Apuntes y ejercicios varios.

- Teoría de Galois
- Análisis funcional
- Variable compleja
- Teoría de Representaciones
- Topología Algebraica

Período magíster en matemáticas, ~2015, 2016, ...

Tarea: Buscar la bibliografía en donde fueron sacados los problemas.
También buscar las clases asociadas (cursos).

Obs. Pido disculpas por la mala caligrafía ... fueron tiempos intensos intelectualmente hablando.

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$K_j = \{x \in \Omega / \ d(x, \partial\Omega) \geq \frac{1}{j}\}$

$K_j^c = \{x / \ d(x, \partial\Omega) < \frac{1}{j}\} \longrightarrow K_j^c = B(\partial\Omega, 1/j)$

$d(x, \partial\Omega) = \inf\{d(x,y) / y \in \partial\Omega\} \leq d(x,y) \quad \forall y \in \partial\Omega$

~~¿d(x, ∂Ω) es una métrica?~~

$(x^n)_{n\in\mathbb{N}}$ tq $x^n \xrightarrow{n\to\infty} x$ Pd: $x \in K_j$

$\frac{1}{j} \leq d(x, \partial\Omega) \leq d(x,y) \quad \forall y \in \partial\Omega$
$\leq d(x, x_j) + d(x_j, y)$

$d(x,y) \leq d(x, x_j) + d(x_j, y)$

K_j ~~$= \{x \in \Omega / d(x,\partial\Omega) \geq 1/j\}$~~.
~~$K_j = \{x \in \Omega$~~

$\forall f \in C_c(\Omega)$

~~sup~~ $supp(f) \subseteq K_n \quad \forall n$.

$x \in supp(f) \Rightarrow \exists r_x > 0$
$B(x, r_x) \subseteq \Omega$
$\sup B(x, r_x) \subseteq \Omega$

$\Rightarrow supp(f) \subseteq B(x_1, r_{x_1}) \cup \dots \cup B(x_s, r_{x_s})$

$\Downarrow$

A probar que $C_c(\Omega) = \bigcup_{n\in\mathbb{N}} C_n$, $C_n = \{f \in C_c(\Omega) / supp(f) \subseteq K_n\}$

Considerar $R_{K_n} : C_n \to C(K_n)$. R_{K_n} aplicación lineal isométrica.

$f \longmapsto f|_{K_n}$

$d(f,g) = \sup_{x\in K_n} |f(x) - g(x)|$

$d(f|_{K_n}, g|_{K_n}) = \sup_{x\in K_n} |f(x) - g(x)|$

$R_{K_n}(C_n) \subseteq C(K_n) \leftarrow$ separable $\Rightarrow$ C_n separable

isometría

$C_c(\Omega) = \bigcup_{n\in\mathbb{N}} C_n$ unión numerable de separables es separable.

Pd: Existe $L/\mathbb{Q}$ Galoisiano, $[L:\mathbb{Q}]=15$

$$\begin{tikzcd} \mathbb{Q}(\theta)=L \arrow[dr, dash, "3"'] & & M=\mathbb{Q}(\alpha) \arrow[dl, dash, "5"] \\ & \mathbb{Q} & \end{tikzcd}$$

$\varphi(2)=1$

$\varphi(16)=\varphi(2^4)=2^3(2-1)$
$=8(1)=8$
$=2^4-2^3=16-8=8$

$\varphi(17)=16$

$\varphi(18)=\varphi(3^2\cdot 2)=\varphi(3^2)\varphi(2)$
$=3(2)=6$

$\varphi(20)=\varphi(5)\varphi(4)$
$=4 \cdot$ x

$\varphi(21)=20$

$\varphi(22)=\varphi(2)\varphi(11)=10$.

$\left.\begin{array}{l} L=\mathbb{Q}(\sqrt[3]{a}) \\ M=\mathbb{Q}(\sqrt[5]{b}) \end{array}\right\}$ no son Galoisianas.

$\varphi(24)=\varphi(3)\varphi(8)=2\cdot\varphi(2^3)=2\cdot 2^2(1)=8$

$\varphi(25)=\varphi(5^2)=\cancel{4}\ 5(4)=20$

$\varphi(26)=$

—o—

Grupos de orden 15.

$15=3\cdot 5$ | $n_5 \mid 3 \Rightarrow n_5\in\{1,3\}$

$n_5\equiv 1 \pmod 5$ ⟶ $n_5=1 \Rightarrow P_5\in Syl_5(G)$ $P_5\trianglelefteq G$

$n_3\mid 5$ $\Rightarrow n_3\in\{1,5\}$ | $\boxed{G=C_3\times C_5}$

$n_3\equiv 1 \pmod 3$

$$\begin{tikzcd} & G \arrow[dl, dash] \arrow[dr, dash] & \\ C_3 \arrow[dr, dash] & & C_5 \arrow[dl, dash] \\ & e & \end{tikzcd} \rightarrow \begin{tikzcd} & L=KM \arrow[dl, dash] \arrow[dr, dash] & \\ K \arrow[dr, dash] & & M \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Q}(\alpha) \arrow[d, dash, "p"] \\ \mathbb{Q} \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{Q}(\alpha,\beta) = L \arrow[dl, dash, "5"'] \arrow[dr, dash, "3"] \arrow[dd, dash, "15"] & \\ \mathbb{Q}(\alpha) \arrow[dr, dash, "3"'] & & \mathbb{Q}(\beta) \arrow[dl, dash, "5"] \\ & \mathbb{Q} & \end{tikzcd}$$

$\varphi(15) = \varphi(3)\varphi(5)$
$= 2 \cdot 4 = 8$

$$\begin{tikzcd} & L \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbb{Q}(\sqrt[15]{2}) \arrow[dr, dash, "15"'] & & \mathbb{Q}(\zeta) \arrow[dl, dash, "\varphi(15)=8"] \\ & \mathbb{Q} & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Q}(\alpha) \arrow[d, dash, "3"] \\ \mathbb{Q} \end{tikzcd}$$

α raiz de $x^3 + ax^2 + bx + c$ → tiene 3 raices reales.

$\alpha = \sqrt[3]{2}\,\omega \qquad \alpha^3 = 2 \Rightarrow \alpha$ raiz de $x^3 - 2$

$\beta = \sqrt[5]{2}\,\rho \quad , \quad \rho^5 = 1 \qquad \delta \quad \delta^{15} = 1$

$\mathbb{Q}(\alpha,\beta) = \mathbb{Q}(\sqrt[3]{2}\omega, \sqrt[5]{2}\rho) \overset{?}{=} \mathbb{Q}(\sqrt[15]{2}, \omega\rho)$

$z = \sqrt[15]{2}\,\omega\rho \Rightarrow z^3 = \sqrt[5]{2}\,\rho^3 \Rightarrow \rho^2 \in L$

$z^5 = \sqrt[3]{2}\,\omega^5 = \sqrt[3]{2}\,\omega^2 \Rightarrow \omega \in L \Rightarrow \sqrt[3]{2} \in L$

$\rho^2 \in L \Rightarrow \rho^{-2} \in L \Rightarrow \sqrt[5]{2}\,\rho^{-1} \in L \Rightarrow \sqrt[5]{2} \in L \Rightarrow \rho \in L$

$\therefore \mathbb{Q}(\sqrt[15]{2}, \omega\rho) \subseteq \cancel{\mathbb{Q}(\sqrt[15]{2}, \omega\rho)}$

tenemos $\zeta = \omega\rho$ tq $\zeta^{15} = (\omega\rho)^{15} = 1$ tenemos que ζ es raiz 15-ava de la unidad

$\cancel{\omega \neq e^{2\pi i k/3}} \qquad \omega = e^{2\pi i/3}, \quad \rho = e^{2\pi i/5}$

$\omega\rho = e^{2\pi i/3 + 2\pi i/5} = e^{2\pi i(1/3 + 1/5)} = e^{2\pi i(8/15)} = e^{2\pi i \cdot 8/15}$

raiz primitiva.

$\therefore \mathbb{Q}(\sqrt[15]{2}, \zeta) = L$

$$\begin{tikzcd} L \arrow[d, dash, "\text{Galoisiana}"] \\ \mathbb{Q} \end{tikzcd}$$

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$L = \mathbb{Q}(\sqrt[3]{2}\omega, \sqrt[5]{2}\rho)$

$z = \sqrt[15]{2}\omega\rho \Rightarrow z^3 = \sqrt[5]{2}\rho^3 \Rightarrow \rho^2 \in L$

$\sqrt[3]{2} \cdot \sqrt[5]{2} = 2^{1/3} \cdot 2^{1/5} = 2^{1/3+1/5} = 2^{5+3/15} = 2^{8/15}$

$$\begin{tikzcd} & G = C_3 \times C_{15} \arrow[dl, dash, "3"'] \arrow[dr, dash, "5"] & \\ C_3 \arrow[dr, dash, "3"', "\text{Gal}"] & & C_{15} \arrow[dl, dash, "5", "\text{Gal}"'] \\ & e & \end{tikzcd}$$

$$\begin{tikzcd} & L = \mathbb{Q}(\alpha,\beta) \arrow[dl, dash, "3"'] \arrow[dr, dash, "5"] & \\ \mathbb{Q}(\alpha) \arrow[dr, dash, "5"'] & & \mathbb{Q}(\beta) \arrow[dl, dash, "3"] \\ & \mathbb{Q} & \end{tikzcd}$$

$\mathbb{Q}(\alpha) = L^{C_3}$

$\mathbb{Q}(\beta) = L^{C_5}$

extensión abeliana

$x^3 - 3x + 1 \in \mathbb{Q}[x]$

Gal C_3

$$\begin{tikzcd} \mathbb{Q}(\zeta) \arrow[d, dash, "30"] \\ \mathbb{Q} \end{tikzcd}$$

$\varphi(\zeta) = 30$

$\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) = (\mathbb{Z}/31\mathbb{Z})^* \cong \mathbb{Z}/30\mathbb{Z}$

$2 \mid |G| \Rightarrow \exists\, \sigma \in G = \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$

$|\sigma| = 2$

$L = \mathbb{Q}(\zeta)^{\langle\sigma\rangle}$

$$\begin{tikzcd} \mathbb{Q}(\zeta) \arrow[d, dash, "2"'] \\ \mathbb{Q}(\zeta)^{\langle\sigma\rangle} = L \arrow[d, dash, "15"', "\text{Galois}"] \\ \mathbb{Q} \end{tikzcd}$$

=

$K \subseteq \mathbb{C}$ cuerpo de descomposición de $x^4-5 \in F=\mathbb{Q}(\sqrt{-5})[x]$

Calcular $Gal(K/F)$

$K/\mathbb{Q}$ $x^4-5 \in \mathbb{Q}[x]$ irreducible. $K=\mathbb{Q}(\sqrt[4]{5},\zeta)$

$\zeta^4=1$

raíces 4ta de la unidad $\{1,-1,i,-i\}$

$$x^4-1=(x^2+1)(x^2-1)=(x-1)(x+1)(x-i)(x+i)$$

$\therefore$ $K=\mathbb{Q}(\sqrt[4]{5},i)$

$\zeta=e^{2\pi i/4}=e^{\pi i/2}=i$

$$\begin{tikzcd}
 & K \arrow[dl, dash, "2"'] \arrow[dd, dash, "8"] \arrow[dr, dash] & \\
\mathbb{R} \supseteq \mathbb{Q}(\sqrt[4]{5}) \arrow[dr, dash, "4"'] & & \mathbb{Q}(i) \arrow[dl, dash, "2"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Q}(\sqrt[4]{5}) \arrow[d, dash] \\
\mathbb{Q}(
\end{tikzcd}$$

$$\sigma: \begin{cases} \sqrt[4]{5} \mapsto \zeta^a\sqrt[4]{5} \\ i \mapsto \pm i \end{cases}$$

$$\begin{tikzcd}
K \arrow[d, dash] \\
\mathbb{Q}(\sqrt{5}\, i) = \mathbb{Q}(\sqrt{-5})
\end{tikzcd}$$

$$x^4-5=(x^2+\sqrt{5})(x^2-\sqrt{5})$$

$(\sqrt{-5})^2=-5 \xrightarrow{(\ )^2}$

raíces de x^4-5, $\sqrt[4]{5}$, $-\sqrt[4]{5}$, $i\sqrt[4]{5}$, $-i\sqrt[4]{5}$

$$\sigma : \begin{cases} \sqrt[4]{5} \mapsto i\sqrt[4]{5} \\ i \mapsto i \end{cases} , \quad \tau : \begin{cases} \sqrt[4]{5} \mapsto \sqrt[4]{5} \\ i \mapsto -i \end{cases}$$

$$z = a + b\sqrt[4]{5} + c(\sqrt[4]{5})^2 + d(\sqrt[4]{5})^3 + ei + fi\sqrt[4]{5} + g\, i(\sqrt[4]{5})^2 + h\, i(\sqrt[4]{5})^3$$

$$\sigma(z) = a + bi\sqrt[4]{5} - c(\sqrt[4]{5})^2 - di(\sqrt[4]{5})^3 + ei - f\sqrt[4]{5} - gi(\sqrt[4]{5})^2 + h(\sqrt[4]{5})^3$$

$$\sigma(\sqrt[4]{5})^2 = (i\sqrt[4]{5})^2 = -(\sqrt[4]{5})^2$$

$$\sigma(\sqrt{-5}) = \sigma(i\sqrt{5}) = \sigma(i(\sqrt[4]{5})^2) = -i(\sqrt[4]{5})^2 = -i\sqrt{5} = -\sqrt{-5}$$

$$\sigma(\sqrt[4]{5})^3 = (i\sqrt[4]{5})^3 = -i(\sqrt[4]{5})^3$$

$$\sigma(i\sqrt[4]{5}) = \sigma(i)\sigma(\sqrt[4]{5}) = i\, i\sqrt[4]{5} = -\sqrt[4]{5}$$

$$\sigma(i(\sqrt[4]{5})^2) = i\,\sigma((\sqrt[4]{5})^2) = i \cdot -(\sqrt[4]{5})^2 = -i(\sqrt[4]{5})^2$$

$$\sigma(i(\sqrt[4]{5})^3) = i\,\sigma((\sqrt[4]{5})^3) = i\cdot(-i(\sqrt[4]{5})^3) = (\sqrt[4]{5})^3$$

$$\sigma(z) = z \iff \sigma(z) - z = 0$$

$$\iff \sqrt[4]{5}(b-f) + (\sqrt[4]{5})^2(c+c) + (\sqrt[4]{5})^3(d-h) + i\sqrt[4]{5}(f-b) + i(\sqrt[4]{5})^2(g+g) + i(\sqrt[4]{5})^3(h+d)$$

$$\begin{cases} b = f \\ c = 0 \\ d = h \\ g = 0 \\ h = -d \end{cases} \quad d = h = 0$$

$$z = a + b(\sqrt[4]{5} + i\sqrt[4]{5}) + ei +$$

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$\sigma\tau = \tau\sigma^3$

$\sqrt{-5} = i\sqrt[2]{5} = i(\sqrt[4]{5})^2$

$\sigma(i(\sqrt[4]{5})^2) = -i(\sqrt[4]{5})^2$

$\sigma^2(i(\sqrt[4]{5})^2) = \sigma(-i(\sqrt[4]{5})^2) = -\sigma(i(\sqrt[4]{5})^2) = i(\sqrt[4]{5})^2$

$\therefore \sigma^2(\sqrt{-5}) = \sqrt{-5}$

$\tau(i(\sqrt[4]{5})^2) = \tau(i)\,\tau((\sqrt[4]{5})^2) = -i(\sqrt[4]{5})^2$ $\qquad 1, \sigma^2, \sigma\tau$

$\sigma\tau(i(\sqrt[4]{5})^2) = \sigma(-i(\sqrt[4]{5})^2) = i(\sqrt[4]{5})^2$

$\therefore \sigma\tau(\sqrt{-5}) = \sqrt{-5}$

$(\sigma\tau)^2 = \sigma\tau\sigma\tau = \tau\sigma^3\sigma\tau = \tau\sigma^4\tau = \tau^2 = 1$

$\sigma^3(i(\sqrt[4]{5})^2) = \sigma(i(\sqrt[4]{5})^2) = -i(\sqrt[4]{5})^2$

$\therefore \sigma^3\tau(i(\sqrt[4]{5})^2) = \sigma^3(-i(\sqrt[4]{5})^2) = i(\sqrt[4]{5})^2$

$\therefore H = \{1, \sigma^2, \sigma\tau, \sigma^3\tau\}$ fijan $\mathbb{Q}(\sqrt{-5})$

$\mathbb{Q}(\sqrt{-5}) = L^H \Rightarrow [L : \mathbb{Q}(\sqrt{-5})] = |H|$

$Gal(L/\mathbb{Q}(\sqrt{-5})) = H \cong V_4$

$(\sigma^3\tau)^2 = \sigma^3\tau\sigma^3\tau = \sigma^3\sigma\tau\tau = \sigma^4\tau^2 = 1\cdot 1 = 1$

$\underline{P3}$ $\zeta_n = e^{2\pi i/n}$, $\alpha \in \mathbb{Q}(\zeta_n)$ | $e^{2\pi i/n} = e^{2\pi i k/n}$

$e^{2\pi i\left(\frac{1}{n}-\frac{k}{n}\right)} = 1$

$m_{\alpha,\mathbb{Q}}(x)$ sólo tiene raíces reales

Pd: $\alpha \in \mathbb{Q}(\cos(2\pi/n))$

$G = Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*$

$\zeta_n = e^{2\pi i/n} = \cos(2\pi/n) + i\,\mathrm{sen}(2\pi/n)$

$\zeta = \cos\left(\frac{2\pi}{n}\right) + i\,\mathrm{sen}\left(\frac{2\pi}{n}\right)$

$\zeta^k = \cos\left(\frac{2\pi k}{n}\right) + i\,\mathrm{sen}\left(\frac{2\pi k}{n}\right)$ $\boxed{k=1}$

$(n,k)=1$

$\zeta^k = \zeta \iff \begin{cases}\cos\left(\frac{2\pi}{n}\right) = \cos\left(\frac{2\pi k}{n}\right)\\ \mathrm{sen}\left(\frac{2\pi}{n}\right) = \mathrm{sen}\left(\frac{2\pi k}{n}\right)\end{cases}$

$\mathbb{Q}(\zeta_n) \;|\; \mathbb{Q}(\alpha) \;|\; \mathbb{Q}$ (Gal, Gal)

$m_{\alpha,\mathbb{Q}}(x) = (x-\alpha)(x-\alpha_2)\cdots(x-\alpha_r)$

$= x^r + \cdots + \alpha\alpha_2\cdots\alpha_r(-1)^r$

$\frac{2\pi}{n} = \arccos\left(\cos\left(\frac{2\pi k}{n}\right)\right) + 2\pi l$

$\frac{2\pi}{n} = \frac{2\pi k}{n} + 2\pi l$

$\frac{1}{n} = \frac{k}{n} + l \Rightarrow 1 = k + nl$

$\cos(2\pi/n)^2 = 1 - \mathrm{sen}(2\pi/n)^2$

$\zeta + \zeta^{-1} = 2\cos(2\pi/n) \Rightarrow$

$\mathbb{Q}(\zeta) \;|\; \mathbb{Q}(\zeta+\zeta^{-1}) \;|\; \mathbb{Q}(\alpha) \;|\; \mathbb{Q}$

$\zeta^{-1} = \zeta$

$\boxed{k=1}$

¿$\zeta \mapsto \zeta^2$ es automorfismo en G?

Si $\alpha \in \mathbb{R} \Rightarrow \bar{\alpha} = \sigma(\alpha) = \alpha$ $\forall \alpha$ raíz

$\Rightarrow \langle\sigma\rangle$ fija a $\mathbb{Q}(\alpha)$

$\tau(\zeta)\zeta = -1$

$\tau(\zeta) = -\frac{1}{\zeta}$

$\mathrm{sen}\left(2\pi\left(\frac{1}{n}-\frac{k}{n}\right)\right) = 0$

$2\pi\left(\frac{1}{n}-\frac{k}{n}\right) = 0$

$\Rightarrow \frac{1}{n} - \frac{k}{n} = 0 \Rightarrow \boxed{k=1}$

Sea $\alpha, \alpha_2, \ldots, \alpha_r$ raíces de $m_{\alpha,\mathbb{Q}}(x)$ $\to \sigma(\alpha_i) = \alpha_i$ $\forall i$

$\mathbb{Q}(\alpha)$ Galoisiana | $\mathbb{Q}(\zeta+\zeta^{-1}) = \mathbb{Q}(\zeta)^{\langle\sigma\rangle}$

$2\pi\left(\frac{1}{n}-\frac{k}{n}\right) = \pi$

$\tau(\zeta+\zeta^{-1}) = \tau(\zeta) + \tau(\zeta)^{-1}$

$\frac{1}{n} - \frac{k}{n} = \frac{1}{2}$

$= \zeta + \zeta^{-1} \to \tau(\zeta) - \zeta = \frac{1}{\tau(\zeta)} - \frac{1}{\zeta} = \frac{\zeta - \tau(\zeta)}{\tau(\zeta)\zeta} \Rightarrow$

$1 - k = \frac{1}{2}n$

$\Rightarrow 1 - \frac{1}{2}n = k$

$$\zeta+\zeta^{-1} = \tau(\zeta+\zeta^{-1}) = \tau(\zeta)+\tau(\zeta)^{-1} \Leftrightarrow \zeta-\tau(\zeta) = \tau(\zeta)^{-1}-\zeta^{-1}$$

$$\Leftrightarrow \zeta-\tau(\zeta) = \frac{1}{\tau(\zeta)}-\frac{1}{\zeta} = \frac{\zeta-\tau(\zeta)}{\zeta\tau(\zeta)} \Rightarrow \zeta\tau(\zeta) = 1$$

$$\Rightarrow \tau(\zeta) = \frac{1}{\zeta} = \bar{\zeta}$$

Norma y traza

K/F finita, $\alpha\in K$. $\begin{matrix} L \\ | \\ K \\ | \\ F \end{matrix}$ Gal. $H \leq Gal(L/F)$ H fija K ($K = L^H$)

$N_{K/F}: K \to F$, $N_{K/F}(\alpha) = \prod_{\sigma} \sigma(\alpha)$ $\quad \sigma: K \to \bar{F}$

$\sigma(K) = L^{\sigma H \sigma^{-1}}$ | Por teo. de Galois $\forall\sigma: \sigma = \tau|_K$, $\tau\in Gal(L/F)$

todas las inmersiones $K \hookrightarrow \bar{F}$ = sobre un conjunto de representantes de H en $Gal(L/F)$

$|Emb(K/F)| = [K:F]$

$\leq$

$|Aut(K/F)|$

Pd: $N_{K/F}(\alpha)\in F$

$$N_{K/F}(\alpha) = \prod_{\alpha} \sigma(\alpha) = \prod_{\substack{\tau\in Gal(L/F) \\ \tau|_K = \sigma}} \tau|_K(\alpha)$$

Sea $m_{\alpha,F}(x)$ polinomio minimal de α, $\sigma(\alpha) = \tau|_K(\alpha)$

K/F finita $\alpha\in K$

$m_\alpha: L \to L$, $m_\alpha(x) = \alpha x$

$N_{K/F}(\alpha) = \det(m_\alpha)$

$\det: L \to K$

$N_{K/F}(\alpha\beta) = \det(m_{\alpha\beta})$
$= \det(m_\alpha \circ m_\beta)$
$= \det(m_\alpha)\det(m_\beta)$
$= N_{K/F}(\alpha) N_{K/F}(\beta)$

(a) $\alpha_1, \dots, \alpha_n$ las raíces de $P_\alpha(x) = \det(xI - L_\alpha)$

— o —

K/F finita, $\alpha \in K$, $L_\alpha : K \to K$, $L_\alpha(x) = \alpha x$ F-lineal

$P_\alpha(x) = \det(xI - L_\alpha)$ $\overset{?}{\exists}$ $\in F[x]$

$L_\alpha(x) = \alpha x$

— · —

$$\begin{array}{c} K \\ | \\ F(\alpha) \\ | \\ F \end{array}$$

Pd: $K = F(\alpha) \Rightarrow P_\alpha(x) = m_{\alpha,F}(x)$

$P_\alpha(\alpha) = \det(\alpha I - L_\alpha)$ | $(\alpha I - L_\alpha)(x) = \alpha I(x) - L_\alpha(x) = 0$.

$\therefore m_{\alpha,F}(x) \mid P_\alpha(x)$.

Sea $\beta \in K$: $P_\alpha(\beta) = 0 \Leftrightarrow \det(\beta I - L_\alpha) = 0 \Rightarrow \beta I - L_\alpha$ singular

$\beta I - L_\alpha$ no inyectiva $\Rightarrow \exists \gamma \in K^\times : (\beta I - L_\alpha)(\gamma) = 0$

~~$\beta\gamma$~~ $0 = \beta\gamma - L_\alpha(\gamma) = \beta\gamma - \alpha\gamma \Rightarrow \beta = \alpha$

$\underset{F\text{-lineal}}{(xI - L_\alpha)} : F(\alpha) \to F(\alpha) \Rightarrow P_\alpha(x)$ tiene grado $[F(\alpha):F] = \deg m_{\alpha F}$

$\therefore P_\alpha = m_{\alpha,F}$

(b) Pd: $P_\alpha(x) = (m_{\alpha,F}(x))^m$, $m = [K : F(\alpha)]$

$P_\alpha(x) = \det(xI - L_\alpha)$ $K = F(\alpha_1, \dots, \alpha_r)$, $\alpha_1 = \alpha$

$\deg P_\alpha = [K:F]$

$P_\alpha|_{F(\alpha)} = m_{\alpha,F}(x)$

$$\begin{array}{c} K \\ | \\ F(\alpha_1, \dots, \alpha_{r-1}) \\ | \\ F(\alpha) \\ | \\ F \end{array}$$

Por definición

$N_{K/F} : K \to F$, $Tr_{K/F} : K \to F$

$N_{K/F}(\alpha) := \det(L_\alpha)$, $Tr_{K/F}(\alpha) := \text{traza}(L_\alpha)$

Pd: $P_\alpha(x) = \prod_{i=1}^{n} (x - \alpha_i) \Rightarrow N_{K/F}(\alpha) = \prod_{i=1}^{n} \alpha_i$

$Tr_{K/F}(\alpha) = \sum_{i=1}^{n} \alpha_i$

Sea $\{e_1, \dots, e_r\}$ F-base de K

—o—

$\deg P_\alpha = [K:F] = n$, $P_\alpha(x) = \prod_{i=1}^{n} (x - \alpha_i)$

$L_\alpha : K \to K$ $\quad P_\alpha(x) = (m_{\alpha,F}(x))^r$, $r = [K : F(\alpha)]$

$N_{K/F}(\alpha) = \prod_\sigma \sigma(\alpha)$ $\quad \sigma \in Emb(K/F)$

Pd: $N_{K/F}(\alpha) \in F \quad \forall \alpha \in K$ $\quad m_{\alpha,F}(x) \in F[x] : m_{\alpha,F}(\sigma(\alpha)) = 0$

deg

$\sigma \in Emb(K/F) \Rightarrow \sigma(\alpha)$ raíz de $m_{\alpha,F}$

$\leq [K:F]$ $\quad \sigma(\alpha) = \tau|_K(\alpha)$

Gal $\left(\begin{array}{c} L \\ |\ \text{Gal} \\ K' \\ |\ \text{sep} \\ F \end{array} \right)$ $\quad \boxed{L^H = K}$

$F \to \bar{F}$ $\quad \therefore \prod_\sigma \sigma(\alpha) \in F$

Pd: $N_{K/F}(\alpha\beta) = \prod_\sigma \sigma(\alpha\beta) = \prod_\sigma \sigma(\alpha)\sigma(\beta) = \prod_\sigma \sigma(\alpha) \prod_\sigma \sigma(\beta)$

$= N_{K/F}(\alpha) N_{K/F}(\beta)$

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$K = F(\sqrt{D}) \rightarrow$ ~~$\sigma(\alpha) =$~~ $\alpha = a + b\sqrt{D}$

$\qquad\qquad\qquad\qquad \sigma(\alpha) = a \pm b\sqrt{D}$

$|$

Galois

(d) $m_\alpha(x) = x^d + a_{d-1}x^{d-1} + \dots + a_1 x + a_0 \in F[x]$, $\alpha \in K$ (K/F)

$n = [K : F]$,

Pd: $d \mid n$

$\partial(m_\alpha(x)) = d \Rightarrow [F(\alpha) : F] = d \wedge F(\alpha) \subseteq K \Rightarrow d \mid n$.

Pd: Hay d conjugados de Galois de α que se repiten n/d veces y

$N_{K/F}(\alpha) = (-1)^n a_0^{n/d}$

$$\text{Gal}\left(\begin{array}{l} K \\ \mid n/d \\ F(\alpha) \\ d \mid \text{sep} \\ F \end{array}\right. \qquad \forall \sigma : F(\alpha) \hookrightarrow \bar{F} \text{ hay } n/d \text{ extensiones a } Gal(K/F)$$

$$\cancel{N_{K/F}(\alpha)} = \prod_{i=1}^{d} (x - \sigma_i(\alpha))^{n/d} = m_\alpha(x)$$

$$= \left(\prod_{i=1}^{d} (x - \sigma_i(\alpha))\right)^{n/d} \quad ?$$

$N_{K/F}(\alpha) = (-1)^{n} a_0^{n/d}$

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L/F Galoisiana $\Rightarrow$ $\alpha \in L$: $m_{\alpha,F}(x) = \prod_{\sigma \in Gal(L/F)} (x - \sigma(\alpha))$

$$\begin{array}{l} L \\ | \\ \alpha \in K \\ |)\text{sep} \\ F \end{array} \qquad m_{\alpha,F}(x) = \prod_{b \in \mathcal{O}(\alpha)} (x - b)$$

$m_{\alpha,F}(x) = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_r)$ $\quad \underline{\alpha_1 = \alpha} \quad \alpha_i \neq \alpha_j$

$\forall \sigma \in Emb(K, \bar{F})$: $m_{\alpha,F}(\sigma(\alpha_i)) = \sigma(m_{\alpha,F}(\alpha_i)) = 0$

$\Rightarrow \sigma(\alpha_i)$ raíz de $m_{\alpha,F}(x)$

$\forall \sigma \in Emb(K, \bar{F})$, $\exists [L:K]$ extensiones $\tau \in Emb(L, \bar{F})$ tq $\tau|_K = \sigma$

Si $\sigma_a(\alpha_i) = \sigma_b(\alpha_i) \Rightarrow \sigma_a \sigma_b^{-1}(\alpha_i) = \alpha_i \Rightarrow \sigma_a \sigma_b^{-1} = 1$ ¿?

$\Rightarrow \sigma_a = \sigma_b$

$\Downarrow$

$\sigma_a \sigma_b^{-1} \in H$, $L^H = F(\alpha_i)$

$\Downarrow$

$\sigma_a \in \sigma_b H$

$Emb(K, \bar{F}) = [K:F]$ $\rightarrow$ K/F separable?

Sea $K = L^H$

$[K:F] \longrightarrow$ separabilidad

$N_{K/F}(\alpha) := \prod_{\sigma \in Emb(K,\bar{F})} \sigma(\alpha) = \prod_{i=1}^{[K:F]} \sigma_i(\alpha)$ $\quad | \quad m_{\alpha,F}(x) = (x - \sigma_1(\alpha)) \dots (x - \sigma_{[K:F]}(\alpha))$ ✓

—o—

K/F Galois, $Gal(K/F) = \langle \sigma \rangle = G$ $\quad |G| = n$

Pd: $\alpha \in K$, $N_{K/F}(\alpha) = 1 \Rightarrow \exists \beta \in K$ $\beta \neq 0$, $\alpha = \beta / \sigma(\beta)$

Sea $G \longrightarrow K^*$

$\sigma^r \longmapsto a_{\sigma^r} = \prod_{i=1}^{r-1} \sigma(\alpha)$ $\qquad \sigma^n = 1 \mapsto a_{\sigma^n} = \prod_{i=0}^{n-1} \sigma^i(\alpha)$

$$\sigma^r\sigma^s = \sigma^{r+s} \longmapsto a_{\sigma^{r+s}} = \prod_{i=0}^{r+s-1}\sigma^i(\alpha) = \prod_{i=0}^{r-1}\sigma^i(\alpha)\prod_{i=r}^{r+s-1}\sigma^i(\alpha)$$

$$= a_{\sigma^r}\prod_{i=0}^{s-1}\sigma^{r+i}(\alpha) = a_{\sigma^r}\,\sigma^r\left(\prod_{i=0}^{s-1}\sigma^i(\alpha)\right)$$

$$= a_{\sigma^r}\,\sigma^r(a_{\sigma^s})$$

T. 90 $\Rightarrow$ $\exists \beta\in K^* : \forall \sigma\in Gal(K/F) : a_{\sigma^r} = \dfrac{\beta}{\sigma^r(\beta)}$

$$\frac{\beta}{\sigma^r(\beta)} = a_{\sigma^r} = \prod_{i=0}^{r-1}\sigma^i(\alpha) = \alpha\prod_{i=1}^{r-1}\sigma^i(\alpha)$$

$$\alpha = \frac{\beta}{\sigma^r(\beta)\prod_{i=1}^{r-1}\sigma^i(\alpha)} = \frac{\beta}{\sigma^r(\beta)\prod_{i=0}^{r-2}\sigma^{i+1}(\alpha)}$$

$\alpha(\alpha^{p^r-1}-1)=0$

$$a_{\sigma^1} = \frac{\beta}{\sigma(\beta)} = \prod_{i=0}^{1-1}\sigma^i(\alpha) = \sigma^0(\alpha) = \alpha.$$

$\alpha^{p^r}-\alpha=0$

$F=\mathbb{F}_p$, $K=\mathbb{F}_{p^r}$, $[\mathbb{F}_{p^r}:\mathbb{F}_p] = |Gal(\mathbb{F}_{p^r}/\mathbb{F}_p)| = r$

$Gal(\mathbb{F}_{p^r}/\mathbb{F}_p) = \langle\sigma\rangle$, $\sigma(\alpha)=\alpha^p$

$$N_{K/F}(\alpha) = \prod_{\sigma\in Gal(\ )}\sigma(\alpha) = \alpha\,\alpha^p\alpha^{p^2}\cdots\alpha^{p^{r-1}} = \alpha^{1+p+p^2+\dots+p^{r-1}}$$

$$= \alpha^{\frac{1-p^r}{1-p}} = (\alpha^{1-p^r})^{1/1-p} = \frac{\alpha}{\alpha^{p^r}} = \left(\frac{\alpha}{\alpha}\right)^{1/1-p} = 1$$

$$1-p^r = (1-p)(1+p+\dots+p^{r-1})$$
$$= 1+p+\dots+p^{r-1}-p-p^2-\dots-p^r = 1-p^r$$

$\exists\beta\in\mathbb{F}_{p^r} : \alpha = \dfrac{\beta}{\beta^p} = \beta^{1-p}$

T-90 dans K/F Galoisienne $[K:F]=n$

K/F Galoisienne, $\sigma \in G = Gal(K/F)$

$\alpha \in K$: $\alpha = \frac{\beta}{\sigma\beta}$ Pq : $N_{K/F}(\alpha) = 1$

$$N_{K/F}(\alpha) = N_{K/F}\left(\frac{\beta}{\sigma\beta}\right) = \prod_{i=1}^{n} \sigma_i\left(\frac{\beta}{\sigma\beta}\right) = \prod_{i=2}$$

$\cancel{\sigma_2}$

$$= \prod_{i=1}^{n} \frac{\sigma_i(\beta)}{\sigma_i\sigma(\beta)} = \prod_{i=1}^{n} \frac{\sigma_i(\beta')}{\sigma_{i+1}} \quad \sigma = \sigma$$

$$N_{K/F}(\alpha) = N_{K/F}\left(\frac{\beta}{\sigma\beta}\right) = N_{K/F}(\beta)\, N_{K/F}(\sigma\beta)^{-1}$$

$$N_{K/F}(\beta) = \prod_{\sigma} \sigma(\beta) \quad , \quad N_{K/F}(\sigma\beta) = \prod_{\tau} \tau(\sigma\beta)$$

$$\tau G = G \qquad\qquad = \prod_{\tau} (\tau\sigma)\beta$$

$$= \prod_{\tau} \tau(\beta) = N_{K/F}(\beta).$$

$\therefore$ $N_{K/F}(\alpha) = 1$

$\alpha \in K$: $\alpha = \beta - \sigma\beta$, $\beta \in K$. $Tr_{K/F}(\alpha) = 0$

$$Tr_{K/F}(\alpha) = Tr_{K/F}(\beta) - Tr_{K/F}(\sigma\beta) = Tr_{K/F}(\beta) - Tr_{K/F}(\beta) = 0$$

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L/K Galois $\quad \mathrm{Gal}(L/K) \longrightarrow L^*$

$\sigma \longmapsto a_\sigma$

$$a_{\sigma\tau} = a_\sigma + \sigma(a_\tau)$$

$\Rightarrow \exists\, b \in L^*$ tal que $a_\sigma = b - \sigma(b) \ \forall\, \sigma \in G = \mathrm{Gal}(L/K)$

__dem.__ Automorfismos de $G = \mathrm{Gal}$ son independientes

$\exists\, c \in L$: $\quad b = \sum_{\sigma\in G} a_\sigma \sigma(c) \neq 0$

$\forall\, \tau \in G$

$$\tau(b) = \sum_{\sigma\in G} \tau(a_\sigma \sigma(c)) = \sum_{\sigma\in G} \tau(a_\sigma)\tau(\sigma(c))$$

$$\sum_{\sigma\in G} \tau(a_\sigma)\, \tau\circ\sigma(c) = \sum_{\sigma\in G} (a_{\tau\sigma} - a_\tau)\, \tau\circ\sigma(c)$$

$$= \sum_{\sigma\in G} (a_{\tau\sigma} - a_\tau)\, \tau(\sigma(c))$$

$$= \sum_{\sigma\in G} a_{\tau\sigma}\tau(\sigma(c)) - a_\tau \tau(\sigma(c))$$

$$= \sum_{\sigma\in G} a_{\tau\sigma}\, \tau\circ\sigma(c) - a_\tau \sum_{\sigma\in G} \tau\circ\sigma(c)$$

$$= b - a_\tau \sum_{\sigma\in G} \tau\circ\sigma(c)$$

$$a_\tau \sum_{\sigma\in G} \tau\circ\sigma(c) = b - \tau(b)$$

$$a_\tau\, \tau\Big(\sum_{\sigma\in G} \sigma(c)\Big) = b - \tau(b)$$

Ahh!

$t = \sum_{\sigma \in G}' \sigma(c') \neq 0$, $\tau \in G = Gal(L/K)$ | Cocodrilos.

$$\tau(t) = \tau\left(\sum_{\sigma \in G} \sigma(c')\right) = \sum_{\sigma \in G} \tau\sigma(c') = t$$

$\Rightarrow \quad c = c'/t \qquad \sum_{\sigma \in G}' \sigma(c) = 1 \longleftarrow \exists c \in L$ tq $tr_{L/K}(c) = 1$

$b = \sum_{\sigma \in G}' a_\sigma \sigma(c)$ se tiene

$$\tau(b) = \sum_{\sigma \in G}' \tau(a_\sigma \sigma(c)) = \sum_{\sigma \in G} \tau(a_\sigma)\, \tau \circ \sigma(c)$$

$$= \sum_{\sigma \in G}' (a_{\tau\sigma} - a_\tau)\, \tau \circ \sigma(c) = \sum_{\sigma \in G} a_{\tau\sigma}\, \tau \circ \sigma(c) - a_\tau \underbrace{\sum_{\sigma \in G} \tau \circ \sigma(c)}_{1}$$

$$= b - a_\tau$$

$$\therefore a_\tau = \cancel{\tau(b)}\, b - \tau(b)$$

L/K A-S $\Leftrightarrow L = K(b)$

$b^p - b \in K$

char $K = p$.

$L = \mathbb{Q}(\alpha)$, α raíz de $f(x) = x^3 + ax + b$

$\alpha_1, \alpha_2, \alpha_3$ raíces de f, $(\alpha_1 = \alpha)$

$$\delta = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)$$

~~M = clausura normal~~

M cdd de f $\Rightarrow$ $M/\mathbb{Q}$ Galoisiana

$$\delta^2 = (\alpha_1 - \alpha_2)^2(\alpha_2 - \alpha_3)^2(\alpha_3 - \alpha_1)^2$$

$$\delta = (\alpha_1\alpha_2 - \alpha_1\alpha_3 - \alpha_2^2 + \alpha_2\alpha_3)(\alpha_3 - \alpha_1)$$

$$= \cancel{\alpha_1\alpha_2\alpha_3} - \alpha_1^2\alpha_2 - \alpha_1\alpha_3^2 + \alpha_1^2\alpha_3 - \alpha_2^2\alpha_3 + \alpha_1\alpha_2^2 + \alpha_2\alpha_3^2 - \cancel{\alpha_1\alpha_2\alpha_3}$$

$=$

$\forall \sigma \in Gal(M/\mathbb{Q})$ $\sigma(\alpha_i)$ raíz de f y actúa transitivamente sobre raíces

$$\therefore \sigma(\delta^2) = \delta^2 \quad \forall \sigma \in Gal(M/\mathbb{Q})$$

$$\therefore \delta^2 \in \mathbb{Q}$$

$\sigma(\delta) = sgn(\sigma)\,\delta$

$\delta \in \mathbb{Q} \Leftrightarrow sgn(\sigma) = 1$

$L/\mathbb{Q}$ Galoisiana $\Leftrightarrow$ raíces de $m_{\alpha,\mathbb{Q}}(x) \in L$

$[L:\mathbb{Q}] \leq 3$ como

$f'(x) = 3x^2 + a$

$= (x-\alpha_2)(x-\alpha_3) + (x-\alpha_1)((x-\alpha_2)(x-\alpha_3))'$

$$f(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \quad = (x-\alpha_2)(x-\alpha_3) + (x-\alpha_1)(x-\alpha_3) + (x-\alpha_1)(x-\alpha_2)$$

$$= (x^2 + (-\alpha_1 - \alpha_2)x + \alpha_1\alpha_2)(x - \alpha_3)$$

$$= (x^2 - (\alpha_1 + \alpha_2)x + \alpha_1\alpha_2)(x - \alpha_3)$$

$f'(\alpha_1) = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)$

$3\alpha_1^2 + a = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)$

$(3\alpha_1^2 + a)(\alpha_2 - \alpha_3) = -\delta$

$$= x^3 - \alpha_3 x^2 - (\alpha_1 + \alpha_2)x^2 + \alpha_3(\alpha_1 + \alpha_2)x + \alpha_1\alpha_2 x - \alpha_1\alpha_2\alpha_3$$

$$= x^3 - (\alpha_1 + \alpha_2 + \alpha_3)x^2 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)x - \alpha_1\alpha_2\alpha_3$$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = a \\ -\alpha_1\alpha_2\alpha_3 = b \end{cases}$$

$\partial(m_{\alpha,\mathbb{Q}}(x)) = \{2, 3\}$

$L/\mathbb{Q}$ Galois $\Leftrightarrow$ $\alpha_1, \alpha_2, \alpha_3 \in L = \mathbb{Q}(\alpha)$

$\Leftrightarrow \delta \in \mathbb{Q}$ L ✓

$$(\alpha_2 - \alpha_3) = -\frac{\delta}{3\alpha_1^2 + a}$$

$\alpha_2 + \alpha_3 = -\alpha_1$

$\alpha_2 - \alpha_3 = -\dfrac{\delta}{3\alpha_1^2 + a}$ $\Rightarrow$ $\alpha_2 = \dfrac{1}{2}\left(-\alpha_1 - \dfrac{\delta}{3\alpha_1^2 + a}\right)$

$L = \mathbb{Q}(\alpha)$ donde $\alpha^3 \in \mathbb{Q} \Rightarrow -3\delta^2$ es un cuadrado en $\mathbb{Q}$.

α raíz de $x^3 - \alpha^3$

$-3\delta^2 = -3(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)$

$f'(x) = 3x^2 + a$

$f'(\alpha_1) = 3\alpha_1^2 + a = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)$

$f'(\alpha_2) = 3\alpha_2^2 + a = (\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)$

$f'(\alpha_3) = 3\alpha_3^2 + a = (\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)$

$-\delta^2 = (3\alpha_1^2 + a)(3\alpha_2^2 + a)(3\alpha_3^3 + a)$

$\sqrt{-3}\,\delta = \zeta \Rightarrow \sigma(\zeta) = \sigma(\sqrt{-3})\,\sigma(\delta)$

$\Rightarrow \underset{\mathbb{Q}}{\underbrace{\alpha^3}} + a\alpha + b = 0 \Rightarrow \alpha \in \mathbb{Q}$

$\mathbb{Q}\ x^3 - \alpha^3 \Rightarrow a = 0,\quad -\delta^2 = 27(\alpha_1\alpha_2\alpha_3)^2$

$-3\delta^2 = +81(\alpha_1\alpha_2\alpha_3)^2$

$= +81(-\alpha^3)^2$

$= +81(\alpha)^6$

$=$ ok funciona!

Gap Dreams
Beach Fossils

Polinomios de grado 3.

$$f(x) = x^3 + ax^2 + bx + c$$

$$\boxed{x = y - \frac{a}{3}}$$ ← cambio de variable mega importante!

$$f(x) = f\left(y - \frac{a}{3}\right) = \left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c$$

$$= y^3 - 3y^2\frac{a}{3} + 3y\frac{a^2}{9} - \frac{a^3}{27} + a\left(y^2 - 2y\frac{a}{3} + \frac{a^2}{9}\right) + by - \frac{ab}{3} + c$$

$$= y^3 - \cancel{y^2}a + \frac{a^2 y}{3} - \frac{a^3}{27} + a\cancel{y^2} - 2y\frac{a^2}{3} + \frac{a^3}{9} + by - \frac{ab}{3} + c$$

$$= y^3 + y\left(\frac{a^2}{3} - \frac{2a^2}{3} + b\right) + \left(c + \frac{2a^3}{27} - \frac{ab}{3}\right)$$

$$= y^3 + y\left(-\frac{a^2}{3} + b\right) + \left(\frac{2a^3}{27} - \frac{ab}{3} + c\right)$$

$$p = -\frac{a^2}{3} + b = \frac{1}{3}(-a^2 + 3b)$$

$$q = \frac{1}{27}(2a^3 - 9ab + 27c)$$

$$\delta = \sqrt{-4p^3 - 27q^2}$$

$$\delta = \sqrt{a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc}$$ → es o no cuadrado en $\mathbb{Q}$

cdd f

$[L : \mathbb{Q}] = 3, 6$

$L = \mathbb{Q}(\theta, \delta)$

$\mathbb{Q}(\theta)$ $\quad$ $\mathbb{Q}(\delta)$

$\mathbb{Q}$

Extensiones compuesto

$$\begin{tikzcd}[column sep=small] & KF' \arrow[dl, dash] \arrow[dr, dash] & \\ K \arrow[dr, dash] \arrow[ddr, dash, bend right, "Gal"'] & & F' \arrow[dl, dash] \arrow[ddl, dash, bend left] \\ & K\cap F' \arrow[d, dash] & \\ & F & \end{tikzcd}$$

$\Rightarrow KF'/_{F'}$ Galois, $Gal(KF'/_{F'}) \cong Gal(K/_{K\cap F'})$

se cumple

ej: $\mathbb{Q}(\omega, \sqrt[3]{2})$, $\omega^3 = 1$

$$\begin{tikzcd}[column sep=small] & \mathbb{Q}(\omega, \sqrt[3]{2}) \arrow[dl, dash] \arrow[dr, dash, "Gal"] & \\ \mathbb{Q}(\omega) \arrow[dr, dash, "Gal"'] & & \mathbb{Q}(\sqrt[3]{2}) \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

- $K/_F$ Galois $\Rightarrow KF'/_{F'}$ Galois ✓
- Pd: $Gal(KF'/_{F'}) \cong Gal(K/_{K\cap F'}) \leq Gal(K/_F)$

$\varphi: Gal(KF'/_{F'}) \longrightarrow \cancel{Gal(K/_{K\cap F'})}\ Gal(K/_F)$

$\sigma \mapsto \sigma|_K$ bien def.

$\sigma|_{F'} = id_{F'} \Rightarrow \sigma|_F = id_F \mid Gal(K/_F) \to K/_F$ normal

$\forall \sigma \in Emb(K, F) \Rightarrow \sigma(K) = K$

$\therefore \sigma$ automorfismo

$\ker(\varphi) = \{\sigma \in Gal(KF'/_{F'}) \mid \sigma|_K = id_K\}$

$\sigma \in \ker(\varphi) \Rightarrow \sigma \in Gal(KF'/_{F'}),\ \sigma|_K = id_K$

$\Rightarrow \sigma|_{F'} = id_{F'},\ \sigma|_K = id$

$\Rightarrow \sigma(z) = z \quad \forall z \in KF'$

$\Rightarrow \sigma = id_{KF'}$

$\therefore \ker(\varphi) = 1$

$\therefore Gal(KF'/_{F'}) \hookrightarrow Gal(K/_F)$

además $Gal(KF'/_{F'})$ fija a $K^H F'$ $\therefore K^H F' = F'$

$\therefore K^H \subseteq F'$

$\therefore \cancel{K^H \cap F'}$

$K^H \subseteq F' \cap K$

$H = \varphi(Gal(KF'/_{F'}))$, K^H cuerpo fijo de H

$\sigma \in H$ fija a K y F' $\Rightarrow K^H \subseteq K$ Galoisiano $K \cap F \subseteq K^H$

$\therefore K^H = F' \cap K$

$Gal(KF'/_{F'}) \ \cancel{\cong Gal(K/_{K\cap F'})}$

IA

$Gal(K/_F)$

$P_\alpha(x) = \det(xI - L_\alpha)$

$xI - L_\alpha : K \to K$

$T_{x,\alpha} = xI - L_\alpha : F(\alpha)^m \to F(\alpha)^m$

$\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$

$m_{\alpha,F}(x) \mid P_\alpha(x)$

$m_{\alpha,F}(x) = \det(xI - L_\alpha)$

$F(\alpha) \leq_F K$

Sea $\bar{x}$ raiz de $P_\alpha(\bar{x})$

$$(xI - L_\alpha)(v_1, \dots, v_m) = ((xI - L_\alpha)v_1, \dots, (xI - L_\alpha)v_m)$$

Sea $\{v_1, \dots, v_m\}$ una $F(\alpha)$-base de $F(\alpha)^m (\cong K)$

$$T_{x,\alpha}(\beta) = T_{x,\alpha}\left(\sum_1^m a_i v_i\right) = \sum_1^m a_i T_{x,\alpha}(v_i)$$

$T := (T_1, \dots, T_m) \qquad T_i : F(\alpha)^m \xrightarrow{} F(\alpha)$

$$(xI - L_\alpha)\left(\sum_1^m a_i v_i\right) = \sum_1^m a_i (xI - L_\alpha) v_i$$

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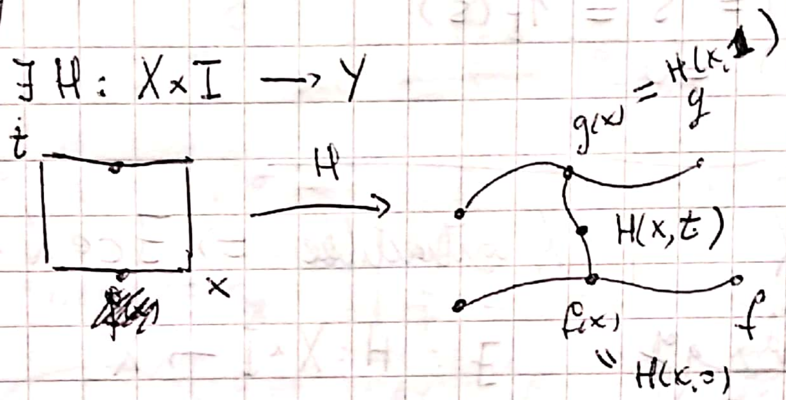
Homotopías

1) Homotopías → funciones (1) → caminos (2)

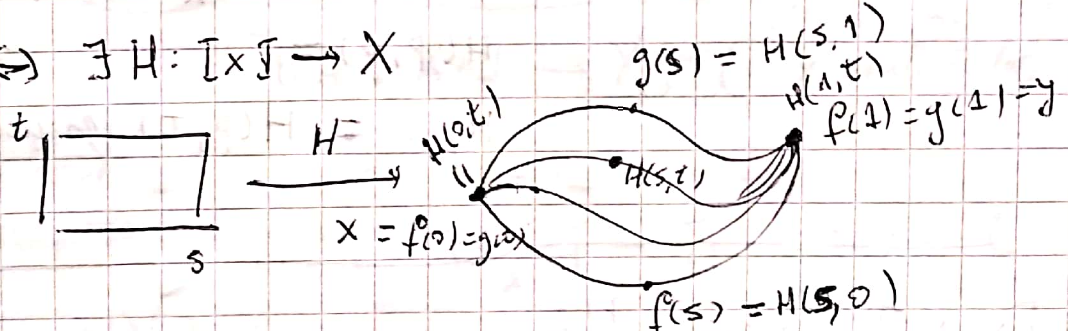
(1) $f, g : X \to Y$

$f \sim g \iff \exists H : X \times I \to Y$



(2) $f, g : I = [0,1] \to X$, $f, g : x \to y$

$f \sim g \iff \exists H : I \times I \to X$



Ef1 : $\sim$ en (1) y (2) son relaciones de equivalencia.

2) Producto de caminos

X - contractible $\iff$ $\cancel{id}$ $i_X : X \to X$ es constante.

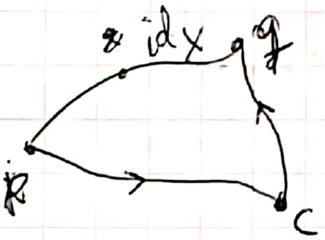
$i_I : I \to I$

$H(s,t) = s(1-t)$

$H(s,0) = s = 1_I(s)$

$H(s,1) = 0$

$x, y \in X$ | X contractible $\Rightarrow \exists c \in X : 1_X \sim c$.



$\exists \; H : X \times I \to X$

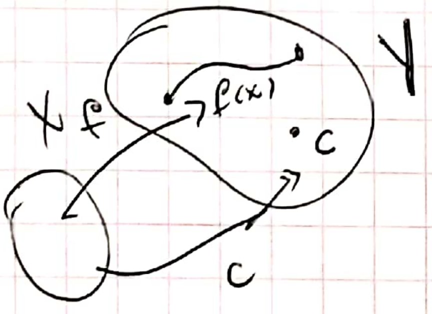
$H(x,0) = id_X(x) = x$

$H(x,1) = \cancel{id_X}\; c$

$H(p,0) = p$

$H_p(t) := H(p,t)$ es un camino entre p y c

$[X, Y]$ Y contractible.



$\exists \; H : Y \times I \to Y$

$X \xrightarrow{f} Y \xrightarrow{1_Y} Y$

$\simeq \qquad \simeq$

$X \xrightarrow{f} Y \xrightarrow{c} Y$

$1_Y \circ f := X \to Y = f$

$\simeq$

$c \circ f : X \to Y$

$= c$

$1_X \sim c$, Y c-c.

$$
\begin{array}{ccccc}
X & \xrightarrow{1_X} & X & \xrightarrow{f} & Y \\
 & & \wr & & \wr \\
X & \xrightarrow{c} & X & \xrightarrow{f_c} & Y
\end{array}
$$

$c \longmapsto f(c)$

$f \circ 1_X \sim f \circ c$

$f \circ H : X \times I \to Y$

$f \circ H(x,0) = f(x)$ $\quad f \sim x \mapsto f(c)$

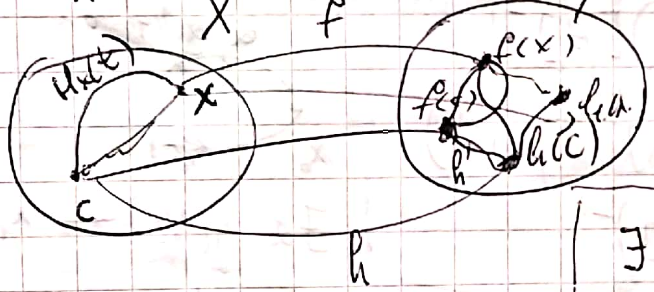
$f \circ H(x,1) = f(c)$ $\quad h \sim x \mapsto h(c)$

$\tilde{f} : X \to Y$, $f(x) = c_x$

$\exists \alpha : f(c) \to h(c)$

$K(x,t) = \alpha(t)$

AF : $f \sim h$



$h(x) = h(c)$ $\forall x \in X$

$K(x,0) = \alpha(0) = f(c)$

$K(x,1) = \alpha(1) = h(c)$

$$\left.\boxed{\begin{array}{c} \exists H : X \times I \to X \\ H(x,0) = x \\ H(x,1) = c \end{array}}\right\} X \text{ contractible}$$

$\forall x \in X$, $\exists K_x : I \to I$, $f(x) \to h(c)$

continue

$K = f \circ H : X \times I \to Y$

$K(x,0) = f(x)$

$K(x,1) = f(c)$

$K : X \times I \to Y$

$K(x,t) = K_x(t)$

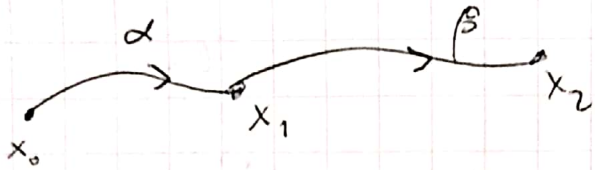
$K(x,0) = K_x(0) = f(x)$

$K(x,1) = K_x(1) = h(c)$

$f : X \to Y$, $h : X \to Y$ } continues

$K : X \times I \to Y$, $K(x,t) = h'($

$K(x,t) = 1$



$\alpha = \gamma g'$

$g' : [0, 1/4] \longrightarrow X$

$g'(s) = g(\tfrac{s}{4})$

$\beta^{-1}\alpha(s) = \begin{cases} \alpha(4t), & t \in [0, 1/4] \end{cases}$

$$\beta\alpha = \gamma(t) = \begin{cases} \alpha(2t), & t \in [0, 1/2] \\ \beta(2t-1), & t \in [1/2, 1] \end{cases}$$

$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$

$[f] \longmapsto [\alpha f \alpha^{-1}] = [\alpha][f][\alpha]^{-1}$

$\hat{\gamma} = \hat{\beta} \circ \hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_2)$

$\hat{\gamma}[f] = \hat{\beta}(\hat{\alpha}[f]) = \hat{\beta}([\alpha][f][\alpha]^{-1})$

$= [\beta]([\alpha][f][\alpha]^{-1})[\beta]^{-1}$

$= ([\beta][\alpha])[f]([\alpha]^{-1}[\beta]^{-1})$

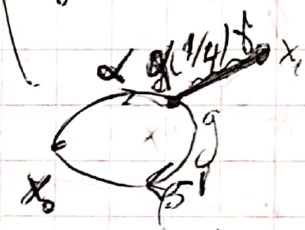
$= [\beta\alpha][f]([\beta][\alpha])^{-1}$

$H(s,t) = \alpha(t)$ $\quad = [\beta\alpha].[f][\beta\alpha]^{-1}$

$= \hat{\gamma}[f]$

Sea $[f][g] \in \pi_1$

$[g] = [\beta^{-1}\alpha]$



$\exists \sigma : g(1/2) \longrightarrow x_1$

Af. $g \sim \beta^{-1}\alpha$

$H(s,t) = \begin{cases} g(s), & s \in [0, 1/2], t = 0 \\ g(st) \end{cases}$

$H(s,0) = g(s)$

$H(s,1) = \beta\cancel{\alpha}(s)$

$\pi_1(X, x_0)$ abeliano $\Rightarrow [f][g] = [g][f]$

$\Rightarrow [f] = [g][f][g]^{-1}$

$\hat{\alpha}[f] = [\alpha][f][\alpha]^{-1}$

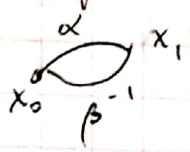
$\hat{\beta}[f] = [\beta][f][\beta]^{-1}$

$\Rightarrow \hat{\alpha} = \hat{\beta} \iff [\alpha][f][\alpha]^{-1} = [\beta][f][\beta]^{-1}$

$\iff [\beta]^{-1}[\alpha][f][\alpha]^{-1}[\beta] = [f]$

$\iff [\beta^{-1}\alpha][f][\alpha^{-1}\beta] = [f]$

$\iff [\beta^{-1}\alpha][f][\beta^{-1}\alpha]^{-1} = [f]$



Scanned with CamScanner

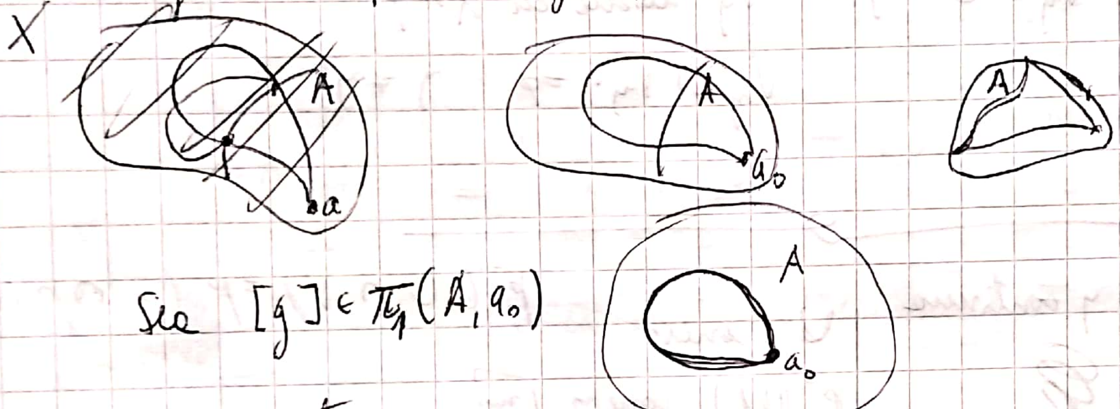
$A \subset X$, $r : X \to A$, $r(a) = a \ \forall a \in A$

r retracción de X ~~sobre~~ A. $a_0 \in A$

Pd: $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ es epi.

$[f] \longmapsto [r \circ f]$

$[g] = [r \circ f]$ algun f.



Sea $[g] \in \pi_1(A, a_0)$

podemos tomar g un loop en A. $\Rightarrow g$ loop en X.

$r \circ g = g$

$A \subset \mathbb{R}^n$, $h : (A, a_0) \to (Y, y_0)$ $(h(a_0) = y_0)$

sea $\tilde{h} : \mathbb{R}^n \to Y$ continua tq $\tilde{h}|_A = h$

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, "h"'] & \mathbb{R}^n \arrow[d, "\tilde{h}"] \\ & Y \end{tikzcd}$$

simp $\Rightarrow h_* = (\tilde{h} \circ i)_* = \tilde{h}_* \circ i_*$

$$\begin{aligned} h_*[\gamma] &= \tilde{h}_* \circ i_*[\gamma] \\ &= \tilde{h}_*([i \circ \gamma]) \\ &= \tilde{h}_*([\gamma]) \\ &= \tilde{h}_*[e_{a_0}] \\ &= [e_{y_0}] \end{aligned}$$

Y top. discreta. $p: X\times Y \longrightarrow X$
$(x,y) \longmapsto x$

p es un <u>cubrimiento</u> de X

$$U \subset X \Rightarrow p^{-1}(U) = U\times Y = U \times \bigcup_{y\in Y}\{y\} = \cancel{U\times} \bigcup_{y\in Y}(U\times\{y\})$$

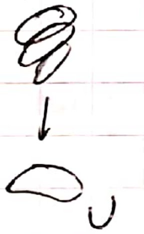
$\{V_y\}_{y\in Y}$, $V_y = U\times\{y\}$ $\quad V_y$ abierto en $X\times Y$

$V_y \cap V_{y'} = \emptyset \quad y\neq y'$

$p: E\to B$ epi y continua

$\{V_\lambda\}_{\lambda\in\Lambda}$

$p^{-1}(U) = \coprod_{\lambda\in\Lambda} V_\lambda$



U conexo.

$p^{-1}(U)$ conexo. $\boxed{\text{no}}$

$$p^{-1}(U\cap V) = p^{-1}(U)\cap p^{-1}(V)$$

$\cancel{X}$ conexo $\Rightarrow p(\cancel{X})$ conexo.

$p(\cancel{X}) = A \cancel{\cup} B$

$p: X\to Y \qquad \cancel{X} \cancel{=} p^{-1}(p(\cancel{X})) = p^{-1}(A\cancel{\cup}B) = p^{-1}(A) \cup p^{-1}(B)$

X conexo $\Rightarrow p^{-1}(A) = \emptyset$ o $p^{-1}(B) = \emptyset$

$p^{-1}(A) = \emptyset \Rightarrow A = \emptyset$

$$p^{-1}(U) = \coprod_{i\in I} V_\lambda \cancel{=} \coprod_{j\in J} W_j \Rightarrow p^{-1}(U) = \coprod_{i\in I} V_i \cup \coprod_{j\in J} W_j$$

Sup que $\{V_i\}_{i\in I} \neq \{W_j\}_{j\in J} \Rightarrow \exists i,\ \forall j: V_i \neq W_j$

$p|_{V_i\cup W_j}: V_i\cup W_j \to U \mid x\in p^{-1}(U) \Rightarrow V_i \cap W_j$ i,j <u>unidos</u>.

$p \quad V_i \quad W_j \quad p$

Como rest. de p a V_i, W_j es homeo

U

$p(V_i\cap W_j)$ abierto en U

como $p(V_i\cap W_j)\neq\emptyset$

$\Rightarrow p(V_i\cap W_j) = U$

$V_i \cap W_j = p|^{-1}_{V_i \cap W_j}(A) \quad A \subset U$ abierto.

Como U conexo $\Rightarrow A = U$

$V_i \cap W_j = p^{-1}|_{V_i \cap W_j}(U)$

$p|_{V_i \cap W_j}(V_i \cap W_j) = U$

$\|$

$p|_{V_i \cap W_j}(V_i) \cap p|_{V_i \cap W_j}(W_j) = U$

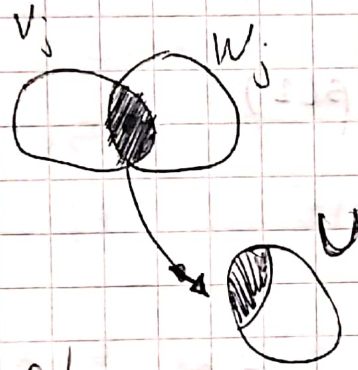
$x \in V_i \cap W_j = p|_{V_i}(U) \cap p|_{W_j}(U)$

$p(x) \in p(V_i \cap W_j) = p(V_i) \cap p(W_j)$

$= \cancel{p|^{-1}_{V_i}(U) \cap p|^{-1}_{W_j}(U)}$

$y \in p(V_i) \cap p(W_j)$

$y = p(v_i) = p(w_j)$



U conexo $\Rightarrow V_i$ es conexo.

$V_i \cap W_j$ es conexo.

$V_i \subset V_i \cup W_j$

abierto y cerrado

Como U es cerrado en $U \Rightarrow V_i$ es abierto y cerrado en $V_i \cup W_j$

$p: E \to B$ cubrimiento, B conexo

$b_0 \in B \Rightarrow \exists U \subset B \quad b_0 \in U \; : \; p^{-1}(U) = \coprod_{i=1}^{k} V_i$

$\forall b \in U \Rightarrow p^{-1}(b)$ tiene k-elementos.

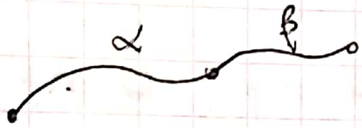
$b \in B, \quad p^{-1}(U_b) = \coprod_{j \in J} V_j$

$U_{b_0} \cup \left(\bigcup_{b \neq b_0} U_b \right) = B$

Sea $A \in \{ A \subset B \,/\, |p^{-1}(U)| = k \}$

$A_k = \{ x \in B \,/\, \exists U \subset B$ vecindad t_q, $x \in U$, $|p^{-1}(x)| = k$

$p: E \to B$ cubrimiento esp.



$\tilde{\alpha} \uparrow$ de α

$\tilde{\beta} \uparrow$ de β

—o—

$\tilde{\alpha}(1) = \tilde{\beta}(0)$

Pd: $\tilde{\beta}\tilde{\alpha} \uparrow$ de $\beta\alpha$.

$p \circ \tilde{\alpha} = \alpha$

$p \circ (\tilde{\beta}\tilde{\alpha})$ es un camino en B

$p \circ (\tilde{\beta}\tilde{\alpha})(0) = p(\tilde{\beta}\tilde{\alpha}(0)) = p \circ \tilde{\alpha}(0) = \alpha(0)$

$p \circ (\tilde{\beta}\tilde{\alpha})(1) = p(\tilde{\beta}\tilde{\alpha}(1)) = p \circ \tilde{\beta}(1) = \beta(1)$

—o—

Sea $e_0 \in p^{-1}(b_0)$, $b_0 = \alpha(0)$

$\Rightarrow \exists\, \tilde{\alpha}$ camino tq $\tilde{\alpha}(0) = e_0$

Sea $e_1 \in p^{-1}(b_1)$, $b_1 = \beta(0)$

$\Rightarrow \tilde{\beta}$ camino tq $\tilde{\beta}(0) = b_1 \Rightarrow b_1 = \tilde{\alpha}(1)$ pto final.

$t \in [0, 1/2] \Rightarrow p \circ (\tilde{\beta}\tilde{\alpha})(t) = p(\tilde{\alpha}\tilde{\beta}(t)) = p(\tilde{\alpha}(t))$
$= p \circ \tilde{\alpha}(t) = \alpha(t)$

$t = 1/2 \Rightarrow p \circ (\tilde{\beta}\tilde{\alpha})(1/2) = p \circ \tilde{\alpha}(1) = p \circ \tilde{\beta}(0)$
$\boxed{= \alpha(1) = \beta(0)}$

$\Rightarrow t \in (1/2, 1] \Rightarrow p \circ (\tilde{\beta}\tilde{\alpha}) = \beta(t)$

$\therefore \forall t$: $\boxed{p \circ (\tilde{\beta}\tilde{\alpha}) = \beta\alpha}$ $\therefore \tilde{\beta}\tilde{\alpha}$ levantamiento de $\beta\alpha$

$g, h : S^1 \to S^1$, $g(z) = z^n$, $h(z) = \frac{1}{z^n}$

$g((\cos\theta, \operatorname{sen}\theta)) = (\cos n\theta, \operatorname{sen} n\theta)$

$\gamma \in \pi_1(S^1, 1) \Rightarrow \gamma(t) = (\cos\theta(t), \operatorname{sen}\theta(t))$

$g \circ \gamma(t) = (\cos n\theta(t), \operatorname{sen} n\theta(t))$

$g_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$

$[\gamma] \mapsto [$

— · —

$g_* : \pi_1(S^1, b_0) \to \pi_1(S^1, b_0)$ (both $\cong \mathbb{Z}$)

$[\gamma] \mapsto [\gamma]^n$

$\pi_1(S^1, 1) \cong \mathbb{Z}$

$[\gamma] \mapsto \tilde{\gamma}(1) = 1$

$\gamma(t) = (\cos\theta(t), \operatorname{sen}\theta(t))$

$\gamma\gamma = \gamma^2$, $\gamma^2(t) = \begin{cases} \gamma(2t), & t \in [0, 1/2] \\ \gamma(2t-1), & t \in [1/2, 1] \end{cases}$

$\gamma(0) = \gamma(1) = (\cos\theta(0), \operatorname{sen}\theta(0)) = (\cos\theta(1), \operatorname{sen}\theta(1))$

$p : E \to B$ cubrimiento. E camino-conexo

$\pi_1(B, b_0) = 0 \Rightarrow p$ homeomorfismo.

p abierta, p sobre. Sea $p(a) = p(b) \in B$

$\exists U \ni p(a), V \ni p(b)$ tq $U = \bigsqcup_{i \in I} A_i$, $V = \bigsqcup_{j \in J} B_j$

$a \in A_i$, $b \in B_j \Rightarrow \exists \tilde{h} : a \to b$ en $E \Rightarrow p \circ \tilde{h}$ loop en $p(a) = p(b)$

$\Rightarrow [p \circ \tilde{h}] = [C_{p(a)}]$

$\tilde{p \circ h}$ en E es un loop punto inicial y final a y b $\therefore a = b$.

$$x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = 0$$

$$|a_{n-1}| + |a_{n-2}| + \dots + |a_1| + |a_0| < 1$$

$$g(x) = 1 + a_{n-1}x + \dots + a_1 x^{n-1} + a_0 x^n$$

$$|g(x)| = |1 + a_{n-1}x + \dots + a_1 x^{n-1} + a_0 x^n|$$

$$\leq 1 + |a_{n-1}||x| + \dots + |a_1||x|^{n-1} + |a_0||x|^n$$

$x \in B^2$: $\leq 1 + |a_{n-1}| + \dots + |a_1| + |a_0| < 2$

$$g(x) = 0 \iff -1 = a_{n-1}x + \dots + a_1 x^{n-1} + a_0 x^n$$

$$\Rightarrow 1 = |\,\dots\,| \leq |a_{n-1}||x| + \dots + |a_1||x|^{n-1} + |a_0||x|^n$$

$$\leq |a_{n-1}| + \dots + |a_1| + |a_0| < 1$$

$(\Rightarrow\Leftarrow)$

$\therefore\ g(x) \neq 0 \quad \forall x \in B^2$

$\Rightarrow$ g tiene todas sus raíces en $(B^2)^c$

$$g(x) = x^n\left(\frac{1}{x^n} + \frac{a_{n-1}}{x^{n-1}} + \dots + \frac{a_1}{x} + a_0\right)$$

#

$f: S^1 \to S^1$, $\boxed{\deg(f) \neq 1}$ | $\deg(f) = a \neq 1$

$\Rightarrow f \sim x^a$

$\downarrow$

$H(x,0) = f(x)$

$H(x,1) = x^a$

$g_a(x) = x^a$, $g_a(1) = 1$

$$\begin{tikzcd} \pi_1(S^1,1) \arrow[d, "f_*"'] \arrow[dr] \arrow[dr, phantom, "\circlearrowright"] & \\ \pi_1(S^1,f(1)) \arrow[r, "\gamma[\alpha]"'] & \pi_1(S^1,1) \end{tikzcd}$$

$f(x) \neq x \quad \forall x$

$g(x) = \frac{f(x)}{x}$, $g(x) \neq 1 \quad \forall x$ $\Rightarrow$ ~~$g: \pi_1(S^1) \to \pi_1(S^1 \setminus \{1\}) \cong \{0\}$~~

~~$\Rightarrow g_* = 0$.~~

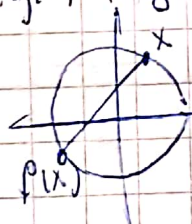
Bilu $\boxed{g_* = 0}$

$$\begin{tikzcd} \pi_1(S^1,1) \arrow[d, "0 = g_*"'] \arrow[dr, "0"] \arrow[dr, phantom, "\circlearrowright"] & \\ \pi_1(S^1,g(1)) \arrow[r, "\gamma[\beta]"'] \arrow[d, "\gamma[\sigma]"'] & \pi(S^1,1) \cong \mathbb{Z} \\ \pi(S^1,f(1)) \arrow[ur] & \end{tikzcd}$$

$\therefore$ ~~$\deg(g) \neq 0$~~ $\Rightarrow f(x) = x\, g(x)$

$\deg = 1 \sim \deg = 0$.

$\sigma: g(1) \to f(1)$



$g_* = 0 \Rightarrow \deg(g) = 0$

$\forall \varepsilon > 0, \ \forall y \ \|y\| < \eta, \ \exists x, \ \|x\| < 2k : \ \|y - \Lambda x\| < \varepsilon$

$y \in Y \Rightarrow \bar{y} = \delta \frac{y}{\|y\|}, \ \|\bar{y}\| = \delta < \eta$

$\delta < \eta \longrightarrow \frac{\delta}{\|y\|} = \frac{\|\bar{y}\|}{\|y\|}$

$\Rightarrow \exists \|x\| < 2k : \ \|\bar{y} - \Lambda x\| < \varepsilon$

$\varepsilon > \|\bar{y} - \Lambda x\| = \|\delta \frac{y}{\|y\|} - \Lambda x\| = \|$

$\Rightarrow \left\| y - \Lambda\left(\frac{\|y\| x}{\delta}\right)\right\| < \frac{\varepsilon \|y\|}{\delta}$

$\delta = \eta / 2k, \quad \left\|\frac{\|y\|x}{\delta}\right\| = \frac{\|y\|}{\delta}\|x\| = \frac{\|y\|}{\|\bar{y}\|}\|x\| < \frac{\|y\|}{\delta} 2k$

$\|y - \Lambda x\| = \left\| \frac{\|y\|}{\delta} \frac{\delta y}{\|y\|} - \Lambda x \right\| = \left\| \frac{\|y\|}{\delta}\left(\delta \frac{y}{\|y\|}\right) - \Lambda x \right\|$

$= \frac{\|y\|}{\delta} \left\| \delta \frac{y}{\|y\|} - \Lambda\left(\frac{\delta x}{\|y\|}\right)\right\| \quad \Big| \ \frac{\|y\|}{\delta} < 1$

$\cancel{\frac{\|y\|}{\delta}} \varepsilon \qquad \frac{\delta \|x\|}{\|y\|} < 2k \Rightarrow \frac{\|x\|}{2k} < \delta^{-1} \|y\|$

$\cancel{\frac{\delta x}{\|y\|} = \|x\| \frac{\delta}{\|y\|} > 1} \Rightarrow \|x\| > \delta^{-1} \|y\|$

$\Lambda : X \to Y$ lineal acotado, X, Y Banach

Λ biyectiva $\Rightarrow \exists \delta > 0$ tq : $\|\Lambda x\| \geq \delta \|x\| \quad \forall x \in X$

dem. Λ epiyectiva $\Rightarrow Y = \bigcup_{k \in \mathbb{N}} \Lambda(kU)$

Y Banach (Baire) $\Rightarrow \exists k : \operatorname{int}(\overline{\Lambda(kU)}) \neq \emptyset$

$y_0 \in \operatorname{int}(\overline{\Lambda(kU)}) \Rightarrow B(y_0, \eta) \subseteq \overline{B(y_0, \eta)} \subseteq \overline{\Lambda(kU)}$

$y_0 ; y_0 + y, \|y\| < \eta \Rightarrow y_0, y_0 + y \in B(y_0, \eta)$

$\Rightarrow \exists x_i', x_i'' \in kU$ tq $\Lambda x_i' \to y_0$

$\Lambda x_i'' \to y_0 + y$

$x_i = x_i'' - x_i'$ tq $\underset{\substack{\downarrow \\ y_0+y}}{\Lambda x_i''} - \underset{\substack{\downarrow \\ y_0}}{\Lambda x_i'} = \Lambda x_i \to y$

$\forall y \in Y, \|y\| < \eta, \forall \varepsilon > 0, \exists \underset{\|x\| < k}{x \in kU} \quad \|\Lambda x - y\| < \varepsilon$

Tomando $\delta = \frac{\eta}{2k}$: $\forall y \in Y, \exists x \in X, \|x\| < \delta^{-1}\|y\|$ and $\|\Lambda x - y\| < \varepsilon$

—o—

T.F.A : $\delta V \subset \Lambda(U) \quad \forall y \in Y, \|y\| < \delta \Rightarrow \exists \|x\| < 1$ tq $\Lambda x = y$

$\Updownarrow \qquad \Lambda$ invertible

$\Lambda^{-1}(\delta V) \subset U \Rightarrow$ dado $x \in \Lambda^{-1}(\delta V) \Rightarrow \|\Lambda x\| < \delta$

$\Rightarrow x \in U \Rightarrow \|x\| < 1$

$\Rightarrow x \in X, \|x\| \geq 1 \Rightarrow \|\Lambda x\| \geq \delta$

$\forall x \in X \Rightarrow \frac{x}{\|x\|} = \bar{x}$ tq $\|\bar{x}\| = 1 \Rightarrow \|\Lambda \bar{x}\| \geq \delta \Rightarrow \left\|\Lambda \frac{x}{\|x\|}\right\| \geq \delta$

$\Rightarrow \|\Lambda x\| \geq \delta \|x\|$ ✓

$f: X \to \mathbb{C}$ $\quad X$ $\mathbb{C}$- espacio vectorial

f f.l.a.

$f(z) = u(z) + iv(z)$, $u,v: X \to \mathbb{R}$ f.l.a.

$f(\alpha z) = \alpha f(z) = \alpha u(z) + i\alpha v(z)$

$f(iz) = u(zi) + iv(zi) = iv(z) - v(z)$

$\shortparallel$

$if(z) = iu(z) -$

$f(z) = u(z) + iv(z)$; $u,v: X \to \mathbb{R}$

$f(z+w) = u(z+w) + iv(z+w) = u(z) + u(w) + i(v(z) + v(w))$

$= (u(z) + iv(z)) + (u(w) + iv(w))$

$= f(z) + f(w)$

$$\Rightarrow \begin{cases} u(z+w) = u(z) + u(w) \\ v(z+w) = v(z) + v(w) \end{cases}$$

$\alpha \in \mathbb{R}$, $f(\alpha z) = \alpha f(z) \to \begin{cases} u(\alpha z) = \alpha u(z) \\ v(\alpha z) = \alpha v(z) \end{cases}$ $\mathbb{R}$-lineales

$f(iz) = if(z) = iu(z) - v(z)$

$\shortparallel$

$u(iz) + iv(iz) \to \begin{cases} u(iz) = -v(z) \\ u(z) = v(iz) \end{cases}$

Además

$f(x) = g(x) - i g(ix)$, $g: X \to \mathbb{R}$ $\mathbb{R}$-lineal.

Hahn-Banach : $F(x) = G(x) - i G(ix)$

es tal que $F|_S = f$ $S \leq X$ sub. esp $\mathbb{C}$-lineal.

Pd: F funcional $\mathbb{C}$-lineal continuo.

$$|F(x)| = |G(x) - iG(ix)| \leq |G(x)| + |G(ix)|$$
$$\leq c\|x\| + c\|xi\|$$
$$= 2c\|x\| \qquad \underline{c>0}$$

$\therefore$ F continuo (acotado)

$$F(x+y) = G(x+y) - iG(i(x+y))$$
$$= G(x) + G(y) - i G(ix+iy) = G(x) + G(y) - iG(ix) - iG(iy)$$
$$= (G(x) - iG(ix)) + (G(y) - iG(iy))$$
$$= F(x) + F(y).$$

Se tiene que $F(ix) = iF(x)$:

$$F(ix) = G(ix) - iG(i^2x) = i($$
$$= G(ix) - iG(-x) = iG(x) + G(ix)$$
$$= i(G(x) - iG(ix)) = iF(x)$$

$\underline{F(ix) = iF(x)}$

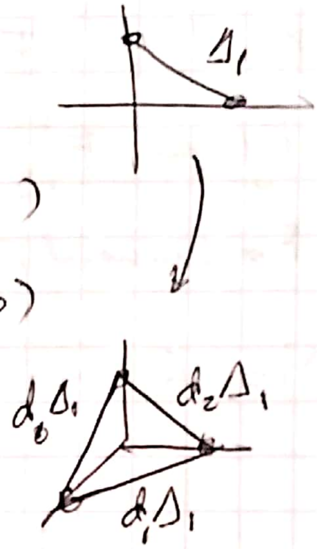
$$\Rightarrow F(\alpha x) = F((a+bi)x) = F(ax + ibx)$$
$$= F(ax) + F(ibx)$$
$$= aF(x) + ibF(x) = (a+bi)F(x)$$
$$= \alpha F(x) \qquad \therefore F \ \underline{\mathbb{C}\text{-lineal}}.$$

$d_i : \Delta_{n-1} \to \Delta_n \qquad i \in \{0, \dots, n\}$

$d_i : \Delta_1 \to \Delta_2 \qquad d_0(x_0, x_1) = (0, x_0, x_1)$

$\Delta_1 \subset \mathbb{R}^2, \quad \Delta_2 \subset \mathbb{R}^3 \qquad d_1(x_0, x_1) = (x_0, 0, x_1)$

$d_2(x_0, x_1) = (x_0, x_1, 0)$



$S_nX = \{\sigma : \Delta_n \to X \text{ continua}\}$

$S_nX \to S_{n-1}X$

$$\begin{tikzcd}
\Delta_n \arrow[d, "\sigma"'] & \Delta_{n-1} \arrow[l, "d_i"'] \arrow[d, "\tau"] \\
X & X \arrow[l]
\end{tikzcd}$$

$d_i^* : S_nX \to S_{n-1}X$

$(\sigma : \Delta_n \to X) \mapsto \sigma \circ d_i : \Delta_{n-1} \to X$

$d_i^*(\sigma) = \sigma \circ d_i$

Pd: $\partial^2 = 0$. C_nX = Grupo ab. libre generado por $S_nX = \mathbb{Z}(S_nX)$

$\partial : C_nX \to C_{n-1}X$, $\quad C_nX \xrightarrow{\partial} C_{n-1}X \xrightarrow{\partial} C_{n-2}X$

$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ d_i$, $\quad \sigma \in S_nX \Leftrightarrow \sigma : \Delta_n \to X$ continua

$d_i : \Delta_{n-1} \to \Delta_n$

$\partial(\partial(\sigma)) = \partial\left(\sum_{i=0}^{n} (-1)^i \sigma \circ d_i\right) = \sum_{i=0}^{n} (-1)^i \partial(\sigma \circ d_i)$

$\sigma \circ d_i : \Delta_{n-1} \to X$
cont.

$$\sigma \circ d_i : \Delta_{n-1} \to X$$

$$\partial(\partial(\sigma)) = \sum_{i=0}^{n} (-1)^i \partial(\sigma \circ d_i) = \sum_{i=0}^{n} (-1)^i \left(\sum_{j=0}^{n-1} (-1)^j (\sigma \circ d_i \circ d_j) \right)$$

$\sigma : \Delta_n \to X$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} (\sigma \circ d_i \circ d_j)$$

$$\sigma \circ d_i = d_i^* : S_n X \to S_{n-1} X$$

$$\partial(\partial(\sigma)) = \partial\left(\sum_{i=0}^{n} (-1)^i d_i^*(\sigma) \right) = \sum_{i=0}^{n} (-1)^i \partial(d_i^*(\sigma))$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} d_j^*(d_i^*(\sigma))$$

$$\sigma \circ d_i \circ d_j = \sigma \circ d_{j+1} \circ d_i$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} d_j^* \circ d_i^* (\sigma) = \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} \sigma \circ d_i \circ d_j$$

$d_0 \circ d_0 : \Delta_{n-2} \to \Delta_n$

$$\sigma \circ d_0 \circ d_0 = \sigma \circ d_1 \circ d_0 \qquad \sigma \circ d_1 \circ d_1 = \sigma \circ d_2 \circ d_1$$
$$\sigma \circ d_0 \circ d_1 = \sigma \circ d_2 \circ d_0$$

$d_0 \circ d_0 : \Delta_0 \to \Delta_2 \qquad d_0 \circ d_0(x_0) = d_0(0, x_0) = (0, 0, x_0)$

$d_0 \circ d_1 : \Delta_0 \to \Delta_2 \qquad d_0 \circ d_1(x_0) = d_0(x_0, 0) = (0, x_0, 0)$

$d_1 \circ d_0 : \Delta_0 \to \Delta_2 \qquad d_1 \circ d_0(x_0) = d_1(0, x_0) = (0, 0, x_0)$

$$C_2 X \xrightarrow{\partial} C_1 X \xrightarrow{\partial} C_0 X \qquad C_2 X = \langle S_2 X \rangle \quad d_i : S_2 X \to S_1 X$$

$$\partial : C_2 X \to C_1 X, \quad \partial(\sigma) = \sum_{i=0}^{2} (-1)^i \sigma \circ d_i = \sigma \circ d_0 - \sigma \circ d_1 + \sigma \circ d_2$$

$x_0 \mapsto (x_0, 0) \mapsto (x_0, 0, 0)$

$$\partial(\partial(\sigma)) = \partial_1(\sigma \circ d_0) - \partial(\sigma \circ d_1) + \partial(\sigma \circ d_2)$$
$$= \sum_{i=0}^{1} (-1)^i (\sigma \circ d_0) \circ d_i = \sigma \circ d_0 \circ d_0 - \sigma \circ d_0 \circ d_1$$

$x_0 \mapsto (x_0, 0) \mapsto (0, x_0, 0)$

$x_0 \mapsto (0, x_0) \mapsto (0, 0,$

$x_0 \mapsto (x_0, 0)$

$$\partial(\partial(\sigma)) = \cancel{\sigma \circ d_0 \circ d_0} - \cancel{\sigma \circ d_0 \circ d_1} - \cancel{\sigma \circ d_1 \circ d_0} + \cancel{\sigma \circ d_1 \circ d_1} + \cancel{\sigma \circ d_2 \circ d_0} - \cancel{\sigma \circ d_2 \circ d_1}$$

$x_0 \mapsto ($

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