COMPLEX NUMBERS

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Complex Numbers

A complex number z is an ordered pair (x, y) of real numbers x and y,

$$\boxed{z = (x, y) = x + iy \text{ or } x + yi}$$

where x is called the real part z, y is the imaginary part of z and $i = \sqrt{-1}$ or $i^2 = -1$. The complex conjugate $\bar{z}$ of the complex number $z = x + yi$ is defined by:

$$\boxed{\bar{z} = x - yi}$$

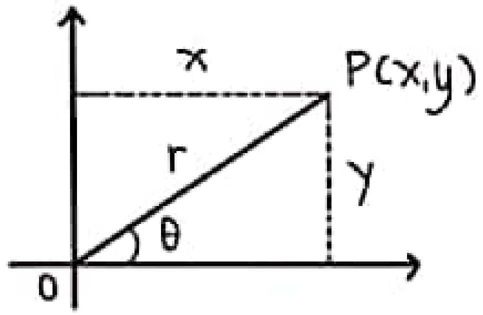
and

$$z \cdot \bar{z} = (x + yi)(x - yi)$$

$$\boxed{z \cdot \bar{z} = x^2 + y^2}$$

Geometric Representation of a Complex Number

A complex number $z = x + yi$ can be considered as a point P with coordinates (x, y) on a rectangular x-y plane called in this case complex plane or the Argand diagram.



Let r be the distance OP and θ be the angle made by OP with the positive x-axis. hence,

$$\boxed{x = r\cos\theta} \; ; \; \boxed{y = r\sin\theta}$$

Forms of a Complex Number

A complex number can be written in the following forms:

a) **Rectangular Form**

$$\boxed{z = x + yi}$$

b) **Trigonometric Form**

$$\boxed{\begin{array}{l} z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta \\ z = r[\cos(\theta + k\cdot 360^\circ) + i\sin(\theta + k\cdot 360^\circ)] \end{array}}$$

$k = 0, 1, 2, 3, \ldots$

where:

$\boxed{r = \sqrt{x^2 + y^2}}$ ← modulus or absolute value of z

$\boxed{\theta = \tan^{-1}\left[\frac{y}{x}\right]}$ ← argument of z in degrees

c) **Polar Form**

$\boxed{z = r\angle\theta}$ ← read as r bar; angle θ

or in general form

$$\boxed{z = r\angle\theta + k\cdot 360^\circ}$$

$k = 0, 1, 2, 3, \ldots$

d) **Exponential Form**

$$\boxed{z = re^{i\theta}}$$

or in general form:

$$\boxed{z = re^{i(\theta + 2k\pi)}}$$

where:

θ is in radian measure

$k = 0, 1, 2, 3, \ldots$

Operations of Complex Numbers in Rectangular Form

Considering the complex numbers; $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the operations are as follows:

a) **Addition:**

The complex number obtained by adding the real parts and imaginary parts of z_1 and z_2, respectively.

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$

$$\boxed{z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)}$$

b) **Subtraction**

The complex number obtained by subtracting the real parts and imaginary parts of z_1 and z_2, respectively.

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$$

$$\boxed{z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)}$$

c) **Multiplication**

The complex number obtained by multiplying the real parts and imaginary parts of z_1 and z_2.

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$$

$$\boxed{z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_2 y_1 + x_1 y_2)}$$

d) **Division**

The complex number obtained by dividing z_2 by z_1

$$\frac{z_2}{z_1} = \frac{x_2 + iy_2}{x_1 + iy_1} = \frac{x_2 + iy_2}{x_1 + iy_1} \cdot \frac{x_1 - iy_1}{x_1 - iy_1}$$

$$\boxed{\frac{z_2}{z_1} = \frac{x_2 x_1 + y_2 y_1}{x_1^2 + y_1^2} + i\,\frac{x_1 y_2 - x_2 y_1}{x_1^2 + y_1^2}}$$

Square root of a Complex Number

To obtain the square root of a complex number $z = x + iy$, consider these:

1) Let $x+iy$ be equal to the square root of the given complex number.
2) Square both sides of the resulting equation
3) Equate the real parts and imaginary parts of the equation, respectively.
4) By simple substitution, solve for the required roots

Sample Problem 1:

Simplify the following:

a) i^{17} b) $i^{12} - i^{10} + i^{9}$ c) $i^{-5} + i^{-8}$

Solution:

a) $i^{17} = (i^4)^4 \cdot i = (1)^4 \cdot i = 1 \cdot i = \boxed{i}$

b) $i^{12} - i^{10} + i^{9} = (i^4)^3 - (i^4)^2 \cdot i^2 + (i^4)^2 \cdot i$
$= (1)^3 - (1)^2 \cdot (-1) + (1)^2 \cdot i$
$= \boxed{2 + i}$

c) $i^{-5} + i^{-8} = \frac{1}{i^5} + \frac{1}{i^8} = \frac{1}{i^4 \cdot i} + \frac{1}{(i^4)^2} = \frac{1}{(1)i} + \frac{1}{(1)^2}$
$= \frac{1}{i} + 1 = \frac{1+i}{i} \cdot \frac{-i}{-i} = \frac{-i - i^2}{-i^2} = \boxed{1 - i}$

Sample Problem 2:

Perform the indicated operations:

a) $(1+2i) + (-3+4i) - (-5-6i)$

b) $(1+2i)(2-3i)(-3+4i)$

c) $\frac{(1+2i) + (6-7i)}{(2-3i) - (-5-6i)}$

d) $\sqrt{-3+4i}$

Solutions:

a) $z = (1+2i) + (-3+4i) - (-5-6i)$
$= (1-3+5) + i(2+4+6) = \boxed{3 + 12i}$

b) $z = (1+2i)(2-3i)(-3+4i) = (2-3i+4i+6)(-3+4i)$

$= (8+i)(-3+4i)$

$= -24-3i+32i-4 = \boxed{-28+29i}$

c) $z = \dfrac{(1+2i)+(6-7i)}{(2-3i)-(-5-6i)} = \dfrac{7-5i}{7+3i} \cdot \dfrac{7-3i}{7-3i}$

$= \dfrac{49-35i-21i-15}{7^2+3^2}$

$= \dfrac{34-56i}{58} = \boxed{0.586-0.966i}$

d) let $x+iy = \sqrt{-3+4i}$

Square both sides:

$x^2+2ixy-y^2 = -3+4i$

$x^2-y^2 = -3 \quad (1)$

$2xy = 4 \quad (2)$

Substitute (2) in (1) $\quad x^2 - \left(\frac{2}{x}\right)^2 = -3$

$x^2 - \frac{4}{x^2} = -3$

$x^4 - 4 = -3x^2$

$x^4+3x^2-4=0$

$(x^2-1)(x^2+4)=0$

$x^2-1=0 \; ; \; x^2+4=0$

$x = \pm 1 \; ; \; x = \pm 2i$ (reject)

For y, using (2)

$y = \frac{2}{\pm 1} = \pm 2$

Hence: $\sqrt{-3+4i} = \pm 1 \pm 2i$

$\sqrt{-3+4i} = \boxed{1+2i}$

$\sqrt{-3+4i} = \boxed{-1-2i}$

Operations OF Complex Numbers in Polar Form

Considering the two complex numbers $z_1 = r_1 \angle \theta_1$ and $z_2 = r_2 \angle \theta_2$, the operations are as follows:

a) Addition/Subtraction

Transform the complex numbers z_1 and z_2 to their equivalent rectangular forms and then perform the indicated operations (add or subtract)

b) Multiplication

The complex number obtained by multiplying z_1 and z_2

$$z_1 \cdot z_2 = r_1 \angle \theta_1 \cdot r_2 \angle \theta_2$$

$$= r_1 (\cos\theta_1 + i\sin\theta_1) \cdot r_2 (\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i (\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1)$$

$$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$\boxed{z_1 \cdot z_2 = r_1 r_2 \angle \theta_1 + \theta_2}$$

c) Division

The complex number obtained by dividing z_2 by z_1

$$\frac{z_2}{z_1} = \frac{r_2 \angle \theta_2}{r_1 \angle \theta_1}$$

$$= \frac{r_2 (\cos\theta_2 + i\sin\theta_2)}{r_1 (\cos\theta_1 + i\sin\theta_1)} \cdot \frac{\cos\theta_1 - i\sin\theta_1}{\cos\theta_1 - i\sin\theta_1}$$

$$= \frac{r_2}{r_1} \left[\frac{(\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1) + i(\sin\theta_2 \cos\theta_1 - \sin\theta_1 \cos\theta_2)}{\cos^2\theta + \sin^2\theta} \right]$$

$$= \frac{r_2}{r_1} [\cos(\theta_2 - \theta_1) + i\sin(\theta_2 - \theta_1)]$$

$$\boxed{\frac{z_2}{z_1} = \frac{r_2}{r_1} \angle \theta_2 - \theta_1}$$

Operations of Complex Numbers in Exponential Form

Considering the two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, the operations are as follows:

a) **Addition / Subtraction**

Transform the complex numbers z_1 and z_2 to their equivalent rectangular forms and then perform the indicated operations (add or subtract)

b) **Multiplication**

the complex number obtained by multiplying z_1 and z_2

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}$$

$$\boxed{z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}}$$

c) **Division**

The complex number obtained by dividing z_2 by z_1

$$\frac{z_2}{z_1} = \frac{r_2 e^{i\theta_2}}{r_1 e^{i\theta_2}}$$

$$\boxed{\frac{z_2}{z_1} = \frac{r_2}{r_1} e^{i(\theta_2 - \theta_1)}}$$

Power of a Complex Number and De Moivre's Theorem

Consider the complex number $z = r\angle\theta$ raise to a certain power n,

$$z^n = [r\angle\theta]^n$$

$$= [r(\cos\theta + i\sin\theta)]^n$$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$\boxed{z^n = r^n \angle n\theta}$$

If $r = 1$ in $z^n = r^n(\cos n\theta + i\sin n\theta)$, then this is known as the De Moivre's Theorem that is,

$$\boxed{z^n = (\cos n\theta + i\sin n\theta) = (\cos\theta + i\sin\theta)^n}$$

If the complex number in rectangular form is raised to a certain power, transform it to its equivalent polar form, then evaluate.

Roots of a Complex Number

To find the roots of a complex number, consider the n^{th} root of it in the general polar or general exponential forms as shown:

$$z^{\frac{1}{n}} = [r\angle\theta + k\cdot 360°]^{\frac{1}{n}}$$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\angle\left[\frac{\theta + k\cdot 360°}{n}\right]$$

Similarly,

$$z^{\frac{1}{n}} = [re^{i(\theta + 2k\pi)}]^{\frac{1}{n}}$$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}e^{i\left[\frac{\theta + 2k\pi}{n}\right]}$$

where:

$$k = 0, 1, 2, 3, \ldots, (n-1)$$

Sample Problem 3:

Perform the indicated operations and express the results in rectangular form correct to three decimal places.

a) $4e^{\frac{\pi}{4}i} + 2\angle{-125°} - (1+3i)$

Solution:

$$\text{let } z = 4e^{\frac{\pi}{4}i} + 2\angle{-125°} - (1+3i)$$

Transform $4e^{\frac{\pi}{4}i}$ and $2\angle{-125°}$ to rectangular form

$$4e^{\frac{\pi}{4}i} = 4\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = 2.828 + 2.828i$$

$$2\angle{-125^\circ} = 2\left[\cos(-125^\circ) + i\sin(-125^\circ)\right] = -1.147 - 1.638i$$

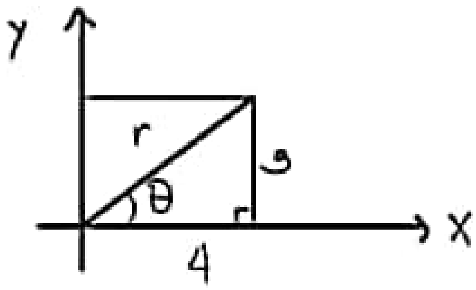
$$z = 2.828 + 2.828i + (-1.147 - 1.638i) - (1+3i)$$

$$\boxed{z = 0.681 - 1.810i}$$

b) $\dfrac{3\angle 120^\circ}{(4+3i)(2e^{\frac{\pi}{3}i})}$

Solution: Let $z = \dfrac{3\angle 120^\circ}{(4+3i)(2e^{\frac{\pi}{3}i})}$

Transform $(4+3i)$ to Polar form.



$r = \sqrt{4^2+3^2}$ $\qquad \theta = \tan^{-1}\left(\frac{3}{4}\right)$

$r = 5$ $\qquad \theta = 36.870^\circ$

$$4+3i = 5\angle 36.870^\circ$$

Transform $2e^{\frac{\pi}{3}i}$ to polar form:

$$2e^{\frac{\pi}{3}i} = 2\angle \frac{\pi}{3} \cdot \frac{180^\circ}{\pi} = 2\angle 60^\circ$$

$$z = \frac{3\angle 120^\circ}{5\angle 36.870^\circ \cdot 2\angle 60^\circ}$$

$$= \frac{3}{(5)(2)} \angle\left[120^\circ - (36.870^\circ + 60^\circ)\right]$$

$$= 0.30\angle 23.130^\circ$$

$$= 0.30\left[\cos(23.130^\circ) + i\sin(23.013^\circ)\right]$$

$$\boxed{z = 0.276 + 0.118i}$$

c) $\left[(1.3e^{1.4i})(1.4\angle 28^\circ)\right]^5$

Solution:

Let $z = \left[(1.3e^{1.4i})(1.4\angle 28^\circ)\right]^5$

Transform $1.3e^{1.4i}$ to polar form:

$$1.3e^{1.4i} = 1.3\angle\left[(1.4)\cdot\frac{180^\circ}{\pi}\right] = 1.3\angle 80.214^\circ$$

$$z = \left[1.3\angle 80.214^\circ \cdot 1.4\angle 28^\circ\right]^5$$

$$= (1.825)^5\angle\left[5(80.214^\circ + 28^\circ)\right]$$

$$= 19.969\angle 541.070^\circ$$

$$= 19.969(\cos 541.070^\circ + i\sin 541.070^\circ)$$

$$\boxed{z = -19.966 - 0.373i}$$

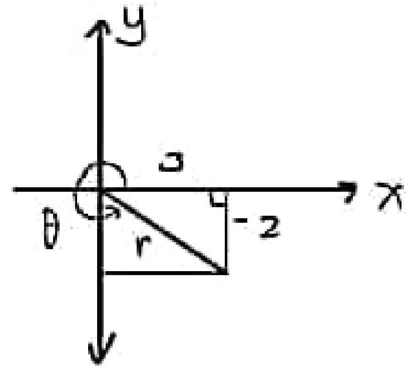
d) $\sqrt[4]{3-2i}$

let $z = \sqrt[4]{3-2i}$

Transform $(3-2i)$ to Polar Form

$r = \sqrt{(3)^2 + (-2)^2}$ $\qquad \theta = 360^\circ - \tan^{-1}\left(\frac{2}{3}\right)$

$r = 3.606$ $\qquad \theta = 326.297^\circ$



$$3-2i = 3.606\angle 326.297^\circ$$

$$z = \left[3.606\angle 326.297^\circ\right]^{\frac{1}{4}}$$

$$z = (3.606)^{\frac{1}{4}}\angle\frac{326.297^\circ + k\cdot 360^\circ}{4}$$

$$z = 1.378\angle\frac{326.297^\circ + k\cdot 360^\circ}{4}$$

For z_1; when $k=0$

$$z_1 = 1.378\angle\frac{326.297^\circ + (0)360^\circ}{4}$$

$$= 1.378\angle 81.574 = 1.378(\cos 81.574 + i\sin 81.574)$$

$$= \boxed{0.202 + 1.363i}$$

For z_2: when k = 1

$$z_2 = 1.378 \angle \frac{326.297^\circ + (1)360^\circ}{4}$$

$$z_2 = 1.378 \angle 171.574^\circ = 1.378(\cos 171.574^\circ + i\sin 171.574^\circ)$$

$$\boxed{z_2 = -1.363 + 0.202i}$$

For z_3, when k = 2:

$$z_3 = 1.378 \angle \frac{326.297^\circ + (2)360^\circ}{4}$$

$$= 1.378 \angle 261.574^\circ = 1.378(\cos 261.574^\circ + i\sin 261.574^\circ)$$

$$\boxed{z_3 = -0.202 - 1.363i}$$

For z_4, when k = 3:

$$z_4 = 1.378 \angle \frac{326.297^\circ + (3)360^\circ}{4}$$

$$z_4 = 1.378 \angle 351.574^\circ = 1.378 \cos 351.574^\circ + i\sin 351.574^\circ$$

$$\boxed{z_4 = 1.363 - 0.202i}$$

Sample Problem 4:

Using De Moivre's Theorem, find the identity of:

a) $\sin 6\theta$ and $\cos 6\theta$

Solution:

$$\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$$

Use Pascal Triangle; n = 6

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$$\cos 6\theta + i\sin 6\theta = \cos^6\theta + 6\cos^5\theta\,(i\sin\theta) + 15\cos^4\theta\,(i\sin\theta)^2 + 20\cos^3\theta\,(i\sin\theta)^3$$
$$+ 15\cos^2\theta\,(i\sin\theta)^4 + 6\cos\theta\,(i\sin\theta)^5 + (i\sin\theta)^6$$

$$\cos 6\theta + i\sin 6\theta = \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta$$
$$+ 15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta$$

$$\therefore \boxed{\begin{aligned} \sin 6\theta &= 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta \\ \cos 6\theta &= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta \end{aligned}}$$

b) $\sin^4\theta$ and $\cos^4\theta$

Solution:
$$z^1 = \cos\theta + i\sin\theta \longrightarrow (1)$$
$$z^{-1} = \cos\theta - i\sin\theta \longrightarrow (2)$$

Eliminate $i\sin\theta$ in (1) and (2)

$$\begin{array}{rl} & z^1 = \cos\theta + i\sin\theta \\ + & z^{-1} = \cos\theta - i\sin\theta \\ \hline & z^1 + z^{-1} = 2\cos\theta \end{array}$$

Eliminate $\cos\theta$ in (1) and (2)

$$\begin{array}{rl} & z^1 = \cos\theta + i\sin\theta \\ - & z^{-1} = \cos\theta - i\sin\theta \\ \hline & z^1 - z^{-1} = 2i\sin\theta \end{array}$$

Similarly,
$$z^n + z^{-n} = 2\cos n\theta \longrightarrow (3)$$
$$z^n - z^{-n} = 2i\sin n\theta \longrightarrow (4)$$

Hence, Using Pascal triangle, $n=4$
$$(2i\sin\theta)^4 = (z^1 - z^{-1})^4$$
$$16i^4\sin^4\theta = z^4 - 4z^3\cdot z^{-1} + 6z^2\cdot z^{-2} - 4z^1\cdot z^{-3} + z^{-4}$$
$$16\sin^4\theta = (z^4 + z^{-4}) - 4(z^2 + z^{-2}) + 6$$

$$16\sin^4\theta = 2\cos 4\theta - 4(2\cos 2\theta) + 6$$

$$\boxed{\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)}$$

$$(2\cos\theta)^4 = (z^1 + z^{-1})^4$$

$$16\cos^4\theta = z^4 + 4z^3\cdot z^{-1} + 6z^2\cdot z^{-2} + 4z^1\cdot z^{-3} + z^{-4}$$

$$= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$$

$$= 2\cos 4\theta + 4(2\cos 2\theta) + 6$$

$$\boxed{\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)}$$

Exponents and Logarithm of Complex Numbers

Sample Problem 5:

Evaluate the following:

a) $\ln(-2+5i)^4$

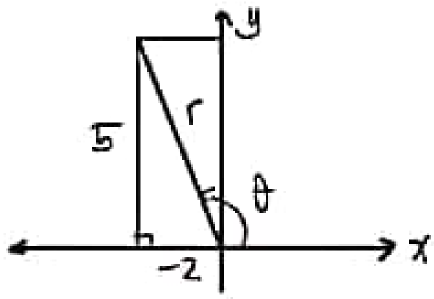
Solution: let $z = \ln(-2+5i)^4$

$$z = 4\ln(-2+5i)$$

Transform $(-2+5i)$ to exponential form:

$r = \sqrt{(-2)^2 + 5^2}$ $\qquad$ $\theta = \pi - \tan^{-1}\left(\frac{5}{2}\right)$

$r = 5.385$ $\qquad$ $\theta = 1.951$ rad



$$-2+5i = 5.385e^{1.95i}$$

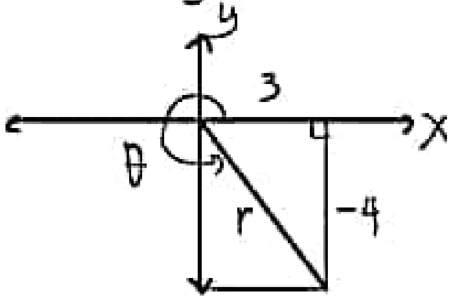
$$= e^{\ln 5.385}e^{1.95i}$$

$$-2+5i = e^{(1.684 + 1.95i)}$$

$$\ln(-2+5i) = \ln e^{(1.684 + 1.95i)} = 1.684 + 1.951i$$

$$z = 4(1.684 + 1.951i)$$

$$\boxed{z = 6.736 + 7.804i}$$



b) $\ln(3-4i)^{(4+i)}$

let $z = \ln(3-4i)^{(4+i)}$

$z = (4+i)\ln(3-4i)$

Transform $(3-4i)$ to Exponential Form:

$r = \sqrt{(3)^2 + (-4)^2}$ $\quad\quad \theta = 2\pi - \tan^{-1}\left(\frac{4}{3}\right)$

$r = 5$ $\quad\quad \theta = 5.356 \text{ rad}$

$3-4i = 5e^{5.356i}$

$= e^{\ln 5} \cdot e^{5.356i}$

$3-4i = e^{(1.609 + 5.356i)}$

$\ln(3-4i) = \ln e^{(1.609+5.356i)} = 1.609 + 5.356i$

$z = (4+i)(1.609+5.356i)$

$= (6.436 + 1.609i + 21.424i - 5.356)$

$\boxed{z = 1.080 + 23.033i}$

c) $(1+2i)^{(2-i)}$

Solution: let $z = (1+2i)^{(2-i)}$

Take the natural logarithm of both sides:

$\ln z = \ln(1+2i)^{(2-i)}$

$\ln z = (2-i)\ln(1+2i)$

Transform $1+2i$ to exponential form:

$r = \sqrt{1^2 + 2^2}$ $\quad\quad \theta = \tan^{-1}\left(\frac{2}{1}\right)$

$r = 2.236$ $\quad\quad \theta = 1.107 \text{ rad}$

$1+2i = 2.236e^{1.107i}$

$= e^{\ln 2.236} e^{1.107i}$

Hence: $\ln(1+2i) = \ln e^{(0.805+1.107i)} = 0.805 + 1.107i$

$$\ln z = (2-i)(0.805 + 1.107i)$$
$$= 1.610 - 0.805i + 2.214i + 1.107$$
$$\ln z = 2.717 + 1.409i$$

Take the inverse natural logarithm of both sides:

$$e^{\ln z} = e^{2.717 + 1.409i}$$
$$z = e^{2.717} \cdot e^{1.409i}$$
$$= 15.135\, e^{1.409i}$$
$$= 15.135 [\cos(1.409) + i \sin(1.409)]$$
$$\boxed{z = 2.438 + 14.937i}$$

d) $\log_{(3+i)}(1+3i)$

Solution: let $z = \log_{(3+i)}(1+3i)$

or $(3+i)^z = 1+3i$

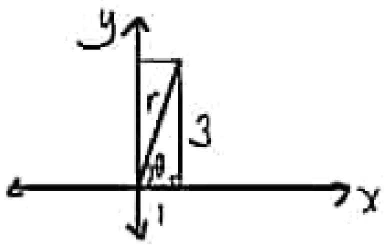
Take natural logarithm, $z \ln(3+i) = \ln(1+3i)$

$$z = \frac{\ln(1+3i)}{\ln(3+i)}$$

Transform $(1+3i)$ to exponential form:

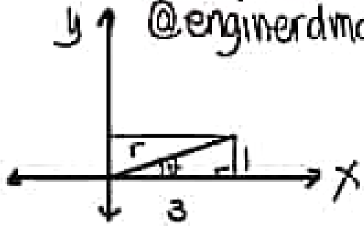
$r = \sqrt{1^2 + 3^2}$ $\qquad \theta = \tan^{-1}\left(\frac{3}{1}\right)$

$r = 3.162$ $\qquad \theta = 1.249$ rad



$$1+3i = 3.162 e^{1.249i}$$
$$= e^{\ln 3.162} e^{1.249i}$$
$$1+3i = e^{(1.151 + 1.249i)}$$
$$\ln(1+3i) = \ln e^{(1.151 + 1.249i)} = 1.151 + 1.249i$$

Transform $(3+i)$ to Exponential form:



$$r = \sqrt{3^2+1^2} \qquad \theta = \tan^{-1}\left(\frac{1}{3}\right)$$

$$r = 3.162 \qquad \theta = 0.322 \text{ rad}$$

$$3+i = 3.162e^{0.322i}$$
$$= e^{\ln 3.162} e^{0.322i}$$
$$3+i = e^{(1.151+0.322i)}$$
$$\ln(3+i) = \ln e^{(1.151+0.322i)} = 1.151+0.322i$$

Hence:

$$z = \frac{1.151+1.249i}{1.151+0.322i} \cdot \frac{1.151-0.322i}{1.151-0.322i}$$
$$= \frac{1.325+1.438i-0.371i+0.402}{1.325+0.104}$$
$$= \frac{1.727+1.067i}{1.429}$$
$$\boxed{z = 1.208+0.747i}$$

Trigonometric functions of Complex Numbers

For the trigonometric functions of complex numbers, consider Euler's identities:

$$e^{i\theta} = \cos\theta + i\sin\theta \longrightarrow (1)$$
$$e^{-i\theta} = \cos\theta - i\sin\theta \longrightarrow (2)$$

Eliminating $\cos\theta$ in (1) and (2) yields

$$\begin{array}{rl} & e^{i\theta} = \cos\theta + i\sin\theta \\ - & e^{-i\theta} = \cos\theta - i\sin\theta \\ \hline & e^{i\theta} - e^{-i\theta} = 2i\sin\theta \end{array}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Eliminating $i\sin\theta$ in (1) and (2)

$$
\begin{array}{rl}
 & e^{i\theta} = \cos\theta + i\sin\theta \\
+ & e^{-i\theta} = \cos\theta - i\sin\theta \\
\hline
 & e^{i\theta} + e^{-i\theta} = 2\cos\theta
\end{array}
$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

let $\theta = z_1 + z_2$

Hence: $$e^{i(z_1+z_2)} = \cos(z_1+z_2) + i\sin(z_1+z_2) \longrightarrow (3)$$

$$e^{-i(z_1+z_2)} = \cos(z_1+z_2) - i\sin(z_1+z_2) \longrightarrow (4)$$

Eliminating $\cos(z_1+z_2)$ in (3) & (4) yields:

$$\sin(z_1+z_2) = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} \longrightarrow (5)$$

Eliminating $i\sin(z_1+z_2)$ in (3) & (4) yields

$$\cos(z_1+z_2) = \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{2} \longrightarrow (6)$$

Similarly, $$e^{i(z_1+z_2)} = e^{iz_1} e^{iz_2}$$

$$= (\cos z_1 + i\sin z_1)(\cos z_2 - i\sin z_2)$$

$$e^{i(z_1+z_2)} = (\cos z_1\cos z_2 - \sin z_1\sin z_2) + i(\sin z_1\cos z_2 + \cos z_1\sin z_2) \longrightarrow (7)$$

and

$$e^{-i(z_1+z_2)} = e^{-iz_1} \cdot e^{-iz_2}$$

$$= (\cos z_1 - i\sin z_1)(\cos z_2 - i\sin z_2)$$

$$e^{-i(z_1+z_2)} = (\cos z_1\cos z_2 - \sin z_1\sin z_2) - i(\sin z_1\cos z_2 + \cos z_1\sin z_2) \longrightarrow (8)$$

Substituting (7) and (8) in (5) results to:

$$\sin(z_1+z_2) = \sin z_1\cos z_2 + \cos z_1\sin z_2$$

Substituting (7) and (8) in (6) results to:

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

$$\theta = z_1 + z_2 = (x + iy)$$

Hence:

$$\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \left[\frac{e^{-y}+e^{y}}{2}\right] + \cos x \left[\frac{e^{-y}-e^{y}}{2i} \cdot \frac{-i}{-i}\right]$$

$$\sin(x+iy) = \sin x \left[\frac{e^{-y}+e^{y}}{2}\right] + i\cos x \left[\frac{e^{y}-e^{-y}}{2}\right]$$

Since

$$\sinh y = \frac{e^{y}-e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^{y}+e^{-y}}{2}$$

and

$$\boxed{\begin{aligned} \sin(x+iy) &= \sin x \cosh y + i \cos x \sinh y \\ \sin(x-iy) &= \sin x \cosh y - i \cos x \sinh y \end{aligned}}$$

Similarly,

$$\cos(x+iy) = \cos \cos iy - \sin x \sin iy$$

$$= \cos x \left[\frac{e^{-y}+e^{y}}{2}\right] - \sin x \left[\frac{e^{-y}-e^{y}}{2i} \cdot \frac{-i}{-i}\right]$$

$$\cos(x+iy) = \cos x \left[\frac{e^{-y}+e^{y}}{2}\right] - i \sin x \left[\frac{e^{y}-e^{-y}}{2}\right]$$

and

$$\boxed{\begin{aligned} \cos(x+iy) &= \cos x \cosh y - i \sin x \sinh y \\ \cos(x-iy) &= \cos x \cosh y + i \sin x \sinh y \end{aligned}}$$

where:

$$\sin iy = i \sinh y$$

$$\cos iy = \cosh y$$

Hyperbolic Functions of Complex Numbers

For hyperbolic functions of complex numbers, consider the following identities:

a) $\sinh iy = i \sin y$

b) $\cosh iy = \cos y$

c) $\sinh(x \pm iy) = \sinh x \cosh iy \pm \cosh x \sinh iy$
$= \sinh x \cos y \pm i \cosh x \sin y$

d) $\cosh(x \pm iy) = \cosh x \cosh iy \pm \sinh x \sinh iy$
$= \cosh x \cos y \pm i \sinh x \sin y$

Sample Problem 6

Evaluate and express in rectangular form correct to three decimal places

a) $\sin(0.423i)$

Solution: let $z = \sin(0.423i)$

$$= i \sinh(0.423)$$

$$= i \, \frac{e^{0.423} - e^{-0.423}}{2}$$

$$\boxed{z = 0.436i}$$

b) $\sin(0.32 + 0.43i)$

Solution: let $z = \sin(0.32 + 0.43i)$

$$= \sin(0.32)\cosh(0.43) + i\cos(0.32)\sinh(0.43)$$

$$= 0.315 \, \frac{e^{0.43} + e^{-0.43}}{2} + i(0.949)\, \frac{e^{0.43} - e^{-0.43}}{2}$$

$$\boxed{z = 0.345 + 0.420i}$$

c) $\cosh(0.94 - 0.49i)$

Solution: let $z = \cosh(0.94 - 0.49i)$

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$$= \cosh(0.94)\cos(0.49) - i\sinh(0.94)\sin(0.49)$$

$$= \frac{e^{0.94}+e^{-0.94}}{2}(0.882) - i\,\frac{e^{0.94}-e^{-0.94}}{2}(0.471)$$

$$\boxed{z = 1.301 - 0.511i}$$

d) $\cos(3.21 - 1.23)$

Solution: let $z = \cos(3.21 - 1.23i)$

$$= \cos(3.21)\cosh(1.23) - i\sin(3.21)\sinh(1.23)$$

$$= (0.998)\,\frac{e^{1.23}+e^{-1.23}}{2} - i(-0.0684)\,\frac{e^{1.23}-e^{-1.23}}{2}$$

$$\boxed{z = 1.853 + 0.107i}$$

Inverse Trigonometric and Inverse Hyperbolic Functions of Complex Numbers

To obtain the value of the inverse trigonometric and inverse hyperbolic functions of complex numbers, consider the following relationships:

a) $\sin^{-1} z = -i \ln\left[iz \pm \sqrt{1-z^2}\right]$

b) $\cos^{-1} z = -i \ln\left[z \pm \sqrt{z^2-1}\right]$

c) $\tan^{-1} z = \frac{i}{2}\ln\left[\frac{i+z}{i-z}\right]$

d) $\sinh^{-1} z = \ln\left[z \pm \sqrt{z^2+1}\right]$

e) $\cosh^{-1} z = \ln\left[z \pm \sqrt{z^2-1}\right]$

f) $\tanh^{-1} z = \frac{1}{2}\ln\left[\frac{1+z}{1-z}\right]$

Sample Problem 7:

Evaluate and express results in rectangular form:

a) $\sin^{-1}(1+2i)$

Solution: let $\theta = \sin^{-1}(1+2i)$

$$= -i\ln\left[i(1+2i) \pm \sqrt{1-(1+2i)^2}\right]$$

$$= -i\ln\left[(i-2) \pm \sqrt{1-(1+4i-4)}\right]$$

$$\theta = -i\ln\left[(-2+i) \pm \sqrt{4-4i}\right]$$

Consider: $\sqrt{4-4i}$

$$x+iy = \sqrt{4-4i}$$

$$x^2 + 2ixy - y^2 = 4-4i$$

$$x^2 - y^2 = 4 \longrightarrow (1)$$

$$2xy = -4$$

$$y = \frac{-2}{x} \longrightarrow (2)$$

Substitute (2) in (1)

$$x^2 - \left(\frac{-2}{x}\right)^2 = 4$$

$$x^4 - 4 = 4x^2$$

$$x^4 - 4x^2 - 4 = 0$$

By quadratic formula:

$$x^2 = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-4)}}{2(1)}$$

$$x^2 = \frac{4 \pm 5.457}{2}$$

Take (+)

$$x^2 = \frac{4 + 5.457}{2}$$

$$x = \pm 2.197$$

$$y = \frac{-2}{\pm 2.197}$$

$$y = \mp 0.910$$

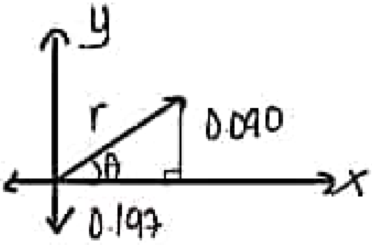
Hence $\sqrt{4-4i} = \pm 2.197 \mp 0.910i$

and $\theta = -\ln\left[(-2+i) \pm (\pm 2.197 \mp 0.910i)\right]$

If (+) $\theta_1 = -i\ln\left[(-2+i) + (2.197 - 0.910i)\right]$

$\theta_1 = -i\ln(0.197 + 0.090i)$

Transform $(0.197 + 0.090i)$ to exponential form:



$r = \sqrt{(0.197)^2 + (0.090)^2}$

$r = 0.217$

$\theta = \tan^{-1}\left(\frac{0.090}{0.197}\right)$

$\theta = 0.428$ rad

$$0.197 + 0.090i = 0.217e^{0.428i}$$

$$= e^{\ln(0.217)}e^{0.4278i}$$

$$= e^{(-1.528 + 0.428i)}$$

$$\ln(0.197 + 0.090i) = \ln e^{(-1.528 + 0.428i)}$$

$$\ln(0.197 + 0.090i) = -1.528 + 0.428i$$

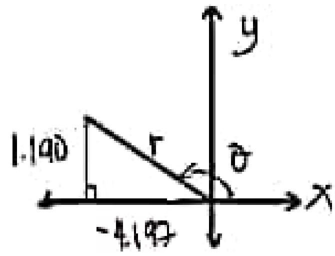
$$\theta_1 = -i(-1.528 + 0.428i)$$

$$\boxed{\theta_1 = 0.428 + 1.528i}$$

If (−) $\theta_2 = -i\ln\left[(-2+i) - (2.197 - 0.090i)\right]$

$\theta_2 = -i\ln(-4.197 + 1.910i)$

Transform $(-4.197 + 1.910i)$ to exponential form:



$r = \sqrt{(-4.197)^2 + (1.910)^2}$

$r = 4.611$

$\theta = \pi - \tan^{-1}\left(\frac{1.190}{4.197}\right)$

$\theta = 2.715$ rad

$$-4.197 + 1.910i = 4.611e^{2.715i}$$

$$= e^{\ln 4.611}e^{2.715i}$$

$$-4.197 + 1.910i = e^{(1.528 + 2.715i)}$$

$$\ln(-4.197+1.910i) = \ln e^{(1.528+2.715i)}$$

$$\ln(-4.197+1.910i) = 1.528+2.715i$$

$$\theta_2 = -i(1.528+2.715i)$$

$$\boxed{\theta_2 = 2.715-1.528i}$$

b) $\cosh^{-1}(2-4i)$

Solution: let $\theta = \cosh^{-1}(2-4i)$

$$= \ln\left[(2-4i) \pm \sqrt{(2-4i)^2-1}\right]$$

$$= \ln\left[(2-4i) \pm \sqrt{4-16i-16-1}\right]$$

$$\theta = \ln\left[(2-4i) \pm \sqrt{-13-16i}\right]$$

Consider $\sqrt{-13-16i}$

$$x+iy = \sqrt{-13-16i}$$

$$x^2+2ixy-y^2 = -13-16i$$

$$x^2-y^2 = -13 \longrightarrow (1)$$

$$2xy = -16$$

$$y = -\frac{8}{x} \longrightarrow (2)$$

Substitute (2) in (1)

$$x^2-\left(-\frac{8}{x}\right)^2 = -13$$

$$x^2-\frac{64}{x^2} = -13$$

$$x^4+13x^2-64=0$$

By Quadratic Formula:

$$x^2 = \frac{-13 \pm \sqrt{(13)^2-4(1)(-64)}}{2(1)}$$

$$x^2 = \frac{-13 \pm 20.616}{2}$$

Take (+)

$$x^2 = \frac{-13+20.616}{2} \quad \therefore x = \pm 1.951$$

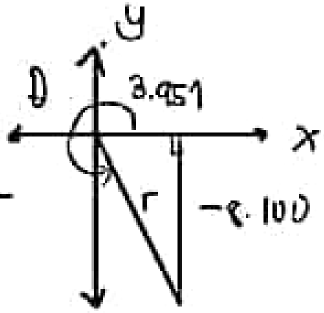
Using (2)
$$y = \frac{-8}{\pm 1.951}$$
$$y = \mp 4.100$$

Hence
$$\sqrt{-13-16i} = \pm 1.951 \mp 4.100i$$

and
$$\theta = \ln[(2-4i) \pm (\pm 1.951 \mp 4.100i)]$$

For (+)
$$\theta_1 = \ln[(2-4i) + (1.951 - 4.100i)]$$
$$\theta_1 = \ln(3.951 - 8.100i)$$

Transform $(3.951 - 8.100i)$ to exponential form



$$r = \sqrt{(3.951)^2 + (-8.100)^2} \qquad \theta = 2\pi - \tan^{-1}\frac{8.100}{3.951}$$
$$r = 9.012 \qquad \theta = 5.166 \text{ rad}$$

$$3.951 - 8.100i = 9.012e^{5.166i}$$
$$= e^{\ln 9.012} e^{5.166i}$$
$$3.951 - 8.100i = e^{(2.199 + 5.166i)}$$
$$\ln(3.951 - 8.100i) = \ln e^{(2.199 + 5.166i)}$$
$$\ln(3.951 - 8.100i) = 2.199 + 5.166i$$
$$\boxed{\theta_1 = 2.199 + 5.166i}$$

If (-)
$$\theta_2 = \ln[(2-4i) - (1.951 - 4.100i)]$$
$$\theta_2 = \ln(0.049 + 0.100i)$$

Transform $0.049 + 0.100i$ to exponential form
$$r = \sqrt{(0.049)^2 + (0.100)^2} \qquad \theta = \tan^{-1}\left(\frac{0.100}{0.049}\right)$$
$$r = 0.111 \qquad \theta = 1.115 \text{ rad}$$
$$0.049 + 0.100i = 0.111e^{1.115i}$$
$$= e^{\ln 0.111} e^{1.115i}$$
$$\ln 0.049 + 0.100i = \ln e^{(-2.198 + 1.115i)}$$
$$\ln(0.049 + 0.100i) = \boxed{-2.198 + 1.115i = \theta_2}$$