Pd: $\{f_k\}_{k\in\mathbb{N}}$ localmente acotado en $\mathbb{D}$.

Sea $K \subseteq \mathbb{D}$ compacto, existe $r>0$ $(r<1)$ tq $K \subseteq B(0,r)$

$$\Rightarrow \forall z \in K : \quad |f_k(z)| = \left|\sum_{n=0}^{\infty} a_{n,k} z^n\right| \leq \sum_{n=0}^{\infty} |a_{n,k}||z|^n$$

$(|z|<r<1)$

$$\leq \sum_{n=0}^{\infty} |a_{n,k}| r^n \quad \forall k \in \mathbb{N}.$$

$\forall k\in\mathbb{N}, \forall n \in \mathbb{N}$, $\quad |a_{n,k}| \leq \sup_{k\geq 0} |a_{n,k}| = M_n < \infty$

$$\Rightarrow \sum_{n=0}^{p} |a_{n,k}| r^n \leq \sum_{n=0}^{p} \sup_{k\geq 0} |a_{n,k}| r^n \; \cancel{\sup_{k\geq 0} \sum_{n=0}^{p} |a_{n,k}| r^n}$$

pero como vimos anteriormente,

$$\limsup_n M_n^{1/n} \leq 1 \overset{\forall\varepsilon>0}{\iff} \exists N_0 \in \mathbb{N}, \forall n \geq N_0 : \sup |a_{n,k}| \leq (1+\varepsilon)^n$$

$$\Rightarrow \sum_{n=0}^{p} \sup_{k\geq 0} |a_{n,k}| r^n = \sum_{n=0}^{N_0} \sup_{k\geq 0} |a_{n,k}| r^n + \sum_{n=N_0+1}^{p} \sup_{k\geq 0} |a_{n,k}| r^n$$

$\forall p \geq N_0$

$$\leq \sum_{n=0}^{N_0} \sup_{k\geq 0} |a_{n,k}| r^n + \sum_{n=N_0+1}^{p} r^n$$

$$\therefore |f_k(z)| \leq \sum_{n=0}^{N_0} \sup |a_{n,k}| r^n + \sum_{n=N_0+1}^{p} r^n \leq \sum_{n=0}^{N_0} \sup |a_{n,k}| r^n + \sum_{n=N_0+1}^{\infty} r^n$$

Como $\sum_{n=N_0+1}^{\infty} r^n < \infty$ $(r<1)$, podemos tomar $\tilde{M} = \sum_{n=0}^{N_0} \sup |a_{n,k}| r^n + \sum_{n=N_0+1}^{\infty} r^n$

$$\therefore \forall k \in \mathbb{N}, \forall z \in K : |f_k(z)| \leq \tilde{M}$$

Así, se tiene que $\{f_k\}_{k\in\mathbb{N}}$ es localmente acotado. $\blacksquare$

Problema 3 $f : \mathbb{C} \to \mathbb{C}$ tq $f(z) = \begin{cases} 1, & Im(z) > 0 \\ 0, & Im(z) = 0 \\ -1, & Im(z) = 0 \end{cases}$.

Sea $\overline{\mathbb{H}}^+ = \{ z \in \mathbb{C} \,/\, Im(z) \geq 0 \}$, $\overline{\mathbb{H}}^- = \{ z \in \mathbb{C} \,/\, Im(z) \leq 0 \}$

$\Rightarrow$ $f = \chi_{\overline{\mathbb{H}}^+} - \chi_{\overline{\mathbb{H}}^-}$, donde $\chi_{\overline{\mathbb{H}}^+}, \chi_{\overline{\mathbb{H}}^-}$ son las funciones características de $\overline{\mathbb{H}}^+$ y $\overline{\mathbb{H}}^-$ respectivamente.

Como $\overline{\mathbb{H}}^{+c} = \mathbb{C} \setminus \overline{\mathbb{H}}^+ = \{ Im(z) < 0 \}$, $\overline{\mathbb{H}}^- = \mathbb{C} \setminus \overline{\mathbb{H}}^- = \{ Im(z) > 0 \}$

son conexos, existen $\{p_n\}_{n \in \mathbb{N}}$, $(q_n)_{n \in \mathbb{N}}$ sucesión de polinomios que convergen en subconjuntos compactos a $\chi_{\overline{\mathbb{H}}^+}|_{\mathbb{H}^+}$ y $\chi_{\overline{\mathbb{H}}^-}|_{\mathbb{H}^-}$ respectivamente (teo de Runge) ya que $\chi_{\overline{\mathbb{H}}^+}|_{\mathbb{H}^+}$, $\chi_{\overline{\mathbb{H}}^-}|_{\mathbb{H}^-}$ son holomorfas.

($\mathbb{H}^+ = \{ z \,/\, Im(z) > 0 \}$, $\mathbb{H}^- = \{ z \,/\, Im(z) < 0 \}$). En particular, ~~convergen puntualmente a~~ $(p_n)_{n \in \mathbb{N}}$, $(q_n)_{n \in \mathbb{N}}$ convergen puntualmente a $\chi_{\overline{\mathbb{H}}^+}|_{\mathbb{H}^+}$ y $\chi_{\overline{\mathbb{H}}^-}|_{\mathbb{H}^-}$.

Considerando $(r_n)_{n \in \mathbb{N}}$ como $r_n = p_n - q_n$ $\forall n \in \mathbb{N}$ (r_n polinomio $\forall n \in \mathbb{N}$) se tiene, $\forall z \in \mathbb{C}$,

$$|f(z) - r_n(z)| = |\chi_{\overline{\mathbb{H}}^+}(z) - \chi_{\overline{\mathbb{H}}^-}(z) - (p_n(z) - q_n(z))|$$

$$\leq |\chi_{\overline{\mathbb{H}}^+}(z) - p_n(z)| + |\chi_{\overline{\mathbb{H}}^-} - q_n(z)|$$

Dado $\varepsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ tq ~~$\forall n \geq N_1, N$~~

$\forall n \geq N_1 \quad : \quad |\chi_{\overline{\mathbb{H}^+}}(z) - p_n(z)| < \varepsilon/2$

$\forall n \geq N_2 \quad : \quad |\chi_{\overline{\mathbb{H}^-}}(z) - q_n(z)| < \varepsilon/2$

si $N_0 = \max\{N_1, N_2\}$ y

$\forall n \geq N_0 \quad : \quad |f(z) - r_n(z)| \leq |\chi_{\overline{\mathbb{H}^+}}(z) - p_n(z)| + |\chi_{\overline{\mathbb{H}^-}}(z) - q_n(z)|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\therefore$ $(r_n)_{n\in\mathbb{N}}$ converge a f puntualmente.

Gregory Mankiw (1988)

principles of economics.

N° Pedido: 330 M278p 1988

P7) $\mathcal{F}$ normal en $H(G)$, $\Omega \subseteq \mathbb{C}$ abierto tq $f(G) \subseteq \Omega$ $\forall f \in \mathcal{F}$.

id: $g: \Omega \to \mathbb{C}$ holomorfa, $g(A)$ acotada $\forall A \subseteq \Omega$ acotado

$\Rightarrow \{ g \circ f \,/\, f \in \mathcal{F} \}$ normal.

Idem. $|g \circ f(z)| = |g(f(z))|$

· $K \subset \Omega$ compacto, $\exists M > 0$ tq $\forall f \in \mathcal{F}$ $\forall z \in K$: $|f(z)| \leq M$

$\Rightarrow f(K) \subseteq \Omega$ acotado $\Rightarrow g(f(K))$ acotado

$\therefore \exists \tilde{M} > 0$ tq $\forall f \in \mathcal{F}$ $\forall x \in K$: $|g \circ f(z)| \leq \tilde{M}$

$\therefore \{ g \circ f \,/\, f \in \mathcal{F} \}$ localmente acotado.

$\therefore$ Montel: $\{ g \circ f \,/\, f \in \mathcal{F} \}$ normal.

P8) $D = \{ z \,/\, |z| < 1 \}$. $\mathcal{F} \subseteq H(D)$

$\mathcal{F}$ normal $\Leftrightarrow \exists (M_n)_{n \in \mathbb{N}}$ $(M_n > 0)$ tq $\limsup \sqrt[n]{M_n} \leq 1$ y si

$$f(z) = \sum_{0}^{\infty} a_n z^n \in \mathcal{F} \Rightarrow |a_n| \leq M_n \quad \forall n \in \mathbb{N}.$$

$\forall f \in H(D) \Rightarrow R^{-1} = \limsup \sqrt[n]{|a_n|}$, $R \leq 1$, $f(z) = \sum_{0}^{\infty} a_n z^n$

$\forall n \in \mathbb{N}$, $A_n = \{ z \in D \,/\, |z| \leq 1 - \frac{1}{n} \} \subseteq D$ compacto.

$\exists M_n > 0$ tq $\forall f$ $\forall z \in A_n$: $|f(z)| \leq M_n$

$a_n = \dfrac{f^{(n)}(0)}{n!}$. $\mathcal{F}^{(n)} = \{ f^{(n)} \,/\, f \in \mathcal{F} \}$ es normal.

$f^{(n)}(0) = a_n n! \Rightarrow |f^{(n)}(0)| = |a_n| n! \leq R_n$, algún $R_n > 0$ de la normalidad

de $\mathcal{F}^{(n)}$. $M_n = R_n / n!$. $d_n = \sup\limits_{k \geq n} \sqrt[n]{M_n}$ decreciente

$M_n = n! \sup\limits_{f \in \mathcal{F}} |a_n|$ | $\tilde{M}_n = M_n / n! = \sup\limits_{f \in \mathcal{F}} |a_n| \geq |a_n|$

$\forall n \in \mathbb{N}$ $\exists f \in \mathcal{F}$: $\tilde{M}_n = \sup\limits_{f} |a_n| \leq |a_n| + 1/n \Rightarrow \sqrt[n]{\tilde{M}_n} \leq \sqrt[n]{|a_n| + 1/n}$

$\limsup \sqrt[n]{\tilde{M}_n} \leq \limsup \sqrt[n]{|a_n| + 1/n}$

$\forall f \in \mathcal{F}: \quad |f^{(n)}(0)| = |a_n|\, n! \Rightarrow |a_n| = \frac{|f^{(n)}(0)|}{n!} \leq \frac{\sup_f |f^{(n)}(0)|}{n!}$

$|f^{(n)}(0)| \leq \frac{\|f\|_C\, n!}{R^{n+1}} \Rightarrow |a_n| \leq \frac{\|f\|_C}{R^{n+1}} \leq \frac{M}{R^{n+1}}$ ~~$\forall R < 1$~~

$\|f\|_C \leq M$ (local acotación $\equiv$ Montel)

$M_n = \frac{M}{R^{n+1}} \Rightarrow \sqrt[n]{M_n} = \frac{\sqrt[n]{M}}{R\sqrt[n]{R}} \xrightarrow{n\to\infty} \frac{1}{R}$

$\therefore \limsup \sqrt[n]{M_n} = \lim \sqrt[n]{M_n} = \frac{1}{R}$

Variable Compleja (Tarea 1)

(1.a) (V ó F). $\Omega = \mathbb{C} \setminus (-\infty, 0] \Rightarrow \exists f : \Omega \to \mathbb{C}$ holomorfa, $[f(z)]^2 = z \ \forall z \in \Omega$

$$[f(z)]^2 = z \Rightarrow 2f(z)f'(z) = 1 \to f'(z) = \frac{1}{2f(z)}$$

f es primitiva de $\frac{1}{2f}$ en $\Omega = \mathbb{C} \setminus (-\infty, 0] \Rightarrow \int_\gamma f' = 0$ (γ → cerrada)

$\Omega = \mathbb{C} \setminus (-\infty, 0]$ estrellado (c/r a $z=1$), (f holo $\Rightarrow f'$ holo).

$$\int_\gamma f' = 0 \Leftrightarrow \frac{1}{2}\int_\gamma \frac{1}{f} = 0 \Leftrightarrow \int_\gamma \frac{1}{f} = 0 \qquad \gamma \subseteq \Omega$$

$$g : \Omega \to \mathbb{C}, \quad g(z) = \frac{1}{2f(z)}$$

$$F(z) = \int_{[1,z]} \frac{1}{2f(w)} dw \quad \text{tq} \quad \frac{F(z) - F(\zeta)}{z - \zeta} = \frac{1}{z-\zeta}\int_{[1,z]} \frac{1}{2f(w)} dw - \frac{1}{z-\zeta}\int_{[1,\zeta]} f'(w)$$

$$= -\frac{1}{z-\zeta}\int_{[z,1]} f'(w)dw - \frac{1}{z-\zeta}\int_{[1,\zeta]} f'(w)dw = -\frac{1}{z-\zeta}\int_{[z,\zeta]} f'(w)dw$$

$$= \frac{1}{z-\zeta}\int_{[\zeta,z]} f'(w)dw$$

$$\frac{F(z) - F(\zeta)}{z-\zeta} - f'(z) = \frac{1}{z-\zeta}\int_{[\zeta,z]} f'(w)dw - \frac{1}{z-\zeta}\int_{[\zeta,z]} f'(z)dw$$

$$\frac{1}{z-\zeta}\int_{[\zeta,z]} dw = \frac{1}{z-\zeta}\int_0^1 ((1-t)\zeta + tz)'dt = \frac{1}{z-\zeta}\int_0^1 (-\zeta + z)dt = 1$$

$\therefore$ F primitiva de $1/2f$ y $F' = f'$

$$\Rightarrow \exists c \in \mathbb{C} : f(z) = F(z) + c = \int_{[1,z]} f'(w)dw + c$$

$f(1) = F(1) + c = 0 + c = c \quad \therefore f(1) = c \to (f(1))^2 = 1 = c^2 \ \therefore c \in \{\pm 1\}$

$$f'(z) = \frac{1}{2f(z)} \iff (2f(z))' = \frac{1}{f(z)}$$

$$\boxed{(f(z))^2 = z}$$

$$f'(1) = \frac{1}{2f(1)} = \frac{1}{2i} = -\frac{1}{2}i \neq 0 \Rightarrow \exists\, V \subseteq \Omega \text{ vecindad de } 1$$

tq $f|_V : V \longrightarrow f(V)$ invertible

$$\begin{tikzcd} 1 \arrow[r, bend left, "f"] & i = f(1) \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} z \arrow[r, bend left, "f"] & w \arrow[l, bend left, "g"] \end{tikzcd}$$

Se tiene $g'(w) = \dfrac{1}{f'(z)} = 2f(z) = 2w$

$$\Rightarrow g(w) = w^2 + d, \quad d \in \mathbb{C}$$

$$f(1) = i \Rightarrow g(i) = 1 + d = 1 \Rightarrow d = 0 \qquad \boxed{g(w) = w^2}$$

$$f : \Omega \longrightarrow \mathbb{C} \qquad (f(1))^2 = 1 \Rightarrow f(1) \in \{\pm i\}$$

$$1 \longrightarrow i, \qquad 1 \longrightarrow -i$$

$$f'(z) = \frac{1}{2f(z)} \neq 0$$

podemos tomar bolas disjuntas. $f'(1) \neq 0$.

$w^2 = z$

$(w + \sqrt{z})(w - \sqrt{z}) = 0$.

$w^2 = re^{i\theta}$

$(w - \sqrt{r}e^{i\theta/2})(w + \sqrt{r}e^{i\theta/2}) = 0$

$w = \sqrt{r}e^{i\theta/2}$

$w = -\sqrt{r}e^{i\theta/2}$

$= \sqrt{r}e^{i\pi}e^{i\theta/2} = \sqrt{r}e^{i(\pi + \frac{\theta}{2})}$

$z = re^{i\theta}, \quad \theta \in (-\pi, \pi), \quad r > 0.$

$$f(z) = r'e^{i\theta'} \Rightarrow r'^2 e^{2i\theta'} = re^{i\theta} \Rightarrow r'^2 = r \Rightarrow r' = \sqrt{r}$$

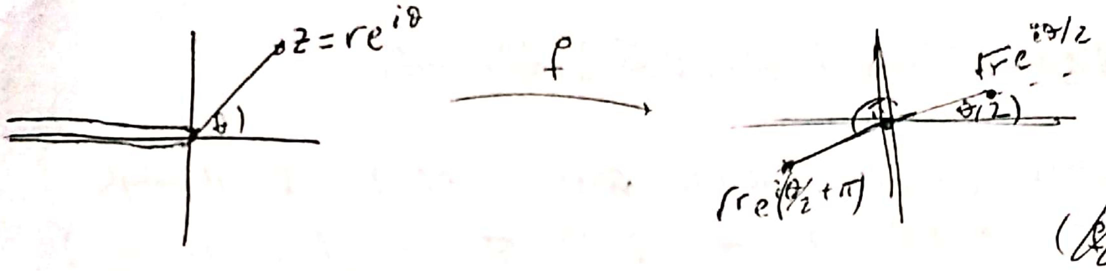
$$\cos(2\theta') + i\,\text{sen}(2\theta') = \cos(\theta) + i\,\text{sen}(\theta)$$

$$\cos(2\theta') = \cos(\theta) \Rightarrow 2\theta' = \theta \Rightarrow \theta' = \frac{\theta}{2}$$

$$\text{sen}(2\theta') = \text{sen}(\theta) \Rightarrow \ldots$$

$$e^{i(2\theta' - \theta)} = 1 \iff 2\theta' - \theta = 2\pi k \iff \theta' = \theta/2 + k\pi$$

$$\left\{ \begin{array}{l} \theta' = \theta/2 \\ \theta' = \theta/2 + k\pi, \quad k \in \mathbb{Z} \end{array} \right.$$



$f(z)=w$

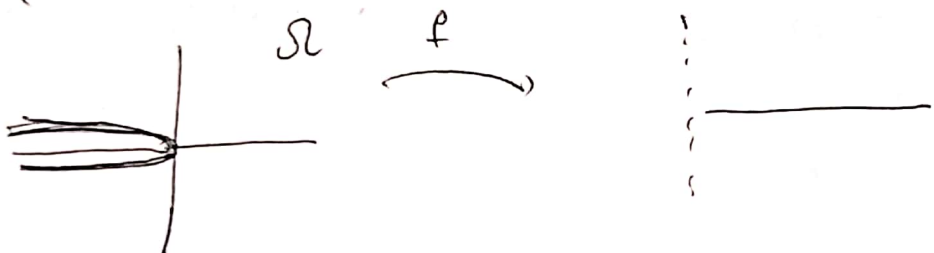
$(f(1))^2=1$

$(f(1))^2=1 \Rightarrow f(1)\in\{1,-1\}$

$i=e^{i\pi/2} \Rightarrow (f(i))^2=i=e^{i\pi/2}$

$f(i)\in\{e^{i\pi/4}, e^{i(\pi/4+\pi)}\}$

$(f(re^{i\theta}))^2 = re^{i\theta} \Rightarrow f(re^{i\theta})\in\{\sqrt{r}e^{i\theta/2}, \sqrt{r}e^{i(\theta/2+\pi)}\}$



$f'(z)=\frac{1}{2f(z)}$, $g'(f(z))=\frac{1}{f'(z)}=2f(z)=2w$

$g'(w)=2w \Rightarrow g(w)=w^2+d$, $f(1)=1=w$

$g(1)=1 \Rightarrow g(1)=1+d=1 \Rightarrow \boxed{d=0}$

$g:\underbrace{\{z\in\mathbb{C} / Re(z)>0\}}_{\Gamma} \to \mathbb{C}$ $g(w)=w^2$

$g(\Gamma)=\Omega$

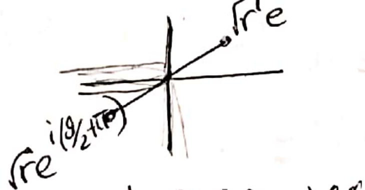
~~$\mathcal{C}$~~

(b) (V ó F). $\Omega \subseteq \mathbb{C}$ región (abierto, conexo, $\neq \emptyset$), $f : \Omega \to \mathbb{C}$ holomorfa tq $(f(z))^2 = z \Rightarrow f(\Omega) \subsetneq$ (semiplano de $\mathbb{C}$).

— o —

Ω conexo $\Rightarrow f(\Omega)$ conexo.

$f(z = re^{i\theta}) = w \Rightarrow w \in \{\sqrt{r}e^{i\theta/2}, \sqrt{r}e^{i(\theta/2+\pi)}\}$ $\quad (e^{-\frac{i\pi}{2}})^2 = e^{-i\pi}$



$2f(z)f'(z) = 1 \Rightarrow f'(z) \neq 0 \quad \forall z \in \Omega$

Sup $0 \in \Omega$: $(f(0))^2 = 0 \Rightarrow f(0) = 0$. $\quad \Big|\; f(\Omega) = \{f(z) / z \in \Omega\}$

Si $f(z) = 0 \Rightarrow z = 0$

$0 \in \Omega \Rightarrow \exists R > 0 : B(0, R) \subseteq \Omega$

$\Omega = B(z_0, R)$, $z_1, z_2 \in \Omega \Rightarrow \forall t \in [0,1] : z_t = (1-t)z_1 + t z_2 \in \Omega$

$(f(z_t))^2 = z_t \quad \forall t \in [0,1]$

$(f(-1))^2 = -1 \Rightarrow f(-1) \in \{i, -i\}$ $\quad -1 = e^{i\pi} \quad f(-1) \in \{e^{i\pi/2}, e^{i(\frac{\pi}{2}+\pi)}\}$ $= e^{i\frac{3\pi}{2}}$

$r > 0$

$(f(-r))^2 = -r \Rightarrow f(-r) \in \{i\sqrt{r}, -i\sqrt{r}\}$
$\quad \mathbb{R}$

Sup $-\alpha \in \Omega$ $\quad \exists R > 0 : -\alpha \in B(-\alpha, R) \subseteq \Omega$ $\quad \{z / \operatorname{Im}(z) > 0\}$

$\alpha \in \mathbb{R}_{>0}$

Sup : $f(-\alpha) = \sqrt{\alpha}i \Rightarrow \exists R' > 0$ tq $f(B(-\alpha, R')) \cap \mathbb{R} \neq \emptyset$.

P2.1 $\alpha, \beta \in (0, 2\pi)$. f es holomorfa $f(0) = 0$.

$f(S_1) = T_1$

$f(S_2) = T_2$

$S_1 = (a, b)$
$S_2 = (c, d)$
$T_1 = (a', b')$
$T_2 = (c', d')$

$$[f'(z_0)] = \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix}$$

$$f(0+h) = f(0) + f'(0)h + \sigma(h)|h| \qquad \sigma(h) \xrightarrow{|h| \to 0} 0$$

$$f(h) = f'(0)h + \sigma(h)|h|$$

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots \qquad f(0) = 0 \Rightarrow a_0 = 0$$

$$f(z) = a_1 z + a_2 z^2 + \dots$$

$$a_i = 0 \ \forall i \geq 2. \quad f(z) = a_1 z = (a_1^1 + a_1^2 i)(x + yi)$$

$$= a_1^1 x + a_1^1 yi + a_1^2 x i - a_1^2 y = (a_1^1 x - a_1^2 y) + (a_1^1 y + a_1^2 x)i$$

$$= \begin{pmatrix} a_1^1 x - a_1^2 y \\ a_1^1 y + a_1^2 x \end{pmatrix} = \begin{pmatrix} a_1^1 & -a_1^2 \\ a_1^2 & a_1^1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\cancel{a_1 = (\alpha, \beta}$ $\quad a_1 = (A, B)$, $\quad f(z) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$= \begin{pmatrix} |a_1| \cos\theta & -|a_1| \operatorname{sen}\theta \\ |a_1| \operatorname{sen}\theta & |a_1| \cos\theta \end{pmatrix} \begin{pmatrix} |z| \cos\varphi \\ |z| \operatorname{sen}\varphi \end{pmatrix} = |a_1||z| \begin{pmatrix} \cos\theta\cos\varphi - \operatorname{sen}\theta \operatorname{sen}\varphi \\ \operatorname{sen}\theta\cos\varphi + \cos\theta \operatorname{sen}\varphi \end{pmatrix}$$

$$= |a_1||z| \begin{pmatrix} \cos(\theta + \varphi) \\ \operatorname{sen}(\theta + \varphi) \end{pmatrix}$$

$$\alpha = \varphi_{S_2} - \varphi_{S_1}, \quad \beta = \varphi_{T_2} - \varphi_{T_1}$$

$$f(S_2) = T_2 \Leftrightarrow |a_1||S_2| \begin{pmatrix} \cos(\theta + \varphi_{S_2}) \\ \operatorname{sen}(\theta + \varphi_{S_2}) \end{pmatrix} = |T_2| \begin{pmatrix} \cos(\varphi_{T_2}) \\ \operatorname{sen}(\varphi_{T_2}) \end{pmatrix}$$

$$f(S_1) = T_1 \Leftrightarrow |a_1||S_1| \begin{pmatrix} \cos(\theta + \varphi_{S_1}) \\ \operatorname{sen}(\theta + \varphi_{S_1}) \end{pmatrix} = |T_1| \begin{pmatrix} \cos(\varphi_{T_1}) \\ \operatorname{sen}(\varphi_{T_1}) \end{pmatrix}$$

$$\Rightarrow \begin{cases} \theta + \varphi_{S_2} = \varphi_{T_2} \\ \theta + \varphi_{S_1} = \varphi_{T_1} \end{cases} \Rightarrow \varphi_{S_2} - \varphi_{S_1} = \varphi_{T_2} - \varphi_{T_1} \Rightarrow \boxed{\alpha = \beta}$$

$$\left| \frac{f(z)-f(w)}{z-w} - \int_0^{2\pi} g_\theta'(z)\,d\theta \right| = \left| \int_0^{2\pi} \left(\frac{g_\theta(z)-g_\theta(w)}{z-w} - g_\theta'(z) \right) d\theta \right|$$

$$\leq \int_0^{2\pi} \left| \frac{g_\theta(z)-g_\theta(w)}{z-w} - g_\theta'(z) \right| d\theta = \frac{1}{|z-w|} \int_0^{2\pi} |g_\theta(z)-g_\theta(w)-(z-w)g_\theta'(z)|\,d\theta$$

$\forall z, w \; tq \; z \neq w$: $\dfrac{g_\theta(z)-g_\theta(w)}{z-w} - g_\theta'(z) = H_{z,w}(\theta)$ continua en $[0,2\pi]$

$\Rightarrow H_{z,w}(\theta)$ uniformemente continua en $[0,2\pi]$.

$\forall \theta \; \forall \varepsilon > 0, \; \exists \delta > 0 \; tq \quad \forall w, \; 0 < |z-w| < \delta \Rightarrow \left| \dfrac{g_\theta(z)-g_\theta(w)}{z-w} - g_\theta'(z) \right| < \dfrac{\varepsilon}{2\pi}$

- $\dfrac{g_\theta(z)-g_\theta(w)}{z-w} - g_\theta'(z) = H_z(w,\theta)$ · continua c/r a w y θ.

$\Rightarrow H_z$ continua en $\mathbb{C}\setminus\{z\} \times [0,2\pi]$.

Pd: $|g_\theta(z)-g_\theta(w)-(z-w)g_\theta'(z)| < \dfrac{\varepsilon}{2\pi}|z-w|$

$$\left| g(\theta,z) - g(\theta,w) - (z-w)\frac{\partial}{\partial z} g(\theta,z) \right| < \frac{\varepsilon}{2\pi}|z-w|$$

$$\left| g(\theta,z) - g(\theta,w) - (z-w)\frac{\partial}{\partial z} g(\theta,z) \right|$$

$$= \left| g(\theta,z) - z\frac{\partial}{\partial z} g(\theta,z) + w\frac{\partial}{\partial z} g(\theta,z) - g(\theta,w) \right|$$

Fijamos $w \in B(z,1)\setminus\{z\} \Rightarrow H_{z,w}(\theta)$ uniformemente continua en $[0,2\pi] \Rightarrow \forall \varepsilon > 0, \; \exists \delta > 0 \; tq \; \forall \varphi, \theta: \; |\theta - \varphi| < \delta \Rightarrow |H_{z,w}(\theta) - H_{z,w}(\varphi)| < \varepsilon$

~~No sirve.~~

Fijemos $\theta_0 \in [0,2\pi]$

$\forall \varepsilon > 0, \; \exists \delta > 0 \; tq \; \forall w \in B(z,\delta)\setminus\{z\}$:

$$\left| \frac{g_\theta(z)-g_\theta(w)}{z-w} - g_\theta'(z) \right| \leq \left| g_\varphi'(z) - g_\theta'(z) \right| + \left| \frac{g_\theta(z)-g_\theta(w)}{z-w} - g_\varphi'(z) \right|$$

$$\leq |g_\varphi'(z) - g_\theta'(z)| + \left| \frac{g_\theta(z)-g_\theta(w)}{z-w} - \frac{g_\varphi(z)-g_\varphi(w)}{z-w} \right| + \left| \frac{g_\varphi(z)-g_\varphi(w)}{z-w} - g_\varphi'(z) \right|$$

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P3) $f:\mathbb{C}\to\mathbb{C}$, $\quad f(z):=\int_0^{2\pi}\exp((\operatorname{sen}\theta)z)\,d\theta$

$$\frac{f(z)-f(w)}{z-w}=\frac{1}{z-w}\left(\int_0^{2\pi}\exp((\operatorname{sen}\theta)z)\,d\theta-\int_0^{2\pi}\exp((\operatorname{sen}\theta)w)\,d\theta\right)$$

$$=\frac{1}{z-w}\int_0^{2\pi}\big(\exp((\operatorname{sen}\theta)z)-\exp((\operatorname{sen}\theta)w)\big)\,d\theta$$

$$=\int_0^{2\pi}\frac{\exp((\operatorname{sen}\theta)z)-\exp((\operatorname{sen}\theta)w)}{z-w}\,d\theta$$

$f(z)=\int_0^{2\pi}g(\theta,z)\,d\theta$, $\quad g_z(\theta)=g(\theta,z)$ ← continua $\forall z\in\mathbb{C}$.

$g_\theta(z)=g(\theta,z)$ ← holomorfa $\forall\theta\in[0,2\pi]$

$$\left|\frac{f(z)-f(w)}{z-w}-\int_0^{2\pi}g_\theta'(z)\,d\theta\right|=\left|\frac{1}{z-w}\int_0^{2\pi}\big(g(\theta,z)-g(\theta,w)\big)\,d\theta-\int_0^{2\pi}g_\theta'(z)\,d\theta\right|$$

$$=\left|\int_0^{2\pi}\left(\frac{g_\theta(z)-g_\theta(w)}{z-w}-g_\theta'(z)\right)d\theta\right| \cancel{\leq} \cancel{\int_0^{2\pi}\left|\frac{g_\theta(z)-g_\theta(w)}{z-w}-g_\theta'(z)\right|d\theta}$$

$$=\frac{1}{|z-w|}\left|\int_0^{2\pi}\big(g_\theta(z)-g_\theta(w)-(z-w)\,g_\theta'(z)\big)\,d\theta\right|$$

$$=\frac{1}{|z-w|}\left|\int_0^{2\pi}\operatorname{Re}\big(g_\theta(z)-g_\theta(w)-(z-w)g_\theta'(z)\big)d\theta+i\int_0^{2\pi}\operatorname{Im}\big(g_\theta(z)-g_\theta(w)-(z-w)g_\theta'(z)\big)d\theta\right.$$

$f:[a,b]\to\mathbb{C}$. $\quad f(\theta)=u(\theta)+iv(\theta)$

$$\left|\int_a^b f(\theta)\,d\theta\right|=\cancel{\int_a^b \operatorname{Re}f(\theta)+i\operatorname{Im}f(\theta)}$$

$$=\left|\int_a^b(u(\theta)+iv(\theta))\,d\theta\right|=\left|\int_a^b u(\theta)\,d\theta+i\int_a^b v(\theta)\,d\theta\right|$$

$$\leq\int_a^b|u(\theta)|\,d\theta+\int_a^b|v(\theta)|\,d\theta=\int_a^b\big(|u(\theta)|+|v(\theta)|\big)\,d\theta$$

P4| f holomorfa en vecindad de $\overline{B}(0,r)$. Encontrar fórmula para

$$\oint_{|z|=r} \overline{f(z)}\,dz$$

$f:\Omega\to\mathbb{C}$ continua, γ curva en Ω. $\quad \int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$

$\gamma(t)=re^{2\pi i t} \qquad \gamma'(t)=2\pi i r e^{2\pi i t}$

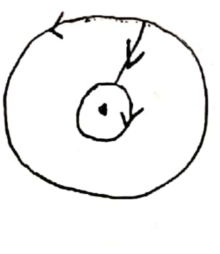
$$\int_{|z|=r}\overline{f(z)}\,dz = \int_0^1 \overline{f(\gamma(t))}\,2\pi i r e^{2\pi i t}\,dt = 2\pi i r\int_0^1 \overline{f(\gamma(t))}\,e^{2\pi i t}\,dt$$

$f(z)=z$: $$\int_{|z|=r}\bar z\,dz = \int_0^{2\pi}\overline{re^{it}}\, rie^{it}\,dt = \int_0^{2\pi} re^{-it}rie^{it}\,dt$$
$$= r^2 i\int_0^{2\pi}dt = 2\pi r^2 i$$

$$\int_{|z|=r}|z|^2\,d\theta = \int_0^{2\pi}|re^{it}|^2\, rie^{it}\,dt = \int_0^{2\pi} r^2 rie^{it}\,dt = r^3 i\int_0^{2\pi}e^{it} = 0.$$

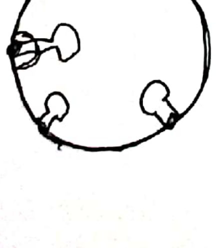
$f(x,y)=|(x,y)|^2 = x^2+y^2$ - dif. en sentido real.

$u=x^2+y^2$, $v=0$. $\quad \frac{\partial u}{\partial x}=2x$. $\frac{\partial u}{\partial y}=2y$



$$\int_{|z|=r}\bar z\,dz = \int_{|z|=r}\frac{|z|^2}{z}\,dz = r^2\int_{|z|=r}\frac{1}{z}\,dz = 2\pi i r^2$$

$$\overline{f(z)} = \frac{|f(z)|^2}{f(z)} = \frac{|z-a_1|^{2m_1}\cdots|z-a_k|^{2m_k}|g(z)|^2}{(z-a_1)^{m_1}\cdots(z-a_k)^{m_k}\,g(z)}$$



$$f(z)=(z-a_1)^{m_1}\cdots(z-a_k)^{m_k}\,g(z)$$

$$\int_{|z|=r} \overline{f(z)}\,dz = \int_{|z|=r} \overline{u(z)+i\,v(z)}\,dz = \int_{|z|=r} u(z)\,dz - i\int_{|z|=r}\overline{v(z)}\,dz$$

$$= \int_{|z|=r} u(z)\,dz - i\int_{|z|=r} v(z)\,dz$$

$w\in B(0,r)$: $$\int_{|z|=r}\overline{f(z)}\,dz = \int_{|z|=r}\frac{\overline{f(z)}\,(z-w)}{z-w}\,dz$$

$$= \int_{|z|=r}\frac{\overline{f(z)}\,z}{z-w}\,dz - w\int_{|z|=r}\frac{\overline{f(z)}}{(z-w)}\,dz$$

— o —

f holomorfa en vecindad de $\overline{B(0,r)}$

$$\Rightarrow \forall z\in B(0,r) : f(z) = \frac{1}{2\pi i}\int_{|w|=r}\frac{f(w)}{w-z}\,dw$$

$$\overline{f(z)} = \frac{i}{2\pi}\,\overline{\int_{|w|=r}\frac{f(w)}{w-z}\,dw}$$

$0<r'<r$ $\quad\forall z\in B(0,r)$, $|z|=r'$: $$\int_{|z|=r'}\overline{f(z)}\,dz = \frac{i}{2\pi}\int_{|z|=r'}\overline{\int_{|w|=r}\frac{f(w)}{w-z}\,dw}\,dz$$

$$\int_{|z|=r'}\overline{f(z)}\,dz = \frac{i}{2\pi}\int_{|z|=r'}\left(\overline{\int_{|w|=r}\frac{f(w)}{w-z}\,dw}\right)dz$$

f continua : $$\int_{|z|=r} f(z)\,dz = \lim_{\substack{r'\to r\\ r'<r}}\int_{|z|=r'} f(z)\,dz$$ $\rightarrow$ demostrar $\boxed{\text{Prd}}$

$$\int_{|z|=r}\overline{f(z)}\,dz = \frac{i}{2\pi}\lim_{r'\to r}\int_{|z|=r'}\overline{\int_{|w|=r}\frac{f(w)}{w-z}\,dw}\,dz$$

$f(z)=z$: $\displaystyle\int_{|z|=r} \bar{z}\, dz = 2\pi r^2 i$

$f(z)=z^n$ $(n\geq 1)$: $\displaystyle\int_{|z|=r} \overline{z^n}\, dz = \int_{|z|=r} \bar{z}^n dz = \int_0^{2\pi} (re^{-it})^n\, rie^{it} dt$

$\displaystyle = r^{n+1} i \int_0^{2\pi} e^{it} e^{-nit} dt = r^{n+1} i \int_0^{2\pi} e^{(1-n)it} dt$

w

$\displaystyle\int_0^{2\pi} e^{(1-n)it} dt = \int_0^{2\pi} \left(\cos((1-n)t) + i \operatorname{sen}((1-n)t)\right) dt$

$\displaystyle = \int_0^{2\pi} \cos((1-n)t)\, dt + i \int_0^{2\pi} \operatorname{sen}((1-n)t)\, dt = \left.\frac{\operatorname{sen}((1-n)t)}{1-n}\right|_0^{2\pi} + i\left[-\frac{\cos((1-n)t)}{1-n}\right]_0^{2\pi}$

$\displaystyle = -i\frac{1}{1-n} + i\frac{1}{1-n}$

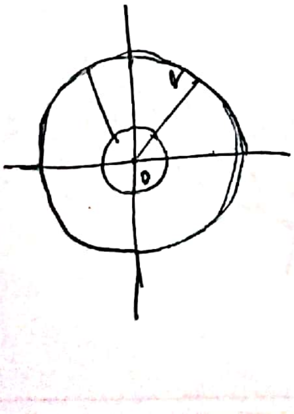
$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

$\displaystyle\int_{|z|=r} \overline{f(z)}\, dz = \bar{a}_n \int_{|z|=r} \bar{z}^n dz + \overline{a_{n-1}} \int_{|z|=r} \bar{z}^{n-1} dz + \cdots + \bar{a}_1 \int_{|z|=r} \bar{z}\, dz + \bar{a}_0 \int_{|z|=r} dz$

$\displaystyle = \bar{a}_1 \int_{|z|=r} \bar{z}\, dz = 2\pi r^2 \bar{a}_1 i = 2\pi r^2 i\, \overline{f'(z_0)}$ $\quad \boxed{f'(z_0)=0}$

~~f) holomorfa en vecindad de $\overline{B(0,r)}$ pint dcos:~~

$\cancel{\displaystyle\int_{|z|=r} \overline{f(z)}\, dz = \int_{|z|=r} \frac{|f(z)|^2}{f(z)}\, dz}$



$p(z)$ polinomio $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

$\overline{p(z)} = \bar{a}_n \bar{z}^n + \cdots + \bar{a}_1 \bar{z} + \bar{a}_0$

$\exists\, g(z) \in \mathbb{C}[x]$: $\overline{p(z)} = g(\bar{z})$

$f(z) = (z-a)(z-b) = z^2 - (a+b)z + ab$, $|a|, |b| < r$

$$\int_{|z|=r} \overline{f(z)}\, dz = 2\pi r^2 \left(\overline{-(a+b)}\right) i = -2\pi r^2 \overline{(a+b)}\, i = -2\pi r^2 \bar{a} i - 2\pi r^2 \bar{b} i$$

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ para $z \in B(0, r')$ $r' < r$

$\sum_{k=0}^{m} a_k z^k$ converge uniforme/. a $\sum_{k=0}^{\infty} a_k z^k$ en $z \in B(0, r'/2)$

$$\int_{|z|=r} \overline{f(z)}\, dz = \int_{|z|=r'/2} \overline{f(z)}\, dz = \int_{|z|=r'/2} \overline{\sum_{n=0}^{\infty} a_n z^n}\, dz$$

$$= \int_{|z|=r'/2} \sum_{n=0}^{\infty} \bar{a}_n \bar{z}^n\, dz = \sum_{n=0}^{\infty} \bar{a}_n \int_{|z|=r'/2} \bar{z}^n\, dz$$

$$= 2\pi i \frac{r'^2}{4} \bar{a}_1 = \frac{\pi r'^2 i}{2} \bar{a}_1 = \frac{\pi r'^2 i}{2} \overline{f'(0)}$$

$f(z)$ holo. $f(z) = \sum_{0}^{\infty} a_n (z - z_0)^n$

$$\overline{f(z)} = \overline{\sum_{0}^{\infty} a_n (z-z_0)^n} = \overline{\lim_{n\to\infty} \sum_{k=0}^{n} a_k (z-z_0)^k} = \lim_{m\to\infty} \sum_{k=0}^{n} \bar{a}_k (\bar{z} - \bar{z}_0)^k$$

$$= \sum_{k=0}^{\infty} \bar{a}_k (\bar{z} - \bar{z}_0)^k$$

$$S^{-1} = \limsup |\bar{a}_k|^{1/n} = \limsup |a_n|^{1/n}$$

$$= R^{-1} \Rightarrow \boxed{S = R}$$

$\therefore$ $\overline{f(z)} = g(\bar{z})$ para g holo en $z \in B(\bar{a}, R)$.

~~$\int_{|z|=r} \overline{f(z)}\, dz = \int_0^{2\pi} \overline{f(re^{it})}\, rie^{it}\, dt$~~

$=$

$$f : \mathbb{C} \to \mathbb{C} \quad , \quad f(z) = \int_0^{2\pi} \exp((\operatorname{sen}\theta) z)\, d\theta .$$

$\forall \theta \in [0, 2\pi]$, $\exp(\operatorname{sen}\theta\, z)$ holomorfa.

$$\exp(\operatorname{sen}\theta\, z) = \sum_{n=0}^{\infty} \frac{(\operatorname{sen}\theta\, z)^n}{n!} = \sum_{n=0}^{\infty} (\operatorname{sen}\theta)^n \frac{z^n}{n!}$$

Radio de convergencia ∞.

$$\int_0^{2\pi} \exp(\operatorname{sen}\theta\, z)\, d\theta = \int_0^{2\pi} \sum_{n=0}^{\infty} (\operatorname{sen}\theta)^n \frac{z^n}{n!}\, d\theta \underset{\text{convergencia uniforme}}{=} \sum_{n=0}^{\infty} \int_0^{2\pi} (\operatorname{sen}\theta)^n \frac{z^n}{n!}\, d\theta$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_0^{2\pi} (\operatorname{sen}\theta)^n\, d\theta$$

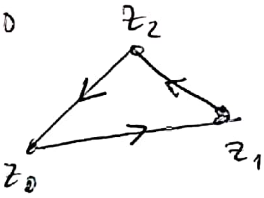
$$S_n(z) = \sum_{k=0}^{n} f_k(z) \quad , \quad S(z) = \sum_{k=0}^{\infty} f_k(z)$$

$$|S_n(z) - S(z)| = \left| \sum_{k=n+1}^{\infty} f_k(z) \right| \leq \sum_{k=n+1}^{\infty} |f_k(z)|$$

Pd: f holomorfa en $\mathbb{C}$.

$T \subseteq \mathbb{C}$ triangulo lleno,

$$\int_{\partial T} f(z)\, dz = \int_{\partial T} \int_0^{2\pi} \exp(\operatorname{sen}\theta\, z)\, d\theta\, dz = \int_0^{2\pi} \underbrace{\int_{\partial T} \exp(\operatorname{sen}\theta\, z)\, dz}_{=0}\, d\theta$$



$\partial T = [z_0, z_1, z_2]$

$$\int_{\partial T} f(z)\, dz = \int_{[z_0, z_1]} f(z)\, dz + \int_{[z_1, z_2]} f(z)\, dz + \int_{[z_2, z_0]} f(z)\, dz$$

$$= \int_0^1 f((1-t)z_0 + t z_1)(z_1 - z_0)\, dt + \int_0^1 f((1-t)z_1 + t z_2)(z_2 - z_1)\, dt + \int_0^1 f((1-t)z_2 + t z_0)(z_0 - z_2$$

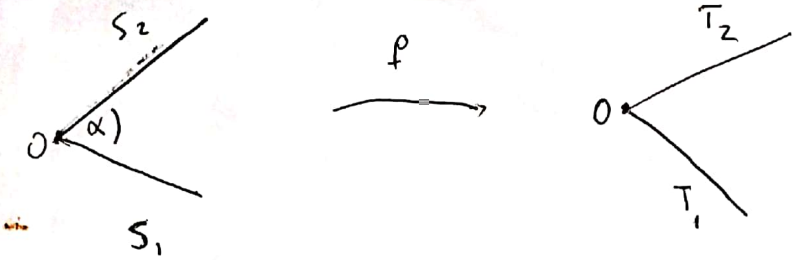
$$\int_0^1 f((1-t)z_0+tz_1)(t_1-t_0)\,dt = z_1\int_0^1 f((1-t)z_0+tz_1)\,dt - z_0\int_0^1 f((1-t)z_0+tz_1)\,dt$$

$$z_1\int_0^1 f((1-t)z_0+tz_1)\,dt = z_1\int_0^1\int_0^{2\pi} \exp(\operatorname{sen}\theta((1-t)z_0+tz_1))\,dt$$

$$= z_1\int_0^1\int_0^{2\pi} \exp(\operatorname{sen}\theta((1-t)(a_0+ib_0)+t(a_1+ib_1)))\,dt$$

$$= z_1\int_0^1\int_0^{2\pi} \exp((\operatorname{sen}\theta(1-t)a_0+ta_1)+i(\operatorname{sen}\theta((1-t)b_0)+tb_1))\,dt$$

$=$ bla!



$f(0)=0$. $\quad f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$

$k = \min\{n \,/\, a_n \neq 0\}$. $\quad f(z) = z^k g(z)$ $\quad , \; g(z) \neq 0 \;\; \forall z \in B(0,r)$

$f(S_1) = T_1 \quad | \quad S_1 = \{ t h_1 \,/\, t \in [0,1] \}$

$f(S_2) = T_2 \quad | \quad S_2 = \{ t h_2 \,/\, t \in [0,1] \}$

$f(th_1) = t^k h_1^k \, g(th_1) \longrightarrow$ constante arg

$f(th_2) = t^k h_2^k \, g(th_2) \longrightarrow$

$$th_1 \cdot th_2 = Re(th_1 \overline{th_2}) = Re(t^2 h_1 \bar{h}_2)$$

$$\arg(f(th_1)) = \arg(h_1^k) + \arg(g(th_1))$$

$$\arg(f(th_2)) = \arg(h_2^k) + \arg(g(th_2))$$

$$\arg(f(th_1)) - \arg(f(th_2)) = k(\arg h_1 - \arg h_2) + \arg(g(th_1)) - \arg(g(th_2))$$

$$\begin{aligned} g(h) &= g(0) + g'(0)h + h\,\varepsilon(h) \\ &= a_k + a_{k+1} h + h\,\varepsilon(h) \end{aligned}$$

$$g(th_1) = a_k + a_{k+1} th_1 + th_1 \varepsilon(th_1) \approx a_k$$

$$g(th_2) = a_k + a_{k+1} th_2 + th_2 \varepsilon(th_2) \approx a_k$$