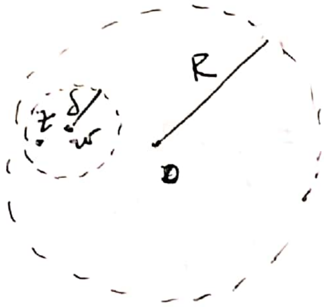
Scanned with CamScanner

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \quad , \quad g(z)=\sum_{n=1}^{\infty} n a_n z^{n-1}$$

Pd: f holomorfa y $f'(z)=g(z)$

Debemos estudiar $\dfrac{f(z)-f(w)}{z-w}-g(w)$

para cuando $z \to w$



$\overline{B(w,\delta)} \subseteq B(0,R)$

$$f(z)=\underbrace{\sum_{n=0}^{N} a_n z^n}_{S_N}+\underbrace{\sum_{n=N+1}^{\infty} a_n z^n}_{T_N}$$

$$\frac{f(z)-f(w)}{z-w}-g(w)=\frac{S_N(z)+T_N(z)-S_N(w)-T_N(w)}{z-w}-g(w)$$

$$=\frac{S_N(z)-S_N(w)}{z-w}+\frac{T_N(z)-T_N(w)}{z-w}-g(w)$$

$$=\frac{S_N(z)-S_N(w)}{z-w}-S_N'(w)+\frac{T_N(z)-T_N(w)}{z-w}+S_N'(w)-g(w)$$

$$\frac{T_N(z)-T_N(w)}{z-w}=\frac{\sum_{n=N+1}^{\infty} a_n z^n-\sum_{n=N+1}^{\infty} a_n w^n}{z-w}=\sum_{k=N+1}^{\infty} a_n\left(\frac{z^k-w^k}{z-w}\right)$$

$$\frac{z^k-w^k}{z-w}=z^{k-1}+z^{k-2}w^1+z^{k-3}w^2+\dots+z^2w^{k-3}+zw^{k-2}+w^{k-1}$$

$z\in \overline{B(w,\delta)}$. Ahora, fijado $w\in B(0,R)$, fijamos $r>0$ tq $0<r<R$ y $|w|<r$. Además $\overline{B(w,\delta)}\subset B(0,r)$

$$\therefore \left|\frac{z^k-w^k}{z-w}\right|=\left|z^{k-1}+z^{k-2}w+z^{k-3}w^2+\dots+z^2w^{k-3}+zw^{k-2}+w^{k-1}\right|$$

$$\leq |z^{k-1}|+|z^{k-2}w|+\dots+|zw^{k-2}|+|w^{k-1}|$$

$$\leq k r^{k-1}$$

$$\left| \frac{T_N(z) - T_N(w)}{z-w} \right| = \left| \sum_{k=N+1}^{\infty} a_k \left(\frac{z^k - w^k}{z-k} \right) \right| \leq \sum_{k=N+1}^{\infty} |a_k| \left| \frac{z^k - w^k}{z + w} \right|$$

$$\leq \sum_{k=N+1}^{\infty} |a_k| k r^{k-1}.$$ Como $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ tiene

radio de convergencia R y $r < R \Rightarrow \exists N_0 \in \mathbb{N}$ tq $\forall n \geq N_0$

$\sum_{k=N+1}^{\infty} |a_k| k r^{k-1} < \varepsilon$ (Previo fijar $\varepsilon > 0$).

$S_N'(w) - g(w) = \sum_{k=\cancel{1}}^{N} k a_k w^{k-1} - \sum_{k=1}^{\infty} k a_n w^{k-1}$. Como $\lim_N S_N'(w) = g(w)$

Existe $N_1 \in \mathbb{N}$ tq $\forall n \geq N_1$: $|S_N'(w) - g(w)| < \varepsilon$

Tomamos $N_2 = \max\{N_1, N_2\}$ para estudiar $\left| \frac{S_N(z) - S_N(w)}{z-w} - S_N'(w) \right|$

Como $S_N(z)$ holomorfa y $\cancel{S_N'(w) = S_N'}$ $\lim_{z \to w} \frac{S_N(z) - S_N(w)}{z-w} = S_N'(w)$

existe $\delta > 0$ tal que $\forall z$, $0 < |z-w| < \delta$

$$\left| \frac{S_N(z) - S_N(w)}{z-w} - S_N'(w) \right| < \varepsilon$$

$\therefore$ $\forall z$, $0 < |z-w| < \delta$ se tiene que

$$\left| \frac{f(z) - f(w)}{z-w} - g(w) \right| \leq \left| \frac{S_N(z) - S_N(w)}{z-w} - S_N'(w) \right| + \left| \frac{T_N(z) - T_N(w)}{z-w} \right|$$

$$+ |S_N'(w) - g(w)|$$

$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

$\therefore$ f holomorfa y $f'(z) = g(z)$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad , \quad g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = f'(z).$$

Inductivamente :

$$f''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$$

$$f'''(z) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n z^{n-3}$$

$$\vdots$$

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\dots(n-k) a_n z^{n-k}$$

$\Rightarrow$ $f(0) = a_0$

$f'(0) = a_1 = 1!\, a_1$

$f^{(2)}(0) = 2 \cdot 1\, a_2 = 2!\, a_2$

$f^{(3)}(0) = 3 \cdot 2 \cdot 1\, a_3 = 3!\, a_3$

$\Rightarrow$ $f^{(k)}(0) = k!\, a_k$

$$\boxed{\therefore\ a_k = \frac{f^{(k)}(0)}{k!}}$$

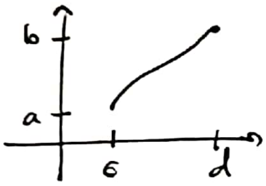
Integración compleja.

Curvas parametrizadas : $\gamma : [a,b] \to \mathbb{C}$ de clase C^1

pto inicial $= \gamma(a)$

pto final $= \gamma(b)$

Sea $h : [c,d] \to [a,b]$ difeo C^1 tal que $h' > 0$



$\tilde{\gamma} = \gamma \circ h : [c,d] \to \mathbb{C}$ reparametrización positiva de γ

$\tilde{\gamma}$ conserva puntos inicial-final.

Obs : suponemos momentaneamente que $\gamma'(t) \neq 0 \quad \forall t \in [a,b]$

$\gamma : [a,b] \to \mathbb{C}$ es llamada C^1 por tramos si es continua y $a = a_0 < a_1 < \dots < a_k = b$ tq $\gamma|_{[a_j, a_{j+1}]}$ es C^1.

Sean $\Omega \subseteq \mathbb{C}$ abierto. $f : \Omega \to \mathbb{C}$ continua, γ curva C^1 cuya imagen está contenida en Ω,

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

Prop. La integral no cambia por reparametrizaciones positivas de la curva.

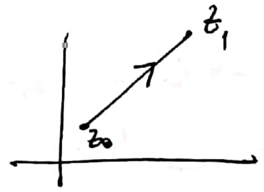
$$\int_{\tilde{\gamma}} f(z)\,dz = \int_c^d f(\tilde{\gamma}(t))\,\tilde{\gamma}'(t)\,dt = \int_c^d f(\gamma(h(t)))\,\gamma'(h(t))\,h'(t)\,dt$$

$$= \int_a^b f(\gamma(s))\,\gamma'(s)\,ds = \int_\gamma f(z)\,dz$$

teo de cambio de variable $s = h(t)$

Ej. $\int_{[z_0,z_1]} f(z)dz = \int_\gamma f(z)dz$

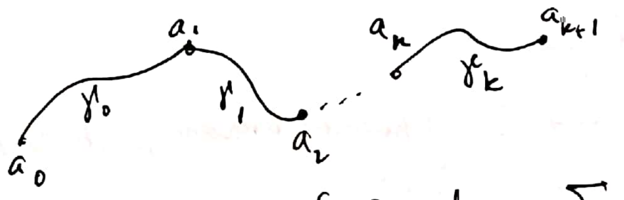
donde



$\gamma : [0,1] \longrightarrow \mathbb{C}$

$\gamma(t) = (1-t)z_0 + t z_1$

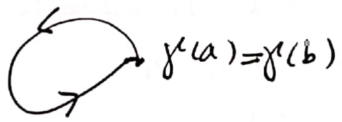
para



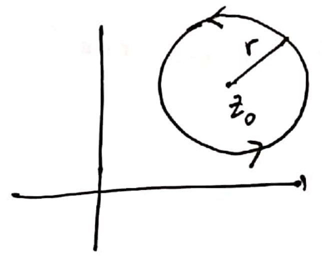
$\gamma_j = \gamma\big|_{[a_j, a_{j+1}]}$ es C^1

$$\int_\gamma f(z)dz = \sum_j \int_{\gamma_j} f(z)dz .$$

Curva cerrada $\gamma : [a,b] \to \mathbb{C}$ tq $\gamma(a) = \gamma(b)$



A veces vamos a querer calcular cosas del tipo $\int_{|z-z_0|=r} f(z)dz$

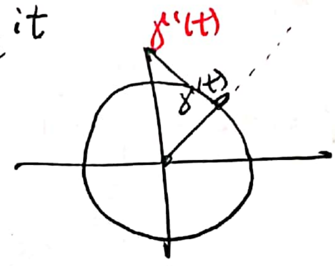


de preferencia con orientación antihoraria.

Un ejemplo: $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$

$$\int_\gamma f(z)dz = \int_{|z-z_0|=r} f(z)dz$$

Ejemplo. $\underbrace{\int_{|z|=r} \frac{1}{z} dz}_{\parallel}$, $\gamma(t) = re^{it}$, $\gamma'(t) = ire^{it}$



$$\int_0^{2\pi} \frac{1}{re^{it}} ire^{it} dt = \int_0^{2\pi} i dt = 2\pi i$$

Propiedades:

$$\int_\gamma c f(z)\,dz = c\int_\gamma f(z)\,dz$$

$$\int_\gamma (f+g)(z)\,dz = \int_\gamma f(z)\,dz + \int_\gamma g(z)\,dz.$$

Longitud de la curva: $\gamma:[a,b]\to\mathbb{C}$, $\ell(\gamma)=\int_a^b|\gamma'(t)|\,dt>0$

Prop. $\left|\int_\gamma f(z)\,dz\right| \leq \ell(\gamma)\,\underbrace{\sup_\gamma|f|}_{\text{significa } \sup\{|f(\gamma(t))| \,/\, t\in[a,b]\}}$

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right| \leq \int_a^b \underbrace{|f(\gamma(t))|}_{\leq \sup_\gamma|f|}\,|\gamma'(t)|\,dt \leq \underbrace{\sup_\gamma|f|\int_a^b|\gamma'(t)|\,dt}_{=\ell(\gamma)\sup_\gamma|f|}$$

def. Una función holomorfa $F:\Omega\to\mathbb{C}$ se llama primitiva de $f:\Omega\to\mathbb{C}$ si $F'=f$.

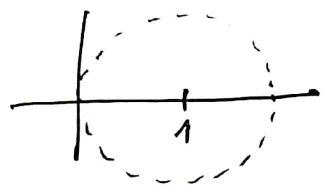
TFC: En el caso anterior, $\gamma:[a,b]\to\mathbb{C}$

$$\int_\gamma f(z)\,dz = F(\gamma(b))-F(\gamma(a)).$$

Corolario: γ curva cerrada $\Rightarrow \int_\gamma f(z)\,dz=0$, siempre y cuando f tenga primitiva.

Ej: $\int_{|z|=r}\frac{1}{z}\,dz = 2\pi i \Rightarrow$ La función $f(z)=\frac{1}{z}$ en el dominio $\Omega=\mathbb{C}\setminus\{0\}$ no tiene primitiva.

Si: $\tilde{\Omega} = \{z \in \mathbb{C} \,/\, |z-1| < 1\}$



$\tilde{f} : \tilde{\Omega} \to \mathbb{C}$ $\quad \tilde{f}(z) = \frac{1}{z}$ si tiene primitiva (A ver después).

__Dem del TFC- complejo__.

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b F'(\gamma(t))\gamma'(t)dt$$

$$= \int_a^b (F\circ\gamma)'(t)\,dt = (F\circ\gamma)(t)\Big|_{t=a}^{t=b} = F\circ\gamma(b) - (F\circ\gamma)(a)$$

TFC usual aplicado a partes real e imaginaria

__Teo de la función inversa.__

Sea $f : \Omega \to \mathbb{C}$ $\underbrace{\text{holomorfa}}_{f' \text{ continua}}$. Si $z_0 \in \Omega$ es tal que $\underbrace{f'(z_0) \neq 0}_{\cancel{f' \text{ continua}}\ y}$.

Entonces $\exists$ vecindad V de z_0 tq

(1) $f|_V : V \to \mathbb{C}$ es inyectiva

(2) La imagen $W = f(V)$ es un conjunto abierto.

(3) $g = (f|_V)^{-1} : W \to V \subseteq \mathbb{C}$ es holomorfa.

(4) $g'(w) = \dfrac{1}{f'(g(w))}$ $\quad \forall w \in W$.

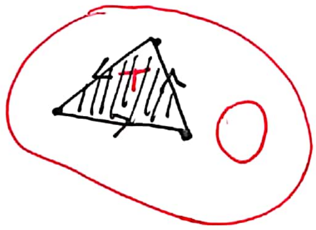
Teorema de Cauchy.

Abierto $\Omega \subseteq \mathbb{C}$ . . . .
(hipótesis topológicas)

$f : \Omega \to \mathbb{C}$ holomorfa $\Rightarrow$ $\forall$ curva cerrada γ con imagen contenida en Ω, $\int_\gamma f(z)dz = 0$. | Ω = disco abierto  ok
$\Omega = \mathbb{C} \setminus \{0\}$ No!

Teo de Goursat (versión débil del teo de Cauchy).

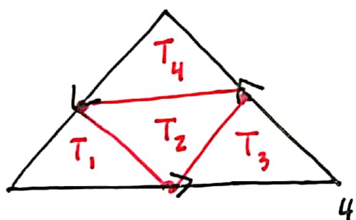
Sea $f : \Omega \to \mathbb{C}$ holomorfa. T un triángulo cerrado "lleno" contenido en Ω.
Sea γ el borde de T recorrido en sentido anti-horario

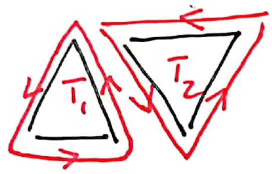


Entonces $\int_\gamma f(z)dz = 0$.

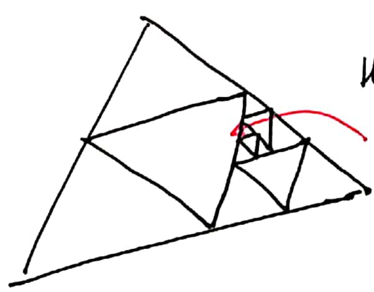
dem: Idea $\quad \gamma = \partial T \qquad T = T_1 \cup T_2 \cup T_3 \cup T_4$

$\gamma_j = \partial^+ T_j$





$$0 \neq \int_\gamma f(z)\,dz = \sum_{j=1}^{4} \underbrace{\int_{\gamma_j} f(z)dz}_{\neq 0 \text{ para algún } j}.$$



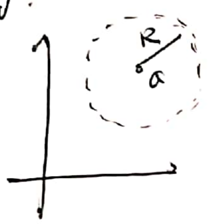
Hacer triángulos recursivamente

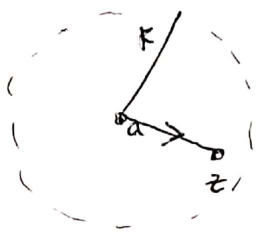
Convergencia a un punto donde f no es holomorfa.

Integración compleja (Da-Silva)

Primitiva: $f:\Omega\to\mathbb{C}$ holomorfa. $F:\Omega\to\mathbb{C}$ primitiva de f ssi F holomorfa y $F'=f$.

Ej. $\Omega = B(a,R)$





$\forall z\in B(a,R)$

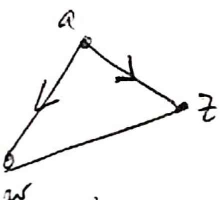
$[a,z]=\{(1-t)a+tz \,/\, t\in[0,1]\}$

Def: $F(z)=\int_{\gamma} f(z)\,dz$

$\gamma=[a,z]$.

$w\in B(a,R)$,

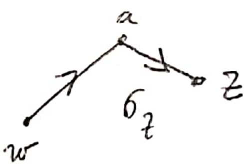
$$\frac{F(z)-F(w)}{z-w}=\frac{\int_{[a,z]} f(z)\,dz-\int_{[a,w]} f(z)\,dz}{z-w}$$



$$\int_{[a,z]} f(z)\,dz-\int_{[a,w]} f(z)\,dz = \,?$$

$$\int_{[a,z]} f(z)\,dz-\int_{[a,w]} f(z)\,dz=\int_{[a,z]} f(z)\,dz+\int_{[w,a]} f(z)\,dz$$

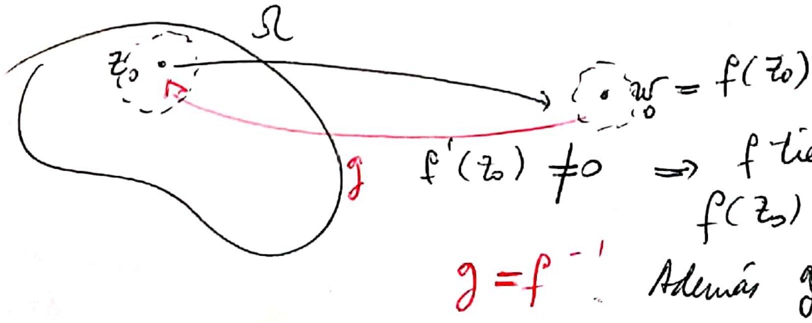
$$=\int_{\sigma_z} f(z)\,dz$$, donde



teo de la función inversa: Falta demostración!

$f:\Omega\to\mathbb{C}$ holomorfa, f' continua

$f\in C^1(\Omega)$



$f'(z_0)\neq 0 \Rightarrow f$ tiene inversa local en $\forall z\in$ vec de w_0

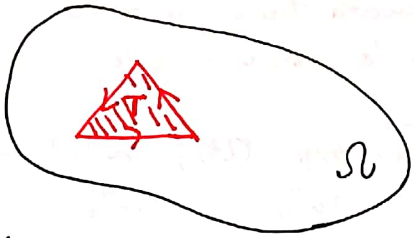
$f(z_0)=w_0$

$g=f^{-1}$. Además $g'(f(z))f'(z)=1$

$\Rightarrow g'(f(z))=1/f'(z)$

Teorema de Goursat.

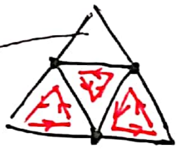
$f: \Omega \to \mathbb{C}$ holomorfa. $T \subset \Omega$ triángulo "lleno".



γ curva cerrada que recorre todo ∂T. Recorrido en sentido antihorario ($\gamma = \partial^+ T$

Af : $\displaystyle\int_\gamma f(z)dz = 0.$

dem. Dado T_0, lo particionamos en 4 subtriángulos ($T = T_0$

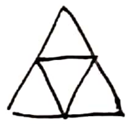


Elegimos el subtriángulo T' que maximiza $\left|\int_{\partial^+ T'} f(z)dz\right|$

Af. $\displaystyle\left|\int_{\partial^+ T'} f(z)dz\right| \geq \frac{1}{4}\left|\int_{\partial^+ T} f(z)dz\right|$

La suma de las 4 integrales (∂^+ subtriángulos)

es $= \displaystyle\int_{\partial^+ T} f(z)dz.$



Llamamos $T' := T_1$. Reempezamos y obtenemos sucesión $T_0 \supset T_1 \supset T_2 \supset \dots$

$I_n = \displaystyle\int_{\partial^+ T_n} f(z)dz \quad , \quad |I_{n+1}| \geq \frac{1}{4}|I_n| \quad \forall n$

$|I_n| \geq \frac{1}{4^n}|I_0|$

$\operatorname{diam}(T_n) = \frac{1}{2^n}\operatorname{diam}(T_0)$

$\displaystyle\bigcap_{n \in \mathbb{N}} T_n = \{z_0\} \quad , \quad z_0 \in \Omega$

f es holomorfa en z_0

$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$

$\displaystyle\lim_{z \to z_0} \psi(z) = 0.$

$$\underbrace{\left|\int_{\partial^+T_n} f(z)\,dz\right|}_{\substack{\| \\ |I_n| \\ \| \\ \frac{1}{4^n}|I_0|}} \leq \left|\underbrace{\int_{\partial^+T_n}\left[f(z_0)+f'(z)(z-z_0)\right]dz}_{=0}\right| + \left|\int_{\partial^+T_n}\Psi(z)(z-z_0)\,dz\right|$$

función tiene primitiva,
y la curva es cerrada.

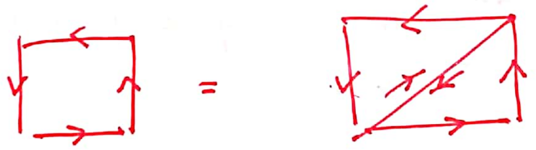
$$\leq \ell(\partial^+T_n)\sup_{z\in T_n}|\Psi(z)|\ \underbrace{\sup_{z\in T_n}|z-z_0|}_{\leq \operatorname{diam}(T_n) = \frac{1}{2^n}\operatorname{diam}(T_0)}$$

$$= \frac{1}{2^n}\ell(\partial^+T_0)$$

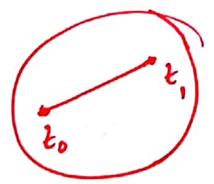
$$\forall n: \quad |I_0| \leq \text{constante}\ \underbrace{\sup_{z\in T_n}|\Psi(z)|}_{\substack{\to 0 \\ \text{cuando } n\to\infty}}$$

$$\therefore\ I_0 = 0$$

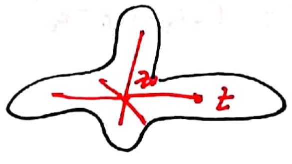
<u>Corolario</u>. El teo de Goursat también se puede aplicar con rectángulos



<u>Def</u>. $C \subseteq \mathbb{C}$ es convexo si $\forall z_0, z_1 \in C$, el segmento $[z_0, z_1] = \{(1-t)z_0 + t z_1 : t\in[0,1]\}$ esté contenido en C



<u>Def</u>. $C \subseteq \mathbb{C}$ es estrellado (Star-Shaped) con respecto a $z_0 \in C$ si $\forall z \in C$,
$[z_0, z] \subset C$



Lema. Sea $\Omega \subseteq \mathbb{C}$ abierto, estrellado con respecto a $z_0 \in \Omega$. Sea $f: \Omega \to \mathbb{C}$ continua. Sea $F(z) := \int_{[z_0, z]} f(w)\,dw$. $(F(z_0) = 0)$

F es holomorfa en $z = z_0$ y $F'(z_0) = f(z_0)$

dem. $F'(z_0) = \lim_{z \to z_0} \dfrac{F(z) - F(z_0)}{z - z_0} = \lim_{z \to z_0} \dfrac{1}{z - z_0} \int_{[z_0, z]} f(w)\,dw$

$$\left| \frac{1}{z-z_0} \int_{[z_0,z]} f(w)\,dw - f(z_0) \right| = \left| \frac{1}{z - z_0} \int_{[z_0,z]} [f(w) - f(z_0)]\,dw \right|$$

$$\leq \cancel{\frac{1}{|z-z_0|} \ell([z_0,z])} \underbrace{\sup_{w \in [z_0,z]} |f(w) - f(z_0)|}_{\xrightarrow[z \to z_0]{} 0}$$

$[z_0, z]$ compacto.

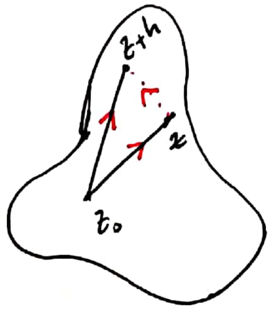
¿Qué pasa en los otros ptos z para $F(z)$?

- **Teorema de Cauchy (1ª Versión)** = Teo de Goursat.
- **Teorema de Cauchy (2ª Versión)**: $f: \Omega \to \mathbb{C}$ holomorfa. Supongamos que Ω estrellado. Entonces f tiene una primitiva en Ω. En particular, $\forall$ curva cerrada γ, $\int_\gamma f(z)\,dz = 0$

dem. $\exists z_0 \in \Omega$ tq Ω es estrellado con respecto a z_0.

$z \in \Omega$: $\quad F(z) := \int_{[z_0, z]} f(w)\,dw$

$$F'(z) = \lim_{h \to 0} \frac{F(z+h) - F(z)}{h}$$



Sea T el triángulo con vértices $z_0, z, z+h$.

Goursat: $\int_{\partial T} f(w)\,dw = 0$

$= -F(z+h) + F(z) + \int_{[z, z+h]} f(w)\,dw$

$$F'(z) = \lim_{h \to 0} \frac{1}{h} \int_{[z, z+h]} f(w)\,dw = f(z)$$

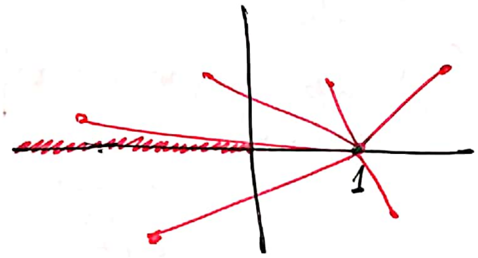
Seno

Es decir, F es primitiva de f. ∎

$$\int_{|z|=r} \frac{dz}{z} = 2\pi i \quad . \quad f(z) = \frac{1}{z} \text{ en el dominio } \mathbb{C} \setminus \{0\} \text{ no tiene primitiva.}$$

Consideremos $\Omega := \mathbb{C} \setminus (-\infty, 0]$ es estrellado c/resp a 1

c/r a z_0



Ω estrellado, $f : \Omega \to \mathbb{C}$ holo

$F(z) = \int_{[z_0, z]} f(w)\,dw$ primitiva de $f(z)$. $\quad F(z_0) = 0$

Busquemos una primitiva de $f : \Omega \to \mathbb{C} \quad z \mapsto \frac{1}{z}$.

$F : \Omega \to \mathbb{C}$ holomorfa tq $F' = f \quad \left(F(1) = 0 \right)$

$g(0) = 1$

Cualquier inversa local g de F satisface

$F'(1) = f(1) = 1 \neq 0$

$\exists$ inversa local en vecindad de $z = 1$

$$z \underset{g}{\overset{F}{\rightleftarrows}} w = F(z)$$

$g'(F(z))$

$\shortparallel$

$$g'(w) = \frac{1}{F'(z)} = z \quad , \quad \boxed{g'(w) = g(w)}$$

$$g(w) = c\,e^{w} \quad , \quad c = 1$$

$$\therefore \; g(w) = e^{w} \quad \text{dominio?}$$

$z = x + iy \in \Omega$

$w = \xi + i\eta$

$$z = e^{w} = e^{\xi}(\cos\eta + i \operatorname{sen}\eta)$$

$$\begin{cases} e^{\xi}\cos\eta = x \\ e^{\xi}\operatorname{sen}\eta = y \end{cases}$$

$$e^{2\xi}(\cos^2\eta + \operatorname{sen}^2\eta) = x^2+y^2$$

$$e^{\xi} = \sqrt{x^2+y^2}$$

$$\xi = \ln\sqrt{x^2+y^2}$$

$$\tan\eta = \frac{y}{x}$$

$$\eta = \arctan\frac{y}{x} = \theta$$

$$F(z) = w$$

$$= \log\sqrt{x^2+y^2} + i\,\underbrace{\eta(x,y)}_{\text{ángulo}}$$

: Rama del logaritmo.

Obs. Podemos crear otras ramas del logaritmo, mediante ~~$F_k(z) = z$~~.

$$F_k(z) = \log\sqrt{x^2+y^2} + i\eta(x,y) + 2\pi i k \quad , \quad k \in \mathbb{Z}.$$

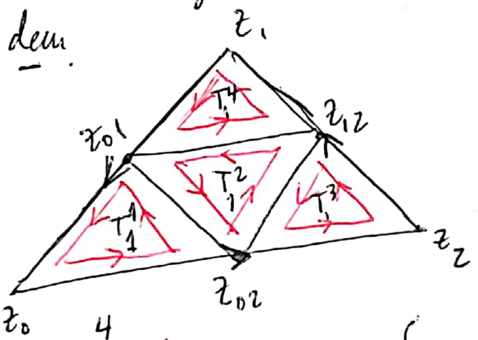
teo. de Goursat

- $f: \Omega \to \mathbb{C}$ holomorfa
- $T \subseteq \Omega$,

Af. $\displaystyle\int_{\gamma} f(z)\,dz = 0$, $\gamma = \partial T$,

$\gamma = \partial^{+} T$ recorrido en sentido antihorario.

dem.



$$\int_{\gamma} f(z)\,dz = \int_{[z_0, z_2]} f(z)\,dz + \int_{[z_2, z_1]} f(z)\,dz + \int_{[z_1, z_0]} f(z)\,dz$$

$$\sum_{j=1}^{4} \int_{\partial^{+} T_1^{j}} f(z)\,dz = \int_{[z_0, z_{02}]} f(z)\,dz + \cancel{\int_{[z_{02}, z_{01}]} f(z)\,dz} + \int_{[z_{01}, z_0]} f(z)\,dz$$

$$+ \cancel{\int_{[z_{02}, z_{12}]} f(z)\,dz} + \cancel{\int_{[z_{12}, z_{01}]} f(z)\,dz} + \cancel{\int_{[z_{01}, z_{02}]} f(z)\,dz}$$

$$+ \int_{[z_{02}, z_2]} f(z)\,dz + \int_{[z_2, z_{12}]} f(z)\,dz + \cancel{\int_{[z_{12}, z_{02}]} f(z)\,dz} + \cancel{\int_{[z_{01}, z_{12}]} f(z)\,dz} + \int_{[z_{12}, z_1]} f(z)\,dz$$

$$+ \int_{[z_1, z_{01}]} f(z)\,dz$$

$$= \left(\int_{[z_0, z_{02}]} f(z)\,dz + \int_{[z_{02}, z_2]} f(z)\,dz \right) + \left(\int_{[z_2, z_{12}]} f(z)\,dz + \int_{[z_{12}, z_1]} f(z)\,dz \right) + \left(\int_{[z_1, z_{01}]} f(z)\,dz + \int_{[z_{01}, z_0]} f(z)\,dz \right)$$

$$= \int_{[z_0, z_2]} f(z)\,dz + \int_{[z_2, z_1]} f(z)\,dz + \int_{[z_1, z_0]} f(z)\,dz = \int_{\gamma} f(z)\,dz .$$

$\therefore \int_\gamma f(z)dz = \sum_{j=1}^{4} \int_{\partial^+ T_j^1} f(z)dz$

existe además $\max_{j=1,2,3,4} \left\{ \left| \int_{\partial^+ T_j^1} f(z)dz \right| \right\}$.

Af. $|I_1| = \max_{j=1,2,3,4} \left\{ \left| \int_{\partial^+ T_j^1} f(z)dz \right| \right\} \geq \frac{1}{4} |I_0|$

$I_0 = \int_\gamma f(z)dz$

dem. Sup $|I_1| < \frac{1}{4} |I_0|$

$|I_1| < \frac{1}{4} |I_0| = \frac{1}{4} \left| \sum_{j=1}^{4} \int_{\partial^+ T_j^1} f(z)dz \right| \leq \frac{1}{4} \sum_{j=1}^{4} \left| \int_{\partial^+ T_j^1} f(z)dz \right|$

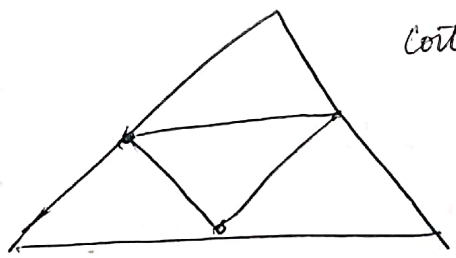
$\leq \frac{1}{4} \sum_{j=1}^{4} |I_1| = |I_1|$ $(\Rightarrow\Leftarrow)$.

- Repetimos el proceso sobre T_1 (tq $I_1 = \int_{\partial^+ T_1} f(z)dz$)
- Obtenemos sucesión de triángulos

$T = T_0 \supset T_1 \supset T_2 \supset T_3 \supset \dots \supset T_n \supset T_{n+1} \supset \dots$

tq $|I_{n+1}| \geq \frac{1}{4} |I_n| \quad \forall n$

Af. $|I_n| \geq \frac{1}{4^n} |I_0|$



corte en ptos medios!

$\text{diam}(T_1) = \frac{1}{2} \text{diam}(T_0)$

$\text{diam}(T_2) = \frac{1}{2} \text{diam}(T_1)$

$\vdots$

$\text{diam}(T_n) = \frac{1}{2} \text{diam}(T_0)$.

T_n compacto $\forall n \in \mathbb{N} \Rightarrow \bigcap_{n\in\mathbb{N}} T_n = \{\alpha\}$, $\alpha \in \Omega$, $\alpha \in T$

$\alpha \in \Omega$ y f es holomorfa en $z=\alpha$

$$f(\alpha+h) = f(\alpha) + f'(\alpha)h + \Psi(h)h \quad , \quad \Psi(h) \xrightarrow{h\to 0} 0$$

$h = z-\alpha$

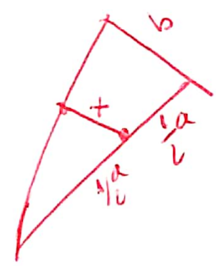
$$f(z) = f(\alpha) + f'(\alpha)(z-\alpha) + \Psi(z-\alpha)(z-\alpha)$$

$$\lim_{z\to\alpha} \Psi(z-\alpha) = 0.$$

$$\left|\int_{\partial^+T_n} f(z)\,dz\right| = \left|\int_{\partial^+T_n} \left[f(\alpha) + f'(\alpha)(z-\alpha) + \Psi(z-\alpha)(z-\alpha)\right] dz\right|$$

$$\le \underbrace{\left|\int_{\partial^+T_n} f(\alpha)\,dz\right|}_{=0} + \underbrace{\left|\int_{\partial^+T_n} (f'(\alpha)(z-\alpha))\,dz\right|}_{=0} + \left|\int_{\partial^+T_n} \Psi(z-\alpha)(z-\alpha)\,dz\right|$$

∂^+T_n curva cerrada
$f(\alpha)$ y $f'(\alpha)(z-\alpha)$
tienen primitiva



$$\frac{\frac{1}{2}a}{x} = \frac{a}{b}$$

$$\boxed{x = \frac{1}{2}b}$$

$$\Rightarrow \underbrace{\left|\int_{\partial^+T_n} f(z)\,dz\right|}_{\ge \frac{1}{4^n}\left|\int_{\partial^+T_0} f(z)\,dz\right|} \le \left|\int_{\partial^+T_n} \Psi(z-\alpha)(z-\alpha)\,dz\right|$$

$$\le \sup_{\partial^+T_n} |\Psi(z-\alpha)(z-\alpha)|\, \ell(\partial^+T_n)$$

$$= \sup_{\partial^+T_n} |\Psi(z-\alpha)|\,|z-\alpha|\; \ell(\partial^+T_n)$$

$$\overset{?}{=} \sup_{\partial^+T_n} |\Psi(z-\alpha)| \underbrace{\sup_{\partial^+T_n} |z-\alpha|}_{\substack{\le diam(T_n) \\ = \frac{1}{2^n} diam(T_0)}} \underbrace{\ell(\partial^+T_n)}_{\frac{1}{2^n}\ell(\partial^+T_0)}$$

- Recurso de variable compleja: $F(z) = \int_{[z_0, z]} f(w)\,dw$

para $f : \Omega \to \mathbb{C}$ continua

Ω estrellado c/r a z_0 $\Rightarrow$ $\forall z \in \Omega : [z_0, z] \subseteq \Omega$

$\Rightarrow$ F b.d. en Ω. $(F : \Omega \to \mathbb{C})$

TFC: $f : I \to \mathbb{R}$ continua $\Rightarrow$ $F(x) = \int_a^x f(t)\,dt$

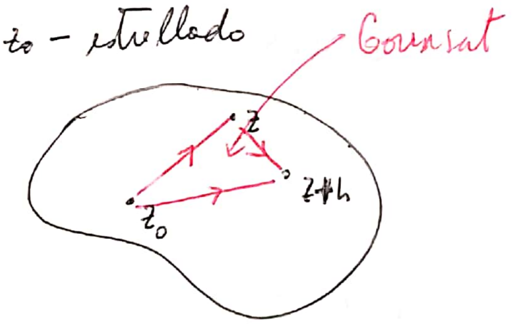
diferenciable en $\mathring{I}$, $F'(x) = f(x)$

Sea. $F : \Omega \to \mathbb{C}$ dif. en $z = z_0$ $\wedge$ $F'(z_0) = f(z_0)$.

obs. Teo Goursat no requiere teo de Cauchy $\left(\int_\gamma f(t)\,dt = 0 \right)$.

Importante: $\Omega \subseteq \mathbb{C}$ estrellado $\Rightarrow$ $f : \Omega \to \mathbb{C}$ holomorfa tiene primitiva.

Ω z_0-estrellado



$$F(z) := \int_{[z_0, z]} f(w)\,dw$$

$T = [z_0, z+h, z]$ Goursat: $\int_{\partial T} f(w)\,dw = 0$.

$$0 = \int_{\partial T} f(w)\,dw = \int_{[z_0, z+h]} f(w)\,dw + \int_{[z+h, z]} f(w)\,dw + \int_{[z, z_0]} f(w)\,dw$$

$$= F(z+h) - \int_{[z, z+h]} f(w)\,dw - F(z)$$

$$G(y) = \int_{[z, y]} f(w)\,dw \qquad G(z) = 0$$

$$\Rightarrow \frac{F(z+h) - F(z)}{h} = \frac{1}{h} \int_{[z, z+h]} f(w)\,dw = \frac{G(z+h) - G(z)}{h}$$

$$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = \lim_{h\to 0} \frac{G(z+h)-G(z)}{h} = G'(z) = f(z)$$

Lema.

$$\therefore \quad F'(z) = f(z).$$

Además, $\forall \gamma \subseteq \Omega$ cerrada, $\int_\gamma f = 0$.

pto final, pto inicial de γ

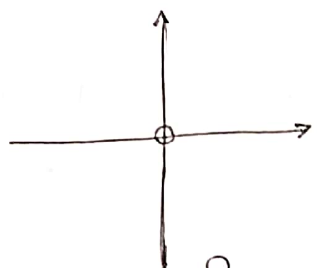
F primitiva de f, $\int_\gamma f(z)\,dz = F(b) - F(a) = 0$.

Exponencial y logaritmo.

$f : \Omega \to \mathbb{C}$, $\Omega = \mathbb{C} \setminus \{0\}$. $f(z) = \frac{1}{z}$

Ej. $\int_{|z|=r} f(z)\,dz = 2\pi i \implies f$ no tiene primitiva en Ω

$\Omega = \mathbb{C} \setminus \{0\}$.



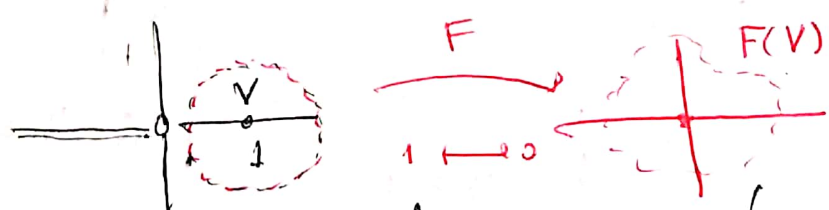
Redefinimos

$\Omega := \mathbb{C} \setminus (-\infty, 0] \implies \Omega$ es estrellado c/r $z = 1$

$\implies F(z) = \int_{[1,z]} f(w)\,dw$ primitiva de $f(z)$. $F(1) = 0$.

$\boxed{F' = f}$ Como $F'(1) = f(1) = 1 \neq 0 \mid F : \Omega \to \mathbb{C}$

$\implies \exists V \subseteq \Omega$ vecindad de $z = 1$ tq $F : V \to \mathbb{C}$ con inversa local.



$g := F^{-1} : F(V) \to V$ holomorfa, $g'(F(z)) = \frac{1}{F'(z)} = \frac{1}{1/z} = z$

$\Rightarrow g'(w) = g'(F(z)) = z = g(w)$. $\cancel{\forall z \in V}$. $\forall w \in F(V)$.

$$\Rightarrow g'(w) = g(w) \Rightarrow g(w) = ce^{w}$$

$g(0) = 1$: $g(0) = ce^{0} = c = 1 \Rightarrow g(w) = e^{w}$

$z = x+iy$, $w = \xi + i\eta$ $\Big|$ $e^{w} = e^{\xi}(\cos\eta + i \operatorname{sen}\eta)$.

$g(w) = z \Rightarrow e^{\xi}(\cos\eta + i\operatorname{sen}\eta) = x+iy \Rightarrow \begin{cases} e^{\xi}\cos\eta = x \\ e^{\xi}\operatorname{sen}\eta = y \end{cases}$

$\Rightarrow (e^{\xi})^2 = x^2+y^2 \Rightarrow e^{\xi} = \sqrt{x^2+y^2} = |z| \Rightarrow \xi = \ln|z|$.

Además $\dfrac{\operatorname{sen}\eta}{\cos\eta} = \dfrac{y}{x} \Rightarrow \tan\eta = \dfrac{y}{x} \Rightarrow \eta = \arctan\dfrac{y}{x} = \arg(z)$

$\therefore w = \ln|z| + i\arg(z)$.

$$e^{w} = e^{\ln|z| + i\arg(z)} = |z|e^{i\arg(z)} = z$$

$\therefore w = \ln(z)$, $z \in \Omega$.

obs. $\cancel{w' = w + 2\pi k i} \Rightarrow$ $F(z) = \ln(z)$ satisface $e^{F(z)} = z$

F se llama ramo del logaritmo. $F'(z) = \dfrac{1}{z}$

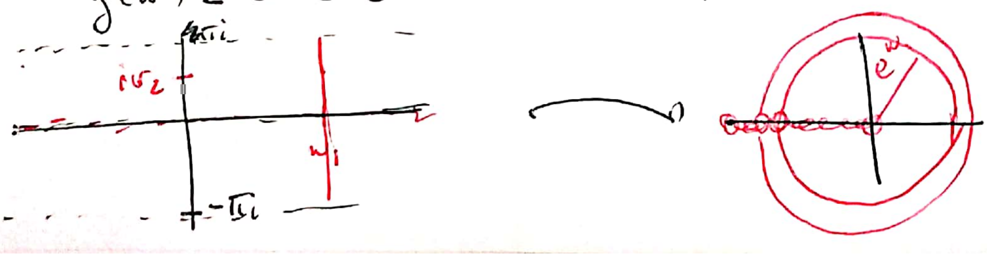
- $\tilde{F}(z) = F(z) + 2\pi k i$ $(k \in \mathbb{Z})$ cumple $e^{\tilde{F}(z)} = e^{F(z)+2\pi k i} = e^{F(z)} = z$

obs. Resolvimos la ecuación $w = e^{z}$

¿Donde se invierte F?

$g : \Gamma \longrightarrow \mathbb{C}$ $g(w) = e^{w}$

$g(w) = e^{w} = e^{w_1 + iw_2} = e^{w_1}(\cos w_2 + i \operatorname{sen} w_2)$



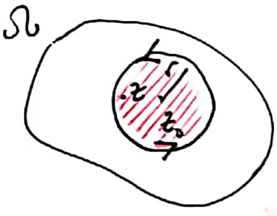
$\therefore \Gamma = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \in (-\pi, \pi)\}$

Tarea. www.mat.uc.cl/~jairo.bochi/ensenanza/MPG3950.htlm.

<u>Fórmula integral de Cauchy</u>.

Teo. (FIC. 1° versión).

$f : \Omega \to \mathbb{C}$ holomorfa. $\Omega \supseteq \overline{B(z_0, r)}$ $\mid \Omega \subseteq \mathbb{C}$ abierto.



Entonces, para todo $z \in B(z_0, r)$, $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_{|w-z_0|=r} \dfrac{f(w)}{w-z}\, dw$

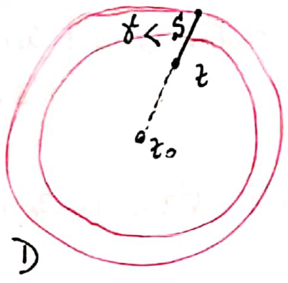
Sup. $\exists\, r' > 0$ tq $z \in \overline{B(z_0, r')} \subset B(z_0, r)$

$\Rightarrow f(z) = \dfrac{1}{2\pi i} \displaystyle\int_{|w-z_0|=r'} \dfrac{f(w)}{w-z}\, dw = \dfrac{1}{2\pi i} \displaystyle\int_{|w-z_0|=r} \dfrac{f(w)}{w-z}$

$\displaystyle\int_{|w-z_0|=r'} \frac{f(w)}{w-z}\, dw = \int_{|w-z_0|=r} \frac{f(w)}{w-z}\, dw$

dem. Sea D disco abierto de centro z_0 tq $\overline{B(z_0, r)} \subset D \subset \Omega$

$\gamma = (|w - z| = r)$



Fijamos $z \in B(z_0, r)$

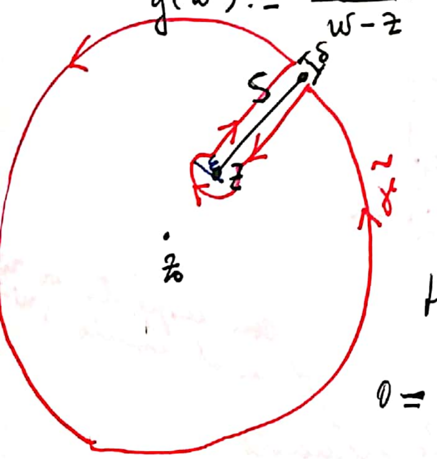
$D \setminus S$ estrellado.

$f : \Omega \to \mathbb{C}$ holomorfa
$D \subseteq \Omega$ ab. tq $\overline{B(z_0, r)} \subset D$. D estrellado
$f|_D : D \to \mathbb{C}$ holomorfa.

S segmento contenido en un radio de D como en el dibujo.

Obs. $z = z_0 \Rightarrow S =$ radio cualquiera de D.

$g(w) := \dfrac{f(w)}{w-z}$ es holomorfa en $D \setminus S$



Cauchy $\Rightarrow \displaystyle\int_{\tilde{\gamma}_{\delta,\varepsilon}} g(w)\, dw = 0 \quad \forall \delta, \varepsilon > 0$ pequeños.

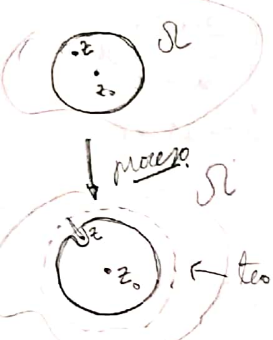
Haciendo $\delta \to 0$, obtenemos

$$0 = \lim_{\delta \to 0} \int_{\tilde{\gamma}_{\delta,\varepsilon}} g(w)\, dw = \underbrace{\int_{|w-z_0|=r} g(w)\, dw}_{\overset{?}{=}\, 2\pi i\, f(z)} - \int_{|w-z|=\varepsilon} g(w)\, dw$$

¡importante hacer esto!

$$g(w) = \frac{f(w)}{w-z} = \frac{f(w)-f(z)}{w-z} + \frac{f(z)}{w-z}$$

$$\int_{|w-z|=\varepsilon} g(w)\,dw = \underbrace{\int_{|w-z|=\varepsilon} \underbrace{\frac{f(w)-f(z)}{w-z}}_{\text{acotada uniformemente como función de } w \text{ en } B(z,\varepsilon)\setminus\{z\}} dw}_{\varepsilon \to 0} + \underbrace{\int_{|w-z|=\varepsilon} \frac{f(z)}{w-z}\,dw}_{= f(z)\,2\pi i}$$



← teo de Cauchy 0 sobre un dominio estrellado

porque $\lim_{w \to z} \frac{f(w)-f(z)}{w-z}$ existe

$\Rightarrow \frac{f(w)-f(z)}{w-z}$ acotada en vecindad $B(z,\varepsilon)\setminus\{z\}$.

¡hey!

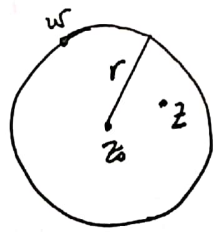
Teorema. Funciones holomorfas son analíticas. Más precisamente:

Sea f holomorfa en una vecindad de $\overline{B(z_0,r)}$. Entonces $f|_{B(z_0,r)}$ tiene expresión en serie de potencias

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad (\text{convergente } \forall z \in B(z_0,r))$$

En particular, el radio de convergencia es $R \geq r$

$$z \longmapsto \frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0} \cdot \frac{1}{1-\underbrace{\left(\frac{z-z_0}{w-z_0}\right)}_{|\cdot|<1}} \to z \neq w$$



$$= \frac{1}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n$$

$$\frac{f(w)}{w-z} = \sum_{n\geq 0}^{\infty} \frac{f(w)}{(w-z_0)^{n+1}} (z-z_0)^n$$ es uniformemente absolutamente convergente.

con respecto a w en el círculo $|w-z_0|$

$z \in B(z_0,r)$

$w \in \partial B(z_0,r)$

$z \in B(z_0,r)$ fijo.

Integramos término a término | Convergencia uniforme $\Rightarrow$ integración término a término

$$\left(\frac{1}{2\pi i} \int_{|w-z_0|=r} \dots \, dw \right)$$

Y así obtenemos:

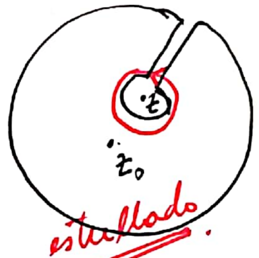
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad , \quad a_n = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}} \, dw$$

Fórmula de Taylor: $a_n = \dfrac{f^{(n)}(z_0)}{n!}$

__Corolario__. De la fórmula integral de Cauchy tenemos

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z)^{n+1}} \, dw \qquad \forall n \geq 1$$

__dem__. Si $z = z_0$ entonces ya está: salió como corolario de la demostración anterior.



estrellado.

Por el caso ya probado:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|w-z|=\epsilon} \frac{f(w)}{(w-z)^{n+1}} \, dw$$

$$= \frac{n!}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z)^{n+1}} \, dw$$

Mismo argumento anterior (ojo de cerradura).

key hole (ojo de cerradura)

observación pendiente estudiar después!

Teorema de Liouville

$f: \mathbb{C} \to \mathbb{C}$ holomorfa y acotada, entonces es constante.

dem. Usando fórmula integral de Cauchy.

Como $\mathbb{C}$ es conexo, es suficiente probar que $f' \equiv 0$

$$f'(z) = \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w^2} \, dw \quad \text{si} \quad r > |z|$$

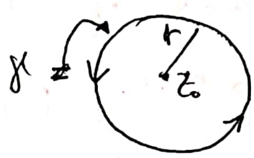
$$|f'(z)| = \left| \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w^2} \, dw \right| \leq \frac{1}{2\pi} 2\pi r \cdot \frac{c}{r^2} = \frac{c}{r} \xrightarrow{r \to \infty} 0$$

$$\therefore \quad f'(z) = 0$$

$\blacksquare$

Estimación de Cauchy:

tenemos. $f^{(n)}(z_0) = \dfrac{n!}{2\pi i}\displaystyle\int_\gamma \dfrac{f(w)}{(w-z_0)^{n+1}}\,dw$



$\gamma(t) = z_0 + re^{it}, \quad t\in[0,2\pi)$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{it})\, rie^{it}}{(re^{it})^{n+1}}\,dt$$

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\int_0^{2\pi} \frac{f(z_0+re^{it})\, rie^{it}}{r^{n+1}e^{(n+1)it}}\,dt\right|$$

$$\leq \frac{n!}{2\pi}\int_0^{2\pi} \frac{|f(z_0+re^{it})|\,r}{r^{n+1}}\,dt = \frac{n!}{2\pi r^n}\int_0^{2\pi} |f(z_0+re^{it})|\,dt$$

$$\leq \frac{n!\,\sup_\gamma \|f\|}{2\pi r^n}\int_0^{2\pi} dt = \frac{n!\,\sup_\gamma |f|}{r^n}$$

Hip inductiva : $f^{(n-1)}(z) = \dfrac{(n-1)!}{2\pi i}\displaystyle\int_\gamma \frac{f(w)}{(w-z)^n}\,dw$

$h \in \mathbb{C}$ tq $z+h \in B(z_0, r)$:

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i h}\left[\int_\gamma \frac{f(w)}{(w-(z+h))^n}\,dw - \int_\gamma \frac{f(w)}{(w-z)^n}\,dw\right]$$

$$= \frac{(n-1)!}{2\pi i h}\int_\gamma f(w)\left[\frac{1}{(w-(z+h))^n} - \frac{1}{(w-z)^n}\right]dw$$

$$\cancel{= \frac{(n-1)!}{2\pi i h}\int_\gamma \frac{f(w)}{(w-z_0)^n}\left[\frac{1}{1-\frac{(z+h)-z_0}{w-z_0}}\right]}$$

$$= \frac{(n-1)!}{2\pi i h}\int_\gamma f(w)\left[\frac{(w-z)^n - (w-(z+h))^n}{(w-(z+h))^n (w-z)^n}\right]dw$$

$$= \frac{(n-1)!}{2\pi i \cancel{h}}\int_\gamma f(w)\underbrace{\left[\frac{\cancel{h}\left[(w-z)^{n-1} + (w-z)^{n-2}(w-(z+h)) + \cdots + (w-z)(w-(z+h))^{n-2} + (w-(z+h))^{n-1}\right]}{(w-(z+h))^n (w-z)^n}\right]}_{\text{continua c/r a } h \text{ (justificar)}}dw$$

$\downarrow \lim_{h\to 0}$

$$\therefore f^{(n)}(z) = \cancel{\lim_{h\to 0}}\ \frac{(n-1)!}{2\pi i}\int_\gamma f(w)\left[\frac{\overbrace{(w-z)^{n-1} + (w-z)^{n-2}(w-z) + \cdots (w-z)^{n-1}}^{= n\text{-veces}}}{(w-z)^n (w-z)^n}\right]$$

$$= \frac{(n-1)!\,n}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,dw = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,dw.$$

$$\therefore \int_\gamma \frac{f(w)}{w-z}\,dw = \int_\gamma \left(\frac{f(w)}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0} \right)^n \right) dw$$

por convergencia uniforme.

$$= \sum_{n=0}^{\infty} \int_\gamma \frac{f(w)}{w-z_0} \left(\frac{z-z_0}{w-z_0} \right)^n dw$$

$$= \sum_{n=0}^{\infty} \Bigg(\underbrace{\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw}_{a_n} \Bigg) (z-z_0)^n$$

$$\therefore f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \forall z \in B(z_0, r)$$

con radio de convergencia $R \geq r$ y $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw$

Como $a_n = \frac{f^{(n)}(z_0)}{n!} \Rightarrow f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw$

$\forall n \in \mathbb{N}$.

— o —

Corolario de la fórmula integral de Cauchy:

$f: \Omega \to \mathbb{C}$ holomorfa, $\overline{B(z_0, r)} \subset \Omega$

$$\Rightarrow f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,dw$$

$\gamma = \partial B(z_0, r)$

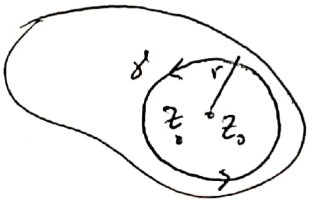
$z = z_0$ siempre se cumple. (Caso anterior).

$$f^{(0)}(z) = f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)}\,dw$$

Fórmula integral de Cauchy.

$f : \Omega \to \mathbb{C}$ holomorfa, $z_0 \in \Omega$. $\overline{B(z_0, r)} \subset \Omega$

$\gamma = \partial \overline{B(z_0, r)}$



(FIC) $\forall z \in B(z_0, r)$: $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw$

Ahora $\forall z \in B(z_0, r) \Rightarrow z \neq w \quad \forall w \in \partial \overline{B(z_0, r)}$

$$\mathbb{C} \ni \frac{f(w)}{w-z} = \frac{f(w)}{w - z_0 + z_0 - z} = f(w) \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}}$$

$$\left| \frac{z - z_0}{w - z_0} \right| = \frac{|z - z_0|}{r} \Rightarrow \frac{f(w)}{w-z} = f(w) \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n$$

importante!

$$\frac{f(w)}{w-z} = \sum_{n=0}^{\infty} \frac{f(w)}{(w - z_0)^{n+1}} (z - z_0)^n \quad w \in \partial \overline{B(z_0, r)}$$

$$F_{z,n}(w) = \frac{f(w)}{(w - z_0)^{n+1}} (z - z_0)^n$$

$$|F_{z,n}(w)| = \left| \frac{f(w)}{(w - z_0)} \right| \left| \frac{z - z_0}{w - z_0} \right|^n \leq \left| \frac{f(w)}{w - z_0} \right|$$

$\frac{f(w)}{w - z_0}$ continua en $\partial \overline{B(z_0, r)} \Rightarrow \exists \sup_{\partial B(z_0, r)} \left\{ \left| \frac{f(w)}{w - z_0} \right| \right\}$

$\forall w \in \partial \overline{B(z_0, r)}$ $\quad \left| \left(\frac{z - z_0}{w - z_0} \right)^n \right| = \frac{|z - z_0|^n}{r^n} = \left(\frac{|z - z_0|}{r} \right)^n \quad \frac{|z - z_0|}{r} < 1$

$\therefore \sum_{n=0}^{\infty} \frac{|z - z_0|^n}{r^n}$ converge. M-test Weierstrass $\rightarrow \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)$ converge uniformemente! $\forall w \in \partial \overline{B(z_0, 1)}$

(Tarea). La integral tiene un error c/resp a la variable de integración.

F. I. Cauchy : $\forall z \in B(z_0, r)$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{|w-z_0|=r} \frac{f(w)}{(w-z)^{n+1}} dw$$ Si f es holomorfa en vecindad de $B(z_0, r)$

Corolario. Estimación de Cauchy

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \left| \oint_{|z_0-w|=r} \frac{f(w)}{(w-z)^{n+1}} dw \right| \leq \frac{n!}{2\pi} 2\pi r \frac{1}{r^{n+1}} \max_{|w-z_0|=r} |f(w)|$$

ya demostrada

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{|w-z_0|=r} |f(w)|$$

arco-convexo

Teo. de Liouville. $f: \mathbb{C} \to \mathbb{C}$ holomorfa y acotada, entonces es constante.

dem. Aplicar la estimación de Cauchy con $n=1$ y $r \to \infty$. ($f' \equiv 0$).

Teo (Teorema Fundamental del Algebra)

Sea $p(z) \in \mathbb{C}[x]$ no constante, entonces $\exists \zeta \in \mathbb{C} : f(\zeta) = 0$

dem. $p(z) = a_0 + a_1 z + \dots + a_n z^n$, $n \geq 1$ $a_n \neq 0$.

$$\frac{p(z)}{z^n} = \frac{a_0}{z^n} + \dots + \frac{a_{n-1}}{z} + a_n \xrightarrow{|z| \to \infty} a_n \quad . \quad \lim_{|z|\to\infty} |p(z)| = \infty$$

$\Uparrow$ Si $p(z) \neq 0 \ \forall z \in \mathbb{C}$, entonces $f(z) := \frac{1}{p(z)}$ está definida en todo $\mathbb{C}$

Además: $\begin{cases} \to \text{es holomorfa en todo } \mathbb{C} \\ \to \text{es acotada } \left(\lim_{|z|\to\infty} p(z) = \infty \to \lim_{|z|\to\infty} f(z) = 0 \right) \end{cases}$

Corolario. (También llamado TFA)

p polinomio de grado $n \Rightarrow p(z) = a(z-\zeta_1)\dots(z-\zeta_n)$

dem. $\Big[$ Inducción sobre n
división de polinomios.

Importante: Aquí empieza a hablar de conexos!

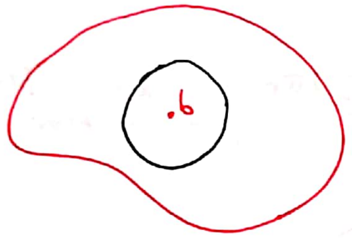
Ya vimos que : Holomorfa $\Rightarrow$ Analítica

Prop. Sea $\Omega \subseteq \mathbb{C}$ abierto conexo. Sea $\begin{cases} f : \Omega \to \mathbb{C} & \text{holomorfa} \\ f \not\equiv 0 \end{cases}$

entonces el conjunto $f^{-1}(0) = \{ z \in \Omega \,/\, f(z) = 0 \}$ no tiene puntos de acumulación en Ω (Los ceros de f son aislados)

Los ceros de funciones holomorfas no nulas no se acumulan

dem:



Sea $b \in \Omega$ t.q $f(b) = 0$

$\exists r > 0$ t.q $\forall z \in B(b,r)$ r: radio de convergencia

$$f(z) = \sum_{n \geq 0}^{\infty} a_n (z-b)^n = a_1(z-b) + a_2(z-b)^2 + \dots$$

obs: $a_0 = f(b) = 0$.

Dos casos: 1) $\forall n \in \mathbb{N}$, $a_n = 0 \Rightarrow f \equiv 0$ en $B(b,r)$

2) $\exists k := \min \{ n \,/\, a_n \neq 0 \} \geq 1$ $g(z) = 0 \Rightarrow a_k = 0$

Definimos $g(z) := a_k + a_{k+1}(z-b) + a_{k+2}(z-b)^2 + \dots$ $a_k \neq 0$

$\forall z \in B(b,r)$: $f(z) = (z-b)^k g(z)$ $g(b) \neq 0$ ($g(b) = 0 \Rightarrow a_k = 0$)

$f(z_0) \neq 0 \Rightarrow \exists r > 0$ pequeño tq $f \neq 0$ en $B(z_0, r)$

Conclusión: $\forall z \in$ vecindad de b, $z \in B(b,r) \setminus \{b\}$ $\boxed{f(z) \neq 0}$

g no puede tener ceros en $B(b,r)$

$A :=$ ptos de acumulación de $f^{-1}(0)$ en Ω

Pd: $A \subseteq \Omega$ abierto:
Sea $w \in A$, como $A = cls(A)$
$\exists \{w_k\}_{k \in \mathbb{N}} \subseteq A$ tq $w_k \to w$
Para R = radio de convergencia
f tiene infinitos ceros en $B(w,R)$
$\Rightarrow$ No se cumple (ii)
$\Rightarrow$ Se cumple (i) $\therefore B(w,R) \subseteq A$

A = cerrado en Ω ($cls\, f^{-1}(0) = f^{-1}(0)$)

A = abierto en Ω

$\Rightarrow A = \emptyset$ o $\underbrace{A = \Omega}_{\Rightarrow f \equiv 0}$

$\boxed{\therefore A = \emptyset}$

obs. k = orden del cero de f.

Corolario de la demostración. Si $f : \Omega \to \mathbb{C}$ es holomorfa y $f^{-1}(0)$ finito

$\exists$ factorización $f(z) = p(z) g(z)$ $\begin{cases} g & \text{holomorfa en } \Omega \\ p & \text{polinomio} \end{cases}$

$$f(z) = (z-a_1)^{k_1}(z-a_2)^{k_2} \dots (z-a_m)^{k_m} g(z) \quad \text{tq} \quad g(a_i) \neq 0$$

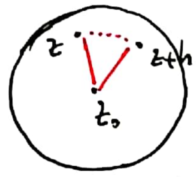
$f^{-1}(0) = \{a_1, \dots, a_m\}$
k_i: orden del cero a_i

Recíproco del teorema de Goursat,

Teo de Morera:

$\Omega \subseteq \mathbb{C}$ abierto, $f : \Omega \to \mathbb{C}$ continua, $\forall T$ triángulo $\subset \Omega$: $\int_{\partial T} f(z)dz = 0$
Entonces, f es holomorfa.

dem. SPG, $\Omega =$ disco $B(z_0, r)$
más poderoso que un conjunto estrellado



$F(z) = \int_{[z_0, z]} f(w)dw$. Entonces F es holomorfa y $F' = f$

$\int_{\partial T} f(z)dz = 0 \Rightarrow \dfrac{F(z+h) - F(z)}{h} = \dfrac{1}{h} \int_{[z+h, z]} f(w)dw \xrightarrow{h \to 0} f(z)$

$f = (\text{holomorfa})'' = \text{holomorfa}$
Corolario F.I. Cauchy

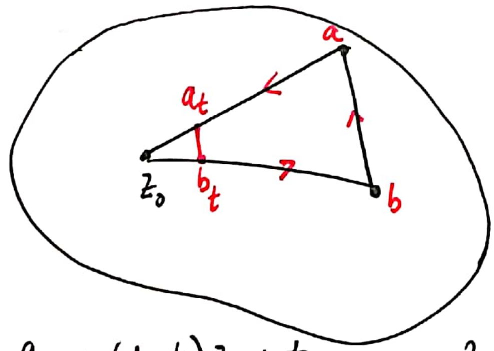
ok.

Ejemplo de Aplicación del teorema de Morera:

Sea $f : \Omega \to \mathbb{C}$ continua. Supongamos que $\exists z_0 \in \Omega$ tq $f|_{\Omega \setminus \{z_0\}}$ es holomorfa. Entonces $f : \Omega \to \mathbb{C}$ es holomorfa.

dem. Sea $T \subseteq \Omega$ triángulo. Hay que probar que $\int_{\partial T} f(z)dz = 0$

Caso 1: z_0 es vértice de T



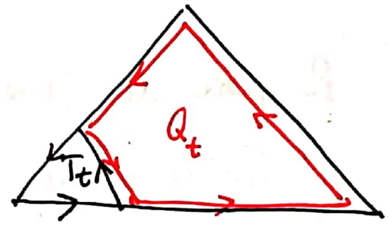
$\int_{\text{curva roja}} f = 0$

$a_t = (1-t)z_0 + ta \xrightarrow[t \to 0]{} z_0$ $t \in (0,1)$.

$b_t = (1-t)z_0 + tb$

Hasta ahora, Teo de Morera es el único que ocupa condición de continuidad de f.

T_t = triángulo con vértices z_0, a_t, b_t

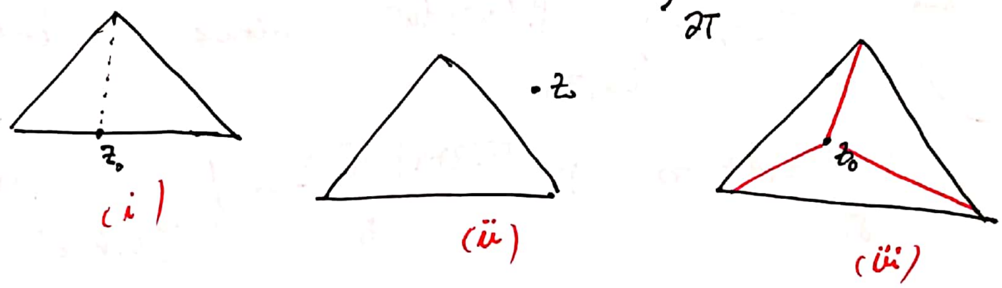


$$\lim_{t \to 0} \int_{T_t} f(z)\,dz = 0 \longrightarrow \int_{\partial T} = \underbrace{\int_{\partial T_t}}_{t \to 0} + \underbrace{\int_{\partial Q_t}}_{=0}$$

$$\int_{\partial T} f = 0$$

ok ✓

Otros casos



Pequeño ejercicio: Teo. de Cauchy (débil)
{ Teo de Green
+ Cauchy - Riemann.

Observación. Hay que demostrar que $\lim_{t \to 0} \int_{T_t} f(z)\,dz = 0$

$$\int_{\partial T_t} f(z)\,dz = \int_{[z_0, a_t]} f(z)\,dz + \int_{[a_t, b_t]} f(z)\,dz + \int_{[b_t, z_0]} f(z)\,dz$$

$$\left| \int_{[z_0, a_t]} f(z)\,dz \right| \leq \sup_{[z_0, a_t]} |f| \; \ell([z_0, a_t]) \xrightarrow{t \to 0} 0$$

Análogo con los demás sumandos

$$\therefore \lim_{t \to 0} \int_{\partial T_t} f(z)\,dz = 0$$

$$\therefore \int_{\partial T} f(z)\,dz = \lim_{t \to 0} \int_{\partial T_t} f(z)\,dz = 0$$

(1) (Teo de Liouville).

$f: \mathbb{C} \to \mathbb{C}$ holomorfa y acotado.

$\exists \alpha \in \mathbb{R}^+ : \ |f(z)| \leq \alpha \quad \forall z \in \mathbb{C} \quad \Rightarrow \quad \sup_{z \in \mathbb{C}} |f| \leq \alpha$

Estimación de Cauchy: $|f'(z)| = \frac{1}{r} \sup_{|w-z|=r} |f(w)| \leq \frac{1}{r} \sup_{w \in \mathbb{C}} |f(w)|$

$\leq \frac{1}{r} \alpha \quad \forall r > 0$

$\therefore |f'(z)| = 0 \quad \forall z \in \mathbb{C}$

$\therefore f' \equiv 0$

$\therefore f$ constante

Pregunta: ¿$f: \mathbb{C} \to \mathbb{C}$ holomorfa y $f' \equiv 0 \Rightarrow f$ constante?

$A := \{ z \in \mathbb{C} \ / \ f'(z) = 0 \} \qquad A = (f')^{-1}(0)$

f' holomorfa $\Rightarrow A$ cerrado.

f holomorfa $\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall z \in B(z_0, R) \quad R > 0.$

$= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots$

$f'(z) = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots$

$f'(z_0) = a_1 = 0$

$f''(z) = 2a_2 + 6a_3(z - z_0) + 12(z - z_0)^2 + \cdots$

$f''(z_0) = 2a_2 = \overset{???}{0}$

~~[illegible]~~ ~~[illegible]~~ $f(0) = a_0$

$A := \{ z \in \mathbb{C} \ / \ f(z) = a_0 \}$. A cerrado. (f continua).

~~[illegible]~~ $\zeta \in A$: $\forall z \in B(\zeta, r)$: $f(z) = \frac{1}{2\pi i} \int_{|w-\zeta|=r} \frac{f(w)}{w - z} \, dw$

Sean $z, w \in \mathbb{C} \Rightarrow \exists \gamma$ tq $z \xrightarrow{\gamma} w$

Como f primitiva de f' $\Rightarrow \int_\gamma f'(\zeta)d\zeta = f(\gamma(1)) - f(\gamma(0))$.
$= f(w) - f(z)$

pero $\int_\gamma f'(\zeta)d\zeta = 0$ ya que $f' \equiv 0$

$\therefore \forall z, w \in \mathbb{C} : f(w) = f(z)$

$\therefore f$ constante.

dos. $f : \Omega \to \mathbb{C}$, Ω abierto y arcoconexo, $f' \equiv 0 \Rightarrow f$ constante.

$=0=$

$f : \Omega \to \mathbb{C}$ ~~conex~~ holomorfa, Ω conexo y abierto $\Rightarrow f : \Omega \to \mathbb{C}$ holomorfa

$A = (f')^{-1}(0)$

$b \in A$ $(f'(b) = 0) \Rightarrow \begin{cases} f'(z) = 0, & \forall z \in B(b, R) \\ f'(z) = 0 & \forall z \in B(b, R) \setminus \{b\} \end{cases}$

R = radio de convergencia.

$A = \mathrm{cls}(A) \Rightarrow A$ abierto y cerrado. $A \subseteq \Omega$

Ω conexo $\Rightarrow A = \emptyset \vee A = \Omega$

Cambiar $A = \{ z \in \mathbb{C} / g(z) = f(z) - c = 0 \}$

$A = g^{-1}(0)$, $f'(z) = g'(z)$ $\forall z \in \Omega$

$b \in \Omega$: $f(z) = a_0 + a_1(z-b) + a_2(z-b)^2 + a_3(z-b)^3 + \dots$, $\forall z \in B(b, R)$

R radio de convergencia.

$f'(z) = a_1 + 2a_2(z-b) + 3a_3(z-b)^2 + \dots$

$f'(z) = 0 \Rightarrow a_1, a_2, \dots = 0$.

$\therefore f(z) = a_1$

$\Omega = \bigcup_{b \in \Omega} B(b, R)$, R = radio de convergencia (NO BASTA CON ESTO).

- $f : \Omega \to \mathbb{C}$ holomorfa (Ω abierto y conexo) tq $f' \equiv 0$ en Ω.

$b \in \Omega \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-b) \quad \forall z \in B(b,R)$

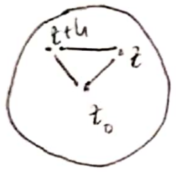
Como $f' \equiv 0 \Rightarrow f(z) = a_1$ en $B(b,R)$

$g(z) := f(z) - a_1$ tiene ~~infinitos~~ $B(b,R) \subseteq g^{-1}(0)$

$\therefore\ g \equiv 0$ en Ω

$\therefore\ f(z) = a_1 \quad \forall z \in \Omega$.

(2) (Teorema de Morera)



$f : \Omega \to \mathbb{C}$ continua.

$$F(z) := \int_{[z_0, z]} f(w)\,dw$$

Af F holomorfa en $B(z_0, r)$ y $F' = f$ $\rightarrow$ F holomorfa $\Downarrow$ $\underline{f \text{ holomorfa}}$.

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_{[z_0, z+h]} f(w)\,dw - \frac{1}{h}\int_{[z_0, z]} f(w)\,dw$$

$$= \frac{1}{h}\int_{[z_0, z+h]} f(w)\,dw + \frac{1}{h}\int_{[z, z_0]} f(w)\,dw = \frac{1}{h}\int_{[z, z+h]} f(w)\,dw$$

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \left| \frac{1}{h}\int_{[z, z+h]} f(w)\,dw - \frac{1}{h}\int_{[z, z+h]} f(z)\,dw \right|$$

$$= \frac{1}{h}\left| \int_{[z, z+h]} (f(w) - f(z))\,dw \right|$$

$$\leq \frac{1}{h} \| f(w) - f(z) \|_{\gamma}\, h \quad \underline{\text{ok}}.$$

$$\int_{[z, z+h]} dw = \int_0^1 h\,dw = h$$

$\gamma(t) = (1-t)z + t(z+h) \Rightarrow \gamma'(t) = -z + z + h$

$f(w) = 1$