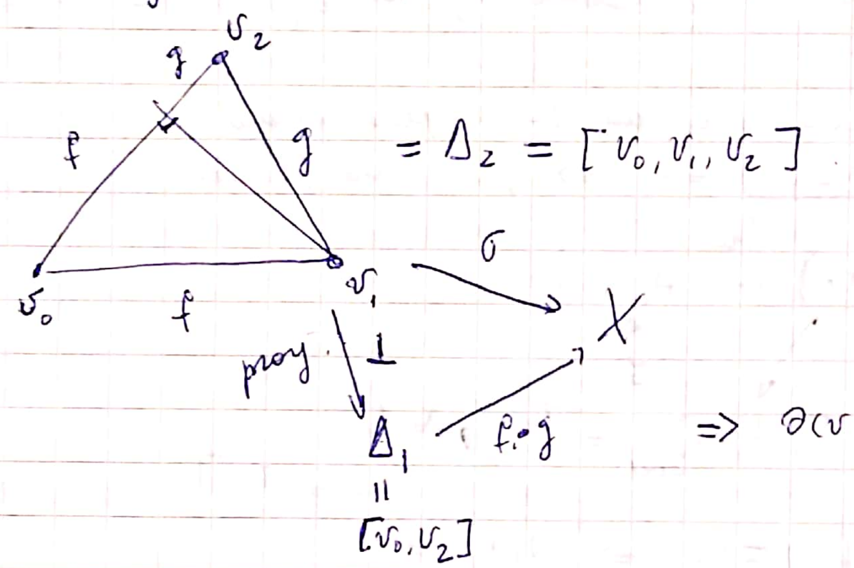
$\therefore$ en homología (en H_1) $\quad 0 = [f] - [g]$

i.e $[f] = [g]$

$\therefore h : \pi_1(X, x_0) \to H_1(X)$ está bien definido.

(3) Sea $f \cdot g$ la curva producto



$\Rightarrow \partial(\sigma) = g - f\cdot g + f$

$\therefore$ en $H_1(X)$, $f \cdot g = f + g$.

Teo X = arco-conexo. Luego h es sobre y su kernel es el conmutador de $\pi_1(X, x)$

Así h induce un isomorfismo entre la abelianización de $\pi_1(X, x_0)$ y $H_1(X)$

dem. (Sobre). Sea $\sum_i n_i \sigma_i \in C_1(X)$ representando un ciclo en $H_1(X)$

- Separándolos (y así, repitiéndolos) podemos asumir $n_i = \pm 1 \ \forall i$.

- Los (-1) los transformamos en $+1$ usando lo hecho anteriormente. Así tenemos $\sum_i \sigma_i$

- Si algún σ_i no es loop $\Rightarrow$ como $\partial(\sum \sigma_i)=0$, tenemos que tener σ_j con un mismo punto de borde.

- Luego $\sigma_i + \sigma_j$ lo cambiamos por $\sigma_i \circ \sigma_j$
- Seguimos ... $\sum_i \sigma_i$ con σ_i loops.
- Como X es arcoconexo, $\gamma_i : x_0 \longrightarrow$ punto base de σ_i

  y como $\gamma_i \circ \sigma_i \circ \overline{\gamma_i} = \gamma_i + \sigma_i - \gamma_i = \sigma_i$

  $\Rightarrow$ todos centrados en x_0

  $\Rightarrow$ sobre.

- $[\pi_1, \pi_1]$ = subgrupo normal más pequeño tal que $\pi_1 / [\pi_1, \pi_1]$ es abeliano.

  $[\pi_1, \pi_1]$ = subgrupo normal más pequeño que contiene a todos los $aba^{-1}b^{-1}$

(G/N abeliano $\Leftrightarrow [G,G] \leq N$)

Prop. universal:

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G/[G,G] \arrow[dl, dashed, "\exists!"] \\ A & \end{tikzcd}$$

A = grupo abeliano.

$\mu_j \geq 0 \ \forall j$; $\mu_j \xrightarrow{*} \mu \Rightarrow \mu_j(E) \to \mu(E) \quad \forall E$, $\mu(\partial E) = 0$.

pre-compacto!

- X H.l.c. μ_j, μ. Borel regular.

(*) Se cumple que $\limsup \mu_k(K) \leq \mu(K) \quad \forall K$ compacto.
$\liminf \mu_k(U) \geq \mu(U) \quad \forall U$ abierto.

dem Sea $K \subseteq X$ compacto $\Rightarrow K \subseteq U \leftarrow$ abierto.
X H.l.c $\Rightarrow \exists f : K \prec f \prec U$ sop compacto $\quad \mu(K) = \inf\{\mu(U) / U \supseteq K\}$ open

$$\mu(U) = \int_U d\mu = \int \chi_U \, d\mu \geq \int f \, d\mu = \lim \int f \, d\mu_j$$
$$= \limsup \int f \, d\mu_j \geq \limsup \int \chi_K \, d\mu_j = \limsup \mu_j(K)$$

$\forall U \supseteq K$ abierto.

$$\inf / \quad \inf_{K \subseteq U} \mu(U) \geq \limsup \mu_k(K)$$

$$\therefore \mu(K) \geq \limsup \mu_k(K)$$

obs Análogo para demostrar que $\liminf \mu_k(U) \geq \mu(U) \quad \forall U \subseteq X$ abierto.

dem. de la afirmación.

$E \subseteq X$. $\mu(\partial E) = 0$, $\partial E = cls(E) - int(E)$

¿porqué? ¿$\bar{E}$ compacto?

$\mu_j(E) =$

$$\limsup \mu_j(\bar{E}) \leq \mu(\bar{E}) = \mu(\partial E \cup int(E)) = \mu(int(E))$$

$\bar{E}$ compacto; $int(E)$ abierto

$$\leq \liminf \mu_j(int(E))$$

$$\therefore \limsup \mu_j(\bar{E}) = \liminf \mu_j(int(E))$$

$$int(E) \subseteq E \subseteq \bar{E} \Rightarrow \mu_j(int(E)) \leq \mu_j(E) \leq \mu_j(\bar{E})$$

$$\therefore \liminf \mu_j(E) = \limsup \mu_j(E).$$

$$\therefore \mu_j(E) \to \mu(E).$$

$P \in \mathcal{L}(H,H)$ es una proyección ~~ortogonal~~

Pd: P proyección ortogonal $\Rightarrow Im(P) = \ker(P)^\perp$

dem. P proyección ortogonal $\Leftrightarrow (Px, y)_H = (x, Py)_H \quad \forall x, y \in H$.

& ~~para~~ $P^2 = P$

$P^2 x = Px$ & $Px = z \in Im(P) \Rightarrow Pz = P^2 x = Px = z \quad \therefore Pz = z \quad \forall z \in Im(P)$.

$\forall y \in \ker(P)$: $(z, y) = (Px, y) = (x, Py) = (x, 0) = 0$.

$\therefore z \in \ker(P)^\perp$

$\therefore Im(P) \subseteq \ker(P)^\perp$

Por otro lado, $H = (Im(P)) \oplus (Im P)^\perp$

$Im(P) \subseteq \ker(P)^\perp \Rightarrow \underbrace{(\ker(P)^\perp)^\perp}_{\ker(P)} \subseteq Im(P)^\perp$

$(Px, Py) = (x, P^2 y) = (x, Py)$

$x \in R(P)^\perp \Rightarrow (x, Pz) = 0 \;\forall z \Rightarrow (Px, z) = 0 \quad \forall z \quad \therefore Px = 0$

$\therefore x \in \ker(P)$.

$\therefore$ ~~Ker~~ $Im(P)^\perp \subseteq \ker(P)$

$\therefore \ker(P)^\perp \subseteq Im(P)$

Pd: $Im(P) \subseteq \ker(P)^\perp$

$\|Px\|^2 = (Px, Px) = (x, P^2 x) = (x, Px)$

$= 0$

$\Rightarrow \|Px\| = 0$

$H = Im(P) \oplus (Im(P))^\perp \Rightarrow Im P \cap (Im P)^\perp = 0$

$\Rightarrow (\ker P)^\perp \cap (Im P)^\perp = 0$

$Px \in (\ker(P))^\perp \quad Px = 0$

$\Downarrow$

$\boxed{x \in \ker P}$

Sea $x \in (Im(P))^\perp \Rightarrow (x, Py) = 0 \quad \forall y \in H$

$= (Px, y) = 0 \quad \forall y \in \ker(P) \Rightarrow$ ~~$Px = 0 \therefore x \in \ker$~~

~~Pd~~:

Pd: $\operatorname{Im}(P) = \ker(P)^{\perp} \Rightarrow (Px, x) = \|Px\|^2$

dem: $\|Px\|^2 = (Px, Px) = (x, P^{*}Px) = (x, P^2x) = (x, Px)$

$H = \operatorname{Im}(P) \oplus \operatorname{Im}(P)^{\perp}$

$\qquad\qquad\qquad\qquad \| \ \ker(P)$

$x = y + z \ / \ y \in \operatorname{Im}(P), \ z \in \ker(P) = \operatorname{Im}(P)^{\perp}$

$\|Px\|^2 = (Px, Px) = (y, y) = (x - z, y) = (x, y) = (x, Px)$.

Af. H espacio de Hilbert, $P: H \to H$ proyección $\to H = \operatorname{Im}(P) \oplus \ker P^{\perp}$

$Px = y \Rightarrow$ ~~$P^2x = Py$~~. $Py = P^2x = Px = y \quad \therefore Py = y$.

$\Rightarrow P(y - x) = 0 \quad \therefore y - x \in \ker P$

$x = y + y - x \Rightarrow H = \operatorname{Im}(P) + \ker(P)$.

$x \in \operatorname{Im} P \cap \ker P \Rightarrow x = Py, \ Px = 0 \Rightarrow Px = P^2y = 0$

$Px = P^2y = Py = x \quad \therefore x = 0 \Rightarrow \operatorname{Im} P \cap \ker P = \{0\}$.

Hilbert

$A \in \mathcal{L}(H)$ A monomorfismo.

$(y_n) \in \operatorname{Im} A$, $y_n \to z$, $y_n = Ax_n$

$(x_n, y_n) = (x_n, Tx_n)$

Algebra |

$f : E^n \to F$ multilineal alternante $(u_1, \dots, u_n), (v_1, \dots, v_n) \in E^n$

$$v_j = \sum_{i=1}^{n} a_{ij} u_i$$

$$f(v_1, \dots, v_n) = f\left(\sum_{i=1}^{n} a_{i1} u_i, \sum_{i=1}^{n} a_{i2} u_i, \dots, \sum_{i=1}^{n} a_{in} u_i \right)$$

para i_1, i.

$$= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \dots \sum_{i_n=1}^{n} a_{i_1,1} a_{i_2,2} \dots a_{i_n,n} f(u_{i_1}, \dots, u_{i_n})$$

$= f(u_{\sigma(1)}, \dots, u_{\sigma(n)})$

$= \operatorname{sgn}(\sigma) f(u_1, \dots, u_n)$

si cualquier persona

cualquier persona

$$= \sum_{i=1}^{n} a_{i1} f\left(u_i, \sum_{i=1}^{n} a_{i2} u_i, \dots, \sum_{i=1}^{n} a_{in} u_i\right)$$

$$= \sum_{i=1}^{n} a_{\sigma(i)1} f\left(u_{\sigma(i)}, \sum_{i=1}^{n} a_{i2} u_i, \dots, \sum_{i=1}^{n} a_{in} u_i\right)$$

$f: E^3 \to F$ ~~bilineal~~ 3-lineal alternante

(u_1, u_2, u_3), $(v_1, v_2, v_3) \in E^3$ $\quad v_j = \sum_{i=1}^{3} a_{ij} u_i$

$f(v_1, v_2, v_3) =$

$\underline{\qquad\qquad}$ o $\underline{\qquad\qquad\qquad}$

$\underline{P2_1}$ $f \in \mathcal{L}(V) \Rightarrow F: V \otimes V \to V \otimes V$

$F(v \otimes w) = f(v) \otimes f(w)$

$M_n(F) \otimes M_m(F) \cong M_{nm}(F \otimes F)$

averiguar.

¿ $\det F$? cuando $\dim V < \infty$.

$f(v_j) = \sum_{i=1}^{n} a_{ij} v_i$

$\mathcal{B} = \{v_i\}_{i=1}^{n}$ base de V

$[f] = \begin{pmatrix} a_{11} & & & \\ a_{21} & \cdots & & \\ \vdots & & & \\ a_{n1} & & & \end{pmatrix} \begin{matrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{matrix}$

$F(v_i \otimes v_j) = f(v_i) \otimes f(v_j)$

$F(v_i \otimes v_j) =$ ~~$\sum a_{ij} v_i \otimes \sum a_{ij} v_i$~~ $\qquad f(v_i) = \sum_{p=1}^{n} a_{pi} v_p$

$$F(v_i \otimes v_j) = \left(\sum_{p=1}^{n} a_{pi} v_p\right) \otimes \left(\sum_{q=1}^{n} a_{qj} v_q\right)$$

$$= \sum_{p=1}^{n} \sum_{q=1}^{n} a_{pi} a_{qj} (v_p \otimes v_q)$$

$[F] = (F(v_1 \otimes v_1) \;\; F(v_1 \otimes v_2) \; \ldots \; F(v_1 \otimes v_n) \;\; F(v_2 \otimes v_1) \; \ldots \; F(v_2 \otimes v_n) \; \ldots \; F(v_n \otimes v_n))$

$F(v_1 \otimes v_1) = \Bigg(\begin{matrix} \sum_{p=1}^{n} \sum_{q=1}^{n} a_{1i} a_{1j} \\ \sum_{p=1}^{n} \sum_{q=1}^{n} a_{1i} a_{2j} \\ \vdots \end{matrix}$

P3| $\{e_i \wedge e_j\}_{1\le i<j\le n}$ base ortonormal de $\mathbb{R}^n$

$\langle\, , \,\rangle$ producto interno en $\mathbb{R}^n \wedge \mathbb{R}^n$

Calcular $\|v\wedge w\|$ para $n=2$, $n=3$

$\|w\| = \sqrt{\beta_1^2 + \beta_2^2}$

$v\cdot w = \alpha_1\beta_1 + \alpha_2\beta_2$

$\cos\theta = \dfrac{\alpha_1\beta_1+\alpha_2\beta_2}{\|v\|\,\|w\|}$

Bla!

$\underline{n=2}$| $\|v\wedge w\|^2 = \langle v\wedge w, v\wedge w\rangle$

$$= \Big\langle \sum_{1\le i<j\le n} \alpha_{ij}(e_i\wedge e_j), \sum_{1\le i<j\le n} \alpha_{ij}(e_i\wedge e_j)\Big\rangle$$

$$= \sum_{1\le i<j\le n}\ \sum_{1\le p<q\le n} \alpha_{ij}\alpha_{pq}\langle e_i\wedge e_j, e_p\wedge e_q\rangle$$

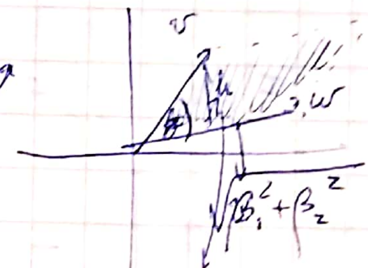
$\underline{n=2}$| $\Rightarrow \langle v\wedge w, v\wedge w\rangle = \sum_{1\le i<j\le 2} \alpha_{ij}\big(\alpha_{12}\langle e_i\wedge e_j, e_1\wedge e_2\rangle\big)$

$$= \alpha_{12}^2\langle e_1\wedge e_2, e_1\wedge e_2\rangle$$

$$= \alpha_{12}^2\|e_1\wedge e_2\| = \alpha_{12}^2$$

$v = \alpha_1 e_1 + \alpha_2 e_2$

$w = \beta_1 e_1 + \beta_2 e_2$



$v\wedge w = (\alpha_1 e_1 + \alpha_2 e_2)\wedge(\beta_1 e_1 + \beta_2 e_2)$

$= (\alpha_1 e_1 + \alpha_2 e_2)\wedge \beta_1 e_1 + (\alpha_1 e_1 + \alpha_2 e_2)\wedge \beta_2 e_2$

$= \alpha_2\beta_1(e_2\wedge e_1) + \alpha_1\beta_2\, e_1\wedge e_2$

$= (\alpha_1\beta_2 - \alpha_2\beta_1)(e_1\wedge e_2)$

$A = \left|\det\begin{pmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{pmatrix}\right|$

$= |\alpha_1\beta_2 - \alpha_2\beta_1|$

$\Rightarrow \langle v\wedge w, v\wedge w\rangle = (\alpha_1\beta_2 - \alpha_2\beta_1)^2$

$\therefore \|v\wedge w\| = |\alpha_1\beta_2 - \alpha_2\beta_1|$

Teoría de Representaciones

$\rho: G \to GL(V)$ $\quad V$ F-espacio vectorial
$\mathcal{B} = \{e_1, \dots, e_n\} \subseteq V$.
$\searrow$ base

$\rho(s): V \to V$ lineal invertible

$$\rho(s)(e_j) = \sum_{i=1}^{n} \alpha_{ij}(s) e_i \quad \forall j = 1, \dots, n. \quad (\alpha_{ij}(s))_{i,j=1}^{n}$$

matriz de $\rho(s)$.

$G = S_3 \quad s = (1,2)$, ρ representación regular de G

$S_3 = \{id, (12), (13), (23), (123), (132)\}$ ✓

$\rho(12)(1) = (12)\,id = (12)$
$\rho(12)(12) = (12)(12) = id$
$\rho(12)(13) = (12)(13) = (132)$
$\rho(12)(23) = (12)(23) = (231) = (123)$
$\rho(12)(123) = (12)(123) = (23)$
$\rho(12)(132) = (12)(132) = (13)$

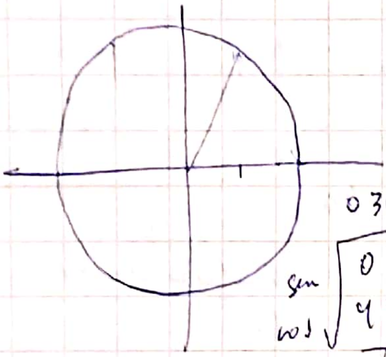
$id \to (1,0,0,0,0,0) = e_1$
$(12) \to (0,1,0,0,0,0) = e_2$
$(13) \to (0,0,1,0,0,0) = e_3$
$(23) \to (0,0,0,1,0,0) = e_4$
$(123) \to (0,0,0,0,1,0) = e_5$
$(132) \to (0,0,0,0,0,1) = e_6$

$\rho(12)(e_1) = e_2$
$\rho(12)(e_2) = e_1$
$\rho(12)(e_3) = e_6$
$\rho(12)(e_4) = e_5$
$\rho(12)(e_5) = e_4$
$\rho(12)(e_6) = e_3$

$$\therefore \quad (\rho(12))_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$G = S_3 \quad , \quad V = \mathbb{R}^2$

ρ representación de $G \cong D_6 \leftarrow$ dihedral



$\omega^3 = 1 \qquad \omega = e^{2\pi i/3} = \cos\left(\frac{2\pi}{3}\right) + i \,\text{sen}\left(\frac{2\pi}{3}\right)$

$= \frac{1}{2} + i\frac{\sqrt{3}}{2} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

	0	30	45	60	90
sen	$\sqrt{0}$	$\sqrt{1}$	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{4}$
cos	$\sqrt{4}$	$\sqrt{3}$	$\sqrt{2}$	$\sqrt{1}$	$\sqrt{0}$
			2		

$\omega^2 = e^{4\pi i/3} = e^{\pi i/3} e^{\pi i} = -e^{\pi i/3}$

$= -\left(\cos\left(\frac{\pi}{3}\right) + i \,\text{sen}\left(\frac{\pi}{3}\right)\right)$

$= -\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

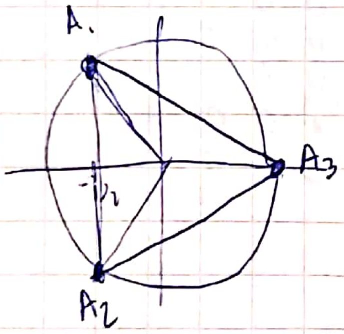
$e^{2\pi i/3} = \cos(\pi - \pi/3) + i \,\text{sen}(\pi - \pi/3) \longrightarrow \cancel{\cos(\pi} \; \text{sen}(\pi)\cos(\pi/3) - \cos(\pi)\,\text{sen}(\pi/3)$

$= -\cos(\pi/3) + i \,\text{sen}\,\pi/3$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{4} - i^2\frac{3}{4} = \frac{1}{4} + \frac{3}{4} = 1.$

$A_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \qquad A_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \qquad , A_3 = 1$



$(12)\,A_1 = A_2$

$(12)\,A_2 = A_1$

$(12)\,A_3 = A_3$

$\rho(12)\,e_2 = \frac{\sqrt{3}}{3}\rho(12)A_1 - \frac{\sqrt{3}}{3}\rho(12)A_2$

$= \frac{\sqrt{3}}{3}A_2 - \frac{\sqrt{3}}{3}A_1 = \frac{\sqrt{3}}{3}\left(-\frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2\right)$

$- \frac{\sqrt{3}}{3}\left(-\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2\right) = -\frac{3}{6}e_2 - \frac{3}{6}e_2 = -e_2$

$A_1 = -\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$

$A_2 = -\frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2$

$A_3 = e_1$

$A_1 + A_2 = -e_1$

$A_1 - A_2 = \sqrt{3}\,e_2$

$\Rightarrow 2A_1 = -e_1 + \sqrt{3}\,e_2$

$\rho(12)\,e_1 = e_1$

$e_1 = -A_1 - A_2 = -\left(-\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2\right) - \left(-\frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2\right) = e_1$

$\rho(12) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$e_2 = \frac{1}{\sqrt{3}}A_1 - \frac{1}{\sqrt{3}}A_2 = \frac{\sqrt{3}}{3}A_1 - \frac{\sqrt{3}}{3}A_2$

$\rho(12)\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}$

ok.

// Topología algebraica

teo 2B.5 $\mathbb{R}$ y $\mathbb{C}$ son las únicas álgebras de división finito-dimensional sobre $\mathbb{R}$ que son conmutativas y tienen identidad.

- La demostración es fácil: tomamos $\mathbb{R}^n$ álgebra de división finita, conmutativa, normada y con identidad.
- $f: S^{n-1} \to S^{n-1}$, $f(x) = \frac{x^2}{|x|^2}$ ← estudiar esa función
- ~~Después~~ Con f se demuestra que $n=2$
- Después asumir que A álgebra de dimensión 2, conmutativa de división, y c/ identidad $\Rightarrow A \cong \mathbb{C}$.

obs. Se necesita ocupar corolario 2B.4 (Hatcher, pág 172).

$f : X \to Y$ homotópica ...

$\Rightarrow \exists\, \tilde{f} : CX \to Y$

$h : I \times X \to Y$ continua tq $h(1,x) = h(1,x')$ $\forall x, x'$

$$\begin{tikzcd} X \arrow[r, hook, "i"', "x \mapsto (0,x)"] \arrow[drr, "f"'] & I \times X \arrow[r, "p"] \arrow[dr, "h"] & I \times X /_{(1,x)\sim(1,x')} = CX \arrow[d, dashed, "\tilde{f}"] \\ & & Y \end{tikzcd}$$

- Ahora si para $f : X \to Y$ $\exists\, \tilde{f} : CX \to Y$ que extiende a f $\Rightarrow f$ es null-homotópica.

dem. tenemos

$$\begin{tikzcd} I \times X \arrow[r, "p"] \arrow[dr, "\tilde{f} \circ p"'] & CX = I \times X /_{(1,x)\sim(1,x')} \arrow[d, "\tilde{f}"] \\ & Y \end{tikzcd} \qquad \tilde{f} \circ p \big|_{\{0\} \times X} = f$$

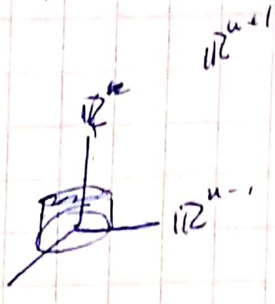
$\tilde{f} \circ p : I \times X \to Y$ es tal que $\tilde{f} \circ p(1,x) = \tilde{f} \circ p(1,x')$ $\forall x, x'$

X puede identificarse con $\{0\} \times X$ $(X \cong \{0\} \times X)$

$\therefore \tilde{f} \circ p$ homotopía

$\therefore f : X \to Y$ null homotópica

Se tiene que $\partial \mathbb{D}^n = S^{n-1}$ | $CS^{n-1} = \mathbb{D}^{n-1}$



$\therefore f$ "null-homotópica" (homotópica a $X \to Y$ constante).

$\therefore \deg(f) = 0$.

Funciones holomorfas. (From Rudin).

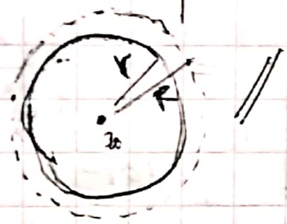
- Conexidad ← importante.

funciones $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$

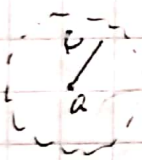
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \longrightarrow \text{relaciones } f' \text{ con } Df : \mathbb{R}^2 \to \mathbb{R}^2$$

$\mathbb{R}^2 \cong \mathbb{C}$.

Serie de potencias : $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ $\qquad \frac{1}{R} = \limsup_{n \to \infty} |c_n|^{1/n}$



$$\frac{1}{R} = \limsup |c_n|^{1/n} \quad , \quad f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$



$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ converge absolutamente en $B(a, R)$

uniformemente en $\overline{B(a,r)}$ $\forall r < R$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$|z| < R \Rightarrow \exists r : |z| < r < R$

$\frac{1}{r} > \frac{1}{R} = \limsup |a_n|^{1/n} \Rightarrow \exists N, \ \forall n \geq N : \frac{1}{r} \geq |a_n|^{1/n}$

$\frac{1}{R} < \frac{1}{r} \Rightarrow |a_n| < \frac{1}{r^n} \quad \forall n \geq N$

$\Rightarrow |a_n z^n| = |a_n| |z|^n < \left(\frac{|z|}{r}\right)^n$

$\Rightarrow \sum_{n \geq N} |a_n z^n| < \sum_{n \geq N} \left(\frac{|z|}{r}\right)^n < \infty \quad \left(\boxed{\frac{|z|}{r} < 1} \right)$

- Importante : M-Test de Weierstrass.

Si $S = \lim \left| \frac{a_n}{a_{n+1}} \right| \Rightarrow R = S$

- $\sum_{n=0}^{\infty} a^n z^n$ $\quad R^{-1} = \limsup |a_n|^{1/n}$ $\quad a_n = a^n$

$R^{-1} = \limsup |a^n|^{1/n} = \limsup |a| = |a|$.

$\lim \left| \frac{a^n}{a^{n+1}} \right| = \lim \frac{1}{|a|} = \frac{1}{|a|}$. $\Rightarrow$ $\underline{R = |a|^{-1}}$.

$\sum_{n=0}^{\infty} a^{n^2} z^n$, $S = \lim \left| \frac{a^{n^2}}{a^{(n+1)^2}} \right| = \lim \left| \frac{a^{n^2}}{a^{n^2+2n+1}} \right| = \lim \frac{1}{|a^{2n+1}|}$

$= \frac{1}{|a|} \lim \frac{1}{|a|^{2n}}$ $\quad S = \begin{cases} \frac{1}{\cancel{|a|}} & , |a| = 1 \\ 0 & , |a| > 1 \\ \infty & , |a| < 1 \end{cases}$

$R^{-1} = \limsup |a^{n^2}|^{1/n} = \limsup |a^n| = \limsup |a|^n$

$|a| = 1$ $\quad \sup\{ |a|^n, |a|^{n+1}, \dots$ $\quad \begin{cases} \infty & , |a| > 1 \\ 1 & , |a| = 1 \\ 0 & , |a| < 0. \end{cases}$

(d) $\sum_{n=0}^{\infty} z^{n!}$ $\quad \cancel{z^{0!}}$ $\quad z + z^2 + z^6 + z^{24} + \dots$

$\quad = z + \sum_{n=1}^{\infty} z^{n!} = z + \sum_{n=0}^{\infty} z^{(n+1)!} = z + \sum_{n=0}^{\infty} z^{n+1} z^{n!}$

$R^{-1} = \limsup |a_n|^{1/n} = \lim$ $\quad a_n \in \{0, 1\}$

$R^{-1} = 1 \Rightarrow \boxed{R = 1}$

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Radio de convergencia de $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)} = \sum_{k=1}^{\infty} a_k z^k \qquad a_k = \begin{cases} \frac{(-1)^n}{n}, & k = n(n+1) \\ 0, & k \neq n(n+1) \end{cases}$$

$$|a_k| = \begin{cases} \frac{1}{n}, & k = n(n+1) \\ 0, & k \neq n(n+1) \end{cases}$$

rompe lim sup.

$$|a_k|^{1/k} = \begin{cases} \frac{1}{n^{1/n(n+1)}} & k = n(n+1) \\ 0 \end{cases}$$

Sup.

— o —

$$f(z) := \int_X \frac{d\mu(\zeta)}{\varphi(\zeta) - z} \qquad \varphi : X \to \mathbb{C} \text{ medible}, \quad z \in \Omega \subseteq \mathbb{C}$$

μ medida compleja.

$\mu : X \to \mathbb{C}$ medida

Hip: $\varphi(X) \cap \Omega = \emptyset$. $\quad\Omega$ abierto.

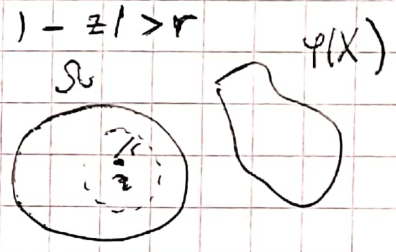
$z \in B(a,r) \subseteq \Omega \qquad \varphi(X) \cap \Omega \neq 0 \to \exists \zeta : \varphi(\zeta) \in \Omega$

$\Rightarrow \exists r' : B(\varphi(\zeta), r') \subseteq \Omega$

$\Rightarrow \exists z \in \Omega : |\varphi(\zeta) - z| < r$

$\varphi(X) \cap \Omega = \emptyset \Rightarrow \forall \zeta \in X \ \forall z \in \Omega : |\varphi(\zeta) - z| > r$

$\forall B(z,r) \subseteq \Omega :$



$$\frac{1}{|\varphi(\zeta) - z|} < \frac{1}{r} \Rightarrow \frac{|z-a|}{|\varphi(\zeta) - z|} < \frac{|z-a|}{r} < 1$$

$$\frac{|z-a|}{|\varphi(\zeta) - z|} = \frac{\left|1 - \frac{a}{z}\right|}{\left|\frac{\varphi(\zeta)}{z} - 1\right|}$$

$$f(z) = \int_{\gamma^*} \frac{dw}{\varphi(w) - z}$$

$f \in H(\Omega) \qquad \Omega \subseteq \mathbb{C}$ abierto.

$\varphi : \gamma^* \to \mathbb{C}$ continua

Af $\displaystyle\sum_{n=0}^{\infty} \frac{(z-a)^n}{(\varphi(\zeta)-a)^{n+1}} = \frac{1}{\varphi(\zeta)-z}$

$$\frac{1}{\varphi(\zeta)-z} = \frac{1}{\varphi(\zeta)}\left(\frac{1}{1-\frac{z}{\varphi(\zeta)}}\right) = \frac{1}{\varphi(\zeta)}\sum_{n=0}^{\infty}\left(\frac{z}{\varphi(\zeta)}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{(z-a)^n}{(\varphi(\zeta)-a)^{n+1}} = \frac{1}{\varphi(\zeta)-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\varphi(\zeta)-a}\right)^n = \frac{1}{\varphi(\zeta)-a}\,\frac{1}{1-\frac{z-a}{\varphi(\zeta)-a}}$$

$$= \frac{1}{\cancel{\varphi(\zeta)-a}}\,\frac{\cancel{\varphi(\zeta)-a}}{\varphi(\zeta)-\cancel{a}-z+\cancel{a}} = \frac{1}{\varphi(\zeta)-z}$$

$$\therefore \sum_{n=0}^{\infty}\frac{(z-a)^n}{(\varphi(\zeta)-a)^{n+1}} = \frac{1}{\varphi(\zeta)-z} \in \mathbb{C} \quad \forall \zeta\in X \quad \forall z\in\Omega$$

convergência uniforme. Importante

$$\Rightarrow \int_X \frac{d\mu(\zeta)}{\varphi(\zeta)-z} = \int_X \sum_{n=0}^{\infty}\frac{(z-a)^n}{(\varphi(\zeta)-a)^{n+1}}\,d\mu(\zeta)$$

$$= \sum_{n=0}^{\infty}\int_X \frac{(z-a)^n}{(\varphi(\zeta)-a)^{n+1}}\,d\mu(\zeta) = f(z)$$

$$= \sum_{n=0}^{\infty}\underbrace{\left(\int_X \frac{d\mu(\zeta)}{(\varphi(\zeta)-a)^{n+1}}\right)}_{c_n}(z-a)^n = f(z)$$

$$c_n = \frac{f^n(a)}{n!} \Rightarrow f^n(a) = n!\int_X \frac{d\mu(\zeta)}{(\varphi(\zeta)-a)^{n+1}}$$

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$$c_n = \int_X \frac{d\mu(\zeta)}{(\varphi(\zeta)-a)^{n+1}} \qquad |c_n| \leq \int_X \frac{d|\mu|(\zeta)}{|\varphi(\zeta)-a|^{n+1}}$$

$$\leq \frac{1}{r^{n+1}} \int_X d|\mu|(\zeta) = \frac{1}{r^{n+1}} |\mu|(X)$$

$$\int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\,\gamma'(t)\,dt$$

$$\left| \int_\gamma f(z)\,dz \right| \leq \int_\gamma |f(z)|\,d|z|$$

$$\left| \int_\gamma f(z)\,dz \right| = \left| \int_\alpha^\beta f(\gamma(t))\,\gamma'(t)\,dt \right| \leq \int_\alpha^\beta |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$\leq \int_\alpha^\beta \|f\|_\infty\, |\gamma'(t)|\,dt = \|f\|_\infty \int_\alpha^\beta |\gamma'(t)|\,dt$$

arcsignar

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\alpha^\beta \frac{\gamma'(s)}{\gamma(s)-z}\,ds$$

$$\varphi(t) = \exp\left(\int_\alpha^t \frac{\gamma'(s)}{\gamma(s)-z}\,ds \right) \Rightarrow \frac{\varphi'(t)}{\varphi(t)} = \frac{\gamma'(t)}{\gamma(t)-z}$$

$$\left(\frac{\varphi(t)}{\gamma(t)-z} \right)' = 0 \Rightarrow \frac{\varphi(t)}{\gamma(t)-z} = k \text{ constante } \quad \forall t.$$

$$t=\alpha: \quad \frac{1}{\gamma(\alpha)-z} = k = \frac{\varphi(t)}{\gamma(t)-z} \Rightarrow \varphi(t) = \frac{\gamma(t)-z}{\gamma(\alpha)-z} \quad (\alpha \leq t \leq \beta).$$

$$\gamma(\alpha) = \gamma(\beta) \Rightarrow \boxed{\varphi(\beta) = 1} \Leftrightarrow \int_\alpha^\beta \frac{\gamma'(s)}{\gamma(s)-z}\,ds \in \mathbb{Z}.$$

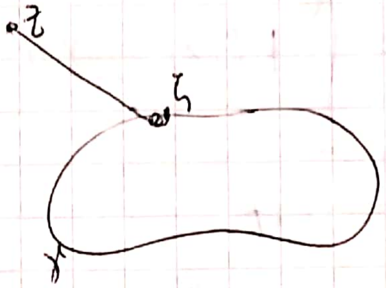
$$\mathrm{Ind}_{\gamma}(z) = \frac{1}{2\pi i}\int_{\alpha}^{\beta} \frac{\gamma'(s)}{\gamma(s) - z}\,ds = \frac{1}{2\pi i}\,\varphi(\beta).$$

$\mathrm{Ind}_{\gamma}(z) \in H(\Omega)$.

$$\mathrm{Ind}_{\gamma}(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{d\mu(\zeta)}{\zeta - z} \in H(\Omega).$$

Ω tiene componentes conexas Ω_i. $\mathrm{Ind}_{\gamma}(\Omega_i) = cte$.

$$|\mathrm{Ind}_{\gamma}(z)| = \frac{1}{2\pi i}\left|\int_{\gamma} \frac{d\mu(\zeta)}{\zeta - z}\right| \leq \int_{\gamma} \frac{d|\mu|(\zeta)}{|\zeta - z|} < 1 \quad |z| \text{ suf. grande.}$$



$\therefore \mathrm{Ind}_{\gamma}(z) = 0 \quad \forall\, |z|$ grande.

Teo de Cauchy: $\int_{\gamma} F'(z)\,dz = 0 \quad \forall$ 

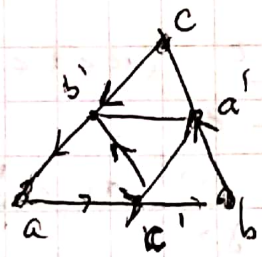
$F \in H(\Omega)$
F' continua en Ω

$$\int_{\gamma} F'(z)\,dz = \int_{\alpha}^{\beta} F'(\gamma(t))\gamma'(t)\,dt = \int_{\alpha}^{\beta} (F\circ\gamma)'(t)\,dt$$

$$= F\circ\gamma(\beta) - F\circ\gamma(\alpha) = 0 \;//$$

$\int_{\alpha}^{\beta} F\,dz = 0$.

$$I = \int_{\partial\Delta} f(z)\,dz$$



$\Delta = \Delta(a, b, c)$

$\{a, c', b'\}$

$\{c', b, a'\} = \{a', c', b\}$

Miércoles - Cálculo 3 $\nearrow$ límite. derivadas
$\searrow$ ec. diferenciales (lineales)

Simón - Bolívar | Manuel + Montt Providencia
Miércoles 2. p.m.

—— ∘ ——

Fórmula integral de Cauchy $\Rightarrow (f \in H(\Omega) \Rightarrow f' \in H(\Omega))$.

importante.

teo. de Cauchy $\xleftrightarrow{\text{converso}}$ teo. de Morera

$f : \Omega \to \mathbb{C}$ continua tq $\int_{\partial\Delta} f(z)dz = 0 \ \forall \Delta \Rightarrow f \in H(\Omega)$.

$\Omega \subseteq \mathbb{C}$ abierto.

—— ∘ ——

$f : \Omega \to \mathbb{C}$ holomorfa Ω convexo $\Rightarrow \exists F : \Omega \to \mathbb{C}$ tq $F' = f$

$a \in \Omega \quad \forall z \in \Omega \quad \exists [a, z] \subseteq \Omega$

$$F(z) = \int_{[a,z]} f(w)dw$$

$z = z'$

a

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0}\left(\int_{[a,z]} f(w)dw - \int_{[a,z_0]} f(w)dw\right)$$

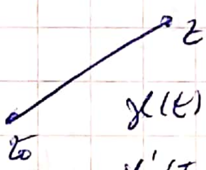
$$= \frac{1}{z - z_0}\left(-\int_{[z,a]} f(w)dw - \int_{[a,z_0]} f(w)dw\right) = \frac{1}{z_0 - z}\int_{[z,z_0]} f(w)dw$$

$$= \frac{1}{z - z_0}\int_{[z_0,z]} f(w)dw$$

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| < \varepsilon \Rightarrow \left|F(z)-F(z_0)-(z-z_0)f(z_0)\right| < \varepsilon|z-z_0|$$

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0}\int_{[z_0,z]} f(w)\,dw - f(z_0)\int_{[z_0,z]}$$

$$\cancel{f(z_0) = \int_\Gamma} \qquad f(z_0) = \frac{1}{z-z_0}\int_{[z_0,z]} dw$$



$$\gamma(t) = (1-t)z_0 + tz$$
$$\gamma'(t) = -z_0 + z$$

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0}\int_{[z_0,z]} \left(f(w)-f(z_0)\right) dw$$

$$\Rightarrow \left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| \leq \frac{1}{|z-z_0|}\int_{[z_0,z]} \left|f(w)-f(z_0)\right| d|w|$$

$$\forall \varepsilon>0,\ \exists \delta>0 : |w-z_0| < \delta \Rightarrow |f(w)-f(z_0)| < \varepsilon$$

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| \leq \frac{1}{|z-z_0|}\int_{[z_0,z]} |f(w)-f(z_0)|\,d|w|$$

$$\leq \frac{1}{|z-z_0|}\int_{[z_0,z]} \varepsilon\, d|w|$$

$$= \varepsilon \frac{1}{|z-z_0|}\int_{[z_0,z]} d|w| = \varepsilon \frac{1}{|z-z_0|}|z-z_0|$$

$$= \varepsilon.$$

Funciones importantes :

- $f(z) = \int_X \frac{d\mu(\zeta)}{\varphi(\zeta) - z}$

- $f(z) = \int_{\gamma} \frac{dw}{w - z}$

- $f(z)\,\mathrm{Ind}_{\gamma}(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z}$

- $F(z) = \int_{[a,z]} f(z)\,dz$.

Teo de la medida.

$\forall f: X \to [0,\infty]$ medible $\exists \{s_n\}_{n\in\mathbb{N}}$: $0 \le s_1 \le s_2 \le \dots \le s_n \le \dots \le f$

$\lim_{n\to\infty} s_n(x) = f(x) \quad \forall x \in X$ → simples.

$$\varphi_n(t) = \begin{cases} k\,\delta_n & , \quad 0 \le t < n \\ n & , \quad n \le t \le \infty \end{cases}$$

Función de Borel.

$0 \le t \le n$, $\quad t - \delta_n \le \varphi_n(t) \le t$

$0 \le \varphi_1 \le \varphi_2 \le \dots \le \varphi_n \le \dots t$, $\quad \varphi_n(t) \xrightarrow{n\to\infty} t \quad \forall t \in [0,\infty]$.

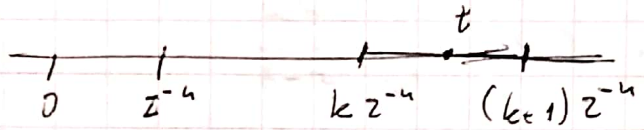
$\Rightarrow \quad s_n = \varphi_n \circ f$

Cumple que ~~$0 \le \varphi_n \le \varphi_{n+1}$ $\forall n$ $\varphi(x) \to f(x)$~~

$0 \le s_n \le s_{n+1} \ \forall n$. $\quad s_n(x) \to f(x)$

$s_n^{-1}(a,b) = f^{-1}(\varphi_n^{-1}(a,b))$.

$f : X \to [0,\infty]$ medible



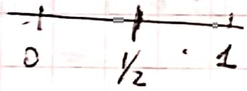
$t \in \mathbb{R}_{\geq 0}$ $\quad |t| - [t] \in [0,1]$ $\quad \exists n \in \mathbb{N}$ tq $\frac{1}{2^n} < |t| - [t]$

$k = [t]$, $\quad k \leq t \leq k+1$

$t \in [0,1]$ $\quad \exists n \in \mathbb{N}$: $\frac{1}{2^n} < t$ $\quad \exists k \in \mathbb{Z}$ tq $\frac{k}{2^n} \leq t < \frac{k+1}{2^n}$

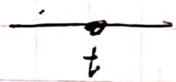
en general para todo $t \in [0,1]$. Después extender $\forall t \in \mathbb{R}$.

$$\varphi_n(t) = \begin{cases} k_n(t)\delta_n & 0 \leq t < n \\ n & n \leq t \leq \infty. \end{cases}$$



$$\frac{k+1}{2^n} - \frac{k}{2^n} = \frac{1}{2^n}.$$

$\exists t, \exists n_0 \in \mathbb{N}$ $\forall k \in \mathbb{N}$ $\quad \frac{k}{2^{n_0}} < t$ $(\Rightarrow \in)$.



$$k\delta_n \leq t < (k+1)\delta_n$$
$$l\delta_n \leq t < (l+1)\delta_n$$
$$(k+l)\delta_n \leq 2t < (k+l+2)\delta_n$$
$$\Rightarrow (k+l)\delta_{n+1} \leq t < (k+l+2)\delta_{n+1}$$

$$\Rightarrow \begin{cases} k\delta_n \leq t < (k+1)\delta_n \\ -(l+1)\delta_n \leq -t < -l\delta_n \end{cases} \Rightarrow (k-l-1)\delta_n \leq 0 < (k+1-l)\delta_n$$

$$\underline{(k+l+2)\delta_{n+1} - (k+l)\delta_{n+1}} \geq 0$$

Basta considerar $\forall n \in \mathbb{N}$: $\mathcal{A}_n = \{A_{n,k}\}_{k \in \mathbb{Z}}$ donde $A_{n,k} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ es una partición de $\mathbb{R}$.

$$\varphi_n(t) = \begin{cases} k_n(t)\,\delta_n & , \quad 0 \le t < n \\ n & , \quad n \le t \le \infty \end{cases}$$

$n \in \mathbb{R}$, $n \in \left[\frac{n2^n}{2^n} , \frac{n2^n+1}{2^n} \right)$ $\quad$ $\underline{k = n2^n}$

$\varphi_n^{-1}(k\delta_n) = [k\delta_n , (k+1)\delta_n)$ $\quad$ | $\quad$ ~~$\varphi_n(\,(t-\delta_n , t]\,)$~~ =

$\rightarrow$ φ_n es borel medible $\forall n$.

$[k\delta_{n+1} , (k+1)\delta_{n+1}) \subseteq [k\delta_n , (k+1)\delta_n)$

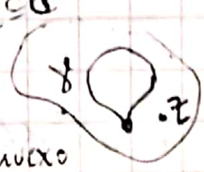
$[\delta_{n+1} , 2\delta_{n+1}) = [\delta_{n+1} , \delta_n]$

$0 \le \varphi_1 \le \varphi_2 \le \dots \le \varphi_n \le \dots \le t$, $\varphi_n(t) \xrightarrow{n\to\infty} t$ $\quad \dots n$

~~Propiedades de las medidas (positivas)~~

Fórmula integral de Cauchy (no convexo)

$\Omega \subseteq \mathbb{C}$



$$f(z)\,\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

Ω convexo

<u>Singularidades</u> $\quad f \in H(\Omega - \{a\})$ $\quad$ a singularidad aislada de f.

$\cancel{f(z)} = \quad g(z) = \dfrac{1}{f(z)-w}$, $z \in D'(a,r)$; $\quad |g| < 1/\delta$ $\Rightarrow$ tiene extensión <u>holomorfa</u>

$g(a) \ne 0$ $\quad | \quad |f(z)-w| > \delta \quad \forall z \in D'(a,r)$

$0 < |g(z)| = \dfrac{1}{|f(z)-w|} < \dfrac{1}{\delta}$ $\quad \Rightarrow |f| < \delta \quad \forall z \in D'(a,\rho)$

$z \in D(a,R)$: $\quad f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad z \in D(a,r)$

$\forall r>0,\ 0<r<R$

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{w-z}\,dw \qquad \forall z \in D(a,r)$$

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t)-z}\gamma'(t)\,dt \quad , \gamma(t) = a + re^{it}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{it})}{a+re^{it}-z} rie^{it}\,dt$$

$$f(z) = \int_X \frac{d\mu(\xi)}{\varphi(\xi)-z}$$

$$c_n = \int_X \frac{d\mu(\xi)}{(\varphi(\xi)-a)^{n+1}}$$

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad , \quad z = a+re^{it} \quad t \in [0,2\pi]$$

$$\Rightarrow \cancel{f(a+re^{it}) = \sum_{n=0}^{\infty} c_n (re^{it})^n = \sum_{n=0}^{\infty} c_n r^n e^{int}}$$

$$\cancel{c_n = \int_{\gamma} \frac{dw}{(z-a)^{n+1}}} \quad \Bigg| \quad \cancel{f^{k}(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{(w-z)^{k+1}}\,dw}$$

$$\cancel{c_n = \frac{f^n(a)}{n!} = \frac{1}{2\pi i n!}\int_{\gamma} \frac{f(w)}{(w-a)^{k+1}}\,dw = \frac{1}{2\pi i n!}\int_0^{2\pi} \frac{f(a+re^{it})}{(re^{it})^{k+1}} rie^{it}\,dt}$$

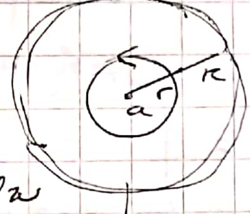
$$\cancel{c_n = \frac{1}{2\pi i n!}\int_0^{2\pi} f(a+re^{it})(re^{it})^{-k}\,dt}$$

$$\cancel{r^k c_n = \frac{1}{2\pi i n!}\int_0^{2\pi} f(a+re^{it}) e^{-kit}\,dt}$$

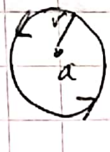
Otra vez: $f(z) = \int_X \frac{d\mu(\zeta)}{\varphi(\zeta)-z}$ es holomorfa en Ω

$$c_n = \int_X \frac{d\mu(\zeta)}{(\varphi(\zeta)-a)^{n+1}} \qquad z \in D(a,r)$$

Ahora: $f \in H(\Omega)$, $D(a,R) \subset \Omega$, consideramos



serie de potencias



$$\gamma(t) = a + re^{it}, \qquad f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

$z \in D(a,r)$

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{it})}{a+re^{it}-z}\, rie^{it}\,dt$$

entonces ocupar anterior con $X = [0,2\pi]$, $\varphi(t) = \gamma(t)$

$$d\mu(t) = f(a+re^{it})\,re^{it}\,dt$$

$$\Rightarrow c_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{it})}{(re^{it})^{n+1}}\, rie^{it}\,dt$$

$$\Rightarrow c_n r^n = \frac{1}{2\pi i}\int_0^{2\pi} f(a+re^{it})\, e^{-nit}\,dt$$

también: ~~$f(z) = \sum_{n=0}$~~ $f(a+re^{it}) = \sum_{n=0}^{\infty} c_n r^n e^{nit}$

$$c_n r^n = \frac{1}{2\pi i}\int_0^{2\pi}\left(\sum_{k=0}^{\infty} c_k r^k e^{kit}\right) e^{-nit}\,dt = \frac{1}{2\pi i}\int_0^{2\pi}\sum_{n=0}^{\infty} c_n r^n\,dt$$

$$= \frac{1}{2\pi i}\int_0^{2\pi}\left(\sum_{k=0}^{\infty} c_k r^k e^{kit}\right) e^{-nit}\,dt = \frac{1}{2\pi i}\int_0^{2\pi}\sum_{k=0}^{\infty} c_k r^k e^{(k-n)it}\,dt$$

$$= \frac{1}{2\pi i}\sum_{k=0}^{\infty} c_k r^k \int_0^{2\pi} e^{(k-n)it}\,dt = \frac{1}{2\pi i}\sum_{k=0}^{\infty} c_k r^k \int_0^{2\pi} e^{kit} e^{-nit}\,dt$$

$$\int_0^{2\pi} |f(a+re^{it})|^2 dt = \int_0^{2\pi} f(a+re^{it})\overline{f(a+re^{it})}\, dt$$

$$= \int_0^{2\pi} \left(\sum_{n=0}^{\infty} c_n r^n e^{int}\right)\left(\sum_{k=0}^{\infty} \overline{c_k}\, r^k e^{-ikt}\right) dt$$

$$= \int_0^{2\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_k r^k e^{ikt}\, \overline{c_{n-k}}\, r^{n-k} e^{-i(n-k)t}\, dt$$

$$= \int_0^{2\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_k \overline{c_{n-k}}\, r^n e^{2ikt} e^{-int}\, dt$$

$$= \int_0^{2\pi} \sum_{n=0}^{\infty} r^n e^{-int} \sum_{k=0}^{n} c_k \overline{c_{n-k}}\, e^{2ikt}\, dt$$

Para quien busca ~~el amor~~ en la vida el amor y ~~la buena~~ compañía, es necesario saber

Cantigo

$$(Hf)(re^{i\theta}) = \begin{cases} f(e^{i\theta}) & r=1 \\ P[f](re^{i\theta}) & 0\le r<1 \end{cases}$$

$\underline{\text{A.L.}}$ $\quad Hf \in C(\bar{U})$

$$|(Hg)(re^{i\theta})| = |P[g](re^{i\theta})| = \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, g(t)\, dt \right|$$

$(0\le r<1)$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(\theta-t)|\,|g(t)|\,dt \le \|g\|_T \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(\theta-t)|\,dt$$

$$= \|g\|_T \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\,dt$$

$$\int_{-\pi}^{\pi} P_r(\theta-t)\,dt = \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\,dt =$$

$$g(e^{i\theta}) = \sum_{-N}^{N} c_n e^{in\theta} \quad \Big| \quad g\in C(T) \qquad T = \partial D(0,1).$$

$$(Hg)(e^{i\theta}) = g(e^{i\theta}) = \sum_{-N}^{N} c_n e^{in\theta} \qquad (r=1)$$

$$r\in[0,1): (Hg)(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, g(t)\, dt = P[g](re^{i\theta})$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\left(\sum_{-N}^{N} c_n t^n\right)dt = \frac{1}{2\pi}\sum_{-N}^{N} c_n \int_{-\pi}^{\pi} P_r(\theta-t)\, t^n\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{-N}^{N} r^{|m|} e^{im(\theta-t)} \sum_{-N}^{N} c_k e^{ikt} = \sum_{-N}^{N} c_n r^{|n|} e^{in\theta} \qquad \underline{\text{ok}}$$

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~~• u armónica, $r>0$ tq: $\overline{D(z,r)} \subset \Omega$~~

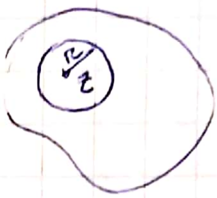
• u ~~sea~~ continua en $\overline{U}$ y armónica en U

$$\Rightarrow \quad u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, u(t)\, dt$$

$z = re^{i\theta}$

$u(re^{i\theta})$ $\quad \theta \in \mathbb{R}$, $0 \leq r < 1$

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta - t)+r^2}\, u(t)\, dt$$



$$u(z+re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-t)+r^2}\, u(z+Re^{it})\, dt$$

$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cancel{\frac{R^2}{R^2+0+0}}\, u(z+Re^{it})\, dt \qquad \boxed{r=0}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(z+Re^{it})\, dt$$

Celulares x comprar: Alcatel Pixi 3 4,0 → 8 MP, 4", 1 GHz Dual core

Alcatel Pixi: White 3. 4,0

Index of a closed curve:

$$\int_\gamma \frac{1}{z-a}\,dz = 2\pi i n \qquad \gamma(t) = a + e^{2\pi i n t}$$

$$\int_\gamma \frac{1}{z-a}\,dz = \int_0^1 \frac{1}{e^{2\pi i n t}}\, 2\pi i n\, e^{2\pi i n t}\,dt = \int_0^1 2\pi i n\,dt = 2\pi i n.$$

Af. $\quad \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} \in \mathbb{Z}, \quad \gamma : [0,1] \longrightarrow \mathbb{C}$, $a \notin \gamma^*$

asumir que γ es suave.

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)-a}\,ds, \quad g(0)=0, \quad g(1) = \int_0^1 \frac{\gamma'(s)}{\gamma(s)-a}\,ds = \int_\gamma \frac{1}{z-a}\,dz.$$

$$g'(t) = \frac{\gamma'(t)}{\gamma(t)-a} \qquad g'(0) = \frac{\gamma'(0)}{\gamma(0)-a} \overset{?}{=} \frac{\gamma'(1)}{\gamma(1)-a}$$

—— o ——

$$P(z) = \alpha(z-\alpha_1)\dots(z-\alpha_n)$$

$$P'(z) = \alpha\Big((z-\alpha_2)\dots(z-\alpha_n) + (z-\alpha_1)\big((z-\alpha_2)\dots(z-\alpha_n)\big)'\Big)$$

$$= \alpha \sum_{j=1}^n \prod_{\substack{k=1\\k\neq j}}^n (z-\alpha_k)$$

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^n \frac{\prod_{k=1, k\neq j}^n (z-\alpha_k)}{\prod_{k=1}^n (z-\alpha_k)} = \sum_{j=1}^n \frac{1}{z-\alpha_j}$$

$$\int_\gamma \frac{P'(z)}{P(z)}\,dz = \sum_{j=1}^n \int_\gamma \frac{1}{z-\alpha_j}\,dz = \sum_{j=1}^n 2\pi i = 2\pi i n.$$

$w = re^{i\theta} \neq 0$ $\quad \gamma$ rectifiable en $\mathbb{C} - \{0\}$ $\quad 1 \to w$

$\gamma(1) = w$

$\gamma(0) = 1$

pd: $\exists k \in \mathbb{Z}$, $\displaystyle\int_\gamma z^{-1} dz = \log r + i\theta + 2\pi i k$.

$$\int_\gamma z^{-1} dz = \log|\gamma(1)| - \cancel{\log|\gamma(0)|} + i\{\arg(\gamma(1)) - \arg(\gamma(0))\}$$

$$= \log|w| + i\{\theta - 0\} = \log r + i\theta$$

$$F_m(z) = \int_\gamma \varphi(w)(w-z)^{-m} dw$$

$$|F_m(z) - F_m(z_0)| = \left|\int_\gamma \varphi(w)(w-z)^{-m} dw - \int_\gamma \varphi(w)(w-z_0)^{-m} dw\right|$$

$$= \left|\int_\gamma \varphi(w)\left[(w-z)^{-m} - (w-z_0)^{-m}\right] dw\right|$$

$$\leq \int_\gamma |\varphi(w)|\left|(w-z)^{-m} - (w-z_0)^{-m}\right||dw|$$

$$\leq \|\varphi\|_\infty \int_\gamma \left|(w-z)^{-m} - (w-z_0)^{-m}\right||dw| \qquad \underline{ok}$$

$f : G \to \mathbb{C}$ $\quad G$ region

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \dots + \frac{1}{z-a_m} + \frac{g'(z)}{g(z)}$$

$$\to \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{m} n(\gamma, a_k) \qquad \underline{ok}!$$

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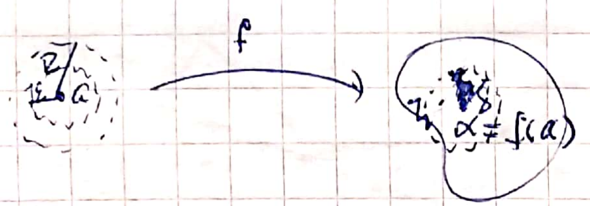


$g(z) = f(z) - \alpha$ tiene a $z = a$ como 0 de orden m

~~$\mathrm{ord}(g, a) = m$~~ $\quad \mathrm{ord}_g(a) = m$

$\Rightarrow \exists\, \delta, \varepsilon > 0 : \; f(z) = \zeta$ tiene m raíces en $B(a, \varepsilon)$ (simples)

donde $0 < |\zeta - \alpha| < \delta$.

Sea $w \in B(\alpha, \delta) \Rightarrow \exists\, \bar{w} \in B(a, \varepsilon) : \; f(\bar{w}) = w$

$\therefore \; B(\alpha, \delta) \subseteq f(B(a, \varepsilon))$

— o —

$\{a_k\}$ ceros de f

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{m} n(\gamma, a_k) \qquad \sigma = f \circ \gamma$$

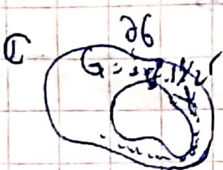
$$n(\sigma, \alpha) = \frac{1}{2\pi i}\int_\sigma \frac{dw}{w - \alpha} = \frac{1}{2\pi i}\int_0^1 \frac{\sigma'(t)\,dt}{\sigma(t) - \alpha} = \frac{1}{2\pi i}\int_0^1 \frac{f'(\gamma(t))\gamma'(t)\,dt}{f(\gamma(t)) - \alpha}$$

$$\int_\gamma \frac{f'(z)}{f(z) - \alpha}\,dz = \int_0^1 \frac{f'(\gamma(t))\,\gamma'(t)}{f(\gamma(t)) - \alpha}\,dt \qquad = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - \alpha}\,dz$$

$$n(\sigma, \alpha) = \frac{1}{2\pi i}\int_\sigma \frac{dw}{w - \alpha} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - \alpha}\,dz = \sum_{k=1}^{m} n(\gamma, a_k)$$

$\{a_k\}$ ceros de $f(z) - \alpha$

— o —



$\gamma : [0, 1] \to \mathbb{C}$ curva suave

$r = d(\{\gamma\}, \partial G)$ $\qquad n(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w - z}$

$H = \{ z \in \mathbb{C} \,/\, n(\gamma, z) = 0 \}$

$\{ z \,/\, d(z, \partial G) < \frac{1}{2} r \} \subset H \qquad d(z, \partial G) < \frac{1}{2} r = \frac{1}{2} d(\{\gamma\}, \partial G)$

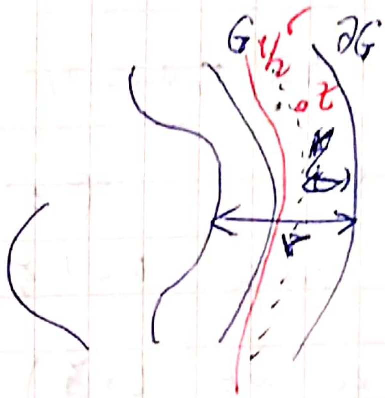
$\varepsilon > 0 \mid d(z, \partial G) < \frac{1}{2} r$

$\qquad \qquad \| $

$\qquad \inf\{ d(z,w) \mid w \in \partial G \}$

$\Rightarrow \exists w \in \partial G : \quad d(z,w) < d(z, \partial G) + \varepsilon$

$\quad \forall\, w \in \{\gamma\} : \quad |z - w| > r$



$\frac{1}{2\pi i} \int_\gamma \frac{1}{w-z}\, dw = n(\gamma, z)$

$\gamma \approx 0 \Rightarrow n(\gamma, z) = 0 \quad \forall z \in \mathbb{C} \setminus G$

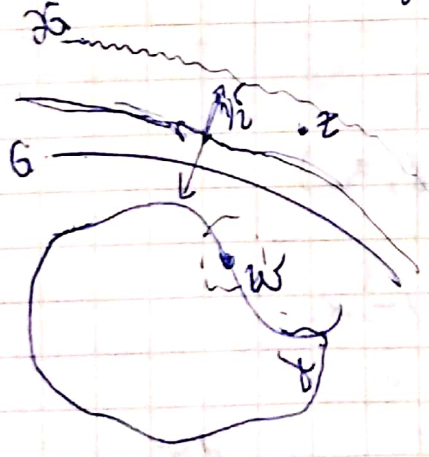
$z \in \partial G \Rightarrow d(z, \partial G) = 0 \qquad \partial G = \overline{G} \setminus G$

cerrado

Sup que $\partial G = \phi \Rightarrow d(z, \partial G) = 0 \qquad \overline{G} = G \Rightarrow \mathbb{C} = G$

$|n(\gamma, z)| = \left| \int_\gamma \frac{dw}{w-z} \right| \leq \int_\gamma \frac{|dw|}{|w-z|} \leq k \int_\gamma |dw| = k\, \ell(\gamma)$

$\leq k\, V(\gamma).$



$|z - w| \geq r/2$

$\forall \varepsilon > 0 \quad \forall w \in \partial G : \quad B(w, \varepsilon) \cap \partial G \neq \phi$

$\exists z_1, z_2 \in \mathbb{C} : z_1 \in G, \ z_2 \notin G$

$\Downarrow \ n(\gamma, z_2) = 0.$

$|z_1 - w| >$

Continuentel altas sport.