Universidad de las Américas.
Algebra II, MAT141
Marzo 28, 2019

Desarrollo Cátedra 1.

Problema 1. (a) Sean $\vec{u} = \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Calcule $\vec{u}^T\vec{v}$, $\vec{v}^T\vec{u}$, $\vec{u}\vec{v}^T$, $\vec{v}\vec{u}^T$

Desarrollo. $\vec{u}^T\vec{v} = \begin{pmatrix} -3 & 2 & -5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (-3a+2b-5c)$

$$\vec{v}^T\vec{u} = \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix} = (-3a+2b-5c)$$

$$\vec{u}\vec{v}^T = \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix}\begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} -3a & -3b & -3c \\ 2a & 2b & 2c \\ -5a & -5b & -5c \end{pmatrix}$$

$$\vec{v}\vec{u}^T = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\begin{pmatrix} -3 & 2 & -5 \end{pmatrix} = \begin{pmatrix} -3a & 2a & -5a \\ -3b & 2b & -5b \\ -3c & 2c & -5c \end{pmatrix}$$

(b) Ocupando propiedades algebraicas del producto y la traspuesta:

$$\begin{aligned} (AB\vec{x})^T &= ((AB)\vec{x})^T \quad \text{(asociatividad)} \\ &= \vec{x}^T(AB)^T \\ &= \vec{x}^T(B^TA^T) \\ &= \vec{x}^TB^TA^T \end{aligned}$$

Por lo tanto: $(AB\vec{x})^T = \vec{x}^TB^TA^T$

Problema 2.

a. Existe A^{-1} siempre y cuando $\det(A) \neq 0$

$$\det(A) = \det\begin{pmatrix} 2 & 1 & 4 \\ 3 & 5 & 7 \\ 1 & 4 & a+1 \end{pmatrix}$$

$$= 2\det\begin{pmatrix} 5 & 7 \\ 4 & a+1 \end{pmatrix} - 1\det\begin{pmatrix} 3 & 7 \\ 1 & a+1 \end{pmatrix} + 4\det\begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$$

$$= 2\left(5(a+1) - 28\right) - 1\left(3(a+1) - 7\right) + 4\left(12 - 5\right)$$

$$= 10(a+1) - 56 - 3(a+1) + 7 + 48 - 20$$

$$= 10a + 10 - 3a - 3 - 21$$

$$= 7a - 14$$

$$\det(A) = 0 \iff 7a - 14 = 0 \iff a = \frac{14}{7} = 2$$

Luego, A^{-1} existe siempre y cuando $a \neq 2$.

b. Ocupamos la fórmula $A^{-1} = \dfrac{1}{\det(A)} \, Adj(A)$, donde

$$Adj(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^T$$

$$a_{11} = (-1)^2 \det\begin{pmatrix} 5 & 7 \\ 4 & 4 \end{pmatrix} = 20 - 28 = -8$$

$$a_{12} = (-1)^3 \det\begin{pmatrix} 3 & 7 \\ 1 & 4 \end{pmatrix} = -1(12 - 7) = -5$$

$$a_{13} = (-1)^4 \det\begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix} = 12 - 5 = 7$$

$$a_{21} = (-1)^3 \det\begin{pmatrix} 1 & 4 \\ 4 & 4 \end{pmatrix} = -(4 - 16) = 12$$

$$a_{22} = (-1)^4 \det\begin{pmatrix} 2 & 4 \\ 1 & 4 \end{pmatrix} = 8 - 4 = 4$$

$$a_{23} = (-1)^5 \det\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} = -(8 - 1) = -7$$

$$a_{31} = (-1)^4 \det\begin{pmatrix} 1 & 4 \\ 5 & 7 \end{pmatrix} = 7 - 20 = -13$$

Problema 3. Sistema de ecuaciones

$$\begin{cases} 2x + 2y = 8000 \\ 3y + z = 10\,500 \\ x + y + z = 7000 \end{cases}$$

x : taza de café

y : sándwich

z : helado.

Desarrollo, Ocupamos Regla de Cramer

Ecuación matricial asociada :

$$\underbrace{\begin{pmatrix} 2 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 8000 \\ 10\,500 \\ 7000 \end{pmatrix}}_{\vec{b}}$$

$$A_1(\vec{b}) = \begin{pmatrix} 8000 & 2 & 0 \\ 10\,500 & 3 & 1 \\ 7000 & 1 & 1 \end{pmatrix}$$

$$A_2(\vec{b}) = \begin{pmatrix} 2 & 8000 & 0 \\ 0 & 10\,500 & 1 \\ 1 & 7000 & 1 \end{pmatrix}$$

$$A_3(\vec{b}) = \begin{pmatrix} 2 & 2 & 8000 \\ 0 & 3 & 10\,500 \\ 1 & 1 & 7000 \end{pmatrix}$$

$$\det\left(A_1(\vec{b})\right) = 8000(3-1) - 2(10\,500 - 7000) = 16000 - 7000 = 9000$$

$$\det\left(A_2(\vec{b})\right) = 2(10\,500 - 7000) - 8000(-1) = 7000 + 8000 = 15000$$

$$\det\left(A_3(\vec{b})\right) = 2(21000 - 10\,500) - 2(-10\,500) + 8000(-3)$$

$$= 21000 + 21000 - 24000 = 18000$$

$$\det(A) = 2(3-1) - 2(-1) = 4 + 2 = 6$$

Así: $x = \dfrac{9000}{6} = 1500$, $y = \dfrac{15000}{6} = 2500$, $z = \dfrac{18000}{6} = 3000$

$$a_{32} = (-1)^5 \det \begin{pmatrix} 2 & 4 \\ 3 & 7 \end{pmatrix} = -1(14 - 12) = -2$$

$$a_{33} = (-1)^6 \det \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} = 10 - 3 = 7$$

Luego: $\mathrm{Adj}(A) = \begin{pmatrix} -8 & -5 & 7 \\ 12 & 4 & -7 \\ -13 & -2 & 7 \end{pmatrix}^T = \begin{pmatrix} -8 & 12 & -13 \\ -5 & 4 & -2 \\ 7 & -7 & 7 \end{pmatrix}$

$$\det(A) = 21 - 14 = 7$$

$$A^{-1} = \frac{1}{7} \begin{pmatrix} -8 & 12 & -13 \\ -5 & 4 & -2 \\ 7 & -7 & 7 \end{pmatrix} = \begin{pmatrix} -8/7 & 12/7 & -13/7 \\ -5/7 & 4/7 & -2/7 \\ 1 & -1 & 1 \end{pmatrix}$$

c. $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ tq $A\vec{x} = \vec{b}$

$$\vec{x} = A^{-1}\vec{b} = \begin{pmatrix} -8/7 & 12/7 & -13/7 \\ -5/7 & 4/7 & -2/7 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} -8/7 + 24/7 - 65/7 \\ -5/7 + 8/7 - 10/7 \\ 1 - 2 + 5 \end{pmatrix} = \begin{pmatrix} -7 \\ -1 \\ 4 \end{pmatrix}$$

$$\therefore \quad \vec{x} = \begin{pmatrix} -7 \\ -1 \\ 4 \end{pmatrix}$$

Segunda forma:

Resolvemos la ecuación matricial $\begin{pmatrix} 2 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8000 \\ 10\,500 \\ 7000 \end{pmatrix}$

mediante el calculo de A^{-1}:

$$\vec{x} = A^{-1}\vec{b}$$

Como $A^{-1} = \dfrac{1}{\det(A)} \cdot Adj(A)$, $Adj(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^T$

$$a_{11} = (-1)^2 \det\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = 3-1 = 2$$

$$a_{12} = (-1)^3 \det\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = -(-1) = 1$$

$$a_{13} = (-1)^4 \det\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} = -3$$

$$a_{21} = (-1)^3 \det\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = -(2-0) = -2$$

$$a_{22} = (-1)^4 \det\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = 2$$

$$a_{23} = (-1)^5 \det\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = -(2-2) = 0$$

$$a_{31} = (-1)^4 \det\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} = 2$$

$$a_{32} = (-1)^5 \det\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = -2$$

$$a_{33} = (-1)^6 \det\begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix} = 6$$

Luego: $A^{-1} = \dfrac{1}{6}\begin{pmatrix} 2 & -2 & 2 \\ 1 & 2 & -2 \\ -3 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 1/3 & -1/3 & 1/3 \\ 1/6 & 1/3 & -1/3 \\ -1/2 & 0 & 1 \end{pmatrix}$

Finalmente: $\vec{x} = \begin{pmatrix} 1/3 & -1/3 & 1/3 \\ 1/6 & 1/3 & -1/3 \\ -1/2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 8000 \\ 10500 \\ 7000 \end{pmatrix} = \begin{pmatrix} \frac{8000}{3} - \frac{10500}{3} + \frac{7000}{3} \\ \frac{8000}{6} + \frac{10500}{3} - \frac{7000}{3} \\ -4000 + 7000 \end{pmatrix}$

$= \begin{pmatrix} 1500 \\ 2500 \\ 3000 \end{pmatrix}$ (o sea, $x = 1500$, $y = 2500$, $z = 3000$)

Problema 4.

Ocupando Regla de Cramer

$$\underbrace{\begin{pmatrix} 1 & h \\ 4 & 8 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 2 \\ k \end{pmatrix}}_{\vec{b}}$$

$$A_1(\vec{b}) = \begin{pmatrix} 2 & h \\ k & 8 \end{pmatrix} \quad , \quad A_2(\vec{b}) = \begin{pmatrix} 1 & 2 \\ 4 & k \end{pmatrix}$$

$$\det(A_1(\vec{b})) = 16 - hk$$

$$\det(A_2(\vec{b})) = k - 8$$

$$\det(A) = 8 - 4h$$

Así, $x = \dfrac{16-hk}{8-4h}$, $y = \dfrac{k-8}{8-4h}$

a. Sistema tiene solución única si $8-4h \neq 0$ $(h \neq 2)$

b. Sistema tiene infinitas soluciones si $h = 2$, $k = 8$

c. Sistema tiene solución nula (sin soluciones) si $h = 2$, $k \neq 8$.