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Integrales dobles

15.1 Integrales dobles sobre rectángulos

Suma de Riemann: $\sum_{i=1}^{n} f(x_i^*)\Delta x$, $\Delta x = \frac{b-a}{n} = x_i - x_{i-1}$

$f:[a,b]\to\mathbb{R}$, $x_i^* \in [x_{i-1}, x_i]$ arbitrario

definición de integral de Riemann: $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

Suma de Riemann para integrales dobles: $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$

$f:[a,b]\times[c,d]\to\mathbb{R}$, $\Delta A = \Delta x \Delta y$, $\Delta x = \frac{b-a}{m} = x_i - x_{i-1}$

$\Delta y = \frac{d-c}{n} = y_j - y_{j-1}$

$(x_{ij}^*, y_{ij}^*) \in R_{ij} = [x_{i-1}, x_i]\times[y_{j-1}, y_j]$, $R = [a,b]\times[c,d]$

definición de integral doble: $\iint_R f(x,y)dA = \lim_{n,m\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$

obs. $f(x,y)\geq 0 \ \forall (x,y)\in R \Rightarrow \iint_R f(x,y)dA$ es el volumen bajo la superficie $z=f(x,y)$ y sobre el rectángulo R.

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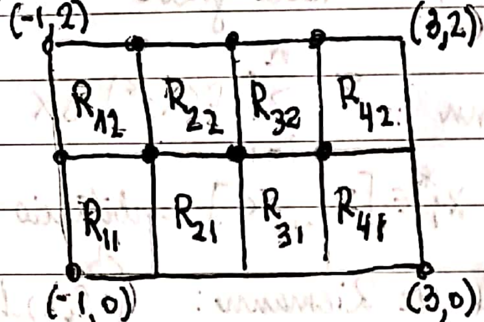
15.1 Ejercicios

Pr. 2. $R=[-1,3]\times[0,2]$. $f(x,y)=y^2-2x^2$ $\qquad R_{ij}$

Estudiar $\iint_R f(x,y)\,dA$ para $m=4$, $n=2$. $(x_{ij}^*, y_{ij}^*) =$

Desarrollo:



$$\Delta x = \frac{3+1}{4} = 1, \quad \Delta y = \frac{2-0}{2} = 1 \Rightarrow \Delta A = \Delta x \Delta y = 1$$

$$\iint_R f(x,y)\,dA \approx f(-1,1)\Delta A + f(0,1)\Delta A + f(1,1)\Delta A + f(2,1)\Delta A$$
$$+ f(-1,2)\Delta A + f(0,2)\Delta A + f(1,2)\Delta A + f(2,2)\Delta A$$
$$= -1+1-1-7+2+4+2-4 = -4$$

Teorema del valor medio para integrales

Caso 1: variable real $\qquad f_{med} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Caso 2 variables: $\qquad f_{med} = \frac{1}{A(R)}\iint_R f(x,y)\,dA$

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Propiedades de las integrales dobles.

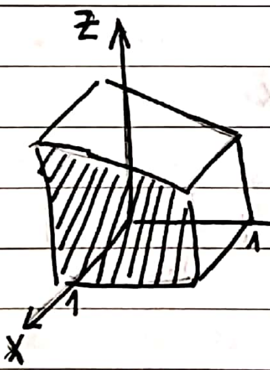
(1) $\iint_R (f(x,y)+g(x,y))\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

(2) $\iint_R c f(x,y)\,dA = c \iint_R f(x,y)\,dA$

(3) $f(x,y) \geq g(x,y) \quad \forall (x,y) \in R \Rightarrow \iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$

15.1 Ejercicios.

P.13. Calcular $\iint_R (4-2y)\,dA$, $R=[0,1]\times[0,1]$



$f(x,y) = 4-2y$

$f(1,0) = 4$, $f(1,1) = 2$

Área trapecio $= \frac{4+2}{2} \cdot 1 = 3$

$\therefore \iint_R (4-2y)\,dA = 3 \cdot 1 = 3.$

P.18. Pd. $0 \leq \iint_R \operatorname{sen}\pi x \cos\pi y\,dA \leq \frac{1}{32}$ para $R=[0,\frac{1}{4}]\times[\frac{1}{4},\frac{1}{2}]$

$\operatorname{sen} 0 = 0 \quad \operatorname{sen}\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\operatorname{sen}\pi x \cos\pi y \geq 0 \Rightarrow \iint_R \operatorname{sen}\pi x \cos\pi y \geq 0$

$\cos\frac{\pi}{4} = \frac{\sqrt{3}}{2} \quad \cos\frac{\pi}{2} = 0$

$0 \leq \iint_R \operatorname{sen}\pi x \cos\pi y\,dA \leq \iint_R \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\,dA = \frac{1}{32}.$ $\square$

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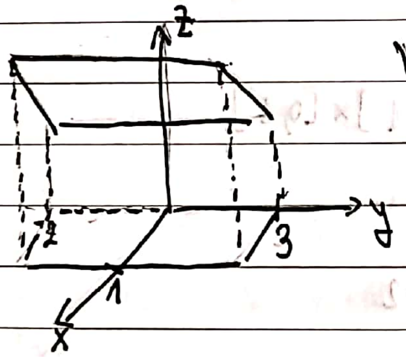
15.2 Integrales iteradas.

Teorema de Fubini: f continua en $R=[a,b]\times[c,d]$

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

Pr. 25 (Sección ejercicios 15.2) Encontrar el volumen del sólido bajo el plano $3x+2y+z=12$ y $R=[0,1]\times[-2,3]$

Desarrollo: $z=f(x,y)=-3x-2y+12$



V = volumen del sólido

$$V=\iint_R f(x,y)\,dA$$

$$=\int_0^1\int_{-2}^{3}(-3x-2y+12)\,dy\,dx$$

15.3 Integrales dobles sobre regiones generales.

f continua en $D=\{(x,y)\mid a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$

$$\iint_D f(x,y)\,dA=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

f continua en $D=\{(x,y)\mid h_1(y)\le x\le h_2(y),\ c\le y\le d\}$

$$\iint_D f(x,y)\,dA=\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

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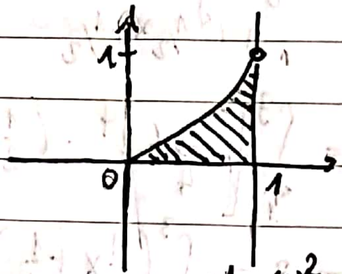
Ejercicio: Resolver problemas 13, 15 (15.3 ejercicios, pág 972)

Pr.13. $D: \begin{cases} y=0 \\ y=x^2 \\ x=1 \end{cases}$

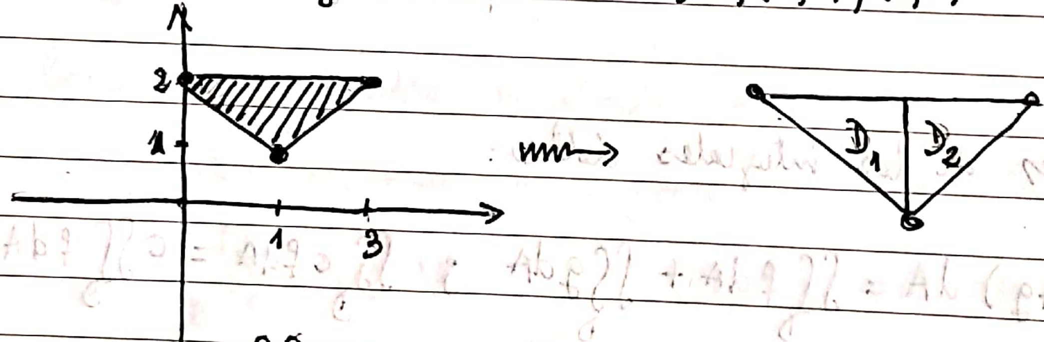


$$\iint_D x\cos y\, dA = \int_0^1\int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 x\left(\int_0^{x^2}\cos y\,dy\right)dx$$

$$= \int_0^1 x\left[\operatorname{sen} y\Big|_0^{x^2}\right]dx = \int_0^1 x\left(\operatorname{sen}x^2 - \operatorname{sen}0\right)dx$$

$$= \int_0^1 x\operatorname{sen}x^2\,dx = -\frac{1}{2}\cos x^2\Big|_0^1 = -\frac{1}{2}\left(\cos 1 + 1\right)$$

Pr. 15 D = triángulo de vértices (0,2), (1,1), (3,2)



Tenemos: $\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$; $f(x,y)=y^3$

Ecuación de la recta: $m_1 = \frac{2-1}{0-1} = -1 \Rightarrow g_1(x) = -x+2$

$m_2 = \frac{2-1}{3-1} = \frac{1}{2} \Rightarrow y-2 = \frac{1}{2}(x-3)$

$g_2(x) = \frac{1}{2}x + \frac{1}{2}$

$$\iint_{D_1} y^3\,dA = \int_0^1\int_{-x+2}^{2} y^3\,dy\,dx = \int_0^1\left(\frac{y^4}{4}\Big|_{-x+2}^{2}\right)dx = \int_0^1\left(4 - \frac{(x-2)^2}{4}\right)dx$$

$$= 4x - \frac{(x-3)^3}{12}\Big|_0^1 = 4 - \frac{2^3}{12} = 4 - \frac{8}{12} = \frac{40}{12} = \frac{8\cdot 5}{3\cdot 4} = \frac{10}{3}$$

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$$\iint_{D_2} y^3\,dA = \int_1^3 \int_{1/2x+1/2}^{2} y^3\,dy\,dx = \int_1^3 \left(\frac{y^4}{4} \Big|_{\frac{1}{2}x+\frac{1}{2}}^{2} \right) dx$$

$$= \int_1^3 \left[\frac{1}{4}\left(2^4 - \left(\frac{1}{2}x+\frac{1}{2}\right)^4\right)\right] dx$$

$$= 8 - \int_1^3 \left(\frac{1}{2}x+\frac{1}{2}\right)^4 dx = 8 - \left(\frac{2}{5}\left(\frac{1}{2}x+\frac{1}{2}\right)^5 \Big|_1^3\right)$$

$$= 8 - \frac{2}{5}\left(2^5 - 1\right)$$

Por lo tanto, $\iint_D y^3\,dA = \frac{10}{3} + 8 - \frac{2}{5}(2^5-1)$

Propiedades de las integrales dobles:

(i) $\iint_D (f+g)\,dA = \iint_D f\,dA + \iint_D g\,dA$; $\iint_D c f\,dA = c\iint_D f\,dA$

(ii) $f \geq g$ en $D \Rightarrow \iint_D f\,dA \geq \iint_D g\,dA$

(iii) $D = D_1 \cup D_2$, $\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$

(iv) $\iint_D dA = \text{area}(D)$

(v) $m \leq f(x,y) \leq M \;\forall (x,y) \in D$, entonces

$$m\,A(D) \leq \iint_D f(x,y)\,dA \leq M\,A(D)$$

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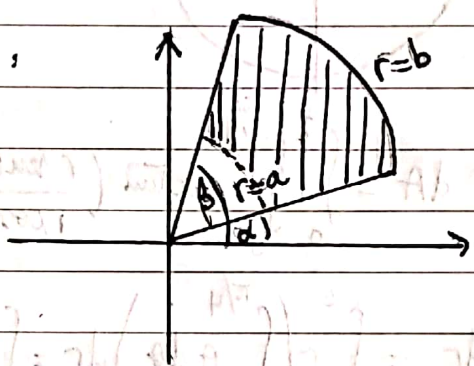
15.4 Integrales dobles en coordenadas polares

Coordenadas polares: $x = r\cos\theta$
$y = r \operatorname{sen}\theta$
$x^2+y^2=r^2$

Rectángulo en coordenadas polares:

$$R=\{(r,\theta) \mid a\leq r\leq b \ , \ \alpha\leq\theta\leq\beta\}$$

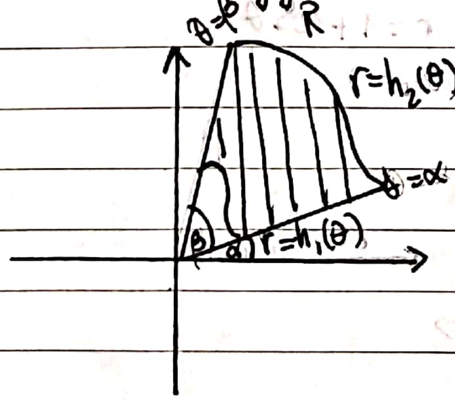
Gráficamente:



Cambio de coordenadas en integrales dobles:

~~f función continua en R polar~~, $0\leq\beta-\alpha\leq 2\pi$;

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\operatorname{sen}\theta)\,r\,dr\,d\theta$$



Para f continua en $D=\{(r,\theta) \mid h_1(\theta)\leq r\leq h_2(\theta),\ \alpha\leq\theta\leq\beta\}$,

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\operatorname{sen}\theta)\,r\,dr\,d\theta$$

Corolario: Cálculo de área en coordenadas polares:

$$A(D)=\iint_D 1\,dA = \int_\alpha^\beta \frac{1}{2}[h(\theta)]^2\,d\theta$$

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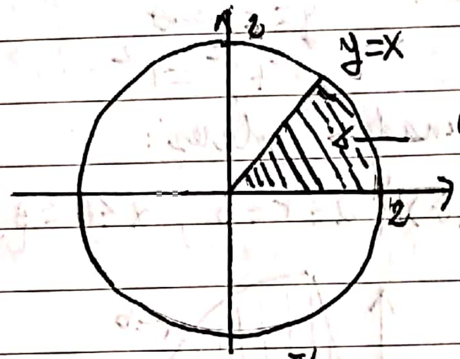
Problemas (sección 15.4 , pág 979)

P.13. $R = \{(x,y) \mid 1 \le x^2+y^2 \le 4,\ 0 \le y \le x\}$



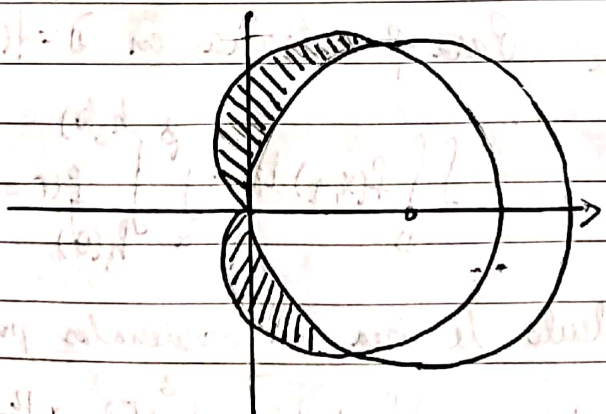
área achurada es la región R

$$\iint_R \arctan(y/x)\, dA = \int_0^2 \int_0^{\pi/4} \arctan\left(\frac{r\,\text{sen}\theta}{r\cos\theta}\right) r\, d\theta\, dr$$

$$= \int_0^2 \int_0^{\pi/4} \theta\, r\, d\theta\, dr = \int_0^2 r \left(\int_0^{\pi/4} \theta\, d\theta \right) dr = \int_0^2 r \left[\frac{\theta^2}{2} \Big|_0^{\pi/4} \right] dr$$

$$= \int_0^2 \frac{\pi^2}{32} r\, dr = \frac{\pi^2}{32} \int_0^2 r\, dr = \frac{\pi^2}{32} \left[\frac{r^2}{2} \Big|_0^2 \right] = \frac{\pi^2}{32} \cdot \frac{4}{2} = \frac{\pi^2}{16}.$$

P.18. A = área limitada por cardioide $r = 1 + \cos\theta$, exterior del círculo $r = 3\cos\theta$



Calculamos los puntos de intersección:

$r = 1 + \cos\theta = 3\cos\theta \iff \cos\theta = 1/2$

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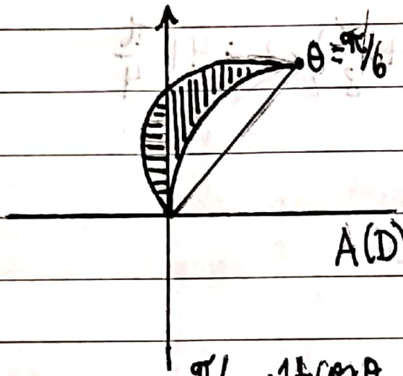
$\cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$

$r = 1+\cos\theta$ son simétricas c/r al eje X
$r = 3\cos\theta$



$A_1 = \pi/3 \leq \theta \leq \pi/2$

$A_2 = \frac{\pi}{2} \leq \theta \leq \pi$

$$A(D) = \iint_D dA = \iint_{D_1} dA + \iint_{D_2} dA$$

$$\iint_{D_1} dA = \int_{\pi/3}^{\pi/2}\int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta = \int_{\pi/3}^{\pi/2} \frac{1}{2}\left[(1+\cos\theta)^2 - (3\cos\theta)^2\right]d\theta$$

$$= \int_{\pi/3}^{\pi/2} \frac{1}{2}\left(1+\cos^2\theta + 2\cos\theta - 9\cos^2\theta\right)d\theta$$

$$= \int_{\pi/3}^{\pi/2} \frac{1}{2}d\theta + \int_{\pi/3}^{\pi/2} -4\cos^2\theta\,d\theta + \int_{\pi/3}^{\pi/2}\cos\theta\,d\theta$$

$$= \frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{3}\right) - 4\int_{\pi/3}^{\pi/2}\frac{1+\cos 2\theta}{2}d\theta + \operatorname{sen}\theta\Big|_{\pi/3}^{\pi/2} = \frac{\pi}{12} - 2\left[\theta + \frac{\operatorname{sen} 2\theta}{2}\Big|_{\pi/3}^{\pi/2}\right] + 1 - \frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{12} - 2\left[\frac{\pi}{2} - \frac{\pi}{3} - \frac{\operatorname{sen}(2\pi/3)}{2}\right] + 1 - \frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{12} - 2\left[\frac{\pi}{6} - \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\right] + 1 - \frac{\sqrt{3}}{2} = \frac{\pi}{12} - \frac{\pi}{3} + \frac{\sqrt{3}}{2} + 1 - \frac{\sqrt{3}}{2} = 1 - \frac{\pi}{4}$$

$$\iint_{D_2} dA = \int_{\pi/2}^{\pi}\int_{0}^{1+\cos\theta} r\,dr\,d\theta = \int_{\pi/2}^{\pi}\frac{(1+\cos\theta)^2}{2}d\theta = \frac{1}{2}\int_{\pi/2}^{\pi}\left(1+2\cos\theta+\cos^2\theta\right)d\theta$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + 2\operatorname{sen}\theta\Big|_{\pi/2}^{\pi} + \frac{1}{2}\int_{\pi/2}^{\pi}(1+\cos 2\theta)\,d\theta\right]$$

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$$= \frac{1}{2}\left[\frac{\pi}{2} - 2 + \frac{1}{2}\left(\theta + \frac{\operatorname{sen} 2\theta}{2}\Big|_{\pi/2}^{\pi}\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} - 2 + \frac{1}{2}\left(\pi - \frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 2 + \frac{\pi}{4}\right] = -1 + \frac{3}{8}\pi$$

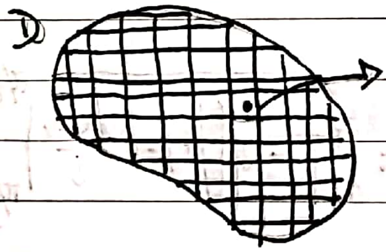
$$\therefore \quad A = 2\left[1 - \frac{\pi}{4} - 1 + \frac{3}{8}\pi\right] = \left(2 + \frac{\pi}{8}\right) \cdot 2 = 4 + \frac{\pi}{4}$$

15.5 Aplicaciones de las integrales dobles.

(i) Densidad y masa : $m = \iint_D \rho(x,y)\,dA$,

m = masa de la placa D

$\rho(x,y)$ = densidad de la placa D en el punto $(x,y) \in D$



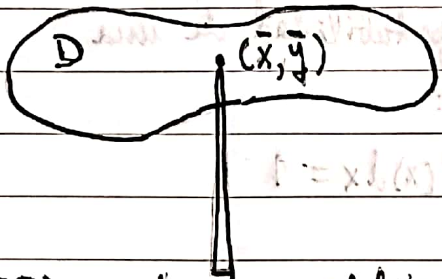
Rectángulo, densidad $\approx \rho(x,y)$

Para el caso de la electricidad , $Q = \iint_D \sigma(x,y)\,dA$

Q = carga total de la placa D

$\sigma(x,y)$ = densidad de carga de la placa en el punto $(x,y) \in D$.

(ii) momento y centro de masa.



$(\bar{x},\bar{y})$ punto de equilibrio de la lámina D

$$\begin{cases} M_x = \text{momento de la lámina eje } X \\ M_y = \text{momento de la lámina eje } Y \end{cases}$$

$$M_x = \iint_D y\,\rho(x,y)\,dA$$

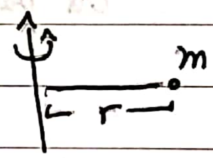
$$M_y = \iint_D x\,\rho(x,y)\,dA$$

$(\bar{x},\bar{y})$ coordenadas del centro de masa de D.

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}, \quad \text{donde } m = \iint_D \rho(x,y)\,dA$$

(iii) Momento de inercia:

Momento de inercia de una partícula : $I = mr^2$



Momento de inercia para una lámina D :

$$I_x = \iint_D y^2 \rho(x,y)\,dA$$

$$I_y = \iint_D x^2 \rho(x,y)\,dA$$

momento de inercia polar : $I_0 = I_x + I_y = \iint_D (x^2+y^2)\rho(x,y)\,dA$

$I = mr^2$, r radio de giro de la partícula de masa m.

$mR^2 = I$, R radio de giro de la lámina de masa m.

Ecuaciones de giro : $m\bar{\bar{y}}^2 = I_x$, $m\bar{\bar{x}}^2 = I_y$

$\bar{\bar{y}}$: radio de giro c/r eje X

$\bar{\bar{x}}$: radio de giro c/r eje Y.

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(iv) Probabilidades:

f función de densidad de probabilidad de una variable aleatoria X, i.e.:

$$\forall x \in \mathbb{R} : f(x) \geq 0 \quad , \quad \int_{-\infty}^{\infty} f(x)\,dx = 1$$

Probabilidad en 2 variables: $P\Big((X,Y) \in D\Big) = \iint_D f(x,y)\,dA$

donde $f(x,y) \geq 0$, $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

En particular, $P\big(a \leq X \leq b,\ c \leq Y \leq d\big) = \int_a^b \int_c^d f(x,y)\,dy\,dx$

(v) Valores esperados:

Caso real: X variable aleatoria con función de densidad f

su media μ es: $\mu = \int_{-\infty}^{\infty} x f(x)\,dx$

Caso 2-variables: X-media e Y-media

$$\mu_1 = \iint_{\mathbb{R}^2} x f(x,y)\,dA \quad , \quad \mu_2 = \iint_{\mathbb{R}^2} y f(x,y)\,dA .$$

Variable aleatoria X normalmente distribuida si

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)/(2\sigma^2)}$$

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μ = media , σ = desviación estandar

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Obs. Cuando X e Y son independientes, $f(x,y) = f_1(x) f_2(y)$.

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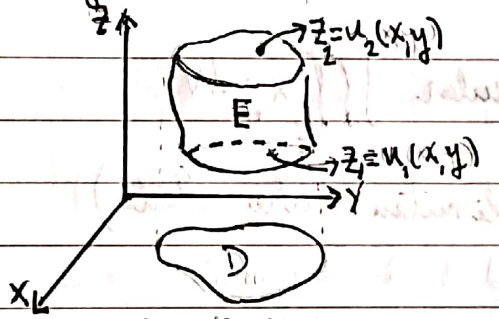
15.6 Integrales triples

Definición de integral triple:

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$$

Teorema de Fubini: $$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Integración en sólidos de tipo 1:



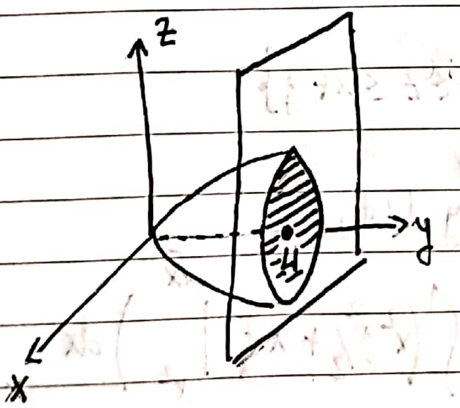
$$E = \{(x,y,z) \mid (x,y)\in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$$

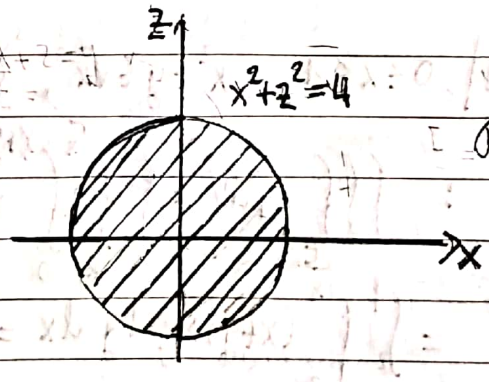
$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

D proyección del sólido
E en el plano XY

Observación: Análogamente se puede proyectar en los planos YZ (tipo 2)
o plano XZ (tipo 3)

Ejemplo (ej. 3, pág 994). E = región limitada por paraboloide $y = x^2+z^2$
y el plano $y=4$





Región de tipo 3.

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Forma alternativa de resolución

$$I = \iiint_E \sqrt{x^2+z^2}\, dV = \int_0^4 \left(\iint_{D_{x,z}} \sqrt{x^2+z^2}\, dA \right) dy$$

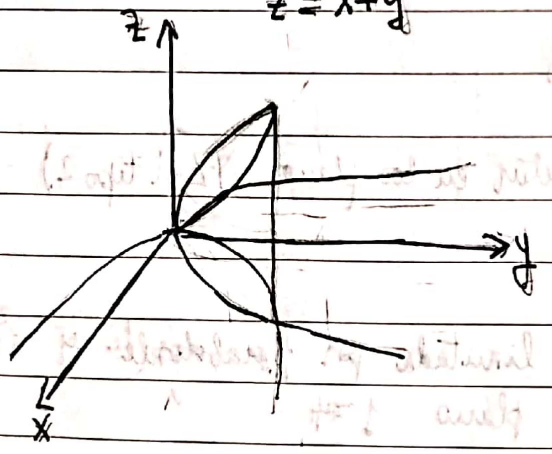
$x = r\cos\theta$
$z = r\,\text{sen}\,\theta$, se tiene que $\displaystyle\iint_{D_{x,z}} \sqrt{x^2+z^2}\, dA = \int_0^{\sqrt{y}} \int_0^{2\pi} \sqrt{r^2}\, r\, d\theta\, dr$

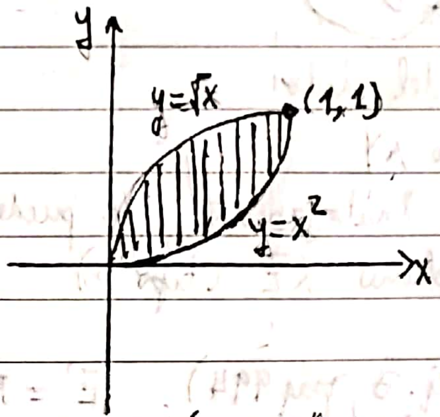
$$= \int_0^{\sqrt{y}} 2\pi r^2 dr = \frac{2\pi}{3} y^{3/2}$$

Por lo tanto, $\displaystyle I = \int_0^4 \frac{2\pi}{3} y^{3/2} dy = \frac{2\pi}{3}\cdot\frac{2}{5} y^{5/2}\Big|_0^4 = \frac{128\pi}{15}$

Pr. 14 (Sección de ejercicios 15.6). Calcular $\iiint_E xy\, dV$

$$E = \begin{cases} y = x^2 \\ x = y^2 \\ z = 0 \\ z = x+y \end{cases}$$ (superficies que limitan dicho sólido))





proyección en el plano XY

$$E = \{(x,y,z) \mid 0 \le x \le 1,\ x^2 \le y \le \sqrt{x},\ 0 \le z \le x+y\}$$

Luego se tiene: $\displaystyle\iiint_E xy\, dV = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy\, dz\, dy\, dx$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)xy\, dy\, dx = \int_0^1 \left(x^2 \frac{y^2}{2} + x\frac{y^3}{3} \Big|_{x^2}^{\sqrt{x}} \right) dx$$

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$$= \int_0^1 \left(\frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx = \frac{x^4}{8} + \frac{2}{21} x^{7/2} - \frac{x^7}{14} - \frac{x^8}{24} \Big|_0^1$$

$$= \frac{1}{8} + \frac{2}{21} - \frac{1}{14} - \frac{1}{24}$$

Aplicaciones de las integrales triples.

(i) Volumen del sólido E : $V(E) = \iiint_E dV$

(ii) Masa del sólido E de densidad $\rho(x,y,z)$, $(x,y,z) \in E$

$$m = \iiint_E \rho(x,y,z)\, dV$$

Su centro de masa es $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$

donde $M_{yz} = \iiint_E x \rho(x,y,z)\, dV$, $M_{xz} = \iiint_E y \rho(x,y,z)\, dV$

$$M_{xy} = \iiint_E z \rho(x,y,z)\, dV$$

Momento de inercia c/r a ejes coordenados:

$I_x = \iiint_E (y^2+z^2) \rho(x,y,z)\, dV$, $I_y = \iiint_E (x^2+z^2) \rho(x,y,z)\, dV$

$$I_z = \iiint_E (x^2+y^2) \rho(x,y,z)\, dV$$

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Carga eléctrica total : $Q = \iiint_E \sigma(x,y,z)\, dV$

donde $\sigma(x,y,z)$ densidad de carga en $(x,y,z) \in E$

Probabilidad de que $(X,Y,Z) \in E$

$$P((X,Y,Z) \in E) = \iiint_E f(x,y,z)\, dV$$

Para X,Y,Z variables aleatorias. f función de densidad de probabilidad

$$f(x,y,z) \geq 0 \quad , \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\, dx\, dy\, dz = 1$$

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15.7 Integrales triples en coordenadas cilíndricas

- coordenadas polares: $x = r\cos\theta$, $y = r\,\text{sen}\,\theta$

  $r^2 = x^2 + y^2$, $\tan(\theta) = y/x$

- coordenadas cilíndricas: $\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z \in \mathbb{R} \end{cases}$

  Un punto (x,y,z) en coordenadas cilíndricas tiene la forma (r, θ, z)

Evaluación de integrales triples en coordenadas cilíndricas

E región de tipo 1, $E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$

D expresable en coordenadas polares:

$$D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

$$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\,\text{sen}\,\theta)}^{u_2(r\cos\theta,\, r\,\text{sen}\,\theta)} f(r\cos\theta, r\,\text{sen}\,\theta, z)\, r\,dz\,dr\,d\theta$$

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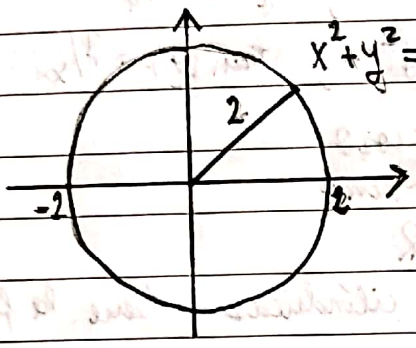
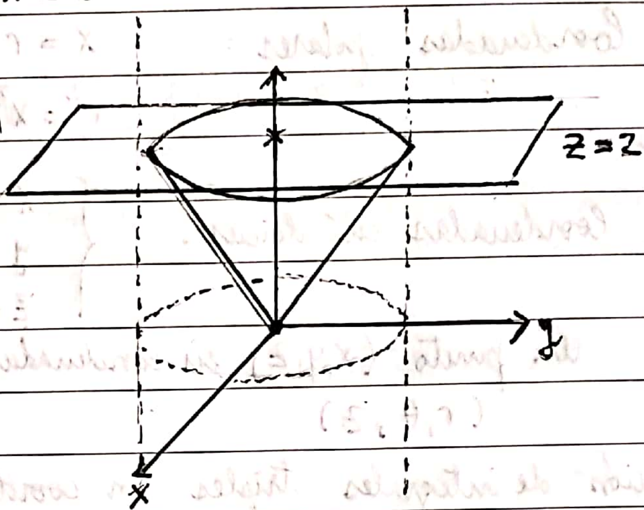
Ejemplo 4 (pag 1003). Evaluar $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$

Desarrollo. $\sqrt{x^2+y^2} \leq z \leq 2$

$-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$

$-2 \leq x \leq 2$





Coordenadas cilíndricas. $\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z \in \mathbb{R} \end{cases}$

$E = \{(r,\theta,z) \mid 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi,\ r \leq z \leq 2\}$

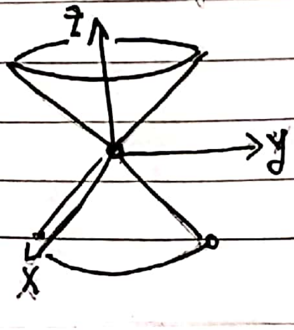
Luego, $\displaystyle\iiint_E (x^2+y^2)\,dz\,dy\,dx = \int_0^{2\pi}\int_0^{2}\int_r^{2} r^3\,dz\,dr\,d\theta = \frac{16}{5}\pi.$

15,7 Ejercicios

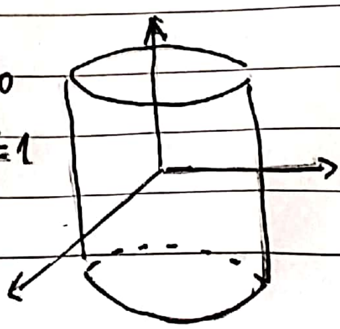
Pr. 21. Calcular $\displaystyle\iiint_E x^2\,dV$, donde $E = \begin{cases} z=0 \\ x^2+y^2=1 \\ z^2=4x^2+4y^2 \end{cases}$

$z^2 = 4x^2+4y^2 \Rightarrow z = \pm 2\sqrt{x^2+y^2}$

Cono



Cilindro $x^2+y^2=1$



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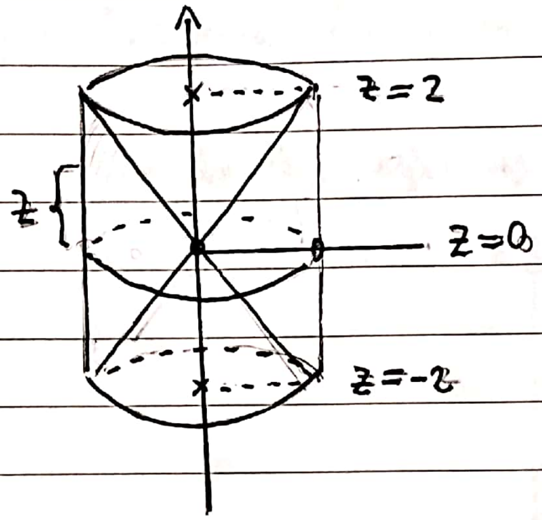
Intersección cono - cilindro : $z^2 = 4(x^2+y^2) = 4 \Rightarrow z = \pm 2$



$z^2 = 4x^2+4y^2 \Rightarrow z^2 = r^2 \Rightarrow z = r \quad , r \geq 0$

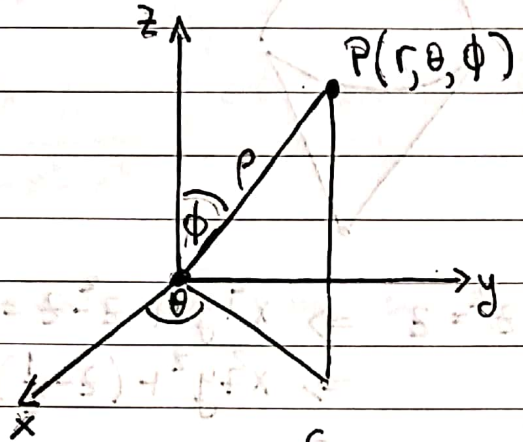
$E = \{(r,\theta,z) \mid 0 \leq \theta \leq 2\pi, \ 0 \leq r \leq 2, \ 0 \leq z \leq r\}$

$$\iiint_E x^2 dV = \int_0^{2\pi} \int_0^2 \int_0^r r^2\cos^2\theta \, (r) \, dz \, dr \, d\theta$$

Pr. 28. Calcular la integral $\displaystyle\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2+y^2} \, dz \, dy \, dx$

15.8 Integrales triples en coordenadas esféricas

Coordenadas esféricas:



$\rho \geq 0$
$0 \leq \theta \leq 2\pi$
$0 \leq \phi \leq \pi$

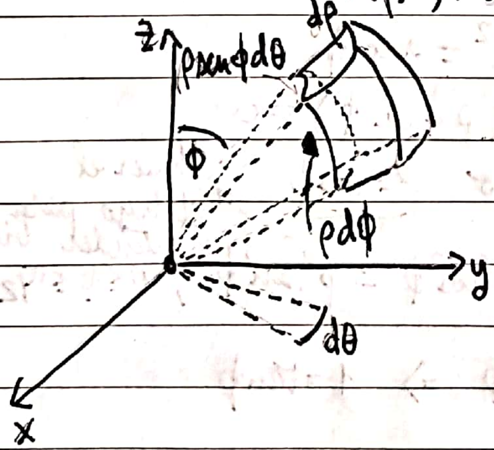
Se tienen las igualdades: $\begin{cases} x = \rho \operatorname{sen}\phi \cos\theta \\ y = \rho \operatorname{sen}\phi \operatorname{sen}\theta \\ z = \rho \cos\phi \end{cases}$

Observar que: $\rho^2 = x^2 + y^2 + z^2$

Para $E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d\}$

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b F(\rho, \theta, \phi)\,d\rho\,d\theta\,d\phi$$

donde $F(\rho, \theta, \phi) = f(\rho \operatorname{sen}\phi \cos\theta, \rho \operatorname{sen}\phi \operatorname{sen}\theta, \rho \cos\theta)\,\rho^2 \operatorname{sen}\phi$



Ejemplo 4. Encuentre el volumen del sólido E limitado por el cono $z=\sqrt{x^2+y^2}$ y la esfera $x^2+y^2+z^2=z$, como se muestra en la figura:

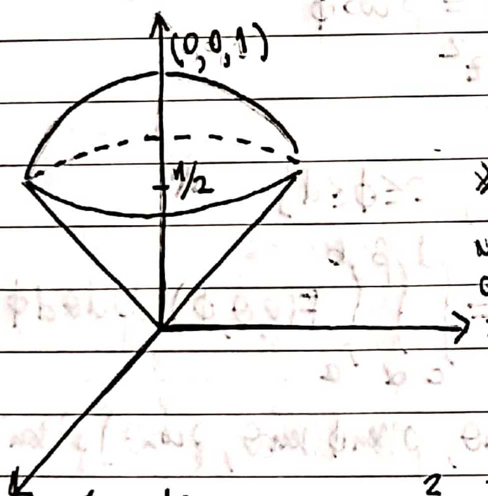
Desarrollo. $x^2+y^2+z^2=z \Rightarrow x^2+y^2+z^2-z=0$

$\Rightarrow x^2+y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4}$

para $z^2=x^2+y^2$, $x^2+y^2+z^2=z \iff 2z^2=z$

$\iff z(2z-1)=0$

$\therefore z=0,\ z=\frac{1}{2}$



$x=\rho \operatorname{sen}\phi \cos\theta$

$y=\rho \operatorname{sen}\phi \operatorname{sen}\theta$

$z=\rho\cos\phi \rightarrow \begin{cases}\rho\cos(\theta)=0\\ \rho\cos(\theta)=1/2\end{cases}$

se tiene que $x^2+y^2+z^2=z \Rightarrow \rho^2=\rho\cos\phi$

$\Rightarrow \rho(\rho-\cos\phi)=0$

$\rho=0$ o $\underline{\rho=\cos\phi}$ → hacer el dibujo para entender bien

$z=\sqrt{x^2+y^2} \Rightarrow \rho\cos\theta=\sqrt{\rho^2\operatorname{sen}^2\phi} \rightarrow \rho^2\cos^2\phi=\rho^2\operatorname{sen}^2\phi$, $0\leq\phi\leq\pi/2$

como $\rho\operatorname{sen}\phi\neq 0 \Rightarrow \rho\cos\phi=\rho\operatorname{sen}\phi \Rightarrow 1=\tan\phi$

$\phi=\frac{\pi}{4}$

$\therefore E=\{(r,\theta,\phi)\mid 0\leq\theta\leq 2\pi,\ 0\leq\phi\leq\pi/4,\ 0\leq\rho\leq\cos\phi\}$

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$$\therefore V = \iiint_E dV = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2 \operatorname{sen}\phi \, d\rho \, d\phi \, d\theta = \frac{\pi}{8}$$

Problema 25. Evaluar $\iiint_E x^2 dV$, donde E es el sólido acotado por el plano XZ y los hemisferios $y=\sqrt{9-x^2-z^2}$, $y=\sqrt{16-x^2-z^2}$

Desarrollo: Coordenadas esféricas
$x = \rho \operatorname{sen}\phi \cos\theta$
$y = \rho \operatorname{sen}\phi \operatorname{sen}\theta$
$z = \rho \cos\phi$

La región $E = \{(\rho,\theta,\phi) \mid 3 \le \rho \le 4,\ 0 \le \theta \le \pi,\ 0 \le \phi \le \pi\}$

$$\iiint_E x^2 dV = \int_0^{\pi}\int_0^{\pi}\int_3^{4} \rho^2 \operatorname{sen}^2\phi \cos^2\theta \,(\rho^2 \operatorname{sen}\phi)\, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi}\int_3^{4} \rho^4 \operatorname{sen}^3\phi \cos^2\theta \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi} \operatorname{sen}^3\phi \cos^2\theta \left[\frac{\rho^5}{5}\right]_3^4 d\phi \, d\theta = \int_0^{\pi}\int_0^{\pi} \frac{1024-243}{5}\left(\operatorname{sen}^3\phi \cos^2\theta\right) d\phi \, d\theta$$

$$= \frac{781}{5}\int_0^{\pi}\int_0^{\pi} \operatorname{sen}^3\phi \cos^2\theta \, d\phi \, d\theta = \frac{781}{5}\int_0^{\pi}\int_0^{\pi} \operatorname{sen}^3\phi \cos^2\theta \, d\theta \, d\phi$$

$$= \frac{781}{5}\left(\int_0^{\pi} \operatorname{sen}^3\phi \, d\phi\right)\left(\int_0^{\pi} \cos^2\theta \, d\theta\right)$$

$$\int_0^{\pi} \cos^2\theta \, d\theta = \int_0^{\pi} \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}\theta + \frac{1}{4}\operatorname{sen}(2\theta)\Big|_0^{\pi} = \frac{1}{2}\pi$$

$$\int \operatorname{sen}^3\phi \, d\phi = \int \operatorname{sen}^2\phi \operatorname{sen}\phi \, d\phi = -\operatorname{sen}^2\phi \cos\phi + \int 2\operatorname{sen}\phi \cos^2\phi \, d\phi$$

$$= -\operatorname{sen}^2\phi \cos\phi + 2\left[\int \operatorname{sen}\phi \, d\phi - \int \operatorname{sen}^3\phi \, d\phi\right]$$

$$= -\operatorname{sen}^3\phi \cos\phi - 2\cos\phi - 2\int \operatorname{sen}^3\phi \, d\phi$$

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$$\int \operatorname{sen}^3\phi \, d\phi = -\frac{1}{3}\left[\operatorname{sen}^2\phi\cos\phi + 2\cos\phi\right] + C$$

$$\int_0^{\pi} \operatorname{sen}^3\phi \, d\phi = -\frac{1}{3}\left[\operatorname{sen}^2\phi\cos\phi + 2\cos\phi\right]_0^{\pi} = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

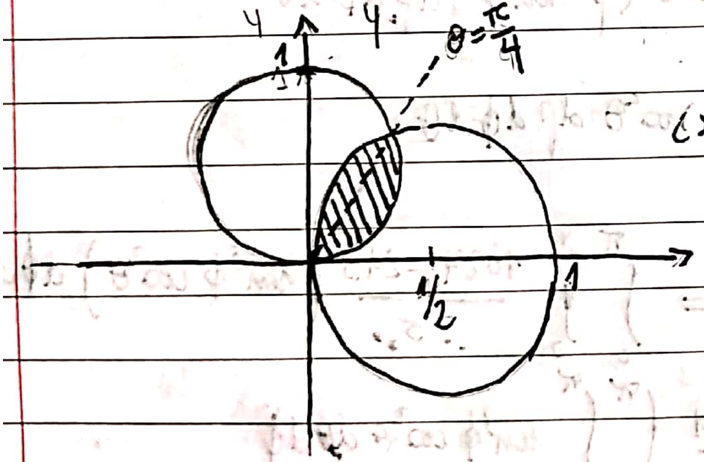
Por lo tanto: $\iiint_E x^2 dV = \frac{781}{5}\cdot\left(\frac{1}{2}\pi\right)\cdot\frac{4}{3} = \frac{1562}{15}\pi$

Problema 17. Calcular el área de la región limitada por los círculos $r = \cos\theta$, $r = \operatorname{sen}\theta$

Desarrollo. $r = \cos\theta = \operatorname{sen}\theta \implies \tan\theta = 1 \implies \theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$0 \leq \theta \leq 2\pi$

$2\pi - \frac{5\pi}{4} = \frac{3\pi}{4}$ $\therefore \tan\theta = 1 \implies \theta = -\frac{3\pi}{4}, \frac{\pi}{4}$.



$(x - \frac{1}{2})^2 + y^2 = 1/4$

$x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4}$

$x^2 + y^2 - x = 0$

$r^2 - r\cos\theta = 0 \iff r(r - \cos\theta) = 0$

$r \neq 0 \quad \therefore \quad r = \cos\theta$

$r = \operatorname{sen}\theta \implies r^2 = r\operatorname{sen}\theta \iff x^2 + y^2 = y$

$x^2 + y^2 - y = 0 \xrightarrow[\text{de cuadrados}]{\text{completación}} x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$

$A = \{(r,\theta) \mid 0 \leq \theta \leq \frac{\pi}{4},\ 0 \leq r \leq \operatorname{sen}\theta\} \cup \{(r,\theta) \mid \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq \cos\theta\}$

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Cálculo del área achurada

$$\iint_A dA = \iint_{A_1} dA + \iint_{A_2} dA$$

$$= \int_0^{\pi/4}\int_{\operatorname{sen}\theta}^{\sqrt{2}/2} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_{\cos\theta}^{\sqrt{2}/2} r\,dr\,d\theta$$

$$\int_0^{\pi/4}\int_0^{\operatorname{sen}\theta} r\,dr\,d\theta = \int_0^{\pi/4} \left.\frac{r^2}{2}\right|_0^{\operatorname{sen}\theta} d\theta = \int_0^{\pi/4} \frac{1}{2}(\operatorname{sen}^2\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/4} \operatorname{sen}^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1-\cos(2\theta)}{2}\,d\theta = \frac{1}{4}\left[\theta - \frac{1}{2}\operatorname{sen}(2\theta)\right]_0^{\pi/4} = \frac{1}{4}\left[\frac{\pi}{4} - \frac{1}{2}\right] = \frac{\pi}{16} - \frac{1}{8}$$

$$\int_{\pi/4}^{\pi/2}\int_0^{\cos\theta} r\,dr\,d\theta = \int_{\pi/4}^{\pi/2} \left.\frac{r^2}{2}\right|_0^{\cos\theta} d\theta = \int_{\pi/4}^{\pi/2} \frac{1}{2}\left[\cos^2\theta\right] d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos^2\theta\,d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{4}\left[\theta + \frac{1}{2}\operatorname{sen}(2\theta)\right]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{4}\left[\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right] = \frac{1}{4}\left[\frac{\pi}{4} - \frac{1}{2}\right] = \frac{\pi}{16} - \frac{1}{8}$$

$\therefore$ Área región achurada $= \frac{\pi}{8} - \frac{1}{4}$

$\cos 2\theta = \cos^2\theta - \operatorname{sen}^2\theta$

$= 2\cos^2\theta - 1$

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Cálculo vectorial

16.1 Campos vectoriales.

Definición. $n=2,3$, $\vec{F}: E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ es un campo vectorial.

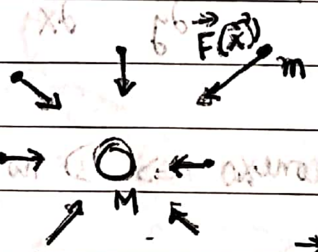
Ejemplo. El campo gradiente de $f: E \to \mathbb{R}$ es $\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}$

f se llama campo escalar.

Obs. $\vec{F}$ conservativo $\iff \vec{F} = \nabla f$, para algún f (función potencial)

Ejemplo (i) Fuerza gravitacional: $\vec{F}(\vec{x}) = \frac{mM}{|\vec{x}|^3}\vec{x}$

donde $G = 6.674 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$



Fuerza central

Campo gravitacional es $E(\vec{x}) = \frac{\vec{F}}{m}$

(ii) Fuerza eléctrica: $\vec{F}(\vec{x}) = \frac{\varepsilon q Q}{|\vec{x}|^3}\vec{x}$, $qQ > 0$ atracción, $qQ < 0$ repulsión

$\varepsilon = \frac{1}{4\pi\varepsilon_0}$ (ε_0 permeabilidad eléctrica del vacío).

P.24. $f(x,y,z) = x\cos(y/z)$. Calcular ∇f

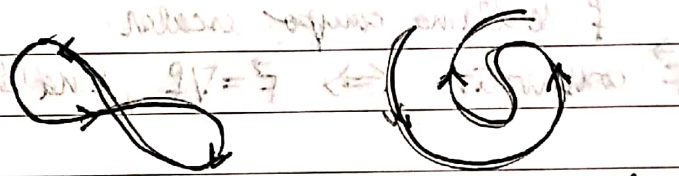
$$\frac{\partial f}{\partial x} = \cos(y/z), \quad \frac{\partial f}{\partial y} = -\frac{x}{z}\,\text{sen}(y/z), \quad \frac{\partial f}{\partial z} = \frac{xy}{z^2}\,\text{sen}(y/z)$$

$$\therefore \nabla f(x,y,z) = \cos(y/z)\,\hat{\imath} - \frac{x}{z}\,\text{sen}(y/z)\,\hat{\jmath} + \frac{xy}{z^2}\,\text{sen}(y/z)\,\hat{k}.$$

- Campos conservativos.

Curva simple: curva $\alpha : I \to \mathbb{R}^n$ sin autointersecciones.

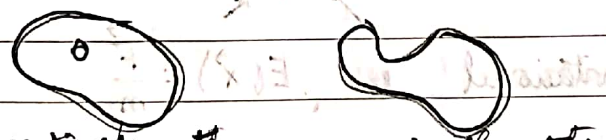
Ejemplo.



Teorema. $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ campo conservativo en D, con derivadas parciales continuas en D, $\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ en D.

Definición. $D \subseteq \mathbb{R}^n$ simplemente conexo $\Leftrightarrow$ D no tiene hoyos.

Ejemplo.



Teorema. $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ en D simplemente conexo con derivadas parciales continuas. Si

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{en } D,$$

entonces $\vec{F}$ es conservativo.

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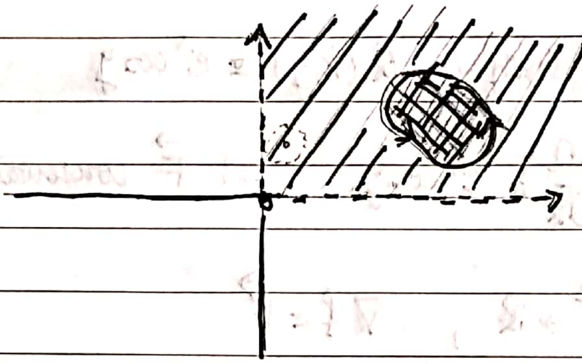
P.29. $A = \{(x,y) \mid x>0,\ y\geq 0\}$

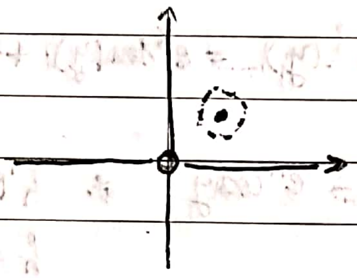


Abierto ✓

Conexo ✓

Simplemente conexo ✓

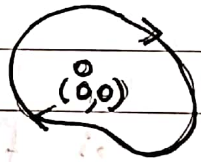
$B = \{(x,y) \mid x \neq 0\}$



Abierto ✓

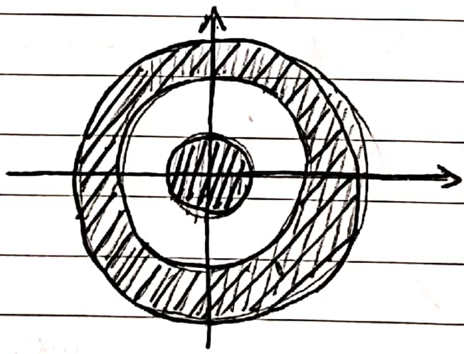
Conexo ✓

Simplemente conexo ✗



$(0,0) \notin B$

$C = \{(x,y) \mid x^2+y^2 \leq 1 \text{ or } 4 \leq x^2+y^2 \leq 9\}$



Abierto ✗

Conexo ✗

Simplemente conexo ✗

P.4. Verificar si $\vec{F}(x,y) = e^x \cos y\,\hat{i} + e^x \operatorname{sen} y\,\hat{j}$ es conservativo

Desarrollo. $P = e^x \cos y$, $Q = e^x \operatorname{sen}(y)$

$\frac{\partial P}{\partial y} = -e^x \operatorname{sen} y$, $\frac{\partial Q}{\partial x} = e^x \operatorname{sen}(y)$ $\therefore$ $\vec{F}$ no conservativo.

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P.5. Repetir problema 4 con $\vec{F}(x,y) = e^x \operatorname{sen} y\,\hat{\imath} + e^x \cos y\,\hat{\jmath}$

Desarrollo. $P(x,y) = e^x \operatorname{sen} y$, $Q(x,y) = e^x \cos y$

$$\frac{\partial P}{\partial y} = e^x \cos y, \quad \frac{\partial Q}{\partial x} = e^x \cos y \quad \therefore \vec{F} \text{ conservativo (en } \mathbb{R}^2)$$

Se tiene que $\exists f : \mathbb{R}^2 \to \mathbb{R}$, $\nabla f = \vec{F}$

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}. \qquad \frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q$$

$$f = \int P\,dx = \int e^x \operatorname{sen} y\,dx + h(y) = e^x \operatorname{sen}(y) + h(y)$$

$$\frac{\partial f}{\partial y} = e^x \cos y + h'(y) = Q = e^x \cos y \quad \therefore h'(y) = 0$$

$h = K$ constante

$$\therefore f(x,y) = e^x \operatorname{sen}(y) + K$$

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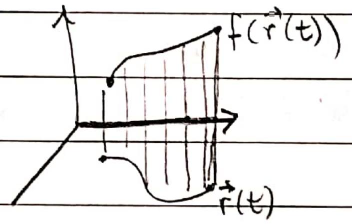
16.2 Integrales de línea para campos escalares.

$f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ $(n=2,3)$ campo escalar continuo

C curva suave definida por $\vec{r}(t)$, $\vec{r}:[a,b] \to \mathbb{R}^n$ tq $\vec{r}(t) \in D$ $\forall t$



$$\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

En particular $s(t) = \int_a^t \|\vec{r}\,'(u)\|\,du$ indica la longitud de la curva definida por $\vec{r}(t)$ en el intervalo $[a,t]$.

TFC $\Rightarrow$ $\dfrac{ds}{dt} = \|\vec{r}\,'(t)\|$

$\int_C f(x,y)\,ds$ es la integral con respecto a la longitud de arco de f.

Corolario:
$$\int_C f(x,y,z)\,dx = \int_a^b f(\vec{r}(t))\,x'(t)\,dt$$
$$\int_C f(x,y,z)\,dy = \int_a^b f(\vec{r}(t))\,y'(t)\,dt$$
$$\int_C f(x,y,z)\,dz = \int_a^b f(\vec{r}(t))\,z'(t)\,dt$$

Para campos escalares P, Q, R, se define:

$$\int_C P\,dx + Q\,dy + R\,dz = \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz$$

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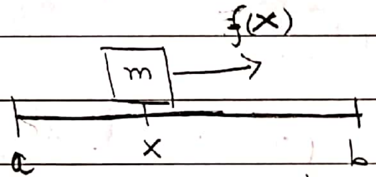
Observación. La integral con respecto a longitud de arco no cambia si cambiamos la orientación del camino C, es decir:

$$\int_{-C} f(x,y,z)\,ds = \int_C f(x,y,z)\,ds$$

Observación. La longitud de arco es $L = \int_C ds = \int_a^b \|\vec{r}\,'(t)\|\,dt$

Integrales de línea para campos vectoriales.

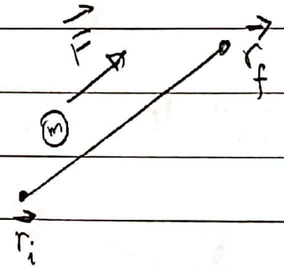
Variable real:



W : Trabajo total de mover la partícula del punto $a \to b$

$$W = \int_a^b f(x)\,dx$$

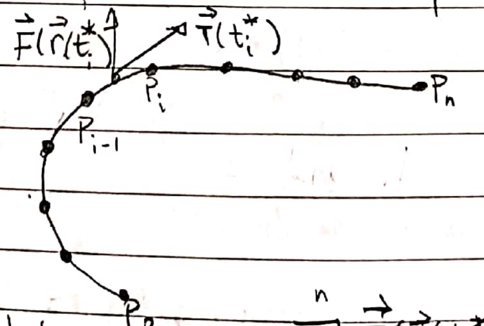
En el espacio:



$\vec{F}$ fuerza constante

$$W = \vec{F}\cdot(\vec{r}_f - \vec{r}_i)$$

Para $\vec{F} = \vec{F}(\vec{r})$ fuerza variable dependiente de la posición de la partícula.



$P_{i-1}P$ subarco de longitud Δs_i

Para W trabajo total, $W = \sum_{i=0}^{n} \vec{F}(\vec{r}(t_i^*))\cdot(\Delta s_i \vec{T}(t_i^*))$ suma de Riemann

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Para $n \to \infty$, $W = \int_C \vec{F} \cdot \vec{T}\, ds$

Como $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$, se tiene que $W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

** El trabajo es la integral de línea con respecto a la longitud de arco de la componente Tangencial de la fuerza **

Definición. $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\, ds$.

Observación. $\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$

Para $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, se tiene que $\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$

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16.2. Ejercicios.

Pr. 2. $I = \int_C xy\, ds$, $C: x = t^2, y = 2t$, $0 \leq t \leq 1$.

Desarrollo. $\vec{r}(t) = (t^2, 2t)$, $\vec{r}\,'(t) = (2t, 2)$, $\|\vec{r}\,'(t)\| = \sqrt{4t^2+4}$

$$I = \int_0^1 t^2(2t)\sqrt{4t^2+4}\, dt = \int_0^1 2t^3\sqrt{4t^2+4}\, dt \underset{(*)}{=}$$

(*) verificar en Sagemath o Python.

Pr. 4. $I = \int_C x \operatorname{sen} y\, ds$, C: segmento que une los puntos $(0,3)$, $(4,6)$

Desarrollo. $\vec{r}(t) = (0,3)(1-t) + (4,6)t$, $0 \leq t \leq 1$

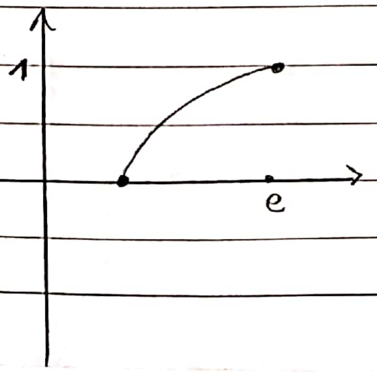
$$= (4t, 3(1-t)+6t) = (4t, 3t+3)$$

$\vec{r}\,'(t) = (4, 3)$, $\|\vec{r}\,'(t)\| = \sqrt{16+9} = 5$

Luego, $I = \int_0^1 4t \operatorname{sen}(3t+3) \cdot 5\, dt = 20\int_0^1 t \operatorname{sen}(3t+3)\, dt$

$\underset{(*)}{=}$

Pr. 6. $I = \int_C xe^y\, dx$, C: curva definida por $x = e^y$ de $(1,0)$ a $(e,1)$



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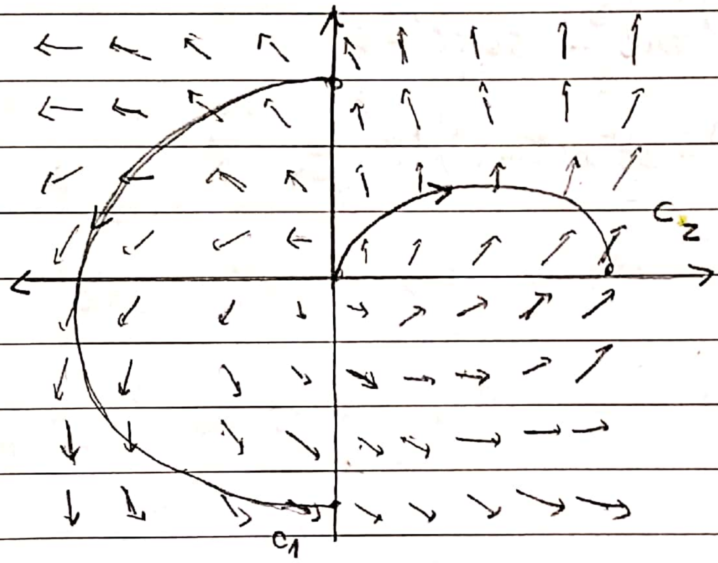
$y=t$, $x=e^t$, $t\in[0,1]$

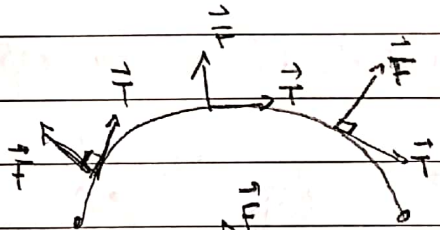
$\vec{r}(t)=(e^t,t)$. $\frac{dx}{dt}=e^t$

Luego, $\int_C xe^y\,dx = \int_0^1 e^t e^t e^t\,dt = \int_0^1 e^{3t}\,dt \underset{(*)}{=}$

P.18.

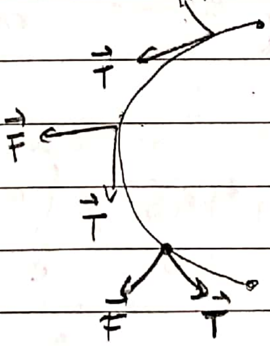


.Sobre C_2:



$\int_C \vec{F}\cdot\vec{T}\,ds = 0$

Sobre C_1:



$\int_C \vec{F}\cdot\vec{T}\,ds > 0$

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P.21 Evaluar $\int_C \vec{F}\cdot d\vec{r}$, donde C está definida por

$$\vec{r}:[0,1]\to\mathbb{R}^3,\quad \vec{r}(t)=t^3\hat{\imath}-t^2\hat{\jmath}+t\hat{k}$$

$$\vec{F}(x,y,z)=\operatorname{sen}x\,\hat{\imath}+\cos y\,\hat{\jmath}+xz\,\hat{k}$$

Desarrollo. $\vec{F}(\vec{r}(t))=\operatorname{sen}t^3\,\hat{\imath}+\cos t^2\,\hat{\jmath}+t^4\,\hat{k}$

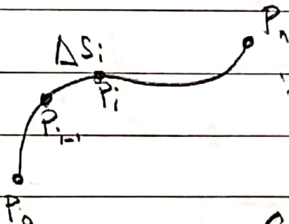
$$\vec{r}\,'(t)=3t^2\hat{\imath}-2t\hat{\jmath}+\hat{k}$$

$$\vec{F}(\vec{r}(t))\circ\vec{r}\,'(t)=3t^2\operatorname{sen}t^3-2t\cos t^2+t^4$$

$$\int_C \vec{F}\cdot d\vec{r}=\int_0^1\left(3t^2\operatorname{sen}t^3-2t^2\cos t+t^4\right)dt \underset{(*)}{=}$$

(*) Interpretación física de la integral de línea con resp. a longitud de arco.

(i) Masa de un alambre de densidad $\rho(x,y,z)$



$$m_i=\rho(x_i^*,y_i^*,z_i^*)\,\Delta S_i$$

$$m=\lim_{n\to\infty}\sum_{i=1}^{n}\rho(x_i^*,y_i^*,z_i^*)\,\Delta S_i=\int_C\rho(x,y,z)\,ds$$

(ii) Coordenadas del centro de masa, $(\bar{x},\bar{y},\bar{z})$

$$\bar{x}=\frac{1}{m}\int_C x\rho(x,y,z)\,ds,\quad \bar{y}=\frac{1}{m}\int_C y\rho(x,y,z)\,ds,\quad \bar{z}=\frac{1}{m}\int_C z\rho(x,y,z)\,ds$$

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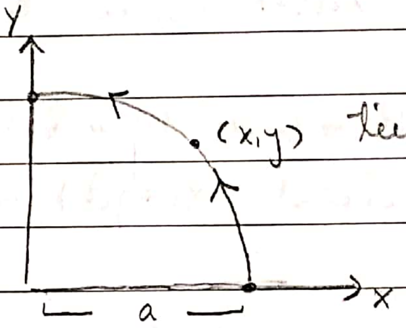
P.34.



(x,y) tiene densidad puntual $\rho(x,y)=kxy$

m : masa del alambre.

$(\bar{x},\bar{y})$: centro de masa del alambre

$$m = \int_C \rho(x,y)\,ds = \int_C \rho(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt \quad , \quad \vec{r}(t) = a(\cos t, \operatorname{sen} t)$$
$$t \in [0, \pi/2]$$

$$= \int_0^{\pi/2} a^2 k (\cos t)(\operatorname{sen} t)\cdot a\,dt = a^3 k \int_0^{\pi/2} (\cos t)(\operatorname{sen} t)\,dt$$

$$= \frac{a^3 k}{2} \int_0^{\pi/2} \operatorname{sen}(2t)\,dt = \frac{a^3 k}{4}\left(-\cos(2t)\right)\Big|_0^{\pi/2} = \frac{a^3 k}{2}$$

$$\bar{x} = \int_C x\,\rho(x,y)\,ds = \int_0^{\pi/2} a(\cos t)\,k\,(\cos t)(\operatorname{sen} t)\,a^3\,dt$$

$$= a^4 k \int_0^{\pi/2} (\cos t)^2 (\operatorname{sen} t)\,dt = -\frac{a^4 k}{3}\Big[(\cos t)^3\Big]_0^{\pi/2} = \frac{a^4 k}{3}$$

$$\bar{y} = \int_C y\,\rho(x,y)\,ds = \int_0^{\pi/2} (a \operatorname{sen} t)\,k\,a^2 (\cos t)(\operatorname{sen} t)\,a\,dt$$

$$= a^4 k \int_0^{\pi/2} (\cos t)(\operatorname{sen} t)^2\,dt = \frac{a^4 k}{3}(\operatorname{sen} t)^3\Big|_0^{\pi/2} = \frac{a^4 k}{3}$$

$\therefore$ Centro de masa : $(\bar{x},\bar{y}) = \left(\frac{2a}{3}, \frac{2a}{3}\right)$

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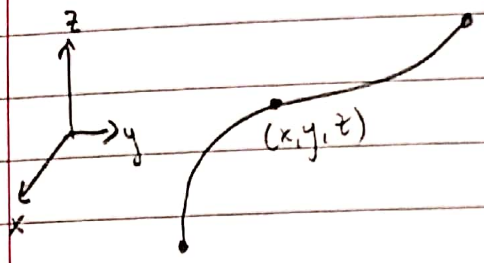
P. 38. Aplicación a la física: Momento de inercia de un alambre



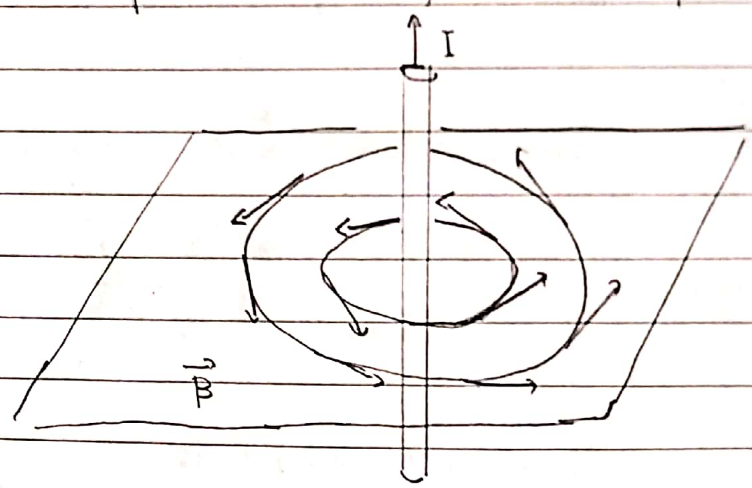
Alambre en el punto (x,y,z) tiene densidad lineal $\rho(x,y,z)$

Momento inercia eje x: $$I_x = \int_C (y^2+z^2)\,\rho(x,y,z)\,ds$$

Momento inercia eje Y: $$I_y = \int_C (x^2+z^2)\,\rho(x,y,z)\,ds$$

Momento inercia eje z: $$I_z = \int_C (x^2+y^2)\,\rho(x,y,z)\,ds$$

P. 48. Aplicación a la física: Ley de Ampere



$\vec{B}$ es el campo magnético generado por la corriente de intensidad I

Ley de Ampere: $$\int_C \vec{B}\cdot d\vec{r} = \mu_0 I$$

μ_0 : permeabilidad eléctrica en el vacío.

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Pd. Para $\|\vec{B}\| = B$, C círculo de radio r, $B = \frac{\mu_0 I}{2\pi r}$

Demostración. $\vec{r}(t) = r(\cos t, \operatorname{sen} t)$.

Se cumple que $\vec{B} = \frac{B}{d}(-y, x)$, para $x^2 + y^2 = d^2$

$$\int_C \vec{B} \cdot d\vec{r} = \int_0^{2\pi} \frac{B}{r}(-r\operatorname{sen} t, r\cos t) \cdot (-r\operatorname{sen} t, r\cos t)\, dt$$

$$= \int_0^{2\pi} \frac{B}{r}\left[r^2\operatorname{sen}^2 t + r^2\cos^2 t\right] dt = Br \int_0^{2\pi} (\operatorname{sen}^2 t + \cos^2 t)\, dt$$

$$= 2Br\pi$$

Por la ley de Ampere, $\int_C \vec{B} \cdot d\vec{r} = \mu_0 I$

Por lo tanto, $B = \frac{\mu_0 I}{2\pi r}$

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16.3 Teorema fundamental de las integrales de línea

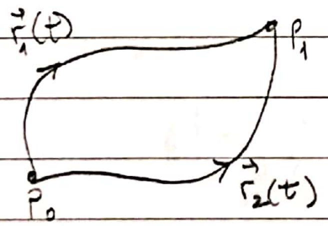
(Variable real) Teo. fundamental del Cálculo: $\int_a^b f'(x)dx = f(b) - f(a)$

Teorema. C suave definida por $\vec{r}(t)$, $a \leq t \leq b$. $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$, ∇f continua en C. Entonces

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Independencia del camino: C_1, C_2 caminos suaves con iguales extremos. Se tiene

$$\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$$



$$\vec{r}_1(a) = \vec{r}_2(a) \quad , \quad \vec{r}_1(b) = \vec{r}_2(b).$$

Corolario: $\oint_C \nabla f \cdot d\vec{r} = 0$

Teorema. $\int_C \vec{F} \cdot d\vec{r}$ es independiente del camino en D ssi $\oint_C \vec{F} \cdot d\vec{r} = 0$ para todo camino $C \subseteq D$ cerrado.

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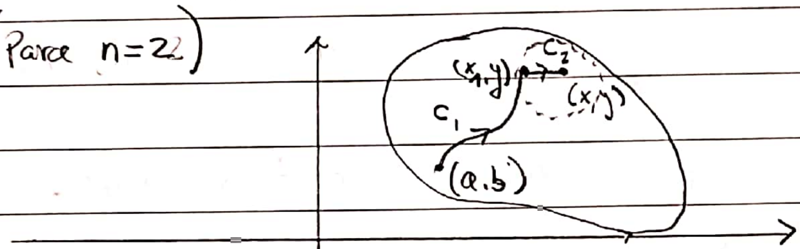
Recordatorio: Conexo, abierto = arco-conexo

$\vec{F}: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ continua, D abierto conexo

Teorema. $\int_C \vec{F} \cdot d\vec{r}$ independiente del camino ssi $\vec{F} = \nabla f$.

Demostración. (Para $n=2$)



Definimos $f(x,y) = \int_{(a,b)}^{(x,y)} \vec{F} \cdot d\vec{r}$

Propiedades: (i) $f(x,y) = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$

(ii) $\frac{\partial}{\partial x} f(x,y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \vec{F} \cdot d\vec{r}$

$$\frac{\partial}{\partial x} \int_{C_2} \vec{F} \cdot d\vec{r} = \frac{\partial}{\partial x} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial x} \int_{C_2} P\,dx + \cancelto{0}{\frac{\partial}{\partial x} \int_{C_2} Q\,dy}$$

$$= \frac{\partial}{\partial x} \int_{x_1}^{x} P(t,y)\,dt = P(x,y)$$

$$\therefore \frac{\partial f}{\partial x} = P(x,y)$$

Análogamente se puede construir un camino para el cual $\frac{\partial f}{\partial y} = Q(x,y)$. □

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