(3) (Métrica en $\mathcal{C}(\Omega)$, $\Omega \subseteq \mathbb{C}$ abierto).

$$d(f,g) := \sum_{n=1}^{\infty} \min\left\{\frac{1}{2^n}, \|f-g\|_{K_n}\right\} < \infty.$$

$$d(f,g) := \sum_{n=1}^{\infty} \min\left\{\frac{1}{2^n}, \|f-g\|_{K_n}\right\} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} - \frac{1}{2}$$

$$= \frac{1}{1-\frac{1}{2}} - \frac{1}{2} = 2 - \frac{1}{2} = \frac{3}{2} < \infty.$$

- $d(f,g) \geq 0 \quad \forall f,g$ ✓
- $f=g$ : $d(f,\cancel{f}) := \sum_{n=1}^{\infty} \min\left\{\frac{1}{2^n}, \|f-\cancel{f}\|_{K_n}\right\} = \sum_{n=1}^{\infty} \min\left\{\frac{1}{2^n}, 0\right\} = 0$

$d(f,g) = 0 \Rightarrow \min\left\{\frac{1}{2^n}, \|f-g\|_{K_n}\right\} = 0 \Rightarrow \|f-g\|_{K_n} = 0 \ \forall n.$

$f = g$ en $K_n$ $\forall n.$ $\Rightarrow$ $f = g$ en $\bigcup_{n\in\mathbb{N}} K_n = \Omega.$

$d(f,g) = \sum_{n=1}^{\infty} \min\left\{\frac{1}{2^n}, \|f-g\|_{K_n}\right\}$, $h \in C(\Omega)$

$\forall n \in \mathbb{N}$ : $\|f-g\|_{K_n} \leq \|f-h\|_{K_n} + \|g-h\|_{K_n}$

$$\min\left\{1/2^n, \|f-g\|_{K_n}\right\} \leq \min\left\{\frac{1}{2^n}, \|f-h\|_{K_n} + \|g-h\|_{K_n}\right\}$$

$$\leq \min\left\{\frac{1}{2^n}, \|f-h\|_{K_n}\right\} + \min\left\{\frac{1}{2^n}, \|g-h\|_{K_n}\right\} \qquad \forall n \in \mathbb{N}.$$

$$\Rightarrow \sum_{n=1}^{\infty} \min\left\{\frac{1}{2^n}, \|f-g\|_{K_n}\right\} \leq \sum_{n=1}^{\infty} \min\left\{\frac{1}{2^n}, \|f-h\|_{K_n}\right\} + \sum_{n=1}^{\infty} \min\left\{\frac{1}{2^n}, \|g-h\right.$$

$$\therefore \quad d(f,g) \leq d(f,h) + d(g,h)$$

- $d(f,g) = d(g,f)$ ✓

(4) $(f_n)_{n\in\mathbb{N}} \in C(\Omega)$ , $f\in C(\Omega)$

$\begin{cases} (1) & d(f_n,f) \xrightarrow{n\to\infty} 0 \\ (2) & f_n \to f \text{ uniformemente en subconjuntos compactos de } \Omega. \end{cases}$

(1) ⇒ (2) $K\subseteq\Omega$ compacto. $\Rightarrow \exists l\in\mathbb{N} : K\subseteq K_l \subset K_{l+1}\subset\cdots$

$$\lim_{n\to\infty} d(f_n,f)=0 \iff \lim_{n\to 0} \sum_{m=1}^{\infty} \min\left\{\frac{1}{2^m}, \|f_n-f\|_{K_m}\right\}=0$$

$$\iff \forall\varepsilon>0,\ \exists N\in\mathbb{N},\ \forall n\geq N : \sum_{m=1}^{\infty} \min\left\{\frac{1}{2^m}, \|f_n-f\|_{K_m}\right\} < \varepsilon$$

$$\Rightarrow \forall m\in\mathbb{N}: \min\left\{\frac{1}{2^m}, \|f_n-f\|_{K_m}\right\}<\varepsilon \quad \forall n\geq N$$

———o———

Basta demostrar que $f_n\to f$ uniformemente en $K_l$ $\forall l\in\mathbb{N}$

$$\forall\varepsilon>0,\ \exists N\in\mathbb{N},\ \forall n\geq 0 : \sum_{m=1}^{\infty} \min\left\{\frac{1}{2^m}, \|f_n-f\|_{K_m}\right\} < \varepsilon$$

$$m=l : \underbrace{\min\left\{\frac{1}{2^l}, \|f_n-f\|_{K_l}\right\}}_{=\|f_n-f\|_{K_l}} < \varepsilon$$

$$\forall m\in\mathbb{N} : \min\left\{\frac{1}{2^m}, \|f_n-f\|_{K_m}\right\} \leq \sum_{m=1}^{\infty} \min\left\{\frac{1}{2^m}, \|f_n-f\|_{K_m}\right\} < \varepsilon$$

Como $\lim_{n\to\infty} d(f_n,f)=0 \Rightarrow \lim_{n\to\infty} \min\left\{\frac{1}{2^m}, \|f_n-f\|_{K_m}\right\}=0\ \forall m\in\mathbb{N}$

$$\therefore \lim_{n\to\infty} \|f_n-f\|_{K_m}=0.$$

Como $\forall K\subseteq\Omega$ $\exists m\in\mathbb{N}$ tq $K\subseteq K_m$, $f_n\to f$ unif en todo compacto de $\Omega$.

(2) ⇒ (1) $f_n\to f$ unif en $K\subseteq\Omega$ $\forall K$ compacto $\Rightarrow f_n\to f$ unif en $K_l$ $\forall l\in\mathbb{N}$.

$$\sum_{m=1}^{\infty} \min\left\{\frac{1}{2^m}, \|f_n-f\|_{K_m}\right\} \leq \sum_{m=1}^{\infty} \|f_n-f\|_{K_m} \leq \sum_{m=1}^{\infty} \frac{1}{2^m} = \frac{3}{2}$$

$\forall\varepsilon>0$ $\exists m_0 = m_0 : \sum_{m\geq m_0} \|f_n-f\|_{K_m} < \varepsilon/2$

Trabajar sobre la suma finita $\|f_n-f\|_{K_1}+\cdots+\|f_n-f\|_{K_{m_0}} < \varepsilon/2$ para $n$ suficientemente grande.

$\Omega \subseteq \mathbb{C}$ abierto

$C(\Omega) = \{ f : \Omega \to \mathbb{C}$ continua $\}$. $\exists$ métrica en $C(\Omega)$ tq $d(f_n, f) \to 0$

$\Leftrightarrow \forall K \subseteq \Omega$ compacto, $f_n \to f$ uniformemente en $K$.

<u>Teo. de Arzelá - Ascoli</u>. (No demostrado en este curso)

clausura compacta

$\mathcal{F} \subset C(\Omega)$ es relativamente compacto $\Leftrightarrow$ (1) y (2)

(1) $\mathcal{F}$ es equicontinuo: $\forall \varepsilon > 0, \exists \delta > 0$ tq $\left.\begin{array}{l} z_1, z_2 \in \Omega \quad f \in \mathcal{F} \\ |z_1 - z_2| < \delta \end{array}\right\} \Rightarrow |f(z_1) - f(z_2)| < \varepsilon$

(2) $\mathcal{F}$ es puntualmente acotado: $\forall z \in \Omega$, el conjunto $\mathcal{F}(z) := \{ f(z) \mid f \in \mathcal{F} \}$ es acotado.

$H(\Omega) \subset C(\Omega)$

" $\{$ funciones holomorfas en $\Omega \}$

<u>Teo. de Weierstrass</u>. $H(\Omega)$ es cerrado.

<u>Teorema de Montel</u>. $\mathcal{F} \subseteq H(\Omega)$ es relativamente compacto $\Leftrightarrow \mathcal{F}$ es localmente acotado,

es decir: $\begin{cases} \forall z \in \Omega \ \exists V \text{ vecindad de } z \text{ contenida en } \Omega, \\ \quad \exists C > 0 \text{ tq} \\ \forall f \in \mathcal{F} \ \forall w \in V, \ |f(w)| \leq C \end{cases}$

<u>Proposición</u>. $\mathcal{F} \subset C(\Omega)$ es localmente acotado $\Leftrightarrow \forall K \subset \Omega$ compacto, $\sup_{f \in \mathcal{F}} \|f\|_K < \infty$. ✓


o sea,
$\forall K \subset \Omega \ \exists C > 0$ tq
$\forall f \in \mathcal{F} \ \forall w \in K : |f(w)| \leq C$

<u>dem</u>. ($\Leftarrow$) trivial

($\Rightarrow$) Usar recubrimiento finito del compacto.

Obs. En el teorema de Montel, $\mathcal{F} \subset H(\Omega)$ relativamente compacto, $\mathcal{F}$ es llamada una familia normal de funciones.

(dem del teo de Montel)

($\Rightarrow$) Si $\mathcal{F} \subset H(\Omega)$ relativamente compacto, entonces por Arzelá-Ascoli, valen (1) y (2)

pero, (1) y (2) $\Rightarrow$ $\mathcal{F}$ localmente acotada.

($\Leftarrow$) Supongamos que $\mathcal{F}$ es localmente acotado ($\subset H(\Omega)$) Hay que probar (1) y (2) (Arzelá-Ascoli)

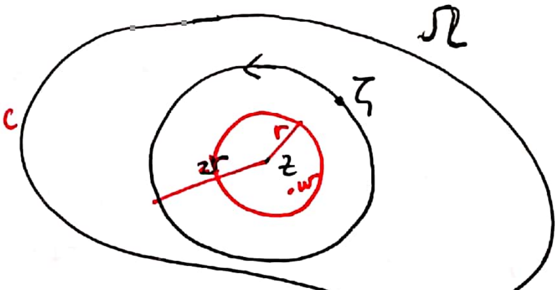
Pero (2) es trivial.

Queda verificar que $\mathcal{F}$ es equicontinuo.

Obs: Estudiar bien la demo del teorema de Montel

Sea $z \in \Omega$, fijemos $r>0$ tal que $\overline{B(z,2r)} \subset \Omega$

$\exists C>0$ tq $f \in \mathcal{F}$
$\zeta \in \overline{B(z,2r)} \Rightarrow |f(\zeta)| \leq C$




Supongamos que $w \in \overline{B(z,r)}$

$$|f(z)-f(w)| = \left| \frac{1}{2\pi i} \int_{|\zeta-z|=2r} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{|\zeta-z|=2r} \frac{f(\zeta)}{\zeta - w} d\zeta \right|$$

$$= \left| \frac{1}{2\pi i} \int_{|\zeta-z|=2r} \underbrace{\frac{(w-z)}{(\zeta-z)(\zeta-w)}}_{|\cdot| \leq \frac{|w-z|}{2r \cdot r}} \underbrace{f(\zeta)}_{|\cdot| \leq C} d\zeta \right| \leq \frac{1}{2\pi} 2\pi \, 2r \, \frac{|w-z|}{2r^2} \cdot C$$

$$= C'|w-z| < \varepsilon$$

$|w-z| < \varepsilon / C'$

$|\zeta - z| = 2r$
$|\zeta - w| \geq r$

Ahora dado $\varepsilon > 0$, sea $\delta = \min\{\frac{\varepsilon}{C'}, r\}$

Entonces $\left.\begin{array}{l} f \in \mathcal{F} \\ w \in B(z,\delta) \end{array}\right\} \rightarrow |f(w)-f(z)| < \varepsilon$

Esto muestra que $\mathcal{F}$ es equicontinuo. ∎

**Ejemplo.** En $\mathbb{R}$ , $f_n(x) = \text{sen}(nx)$

$\mathcal{F} = \{f_1, f_2, \dots\}$ → localmente acotado → puntualmente acotado → No es equicontinuo

→ No es relativamente compacto.

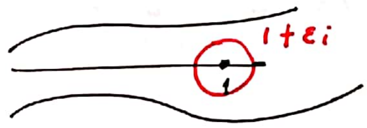
* Montel no tiene análogo real.

$$\text{sen}\, z = \frac{e^{iz} - e^{-iz}}{2i}$$

$f_n(z) = \text{sen}(nz)$ | $e^{-i(1+i\varepsilon)n}$ $e^{\varepsilon n}$

$f_n \in H(\Omega)$

$\Omega \supset$ eje real

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**Teorema de convergencia de Vitali.**

$(f_n)$ localmente acotada en $H(\Omega)$
$\forall z \in \Omega$ $\exists r > 0$, $\exists c > 0$ tq $B(z,r) \subset \Omega$ y $\forall n \in \mathbb{N}$
$\|f_n\|_{B(z,r)} \leq c$

$\Omega \subseteq \mathbb{C}$ abierto conexo. $(f_n)_n$ sucesión localmente acotada en $H(\Omega)$

$E := \{z \in \Omega ;\ \exists \lim_{n\to\infty} f_n(z)\}$ → todos los límites puntuales. Supongamos que $E$ tiene punto de acumulación en $\Omega$, entonces $E = \Omega$ y $\exists f \in H(\Omega)$ tal que $f_n \to f$ uniformemente en compactos.

Vitali no tiene análogo real:

$f_n(x) = \sin(2^n x)$

$E \supset \left\{ \frac{p}{2^n}\pi \ / \ \begin{matrix} p \in \mathbb{Z} \\ k \in \mathbb{N} \end{matrix} \right\}$ denso. $\text{sen}\left(2^n \frac{\pi}{2^k}\right) = 0$ suficientemente grande.

**obs.** En espacio métrico compacto cualquiera, sucesión $(x_n)$ convergente $\Longleftrightarrow$ toda subsucesión $(x_{n_i})$ es convergente y tienen el mismo límite.

<u>dem. del teo. de Vitali</u>. Supongamos que $(f_n)$ no es convergente en el espacio métrico $(H(\Omega), d)$. Por Montel, $\{f_n / n \in \mathbb{N}\}$ es relativamente compacto. Por lo tanto, $\exists$ dos subsucesiones convergentes pero con límites distintos.

$$f_{n_i} \xrightarrow{i\to\infty} f \in H(\Omega)$$

pendiente!

$$f_{m_j} \xrightarrow[j\to\infty]{} g \in H(\Omega)$$

con $f = g$ en $E$

$$f(z) = \lim_{n\to\infty} f_n(z) = g(z)$$

//

(1) $\mathcal{F} \subseteq C(\Omega)$ localmente acotado $\Leftrightarrow$ $\forall K \subset \Omega$ compacto, $\sup_{f\in\mathcal{F}} \|f\|_K < \infty$.

($\Leftarrow$) $z_0 \in \Omega \Rightarrow \exists V_{z_0} \subset \Omega$ vecindad de $z_0$, $V_{z_0} = B(z_0, r)$ tq $\overline{V}_{z_0} \in \Omega$. como $\sup_{f\in\mathcal{F}} \|f\|_{\overline{V}_{z_0}} < \infty$

$$\Rightarrow \forall f \in \mathcal{F} : \|f\|_{\overline{V}_{z_0}} \leq \sup_{f\in\mathcal{F}} \|f\|_{\overline{V}_{z_0}} < \infty$$

$\forall f \in \mathcal{F}$

$$\|f\|_{\overline{V}_{z_0}} = \sup\{|f(w)| \,/\, w \in \overline{V}_{z_0}\} \leq \sup_{f\in\mathcal{F}} \|f\|_{\overline{V}_{z_0}} < \infty$$

$$\Rightarrow \begin{matrix}\forall f \in \mathcal{F} \\ \forall w \in \overline{V}_{z_0}\end{matrix} : |f(w)| \leq \sup_{f\in\mathcal{F}} \|f\|_{\overline{V}_{z_0}} < \infty$$

~~como $\forall w \in \overline{V}_{z_0}$ ... $\sup \|f\|_{\overline{V}_{z_0}}$ ...~~ $V_{z_0} \subset \overline{V}_{z_0}$

$$\Rightarrow \begin{matrix}\forall f \in \mathcal{F} \\ \forall w \in V_{z_0}\end{matrix} : |f(w)| \leq \sup_{f\in\mathcal{F}} \|f\|_{\overline{V}_{z_0}} < \infty$$

Tomando $C = \sup_{f\in\mathcal{F}} \|f\|_{\overline{V}_{z_0}}$ se cumple la condición.

($\Rightarrow$) $K \subset \Omega$ compacto.

$\mathcal{F} \subset C(\Omega)$ localmente acotado $\Rightarrow \forall w \in K$, $\exists r_w > 0$ tq $B(w, r_w) \subset \Omega$, $\exists C_w > 0$ $\forall \zeta \in B(w, r_w)$ $\forall f \in \mathcal{F}$, $|f(\zeta)| \leq C_w$

$$\therefore K \subset \bigcup B(w, r_w)$$

$K$ compacto $\Rightarrow K \subset B(w_1, r_{w_1}) \cup \cdots \cup B(w_k, r_{w_k})$ $(k \in \mathbb{N})$

$\forall f \in \mathcal{F}$ $\forall \zeta \in B(w, r_w)$ ; $|f(\zeta)| \leq C_w$

$C := \max\{C_{w_j} \,/\, j = 1, \ldots, k\} \Rightarrow \forall \zeta \in K$ ; $\forall f \in \mathcal{F} : |f(\zeta)| \leq C$

$$\therefore \forall f \in \mathcal{F} : \|f\|_K \leq C$$

$$\therefore \sup_{f\in\mathcal{F}} \|f\|_K \leq C < \infty$$

(Teo. de Montel)

(2) $\mathcal{F} \subset H(\Omega)$ equicontinua y puntualmente acotada.

Pd: $\mathcal{F}$ localmente acotada

dem. $z_0 \in \Omega$. $\exists M > 0$ tq $\forall f \in \mathcal{F} : |f(z_0)| \leq M$

$\exists \delta > 0$ tq $\forall z \in B(z_0, \delta)$ $\forall f \in \mathcal{F}$ : $|f(z) - f(z_0)| < 1$

Tomando $\delta' = \frac{\delta}{2}$, $\overline{B(z_0, \delta')} \subset B(z_0, \delta) \subset \Omega$

$\forall z \in \overline{B(z_0, \delta')}$ $\forall f \in \mathcal{F}$ : $|f(z) - f(z_0)| < 1$

$|f(z)| < |f(z_0)| + 1 \leq M + 1$

$M' = M + 1 \Rightarrow \forall f \in \mathcal{F}$ : $\sup_{\overline{B(z_0, \delta')}} |f(z)| = \|f\|_{\overline{B(z_0, \delta')}} \leq M' < \infty$

$\therefore \sup_{f \in \mathcal{F}} \|f\|_{\overline{B(z_0, \delta')}} \leq M'$ //

Obs. Demo vale para $\mathcal{F} \subset C(\Omega)$.

- $\mathcal{F} \subset H(\Omega)$. $\mathcal{F}$ localmente acotada $\Rightarrow$ $\mathcal{F}$ puntualmente acotada.

$z \in \Omega$. $\mathcal{F}$ loc. acotada $\Rightarrow$ $\exists r > 0$, $\exists M > 0$, $\overline{B(z, r)} \subset \Omega$,

$\|f\|_{\overline{B(z,r)}} \leq M$ $\forall f \in \mathcal{F}$

$\|f\|_{\overline{B(z,r)}} \leq M$ $\forall f \in \mathcal{F}$ $\Rightarrow$ $\forall f \in \mathcal{F}$ $\forall w \in \overline{B(z, r)}$ : $|f(w)| \leq M$

$(w = z)$ $\forall f \in \mathcal{F}$ : $|f(z)| \leq M$

$\therefore \mathcal{F}(z)$ acotado.

$\therefore \mathcal{F}$ puntualmente acotado.

(3) (teo. de convergencia de Vitali).

$\Omega \subseteq \mathbb{C}$ abierto conexo. $(f_n)_n \subseteq H(\Omega)$ loc. acotada.

$E := \{ z \in \Omega \,/\, \lim_n f_n(z) \in \mathbb{C} \}$.

Pd: $E$ tiene ptos de acumulación $\Rightarrow$ $E = \Omega$ y $\exists f \in H(\Omega)$ tal que $d(f_n, f) \xrightarrow{n\to\infty} 0$ ($d$ métrica en $C(\Omega)$)

dem. $(z_n)_n$ suc. en $E$ tq $z_n \xrightarrow{n\to\infty} z$

$\forall n \in \mathbb{N}$ $\exists r_n > 0$, $\exists c_n > 0$ tq $B(z_n, r_n) \subset \Omega$, $\forall k \in \mathbb{N}$ : $\| f_k \|_{\overline{B(z_n, r_n)}} \leq c_n$

$\| f_k \|_{\overline{B(z_n, r_n)}} \leq c_n \iff \forall k \in \mathbb{N}$ $\forall w \in \overline{B(z_n, r_n)}$ : $| f_k(w) | \leq c_n$

$\{f_n\}_n$ localmente acotada $\underset{\text{Montel}}{\iff}$ $\{f_n\}_n$ rel. compacto $\underset{\text{A.A}}{\iff}$ $\{f_n\}_n$ equicontinuo pte. acotado.

$\{f_n\}_n$ equicontinuo: $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall z, z' \in \Omega$ $\forall n \in \mathbb{N}$
$|z - z'| < \delta \Rightarrow |f_n(z) - f_n(z')| < \varepsilon$

~~[illegible]~~

$\forall k \in \mathbb{N}$: $f_k(z_n) \xrightarrow{n\to\infty} f_k(z)$, $f_k(z_n) \xrightarrow{k\to\infty} c_n$ | $\exists N \in \mathbb{N}$, $\forall n \geq N$ : $|z_n - z| < \delta$
$\Rightarrow \forall k \in \mathbb{N}$ : $|f_k(z_n) - f_k(z)| < \varepsilon$

$(c_n)_{n \in \mathbb{N}}$, $c_n = \lim_k f_k(z_n)$.

$|c_n - c_m| = | c_n - f_k(z_n) + f_k(z_n) - c_m |$

$\leq | c_n - f_k(z_n) | + | f_k(z_n) - c_m | \leq | c_n - f_k(z_n) | + | f_k(z_n) - f_k(z_m) | + | f_k(z_m) - c_m |$

$\forall \varepsilon > 0$, $\exists k_1, k_2 \in \mathbb{N}$ tq $\forall k \geq k_1$ : $| c_n - f_k(z_n) | < \varepsilon/3$
$\forall k \geq k_2$ : $| c_m - f_k(z_m) | < \varepsilon/3$

$k_0 = \max\{k_1, k_2\}$. ~~[illegible]~~

~~[illegible]~~ $\forall k \geq k_0$

$\therefore \; |c_n - c_m| < \frac{2\varepsilon}{3} + |f_k(z_n) - f_k(z_m)|$

$\exists N \in \mathbb{N}$ tq $\forall n, m \geq N : |z_n - z_m| < \delta$

$$\forall n, m \geq N : |c_n - c_m| < \frac{2\varepsilon}{3} + \varepsilon = \varepsilon$$

$$\therefore (c_n)_{n \in \mathbb{N}} \text{ de Cauchy}$$

$\exists c \in \mathbb{C} : c_n \xrightarrow{n \to \infty} c$

$$|c - f_k(z)| = |c - c_n + c_n - f_k(z)| \leq |c - c_n| + |c_n - f_k(z)|$$
$$\leq |c - c_n| + |c_n - f_k(z_n)| + |f_k(z_n) - f_k(z)|$$

$\forall \varepsilon > 0 \; \exists N_1, N_2 \in \mathbb{N}$ tq $\forall n \geq N_1 : |c - c_n| < \varepsilon/3$

$\cancel{\forall k}$ $\forall n \geq N_2 : |f_k(z_n) - f_k(z)| < \varepsilon \quad \forall k$

$(|z_n - z| < \delta)$

$\therefore \forall k \in \mathbb{N} \quad \forall n \geq \max\{N_1, N_2\}$

$$|c - f_k(z)| < \varepsilon/3 + |c_n - f_k(z_n)| + \varepsilon/3$$

$\exists k_0 \in \mathbb{N} \quad \forall k \geq k_0 : |c_n - f_k(z_n)| < \varepsilon/3$

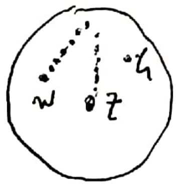
$$\therefore \; |c - f_k(z)| < \varepsilon \quad \forall k \geq k_0$$

~~Cond~~ Con lo anterior, $\lim_k f_k(z) \in \mathbb{C}$ ($\Leftrightarrow z \in E$)

$\therefore E$ cerrado.

Pd: $E \subseteq \Omega$ abierto. $z \in E \Rightarrow \exists r > 0 : B(z, r) \subseteq \Omega$

$\exists w \in E$ tq $w \in B(z, r)$, $\lim_k f_k(w) \in \mathbb{C}$

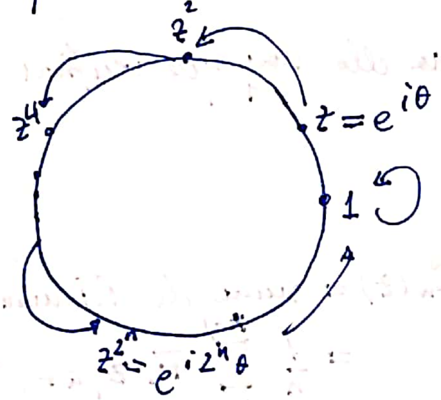


Ejemplo. $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ define una función holomorfa en $\mathbb{D} = \{ z / |z| < 1 \}$

$$R^{-1} = \limsup_{n\to\infty} \underbrace{|a_n|^{1/n}}_{0 \text{ ó } 1} = 1 = \inf_n \sup_{k \geq n} |a_n|^{1/n}$$

Af. Todos los puntos de $\partial \mathbb{D}$ son singulares.

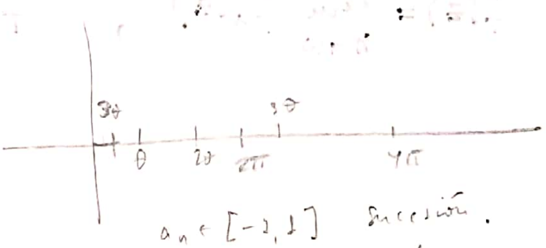
Dem.




$f(1) = 1 + 1 + 1 + \dots = \infty$

$z = e^{i\theta}$, $z^m = e^{mi\theta}$ $\boxed{= \cos(m\theta)} + i \operatorname{sen}(m\theta)$

$a_n = \cos(n\theta)$, $\theta \in [0, 2\pi)$




$a_n \in [-1, 1]$ sucesión.

$\exists (a_{n_k})$ convergente

obvio!

$2^m \theta = 2\pi k$, $k \in \mathbb{Z}$ $\Rightarrow \theta = \dfrac{2\pi k}{2^m}$

$z = e^{i\theta}$

$z^{2^n} = 1 \quad \forall n \geq m \Rightarrow \dfrac{2\pi(a 2^m + b 2^m)}{2^m} = 2\pi a + b$, $a, b \in \mathbb{Z}$

$D = \left\{ \exp\left( \dfrac{2\pi k}{2^m} i \right), \; \begin{matrix} k \in \mathbb{Z} \\ m \in \mathbb{N} \end{matrix} \right\}$ $\quad D \subseteq \mathbb{S}^1$ denso en $\mathbb{S}^1$

contiene un subgrupo denso.

$z \in D$ : $\sum_{n=0}^{\infty} z^{2^n} = \text{algo} + 1 + 1 + 1 + \dots$

Af. $z \in D \Rightarrow \lim_{r \to 1^-} |f(rz)| = \infty$

La afirmación implica que todos los puntos son singulares.

$$f(rz) = \sum_{n=0}^{\infty} r^{2^n} \underbrace{z^{2^n}}_{\substack{=1 \\ \forall n \geq m}} = \underbrace{\text{algo}}_{\substack{\in \mathbb{C} \\ |\cdot| \leq m}} + \underbrace{r^{2^m}}_{>0} + \underbrace{r^{2^{m+1}}}_{>0} + \dots$$

$r < 1$

**Teorema.** $\Omega \subseteq \mathbb{C}$ abierto, $F : \Omega \times [a,b] \to \mathbb{C}$ continua. Si $\forall s \in [a,b]$, la función $F(\cdot, s)$ es holomorfa, entonces la función

$$f(z) := \int_a^b F(z,s)\, ds$$

es holomorfa en $\Omega$.

dem. Queremos ocupar teo. de Morera. Para ello hay que verificar que $f$ es continua.

dem 1. Morera + Fubini ✓ fácil

dem 2. $f(z) = \lim_{n\to\infty} f_n(z)$, $f_n(z) =$ suma de Riemann

$$= \frac{1}{n} \sum_{j=0}^{n-1} F\left(z, a + \left(\frac{b-a}{n}\right) j\right)$$

$=$ función holomorfa.

Ejercicio. $f_n \to f$ uniformemente en subconjuntos compactos.

pendiente!

$z \in D$ disco cerrado

Shakarchi:

$$|f_n(z) - f(z)| = \left| \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} F(z, k/n), F(z, s) \right|$$

$$< \sum_{1}^{n} \int_{(k-1)/n}^{k/n} |F(z, k/n) - F(z,s)|\, ds$$

$$< \sum_{1}^{n} \varepsilon/n = \varepsilon$$

Consecuencia: Weierstrass $\Rightarrow f$ holomorfa.

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## Extensión de Funciones Holomorfas

Sea $\Omega \subseteq \mathbb{C}$ abierto, $f : \Omega \to \mathbb{C}$ holomorfa


Def. $p \in \partial\Omega$ es punto regular si $\exists$ vecindad $V \ni p$ y extensión holomorfa $g : \Omega \cup V \to \mathbb{C}$ de $f$.

Si $p \in \partial\Omega$ no es regular $\Rightarrow p$ se llama singular.

Ejemplo. $f(z) = 1 + z + z^2 + \cdots$ es holomorfa en $\mathbb{D} = \{ z \mid |z| < 1 \}$

$f(z) = \dfrac{1}{1-z}$, $z \in \mathbb{D}$. $\mathbb{S}^1 = \{ z \mid |z| = 1 \}$

$\{\text{puntos regulares}\} = \mathbb{S}^1 \setminus \{1\}$

$\{\text{puntos singulares}\} = \{1\}$

No olvidar la serie $f(z) = \sum_0^\infty z^{2^n}$

Otro ejemplo. Fijemos $\lambda \in (0,1)$. Sea $f(z) := \sum_0^\infty \lambda^n z^{2^n}$.

(0) $f$ es holomorfa en $\mathbb{D}$

Afirmación. (1) $f$ se extiende a una función continua en $\overline{\mathbb{D}}$.

(2) Todos los puntos de $\partial\mathbb{D}$ son singulares.

(0) Radio de convergencia $=1$

$(\lambda^n)^{1/2^n} = \lambda^{n/2^n} \xrightarrow{n\to\infty} \lambda^0 = 1$

dem. (1) es trivial, pues $\forall n \in \mathbb{N}$,

la $z \in \overline{\mathbb{D}} \mapsto z^{2^n}$ es continua, $|z^{2^n}| \leq 1$

$\sum_{n=0}^\infty \lambda^n \cdot 1$ es convergente (test M de Wierstrass)

Recordatorio (M-test de Wierstrass).

$\forall n \in \mathbb{N}$, $u_n : X \to \mathbb{C}$ tq.

$|u_n(x)| \leq M_n$ $\forall x \in X$ y

$\sum_1^\infty M_n < \infty \Rightarrow \sum_1^\infty u_n(x)$ uniformemente convergente

$$z f'(z) = \sum_{n=0}^\infty \lambda^n 2^n z^{2^n} = \sum_{n=0}^\infty \underbrace{(2\lambda)}_{\geq 1}{}^n z^{2^n}$$

Si $\lambda \geq \frac{1}{2}$ entonces el mismo argumento anterior muestra que

$$\operatorname{Sing}(z \mapsto z f'(z)) = \partial\mathbb{D}$$

Buen argumento!

y por lo tanto,

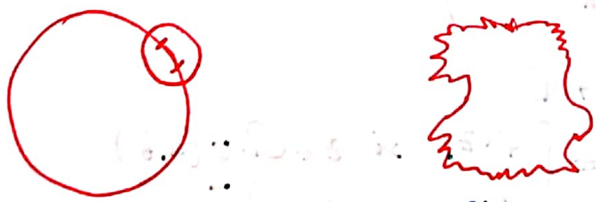
$$\operatorname{Sing}(f) = \partial\mathbb{D}$$

ok

sup $|z|=1$ es regular $\Rightarrow f(z) \in \mathbb{C}$
$\Rightarrow z f'(z) \in \mathbb{C}$ $(\Rightarrow\Leftarrow)$

Si $\lambda \in [1/4; 1/2)$, entonces considero $z \frac{d}{dz}(z f'(z))$, etc.

obs. La función continua $f|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \mathbb{C}$ es una curva que no es real diferenciable en ningún punto.



obs. $0 < \lambda < 1$. La función $f(z) = \sum_{n=0}^\infty \lambda^n z^{n^2}$ (holomorfa en $\mathbb{D}$)

- tiene $\operatorname{sing}(f) = \partial\mathbb{D}$
- tiene extensión $C^\infty$ al $\overline{\mathbb{D}}$

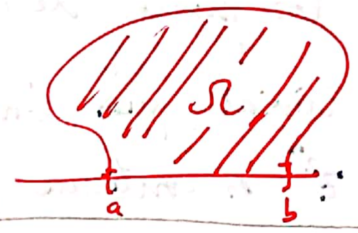
# Principio de Reflexión de Schwarz.

**Teorema.** Supongamos que: $\Omega \subset \{ z \in \mathbb{C},\ \mathrm{Im}(z) > 0 \}$ abierto y ~~[illegible]~~ $\overline{\Omega} \cap \mathbb{R} = [a,b]$.

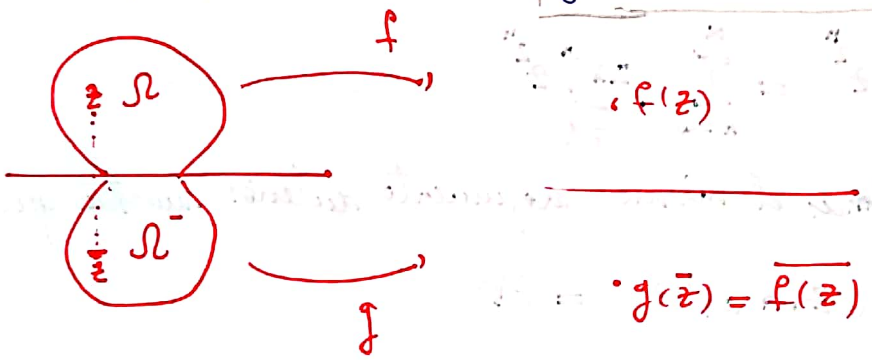
Sea $f : \Omega \to \mathbb{C}$ holomorfa que se extiende continuamente a $\overline{\Omega}$.

Además $\forall x \in (a,b)$, $f(x) \in \mathbb{R}$.

Entonces todos los puntos de $(a,b)$ son regulares.




**dem.** $\Omega^- := \{ \bar{z} \ / \ z \in \Omega \}$. Sea $\boxed{g : \Omega^- \longrightarrow \mathbb{C},\ g(z) = \overline{f(\bar{z})}}$




$z \in \mathbb{R}$, $g(z) = \overline{f(\bar{z})} = \overline{f(z)}$
$= f(z)$

**Af 1.** $g$ es holomorfa en $\mathring{\Omega}^-$

$$g'(z) = \lim_{h \to 0} \frac{g(z+h) - g(z)}{h} = \lim_{h \to 0} \frac{\overline{f(\overline{z+h})} - \overline{f(\bar{z})}}{h}$$

$$= \overline{\left( \lim_{\substack{h \to 0 \\ \bar{h} \to 0}} \frac{f(\bar{z} + \bar{h}) - f(\bar{z})}{\bar{h}} \right)} = \overline{f'(\bar{z})}$$

Sea $F : \Omega \cup (a,b) \cup \Omega^- \longrightarrow \mathbb{C}$

definida por $F(z) := \begin{cases} f(z) & \text{si } z \in \Omega \cup (a,b) \\ g(z) & z \in \Omega^- \end{cases}$ Gluing lemma?

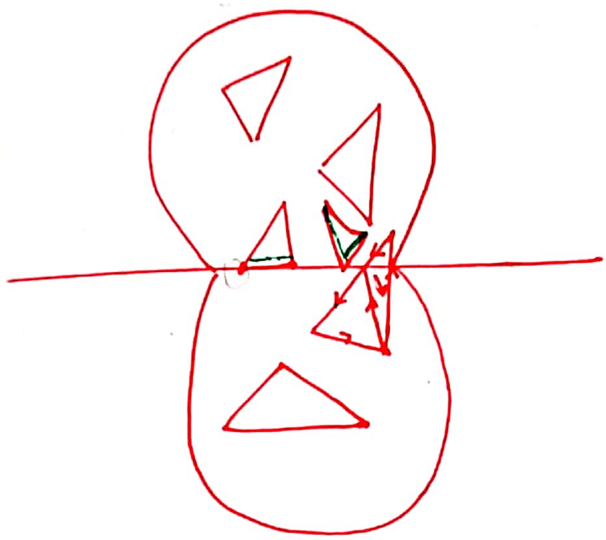
$F$ es continua

**Af 2.** $F$ es holomorfa.

$F : \Omega \cup (a,b) \cup \Omega^- \longrightarrow \mathbb{C}$

$F(z) = \begin{cases} f(z), & z \in \Omega \cup (a,b) \\ g(z), & z \in \Omega^- \end{cases}$

dem. de la afirmación? Morera

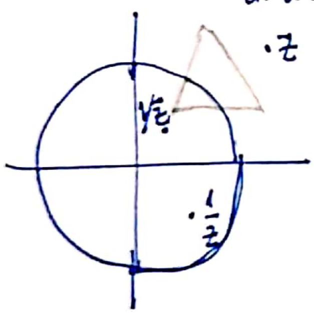


Ejercicio. Si $f:\mathbb{D}\to\mathbb{C}$ es holomorfa, se extiende continuamente al $\overline{\mathbb{D}}$,

supongamos:

$\forall z\in\partial\mathbb{D}$, $f(z)\in\mathbb{R}$

Entonces hay extensión holomorfa $F:\mathbb{C}\to\mathbb{C}$




La extensión es $F(z):=\overline{f\left(\frac{1}{\bar z}\right)}$

$f:\mathbb{D}\to\mathbb{C}$ holomorfa, se es

$\overline{\mathbb{D}}\cap\mathbb{R}=[$

$h(z):\mathbb{D}^+\to\mathbb{C}$ $\quad h(z)=\frac{1}{z}$ $\quad \overline{\mathbb{D}^+}\cap\mathbb{R}=[-1,1]$

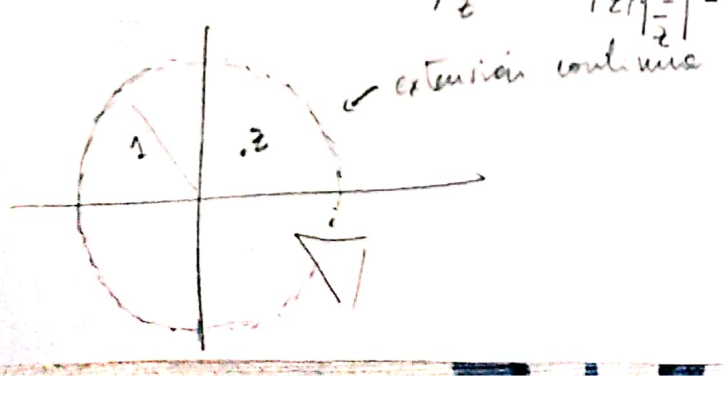
$|z|^2=z\cdot\bar z$

$1/\bar z$ $\quad |z|\left|\frac{1}{\bar z}\right|=1$

$g:\mathbb{D}^-\to\mathbb{C}$

$g(z)=\overline{f\left(\frac{1}{\bar z}\right)}=\overline{f\circ h(\bar z)}$

$f\circ h$ holomorfa en $\mathbb{D}\setminus\{0\}$




$F:\mathbb{D}\cup\partial\mathbb{D}\cup\mathbb{D}^-\to\mathbb{C}$

$F(z)=\begin{cases} f(z), & z\in\mathbb{D}\cup\partial\mathbb{D}\\ g(z), & z\in\mathbb{D}^-\end{cases}$

( Extensión de funciones holomorfas. Principio de Reflexión de Schwarz.)

(1) $F : \Omega \times [a,b] \longrightarrow \mathbb{C}$ continua ( $\Omega \subseteq \mathbb{C}$ abierto )

$F(\cdot, s)$ holomorfa $\forall s \in [a,b]$ , $f(z) := \int_a^b F(z,s)ds$ , $z \in \Omega$.

Pd: $f$ continua en $\Omega$.

$$|f(z) - f(z_0)| = \left| \int_a^b \big(F(z,s) - F(z_0,s)\big) ds \right|$$

$z_0 \in \Omega$ , $\exists r>0$ tq $D_r = \overline{B(z_0,r)} \subset \Omega$ $\Rightarrow$ $F$ uniformemente continua en $D_r \times [a,b]$

$\forall \varepsilon > 0, \exists \delta > 0$ tq $\forall (z,s), (z',s') \in D_r \times [a,b]$ , $\|z - z'\| + |s-s'| < \delta$

$\Rightarrow |F(z,s) - F(z_0', s)|$

$$\forall z : \int_a^b F(z,s)ds = \int_a^b u(z,s)ds + i \int_a^b v(z,s)ds$$

$F = (u,v)$ continua $\Leftrightarrow$ $u,v : \Omega \times [a,b] \to \mathbb{R}$ continua

$\forall z \in \Omega$ , $u_z : \{z\} \times [a,b] \longrightarrow \mathbb{R}$ continua $\Rightarrow$ $u(\circ, s)$ continua

$\therefore$ $F(\cdot, s)$ continua en $[a,b]$

$$|f(z) - f(z_0)| = \left| \int_a^b F(z,s)ds - \int_a^b F(z_0,s)ds \right|$$

$$= \left| \int_a^b F(z,s)ds - \Phi(t) \right| + \left| \Phi(t) - \int_a^b F(z_0,s)ds \right| , \quad \Phi_z(t) = \int_a^t F(z,s)ds$$

$$F(z,s) = u(z,s) + i v(z,s)$$

$$|f(z) - f(z_0)| = \left| \int_a^b F(z,s)ds - \int_a^b F(z_0,s)ds \right|$$

$$= \left| \int_a^b u(z,s)ds + i\int_a^b v(z,s)ds - \int_a^b u(z_0,s)ds - i\int_a^b v(z_0,s)ds \right|$$

$$\leq \left| \int_a^b (u(z,s) - u(z_0,s))ds \right| + \left| \int_a^b (v(z,s) - v(z_0,s))ds \right|$$

$$\leq \int_a^b |u(z,s) - u(z_0,s)| ds + \int_a^b |v(z,s) - v(z_0,s)| ds$$

$u, v$ continuas, $\therefore \forall \varepsilon > 0, \exists \delta_1 > 0$ tq $\forall z \in \Omega : |z - z_0| < \delta_1$

$\Rightarrow |u(z,s) - u(z_0,s)| < \varepsilon/2(b-a)$

$\exists \delta_2 > 0, \forall z \in \Omega : |z - z_0| < \delta_2 \quad : |v(z,s) - v(z_0,s)| < \varepsilon/2$

$\delta = \min\{\delta_1, \delta_2\} \quad \forall z \in \Omega, |z - z_0| < \delta \quad : \quad |f(z) - f(z_0)| < \varepsilon$

$\therefore f(z) = \int_a^b F(z,s)ds$ continua en $\Omega \times [a,b]$

<u>Af 2</u>. $f$ holomorfa

$\forall n \in \mathbb{N} : \quad f_n(z) = \sum_{j=0}^{n} F(z, \frac{b-a}{n}j + a) \frac{b-a}{n}$

$\forall z \in \Omega : \quad \lim_{n \to \infty} f_n(z) = f(z), \quad \forall n : f_n$ holomorfa en $\Omega$

Pd: $f_n \to f$ uniformemente en subconjuntos compactos.

$\Leftrightarrow \forall K \subseteq \Omega$ compacto, $\| f_n - f \|_K \xrightarrow{n \to \infty} 0$

Primero con discos. Sea $D \subseteq \Omega$ disco cerrado ~~$\lim_{n\to\infty} f_n(z) - f(z)$ unif en $D$~~

~~$\forall \varepsilon > 0 \exists N \in \mathbb{N}$~~ tq $F(z,s)$ uniformemente continua en $D \times [a,b]$

$\forall \varepsilon > 0, \exists \delta > 0$ tq $\forall s_1, s_2 : |s_1 - s_2| < \delta \Rightarrow \sup_{z \in D} |F(z,s_1) - F(z,s_2)|$

(2) (Extensiones holomorfas).

$f(z)=1+z+z^2+z^3+\cdots$ , $f(z)=\dfrac{1}{1-z}$ , $z\in B(0,1)=\Omega$

$z\neq 1$ , $|z|=1 \Rightarrow f(z)=\dfrac{1}{1-z}$

$\forall z\in \partial B(0,1)\setminus\{1\}$ , $g(z):=\dfrac{1}{1-z}$

$g: V\cup\Omega \to \mathbb{C}$ holomorfa.

$g|_{\Omega}=f$. efectivamente.

$\partial B(0,1)\setminus\{1\}$ puntos regulares

$\{1\}$ punto singular.

(3) (Principio de reflexión de Schwarz).

P.R.S (1° versión).

$\Omega\subset\{z\in\mathbb{C} / \operatorname{Im}(z)>0\}$ , abierto, $\overline{\Omega}\cap\mathbb{R}=[a,b]$

$f:\Omega\to\mathbb{C}$ holomorfa que se extiende continuamente a $\overline{\Omega}$

$\forall x\in(a,b)$ , $f(x)\in\mathbb{R}$

-o-

$\Omega^-=\{\bar z / z\in\Omega\}$ , $g:\Omega^-\to\mathbb{C}$

$g(z)=\overline{f(\bar z)}$

$g:\Omega^-\to\mathbb{C}$ holomorfa.

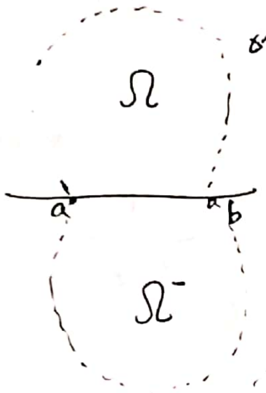
$$\frac{g(z+h)-g(z)}{h}=\frac{\overline{f(\overline{z+h})-f(\bar z)}}{h}=\overline{\frac{f(\bar z+\bar h)-f(\bar z)}{\bar h}}\xrightarrow{h\to 0}\overline{f'(\bar z)}$$

Ahora definir $F:\Omega\cup(a,b)\cup\Omega^-\to\mathbb{C}$ , $F(z)=\begin{cases} f(z), & z\in\Omega\cup(a,b)\\ g(z), & z\in\Omega^-\end{cases}$

Af 1. F continua | F continua $F^{-1}(O)=f^{-1}(O)\cup g^{-1}(O)$ abiertos, cuando $O\subseteq\mathbb{C}$ abierto

Af 2. F holomorfa.

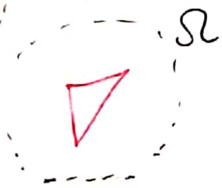
dem. F holomorfa por teo. de Morera




✓ $\Omega \cup (a,b) \cup \Omega^-$

Calcular $\int_{\partial T} F(z)\,dz = ?$ $\quad T \subseteq \Omega \cup (a,b) \cup \Omega^-$ triángulo.

Caso 1.

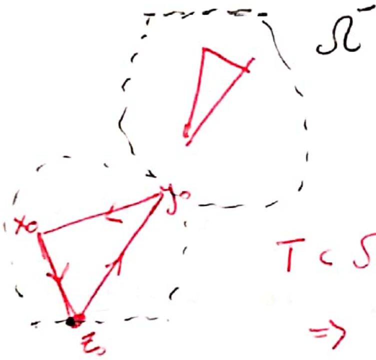



holo en Ω

$$T \subset \Omega \Rightarrow \int_{\partial T} F(z)\,dz = \int_{\partial T} f(z)\,dz = 0$$

Caso 2.

holo en $\Omega^-$



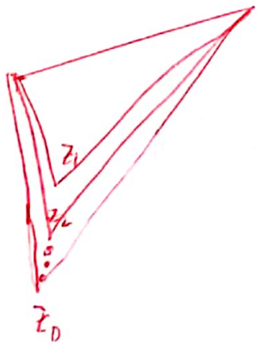

$$T \subset \Omega^- \Rightarrow \int_{\partial T} F(z)\,dz = \int_{\partial T} g(z)\,dz = 0$$

Caso 3.

$T \subset \Omega \cup (a,b)$, $z_0 \in (a,b)$

$\Rightarrow \exists (T_n) \subset \Omega$ triángulos tales que

$$\lim_{n\to\infty} T_n = T$$




$$\int_{\partial T} F(z)\,dz = \int_{[z_0,y_0]} F(z)\,dz + \int_{[y_0,\cancel{x_0}]} F(z)\,dz + \int_{[x_0,y_0]} F(z)\,dz$$

$$\int_{[z_0,y_0]} F(z)\,dz = \int_0^1 F((1-t)z_0 + ty_0)(y_0 - z_0)\,dt$$

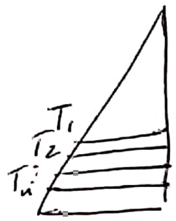
continuidad de $\int_0^1 F(z,t)\,dt$

$$= \int_0^1 F((1-t)\lim_{n\to\infty} z_n + ty_0)(y_0 - \lim z_n)\,dt = \lim_{n\to\infty} \int_0^1 F((1-t)z_n + ty_0)(y_0 - z_n)\,dt$$

$$\therefore \int_{\partial T} F(z)dz = \lim_{n\to\infty} \int_{\partial T_n} F(z)dz = \lim_{n\to\infty} \int_{\partial T_n} f(z)dz = 0$$

<u>Caso 4</u>

igual que caso 3 (mismo procedimiento)




$(T_n)_n \subset \Omega$

<u>Caso 5</u>


$T = T_1 + T_2 + T_3$

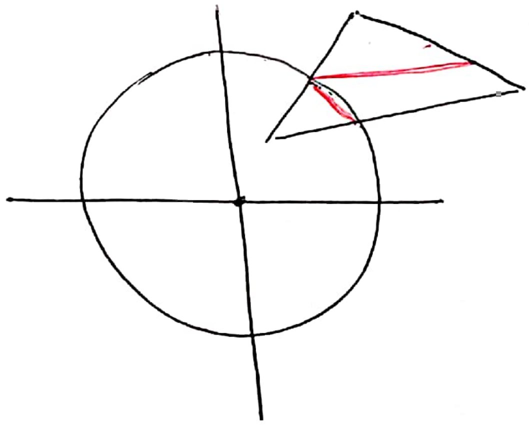
Hacer procedimiento de 3 y 4.

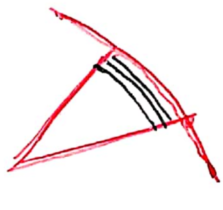
<u>2° Version (P.R.S.)</u>

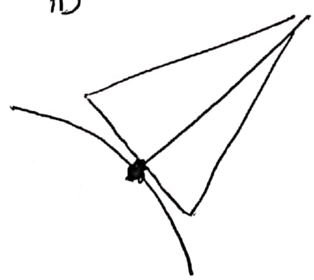
$f : \mathbb{D} \to \mathbb{C}$ holomorfa, se extiende continuamente a $\overline{\mathbb{D}}$,

$f(\partial \mathbb{D}) \subseteq \mathbb{R}$.

<u>Af.</u> $\exists F : \mathbb{C} \to \mathbb{C}$ holomorfa tal que $F_{\overline{\mathbb{D}}} = f$.







Teorema de Runge.

Motivación.

Teorema de aproximación de Weierstrass: Si $K \subset \mathbb{R}^d$ es compacto, entonces $\forall f: K \to \mathbb{R}$ continua. $\exists$ sucesión de polinomios $p_n: \mathbb{R}^d \to \mathbb{R}$ tal que $p_n \to p$ uniformemente en $K$. (Demostrado en curso Análisis 1)

¿Qué pasa con variable compleja?

$K \subset \mathbb{C}$ compacto, $f: K \to \mathbb{C}$ continua ¿$\exists p_n : \mathbb{C} \to \mathbb{C}$ polinomios tal que $p_n \to f$ uniformemente?

Resp: No

Ejemplo 1 $K = \{ z \in \mathbb{C} \mid |z| = 1 \}$, $f(z) = \bar{z}$

$$\oint_{|z|=1} f \, dz = 2\pi i \neq 0. \quad p_n \to f \text{ uniforme} \Rightarrow \int_{|z|=1} f = \lim \underbrace{\int_{|z|} p_n}_{=0} = 0.$$

$(\to\Leftarrow)$.

¿Y si suponemos $f$ holomorfa?

Resp. No!

Ejemplo 2. $f(z) = \frac{1}{z}$ bien definida y holomorfa en $\Omega = \mathbb{C} \setminus \{0\}$.

$K \subset \Omega$, $K = \{ z \in \mathbb{C} \mid \frac{1}{2} \leq |z| \leq 2 \}$ ($K$ compacto), entonces no existe sucesión de polinomios $(p_n)_n$ tal que $p_n \to f$ unif en $K$.

De hecho, $|z| = 1 \Rightarrow f(z) = \frac{1}{z} = \bar{z}$.

$$\int_{|z|=1} f \, dz = \int_{|z|=1} \bar{z} \, dz = 2\pi i$$

* funciones $f: \Omega \to \mathbb{C}$ continuas y/o holomorfas no siempre se pueden aproximar por polinomios (uniformemente en compactos).

# Teo de Runge ≈ Teo de aproximación de Weierstrass.

Funciones racionales: Si $p, q : \mathbb{C} \to \mathbb{C}$ polinomios, $q \not\equiv 0$,
entonces la función $f : \mathbb{C} \setminus q^{-1}(0) \to \mathbb{C}$

$$f(z) = \frac{p(z)}{q(z)}$$ es llamada "función racional".

Si $p, q$ no tiene ceros comunes, entonces $q^{-1}(0)$ es "el conjunto de polos de $f$".

**Teorema de Runge.** Sea $\Omega \subseteq \mathbb{C}$ abierto, $K \subset \Omega$ compacto. Sea $f : \Omega \to \mathbb{C}$ holomorfa. (★)

(1) $\exists$ sucesión $(g_n)_n$ de funciones racionales, cada una de ellas con polos en $\mathbb{C} \setminus K$ tales que $g_n \to f$ uniformemente en $K$.

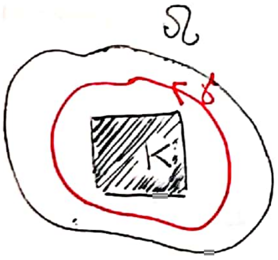
$K$ compacto $\Rightarrow \mathbb{C} \setminus K$ abierto.

(2) Si $\mathbb{C} \setminus K$ es conexo, entonces es posible elegir $g_n$ = polinomios

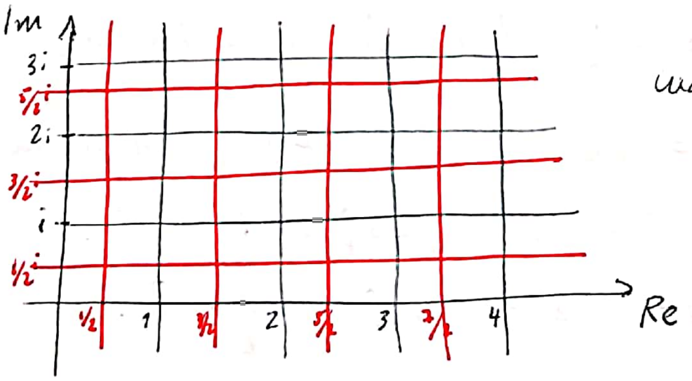
(∃! segmentos (ver Stakarchi))

**Lema.** Suponiendo (★). Existen curvas $\gamma_1, \dots, \gamma_n$ en $\Omega \setminus K$ tales que $\forall z \in K$, no necesariamente cerradas.

$$\boxed{f(z) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w - z}\, dw}$$




Dem. Consideremos la descomposición de $\mathbb{C}$ en cuadrados diádicos de lado $2^{-m}$, donde $m$ es tal que




cuadrados de lado $2^{-m}$

$Q$ = cuadrado de la partición $\mathcal{D}$ $[= \mathcal{D}_m]$

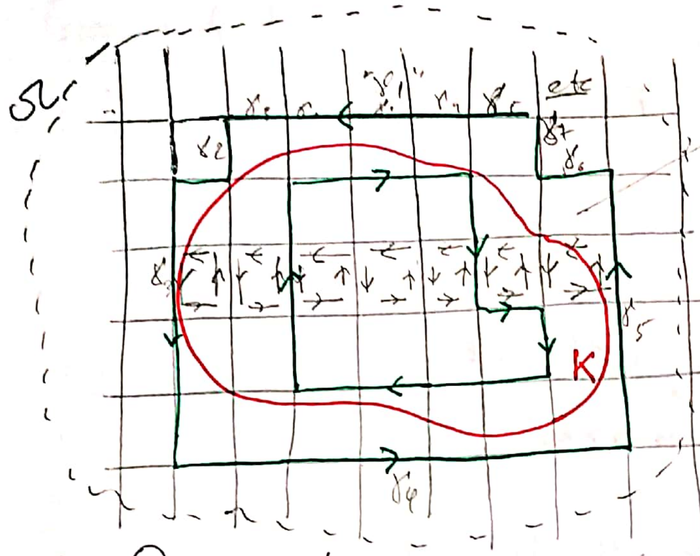
$Q \cap K \Rightarrow Q \subset \Omega$

¿Qué quiere decir exactamente?

$f : \Omega \to \mathbb{C}$ holo, $K \subseteq \Omega$ compacto

$\forall z \in K : f(z) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w - z}\, dw$

$0 < \delta < \frac{1}{2} d(K, \mathbb{C} \setminus \Omega)$.




Cantidad de cuadrados que intersectan a $K$ es finita

El tamaño del reticulado se elije de tal manera que $\gamma_j \in \Omega \setminus K$, $\gamma_j$ no intersecta a $K$.

$$K \subset \bigcup_{Q_i \in \mathcal{D}_m} \text{int}\, Q_i \Rightarrow K \subset \text{int}\, Q_1 \cup \dots \cup \text{int}\, Q_p \subset Q_1 \cup \dots \cup Q_p$$

Son finitos usando compacidad.

$Q \in \mathcal{D} \Rightarrow$ orientamos $\partial Q = \square$

Sean $Q_1, Q_2, Q_3, \dots, Q_p$ los cuadrados que intersectan $K$. Sean $\gamma_1, \dots, \gamma_n$ los lados (orientados) de los $Q_j$'s que pertenecen a uno solo de los $Q_j$'s

$$\frac{1}{2\pi i}\int_{\partial Q_j} \frac{f(w)}{w-z}\,dw = \begin{cases} f(z), & z \in \text{int}(Q_j) \\ 0, & z \in \mathbb{C}\setminus \overline{Q_j} \end{cases}$$

fórmula de Cauchy para cuadrados.

teorema de Cauchy

$$z \in \bigsqcup_{j=1}^{p} \text{int}(Q_j) \Rightarrow f(z) = \sum_{j=1}^{p} \frac{1}{2\pi i}\int_{\partial Q_j} \frac{f(w)}{w-z}\,dw \quad \text{ok}$$

(unión disjunta)

holomorfa

$$f(z) = \sum_{j=1}^{n} \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(w)}{w-z}\,dw \qquad \forall z \in K \setminus \bigcup_{j=1}^{p} \partial Q_j.$$

En particular, ok. la igualdad se cumple $\forall z \in K$ (por continuidad).

$\forall k$ $z_k \in K \setminus \bigcup \partial Q_j$, $z_k \to z \in \partial Q_j$

$$f(z_k) = \sum_{j=1}^{n} \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(w)}{w-z_k}\,dw \longrightarrow \sum_{j=1}^{n} \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(w)}{w-z}\,dw$$

(obs. El resultado anterior vale para módulo teo de Cauchy con cuadrados)

Dem de la parte 1 del teo de Runge.

Empezamos con el lema. Reemplazamos cada $\int_{\gamma_j}$ por una suma de Riemann (las cuales son funciones racionales de $z$).

$$\int_{\gamma_j} \underbrace{\frac{f(w)}{w-z}}\,dw$$

función racional de $z$

$$\int_{\gamma_j} \frac{f(w)}{w-z}\,dw = \sum_{l=0}^{n} \frac{f(\gamma_j(\frac{b-a}{n}l+a))}{\gamma_j(\frac{b-a}{n}l+a) - z}$$

$\forall z \in K$,

$$\left| \int_0^1 F_z(t)\,dt - \underbrace{\frac{1}{n}\sum_{j=0}^{n-1} F_z\left(\frac{j}{n}\right)}_{\int \text{escalera}} \right| \leq \int_0^1 | F_z - \text{escalera} |\, dt$$

$$= \sum_{j=0}^{n-1} \int_{j/n}^{(j+1)/n} |\underbrace{F_z(t) - F_z(j/n)}_{< \varepsilon \text{ si } n \geq n_0}|\, dt < \varepsilon$$

Obs. Los pasos de la demostración del teorema de Weierstrass:

$f : K \subseteq \mathbb{R}^d \to \mathbb{R}$ continua

1) Extender $f$ a una función continua $g : \mathbb{R}^d \to \mathbb{R}$ con soporte compacto.
2) "Gaussian blur"
   Se aproxima $g$ por $g * K_\varepsilon$ = convolución con el núcleo del calor.
3) Reemplazamos la exponencial $e^t$ en la fórmula por un polinomio

$$1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!}$$

Fórmula de Cauchy para cuadrados.

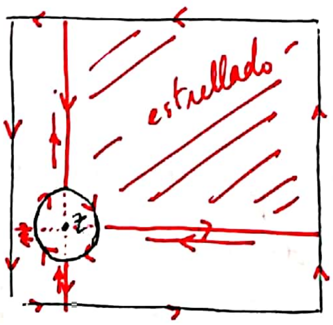
Sea $Q$ = cuadrado $\subset \Omega$ = abierto

$\gamma = \partial Q$ orientado con sentido antihorario.

$f : \Omega \to \mathbb{C}$ holomorfa,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z}\, dw$$

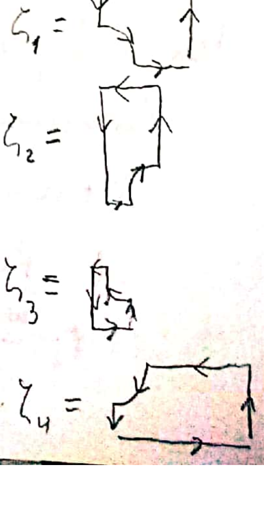
dem.




$\boxed{n = 4?}$

Cauchy: $0 = \sum_{i=1}^{n} \int_{\zeta_i} \frac{f(w)}{w - z}\, dw$

$$= \int_{\partial Q} \cdots - \underbrace{\int_{\partial \text{círculo}} \cdots}_{f(z)}$$




ok

Ejercicio: · Generalizar para otros polígonos
· Generalizar para otras curvas.

# Teorema de Runge.

(1) $F : \Omega \times [0,1] \to \mathbb{C}$ , $\Omega \subseteq$ abierto.

(i) $F(z,s)$ holomorfa en $z$ $\forall s \in [0,1]$

(ii) $F$ continua en $\Omega \times [0,1]$

Af. $f : \Omega \to \mathbb{C}$ , $f(z) := \int_0^1 F(z,s)\,ds$ es holomorfa.

dem. Truco : sumas de Riemann , $\int_0^1 F(z,s)\,ds = \lim_{n\to\infty} \sum_{j=0}^{n-1} F(z, \frac{1}{n}j)$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} F(z, \tfrac{j}{n})$$

$\forall n \in \mathbb{N}$, $f_n(z) := \frac{1}{n}\sum_{j=1}^{n} F(z, \frac{j}{n})$ . $f_n$ continua en $\Omega$.

$\lim_{n\to\infty} f_n(z) = f(z)$ (puntualmente).

$\forall n \in \mathbb{N}$, $f_n$ holomorfa en $\Omega$.

Pd: $f_n \to f$ uniformemente en $K \subset \Omega$ compacto, $\forall K$.

dem $K \subseteq \Omega$ compacto. $F$ uniformemente continua en $K \times [0,1]$

$\forall \varepsilon > 0$, $\exists \delta > 0$ tq $\forall (z,s_1), (z,s_2) \in K \times [0,1]$ tq $|s_1 - s_2| < \delta$
$\Rightarrow |F(z,s_1) - F(z,s_2)| < \varepsilon$

$\forall z \in K$ : $|F(z,s_1) - F(z,s_2)| < \varepsilon$

$\Rightarrow \forall z \in K$ : $|s_1 - s_2| < \delta \Rightarrow \sup_{z\in K} |F(z,s_1) - F(z,s_2)| < \varepsilon$ (*)

$z \in K$,

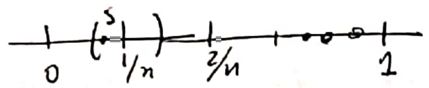
$$|f_n(z) - f(z)| = \left|\frac{1}{n}\sum_{j=1}^{n} F(z, j/n) - \int_0^1 F(z,s)\,ds\right|$$

$$= \left|\sum_{j=1}^{n} \int_{j-1/n}^{j/n} F(z, j/n)\,ds - \sum_{j=1}^{n} \int_{j-1/n}^{j} F(z,s)\,ds\right|$$

$$= \left|\sum_{j=1}^{n} \int_{j-1/n}^{j/n} (F(z, j/n) - F(z,s))\,ds\right| \leq \sum_{j=1}^{n} \int_{j-1/n}^{j/n} |F(z, j/n) - F(z,s)|\,ds$$

$j=1$ : $\displaystyle\int_0^{1/n} |F(z,1/n) - F(z,s)|\,ds$

$\forall \varepsilon>0, \ \exists \delta_0>0$ tq $\forall s: |1/n - s| < \delta_0 \Rightarrow |F(z,1/n)-F(z,s)|<\varepsilon \quad \forall z\in K.$

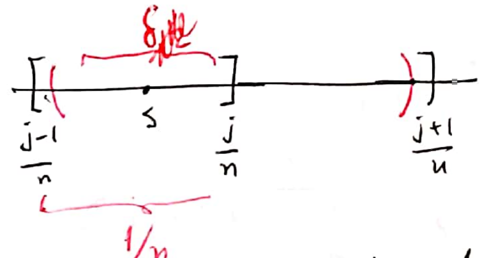


$\displaystyle\int_0^{1/n} |F(z,1/n) - F(z,s)|\,ds < \frac{\varepsilon}{n}$

$j\in\{1,\dots,n\}$

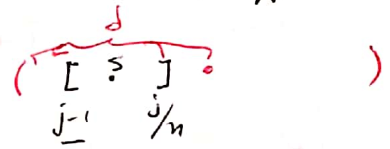
$\exists \delta_{j-1}>0, \quad \forall s: |j/n - s| < \delta_{j-1} \Rightarrow |F(z,1/n)-F(z,s)|<\varepsilon$

$s\in[(j-1)/n, j/n] \Rightarrow$



Delta ya fijado en (A), $\Rightarrow \exists N\in\mathbb{N}$ tq $\frac{1}{N}<\delta$

$\forall n\geq N \Rightarrow \frac{1}{n}<\delta \Rightarrow$



$\Rightarrow \cancel{\sum} \displaystyle\int_{(j-1)/n}^{j/n} |F(z,j/n)-F(z,s)|\,ds < \varepsilon/n \quad \forall z\in K$

$\therefore \ \forall n\geq N \Rightarrow |f_n(z)-f(z)|<\varepsilon \quad \forall z\in K$

$\therefore \ f_n \to f$ uniformemente en $K$

(2) Lema. $\gamma$ segmento, $\gamma\subseteq\Omega\setminus K$ ($K$ compacto, $K\subseteq\Omega$)

$\Rightarrow \exists (q_n)_{n\in\mathbb{N}}$ funciones racionales, $\{$polos de $q_n\}\subseteq\gamma$.

$q_n \longrightarrow \displaystyle\int_\gamma \frac{f(\zeta)}{(\zeta-z)}\,d\zeta$ uniformemente en $K$.

dem. $\displaystyle\int_\gamma \frac{f(\zeta)}{(\zeta-z)}\,d\zeta = \int_0^1 \frac{f(\gamma(t))}{\gamma(t)-z}\gamma'(t)\,dt$

$F(z,t) := \dfrac{f(\gamma(t))}{\gamma(t)-z}\gamma'(t)$ . $F(z,t)$ ~~holomorfa en~~ continua $\Omega\times[0,1]$

# Teorema de Runge

$K \subset \Omega \subseteq \mathbb{C}$ , $K$ compacto.
$\Omega$ abierto.

$f : \Omega \to \mathbb{C}$ holomorfa

(1) $\exists (g_n)_n$ racionales tq
- Los polos de $g_n$ estan en $\mathbb{C} \setminus \Omega$ $\forall n$.
- $g_n \to g$ uniformemente en $K$

(2) Si $\mathbb{C} \setminus K$ es conexo, entonces es posible elegir los $g_n$ polinomios.

dem. (1) Demostrado. (2) pendiente!

(2) Para demostrar parte (2), es suficiente demostrar:

Lema. $K \subseteq \mathbb{C}$ es compacto y $\mathbb{C} \setminus K$ es conexo, entonces para toda $g$ racional con polos en $\mathbb{C} \setminus K$, $\exists$ polinomios $p_n$ tq $p_n \xrightarrow{n \to \infty} g$ uniformemente en $K$.

( $\forall \varepsilon > 0$, $\exists$ polinomio $p$ tq $\|g - p\|_K < \varepsilon$ )

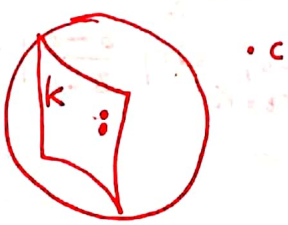
dem. del lema. Es suficiente considerar las funciones racionales $g_c(z) := \frac{1}{z - c}$, donde $c \notin K$.

La aproximación de las demás funciones racionales por polinomios, será posible usando fracciones parciales.

$B := \{ c \in \mathbb{C} \setminus K : \exists p_n \text{ polinomio tq } p_n \to g_c \text{ unif. en } K \}$.

Hay que probar que $B = \mathbb{C} \setminus K$.

1) $B \neq \emptyset$ $r := \sup_{z \in K} |z|$. Sea $c$ tal que $|c| > r$. Afirmación: $c \in B$



$$g_c(z) = \frac{1}{z-c} \qquad z \in K, \qquad |z| \le r < |c|$$

$$= -\frac{1}{c}\,\frac{1}{1-\frac{z}{c}} = -\frac{1}{c}\sum_{n=0}^{\infty}\left(\frac{z}{c}\right)^n \qquad \left|\frac{z}{c}\right| < 1$$

$$= \sum_{n=0}^{\infty} -\frac{1}{c^{n+1}} z^n, \qquad |z| < |c|$$

radio de convergencia de la serie.

Convergencia uniforme en el disco $\overline{B(0,r)}$, y en particular en $K$.

2) $B$ es cerrado en $\mathbb{C}\setminus K$, es decir,

$$\underset{\substack{\cap \\ B}}{c_j} \xrightarrow[j\to\infty]{} c \in \mathbb{C}\setminus K \Rightarrow c \in B$$

De hecho,

$$g_{c_j} \xrightarrow{j\to\infty} g \quad \text{uniformemente en } K.$$

3) $B$ es abierto.

Sea $c \in B$ , $b \in \mathbb{C}\setminus K$

$$g_b(z) = \frac{1}{z-b} = \frac{1}{(z-c)-(b-c)} = \frac{1}{z-c}\,\frac{1}{1-\frac{b-c}{z-c}}$$

$$g_b(z) = \frac{1}{z-c}\sum_{n=0}^{\infty}\left(\frac{b-c}{z-c}\right)^n = \sum \frac{(b-c)^n}{(z-c)^{n+1}}$$

$$g_b(z) = \sum_{n=0}^{\infty}(b-c)\big(g_c(z)\big)^{n+1}, \quad \text{convergencia uniforme sobre } K$$

Converge uniformemente si $\left|\frac{b-c}{z-c}\right| \le \frac{1}{2}$

tomamos $b \in \mathbb{C}\setminus K$ tal que $|b-c| \le \frac{1}{2} d(c,K)$

$$|b-c| \le \tfrac{1}{2} d(c,K) \le \tfrac{1}{2}|z-c| \;\; \forall z\in K \Rightarrow \left|\frac{b-c}{z-c}\right| \le \frac{1}{2}$$

Trunco la serie, aproximo $g_c$ por polinomios, y listo.

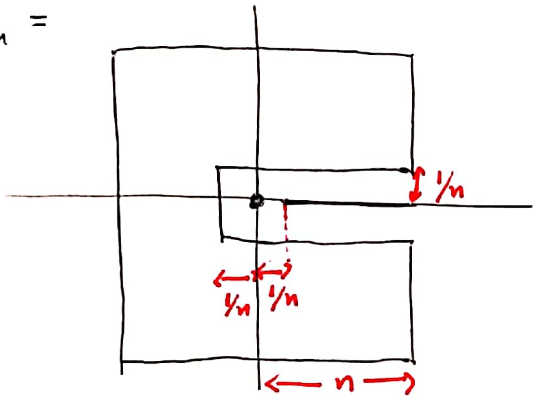
Ejemplo de límite puntual de funciones holomorfas que no es continuo.

Afirmación. Existe una sucesión de polinomios $(p_n)$ que converge puntualmente en todo $\mathbb{C}$.

Consideremos, $\chi_0(z) = \begin{cases} 1, & z=0 \\ 0, & z \neq 0 \end{cases}$

$\forall n \geq 2$ entero

$K_n =$




$K_n$ es compacto

$K_2 \subset K_3 \subset K_4 \subset \ldots$

$\mathbb{C} \setminus K_n$ es conexo

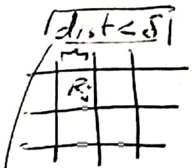
$\bigcup_{n=2}^{\infty} K_n = \mathbb{C}$

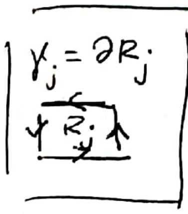
$\forall n \in \mathbb{N}$, $\exists$ vecindad $\Omega_n \supset K_n$ tal que $\chi_0|_{\Omega_n}$ es holomorfa. Por el teo de Runge, $\exists$ polinomio $p_n$ tal que $\|p_n - \chi_0\|_{K_n} < 1/n$ ◻

Obs. Si $U$ es una vecindad del 0, entonces la familia $\{p_n|_U, n \geq 2\}$ no es uniformemente acotada (por Vitali)

Conclusión. $\|p_n\|_{\bar{U}} \xrightarrow{n \to \infty} \infty$.

# Teorema de Runge.




(1) $0 < \delta < \frac{1}{2} d(K, \mathbb{C}\setminus\Omega)$. $\mathcal{R}$ reticulado en $\mathbb{C}$ tq

$K$ compacto $\Rightarrow K \subset \bigcup_{j=1}^{m} R_j$

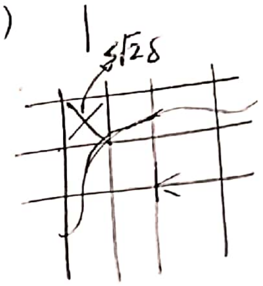
Af. $d(z,K) < \sqrt{2}\,\delta$. $\forall z \in R_j$

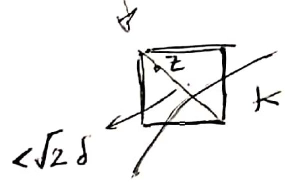
$\forall w \in K$, $d(z,K) \leq d(z,w)$

$z \in K \Rightarrow d(z,K) = 0$.

$z \notin K \Rightarrow d(z,K) < \sqrt{2}\,\delta$

(ver dibujo).



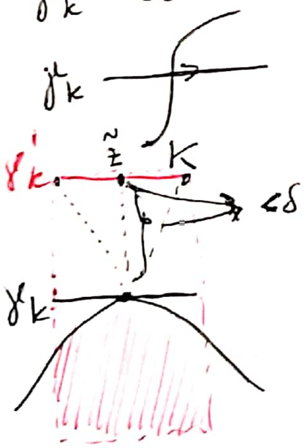



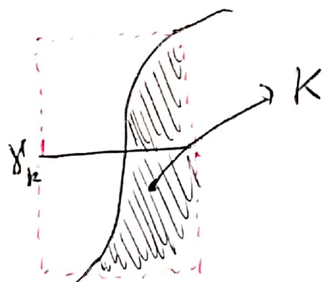

$2\delta < d(K, \mathbb{C}\setminus\Omega) \Rightarrow \forall z \in K, \forall w \in \mathbb{C}\setminus\Omega, \quad 2\delta < d(z,w)$

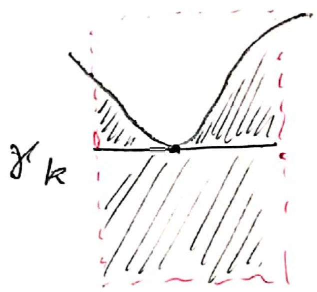
$d(z,w) < \sqrt{2}\,\delta$ $\Rightarrow z \in \Omega \Rightarrow R_j \subset \Omega$.
$\forall w \in K$

(2) Demostrado que $\sum_{k=1}^{n} \int_{\gamma_k} f = \sum_{j=1}^{m} \int_{\partial R_j} f$ ~~[illegible]~~ $f : \Omega \longrightarrow \mathbb{C}$ holomorfa.

Af. $\gamma_k \in \Omega\setminus K \quad \forall k$




$z \in K \cap \gamma_k \neq \emptyset \Rightarrow d(z,K) = 0$

Def: $\gamma_k$ segmentos que pertenecen a uno y sólo un lado de algún cuadrado $R_j$ $j \in \{1, \dots, m\}$

$F(z,t)$ uniformemente continua en $K \times [0,1]$:

$\forall \varepsilon > 0,\ \exists \delta > 0$ tq $\forall s_1, s_2 \in [0,1]$ : $|s_1 - s_2| < \delta \Rightarrow \sup_{z\in K} |F(z,s_1) - F(z,s_2)| < \varepsilon$

Sumas de Riemann: $\displaystyle\int_0^1 F(z,s)\,ds = \lim_{n\to\infty} \sum_{j=1}^{n}{}' F(z, j/n) \quad \forall z \in \Omega.$

$$\sum_{j=1}^{n}{}' F(z, j/n) = \sum_{j=1}^{n}{}' \frac{f(\gamma(j/n))}{\gamma(j/n) - z}\gamma'(j/n)$$

$$q_n(z) = \sum_{j=1}^{n}{}' \frac{f(\gamma(j/n))}{\gamma(j/n) - z}\gamma'(j/n) \qquad \text{(función racional)}$$

Por dem en (1), $q_n \xrightarrow{n\to\infty} \displaystyle\int_0^1 F(z,s)\,ds$ uniformemente en $K$

$\therefore\ q_n \xrightarrow{n\to\infty} \displaystyle\int_0^1 F(z,s)\,ds$ uniformemente en $K$.

obs: $q_n$ tiene polos en $\gamma$

(3) Teorema de Runge (primera parte).

$f : \Omega \to \mathbb{C}$ holomorfa, $K \subseteq \Omega$ compacto. $\exists (g_n)_{n\in\mathbb{N}}$ funciones racionales tq $\{$polos de $g_n\} \subseteq \ \ \setminus K$, $g_n \xrightarrow{n\to\infty} f$ uniformemente en $K$.

dem Por lema visto en clases

$\exists\ \gamma_1, \dots, \gamma_m$ segmentos en $\Omega \setminus K$ tq

$$\forall z \in K : \quad f(z) = \sum_{j=1}^{m}{}' \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w - z}\,dw \qquad \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(w)}{w-z}\,dw$$

Por (2), $\forall j \in \{1, \dots, m\}$ $\exists (q_{n,j})_{n\in\mathbb{N}}$ funciones racionales $\{$polos de $q_{n,j}\} \subseteq \gamma_j$ tq $q_{n,j} \xrightarrow{n\to\infty} (\ )$ uniformemente en $K$

$q_n := \displaystyle\sum_{j=1}^{m}{}' q_{n,j}$, $\forall n \in \mathbb{N}$ : $q_n$ función racional, $\{$polos de $q_n\} \subset \displaystyle\bigcup_{j=1}^{n} \gamma_j \subseteq \Omega \setminus K$

$\underline{\text{Af}}$. $q_n \longrightarrow f$ uniformemente en $K$.

$z \in K$,

$$|q_n(z) - f(z)| = \left| \sum_{j=1}^{m} q_{n,j}(z) - \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w-z}\, dw \right|$$

$$\leq \sum_{j=1}^{m} \left| q_{n,j}(z) - \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w-z}\, dw \right|$$

$\forall \varepsilon > 0$, $\exists N_j \in \mathbb{N}$ tq $\forall n \geq N_j$ $\forall z \in K$

$$\left| q_{n,j}(z) - \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w-z}\, dw \right| < \varepsilon/m$$

$N = \max\{N_j\}$, $\forall n \geq N$ $\forall z \in K$

$$|q_n(z) - f(z)| \leq \sum_{j=1}^{m} \left| q_{n,j}(z) - \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w-z}\, dw \right|$$

$$< \sum_{j=1}^{m} \varepsilon/m = \frac{\varepsilon}{m} \sum_{j=1}^{m} = \varepsilon.$$

Problemas resueltos
(Stein - Shakarchi)

P7) Factores de Blaschke

(a) $z, w \in \mathbb{C}$ ; $z\bar{w} \neq 1$ ~~PD~~ : $|z|, |w| < 1 \Rightarrow \left|\frac{w-z}{1-\bar{w}z}\right| < 1$

$$\left|\frac{w-z}{1-\bar{w}z}\right| < 1 \iff \left|\frac{w-z}{1-\bar{w}z}\right|^2 < 1 \iff 1 > \frac{(w-z)(\bar{w}-\bar{z})}{(1-\bar{w}z)(1-w\bar{z})} = \frac{|w|^2 - w\bar{z} - z\bar{w} + |z|^2}{1 - w\bar{z} - \bar{w}z + |zw|^2} = \frac{|w|^2 - 2Re(z\bar{w}) + |z|^2}{1 - 2Re(z\bar{w}) + |zw|^2}$$

$$\Rightarrow (w-z)(\bar{w}-\bar{z}) < (1-\bar{w}z)(1-w\bar{z}).$$

$$\iff (z-w)(\bar{z}-\bar{w}) < (1-\bar{w}z)(1-w\bar{z})$$

$$\left|\frac{w-z}{1-\bar{w}z}\right| = \left|\frac{w-re^{i\theta}}{1-\bar{w}re^{i\theta}}\right| = \left|\frac{we^{-i\theta}-r}{e^{-i\theta}-\bar{w}r}\right| = \left|\frac{we^{-i\theta}-r}{1-\bar{w}e^{i\theta}r}\right| \quad (*)$$

$$= \left|\frac{we^{-i\theta}-r}{1-\overline{we^{-i\theta}}\,r}\right|$$

$|w| < 1 \Rightarrow |we^{-i\theta}| < 1$

$\varphi_\theta : \mathbb{D} \to \mathbb{D}$

$w \mapsto we^{-i\theta}$ biyección.

$u \in \mathbb{D} \Rightarrow u = re^{i\tilde{\theta}} \Rightarrow u = re^{i\tilde{\theta}-\theta}e^{i\theta}$, $w = re^{i\tilde{\theta}-\theta} \in \mathbb{D}$.

$0 \leq r < 1$ $\qquad we^{i\theta} = u$.

$\varphi_\theta$ inyectiva ✓.

Sup $z \in \mathbb{R}$ $(z = r)$

$$(r-w)(r-\bar{w}) = r^2 - r\bar{w} - rw + |w|^2 = r^2 - 2r\,Re(w) + |w|^2$$

$$(1-\bar{w}r)(1-wr) = 1 - rw - r\bar{w} + r^2|w|^2 = 1 - 2r\,Re(w) + r^2|w|^2$$

$$(1-\bar{w}r)(1-wr) - (r-w)(r-\bar{w}) = 1 - r^2 + r^2|w|^2 - |w|^2 = 1 - r^2 - |w|^2(1-r^2)$$

$$= (1-r^2)(1-|w|^2) \geq 0. \quad (**)$$

$(*) + (**)$ : $\left|\frac{w-z}{1-\bar{w}z}\right| < 1 \quad \forall\, |z|, |w| < 1$

Pd: $\left|\frac{w-z}{1-\bar{w}z}\right| = 1$ si $|z|=1$ ó $|w|=1$

$z\in\mathbb{R}$, $z=1$ $\forall w\in\mathbb{D}$: $\left|\frac{w-1}{1-\bar{w}\cdot 1}\right| = \left|\frac{w-1}{1-\bar{w}}\right| = \frac{|w-1|}{|1-\bar{w}|} = \frac{|w-1|}{|1-w|} = 1$

Tomamos la desigualdad $(r-w)(r-\bar{w}) \leq (1-rw)(1-r\bar{w})$

con $|w|=1$ $\Updownarrow$

$(1-r^2)(1-|w|) \geq 0$

$\therefore (r-w)(r-\bar{w}) = (1-rw)(1-r\bar{w})$

$\therefore \left|\frac{w-r}{1-\bar{w}r}\right| = 1$

$\therefore \forall z\in\mathbb{D}: \left|\frac{w-z}{1-\bar{w}z}\right| = 1$

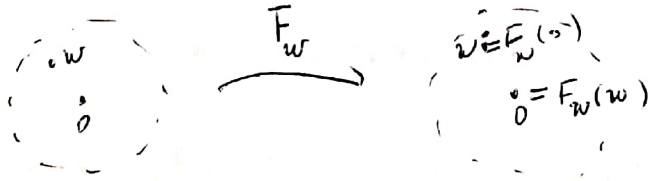
(b) Estudiar la función $F_w: z \longmapsto \frac{w-z}{1-\bar{w}z}$ $(w\in\mathbb{D}\ (|w|<1))$

(i) Por (a) $\forall z\in\mathbb{D}$, $|F_w(z)| = \left|\frac{w-z}{1-\bar{w}z}\right| < 1$ $\therefore |F_w(z)|<1$

$\therefore F_w(z)\in\mathbb{D}$.

$F_w$ holomorfa porque $F_w = \frac{F_w^1}{F_w^2}$, $F_w^1(z) = w-z$, $F_w^2(z) = 1-\bar{w}z$ holomorfas y $F_w^2(z)\neq 0$ $\forall z\in\mathbb{D}$.

(ii) $F_w(0) = \frac{w-0}{1-\bar{w}\cdot 0} = w$. $F_w(w) = \frac{w-w}{1-|w|^2} = 0$.




¿$F_w$ permutará otros puntos?

$F_w(a) = b$. $F_w(b) = \frac{w-b}{1-\bar{w}b} = \frac{w - \frac{w-a}{1-\bar{w}a}}{1-\bar{w}\frac{w-a}{1-\bar{w}a}} = \frac{\cancel{w}-|w|^2a-\cancel{w}+a}{1-\cancel{\bar{w}a}-|w|^2+\cancel{\bar{w}a}} = \frac{a\cancel{(1-|w|^2)}}{\cancel{1-|w|^2}}$

$= a$ $\therefore$ $F_w$ permuta todos los puntos.

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$\therefore\ F_w \circ F_w = id_{\mathbb{D}}$ ($F$ biyectiva).

(iii) $|z| = 1 \Rightarrow |F_w(z)| = 1$ por parte (a).

—o—

P9) Encontrar fórmulas para las ec's de Cauchy-Riemann en coordenadas polares.

$f = u + iv$ holomorfa. Ec's C-R: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

$u = u(x,y)$
$v = v(x,y)$

$x = r\cos\theta$
$y = r\,\mathrm{sen}\,\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\mathrm{sen}\,\theta$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\mathrm{sen}\,\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\,\mathrm{sen}\,\theta) + \frac{\partial u}{\partial y}(r\cos\theta) = -r\frac{\partial u}{\partial x}\mathrm{sen}\,\theta + r\frac{\partial u}{\partial y}\cos\theta$$

$$= r\left(\frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}\mathrm{sen}\,\theta\right)$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\,\mathrm{sen}\,\theta) + \frac{\partial v}{\partial y}(r\cos\theta) = r\left(\frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\mathrm{sen}\,\theta\right)$$

$$= r\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\mathrm{sen}\,\theta\right)$$

$$\frac{\partial u}{\partial \theta} = r\left(\frac{\partial u}{\partial x}\mathrm{sen}\,\theta + \frac{\partial v}{\partial x}\cos\theta\right)$$

$$= -r\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\mathrm{sen}\,\theta\right)$$

$$\therefore \left\{ \begin{array}{l} \dfrac{\partial v}{\partial \theta} = r\dfrac{\partial u}{\partial r} \\[2ex] \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r} \end{array} \right.$$

Eq's de Cauchy Riemann en coordenadas polares.

$\dfrac{\partial u}{\partial r}$, $\dfrac{\partial u}{\partial \theta}$, $\dfrac{\partial v}{\partial r}$, $\dfrac{\partial v}{\partial \theta}$ son funciones continuas!

Aplicar a $\log(z) = \log(r) + i\theta$, $z = re^{i\theta}$, $r > 0$, $-\pi < \theta < \pi$

$f(r,\theta) = \log(r) + i\theta$.  $u(r,\theta) = \log r$
$v(r,\theta) = \cancel{i}\theta$.

$\frac{\partial u}{\partial r} = \frac{1}{r}$, $\frac{\partial u}{\partial \theta} = 0$, $\frac{\partial v}{\partial r} = 0$, $\frac{\partial v}{\partial \theta} = 1$ $\left| \begin{array}{l} \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \end{array} \right.$ ✓

—o—

P13| $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ holomorfa.

(a) $Re(f)$ constante.  $f = u + iv$ ; $u, v : \Omega \to \mathbb{C}$
$u = Re(f)$.

$u$ cte $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$

$\therefore$ $v$ constante. $(v = v(x,y))$.

$\therefore f$ constante

(b) $Im(f)$ constante (análogo parte (a)).

(c) $|f|$ cte. $|f| = u^2 + v^2 = k$, $k \in \mathbb{C}$.

$$\left\{ \begin{array}{l} 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \\ 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \end{array} \right. \Rightarrow \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

$$= \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0.$$

$$\left| \frac{f(u+h) - f(u)}{h} \right| \leq \frac{|f(u+h)| + |f(u)|}{h}$$

$|f| = k \Rightarrow \forall z \in \Omega$, $|f(z)| = k \Rightarrow f(z) = \cancel{\ldots} = ke^{i\theta}$

~~[illegible]~~ $f(r,\theta) = k\cos\theta + ik\,\text{sen}\,\theta$.  $u(r,\theta) = k\cos\theta$
$v(r,\theta) = k\,\text{sen}\,\theta$.

$\frac{\partial u}{\partial r} = 0$, $\frac{\partial u}{\partial \theta} = -k\,\text{sen}\,\theta$, $\frac{\partial v}{\partial r} = 0$, $\frac{\partial v}{\partial \theta} = k\cos\theta$

Ec's Cauchy - Riemann : $k = 0$.

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$$2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0 \Rightarrow 2u\frac{\partial u}{\partial x}-2v\frac{\partial u}{\partial y}=0$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}-v\frac{\partial^2 u}{\partial x\partial y}=0$$

$$\begin{cases} 2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0 \\ 2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=0\end{cases} \Rightarrow \begin{cases} 2u\frac{\partial u}{\partial x}-2v\frac{\partial u}{\partial y}=0 \quad /\cdot v \\ 2v\frac{\partial u}{\partial x}+2u\frac{\partial u}{\partial y}=0 \quad /\cdot -u\end{cases}$$

$$\Rightarrow \begin{cases} 2uv\frac{\partial u}{\partial x}-2v^2\frac{\partial u}{\partial y}=0 \\ -2uv\frac{\partial u}{\partial x}-2u^2\frac{\partial u}{\partial y}=0\end{cases} \Rightarrow \Big\} \; 2(u^2+v^2)\frac{\partial u}{\partial y}=0 \Rightarrow \frac{\partial u}{\partial y}=0.$$

$$\Rightarrow \frac{\partial u}{\partial x}=0 \qquad \therefore \underline{u, v \text{ constantes}}.$$

P18| $f$ tiene expansión en serie de potencias alrededor del origen. Demostrar que $f$ tiene expansión en serie de potencias alrededor de cualquier punto en su disco de convergencia.

dem $\quad f(z)=\sum_{n=0}^{\infty} a_n z^n, \qquad z\in B(0,R)$

Sea $z_0\in B(0,R), \qquad z=z_0+(z-z_0)$

$\forall n\in\mathbb{N}, \; z^n=(z_0+(z-z_0))^n=\sum_{j=0}^{n}\binom{n}{j}z_0^j(z-z_0)^{n-j}$

$$f(z)=\sum_{n=0}^{\infty}a_n z^n=\sum_{n=0}^{\infty}a_n\sum_{j=0}^{n}\binom{n}{j}z_0^j(z-z_0)^{n-j}= \cancel{\sum_{n=0}^{\infty}\sum_{j=0}^{n}\cdots}$$

...

**P24** $\gamma:[a,b]\to\mathbb{C}$ curva suave. $f$ continua

Pd: $\displaystyle\int_\gamma f(z)\,dz = -\int_{\gamma^-} f(z)\,dz$

$\gamma^-:[a,b]\to\mathbb{C}$, $\gamma^-(t)=\gamma(1-t)$

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma^-(t))(\gamma^-(t))'dt = -\int_0^1 f(\gamma(1-t))\gamma'(1-t)\,dt$$

$1-t=s \quad ds=-dt \quad \begin{cases} t=0 \to s=1 \\ t=1 \to s=0.\end{cases}$

$$\therefore \int_\gamma f(z)dz = \int_1^0 f(\gamma(s))\gamma'(s)\,ds = -\int_0^1 f(\gamma(s))\gamma'(s)ds$$

$$= -\int_\gamma f(z)dz.$$

**P26** $f$ continua en $\Omega$ región ($\Omega\subseteq\mathbb{C}$). Sup. que $f$ tiene dos primitivas. Por demostrar que difieren de una constante.

$F,\tilde{F}$ primitivas de $f \Rightarrow F'=\tilde{F}'=f \Leftrightarrow (F-\tilde{F})'=0.$

Pd: $\exists\,\alpha\in\mathbb{C}$, $F=\tilde{F}+\alpha$.

$F:\Omega\to\mathbb{C}$ holomorfa tal que $F'=0$. $F(z_0)=w_0 \quad z_0\in\Omega$

$A=\{z\in\Omega \mid F(z)=w_0\} = F^{-1}(w_0) \to A\subseteq\Omega$ cerrado.

$\cap$

$\Omega$.

$w\in A \Rightarrow \exists\, r>0: \ B(w,r)\subseteq\Omega. \quad \zeta\in B(w,r)$

def: $h(t)=F((1-t)w+t\zeta)$

$t\in[0,1]$ pero para efectos de derivar, puede tomarse un abierto $(-\varepsilon,1+\varepsilon)$ $(\varepsilon>0)$

$t\neq s$, $\displaystyle\frac{h(t)-h(s)}{t-s} = \frac{F((1-t)w+t\zeta)-F((1-s)w+s\zeta)}{t-s}$

$$= \frac{F((1-t)w+t\zeta)-F((1-s)w+s\zeta)}{(1-t)w+t\zeta-((1-s)w+s\zeta)}\cdot\frac{(1-t)w+t\zeta-((1-s)w+s\zeta)}{t-s}$$

$$\frac{h(t)-h(s)}{t-s} = \frac{F((1-t)w+t\zeta) - F((1-s)w+s\zeta)}{(1-t)w+t\zeta-((1-s)w+s\zeta)} \cdot \frac{w(s-t)+\zeta(t-s)}{t-s}$$

$$\lim_{t\to s} \frac{h(t)-h(s)}{t-s} = \underbrace{F'((1-s)w+s\zeta)}_{=0}(\zeta-w) = 0$$

$\therefore \quad h'(s)=0 \quad \forall s$

$h$ constante. $\quad h(0) = F(w) = w_0 \quad \therefore \quad F(\zeta) = F(1) = w_0$

$\therefore \quad \zeta \in A$

$\therefore \quad B(w,r) \subseteq A \quad$ (A abierto

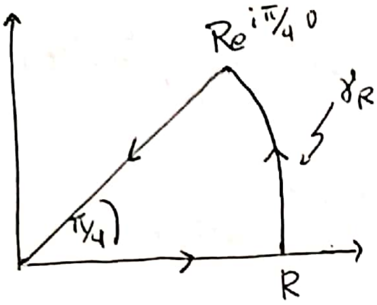
$\Omega$ conexo $\Rightarrow$ $\cancel{B(w,r)}$ $\quad A=\Omega$ ó $A=\phi$

Como $A \neq 0 \Rightarrow \boxed{A=\Omega}$

- Aplicar lo anterior a $F-\tilde{F}$.

$\boxtimes$

P1| Demostrar que $\int^{\infty} sen(x^2)\,dx = \frac{\sqrt{2\pi}}{4}$




$f(z) = e^{-z^2}$, $\int_{\gamma_R} f(z)\,dz = 0$

$\gamma_R = \gamma_R^1 + \gamma_R^2 + \gamma_R^3$

$\gamma_R^1 = [0,R] = tR,\ 0 \le t \le 1$ $\quad \gamma_R^2(t) = Re^{i\theta}$, $0 \le \theta \le \pi/4$

$\gamma_R^3(t) = te^{i\pi/4}$, $t \in [0,R]$

$$0 = \int_{\gamma_R} f(z)\,dz = \int_0^1 f(tR)\,R\,dt + \int_0^{\pi/4} f(Re^{it})\,Rie^{it}\,dt - \int_0 f(te^{i\pi/4})\,e^{i\pi/4}\,dt$$

$$\int_0^1 f(tR)\,R\,dt \underset{\substack{tR=s\\ ds=Rdt}}{=} \int_0^R f(s)\,ds = \int_0^R e^{-x^2}dx = \int_0^R e^{(ix)^2}dx \xrightarrow{R\to\infty} \frac{\sqrt{\pi}}{2}$$

$$\int_0^R f(te^{i\pi/4})\,e^{i\pi/4}\,dt = e^{i\pi/4}\int_0^R e^{-(te^{i\pi/4})^2}dt = e^{i\pi/4}\int_0^R e^{-t^2 i}\,dt$$

$$= e^{i\pi/4}\left(\int_0^R \left(\cos(-t^2) + i\,sen(-t^2)\right)dt\right)$$

$$= e^{i\pi/4}\left(\int_0^R \cos(-t^2)\,dt + i\int_0^R sen(-t^2)\,dt\right)$$

$$= e^{i\pi/4}\int_0^R \cos(-t^2)\,dt + ie^{i\pi/4}\int_0^R sen(-t^2)\,dt$$

| | 0 | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
|---|---|---|---|---|---|
| s | 0 | 1 | 2 | 3 | 4 |
| c | 4 | 3 | 2 | 1 | 0 |

$\sqrt{\ }$ / 2

$$= \left(\frac{\sqrt2}{2} + i\frac{\sqrt2}{2}\right)\int_0^R \cos(-t^2)\,dt + \left(-\frac{\sqrt2}{2} + i\frac{\sqrt2}{2}\right)\int_0^R sen(-t^2)\,dt$$

$$= \frac{\sqrt2}{2}\left(\int_0^R \cos(-t^2)\,dt - \int_0^R sen(-t^2)\,dt\right) + i\frac{\sqrt2}{2}\left(\int_0^R \cos(-t^2)\,dt + \int_0^R sen(-t^2)\,dt\right)$$

$$= \frac{\sqrt2}{2}\left(\int_0^R \cos(t^2)\,dt + \int_0^R sen(t^2)\,dt\right) + \frac{i\sqrt2}{2}\left(\int_0^R \cos(t^2)\,dt - \int_0^R sen(t^2)\,dt\right)$$

$$\left|\int_0^{\pi/4} f(Re^{it})\,Rie^{it}\,dt\right| \le \int_0^{\pi/4} |f(Re^{it})|\,R\,dt = R\int_0^{\pi/4} |f(Re^{it})|\,dt \le R\int_0^{2\pi} |f(Re^{it})|\,dt$$

$$|f(Re^{it})| = |e^{-(Re^{it})^2}| = |e^{-R^2 e^{2it}}|$$

$$= |e^{-R^2(\cos(2t)+i\,\text{sen}(2t))}| = |e^{-R^2\cos(2t)} e^{-R^2 i\,\text{sen}(2t)}|$$

$$= e^{-R^2\cos(2t)} = \frac{1}{e^{R^2\cos(2t)}} = e^{-R^2(\cos^2(t)-\text{sen}^2(t))} = e^{R^2\text{sen}^2(t)} e^{-R^2\cos^2(t)}$$

$$\int_0^{\pi/4} f(Re^{it}) Ri e^{it} dt = \int_0^{\pi/4} e^{-(Re^{it})^2} Rie^{it} dt = \int_0^{\pi/4} e^{-R^2 e^{2it}} Rie^{it} dt$$

$$= \int_0^{\pi/4} e^{-R^2(\cos(2t)+i\,\text{sen}(2t))} Ri(\cos(t)+i\,\text{sen}(t)) dt$$

$$= \int_0^{\pi/4} e^{-R^2(\cos(2t)+i\,\text{sen}(2t))} (-R\,\text{sen}(t) + Ri\cos(t)) dt$$

$$= \int_0^{\pi/4} e^{-R^2\cos(2t)} (\cos(-R^2\text{sen}(2t)) + i\,\text{sen}(-R^2\text{sen}(2t)))(-R\,\text{sen}(t) + iR\cos(t)) dt$$

$$= \int_0^{\pi/4} e^{-R^2\cos(2t)} \Big(-R\cos(-R^2\text{sen}(2t))\text{sen}(t) - R\,\text{sen}(-R^2\text{sen}(2t))\cos(t)$$
$$+ Ri(\cos(-R^2\text{sen}(2t))\cos(t) - \text{sen}(-R^2\text{sen}(2t))\text{sen}(t))\Big) dt$$

$$\left|\int_0^{\pi/4} f(Re^{it}) Rie^{it} dt\right| \leq \int_0^{\pi/4} \frac{R}{e^{R^2\cos(2t)}} dt \quad , \forall t \neq \pi/4 \quad \frac{R}{e^{R^2\cos(2t)}} \xrightarrow{R\to\infty} 0$$

integrable. $\forall R$

$\forall \varepsilon > 0 \ \exists \delta$ tq : $\displaystyle\int_{\pi/4-\delta}^{\pi/4} \frac{R}{e^{R^2\cos(2t)}} dt < \varepsilon/2$

$$\int_0^{\pi/4} \frac{R}{e^{R^2\cos(2t)}} dt = \underbrace{\int_0^{\pi/4-\delta} \frac{R}{e^{R^2\cos(2t)}} dt}_{\substack{\downarrow \text{ uniformemente} \\ 0}} + \underbrace{\int_{\pi/4-\delta}^{\pi/4} \frac{R}{e^{R^2\cos(2t)}} dt}_{<\varepsilon/2} \xrightarrow{R\to\infty} 0$$

Teo. de Egoroff (~)

$$\therefore \int_0^{\pi/4} f(Re^{it}) Rie^{it} dt \xrightarrow[R\to\infty]{} 0$$

Cuando $R \to \infty$

$$0 = \frac{\sqrt{\pi}}{2} - \frac{\sqrt{2}}{2}\left( \int_0^\infty \cos(t^2)dt + \int_0^\infty \operatorname{sen}(t^2)dt \right) + i\frac{\sqrt{2}}{2}\left( \int_0^\infty \cos(t^2)dt - \int_0^\infty \operatorname{sen}(t^2)dt \right)$$

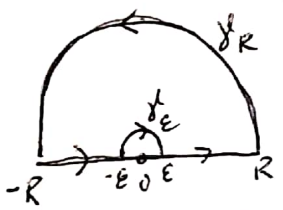
$$\therefore \int_0^\infty \cos(t^2)dt = \int_0^\infty \operatorname{sen}(t^2)dt \leftarrow$$

$$\Rightarrow \sqrt{2}\int_0^\infty \operatorname{sen}(t^2)dt = \frac{\sqrt{\pi}}{2} \Rightarrow \int_0^\infty \operatorname{sen}(t^2)dt = \frac{\sqrt{\pi}}{2\sqrt{2}} = \frac{\sqrt{2}\sqrt{\pi}}{2\,2} = \frac{\sqrt{2\pi}}{4} \blacksquare$$

P2] Pd: $\displaystyle\int_0^\infty \frac{\operatorname{sen} x}{x}dx = \frac{\pi}{2}$

dem. Hint: $\displaystyle\frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x}dx \ldots$

$$\frac{e^{ix}-1}{x} = \frac{\cos(x) + i\operatorname{sen} x - 1}{x} = \frac{\cos x - 1}{x} + i\frac{\operatorname{sen} x}{x}$$




$\gamma = [\varepsilon, R] + \gamma_R + [-R, -\varepsilon] + \gamma_\varepsilon$, T.C: $\displaystyle\int_\gamma \frac{e^{iz}-1}{z}dz = 0.$

$$0 = \int_{[\varepsilon,R]} \frac{e^{iz}-1}{z}dz + \int_{\gamma_R} \frac{e^{iz}-1}{z}dz + \int_{[-R,-\varepsilon]} \frac{e^{iz}-1}{z}dz + \int_{\gamma_\varepsilon} \frac{e^{iz}-1}{z}dz$$

$$\int_{[\varepsilon,R]} \frac{e^{iz}-1}{z}dz = \int_\varepsilon^R \frac{e^{ix}-1}{x}dx = \int_\varepsilon^R \frac{\cos x - 1}{x} + i\int_\varepsilon^R \frac{\operatorname{sen} x}{x}$$

$$\int_{[-R,-\varepsilon]} \frac{e^{iz}-1}{z}dz = \int_{-R}^{-\varepsilon} \frac{e^{ix}-1}{x}dx = \underbrace{\int_{-R}^{-\varepsilon} \frac{\cos x - 1}{x}dx}_{(1)} + i\underbrace{\int_{-R}^{-\varepsilon} \frac{\operatorname{sen} x}{x}dx}_{(2)}$$

$$y = -x \Rightarrow dy = -dx : \ (1) = \int_R^\varepsilon \frac{\cos(-y)-1}{-y}(-dy) = \int_R^\varepsilon \frac{\cos(y)-1}{y}dy = -\int_\varepsilon^R \frac{\cos(x)-1}{x}dx$$

$$(2) = \int_R^\varepsilon \frac{\operatorname{sen}(-y)}{-y}(-dy) = -\int_R^\varepsilon \frac{\operatorname{sen}(y)}{y}dy = \int_\varepsilon^R \frac{\operatorname{sen} x}{x}dx$$

$$\int_{\gamma_R} \frac{e^{it}-1}{z}\,dz = \int_0^{\pi} \frac{\exp(iRe^{it})-1}{Re^{it}} \cdot Rie^{it}\,dt = i\int_0^{\pi} (\exp(iRe^{it})-1)\,dt$$

$$\exp(iRe^{it})-1 = \exp(iR(\cos t + i\,\mathrm{sen}\,t))-1 = \exp(-R\,\mathrm{sen}\,t + iR\cos t)-1$$
$$= -1 + \exp(-R\,\mathrm{sen}\,t)\exp(iR\cos t)$$

$$\left|\int_{\gamma_R} \frac{e^{iz}-1}{z}\,dz\right| \le \int_0^{\pi} |\exp(iRe^{it})-1|\,dt = \int_0^{\pi} |-1+\exp(-R\,\mathrm{sen}\,t)\exp(iR\cos t)|\,dt$$

$$= \int_0^{\pi/2} |-1+\exp(-R\,\mathrm{sen}\,t)\exp(iR\cos t)|\,dt + \int_{\pi/2}^{\pi} |-1+\exp(-R\,\mathrm{sen}\,t)\exp(iR\cos t)|\,dt$$

$\mathrm{sen}(t+\pi/2) = \mathrm{sen}\,t\cos\frac{\pi}{2} + \mathrm{sen}\frac{\pi}{2}\cos t$

$\cos(t+\pi/2) = \cos t\cos\pi/2 - \mathrm{sen}\,t\,\mathrm{sen}\,\pi/2$

$$= \int_0^{\pi/2} |-1+\exp(-R\,\mathrm{sen}\,t)\exp(iR\cos t)|\,dt + \int_0^{\pi/2} |-1+\exp(-R\cos t)\exp(-iR\,\mathrm{sen}\,t)|\,dt$$

$$|\exp(-R\,\mathrm{sen}\,t)\exp(iR\cos t)| = \exp(-R\,\mathrm{sen}\,t) \xrightarrow[\forall t\neq 0]{R\to\infty} 0$$

$$\int_0^{\pi/2} |-1+\exp(-R\,\mathrm{sen}\,t)\exp(iR\cos t)|\,dt \le \frac{\pi}{2} + \int_0^{\pi/2} \exp(-R\,\mathrm{sen}\,t)\,dt$$

$$\int_0^{\delta} |-1+\exp(-R\,\mathrm{sen}\,t)\exp(iR\cos t)|\,dt + \int_{\delta}^{\pi/2} |-1+\exp(-R\,\mathrm{sen}\,t)\exp(iR\cos t)|\,dt$$

$$\int_{\sigma} \frac{e^{iz}-1}{z}\,dz = e^{i0}-1 = 0.$$

$$\left|\frac{e^{iz}-1}{z}\right| \le \frac{|e^{iz}|+1}{|z|} = \frac{\exp(-R\,\mathrm{sen}(t))+1}{R}$$

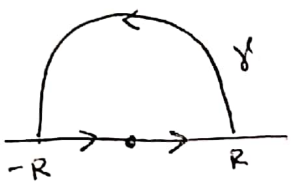
$$\int_{\gamma_R} \frac{e^{iz}-1}{z}\,dz = i\int_0^{\pi} (\exp(iRe^{it})-1)\,dt$$
$$= i\int_0^{\pi} \exp(iRe^{it}) - i\pi$$

$$\left|i\int_0^{\pi} \exp(iRe^{it})\,dt\right| \le \int_0^{\pi} \exp(-R\,\mathrm{sen}\,t)\,dt \xrightarrow{R\to\infty} 0$$

$$\therefore \int_{\gamma_R} \frac{e^{it}-1}{z}\,dz \xrightarrow{R\to\infty} i\pi$$

24] Demostrar que $\forall \xi \in \mathbb{C}$, $e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi} dx$

$f(z) = \exp(-\pi z^2)\exp(2\pi i z \xi)$, $\int_\gamma f(z)dz = 0$.



$$0 = \int_{-R}^{R} f(x)dx + \int_0^\pi f(Re^{it})\, Rie^{it} dt$$

$$\int_0^\pi f(Re^{it})\, Rie^{it} dt = \int_0^\pi \exp(-R^2\pi e^{2it})\, \underline{\exp(2\pi i Re^{it}\xi)}\, Rie^{it} dt$$

$$\left|\int_0^\pi f(Re^{it}) Re^{it} dt\right| \leq \int_0^\pi R\,|f(Re^{it})|\,dt$$

$$= \int_0^\pi R \exp(-R^2\pi\cos 2t)\exp(-2\pi R\,\mathrm{sen}(t)\,Re(\xi))\,dt$$

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi} dx = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi x Im(\xi)} e^{2\pi i x Re(\xi)} dx$$

$$2\pi i Re^{it}\xi = 2\pi i R(\cos t + i\,\mathrm{sen}\,t)(Re(\xi) + i Im(\xi))$$
$$= 2\pi i R\big((\cos(t)Re(\xi) - \mathrm{sen}(t)Im(\xi)) + i(\cos(t)Im(\xi) + \mathrm{sen}(t)Re(\xi))\big)$$
$$= -2\pi R(\cos(t)Im(\xi) + \mathrm{sen}(t)Re(\xi)) + \dots$$

$$\int_0^1 e^{-\pi x^2} e^{2\pi i x\xi} dx \quad \Big| \quad y = \pi x \to dy = \pi dx$$

$$= \int_0^1 e^{-\pi x^2 + 2\pi i x\xi} dx = \int_0^1 e^{-\pi(x^2 + 2ix\xi)} dx = \int_0^1 e^{-\pi(x^2 \mp 2xi\xi \pm \xi^2) \pm \pi\xi^2} dx$$

$$= e^{-\pi\xi^2}\int_0^1 e^{-\pi(x^2 - 2xi\xi - \xi^2)} dx = e^{-\pi\xi^2}\int_0^1 e^{-\pi(x - i\xi)^2} dx$$

$$\int_{\gamma_\varepsilon} \frac{e^{iz}-1}{z}\,dz = \int_{-\pi}^{-2\pi} \frac{\exp(i\varepsilon e^{-it})-1}{\varepsilon e^{-it}}\,(-\varepsilon i e^{-it})\,dt = -i\int_{-\pi}^{-2\pi} (\exp(i\varepsilon e^{-it})-1)\,dt$$

$$= -i\int_{-\pi}^{-2\pi} \exp(i\varepsilon e^{-it})\,dt + i\int_{-\pi}^{-2\pi} dt = -i\int_{-\pi}^{-2\pi} \exp(i\varepsilon e^{-it})\,dt - \pi i$$

$$= \left|-i\int_{-\pi}^{-2\pi} \exp(i\varepsilon e^{-it})\,dt\right| = \int_{-\pi}^{-2\pi} \exp(\varepsilon \operatorname{sen} t)\,dt$$

$f(z) = \frac{e^{iz}-1}{z}$ holomorfa en $\mathbb{C}\setminus\{0\}$

$$f(z) = \frac{1}{z}\left(\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - 1\right) = \frac{1}{z}\sum_{n=1}^{\infty}\frac{(iz)^n}{n!} = \frac{1}{z}\left(iz + \sum_{n=2}^{\infty}\frac{(iz)^n}{n!}\right) = i + \underbrace{\sum_{n=2}^{\infty}\frac{i(iz)}{n!}}_{\text{holomorfa}}$$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{e^{iz}-1}{z} = i\lim_{z\to 0}\frac{e^{iz}-1}{iz} = i(e^{z})'\big|_{z=0} = i$$

$\exists M>0$ tq $|f(z)| \leq M \quad \forall z\in \overline{B(0,1)}$

$$\therefore \lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon}\frac{e^{iz}-1}{z}\,dz = 0.$$

Por lo tanto:

$$0 = 2i\int_{\varepsilon}^{R}\frac{\operatorname{sen} x}{x}\,dx + \int_{\gamma_R}\frac{e^{iz}-1}{z}\,dz + \int_{\gamma_\varepsilon}\frac{e^{iz}-1}{z}\,dz$$

$$\xrightarrow[\varepsilon\to 0]{R\to\infty} 2i\int_0^{\infty}\frac{\operatorname{sen} x}{x}\,dx - i\pi + 0 = 0 \Rightarrow \int_0^{\infty}\frac{\operatorname{sen} x}{x}\,dx = \frac{\pi}{2}$$

$\therefore \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi}\, dx = e^{-\pi \xi^2} \int_{-\infty}^{\infty} e^{-\pi (x - i\xi)^2}\, dx$

?d: $\int_{-\infty}^{\infty} e^{-\pi (x - i\xi)^2}\, dx = 1$

$\int_{-\infty}^{\infty} e^{-\pi (x - i\xi)^2}\, dx = \int_{-\infty}^{\infty} e^{-(\sqrt{\pi}x - i\sqrt{\pi}\xi)^2}\, dx$ | $y = \sqrt{\pi}x - i\sqrt{\pi}\xi$, $dy = \sqrt{\pi}\, dx$

$\cancel{\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x (Re(\xi) + i Im(\xi))}\, dx = \int_{-\infty}^{\infty} e^{-\pi x^2 + 2\pi x Im(\xi)} e}$ ¿se puede hacer sustitución?

$\int_{-R}^{R} e^{-\pi (x - i\xi)^2}\, dx = \cancel{\int_{-\sqrt{\pi}R - i\sqrt{\pi}\xi}}$

$x = (1-t)(-R) + tR$, $x = (1-t)(-R) + tR$

$x = -R + 2tR$

$dx = 2R\, dt$

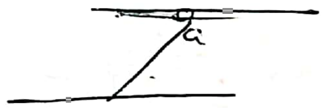
$x = -R \to t = 0$

$x = R \to t = 1$

$\cancel{\int_{-R}^{R} e^{-\pi((1-t)(-R) + tR - i\xi)^2}\, 2R\, dt}$

$\int_{-R}^{R} e^{-\pi (x - i\xi)^2}\, dx = \int_{0}^{1} e^{-\pi((1-t)(-R) + tR - i\xi)^2}\, 2R\, dt$

$= \cancel{\int_{0}^{1} e^{-\pi((1-t)(-R - i\xi) + t(R - i\xi))^2}\, 2R\, dt}$



$\int_{\gamma} e^{-\pi z^2}\, dz = ??$

$\cancel{\gamma = (1-t)}$

$\cancel{\gamma(t) = tx + ai}$, $a \in \mathbb{C}$

$\gamma(t) = t + a$, $a \in \mathbb{C}$

$\int_{-\infty}^{\infty} e^{-\pi (t + a)^2}\, dt$

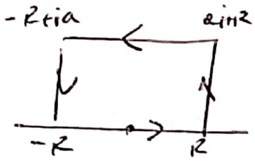
$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi}\, dx = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x (Re(\xi) + i Im(\xi))}\, dx \qquad x = \frac{k}{Re(\xi)},\ k \in \mathbb{Z}$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi x Im(\xi) + 2\pi i x Re(\xi)}\, dx = \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi x Im(\xi)} e^{2\pi i x Re(\xi)}\, dx$$

$$= \cancel{\int_{-\infty}^{\infty} e^{-\pi(x^2 + 2ix Re(\xi) - Re(\xi)^2)} e^{-2\pi x Im(\xi)}\, dx}$$

$$= \cancel{\int_{-\infty}^{\infty} e^{-\pi(x + iRe(\xi))^2} e^{-2\pi x Im(\xi)}\, dx}$$

$$= \int_{-\infty}^{\infty} e^{-\pi(x^2+2x}$$




$$\therefore \int_{[-R,R]} e^{-x^2}\, dx = \int_{[-R+ia,\, R+ia]} e^{-z^2}\, dz$$

$$\int_\gamma e^{-z^2}\, dz = 0 = \int_{-R}^{R} e^{-x^2}\, dx + \int_{[R,\, ai+R]} e^{-z^2}\, dz + \int_{[ai+R,\, -R+ai]} e^{-z^2}\, dz$$

$$+ \int_{[-R+ai,\, -R]} e^{-z^2}\, dz$$

$$\therefore \int_{-\infty}^{\infty} e^{-\pi(x - i\xi)^2}\, dx = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1$$

$$\int_{[R,\, ai+R]} e^{-z^2}\, dz = \int_0^1 e^{-((1-t)R + t(ai+R))^2} ai\, dt$$

$$\int_{[-R+ai,\, -R]} e^{-z^2}\, dz = \int_0^1 e^{-((1-t)(-R+ai) - tR)^2} (-ai)\, dt$$

$$= \int_1^0 e^{-((1-t)(-R+ai) - tR)^2} ai\, dt \underset{s=1-t}{\overset{dt=-ds}{=}} -\int_0^1 e^{-(-s(R+ai) + (-1+s)R)^2} ai\, ds$$

$$= -\int_0^1 e^{-((1-s)R + s(R+ai))} ai\, ds \longrightarrow$$

No?

FUNCIONA!

$$\int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi x\, Im(\xi)} e^{2\pi i x\, Re(\xi)} dx = \int_{-\infty}^{\infty} e^{-\pi(x^2+2x\, Im(\xi)+ Im(\xi)^2)+\pi Im(\xi)^2} e^{2\pi i x\, Re(\xi)} dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi(x+Im(\xi))^2} e^{\pi Im(\xi)^2 + 2\pi i x\, Re(\xi)} dx = e^{\pi Im(\xi)^2}\int_{-\infty}^{\infty} e^{-\pi(x+Im(\xi))^2} e^{2\pi i x\, Re(\xi)} dx$$

$$= e^{\pi Im(\xi)^2}\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i (x - Im(\xi))\, Re(\xi)} dx$$

$$= e^{\pi Im(\xi)^2 - 2\pi i\, Re(\xi)\, Im(\xi)} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\, Re(\xi)} dx$$

$$= e^{\pi Im(\xi)^2 - 2\pi i\, Re(\xi)\, Im(\xi)} \int_{-\infty}^{\infty} e^{-\pi(x^2 - 2ix\, Re(\xi) - Re(\xi)^2) - \pi Re(\xi)^2} dx$$

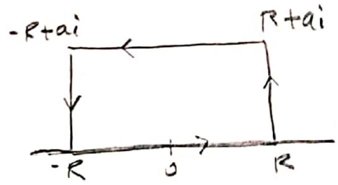
$$= e^{\pi Im(\xi)^2 - 2\pi i\, Re(\xi)\, Im(\xi) - \pi Re(\xi)^2}\int_{-\infty}^{\infty} e^{-\pi(x - i\, Re(\xi))^2} dx$$

$$= e^{-\pi(Re(\xi)^2 + 2i\, Re(\xi)\, Im(\xi) - Im(\xi)^2)} \int_{-\infty}^{\infty} e^{-\pi(x-i\, Re(\xi))^2}dx$$

$$= e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi(x-i\, Re(\xi))^2} dx ,$$

$(1-t)(R+ai)+t(-R+ai)$
$= R+ai-tR-\cancel{tai}-tR+\cancel{tai}$
$= R+ai-2tR$
$(1-t)(-R+ai)-tR = -R+ai+\cancel{tR}-ait-\cancel{tR}$
$= -R+ai-ait$

Hay que calcular $\int_{-\infty}^{\infty} e^{-\pi(x-i\, Re(\xi))^2}dx$




$$\gamma = [-R,R]+[R,R+ai]+[R+ai,-R+ai]+[-R+ai,-R]$$

$$\int_\gamma e^{-\pi z^2}dz = 0$$

$$0 = \int_\gamma e^{-\pi z^2}dz = \int_{[-R,R]} e^{-\pi z^2}dz + \int_{[R,R+ai]} e^{-\pi z^2}dz + \int_{[R+ai,-R+ai]} e^{-\pi z^2}dz + \int_{[-R+ai,-R]} e^{-\pi z^2}dz$$

$$= \int_{-R}^{R} e^{-\pi x^2}dx + \int_0^1 e^{-\pi((1-t)R+t(R+ai))^2} ai\, dt + \int_0^1 e^{-\pi((1-t)(R+ai)+t(-R+ai))^2}(-2R)\,dt$$
$$+ \int_0^1 e^{-\pi((1-t)(-R+ai)-tR)^2}(-ai)\,dt$$

$$-\int_0^1 e^{-(s(R+ai)+(-1+s)R)^2} ai\,ds = -\int_0^1 e^{-(sR+\underbrace{sai-R}_{sai-R}-sR)^2} ai\,ds$$

$$= \cancel{-\int_0^1 e^{(1-(1-s))R}}$$

$$(1-t)R+t(ai+R) = R-\cancel{tR}+ait+\cancel{Rt} = R+ait$$

$$(1-t)R+t(ai+R) = R-\cancel{tR}+tai+\cancel{tR} = R+tai$$

$$\int_{[-R+ai,-R]} e^{-z^2}dz = \int_0^1 e^{-((1-t)(ai-R)-tR)^2}(-ai)dt = \int_0^1 e^{-(ai-R-ait)^2}(-ai)dt$$

$$dt=-ds \qquad s=1-t$$

$$= \int_1^0 e^{-(ai-R-ait)^2} ai\,dt = -\int_0^1 e^{-(ai-R-ai(1-s))^2} ai\,ds$$

$$= -\int_0^1 e^{-(ai-R-ai+ais)^2} ai\,ds = -\int_0^1 e^{-(-R+ais)^2} ai\,ds$$

---

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi} dx = \int_0^{\infty} e^{-\pi(x^2-2ix\xi)} dx = \int_0^{\infty} e^{-\pi(x^2-2ix\xi-\xi^2+\xi^2)} dx$$

$$= \int_{\infty}^{\infty} e^{-\pi(x^2-2ix\xi-\xi^2)-\pi\xi^2} dx = \int_{\infty}^{\infty} e^{-\pi(x^2-2ix\xi-\xi^2)} e^{-\pi\xi^2} dx$$

$$= e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi(x-i\xi)^2} dx = e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-(\sqrt{\pi}x-i\sqrt{\pi}\xi)} dx$$

$$\int_{-\infty}^{\infty} e^{-(\sqrt{\pi}x-i\sqrt{\pi}\xi)^2} dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(x-i\sqrt{\pi}\xi)^2} dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(x+\operatorname{Im}(\xi)\sqrt{\pi}-i\sqrt{\pi}\operatorname{Re}(\xi))^2} dx$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(x-i\sqrt{\pi}\operatorname{Re}(\xi))^2} dx$$

$$\int_0^1 e^{-\pi((1-t)R+t(R+ai))^2} ai\,dt = \int_0^1 e^{-\pi(R+ait)^2} ai\,dt$$

$$\int_0^1 e^{-\pi((1-t)(-R+ai)-tR)^2}(-ai)\,dt = \int_0^1 e^{-\pi(-R+ai-ait)^2}(-ai)\,dt$$

$$\int_{-\infty}^{\infty} e^{-\pi(x-iRe(\xi))^2}dx = \int_{-\infty}^{\infty} e^{-\pi(x^2-2ixRe(\xi)-Re(\xi)^2)}dx$$

$$= \int_{-\infty} e^{-\pi x^2} e^{2\pi i x Re(\xi)} e^{\pi Re(\xi)^2} dx = e^{\pi Re(\xi)^2}\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x Re(\xi)}\, dx$$

Sup $\xi \in i\mathbb{R}$ : $Re(\xi)=0$ , $e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi(x-i\cancel{Re(\xi)})^2}dx = e^{-\pi\xi^2}\underbrace{\int_{-\infty}^{\infty} e^{-\pi x^2}dx}_{=1}$

$$= e^{-\pi\xi^2}$$

$\xi \in \mathbb{R} \Rightarrow i\xi \in i\mathbb{R}$ : $Re(i\xi)=0$

$$e^{-\pi(i\xi)^2}\int_{-\infty}^{\infty} e^{-\pi(x-iRe(i\xi))^2}dx = e^{\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi x^2}dx = e^{\pi\xi^2}$$

$\xi \in \mathbb{C}$ , $i\xi = i(Re(\xi)+iIm(\xi)) = iRe(\xi) - Im(\xi)$

$$f(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi}dx = e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi(x-iRe(\xi))^2}dx$$

$f(i\xi) = e^{-\pi(i\xi)^2}\int_{-\infty}^{\infty} e^{-\pi(x-iRe(i\xi))}dx$ , $\xi \in \mathbb{R} \Rightarrow f(i\xi) = e^{\pi\xi^2} = e^{-\pi(i\xi)^2}$

$\forall \xi \in \mathbb{C}$ , $\exists \sigma \in \mathbb{C}$ tq $\xi\sigma \in i\mathbb{R}$ , $\xi\sigma \in \{|\sigma||\xi|i, -|\sigma||\xi|i\}$

$\xi\sigma = |\xi||\sigma|i$

$f(\xi\sigma) = f(|\xi||\sigma|i) = e^{\pi|\xi|^2|\sigma|^2} = e^{-\pi(|\xi||\sigma|i)^2} =$

$$f(\xi)=\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi}\,dx = e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi(x-i\xi)^2}\,dx \qquad \forall \xi\in\mathbb{C}$$

$\xi\in i\mathbb{R} \Rightarrow i\xi\in\mathbb{R}$

$$f(\xi)=e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi(x-i\xi)^2}\,dx = e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = e^{-\pi\xi^2}$$

$$e^{-\pi\xi^2}=e^{-\pi(Re(\xi)+i\,Im(\xi))^2}=e^{-\pi(Re(\xi)^2+2i\,Re(\xi)\,Im(\xi)-Im(\xi)^2)}$$

$$=e^{-\pi(Re(\xi)^2-Im(\xi)^2+2i\,Re(\xi)\,Im(\xi))}$$

$$=e^{-\pi(Re(\xi)^2-Im(\xi)^2)}\,e^{-\pi(2i\,Re(\xi)\,Im(\xi))}$$

$\forall \xi\in\mathbb{C},\ \exists\sigma\in\mathbb{C}$ tq $\xi+\sigma\in i\mathbb{R}$

$$f(\xi+\sigma)=e^{-\pi(\xi+\sigma)^2}=e^{-\pi\xi^2}e^{-\pi\sigma^2}e^{-2\pi\xi\sigma}.$$

$e^{-\pi(x-i\xi)}$ holomorfa en $\xi$ $\forall x\in\mathbb{R}$

$f:\mathbb{C}\to\mathbb{C}$, $\quad f(\xi)=e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi(x-i\xi)^2}dx$

$g:\mathbb{C}\to\mathbb{C}$, $\quad g(\xi)=e^{-\pi\xi^2}$

. $f,g$ holomorfas tq $f(z)=g(z)$ $\forall z\in i\mathbb{R} \Rightarrow f=g$ en $\mathbb{C}$

$\therefore\quad f(\xi)=e^{-\pi\xi^2}\quad \forall\xi\in\mathbb{C}.$

P5) $f \in C^1(\Omega)$ (derivada continua). $T \subset \Omega$ triángulo.

Pd: $\int_{\partial T} f(z)dz = 0$ ocupando teo de Green.

Teo de Green. $F = (f,g) : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ $F \in C^1(\Omega)$,

$$\int_{\partial T} f dx + g dy = \int_T \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy$$

$$\int_{\partial T} f dx + g dy = \int_{\partial T} F \cdot dz$$

$f : \Omega \to \Omega$ ; $f = (u, v)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\int_{\partial T} f(z) dz = \int_T \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy \quad , \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$g = u$, $f = v$

$$\int_{\partial T} v dx + u dy = \int_T \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy = \int_T 0 \, dx dy = 0.$$

$$\int_{\partial T} f dz = \int_{\partial T} (u + iv)(dx + i dy) = \int_{\partial T} (u dx + i u dy + i v dx - v dy)$$

$$= \int_{\partial T} u dx - v dy + i \int_{\partial T} v dx + u dy$$

$$= \iint_T \left( -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy + i \iint_T \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy$$

$$= 0 + 0 = 0$$

$\gamma_1 = \gamma_1(t)$
$\gamma_2 = \gamma_2(t)$

$\gamma = \gamma_1 + i\gamma_2$

$u = u(\gamma(t))$
$v = v(\gamma(t))$

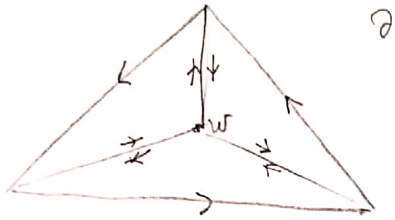
$$\int_{\partial T} f(z) dz = \int_0^1 f(\gamma(t)) \gamma'(t) dt = \int_0^1 (u + iv)(\gamma_1' + i\gamma_2') dt$$

$$= \int_0^1 (u\gamma_1' - v\gamma_2') + i(u\gamma_2' + v\gamma_1') dt = \int_0^1 (u\gamma_1 - v\gamma_2) dt + i \int_0^1 (u\gamma_2' + v\gamma_1') dt$$

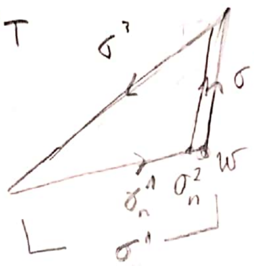
**P6** $\Omega \subseteq \mathbb{C}$ conjunto abierto, $T \subseteq \Omega$ triángulo. $f : \Omega \to \mathbb{C}$ holomorfa en $\Omega \setminus \{w\}$, $w \in T$.

Pd: $f$ acotada en vecindad de $w \Rightarrow \int_{\partial T} f(z)\,dz = 0$

dem: $\exists V_w \subseteq \Omega$ vecindad de $w$ tq $|f(z)| \leq M \quad \forall z \in V_w$




$\partial T = \partial^+ T$




$\{\sigma_n^2\}_{n \in \mathbb{N}}$

$\sigma_n^2 : [0,1] \to \mathbb{C}$

$\sigma_n^2 \xrightarrow{n \to \infty} \sigma^2$ uniformemente en $[0,1]$

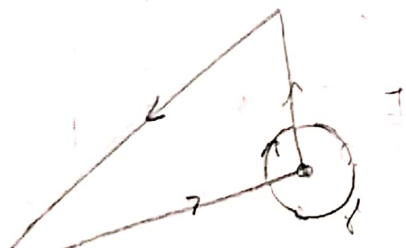
$T_n = \sigma_n^1 + \sigma_n^2 + \sigma^3$, $\int_{\partial T_n} f(z)\,dz = 0$

$|f(z)| \leq M \Rightarrow |(z-w) f(z)| \leq M(z-w) \quad \forall z \in \Omega$

$\lim_{z \to w} |(z-w) f(z)| \leq \lim_{z \to w} M|z-w| = 0 \Rightarrow \lim_{z \to w} (z-w) f(z) = 0$

$F(z) = (z-w) f(z)$ holomorfa en $\Omega \setminus \{w\}$

$$\lim_{\substack{z \to w \\ z \neq w}} \frac{F(z) - F(w)}{z-w} = \lim_{z \to w} \frac{(z-w) f(z)}{z-w} = \lim_{z \to w} f(z)$$

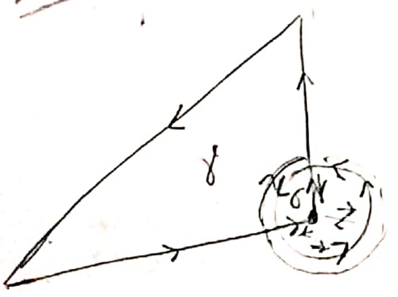



$$\left| \int_\gamma f(z)\,dz \right| = \left| \int_0^{2\pi} f(\gamma(t))\gamma'(t)\,dt \right| \leq M \int_0^{2\pi} |\gamma'(t)|\,dt$$

$\gamma = w + Re^{i\theta}$

$\boxed{\gamma \subset V_w}$

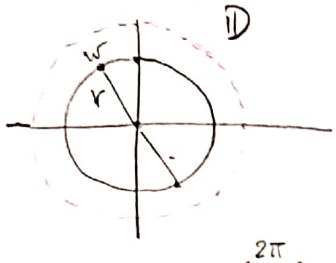
$$= M \int_0^{2\pi} |Ri|\,dt = 2MR\pi \xrightarrow{R \to 0} 0$$




$$\int_{\partial T} f(z)\,dz = \cancel{\int_\gamma f(z)\,dz}^{\,0} + \int_{\sigma = \sigma(R)} f(z)\,dz = \int_{\sigma = \sigma(R)} f(z)\,dz$$

$$\int_{\sigma = \sigma(R)} f(z)\,dz \xrightarrow{R \to 0} 0 \qquad \therefore \int_{\partial T} f(z)\,dz = 0$$

P7] $f:\mathbb{D}\to\mathbb{C}$ holomorfa. $d=\sup\{|f(z)-f(w)| \,/\, z,w\in\mathbb{D}\}$ diámetro de $f(\mathbb{D})$. Se tiene que $2|f'(0)|\le d$.




$r<1$, $\quad f'(0)=\frac{1}{2\pi i}\int_{|w|=r}\frac{f(w)}{w^2}\,dw.$

$$f'(0)=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(re^{i\theta})}{r^2e^{2i\theta}}\,rie^{i\theta}\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(re^{i\theta})}{re^{i\theta}}\,d\theta$$

$\gamma(t)=-re^{i\theta}$, $\theta\in[0,2\pi]$

$$f'(0)=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(-re^{i\theta})}{(re^{i\theta})^2}(-rie^{i\theta})\,d\theta=-\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(-re^{i\theta})}{(re^{i\theta})^2}(rie^{i\theta})\,d\theta$$

$$=-\frac{1}{2\pi i}\int_{|w|=r}\frac{f(-w)}{w^2}\,dw \qquad \therefore\ 2f'(0)=\frac{1}{2\pi i}\int_{|w|=r}\frac{f(w)-f(-w)}{w^2}\,dw$$

$$=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(re^{i\theta})-f(-re^{i\theta})}{(re^{i\theta})^2}(rie^{i\theta})\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(re^{i\theta})-f(-re^{i\theta})}{re^{i\theta}}\,d\theta$$

$$\Rightarrow 2|f'(0)|\le\sup_{\theta\in[0,2\pi]}|f(re^{i\theta})-f(-re^{i\theta})|\,\frac{1}{r}\qquad\forall r<1$$

$$\le\sup_{z,w\in\mathbb{D}}|f(z)-f(w)|\,\frac{1}{r}\qquad\forall\, 0<r<1$$

$$\therefore\ 2|f'(0)|\le\sup_{z,w\in\mathbb{D}}|f(z)-f(w)|$$

obs. la igualdad se alcanza cuando $f$ es lineal, $f(z)=a_0+a_1z$

$f'(z)=a_1$, $\quad f'(0)=a_1$. $\quad |f(z)-f(w)|=|a_1||z-w|$

$$2|f'(0)|\le\sup_{z,w\in\mathbb{D}}|f(z)-f(w)|\ \Rightarrow\ 2|a_1|\le\sup_{z,w\in\mathbb{D}}|a_1||z-w|=|a_1|\overbrace{\sup_{z,w\in\mathbb{D}}|z-w|}^{=2}$$

$$\therefore\ 2|a_1|=2|a_1|$$

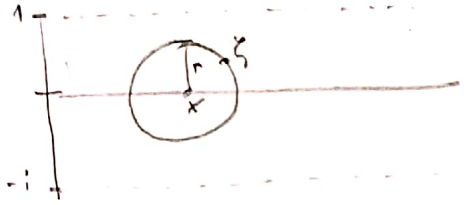
P8) $f : \Omega \to \mathbb{C}$, holomorfa tq $\Omega = \{ z \in \mathbb{C} \mid -1 < \operatorname{Im}(z) < 1 \}$

$|f(z)| \leq A(1+|z|)^\eta$, $\eta \in \mathbb{R}$ (fijo).

Pd: $\forall n \in \mathbb{N}$, $\exists A_n \geq 0$ tq $|f^{(n)}(x)| \leq A_n (1+|x|)^\eta \quad \forall x \in \mathbb{R}$

dem. $|f(x)| \leq A(1+|x|)^\eta \quad \forall x \in \mathbb{R}$.

$$f^{(n)}(x) = \frac{n!}{2\pi i} \int_{|\zeta - x| = r} \frac{f(\zeta)}{(\zeta - x)^{n+1}} \, d\zeta$$



$$\overline{B(x,r)} \subset \Omega$$

$$f^{(n)}(x) = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(x + re^{i\theta})}{(re^{i\theta})^{n+1}} \, r i e^{i\theta} \, d\theta = \frac{n!}{2\pi} \int_0^{2\pi} \frac{f(x+re^{i\theta})}{(re^{i\theta})^n} \, d\theta$$

$$|f^{(n)}(x)| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(x+re^{i\theta})|}{r^n} \, d\theta \leq \frac{n!}{2\pi r^n} \sup_{|\zeta - x| = r} |f(\zeta)| \, 2\pi$$

$$\Rightarrow |f^{(n)}(x)| \leq \frac{n! \sup_{|\zeta - x| = r} |f(\zeta)|}{r^n} \leq \underbrace{\frac{n!}{r^n} \sup_{|\zeta - x| = r} |f(\zeta)|}_{\in \mathbb{R}^+} \underbrace{(1+|x|)^\eta}_{\geq 1}$$

$$\therefore |f^{(n)}(x)| \leq \frac{n!}{r^n} \sup_{|\zeta - x| = r} |f(\zeta)| \, (1+|x|)^\eta$$

Tomar $A_n := \frac{n!}{r^n} \sup_{|\zeta - x| = r} |f(\zeta)|$ tal que

$$|f^{(n)}(x)| \leq A_n (1+|x|)^\eta$$

∎

P9| $\Omega \subseteq \mathbb{C}$ abierto acotado. $\varphi : \Omega \to \Omega$ holomorfa.

Pd: $\exists z_0 \in \Omega$ tq $\varphi(z_0) = z_0$, $\varphi'(z_0) = 1 \Rightarrow \varphi$ lineal

dem. $\sigma(z) = \varphi(z) - z_0$ holomorfa $\Omega \to \Omega$, $\sigma(z_0) = 0$

$\sigma'(z) = \varphi'(z) \Rightarrow \sigma'(z_0) = \varphi'(z_0) = 1$

Sup que es verdad para $z_0 = 0$, $\varphi(0) = 0$, $\varphi'(0) = 1$

$z_0 \neq 0$: $\sigma(z) := \varphi(z_0 + z) - z_0$, $\sigma(0) = \varphi(z_0) - z_0 = 0$

$\sigma'(z) = \varphi'(z_0 + z)$, $\sigma'(0) = \varphi'(z_0) = 1$

$\Rightarrow \varphi(z_0 + z) = \sigma(z) - z \Rightarrow \varphi(z) = \sigma(z - z_0) - (z - z_0)$ lineal.

Supongamos que $z_0 = 0$

expansión de $\varphi$ en vecindad de 0

$\varphi(z) = z + a_n z^n + \mathcal{O}(z^{n+1})$. $\varphi_2(z) = \varphi \circ \varphi(z) = \varphi(\varphi(z))$

$= \varphi(z) + a_n(\varphi(z))^n + \mathcal{O}((\varphi(z))^{n+1}) = z + a_n z^n + \mathcal{O}(z^{n+1}) + a_n(z + a_n z^n + \mathcal{O}(z^{n+1}))^n + \mathcal{O}((\varphi(z))^n$

$= z + a_n z^n + \mathcal{O}(z^{n+1}) + a_n(z + a_n z^n + \mathcal{O}(z^{n+1}))^n + \mathcal{O}((z + a_n z^n + \mathcal{O}(z^{n+1}))^{n+1})$

$= z + a_n z^n + \mathcal{O}(z^{n+1}) + a_n(z^n + \mathcal{O}(z^{n+1})) + \mathcal{O}(z^{n+1})$

$= z + a_n z^n + \mathcal{O}(z^{n+1}) + a_n z^n + \mathcal{O}(z^{n+1}) + \mathcal{O}(z^{n+1}) = z + 2a_n z^n + \mathcal{O}(z^{n+1})$

Por inducción, $\forall k \in \mathbb{N}$, $\varphi_k(z) = z + k a_n z^n + \mathcal{O}(z^{n+1})$

$\varphi_k'(z) = 1 + kn a_n z^{n-1} + \mathcal{O}(z^n)$

⋮

Averiguar más sobre "$\mathcal{O}(z)$".

P11. f holomorfa en $D_{R_0}$ (centro 0 y radio $R_0$)

(a) $\forall R,\ 0<R<R_0,\ |z|<R$ : $f(z)=\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\, Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi$

(b) Demostrar que $Re\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right)=\frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}$

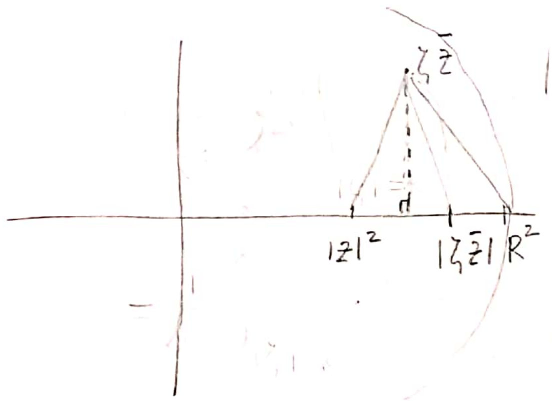
dem. (a) $w=R^2/\bar z$ , $\int_{|\zeta|=r}\frac{f(\zeta)}{\zeta-w}\,d\zeta = f(w)\,2\pi i$

$|w|=R^2/|\bar z| > R^2/R > R$ ($|z|<R \Rightarrow \frac{1}{|z|}>1/R$)

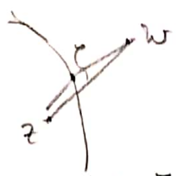
$\Rightarrow \frac{f(\zeta)}{\zeta-w}$ holomorfa en $\underset{B(0,R)}{B(0,|w|)\,\cup\!\!\!\!\cup}$ $\Rightarrow \int_{|\zeta|=R}\frac{f(\zeta)}{\zeta-w}\,d\zeta=0$ (Teorema de Cauchy)

$$\left|\frac{\zeta-z}{\zeta-w}\right| = \left|\frac{\zeta-z}{\zeta-R^2/\bar z}\right| = \left|\frac{\zeta-z}{\frac{\zeta\bar z-R^2}{\bar z}}\right| = \left|\frac{\zeta\bar z-\bar z z}{\zeta\bar z-R^2}\right| = \left|\frac{\zeta\bar z-|z|^2}{\zeta\bar z-R^2}\right|<1$$

$|\zeta\bar z|=|\zeta||\bar z|<R^2$   $|\zeta\bar z|=R|z|>|z|^2$




$|\zeta\bar z-R^2|>|\zeta\bar z-|z|^2|$




$\left|\frac{w-\bar z}{w-z}\right|<1$

$$Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)=\frac{1}{2}\left[\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}+\frac{\overline{Re^{i\varphi}}+\bar z}{\overline{Re^{i\varphi}}-\bar z}\right]$$

$$=\frac{1}{2}\left[\frac{\zeta+z}{\zeta-z}+\frac{\bar\zeta+\bar z}{\bar\zeta-\bar z}\right]=\frac{1}{2}\left[\frac{\zeta+z}{\zeta-z}+\frac{\zeta\bar\zeta+\zeta\bar z}{\zeta\bar\zeta-\zeta\bar z}\right]$$

$f(z)=\frac{1}{2\pi i}\int_{|\zeta|=R}\frac{f(\zeta)}{\zeta-z}\,d\zeta$ , $0=\int_{|\zeta|=R}\frac{f(\zeta)}{\zeta-w}\,d\zeta=\int_{|\zeta|=R}\frac{f(\zeta)}{\zeta-\frac{R^2}{\bar z}}\,d\zeta=\int_{|\zeta|=R}\frac{f(\zeta)\bar z}{\zeta\bar z-R^2}\,d\zeta$

$$=\frac{1}{2}\left[\frac{\zeta+z}{\zeta-z}-\frac{\zeta\bar z+R^2}{\zeta\bar z-R^2}\right]$$

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$$\int_0^{2\pi} f(\zeta)\ \frac{1}{2}\left[\frac{\zeta+z}{\zeta-z}+\frac{\bar\zeta+\bar z}{\bar\zeta-\bar z}\right]d\varphi = \frac{1}{2}\int_0^{2\pi} f(\zeta)\left[\frac{\zeta+z}{\zeta-z}-\frac{\zeta\bar z+R^2}{\zeta\bar z-R^2}\right]d\varphi$$

$$=\frac{1}{2}\int_0^{2\pi} f(\zeta)\frac{\zeta+z}{\zeta-z}d\varphi-\frac{1}{2}\int_0^{2\pi} f(\zeta)\frac{\zeta\bar z+R^2}{\zeta\bar z-R^2}d\varphi$$

$$\int_0^{2\pi} f(\zeta)\frac{\zeta+z}{\zeta-z}d\varphi=\int_0^{2\pi}\frac{f(\zeta)\zeta}{\zeta-z}d\varphi+z\int_0^{2\pi}\frac{f(\zeta)}{\zeta-z}d\varphi$$

$$=\frac{2\pi i}{i}\left(\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\varphi})}{Re^{i\varphi}-z}Rie^{i\varphi}d\varphi\right)+z\int_0^{2\pi}\frac{f(\zeta)}{\zeta-z}d\varphi=2\pi f(z)+z\int_0^{2\pi}\frac{f(\zeta)}{\zeta-z}d\zeta$$

$$\int_0^{2\pi} f(\zeta)\frac{\zeta\bar z+R^2}{\zeta\bar z-R^2}d\varphi=\int_0^{2\pi}\frac{f(\zeta)\zeta\bar z}{\zeta\bar z-R^2}d\varphi+R^2\int_0^{2\pi}\frac{f(\zeta)}{\zeta\bar z-R^2}d\varphi$$

$$=\frac{\bar z}{i}\int_0^{2\pi}\frac{f(Re^{i\varphi})(Re^{i\varphi})i\,d\varphi}{Re^{i\varphi}\bar z-R^2}+R^2\int_0^{2\pi}\frac{f(\zeta)}{\zeta\bar z-R^2}d\varphi=\frac{\bar z}{i}\int_{|\zeta|=R}\frac{f(\zeta)}{\zeta\bar z-R^2}d\zeta+R^2\int_0^{2\pi}\frac{f(\zeta)}{\zeta\bar z-R^2}d\varphi$$

(the first term of the last expression is marked $0$)

$$\int_0^{2\pi} f(\zeta)\ \frac{1}{2}\left[\frac{\zeta+z}{\zeta-z}+\frac{\bar\zeta+\bar z}{\bar\zeta-\bar z}\right]d\varphi=\frac{1}{2}\left(2\pi f(z)+z\int_0^{2\pi}\frac{f(\zeta)}{\zeta-z}d\zeta\right)-\frac{1}{2}R^2\int_0^{2\pi}\frac{f(\zeta)}{\zeta\bar z-R^2}d\varphi$$

$$=\pi f(z)+\frac{1}{2}\left[\int_0^{2\pi}\frac{f(\zeta)z}{\zeta-z}d\varphi-\int_0^{2\pi}\frac{f(\zeta)R^2}{\zeta\bar z-R^2}d\varphi\right]$$

$$\int_0^{2\pi}\frac{f(\zeta)z}{\zeta-z}d\varphi-\int_0^{2\pi}\frac{f(\zeta)R^2}{\zeta\bar z-R^2}d\varphi=\int_0^{2\pi}\frac{f(\zeta)\zeta z\bar z-\cancel{f(\zeta)zR^2}-f(\zeta)\zeta R^2+\cancel{f(\zeta)zR^2}}{(\zeta-z)(\zeta\bar z-R^2)}d\varphi$$

$$=\int_0^{2\pi} f(\zeta)\frac{\zeta z\bar z-\zeta R^2}{(\zeta-z)(\zeta\bar z-R^2)}d\varphi=\int_0^{2\pi} f(\zeta)\frac{\zeta z-\zeta w}{(\zeta-z)(\zeta-w)}d\varphi=\int_0^{2\pi} f(\zeta)\zeta\frac{z-w}{(\zeta-z)(\zeta-w)}d\varphi$$

$$=\int_0^{2\pi} f(\zeta)\zeta\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-w}\right)d\varphi=\int_0^{2\pi}\frac{f(\zeta)\zeta}{\zeta-z}d\varphi-\cancel{\int_0^{2\pi}\frac{f(\zeta)\zeta}{\zeta-w}d\varphi}=2\pi f(z)$$

$$=2\pi\left(\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\varphi})Re^{i\varphi}i}{Re^{i\varphi}-z}d\varphi\right)=2\pi f(z)$$

$$\therefore \int_0^{2\pi} f(Re^{i\varphi}) \, Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right) d\varphi = \pi f(z) + \frac{1}{2}\left[2\pi f(z)\right] = \pi f(z) + \pi f(z)$$

$$\therefore \quad f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi}) \, Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right) d\varphi$$

(b) $$Re\left(\frac{Re^{i\varphi}+r}{Re^{i\varphi}-r}\right) = \frac{1}{2}\left(\frac{Re^{i\varphi}+r}{Re^{i\varphi}-r} + \frac{Re^{-i\varphi}+r}{Re^{-i\varphi}-r}\right)$$

$$= \frac{1}{2}\left(\frac{R^2 - \cancel{Rre^{i\varphi}} + \cancel{Rre^{-i\varphi}} - r^2 + R^2 + \cancel{Rre^{i\varphi}} - \cancel{Rre^{-i\varphi}} - r^2}{R^2 - Rre^{i\varphi} - Rre^{-i\varphi} + r^2}\right)$$

$$= \frac{1}{2}\left(\frac{2R^2 - 2r^2}{R^2 - Rr(e^{i\varphi}+e^{-i\varphi}) + r^2}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\varphi + r^2}$$

$$\therefore \quad Re\left(\frac{Re^{i\varphi}+r}{Re^{i\varphi}-r}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\varphi + r^2}$$

P12 $\quad u: \mathbb{D} \to \mathbb{R}$, $u \in C^2(\mathbb{D})$ armónica ($\Delta u = 0$)

a) $\exists f: \mathbb{D} \to \mathbb{C}$ holomorfa tq $Re(f) = u$ (Hint: $f'(z) = 2\dfrac{\partial u}{\partial z}$)

Recordatorio: $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$, $\dfrac{\partial}{\partial \bar z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$

$f$ holomorfa $\Rightarrow \dfrac{\partial f}{\partial x} = -i\dfrac{\partial f}{\partial y} \iff \dfrac{\partial f}{\partial \bar z} = 0$

$f$ holomorfa $\Rightarrow f'(z_0) = \dfrac{\partial f}{\partial z}(z_0) \quad \forall z_0$

$$\frac{\partial}{\partial z} f = \frac{1}{2}\left(\frac{\partial}{\partial x} f - i\frac{\partial}{\partial y} f\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(2\frac{\partial u}{\partial x} - 2i\frac{\partial u}{\partial y}\right) =$$

$$= 2\left(\frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)\right) = 2\frac{\partial u}{\partial z}$$

(b) (Fórmula integral de Poisson).

$u : \overline{\mathbb{D}} \to \mathbb{R}$ continua, $u|_{\mathbb{D}}$ armónica.

Pd: $\forall z = re^{i\theta}$ : $u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)\, u(\varphi)\, d\varphi$

donde $P_r(\gamma)$ "núcleo de Poisson" : $P_r(\gamma) = \frac{1-r^2}{1-2r\cos\gamma + r^2}$

dem. $u|_{\mathbb{D}}$ armónica $\Rightarrow \exists\, f : \mathbb{D} \to \mathbb{C}$ holomorfa tal que $Re(f) = u|_{\mathbb{D}}$

Problema 11 : $f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\, Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right) d\varphi$

Para $0<R<1$, $\forall |z|<R$

$\forall z = re^{i\theta}$, $r<R$ : $u(re^{i\theta}) = Re\, f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\varphi})\, Re\left(\frac{Re^{i\varphi}+re^{i\theta}}{Re^{i\varphi}-re^{i\theta}}\right) d\varphi$

$$\Rightarrow u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\varphi})\, Re\left(\frac{Re^{i(\varphi-\theta)}+r}{Re^{i(\varphi-\theta)}-r}\right) d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\varphi})\, \frac{R^2-r^2}{R^2-2Rr\cos(\varphi-\theta)+r^2}\, d\varphi$$

Por continuidad de $u$ en $\partial\mathbb{D}$,

$$u(re^{i\theta}) = \frac{1}{2\pi}\lim_{R\to 1^-}\int_0^{2\pi} u(Re^{i\varphi})\, \frac{R^2-r^2}{R^2-2Rr\cos(\varphi-\theta)+r^2}\, d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})\, \frac{R^2-r^2}{R^2-2Rr\cos(\varphi-\theta)+r^2}\, d\varphi \qquad \blacksquare$$

$\forall r<1$ (i.e, $\forall z \in \mathbb{D}$).

$g : \mathbb{D} \to \mathbb{C}$, $\quad g(z) := 2 \dfrac{\partial u}{\partial z}$ . Pd: $g$ holomorfa

$$g(z) = 2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \to g \text{ continua. } \quad g = p + iq, \quad p = \frac{\partial u}{\partial x}, \quad q = -\frac{\partial u}{\partial y}$$

$$\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial p}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial q}{\partial x} = -\frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial q}{\partial y} = -\frac{\partial^2 q}{\partial y^2} \quad \text{continuas}$$

$$\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial x^2} \underset{\substack{\downarrow \\ u \text{ armónica}}}{=} -\frac{\partial^2 u}{\partial y} = \frac{\partial q}{\partial y}, \quad \frac{\partial q}{\partial x} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial p}{\partial y}$$

$$\therefore \frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}, \quad \frac{\partial q}{\partial x} = -\frac{\partial p}{\partial y} \quad \text{cc's \underline{Cauchy-Riemann}}$$

$\therefore$ $g$ holomorfa.

Como $g : \mathbb{D} \to \mathbb{C}$ holomorfa y $\mathbb{D}$ estrellado $\Rightarrow \exists F : \mathbb{D} \to \mathbb{C}$ primitiva de $g$ tq $F' = g$.

$$F(z) = \int_{[0,z]} g(\zeta)\, d\zeta = 2\int_{[0,z]} \frac{\partial u}{\partial z} d\zeta$$

$g$ holomorfa $\Rightarrow$ $g'$ holomorfa, $\quad g' = 2\dfrac{\partial u}{\partial z} =$

$$F(z) = \int_{[0,z]} \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) d\zeta = \int_0^1 \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)\left(\frac{\partial}{\partial t}\gamma_1 + i\frac{\partial}{\partial t}\gamma_2\right) dt$$

$$= \int_0^1 \left(\frac{\partial u}{\partial x}\frac{\partial \gamma_1}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial \gamma_2}{\partial t}\right) dt + i\int_0^1 \left(\frac{\partial u}{\partial x}\frac{\partial \gamma_2}{\partial t} - \frac{\partial u}{\partial y}\frac{\partial \gamma_1}{\partial t}\right) dt,$$

$$\Rightarrow \operatorname{Re}(F(z)) = \int_0^1 \left(\frac{\partial u}{\partial x}\frac{\partial \gamma_1}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial \gamma_2}{\partial t}\right) dt = \int_0^1 \frac{\partial}{\partial t}(u \circ \gamma)\, dt$$

$$= u(\gamma(1)) - u(\gamma(0))$$

$$= u(z) - \underbrace{u(0)}_{K \in \mathbb{C}} \qquad \therefore \quad \operatorname{Re}(F(z)) = u(z) + K \quad \forall z \in \mathbb{D}$$

$$\operatorname{Re}(F) = u + K$$

P13 $f: \mathbb{C} \to \mathbb{C}$ holomorfa. $\forall z_0 \in \mathbb{C}$, $\exists n \in \mathbb{N}$ $(n = n(z_0))$, tq $c_n = 0$

donde $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$

Pd: $f$ es un polinomio.

dem: $z_0 \in \mathbb{C}$, $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ expansión analítica de $f$ en una vecindad de $z_0$

$\forall n \in \mathbb{N}$, $c_n = \dfrac{f^{n}(z_0)}{n!} = 0 \iff f^{(n)}(z_0) = 0$

$\xi : \mathbb{C} \longrightarrow \mathbb{N}$

$z \longmapsto n$, $n$ es tal que $f^{(n)}(z) = 0$.

Sup que si $\forall z \neq w$ en $\mathbb{C}$ $\Rightarrow \xi(z) \neq \xi(w)$ $\Rightarrow$

$\Rightarrow \xi : \mathbb{C} \to \mathbb{N}$ inyectiva $(\Rightarrow\Leftarrow)$ ($\mathbb{C}$ no numerable)

Además $\mathbb{C} = \bigcup_{n \in f(\mathbb{N})} \xi^{-1}(n)$ $\Rightarrow$ $\exists n \in f(\mathbb{N})$ tq $f^{-1}(n)$ no numerable

$\Rightarrow \forall w \in f^{-1}(n) : f^{(n)}(w) = 0$ En particular $f^{-1}(n)$ posee punto de acumulación

$\therefore f^{(n)}(z) = 0 \quad \forall z \in \mathbb{C}$

$\therefore f^{(k)}(z) = 0 \quad \forall z \in \mathbb{C}, \forall k \geq n$

Así, $f(z) = f(z_0) + f'(z_0)(z - z_0) + \dots + \dfrac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)$ ☐

P15 | $f : \overline{\mathbb{D}} \to \mathbb{C}$ continua, $f(z) \neq 0$. $f|_{\mathbb{D}}$ holomorfa

$|f(\partial \mathbb{D})| = 1$. Demostrar que $f$ es constante.

dem. $g(z) = \dfrac{1}{\overline{f(1/\bar{z})}}$ $\forall |z| \geq 1$. $z \in \partial\mathbb{D}$, $z = e^{i\theta}$

$$g(e^{i\theta}) = \frac{1}{\overline{f(1/e^{-i\theta})}} = \frac{1}{\overline{f(e^{i\theta})}}$$

$f(e^{i\theta}) = e^{i\varphi} \Rightarrow g(e^{i\theta}) = \dfrac{1}{e^{-i\varphi}} = e^{i\varphi} = f(e^{i\theta})$. $f = g$ en $\partial\mathbb{D}$

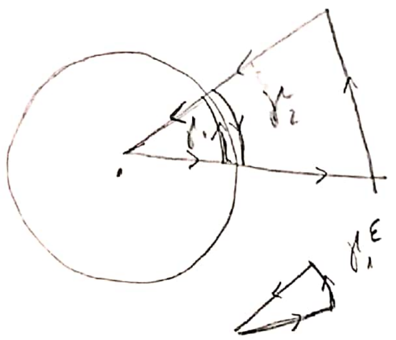
$g$ holomorfa? $z_0, z \in \mathbb{C}$ tq $|z_0|, |z| > 1 \Rightarrow \dfrac{1}{\bar{z}_0}, \dfrac{1}{\bar{z}} \in \mathbb{D}$

$f$ holomorfa en $z_0$:

$$f(1/\bar{z}) = \sum a_n (1/\bar{z} - 1/\bar{z}_0)^n \Rightarrow \overline{f(1/\bar{z})} = \sum \bar{a}_n (1/z - 1/z_0)^n$$

$\therefore$ $g(z) = \dfrac{1}{\overline{f(1/\bar{z})}}$ holomorfa en $z_0$

$$F(z) = \begin{cases} f(z), & z \in \overline{\mathbb{D}} \\ g(z), & |z| > 1 \end{cases} \quad \text{holomorfa}$$



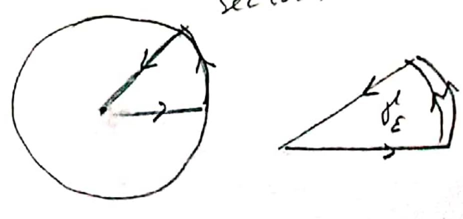

$$\int_{\partial T} F(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} g(z)\,dz.$$

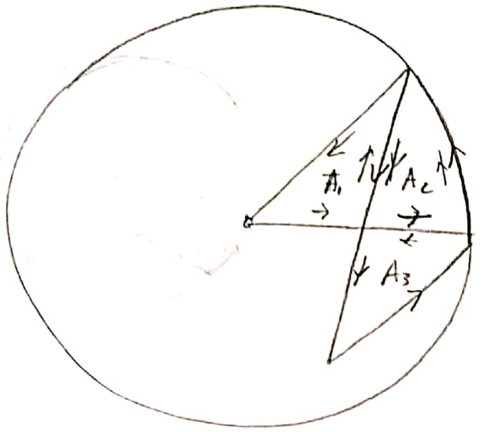
$$\int_{\gamma_1} f\,dz = \lim_{\varepsilon \to 0} \int_{\gamma_1^\varepsilon} f(z)\,dz = 0$$

$$\int_{\gamma_2} g(z)\,dz = \lim_{\varepsilon \to 0} \int_{\gamma_2^\varepsilon} g(z)\,dz = 0$$

sectores circulares

Podemos calcular $\int_{\gamma} f(z)\,dz$, donde






$\gamma = \partial A_2 + \partial A_3$ , $\tilde{\gamma} = \partial($sección circular$)$

$\gamma = \tilde{\gamma} + A_3 - A_1$ dc.

$$\therefore \int_\gamma f(z)dz = \int_{\partial A_3} f(z)dz + \underbrace{\int_{\partial A_2} f(z)dz}$$

$$= \underbrace{\int_{\partial A_1} f(z)dz}_{=0} + \int_{\partial A_2} f(z)dz = \int_{\tilde{\gamma}} f(z)dz = 0$$

$F : \mathbb{C} \to \mathbb{C}$ holomorfa tal que $|F(z)| = 1$ $\forall z \in \partial\mathbb{D}$

Como $F$ acotada $\Rightarrow$ $F$ constante $\therefore$ $f$ constante. ☒

Problemas adicionales.

[P1] $f(z)=\sum_{n=0}^{\infty} z^{2^n}$ , $|z|<1$ .

$R^{-1}=\limsup |a_n|^{1/n}$ , $a_n\in\{0,1\}$ $\forall n$: $\boxed{R=1}$

f no puede ser extendida analíticamente a $\overline{\mathbb{D}}$.

... def $\forall p,k\in\mathbb{Z}$ : $\theta=\theta(p,k)=2\pi p/2^k$

$$z=re^{i\theta}=re^{i2\pi p/2^k}=r\left(\cos\left(\frac{2\pi p}{2^k}\right)+i\,\mathrm{sen}\left(\frac{2\pi p}{2^k}\right)\right)$$

$$f(z)=\sum_{n=0}^{\infty} z^{2^n}=\sum_{n=0}^{\infty} r^{2^n}(e^{i\theta})^{2^n}=\sum_{n=0}^{\infty} r^{2^n}e^{i2^n\theta}$$

$2^n\theta=2^n\cdot 2\pi p/2^k$ : $f(z)=\sum_{n=0}^{k-1} z^{2^n}+r^{2^k}+r^{2^{k+1}}+r^{2^{k+2}}+\dots$

$$f(z)=\sum_{n=0}^{k-1} z^{2^n}+r^{2^k}(1+r^2+r^4+r^8+\dots)$$

$$Re(f(z))=\underset{\mathbb{R}}{\sigma_r}+\sum_{m=0}^{\infty} r^{2^{k+m}}$$

$\forall M>0$ $\exists r_0\in(0,1)$ tq $Re(f(z))\geq M$

$\therefore$ $\lim_{r\to 1} Re(f(z))=\infty$

$\therefore$ $\lim_{r\to 1}|f(re^{i\theta})|=\infty$.

P21 $\quad F(z) = \sum_{n=1}^{\infty} d(n) z^n \quad , \quad |z| < 1$

$d(n)$: nº de divisores de $n$.

Pd: $\displaystyle \sum_{n=1}^{\infty} d(n) z^n = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$

dem $\displaystyle \sum_{n=1}^{\infty} z^n \frac{1}{1-z^n} = \sum_{n=1}^{\infty} z^n \sum_{m=0}^{\infty} (z^n)^m = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (z^n)^{m+1} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} z^{nm+n}$

$n=1$: $\displaystyle z \frac{1}{1-z} = z\left( \sum_{m=0}^{\infty} z^m \right) = z + z^2 + z^3 + z^4 + \dots$

$n=2$: $\displaystyle z^2 \frac{1}{1-z^2} = z^2 \left( \sum_{m=0}^{\infty} z^{2m} \right) = z^2 + z^4 + z^6 +$

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Tarea: Buscar qué significa 'rate'.

**Prop.** Si $\Omega \subseteq \mathbb{C}$ abierto conexo, $f : \Omega \to \mathbb{C}$ holomorfa tiene cero aislado en $z_0 \in \Omega$, entonces existen únicos $n \geq 1$ entero y $M : \Omega \to \mathbb{C}$ holomorfa tales que

$$\begin{cases} f(z) = (z-z_0)^n M(z) \\ M(z_0) \neq 0 \end{cases}$$

$f(z) = (z-z_0)^n M(z) \quad \forall z \in B(z_0,R)$

$\Rightarrow M(z) = \dfrac{f(z)}{(z-z_0)^n} \quad \forall z \in B(z_0,R) \setminus \{z_0\}$

$\dfrac{f(z)}{(z-z_0)^n}$ holomorfa en $\Omega \setminus \{z_0\}$

$$f(z) = \underbrace{a_n}_{\neq 0}(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \dots$$

$$M(z) = \begin{cases} a_n + a_{n+1}(z-z_0) + \dots, & z \in B(z_0,R) \\ \dfrac{f(z)}{(z-z_0)^n}, & z \in \Omega \setminus \{z_0\} \end{cases}$$

$n$ : orden o multiplicidad del cero.

Si $n = 1$ decimos que $z_0$ es cero simple de $f$.

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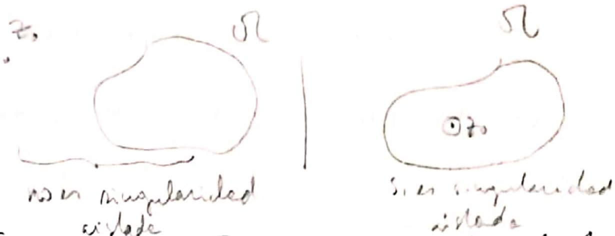



## Singularidades.

**Def.** $\Omega \subseteq \mathbb{C}$ abierto, decimos que un punto $z_0 \in \mathbb{C} \setminus \Omega$ es singularidad aislada de una función holomorfa $f : \Omega \to \mathbb{C}$ si $\Omega \cup \{z_0\}$ es conjunto abierto, o sea,

$\exists r > 0$ tq $B(z_0, r) \subseteq \Omega \cup \{z_0\}$

$\forall w \in B(z_0,r),\ w \neq z_0 \Rightarrow w \in \Omega$

$$\exists r > 0 \text{ tq } \Omega \cap B(z_0, r) = B(z_0, r) \setminus \{z_0\} \checkmark$$

$B(z_0,r) \subseteq \Omega \cup \{z_0\} \Rightarrow B(z_0,r) \cap \Omega = B(z_0,r) \setminus \{z_0\}$

$B(z_0,r) \subseteq \Omega \cup \{z_0\} \Leftrightarrow B(z_0,r) \cap (\Omega \cup z_0) = B(z_0,r) \mid \cap \{z_0\}^c$

**Def.** Una singularidad aislada de $f : \Omega \to \mathbb{C}$ es llamada singularidad removible si la función $f$ se extiende a una función holomorfa en $\Omega \setminus \{z_0\}$. debiera ser $\Omega \cup \{z_0\}$

$B(z_0,r) \cap (\Omega \setminus \{z_0\} \cup \Omega_1 \setminus \{z_0\}) = B(z_0,r) \setminus \{z_0\}$

$B(z_0,r) \cap \Omega = B(z_0,r) \setminus \{z_0\}$

**Prop.** $z_0$ singularidad removible de $f \iff \lim_{z \to z_0} (z - z_0) f(z) = 0$.

$z_0$ singularidad removible de $f \Rightarrow \exists g : \Omega \cup \{z_0\} \to \mathbb{C}$ holo tq $g|_{\Omega} = f$
$f : \Omega \to \mathbb{C}$

**dem.** $(\Rightarrow)$ trivial

$\forall z \neq z_0$ : $(z - z_0) f(z) = (z - z_0) g(z)$
$z \in \Omega$

$(\Leftarrow)$ Sea $g(z) = \begin{cases} (z - z_0) f(z) & , z \in \Omega \\ 0 & , z = z_0 \end{cases}$

$g$ acotada en vecindad de $z_0$

$z \to z_0 \Rightarrow (z - z_0) g(z) \to 0$

$g$ es continua en $\Omega \cup \{z_0\}$ y holomorfa en $\Omega$. $f$ acotada en $V_{z_0} \setminus \{z_0\}$.

Luego por el teo. de Morera, $g$ es holomorfa en $\Omega \cup \{z_0\}$

$g(z) = (z - z_0) f(z)$ holo en $\Omega \cup \{z_0\}$, $g(z_0) = 0$ | $g(z) = a_1 (z - z_0) + a_2 (z - z_0)^2 + \dots$

$h(z) = \dfrac{g(z)}{(z - z_0)} : \Omega \to \mathbb{C}$ holo. $\quad h(z) = \begin{cases} \dfrac{g(z)}{(z - z_0)} & , z \in \Omega \\ a_1 & , z = z_0 \end{cases}$

Por la proposición precedente

Morera: $h$ holomorfa en $\Omega \cup \{z_0\}$ y $h|_{\Omega} = f$

$$g(z) = (z - z_0)^n h(z)$$

$\hookrightarrow$ holomorfa en $\Omega \cup \{z_0\}$

$\Rightarrow f(z) = \underbrace{(z - z_0)^{n-1} h(z)}_{\text{holomorfa en } \Omega \cup \{z_0\}}$ , $z \in \Omega$. $z \in B(z_0, R) \setminus \{z_0\}$ ✓

✓

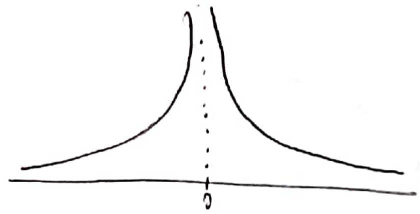
Hasta acá: la función buscada es $h(z) = (z - z_0)^{n-1} h(z)$

**Obs.** La proposición no tiene análogo real:

$f(x) = \dfrac{1}{\sqrt{|x|}}$ , $x_0 = 0$

$(x - 0) \dfrac{1}{\sqrt{|x|}} = \dfrac{x}{\sqrt{|x|}} \xrightarrow{x \to 0} 0$

$\dfrac{1}{\sqrt{|x|}}$




$$\frac{x - 0}{\sqrt{|x|}} \longrightarrow 0$$ pero no es real removible. ok.

**Def.** Una singularidad aislada $z_0$ de $f$ es llamada polo de $f$ si

$$\lim_{z \to z_0} |f(z)| = \infty$$ ✓

En caso de $z = z_0$ singularidad removible $|f(z)| \leq K$ para $z \in B(z_0, \delta) \setminus \{z_0\}$.

**Ej.** $f(z) = \dfrac{1}{z}$ en $z_0 = 0$.

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$\lim_{z \to z_0} |f(z)| = \infty \iff \forall M > 0\ \exists \delta > 0$ tq $\forall z \in B(z_0, \delta) \setminus \{z_0\}, |f(z)| \geq M$

Si $z_0$ es polo de $f$, entonces la función siguiente es holomorfa en $\Omega \cup \{z_0\}$.

$z_0$ polo de $f \Rightarrow \exists \delta > 0$ tq $\forall z \in B(z_0, \delta) \setminus \{z_0\}, |f(z)| > 1$
$\Rightarrow g : B(z_0, \delta) \to \mathbb{C}$ como $g(z) = \begin{cases} 1/f(z), & z \neq z_0 \\ 0, & z = z_0 \end{cases}$
$\lim_{z \to z_0} |g(z)| = 0$
$\therefore \lim_{z \to z_0} g(z) = 0$

Sup $z_0$ polo de $f$
$w_0$ cero de $f$
$f(z) = (z - w_0) \tilde{f}(z)$ $\tilde{f} : \Omega \to \mathbb{C}$
$\frac{h(z)}{(z-z_0)^m} \Rightarrow h(z) = (z - z_0)^m (z - w_0)^n \tilde{f}(z)$

$$g : \Omega \cup \{z_0\} \to \mathbb{C}$$

$$g(z) = \begin{cases} 1/f(z), & z \in \Omega \\ 0, & z = z_0 \end{cases}$$

y tiene 0 aislado en $z_0$. Luego $\exists!\, n$, $\exists!\, h(z)$
(tomar $B(z_0, \delta') \subseteq B(z_0, \delta)$)

$$g(z) = (z - z_0)^n h(z) \quad h(z_0) \neq 0$$

$\Rightarrow \hat{h}(z) = \frac{1}{h(z)}$ ↳ holomorfa en $\Omega \cup \{z_0\}$ No! ↘ holo en $B(z_0, \delta')$

por lo tanto:
$$f(z) = \frac{h(z)}{(z - z_0)^n} \quad \forall z \in \Omega. \quad \forall z \in B(z_0, \delta')$$

$\tilde{h}(z) = (z - z_0)^n f(z)$
holo en $\Omega$. $\tilde{h} = h$ en $B(z_0, \delta)$
$\tilde{h}(z)$ tiene expansión en serie de potencias en $z = z_0$
$\therefore \tilde{h}$ holo en $\Omega \cup \{z_0\}$!

$n$: orden o multiplicidad del polo

primero hacemos la construcción para vec de $z_0$ y después extendemos a todo $\Omega \cup \{z_0\}$

$f : \Omega \to \mathbb{C}$ holomorfa con polo en $z = z_0$
$\iff \exists$ vecindad de $z_0$ tq
$g : V_{z_0} \to \mathbb{C}$
$g(z) = \begin{cases} 1/f(z) & z \neq z_0 \\ 0, & z = z_0 \end{cases}$
es holomorfa.

Prop. Si $z_0$ es un polo de orden $n$ de $f$, entonces

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \dots + \frac{a_{-1}}{(z - z_0)} + \varphi(z)$$

donde $\begin{cases} a_{-n} \neq 0 \\ \varphi : \Omega \cup \{z_0\} \to \mathbb{C} \text{ es holomorfa.} \end{cases}$

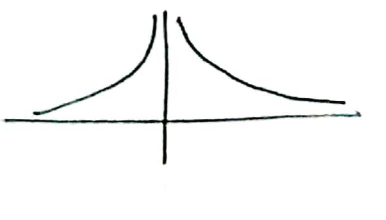
No es en una vecindad de $z_0$ (pequeña)

dem trivial.

Corolario. Si $z_0$ es un polo de $f$, entonces $|f(z)| \xrightarrow[z \to z_0]{} \infty$

con velocidad polinomial, esto es, $|f(z)| = O(|z - z_0|^{-n})$

obs. El corolario no tiene análogo real



$f(x) = e^{\frac{1}{x^2}}$

notemos que $f(z) = e^{1/z^2}$ es holo en $\mathbb{C} \setminus \{0\}$, $z_0 = 0$ no es polo, de hecho, si $z = iy$

$$f(z) = e^{1/(iy)^2} = e^{-1/y^2} \xrightarrow[y \to 0]{} 0$$

<u>Def</u>. Una singularidad aislada $z_0$ de $f$ es llamada singularidad esencial de $f$ si no es singularidad removible o polo.

Equivalentemente, $f(z)$ tiene al menos dos puntos de acumulación cuando $z \to z_0$.

<u>Teorema (Casorati - Weierstrass)</u>. (¡Pendiente!)

Si $z_0$ es singularidad esencial de $f$, entonces el conjunto de puntos de acumulación de $f(z)$ cuando $z \to z_0$ es todo $\mathbb{C}$.

<u>dem</u>. Supongamos que $\exists b \in \mathbb{C}$ que no es punto de acumulación, o sea, $\exists \varepsilon > 0$, $\exists \delta > 0$ tq $z \in B(z_0, \delta) \setminus \{z_0\} \Rightarrow f(z) \notin B(b, \varepsilon)$

Sea $g(z) = \dfrac{1}{f(z) - b}$, holomorfa en $U$
y acotada en $U \cup \{z_0\}$

$\Rightarrow z_0$ es singularidad removible de $g$, luego $g(z_0)$ está definido.

$\Rightarrow f(z) = b + \dfrac{1}{g(z)}$
- Si $g(z_0) \neq 0 \Rightarrow z_0$ es singularidad removible de $f$. ($\Rightarrow\Leftarrow$)
- Si $g(z_0) = 0$ de orden $n$, entonces $f$ tiene un polo de orden $n$ ($\Rightarrow\Leftarrow$)

$$f(z)= \frac{a_{-n}}{(z-z_0)^n}+\dots+\frac{a_{-1}}{(z-z_0)}+g(z) \quad \forall z\in V_{z_0}$$

# Residuos

$$\tilde g(z) := f(z)-\left(\frac{a_{-n}}{(z-z_0)^{n}}+\dots+\frac{a_{-1}}{(z-z_0)}\right) \text{ holo en } \Omega$$

Sea $f$ holomorfa con polo en $z_0$. Por lo tanto: $\tilde g(z)$ tiene expansion en serie de potencias en vec de $z=z_0$

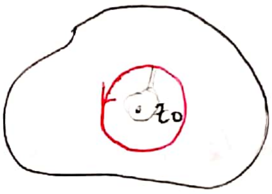
$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{(z-z_0)} + g(z)$$

En otras palabras, se puede extender $g$ holomorfamente a $\Omega\cup\{z_0\}$

donde: $n\geq 1$ es el orden del polo.
$a_{-n},\dots,a_{-1}\in\mathbb{C}$
$a_{-n}\neq 0$.

Obs. $f:\Omega\longrightarrow\mathbb{C}$ holomorfa.

$g$ es holomorfa en $\operatorname{dom} g = \underbrace{\overbrace{\operatorname{dom} f}^{=\Omega}\cup\{z_0\}}_{}$ abierto por hipótesis.




Supongamos $r>0$ tq $\overline{B(z_0,r)}\subset\Omega\cup\{z_0\}$.

$$\oint_{|z-z_0|=r} f(z)dz = \oint_{|z-z_0|}\frac{a_{-n}}{(z-z_0)^n}dz+\dots+\oint_{|z-z_0|=r}\frac{a_{-1}}{(z-z_0)}dz + \underbrace{\cancel{\oint_{|z-z_0|=r} g(z)dz}}_{0\ (\text{Cauchy})}$$

$$\oint_{|z-z_0|=r}\frac{1}{(z-z_0)^k}dz = \int_0^{2\pi}\frac{ire^{it}}{(re^{it})^k}dt = ir^{-k+1}\int_0^{2\pi} e^{i(-k+1)t}dt = \begin{cases} 0, & -k+1\neq 0\\ 2\pi i, & -k+1=0.\end{cases}$$

$$\therefore \oint_{|z-z_0|=r} f(z)dz = a_{-1}(2\pi i)$$

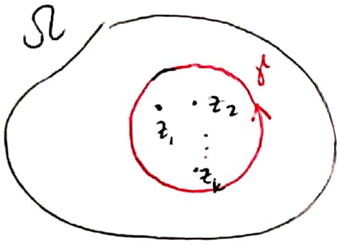
Def. El residuo de $f$ en $z_0$ es $\operatorname{Res}(f,z_0) := a_{-1}$

$\Omega \setminus S$ es $a$-estrellado

$\int_{\gamma} f(z) = 0 \Rightarrow \int_{\gamma'} f(z)dz = \int_{\gamma_\varepsilon} f(z)dz$

## Teorema de los residuos

Sean $B \subseteq \mathbb{C}$ bola abierta, $z_1, \dots, z_k \in B$. $f$ holomorfa en (vecindad de $\overline{B}$) $\setminus \{z_1, \dots, z_k\}$. $f$ tiene polos en $z_1, \dots, z_k$.



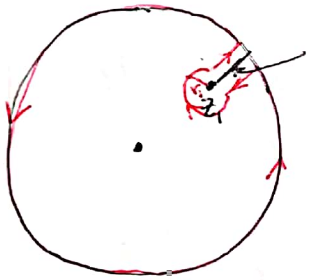
$\gamma$ el borde de $B$ recorrido en sentido anti-horario.

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(f, z_k) \qquad f(z) = \frac{h(z)}{(z-z_0)^n}$$

$h$ holo en vec de $z_0$.

Dem. Caso 1: $k=1$ (único polo) $z_1$ = centro de la bola. Ya está hecho.

Caso $k=1$ único polo, pero distinto del centro



$S$ segmento. $B \setminus S$ estrellado, contenido en $\Omega$.

$$\overset{\text{Cauchy}}{\Rightarrow} \int_{\eta_t} f(z)\,dz = 0.$$

obs. $\Omega$ vecindad de $B$.

Haciendo $t \downarrow 0$ nos obtiene:

$$\int_\gamma f(z)\,dz - \underbrace{\oint_{|z-z_0|=r_0} f(z)\,dz}_{=2\pi i \,\mathrm{Res}(f, z_0)} = 0$$

$$\therefore \int_\gamma f(z)\,dz = 2\pi i \,\mathrm{Res}(f, z_0).$$

$$\mathrm{Res}(f, z_0) = \frac{1}{2\pi i}\int_\gamma f(z)\,dz.$$

Inducción sobre $k$.

Supongamos que el teorema está probado para un valor $k$, y probamos para $k+1$,

$$f(z) = \underbrace{\frac{a_{-n}}{(z-z_{k+1})^n} + \dots + \frac{a_{-1}}{(z-z_{k+1})}}_{=Q(z)\text{: tiene polo en } z_{k+1}} + g(z)$$

$g(z) \longrightarrow$ tiene polos en $z_1, \dots, z_k$

$$\int_\gamma f(z)dz = \int_\gamma Q(z)dz + \int_\gamma g(z)dz = 2\pi i a_{-1} + 2\pi i \sum_{j=1}^{k} Res(g, z_j)$$

<u>Ahora</u>: $a_{-1} = Res(f, z_{k+1})$ (por def).

$(*)$ $Res(g, z_j) = Res(f, z_j)$ $\forall j \in \{1, \dots, k\}$

$\forall j \in \{1, \dots, k\}$ Sea $\varepsilon < \min_{j,l} \{|z_j - z_l|\}$

$$Res(f, z_j) - Res(g, z_j) = \frac{1}{2\pi i} \oint_{|z-z_j|=\varepsilon} f(z)dz - \frac{1}{2\pi i} \oint_{|z-z_j|=\varepsilon} g(z)dz$$

$$= \frac{1}{2\pi i} \oint_{|z-z_j|=\varepsilon} (f(z) - g(z))dz = \frac{1}{2\pi i} \oint_{|z-z_j|=\varepsilon} Q(z)dz = 0$$ (Por teo de Cauchy)