VARIABLE COMPLEJA

$\mathbb{C} := \mathbb{R}[i] = \{a+bi \,/\, a,b \in \mathbb{R}\} = \mathbb{R} \oplus i\mathbb{R}$

i raíz de x^2+1.

$\overline{a+bi} = a-bi$

$z \in \mathbb{C} : |z| = \sqrt{z\bar{z}}$

Principio de Lefschetz: Todo cuerpo algebraicamente cerrado de característica 0 y de la misma cardinalidad que $\mathbb{C}$ es isomorfo a $\mathbb{C}$.

$\mathbb{C}$ es algebraicamente cerrado.

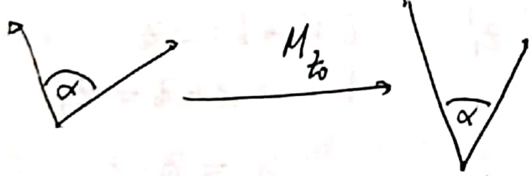
$z_0 = a_0 + b_0 i \;;\; M_{z_0} : \mathbb{C} \to \mathbb{C} \,,\; M_{z_0}(z) = z_0 z \quad (z_0 \neq 0)$

Para $z = a+bi$, $\quad M_{z_0} : \begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} a_0 a - b_0 b \\ a_0 b + b_0 a \end{pmatrix} = \begin{pmatrix} a_0 & -b_0 \\ b_0 & a_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

$$z_0 = |z_0| \left(\frac{z_0}{|z_0|} \right) \qquad = \underset{\begin{pmatrix} |z_0| & \\ & |z_0| \end{pmatrix}}{\overset{\|}{|z_0|}} \underbrace{\begin{pmatrix} \frac{a_0}{|z_0|} & -\frac{b_0}{|z_0|} \\ \frac{b_0}{|z_0|} & \frac{a_0}{|z_0|} \end{pmatrix}}_{\text{determinante } 1} \begin{pmatrix} a \\ b \end{pmatrix}$$

existe $\theta \in \mathbb{R}$ tal que $\cos\theta = \frac{a_0}{|z_0|}$, $\operatorname{sen}\theta = -\frac{b_0}{|z_0|}$ $\left(\left(\frac{a_0}{|z_0|}\right)^2 + \left(\frac{-b_0}{|z_0|}\right)^2 = 1 \right)$

Af. M_{z_0} es conforme (preserva ángulo y orientación)



Ejercicio. Toda matriz que preserva ángulos (conforme) ~~[illegible]~~ proviene de la multiplicación por un complejo.

Obs. anticonforme = preserva ángulo e invierte orientación.
= conforme compuesta con la conjugación compleja.

Grupo afín complejo: $\{z \mapsto az+b \mid a,b \in \mathbb{C},\ a \neq 0\}$

$\overline{\mathbb{C}}$ = esfera de Riemann.

$= \mathbb{C} \cup \{\infty\}$. provisto del atlas $\left\{ \begin{array}{l} \mathbb{C} \xrightarrow{z} \mathbb{C} \\ (\mathbb{C}\setminus\{0\}) \cup \infty \xrightarrow{1/z} \mathbb{C} \end{array} \right\}$

Grupo de las transformaciones de Moebius:

$$\left\{ z \mapsto \frac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{C},\ ad-bc \neq 0 \right\}$$

$$M(z) = \frac{az+b}{cz+d} = y \quad , \quad z = \frac{dy-b}{-cy+a}$$

$$M(\infty) = \infty$$
$$M(z) = az+b$$

$$\begin{tikzcd} \tfrac{1}{\zeta} \quad \overline{\mathbb{C}}\setminus\{0\} \arrow[r, "M"] \arrow[d, "1/z"'] & \overline{\mathbb{C}}\setminus\{0\} \quad a\tfrac{1}{\zeta}+b \arrow[d, "1/z"] \\ \mathbb{C} \arrow[r] & \mathbb{C} \end{tikzcd}$$

ζ

$$\frac{1}{a\frac{1}{\zeta}+b} = \frac{\zeta}{a+b\zeta} := N(\zeta) = \frac{1}{a}\zeta\left(\frac{1}{1+\frac{b}{a}\zeta}\right)$$

$a=1 \Rightarrow D_\infty M : T_\infty\overline{\mathbb{C}} \longrightarrow T_\infty\overline{\mathbb{C}}$ es la identidad $\boxed{M(z) = z+b}$

$$\begin{tikzcd} & 0,1 \arrow[dl, "M"] \arrow[dr, "\tilde{M}"'] & \\ z_0, z_1 \arrow[rr, squiggly, "\tilde{M}\circ M^{-1}"] & & z_0', z_1' \end{tikzcd}$$

$0 \longmapsto b := z_0$

$1 \longmapsto a + z_0 = z_1$

$a = z_1 - z_0$

$$M(z) = (z_1 - z_0)z + z_0$$
$$M(0) = z_0$$
$$M(1) = z_1$$

. El grupo afín complejo es 2-transitivo

• Pd: El grupo de las transformaciones de Moebius es 3-transitivo.

$$0, 1, \infty \rightsquigarrow z_0, z_1, z_\infty.$$

z_0, z_1, z_∞

$$M(z) = \frac{az+b}{cz+d} \quad , \quad M(z) = \frac{z_1 - z_\infty}{z_1 - z_0} \cdot \frac{z - z_0}{z - z_\infty}$$

Ejercicio. Demostrar que el grupo de las transformaciones de Moebius es 3-transitivo.

$a, b, c, d \in \mathbb{C}$, distintos dos a dos.

$$R(a,b,c,d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$$

$$\parallel$$

$$R(a,b;c,d)$$

Af. $R(M(a), M(b), M(c), M(d)) = R(a,b,c,d)$

Ejercicio. Demostrar que R se extiende continuamente a $(\overline{\mathbb{C}})^4 \setminus \Delta$

$$\Delta = \left\{ (a,b,c,d) \in \overline{\mathbb{C}}^4 \;\middle|\; \begin{array}{l} a=b \text{ o } b=c \\ \text{o } c=d \text{ o } a=d \\ a=c \;,\; b=d \end{array} \right\}$$

Disco unitario: $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$

$$\left\{ z \mapsto \lambda \frac{z-a}{1-\bar{a}z} \;\middle|\; |\lambda| = 1, a \in \mathbb{D} \right\}$$

$$\parallel$$

Grupo de automorfismos de $\mathbb{D}$ = {transformaciones de Moebius que preservan $\mathbb{D}$}

$\left(\dfrac{|dz|}{1-|z|^2} \right)$ = {grupo de isometrías que preservan la orientación de $\mathbb{D}$}

Plano proyectivo complejo:

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

Acción de $\mathbb{C}^*$ en $\mathbb{C}^2$, $\lambda(z,w) = (\lambda z, \lambda w)$

Órbitas: "rectas" que pasan por $\mathbb{C}^2$

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{C}^2 \setminus \{(0,0)\} \big/ \mathbb{C}^*$$

$[z:w]$ = clase de (z,w) $(\in \mathbb{C}^2 \setminus \{(0,0)\})$

$$\zeta : \mathbb{P}^1(\mathbb{C}) \setminus \{[1:0]\} \longrightarrow \mathbb{C}$$
$$[z:w] \longmapsto \frac{z}{w}$$

biyección con inversa
$$\mathbb{C} \ni \zeta \longmapsto [\zeta : 1]$$

$$\xi : \mathbb{P}^1(\mathbb{C}) \setminus \{[0:1]\} \longrightarrow \mathbb{C}$$
$$[z:w] \longmapsto \frac{w}{z}$$
(inversa $\xi \mapsto [1:\xi]$)

$\zeta \cdot \xi = 1$ en $\mathbb{P}^1(\mathbb{C}) \setminus \{[1:0], [0:1]\}$

$L : \mathbb{C}^2 \to \mathbb{C}^2$ lineal compleja invertible $(\in GL(2,\mathbb{C}))$

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad L\begin{pmatrix} \lambda z \\ \lambda w \end{pmatrix} = \lambda \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} = \lambda L \begin{pmatrix} z \\ w \end{pmatrix}$$

$$\begin{array}{ccc} \mathbb{C}^2 \setminus \{(0,0)\} & \longrightarrow & \mathbb{C}^2 \setminus \{(0,0)\} \\ \downarrow & & \downarrow \\ \mathbb{P}^1(\mathbb{C}) & \longrightarrow & \mathbb{P}^1(\mathbb{C}) \end{array}$$

(???)

$$[z:w] \longmapsto [Az + Bw : Cz + Dw], \quad \zeta \longmapsto \frac{A\zeta + B}{C\zeta + D}$$

Tomamos la acción de $\mathbb{C}^*$ en $\mathbb{C}^2 \setminus \{(0,0)\}$

$$S^3 := \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$$

Acción de S^1 en S^3, $\lambda(z,w) = (\lambda z, \lambda w)$

$$\underset{\underset{\mathbb{P}^1(\mathbb{C})}{\shortparallel}}{\mathbb{C}^2 \setminus \{(0,0)\} / \mathbb{C}^*} = S^3 / S^1$$

$$\begin{tikzcd}
S^3 (\subseteq \mathbb{C}^2 = \mathbb{R}^4) \arrow[d] \arrow[dr, "p"] & \\
\mathbb{P}^1(\mathbb{C}) \arrow[r, dashed, leftrightarrow, "\tilde{p}"] & S^2 \subseteq \mathbb{C} \times \mathbb{R} \; (\cong \mathbb{R}^3)
\end{tikzcd}$$

$$p(z,w) = (2z\bar{w}, |z|^2 - |w|^2)$$

$$(z,w) \in S^3, \quad |z|^2 + |w|^2 = 1$$

$$|2z\bar{w}|^2 + (|z|^2 - |w|^2)^2 = 4|z|^2|w|^2 + |z|^4 - 2|z|^2|w|^2 + |w|^4 = (|z|^2 + |w|^2)^2 = 1$$

$(z,w) \in S^3$, $\lambda \in S^1$

$$p(\lambda z, \lambda w) = (2\lambda z \overline{\lambda w}, |\lambda z|^2 - |\lambda w|^2) = p(z,w)$$

$$(\zeta, h) \in S^2 \subseteq \mathbb{C} \times \mathbb{R}, \quad |\zeta|^2 + h^2 = 1$$

$z \in \mathbb{R}$, $w \in \mathbb{C}$, $(z,w) \in S^3$, $(z^2 + |w|^2 = 1)$
$(z \geq 0)$

$$p(z,w) = (\zeta, h)$$

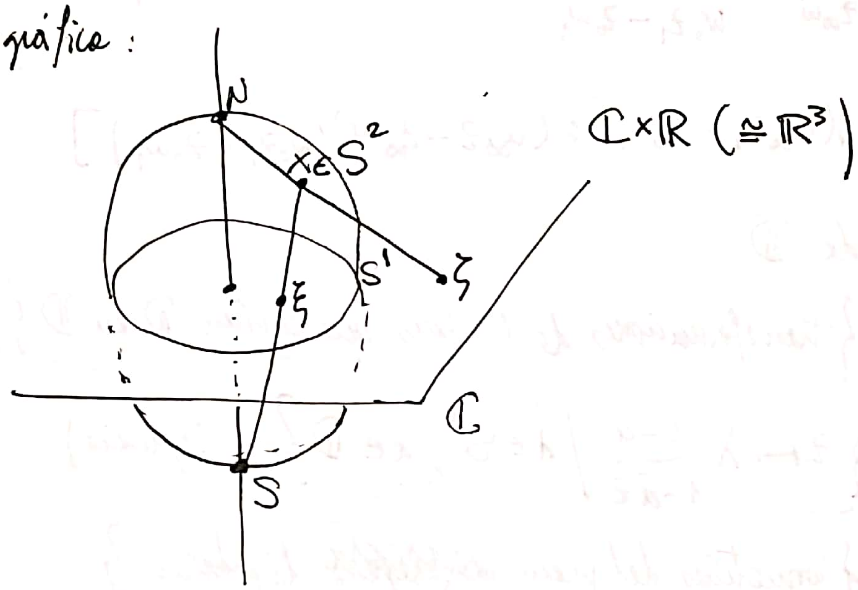
$$2z\bar{w} = \zeta, \quad z^2 - |w|^2 = h$$

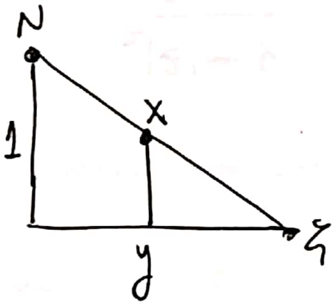
$$|w|^2 = z^2 + h = 1 - z^2 \quad \Big\} \quad z^2 = \frac{1-h}{2} \Rightarrow z = \sqrt{\frac{1-h}{2}}$$

$$w = \frac{\bar{\xi}}{2z}$$

$$[z:w] = \tilde{P}^{-1}(\zeta, h)$$

Proyección estereográfica:

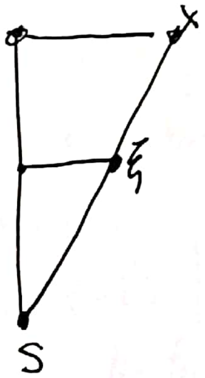




$$x = (y, h) \in S^2, \quad |y|^2 + h^2 = 1$$

$$\frac{\zeta}{1} = \frac{\zeta - y}{h}, \quad \frac{\xi}{1} = \frac{y}{1+h}$$

$$\zeta = \frac{y}{1-h}, \quad \xi = \frac{y}{1+h}$$



$$\zeta \cdot \xi = \frac{y^2}{1-h^2}$$ es de norma 1

$$\zeta \cdot \bar{\xi} = \frac{|y|^2}{1-h^2} = 1$$

El grupo de las transformaciones de Moebius es 3-transitivo.

$[z_0 : w_0]$, $[z_1 : w_1]$, $[z_\infty : w_\infty] \in \mathbb{P}^1(\mathbb{C})$

$$\frac{w_0 z - z_0 w}{w_\infty z - z_\infty w} \cdot \frac{w_\infty z_1 - z_\infty w_1}{w_0 z_1 - z_0 w_1}$$

$$[(w_0 z - z_0 w)(w_\infty z_1 - z_\infty w_1) : (w_\infty z - z_\infty w)(w_0 z_1 - z_0 w_1)]$$

Automorfismos de $\mathbb{D}$

$= \{$transformaciones de Moebius que envían $\mathbb{D}$ en $\mathbb{D}\}$

$= \left\{ z \mapsto \lambda \dfrac{z-a}{1-\bar{a}z} \;\middle|\; \lambda \in S^1, a \in \mathbb{D} \right\}$ (ejercicio)

$= \{$isometrías del plano ~~[illegible]~~ hiperbólico$\}$ $\qquad \dfrac{2\,dz}{1-|z|^2}$

$U \subseteq \mathbb{C}$ abierto.

$f : U \to \mathbb{C}$ es diferenciable en $z_0 \in U$, en el sentido complejo ssi

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

existe.

Notación: Si el límite anterior existe, se denotará por $f'(z_0)$

$$\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) = o(1)$$

$h := z - z_0$, $f(z_0+h) - f(z_0) = f'(z_0)h + O(|h|)$

$f(z) = u(z) + iv(z)$; $f'(z_0) = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$

$$\partial_x u = \partial_y v \quad , \quad -\partial_x v = \partial_y u \quad \text{(ec's de Cauchy-Riemann)}$$

Si u, v son de clase C^2,

$$\partial^2 u = \partial^2_{xy} v = -\partial_y^2 u \quad , \quad \partial_x^2 u + \partial_y^2 u = 0 \qquad \boxed{\Delta u = 0}$$

$$\boxed{\Delta = \partial_x^2 + \partial_y^2}$$

Análogamente, $\Delta v = 0$

u, v son funciones armónicas ($\Leftrightarrow \Delta u, \Delta v = 0$)

$$u(z_0) = \int_{\partial B(z_0, r)} u(z)\, dz \quad , \text{ para } \overline{B(z_0, r)} \subseteq U$$

Retrocedamos un poco...

__Lema__. Para toda $L : \mathbb{R}^2 \to \mathbb{R}^2$, existen $a, b \in \mathbb{R}^2$ tales que $L(z) = az + b\bar{z}$

__Dem.__ $z = x + iy$

$L(z) = Ax + By + i(Cx + Dy)$

$a = \alpha + i\beta$, $b = \gamma + i\delta$ | $az + b\bar{z} = \alpha x - \beta y + i(\alpha y + \beta x) + \gamma x + \delta y + i(-\gamma y + \delta x)$

Con ello tenemos: $A = \alpha + \gamma$, $B = -\beta + \delta$

$C = \beta + \delta$, $D = \alpha - \gamma$.

$$\alpha = \frac{A+D}{2}, \quad \gamma = \frac{A-D}{2}$$
$$\beta = \frac{C-B}{2}, \quad \delta = \frac{B+C}{2}$$

Obs. Además, a,b están unicamente determinados por L.

Obs. Para el caso de las 1-formas:

$$\begin{cases} dz = dx + i\,dy \\ d\bar{z} = dx - i\,dy \end{cases}$$

Si ω es 1-forma diferenciable, $\omega(x) = f(x,y)\,dx + g(x,y)\,dy$, se escribe como $\omega(x) = a(x,y)\,dz + b(x,y)\,d\bar{z}$.

Sea $F: U \to \mathbb{C} = \mathbb{R}^2$, $F(x,y) = (u(x,y), v(x,y))$

$$\begin{aligned} dF(x,y) &= \left((\partial_x u)dx + (\partial_y u)dy,\ (\partial_x v)dx + (\partial_y v)dy\right) \\ &= (\partial_x u)dx + (\partial_y u)dy + i(\partial_x v)dx + i(\partial_y v)dy \\ &= \left(\frac{\partial_x u + \partial_y v}{2} + i\left(\frac{\partial_x v - \partial_y u}{2}\right)\right)dz \\ &\quad + \left(\frac{\partial_x u - \partial_y v}{2} + i\left(\frac{\partial_y u + \partial_x v}{2}\right)\right)d\bar{z}. \end{aligned}$$

Obs. Para que F sea holomorfa, el factor que acompaña a $d\bar{z}$ (anticonforme) debe ser 0 (Cauchy-Riemann).

$$dF(x,y) = \frac{1}{2}(\partial_x F - i\partial_y F)\,dz + \frac{1}{2}(\partial_x F + i\,\partial_y F)\,d\bar{z}.$$

Tenemos $\partial F := \frac{1}{2}(\partial_x F - i\partial_y F)$

$$\boxed{\begin{aligned} \partial &= \frac{1}{2}(\partial_x - i\partial_y) \\ \bar{\partial} &= \frac{1}{2}(\partial_x + i\partial_y) \end{aligned}}$$

Def. F es diferenciable en z_0 si $\bar{\partial}_{z_0} F = 0$

Para F, G: $\quad (aF+bG)'(z_0) = aF'(z_0) + bG'(z_0)$

$$\bar{\partial}(aF+bG) = a\bar{\partial}F + b\bar{\partial}G.$$

F, G diferenciables en z_0 $\Rightarrow$ $F\cdot G$ es diferenciable

Además, $\quad \partial(F\cdot G) = \partial F\cdot G + F\cdot\partial G$

$$\bar{\partial}(F\cdot G) = \bar{\partial}F\cdot G + F\bar{\partial}G.$$

$F = u+iv$
$G = c+id$ $\quad , \quad F\cdot G = uc - vd + i(ud+vc)$

$$\frac{1}{2}(\partial_x(FG) - i\partial_y(FG))$$

$$= \frac{1}{2}\Big((\partial_x u)c + u(\partial_x c) - (\partial_x v)d - v(\partial_x d) + i\big((\partial_x u)d + u(\partial_x d) + (\partial_x v)c + v(\partial_x c)\big)$$

$$- i\big((\partial_y u)c + u(\partial_y v) - (\partial_y v)d - v(\partial_y d)\big) + i\big((\partial_y u)d + u(\partial_y d) + (\partial_y v)c + v(\partial_y c)\big)\Big)$$

$$F(z)G(z) - F(z_0)G(z_0) = (F(z) - F(z_0))G(z) + F(z_0)(G(z) - G(z_0))$$

$$\bar{\partial}F(z_0) = \lim \frac{\bar{F}(z) - \bar{F}(z_0)}{z - z_0}$$

$$\boxed{\overline{\partial F} = \bar\partial \bar F}$$

$$\bar\partial F = \overline{\partial \bar F}$$

$$\bar\partial(F\cdot G) = \overline{\partial(\bar F\cdot \bar G)} = \overline{\partial(\bar F)\bar G + \bar F \partial \bar G} = (\overline{\partial \bar F})\bar{\bar G} + \bar{\bar F}\overline{\partial \bar G}$$
$$= (\overline{\partial \bar F})G + F(\overline{\partial \bar G}) = (\bar\partial F)G + F(\bar\partial G).$$

$$\bar\partial \bar F = \frac{1}{2}(\partial_x \bar F + i\partial_y \bar F) = \frac{1}{2}(\partial_x u - i\partial_x v + i\partial_y u + \partial_y v)$$
$$= \overline{\frac{1}{2}(\underbrace{\partial_x u + i\partial_x v}_{\partial_x F} \underbrace{- i\partial_y u + \partial_y v}_{-i\partial_y F})}.$$

Para $G(z_0) \neq 0$

$$\left(\frac{F}{G}\right)'(z_0) = \frac{F'G - FG'}{G^2}$$

$$\partial\left(\frac{F}{G}\right)$$

$$\bar\partial\left(\frac{F}{G}\right) = \overline{\partial\left(\frac{\bar F}{\bar G}\right)} = \overline{\left(\frac{(\partial \bar F)\cdot \bar G - \bar F(\partial \bar G)}{\bar G^2}\right)}$$

$$= \frac{\overline{\partial \bar F}\cdot G - F\,\overline{\partial \bar G}}{G^2} = \frac{\bar\partial F\cdot G - F\,\bar\partial G}{G^2}.\ \blacksquare$$

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Ejemplos.

z^n es diferenciable.

$$\frac{z^n - z_0^n}{z - z_0} = z^{n-1} + z^{n-2} z_0 + \dots + z_0^{n-1}$$

$$\therefore \quad \frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1} = \left(z^{n-1} - z_0^{n-1}\right) + \left(z^{n-2} - z_0^{n-2}\right) z_0 + \dots + \left(z^{n-1} - z_0^{n-1}\right)$$

$$\partial z^n = n z^{n-1} \qquad \partial \bar{z}^n = \overline{\bar{\partial} z^n} = 0 \qquad \partial(z^n \bar{z}^m) = n z^{n-1} \bar{z}^m$$

$$\bar{\partial} z^n = 0 \qquad \partial(z\bar{z}) = \bar{z} \qquad \bar{\partial}(z^n \bar{z}^m) = \overline{\partial(\bar{z}^n z^m)} = \overline{m z^{m-1} \bar{z}^n}$$

$$= m z^n \bar{z}^{m-1}$$

$$= z^n (m \bar{z}^{m-1})$$

$$DF(z_0) = (\partial F(z_0))\, z + (\bar{\partial} F(z_0))\, \bar{z}$$

Si F es diferenciable en z_0, entonces $F'(z_0) = \partial F(z_0)$.

Repaso.

$\mathbb{C}=\mathbb{R}^2$, $F: U \longrightarrow \mathbb{C}$

$U \subseteq \mathbb{R}^2$

$$\partial F = \frac{1}{2}(\partial_x - i\partial_y) \,,\quad \bar{\partial} F = \frac{1}{2}(\partial_x + i\partial_y)$$

$$D_z F(v) = \partial F(z_0)\cdot v + \bar{\partial} F(z_0)\bar{v}$$

$$dF = \partial F\, dz + \bar{\partial} F\, d\bar{z}$$

$$dz = dx + i\,dy \,,\quad d\bar{z} = dx - i\,dy$$

- F holomorfa en z_0 , $\partial F(z_0) = F'(z_0)$
  $\bar{\partial} F(z_0) = 0$

$$F(z_0+h) = F(z_0) + \partial F(z_0) h + \bar{\partial} F(z_0)\bar{h} + o(|h|)$$

$$F(z_0+h)G(z_0+h) = F(z_0)G(z_0) + F(z_0)\partial G(z_0) h + F(z_0)\bar{\partial} G(z_0)\bar{h} + G(z_0)\partial F(z_0) h + G(z_0)\bar{\partial} F(z_0)\bar{h} + o(|h|)$$

$$= F(z_0)G(z_0) + \underbrace{(F(z_0)\partial G(z_0) + \partial F(z_0) G(z_0))}_{\partial(FG)(z_0)} h + \underbrace{(F(z_0)\bar{\partial} G(z_0) + \bar{\partial} F(z_0)\cdot G(z_0))}_{\bar{\partial}(FG)(z_0)}\bar{h} + o(|h|)$$

$$\partial \frac{1}{F} = -\frac{\partial F}{F^2} \,,\quad \partial\left(\frac{F}{G}\right) = \frac{(\partial F)G - F(\partial G)}{G^2}$$

$$(F\circ G)(z_0+h) = F\left(G(z_0) + \partial G(z_0) h + \bar{\partial} G(z_0)\bar{h} + o(|h|)\right)$$

$$= F\circ G(z_0) + \partial F(G(z_0))(\partial G(z_0) h + \bar{\partial} G(z_0)\bar{h}) + \bar{\partial} F(G(z_0))\overline{(\partial G(z_0) h + \bar{\partial} G(z_0)\bar{h})} + o(|h|)$$

$$= F\circ G(z_0) + (\partial F(G(z_0))\cdot \partial F(z_0) + \bar{\partial} F(G(z_0))\overline{\bar{\partial} G(z_0)})h + (\partial F(G(z_0))\cdot \bar{\partial} G(z_0) + \bar{\partial} F(G(z_0))\cdot \overline{\partial G(z_0)})\bar{h} + o(|h|)$$

$$\partial(F\circ G) = \partial F\circ G\cdot\partial G + \bar\partial F\circ G\cdot\partial\bar G$$
$$\bar\partial(F\circ G) = \bar\partial F\circ G\cdot\bar\partial\bar G + \partial F\circ G\cdot\bar\partial G$$

Teo. U simplemente conexo, $\bar U$ compacto, ∂U curva C^∞.
$f:\bar U\to\mathbb{C}$ continua y analítica en U. Entonces $\int_{\partial U} f(z)\,dz=0$

$\gamma:[a,b]\to\partial U$ de clase C^1, $\gamma(a)=\gamma(b)$

$$\int_{\partial U} f(z)\,dz = \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$$

$df=\partial f\,dz+\bar\partial f\,d\bar z$

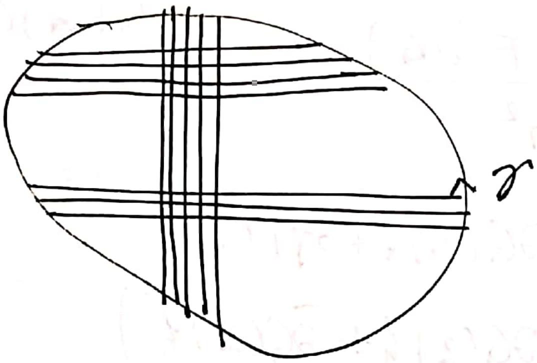
$\partial f\,dz\wedge dz$

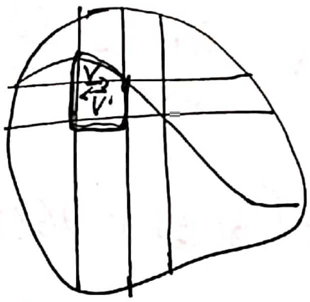
Dem 1. Teorema de Stokes $\Rightarrow \int_{\partial U} f(z)\,dz = \int_U d(f(z)\,dz)$

$dz=dx+i\,dy$

$$dz\wedge dz = (dx+i\,dy)\wedge(dx+i\,dy) = \cancel{dx\wedge dx}+i\,dx\wedge dy + i\,dy\wedge dx - \cancel{dy\wedge dy}$$
$$= i(dx\wedge dy) - i(dx\wedge dy) = 0.$$

Dem 2.





$$\int_{\partial U} f(z)\,dz \approx \sum_V \int_{\partial V} f(x_V)+f'(x_V)\,h\,dh$$

$g(z) = az+b$; $a,b \in \mathbb{C}$, $a \neq 0$

$$\int_{\partial V} g(h)\,dh = \int_0^1 g(\gamma(t))\gamma'(t)\,dt = \int_0^1 a\,\gamma(t)\gamma'(t) + b\gamma'(t)\,dt$$

$\gamma : [0,1] \to \partial V$. $\int_0^1 b\gamma'(t)\,dt = b\int_0^1 \gamma'(t)\,dt = b(\gamma(1) - \gamma(0)) = 0$

$\gamma(0) = \gamma(1)$

$\gamma(t) = u(t) + iv$

$\gamma(t)\gamma'(t) = (u(t)+iv(t))(u'(t)+iv'(t)) = u(t)u'(t) - v(t)v'(t)$
$+ i(v(t)u'(t) + v'(t)u(t))$

Así, $\int_0^1 \gamma(t)\gamma'(t)\,dt = \left(\frac{1}{2}u^2 - \frac{1}{2}v^2 + iuv\right)\Big|_0^1 = 0$

Vamos a calcular $\int_{\partial B(0,r)} \frac{1}{z}\,dz$. Para ello consideremos el camino cerrado siguiente:

$$\gamma : [0,1] \longrightarrow \partial B(0,r)$$
$$\theta \longmapsto r\exp(2\pi i\theta)$$

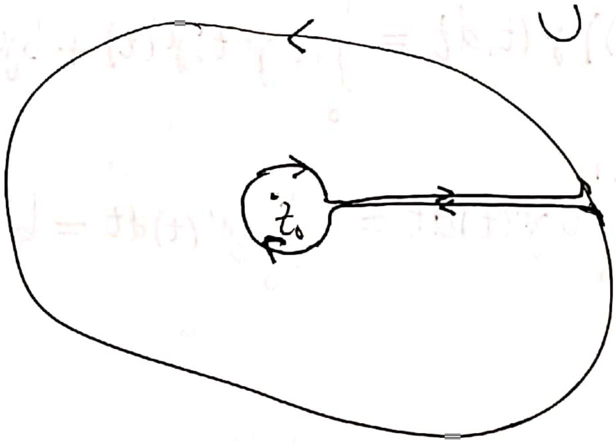
$$\Rightarrow \int_{\partial B(0,r)} \frac{1}{z}\,dz = \int_0^1 \frac{1}{r\exp(2\pi i\theta)}\, r\cdot 2\pi i \exp(2\pi i\theta)\,d\theta = 2\pi i$$

• Fórmula integral de Cauchy.

Sea $z_0 \in U$,

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(z)}{z - z_0}\,dz$$

Como en U el único punto conflictivo es $z=z_0$ podemos hacer lo siguiente:



$$\int_{\partial U} \frac{f(z)}{z-z_0}\,dz = \int_{\partial(U\setminus B(z_0,r))} \frac{f(z)}{z-z_0}\,dz + \int_{\partial B(z_0,r)} \frac{f(z)}{z-z_0}\,dz$$

$$\int_{\partial B(z_0,r)} \frac{f(z)}{z-z_0}\,dz = \int_{\partial B(z_0,r)} \frac{f(z_0)}{z-z_0}\,dz + \cancelto{0}{\int_{\partial B(z_0,r)} f'(z_0)\,dz} + o(1)$$

$$= f(z_0)\int_{\partial B(z_0,r)} \frac{1}{z-z_0}\,dz = 2\pi i\, f(z_0) + o(1)$$

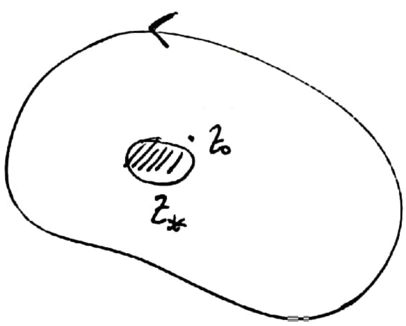
$f(z) = f(z_0) + f'(z_0)(z-z_0) + o(|z-z_0|)$

$$\int_{\partial U} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0) + o(1)$$

$$\therefore \int_{\partial U} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

—·—

Sea $z_* \in U$



$$\partial_{z_0}\int_{\partial U}\frac{f(z)}{z-z_0}dz=\int_{\partial U}\partial_{z_0}\left(\frac{f(z)}{z-z_0}\right)dz$$

$$\Rightarrow f'(z_0)=\frac{1}{2\pi i}\int_{\partial U}\frac{f(z)}{(z-z_0)^2}dz$$

$$f''(z_0)=\frac{1}{2\pi i}\cdot 2\int_{\partial U}\frac{f(z)}{(z-z_0)^3}dz$$

$$\therefore \forall n\in\mathbb{N}:\quad f^{(n)}(z_0)=\frac{1}{2\pi i}n!\int_{\partial U}\frac{f(z)}{(z-z_0)^{n+1}}dz$$

<u>Corolario</u>. Toda función analítica es infinitamente diferenciable en el sentido complejo.

<u>Def</u>. Un abierto U de $\overline{\mathbb{C}}$ es simplemente conexo si es conexo y ∂U es conexo.

<u>Ejemplo</u>. $U_0=(\mathbb{C}\setminus\mathbb{R})\cup(0,1)$

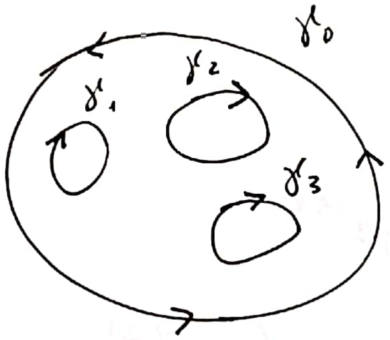
$\partial U_0=(-\infty,0]\cup[1,\infty)\cup\{\infty\}$

U_0 es simplemente conexo.

<u>Teo</u>. $U\subseteq\mathbb{C}$ abierto acotado, U simplemente conexo. ∂U C^1 por pedazos, entonces

$$\int_{\partial U}f(z)dz=0$$

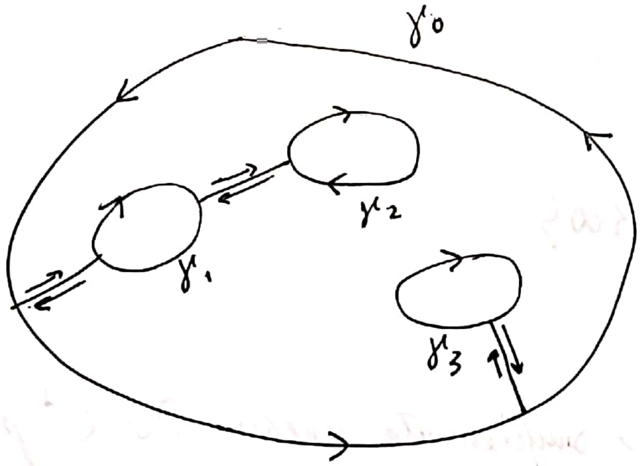
$U \subseteq \mathbb{C}$ abierto acotado $\overset{\text{conexo}}{\vee}$, ∂U = unión disjunta de curvas de Jordan $\gamma_0, \gamma_1, \dots, \gamma_n$ C^1 por pedazos.



$$\partial U = \underset{\substack{\uparrow \\ \text{orientado positivamente}}}{\gamma_0} \sqcup \gamma_1 \underbrace{\sqcup \gamma_2 \sqcup \dots \sqcup \gamma_n}_{\text{orientado negativamente}}$$

Teorema. $\displaystyle\int_{\partial U} f(z)dz = 0 \iff \int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz + \dots + \int_{\gamma_n} f(z)dz$

dem. Es bastante fácil, solo basta hacer lo siguiente



y luego ocupamos el teorema anterior.

$\cdot \int_{\partial B(0,r)} \frac{1}{z} dz = 2\pi i \Rightarrow \mathbb{C}^*$ no es simplemente conexo.

Fórmula de Leibnitz $\Rightarrow$

$g: \gamma \times B \to \mathbb{C}$ continua y para $z \in \gamma$, $w \mapsto g(z,w)$ es C^1 en el sentido real:

$$\partial_w \int_\gamma g(z,w)\,dw = \int_\gamma \partial_w g(z,w)\,dw$$

Así, para $f(z_0) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(z)}{z - z_0}$ (fórmula integral de Cauchy)

$$\Rightarrow f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial U} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$g: \partial U \to \mathbb{C}$ continua, $f(z_0) := \int_{\partial U} \frac{g(z)}{z - z_0} dz$ (holomorfa en el interior)

Para $U = B(0,r)$

$\gamma: [0,1] \to \mathbb{C}$

$\theta \mapsto r\exp(2\pi i \theta)$

$z_0 = 0$,

$$\int_{\partial B(0,r)} \frac{f(z)}{z^{n+1}} dz = \int_0^1 \frac{f(r\exp(2\pi i\theta))}{r^{n+1}\exp(i(n+1)2\pi\theta)} 2\pi i r \exp(2\pi i \theta)\, d\theta$$

$$= 2\pi i \int_0^1 \frac{f(r\exp(2\pi i\theta))}{(r\exp(2\pi i\theta))^n} d\theta$$

$$f^{(n)}(z_0) = n! \int_0^1 \frac{f(r\exp(2\pi i\theta))}{(r\exp(2\pi i\theta))^n} \, d\theta$$

$$f(r\exp(2\pi i\theta))\, r^n \left(\cos(2\pi n\theta) + i\,\mathrm{sen}(-2\pi n\theta)\right)$$

$F: \mathbb{R} \to \mathbb{C}$

$F(\theta) = f(r\exp(2\pi i\theta))$

λ_r = medida de probabilidad uniforme en $\partial B(0,r)$

$$\frac{f^{(n)}(z_0)}{n!} = \int f(z) z^{-1} \, d\lambda_r \longrightarrow \text{n-ésimo coef de Fourier}$$

$n \in \mathbb{Z}$, $\quad a_n = \int f(z) z^{-n} \, d\lambda_r \quad ; \; f(z) = \sum_{n\in\mathbb{Z}}' a_n z^n.$

$f(z_0) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(z)}{z - z_0} dz$. Sea $r > 0$ tal que $B(z_0, r) \subseteq U$ y tal que f está definida y es continua en el borde.

$$\frac{1}{z-w} = \frac{1}{z} \cdot \frac{1}{1 - \frac{w}{z}} = \frac{1}{z}\left(1 + \frac{w}{z} + \frac{w^2}{z^2} + \cdots\right)$$

$|w| < |z|$

Reemplazando en la fórmula integral de Cauchy

$$f(w) = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{f(z)}{z} \sum_{n=0}^{\infty}{}' \left(\frac{w}{z}\right)^n dz = \frac{1}{2\pi i} \sum_{n=0}^{\infty}{}' \left(\int_{\partial B(0,r)} \frac{f(z)}{z^{n+1}} dz \right) w^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{f^{(n)}(0)}{n!} \right) w^n$$

(Very Good!)

Ahora para el caso general:

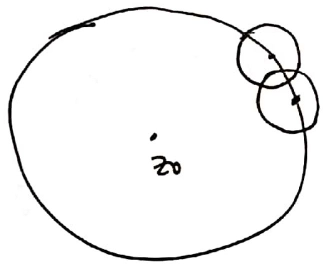
$B(z_0,r) \subseteq U$

$$f(w) = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(z)}{z-w} dz$$

considerando $\displaystyle \frac{1}{z-w} = \frac{1}{(z-z_0)-(w-z_0)} = \frac{1}{z-z_0} \cdot \frac{1}{1-\left(\frac{w-z_0}{z-z_0}\right)}$

procedemos como en la parte anterior!

Si r es el radio de convergencia de f en z_0, entonces f tiene una singularidad en $\partial B(z_0,r)$. En caso contrario, puede extenderse de manera holomorfa en vecindades de $z \in \partial B(z_0,r)$ como lo muestra el dibujo



Sea f holomorfa en z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

* Si $\forall n \geq 0$, $a_n = 0 \Rightarrow f(z) \equiv 0$

* Si no, $m := \min\{n \geq 0 : a_n \neq 0\}$

$$f(z) = (z-z_0)^m \left(\sum_{l=0}^{\infty} a_{l+m} (z-z_0)^l \right)$$

$$f(z) = (z-z_0)^m \left(a_m + a_{m+1}(z-z_0)^m + \dots \right)$$

Consecuencias.

- Los ceros de una función no nula son aislados.
- (Principio de la identidad). Dos funciones holomorfas que en un dominio conexo coinciden en un conjunto con un punto de acumulación, son iguales.

—o—

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\partial U} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$$\left| \frac{f^{(n)}(z_0)}{n!} \right| \leq \frac{l(\partial U)}{2\pi} \sup_{\partial U} |f| \cdot \operatorname{dist}(z_0, \partial U)^{-(n+1)}$$

Estimación de Cauchy.

$U \subseteq \mathbb{C}$ abierto simplemente conexo, ∂U curva de Jordan C^1 a pedazos.

$f : \overline{U} \to \mathbb{C}$ continua y holomorfa en U

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(z)}{z - z_0} \, dz$$

Para $U = B(a,r)$ (el disco debe estar contenido en el dominio, puede ser maximal)

$$f(z_0) = \sum_{n=0}^{\infty} \left(\frac{f^{(n)}(a)}{n!} \right) z^n$$

Ejemplo. $z \mapsto \frac{1}{z-1} + \frac{1}{z+1}$ holomorfa pero con singularidades en 1 y -1.

$$f(z) = \sum_{\frac{p}{q} \in \mathbb{Q} \cap [0,1)} \frac{2^{-q}}{z - \exp(2\pi i \frac{p}{q})}$$ holomorfa pero todo $\partial B(0,1)$ "es singularidad".

$z_0 \in U$, f no constante:

$$f(z) = (z - z_0)^m g(z) \quad , \; g(z_0) \neq 0$$

$\Rightarrow \exists r > 0$ tal que el único cero de f en $B(z_0, r)$ es z_0.

(Los ceros de una función holomorfa son aislados)

$\Leftrightarrow$ Principio de la identidad.

Obs. Lo anterior es válido para abiertos conexos.

Con las mismas hipótesis que en la primera parte:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial B(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

$$\gamma : [0,1] \longrightarrow \partial B(z_0, r)$$
$$\theta \mapsto z_0 + \exp(2\pi i \theta)$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_0^1 \frac{f(z_0 + r\exp(2\pi i\theta))}{(r\exp(2\pi i \theta))^{n+1}} 2\pi i \, r\exp(2\pi i\theta) \, d\theta$$

λ_r = medida de probabilidad uniforme en $\partial B(z_0, r)$

$$\frac{f^{(n)}(z_0)}{n!} = \int f(w)(w - z_0)^{-n} \, d\lambda_r(w)$$

(integral en el sentido de Lebesgue)

• El valor de f en un punto es el promedio de los valores en $\partial B(z_0, r)$

Principio del máximo: $\sup_U |f| \leq \sup_{\partial U} |f|$

Supongamos que existe $z_0 \in U$ tal que $|f(z_0)| = \sup_{\partial U} |f|$

Podemos tomar $\rho > 0$ t.q. $B(z_0, \rho) \subseteq U$

$$f(z_0) = \int f(z) \, d\lambda_r$$

$$|f(z_0)| \leq \int |f(z)| \, d\lambda_r \leq \sup_{\partial B(z_0, r)} |f| \leq \sup_{\partial U} |f|$$

$$\Rightarrow \forall z \in \partial B(z_0, r) : |f(z)| = \sup_{\partial U} |f|$$

$$|f(z_0)| = |f(z)| \quad \forall z \in B(z_0, r)$$

Falta algo (queda pendiente)

$U = B(z_0, r)$

$$\frac{1}{n!} f^{(n)}(z_0) = \int f(z)(z-z_0)^{-n} \, d\lambda_r$$

$$\frac{1}{n!} |f^{(n)}(z_0)| \leq \frac{\sup_{z \in \partial B(z_0,r)} |f(z)|}{r^n}$$

$$\sup_{\partial B(z_0,r)} |f| \geq r^n |f^{(n)}(z_0)| \frac{1}{n!} \qquad (*)$$

Dem. del teorema fundamental del álgebra.

$p(z) \in \mathbb{C}[z]$ de grado ≥ 1 tal que no tiene ceros en $\mathbb{C}$.

$$f : \mathbb{C} \to \mathbb{C} \qquad \text{\underline{f entera}}.$$
$$z \mapsto \frac{1}{p(z)}$$

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_1 z + a_0 \quad , \; a_d \neq 0 .$$

$(|z| \geq R \geq 1)$

$$= a_d z^d \left(1 + \frac{a_{d-1}}{a_d} \frac{1}{z} + \frac{a_{d-2}}{a_d} \frac{1}{z^2} + \dots + \frac{a_0}{a_d} \frac{1}{z^d} \right)$$

$$R = \max \left\{ 2 \frac{|a_{d-1}| + \dots + |a_0|}{|a_d|} , 1 \right\}$$

$$|p(z)| \geq |a_d| |z|^d \left(1 - \left| \frac{a_{d-1}}{a_d} \right| \frac{1}{|z|} - \dots - \left| \frac{a_0}{a_d} \right| \frac{1}{|z|^d} \right)$$

$$\geq |a_d| |z|^d \left(1 - \frac{1}{R} \frac{|a_0| + \dots + |a_{d-1}|}{|a_d|} \right)$$

$$\geq \frac{1}{2} |a_d| |z|^d$$

$$|f(z)| \le \frac{2}{|a_d|} \frac{1}{|z|^d} \xrightarrow[z\to\infty]{} 0$$

$(*) \Rightarrow$ para todo n, $f^{(n)}(0)=0$

Principio de la identidad $\Rightarrow f \equiv 0$ $(\Rightarrow\Leftarrow)$.

<u>Afirmación</u>. $\mathbb{C}$ es algebraicamente cerrado.

~~p~~ <u>dem</u>. $p(z) \in \mathbb{C}[z]$, no constante.

Sea z_0 un cero de $P(z)$

$$P(z) = a_d z^d + \dots + a_0$$

$$w = z - z_0$$

$$P(z) = a_d (w+z_0)^d + \dots + a_0 = b_d w^d + \dots + b_0 \quad , \quad b_0 = P(z_0)$$

$$P(z) = P(z_0) + (z-z_0)Q(z)$$

Q polinomio de grado $d-1$

z_0 cero de $P \Rightarrow P(z) = (z-z_0)\, Q(z)$

$$P(z) = a \prod_{i=1}^{d} (z - z_i)$$

Fórmula del promedio (valor medio)

$f: U \to \mathbb{R}$ holomorfa, $\overline{B(p,r)} \subseteq U$

$$f(p) = \int f(z) d\lambda_{(p,r)}(z)$$

$\lambda_{(p,r)}$ medida de probabilidad uniforme en $\partial B(p,r)$

$$\left(\frac{f^{(n)}(p)}{n!} = \int f(p) z^{-n} d\lambda_{(p,r)}(z) \right)$$

Principio del maximo. U abierto conexo, $f: U \to \mathbb{R}$ tq $\sup |f| = |f(p)|$, para algún $p \in U$.

$\Rightarrow f$ es constante.

dem. $r > 0$ tal que $\overline{B(p,r)} \subseteq U$,

$$f(p) = \int f(z) d\lambda_{(r,p)}(z)$$

$$\Rightarrow |f(p)| \leq \int |f(z)| d\lambda_{(r,p)}(z) \leq \int |f(p)| d\lambda_{(r,p)}(z) = |f(p)|$$

$$\Rightarrow \forall z \in B(p,r) : |f(z)| = |f(p)| = \rho$$

Existe $\varphi: [0,1] \to \mathbb{R}$ tal que

$$f(p + r\exp(2\pi i\theta)) = \rho \exp(2\pi i \varphi(\theta))$$

$$f(p) = \int_0^1 f(p + r\exp(2\pi i\theta)) d\theta = \rho \int_0^1 \exp(2\pi i \varphi(\theta)) d\theta$$

$$\int_0^1 \exp(2\pi i \varphi(\theta)) d\theta = \int_0^1 \cos(2\pi \varphi(\theta)) d\theta + i \int_0^1 \operatorname{sen}(2\pi \varphi(\theta)) d\theta$$

Ocupando la desigualdad de Cauchy-Schwarz:

$$\int \cos(2\pi\varphi(\theta))\,d\theta \leq \left(\int \cos(2\pi\varphi(\theta))^2\,d\theta\right)^{1/2}$$

$$\int \operatorname{sen}(2\pi\varphi(\theta))\,d\theta \leq \left(\int \operatorname{sen}(2\pi\varphi(\theta))^2\,d\theta\right)^{1/2}$$

$$\therefore\ 1 = \left|\int_0^1 \exp(2\pi i\varphi(\theta))\right|^2 \leq 1$$

(La desigualdad de Cauchy-Schwarz alcanza la igualdad cuando los vectores son paralelos.)

$$\int 1\cdot\cos(2\pi\varphi(\theta))\,d\theta \leq \left(\int 1^2\,d\theta\right)^{1/2}\left(\int \cos(2\pi\varphi(\theta))^2\,d\theta\right)^{1/2}$$

etc...

$\therefore\ \varphi$ constante.

—·—

<u>Teorema</u>. $\mathbb{C}$ es algebraicamente cerrado.

<u>dem</u>. $n \geq 0$,

$$\frac{f^{(n)}(p)}{n!} = \int f(p)\,z^{-n}\,d\lambda_{(p,r)}(z)$$

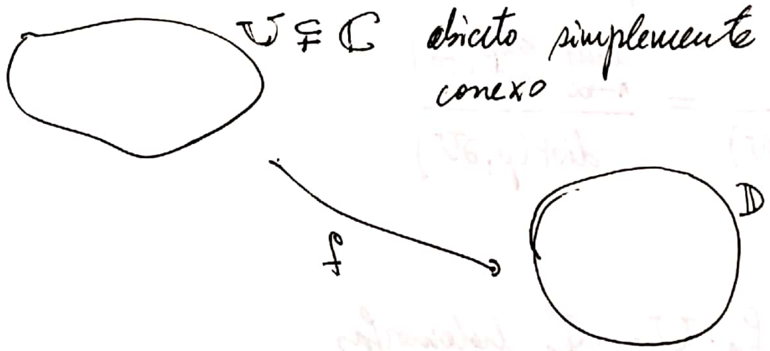
$$\left|\frac{f^{(n)}(p)}{n!}\right| \leq \frac{\sup_{\partial B(p,r)}|f|}{r^n}$$

$$\sup_{\partial B(p,r)}|f| \geq \left|\frac{f^{(n)}(p)}{n!}\right| r^n$$

(Relación super importante!) :)

Teorema (de Liouville)

Toda función holomorfa $f : \mathbb{C} \to \mathbb{D}$ es constante.



Objetivo: demostrar que $U \cong \mathbb{D}$

$\rightarrow$

Límites uniformes de funciones holomorfas

$U \subseteq \mathbb{C}$ abierto conexo.

$(f_n)_{n \geq 1}$ funciones holomorfas tq, $f_n : U \to \mathbb{C}$ convergen uniformemente a una función $f : U \to \mathbb{C}$

Sea $p \in U$, $r > 0$: $\overline{B(p,r)} \subseteq U$, $z_0 \in B(p,r)$

$$f(z_0) = \lim_{n\to\infty} f_n(z_0) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial B(p,r)} \frac{f_n(z)}{z - z_0} dz$$

$$= \frac{1}{2\pi i} \int_{\partial B(p,r)} \lim_{n\to\infty} \frac{f_n(z)}{z-z_0} dz \quad \text{(ya que el límite es uniforme)}$$

$$= \frac{1}{2\pi i} \int_{\partial B(p,r)} \frac{f(z)}{z-z_0} dz$$

Para tal f vale la formula integral de Cauchy $\Rightarrow$ f es holomorfa en toda $B(p,r)$

$\therefore f$ holomorfa en U

Ahora,

$$|f(p)| \leq \left| \int f(z) z^{-1} d\gamma_{p,r}(z) \right| \leq \frac{\sup_{B(p,r)} |f|}{r}$$

$$\Rightarrow |f'(p)| \leq \frac{\sup_U |f|}{dist(p, \partial U)} = \frac{\lim_{n\to\infty} \sup_U |f_n|}{dist(p,\partial U)}$$

— o —

U abierto conexo, $(f_n)_{n\geq 1}$, $f_n : U \to \mathbb{C}$ holomorfas

$$\forall n \in \mathbb{N}: \quad \sup_U |f_n| \leq M$$

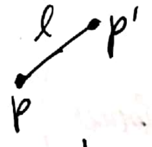
Pd: $(f_n)_{n\geq 1}$ es equicontinua en cualquier compacto.

dem. Sea $K \subseteq U$ compacto.

Existe $r > 0$ tal que $\overline{B(K,r)} =: K' \subseteq U$ | $B(K,r) = \bigcup_{p\in K} B(p,r)$

Fórmula integral de Cauchy $\Rightarrow \forall n \geq 1$, $p \in K'$, $|f_n'(p)| \leq \dfrac{M}{dist(K', \partial U)}$

Sean $p, p' \in K$, $|p - p'| \leq r$



$$l \subseteq K' \qquad f_n(p) - f_n(p') = \int_l f'(z)\, dz$$

$$\Rightarrow |f_n(p) - f_n(p')| \leq |p - p'| \cdot \frac{M}{dist(K', \partial U)}$$

$\therefore (f_n|_K)_{n\geq 1}$ es uniformemente Lipschitziana.

Teorema (de Montel).

U abierto conexo, $(f_n)_{n\geq 1}$, $f_n : U \to B(0,M)$. Entonces $(f_n)_{n\geq 1}$ es normal.

Repaso:

Fórmula integral de Cauchy $\Rightarrow$ todo límite uniforme de funciones holomorfas es holomorfo.

$U \subseteq \mathbb{C}$ abierto. $(f_n)_{n\geq 1}$ holomorfas, $f_n : U \to \mathbb{C}$, tales que existe $M>0$ independiente de n tal que

$$\sup_U |f_n| < M \qquad (*)$$

$\Rightarrow (f_n)_{n\geq 1}$ es localmente-uniformemente Lipschitz.

Familia normal (Familia precompacta en este contexto) := Familia de funciones holomorfas tal que toda sucesión admite una subsucesión convergente.

· Teorema de Montel.

Toda familia uniformemente acotada de funciones holomorfas es normal.

dem. Lo anterior (*) + Arzelà-Ascoli. ∎

Series de Laurent

$r' > r > 0$

$A(r,r') := \{z \in \mathbb{C} \mid r < |z| < r'\}$

$f : \overline{A(r,r')} \longrightarrow \mathbb{C}$ continua y holomorfa en $A(r,r')$

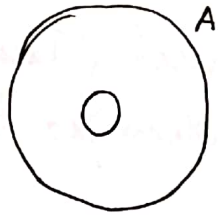
$$f(z) = \frac{1}{2\pi i} \int_{\partial A(r,r')} \frac{f(w)}{z-w}\, dw = \frac{1}{2\pi i}\left[\int_{\partial B(0,r)} \frac{f(w)}{z-w}\, dw - \int_{\partial B(0,r)} \frac{f(w)}{z-w}\, dw \right]$$

$$v(z) := \frac{1}{2\pi i}\int_{\partial B(0,r')} \frac{f(w)}{w-z}\,dw = \sum_{n=0}^{\infty}{}' a_n z^n$$ con radio de convergencia $\geq r'$.

Ejemplo. $f(z) = \frac{1}{z}$

$$v(z) = \frac{1}{2\pi i}\int_{\partial B(0,r')} \frac{\frac{1}{w}}{w-z}\,dw = \frac{1}{2\pi i}\frac{1}{z}\int_{B(0,r')}\left(\frac{1}{w-z}-\frac{1}{w}\right)dw = 0 \quad \text{(ejercicio)}$$

para $r'=1$: $f|_{S^1}(w) = \sum_{n\in\mathbb{Z}}{}' a_n z^n$.



A(r,r')

Sea $$u(z) := \frac{1}{2\pi i}\int_{\partial B(0,r)} \frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i}\int_{\partial B(0,r)} \frac{f(w)}{w-\frac{1}{\zeta}}\,dw$$

$\zeta = \frac{1}{z}$, $\xi = \frac{1}{w}$

$|\zeta| < \frac{1}{r}$

$$= \frac{1}{2\pi i}\int_{\partial B(0,r)} \frac{f(\frac{1}{\xi})}{\frac{1}{\xi}-\frac{1}{\zeta}}\,d\left(\frac{1}{\xi}\right) = \dots$$

Considerando el camino $\gamma : [0,1] \longrightarrow B(0,r)$

$\theta \longmapsto r\exp(2\pi i\theta)$

$$\dots = \frac{1}{2\pi i}\int_0^1 \frac{f(r\exp(2\pi i\theta))}{r\exp(2\pi i\theta)-\frac{1}{\zeta}}\, 2\pi i\exp(2\pi i\theta)\,d\theta = \int_0^1 \frac{f\left(\frac{1}{\xi(\theta)}\right)}{\frac{1}{\xi(\theta)}-\frac{1}{\zeta}}\,\frac{1}{\xi(\theta)}\,d\theta$$

$$\left(\begin{array}{l} w = r\exp(2\pi i\theta) \\ \xi(\theta) = \frac{1}{r}\exp(-2\pi i\theta) \end{array}\right)$$

$$= \int_{\partial B(0,\frac{1}{r})} \frac{f(\frac{1}{\xi})}{\frac{1}{\xi}-\frac{1}{\zeta}}\,\frac{1}{\xi}\left(-\frac{1}{2\pi i\xi}\right)d\xi$$

$\xi'(\theta) = -2\pi i\,\xi(\theta)$

$$= -\frac{1}{2\pi i}\int_{\partial B(0,\frac{1}{r})} \frac{f(\frac{1}{\xi})}{\frac{1}{\xi}-\frac{1}{\zeta}}\,\frac{1}{\xi^2}\,d\xi$$

Notar que $\frac{1}{\xi^2}\left(\frac{1}{\xi}-\frac{1}{\zeta}\right) = \frac{1}{\xi}+\frac{1}{\zeta-\xi}$, la integral queda

$$-\frac{1}{2\pi i}\int_{\partial B(0,\frac{1}{r})} \frac{f(1/\xi)}{\frac{1}{\xi}-\frac{1}{\zeta}}\,\frac{1}{\xi^2}\,d\xi = -\frac{1}{2\pi i}\int\left(\frac{f(1/\xi)}{\xi} - \frac{f(1/\xi)}{\xi-\zeta}\right)d\xi$$

$$\therefore\ u(z) = \frac{1}{2\pi i}\int_{\partial B(0,r)}\frac{f(w)}{w-z}\,dw = -\frac{1}{2\pi i}\int_{\partial B(0,\frac{1}{r})}\frac{f(1/\xi)}{\xi}\,d\xi + \frac{1}{2\pi i}\int_{\partial B(0,\frac{1}{r})}\frac{f(1/\xi)}{\xi-\zeta}\,d\xi$$

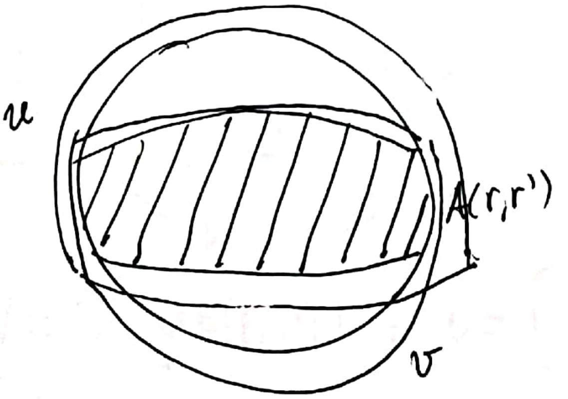
$$= \sum_{m=1}^{\infty}{}' \, b_m \zeta^m \qquad \forall \zeta \in B(0,\tfrac{1}{r})$$

$$= \sum_{m=1}^{-\infty}{}' \, b_{-m} z^m$$

$\therefore\ \forall z \in A(r,r')$, $f(z) = v(z) - u(z) = \sum'_{n\in\mathbb{Z}} A_n z^n$, donde $A_n = \begin{cases} a_n & n \geq 0 \\ -b_n, & n \leq 1 \end{cases}$.

En resumen:

Esfera de Riemann:



$f = u - v$

$\mathrm{ord}(f,p) = n \geq 0$ finito $\Rightarrow f(z) = (z-p)^n g(z)$ (Singularidad removible)

$\mathrm{ord}(f,p) < 0$ finito

$n := -\mathrm{ord}(f,p)$, $f(z) = \frac{1}{(z-p)^n} g(z)$ (Polo de orden n)

Cuando $\mathrm{ord}(f,p)$ finito : $f(z) = (z-p)^{\mathrm{ord}(f,p)} g(z)$

$\mathrm{ord}(f,p) = -\infty$ (Singularidad esencial)

En particular, $f(B(p,\rho_0)\setminus\{p\})$ es denso en $\mathbb{C}$.

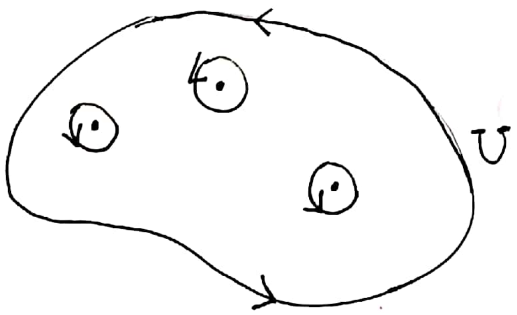
Residuos

$$\mathrm{Res}(f,p) = \frac{1}{2\pi i}\int_{\partial B(p,r)} f(z)\,dz = \int f(z)\, z\, d\lambda_{(p,r)}(z)$$

$(0<r\ll 1)$

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \quad , \quad \int f(z) z\, d\lambda_{(p,r)}(z) = \int \sum_{n\in\mathbb{Z}} a_n z^{n+1} d\lambda_{(p,r)}(z) = a_{-1}$$

$$\forall N \neq 0 \text{ entero}, r>0 : \int z^N d\lambda_{(p,r)}(z) = 0$$



$$\mathrm{Res}(f,U) := \frac{1}{2\pi i}\int_{\partial U} f(z)\,dz$$

$$V := U \setminus \bigcup_i B(p_i,r_i) \ ; \ 0 = \int_{\partial V} f(z)\,dz = \int_{\partial U} f(z)\,dz - \sum_i \int_{\partial B(p_i,r_i)} f(z)\,dz$$

Vamos a tomar el estudio anterior y llevarlo al caso extremo.

$\rho > 0$

$f : B(0,\rho)\setminus\{0\} \to \mathbb{C}$. Sea $\rho_0 \in (0,\rho)$

f definida y holomorfa en $A(\rho_0,\rho)$: $f(z) = \sum_{n\in\mathbb{Z}}' a_n z^n$ en $A(\rho_0,\rho)$

$f(z) = \sum_{n\in\mathbb{Z}}' a_n' z^n$ en $A(\rho_0',\rho)$

$$\sum_{n\in\mathbb{Z}}' (a_n - a_n') z^n \equiv 0 \quad \text{en } A(\rho_0,\rho)$$

Podemos notar que $f(z) = u(z) - v(z) = \tilde{u}(z) - \tilde{v}(z)$

$$\Delta(z) = u(z) - \tilde{u}(z) = v(z) - \tilde{v}(z) = 0$$

$$\therefore \begin{cases} u(z) = \tilde{u}(z) \\ v(z) = \tilde{v}(z) \end{cases}$$

$$f(z) = \sum_{n\in\mathbb{Z}}' a_n z^n \quad \text{en } B(0,\rho)\setminus\{0\}$$

Def. $\operatorname{ord}(f,0) = \inf\{n\in\mathbb{Z} / a_n \neq 0\}$

$\operatorname{ord}(0,p) = \infty$

Af. $\operatorname{ord}(\cdot,p)$ es una valuación.

$\operatorname{ord}(fg,p) = \operatorname{ord}(f,p) + \operatorname{ord}(g,p)$

$\operatorname{ord}(f+g,p) \geq \min\{\operatorname{ord}(f,p), \operatorname{ord}(g,p)\}$

Así tenemos $Res(f, U) = \sum_i{}' Res(f, p_i)$

Repaso. $f: \overline{A(r,r')} \to \mathbb{C}$ continua, holomorfa en $A(r,r')$

$$\Rightarrow f(z) = \sum_{n\in\mathbb{Z}}' a_n z^n$$

a_n está únicamente determinados por f.

$f(z) = u^+(z) + u^-(z)$, $u^+(z)$ holomorfa en $B(0,r)$
$u^-(z)$ holomorfa en $\mathbb{C} \setminus \overline{B(0,r)}$
tq $u^-(z) \to 0$ cuando $z \to \infty$

u^+, u^- están únicamente determinados por f.

$\rho > 0$, $f: B(p,\rho) \setminus \{p\} \to \mathbb{C}$ holomorfa

$$\Rightarrow f(z) = \sum_{n\in\mathbb{Z}}' a_n (z-p)^n$$

$\operatorname{ord}(f,p) = \inf\{n \in \mathbb{Z} \mid a_n \neq 0\}$ $(\inf \emptyset = \infty, \operatorname{ord}(0,p) = \infty)$

$$\operatorname{ord}(f,p) = \begin{cases} \geq 1, & p \text{ es cero de orden } = \operatorname{ord}(f,p) \\ \leq -1, > -\infty & p \text{ es un polo de orden } = -\operatorname{ord}(f,p) \\ -\infty, & \text{singularidad esencial.} \end{cases}$$

$\operatorname{ord}(f,p)$ es finito $\Rightarrow f(z) = (z-p)^{\operatorname{ord}(f,p)} \underbrace{\sum_{n\geq 0} a_{n-\operatorname{ord}(f,p)} (z-p)^n}_{g(z)}$

$g(p) \neq 0$. g se extiende a una función holomorfa en $B(p,\rho)$

~~g(...)~~

$\Rightarrow \exists \rho' \in (0,\rho)$ tq $\frac{1}{g}$ está definida y es holomorfa en $B(p,\rho')$

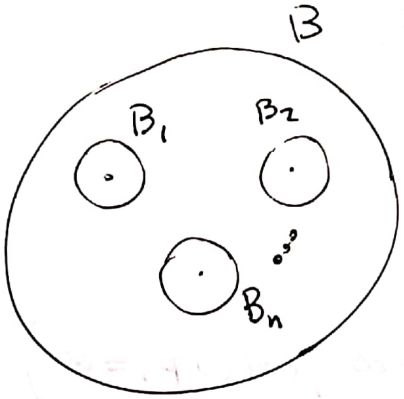
en $B(p,\rho)\setminus\{p\}$

$$\frac{1}{f(z)} = (z-p)^{-\operatorname{ord}(f,p)}\left(\frac{1}{g}\right)(z)$$

Si $\operatorname{ord}(f,p)<0$, entonces $\frac{1}{f}$ se extiende a una función holomorfa en $B(0,\rho')$. Además, p es un cero de orden $-\operatorname{ord}(f,p)$ de $\frac{1}{f}$.

— ◦ —

Series de Mittag-Leffler



$p\in\mathbb{C}$, $R>0$, $B=B(p,R)$

$p_1,\dots,p_n\in B(p,R)$

$R_1,\dots,R_n>0$: $B_i:=B(p_i,R_i)$

$\overline{B}_1,\dots,\overline{B}_n\subseteq B$

$\overline{B}_1,\dots,\overline{B}_n$ disjuntas dos a dos

$X:=B\setminus(\overline{B}_1\cup\dots\cup\overline{B}_n)$

Sea $f:\overline{B}\setminus(B_1\cup\dots\cup B_n)\to\mathbb{C}$ continua, holomorfa en $B\setminus(\overline{B}_1\cup\dots\cup\overline{B}_n)$

Pd: Existe $u:B\to\mathbb{C}$ holomorfa y $u_i:\mathbb{C}\setminus\overline{B}_i\to\mathbb{C}$ holomorfa, tal que $u_i(z)\to 0$, cuando $z\to\infty$, tq $f=u+\sum_{i=1}^{n}u_i$ y la descomposición es única.

$\partial X=\partial B\cup\partial B_1\cup\dots\cup\partial B_n$

$z\in X$, ocupando la fórmula integral de Cauchy

$$2\pi i f(z)=\int_{\partial X}\frac{f(w)}{w-z}\,dw=\int_{\partial B}\frac{f(w)}{w-z}\,dw-\sum_{i=1}^{n}\int_{\partial B_i}\frac{f(w)}{w-z}\,dw$$

$$u(z) := \frac{1}{2\pi i}\int_{\partial B} \frac{f(w)}{z-w}\,dw \quad , \quad u : B \to \mathbb{C} \text{ holomorfa}$$

$$u_i : \mathbb{C}\setminus B_i \longrightarrow \mathbb{C} \quad , \quad u_i(z) := -\frac{1}{2\pi i}\int_{\partial B_i} \frac{f(w)}{w-z}\,dw$$

$$\zeta := \frac{1}{z-p_i} \quad , \quad \xi := \frac{1}{w-p_i} \quad , \quad w = p_i + \frac{1}{\xi} \quad . \quad dw = -\frac{1}{\xi^2}\,d\xi$$

$$\hat{B}_i = \{\xi \in \mathbb{C} / p_i + \tfrac{1}{\xi} \in B_i\}$$

$$u_i\left(\frac{1}{\zeta}\right) = -\frac{1}{2\pi i}\int_{\partial\hat{B}_i} \frac{f(p_i+\frac{1}{\xi})}{p_i+\frac{1}{\xi}-(p_i+\frac{1}{\zeta})}\left(-\frac{1}{\xi^2}\right)d\xi$$

$$= \frac{1}{2\pi i}\int_{\partial\hat{B}_i} f\left(p_i+\frac{1}{\xi}\right)\frac{\zeta}{\xi(\zeta-\xi)}\,d\xi = \frac{1}{2\pi i}\int_{\partial\hat{B}_i}\frac{f(p_i+\frac{1}{\xi})}{\xi}\,d\xi$$

$$- \frac{1}{2\pi i}\int_{\partial\hat{B}_i}\frac{f(p_i+\frac{1}{\xi})}{\xi-\zeta}\,d\xi$$

$$u_i\left(\frac{1}{\zeta}\right) = \sum_{n=1}^{\infty} a_{i,n}\zeta^n \quad \text{en} \quad \{\zeta \in \mathbb{C} / \, |p_i + \tfrac{1}{\zeta} - p_i| > R_i\}$$

$$\shortparallel$$

$$\{\zeta \in \mathbb{C} / |\zeta| < R_i^{-1}\}$$

$$u_i(z) = \sum_{n=1}^{\infty} a_{i,n} z^{-n} \quad \text{en} \quad \{z \in \mathbb{C} / \, |z - p_i| > R_i\}$$

Unicidad :

$$u + u_1 + \cdots + u_n = v + v_1 + \cdots + v_n$$

$$\Rightarrow \quad u - v = (v_1 - u_1) + \cdots + (v_n - u_n)$$

Teorema de Liouville $\Rightarrow$ $u - v = 0$

$$(u_1 - v_1) + \cdots + (u_n - v_n) = 0$$

$$(u_1 - v_1) + \cdots + (u_i - v_i) = u_{i+1} - v_{i+1} \qquad \overline{B}_i \qquad \overline{B}_{i+1}$$

Se procede por inducción.

$D_1, \ldots, D_m$ bolas en $\overline{\mathbb{C}}$ (esfera de Riemann)

$\overline{D}_1, \ldots, \overline{D}_m$ disjuntas dos a dos

$$X := \overline{\mathbb{C}} \setminus (\overline{D}_1 \cup \ldots \cup \overline{D}_m)$$

$f : \overline{X} \to \mathbb{C}$ continua, holomorfa en X. Dado $p \notin X$, existe una única descomposición

$$f = f(p) + v_1 + \cdots + v_m$$

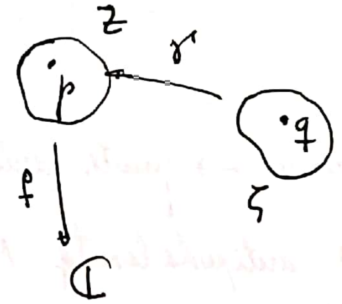
$$v_i : \overline{\mathbb{C}} \setminus \overline{D}_i \to \mathbb{C} \text{ holomorfa}, \quad v_i(p) = 0.$$

—o—

Para $r > 0$ pequeño

$$\operatorname{Res}(f, p) := \frac{1}{2\pi i} \int_{\partial B(p,r)} f(z)\,dz$$

$q \in \mathbb{C}$, g holomorfa en una vecindad de q, tal que $g(q)=p$, $g'(q)\neq 0$



$$\frac{1}{2\pi i}\int_{\partial B(q,r)} (f\circ g)(\zeta)\,d\zeta = \dots$$

$$g_*(f(z)dz) = (f\circ g)(\zeta)\,g'(\zeta)\,d\zeta$$

$$\int_{\partial B(q,r)} (f\circ g)(\zeta)\,d\zeta = \int_0^1 (f\circ g)(\gamma(\theta))\,g'(\zeta)\,\gamma'(\theta)\,d\theta$$

$$= \int_0^1 f(g\circ\gamma(\theta))(g\circ\gamma)'(\theta)\,d\theta$$

$$= \int_{\partial(g(\partial B(q,r)))} f(z)\,dz$$

Desarrollo prueba 1

$\boxed{1}$ Simetrías esféricas → puntos antipodales $(0,\infty)$ en antipodales $(\infty,0)$

Af. $\exists \zeta,\zeta' \in \overline{\mathbb{C}}$ antipodales tq $M(\zeta), M(\zeta')$ son antipodales.

Si se cumpliera la afirmación,

$$h_1(\zeta)=0 \ , \ h_1(\zeta')=\infty$$
$$h_2^{-1}(0)=M(\zeta) \ , \ h_2^{-1}(\infty)=M(\zeta')$$

Así, $h_2\circ M\circ h_1^{-1}(0)=0$

$h_2\circ M\circ h_1^{-1}(\infty)=\infty$ } → como fija a ∞, entonces es afín : $az+b$

Como fija a 0, es dilatación, $az \Rightarrow h_2\circ M\circ h_1^{-1}(z)=\lambda z$

Veamos la afirmación entonces: $\zeta'=-\frac{1}{\overline{\zeta}}$ (si ζ,ζ' son antipodales)

$$M(\zeta)=-\frac{1}{\overline{M(\zeta')}}=-\frac{1}{\overline{M\left(-\frac{1}{\overline{\zeta}}\right)}} \qquad \left(M(z)=\frac{Az+B}{Cz+D} \ , \ AD-BC\neq 0\right)$$

$$\Rightarrow M(\zeta)\overline{M\left(-\frac{1}{\overline{\zeta}}\right)}=-1 \quad \Big| \quad \frac{A\zeta+B}{C\zeta+D}\cdot\overline{\frac{-A\frac{1}{\overline{\zeta}}+B}{-C\frac{1}{\overline{\zeta}}+D}}=-1$$

$$\Rightarrow \frac{(A\zeta+B)(B\overline{\zeta}-A)}{(C\zeta+D)(D\overline{\zeta}-C)}=-1$$

Queda,

$$\frac{AB|\zeta|^2-A^2\zeta+B^2\overline{\zeta}-AB}{CD|\zeta|^2-C^2\zeta+D^2\overline{\zeta}-CD}=-1$$

$$\Leftrightarrow AB|\zeta|^2-A^2\zeta+B^2\overline{\zeta}-AB=-CD|\zeta|^2+C^2\zeta-D^2\overline{\zeta}+CD$$

$$(AB+CD)|\zeta|^2-(A^2+C^2)\zeta+(B^2+D^2)\overline{\zeta}-(AB+CD)=0$$

$\alpha=AB+CD$

$\beta=A^2+C^2$

$\gamma=B^2+D^2$

se tiene $\alpha(|\zeta|^2-1)-\beta\zeta+\gamma\overline{\zeta}=0$

$\Leftrightarrow \alpha\zeta\overline{\zeta}-\beta\zeta+\gamma\overline{\zeta}-\alpha=0$

Nos gustaría $(a\zeta-b)(c\bar\zeta-d)=ac\zeta\bar\zeta-ad\zeta-bc\bar\zeta+bd$

Buscaríamos : $\begin{cases} ac=\alpha \\ -ad=-\beta \\ -bc=\gamma \\ bd=-\alpha \end{cases} \rightarrow \begin{array}{l} \alpha^2=\beta\gamma \\ A^2B^2+C^2D^2+2ABCD=A^2B^2+C^2D^2+A^2D^2+B^2C^2 \\ ¿(AD-BC)^2=0? \end{array}$

Queremos que M envíe $\underbrace{\text{el } 0 \text{ en } \infty}_{\frac{az+b}{cz+d}}$ y el $\underbrace{\infty \text{ en } 0}_{a=0}$.

$M(z)=\lambda\cdot\frac{1}{z}$

Y quedan fijos por simetrías $\longrightarrow \lambda=-1$, $M(z)=-\frac{1}{z}$

De nuevo, faltó conjugar (*) :

$$\frac{A\zeta+B}{C\zeta+D}=\frac{-A\frac{1}{\bar\zeta}+B}{-C\frac{1}{\bar\zeta}+D}=-1 \iff \frac{A\zeta+B}{C\zeta+D}\cdot\frac{-\bar A\frac{1}{\zeta}+\bar B}{-\bar C\frac{1}{\zeta}+\bar D}=-1$$

$$=\frac{A\zeta+B}{C\zeta+D}\cdot\frac{\bar B\zeta-\bar A}{\bar D\zeta-\bar C}=-1$$

$\iff (A\zeta+B)(\bar B\zeta-\bar A)=-(C\zeta+D)(\bar D\zeta-\bar C)$

$\iff A\bar B\zeta^2+(-|A|^2+|B|^2)\zeta-\bar AB=-C\bar D\zeta^2+(|C|^2-|D|^2)\zeta+\bar CD$

$\iff (A\bar B+C\bar D)\zeta^2+(-|A|^2+|B|^2-|C|^2+|D|^2)\zeta+(-\bar AB-\bar CD)=0$

(i) Si $A\bar B+C\bar D=0 \Rightarrow A\bar B=-C\bar D$ $\quad (-\bar AB-\bar CD)=0.$

$$\frac{A}{C}=-\overline{\left(\frac{D}{B}\right)}$$

$\underbrace{\frac{A}{C}}_{\text{imagen de }\infty}$ es antipodal con $\underbrace{\frac{B}{D}}_{\text{imagen de }0}$ $\Big\}$ con esto se demuestra la afirmación.

Si $A\bar B+C\bar D=0 \rightarrow$ cuadrática, hay solución.

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$\boxed{2}$ $f_0(z)=z$

$f_n(z) = \dfrac{f_{n-1}(z)^2+1}{2}$, $\forall n \geq 1$

(i) $(f_n|_{B(0,1)})_{n\in\mathbb{N}}$ es normal.

~~Por inducción: ...~~

$|z|<1$

Por inducción, $|f_n(z)|<1 \rightarrow$ teo. de Montel $\Rightarrow$ familia normal.

(ii) $r>1$: $(f_n|_{B(0,r)})_{n\in\mathbb{N}}$

$z=1+w \rightarrow g_0(w)=f_0(1+w)=1+w$. $\quad g_n(w)=f_n(1+w)-1$

$= \dfrac{(f_{n-1}(1+w))^2+1}{2} = \dfrac{(g_{n-1}(w)+1)^2+1}{2} - 1$

$g_1(w) = \dfrac{(1+w)^2+1}{2} - 1 = \dfrac{2w+w^2}{2} = w + \dfrac{w^2}{2}$

$g_2(w) = w + \dfrac{w^2}{2} + \dfrac{\left(w+\frac{w^2}{2}\right)^2}{2} = w + \dfrac{w^2}{2} + \dfrac{w^2}{2} + \dfrac{w^3}{2} + \dfrac{w^4}{4} = w + w^2 + \dfrac{w^3}{2} + \dfrac{w^4}{4}$

$= g_{n-1}(w) + \dfrac{g_{n-1}(w)^2}{2}$

$g_3(z) = w + w^2 + \dfrac{w^3}{2} + \dfrac{w^4}{4} + \dfrac{\left(w+w^2+\frac{w^3}{2}+\frac{w^4}{4}\right)^2}{2}$

$= w + w^2 + \dfrac{w^3}{2} + \dfrac{w^4}{4} + \dfrac{w^2+2w^3+\dots}{2}$

$= w + \dfrac{3}{2}w^2$

$g_n(w) = w + \dfrac{n}{2}w^2 + \dots \rightarrow$ por lo tanto, no puede ser normal.

$\boxed{3}$ $R([x:y]) = [P(x,y):Q(x,y)]$

a) $R([x:y]) = 0 \Leftrightarrow Q(x,y) = 0$

$\hookrightarrow$ conjunto finito de puntos donde R es ∞ (polos: ceros de Q)

Y fuera de ese conjunto: $\mathbb{C}\setminus R^{-1}(\infty)$:

En c/u de esos puntos es holomorfa, pues el denominador $\neq 0$

(cociente de holomorfas es holomorfa)

b) $\overline{\mathbb{C}}$

$$\begin{tikzcd} & \infty \arrow[dl, "M"'] \arrow[dr, "\tilde{M}"] & \\ p \arrow[rr, bend right, "\tilde{M}\circ M^{-1}"'] & & \tilde{p} \end{tikzcd}$$

$$\overset{\shortparallel}{h}$$

$M, \tilde{M}$ tales que $M(\infty), \tilde{M}(\infty) \in \mathbb{C}$

Pd: $\mathrm{ord}(R\circ M^{-1}, M(\infty)) = \mathrm{ord}(R\circ M^{-1}, \tilde{M}(\infty))$

$$h := \tilde{M}\circ M^{-1},\ Q = R\circ M^{-1} \mid \mathrm{ord}(Q,p) = \mathrm{ord}(Q\circ h^{-1}, \overset{h(p)}{\overset{\shortparallel}{\tilde{p}}})$$

$$Q(z) = \sum_{n\in\mathbb{Z}} a_n (z-p)^n,\quad Q\circ h^{-1}(w) \Rightarrow \sum_{n\in\mathbb{Z}} b_n (w-h(p))^n$$

$$\rightarrow w = h(z),\ h(z) = \frac{Az+B}{Cz+D}$$

Mejor: Llamemos $n = \mathrm{ord}(Q,p)$, $\tilde{n} = \mathrm{ord}(Q\circ h^{-1}, \tilde{p})$. Así

$$Q(z) = (z-p)^n g(z)$$

$$Q\circ h^{-1}(w) = (w-h(p))\tilde{g}(w) = (h(z)-h(p))\tilde{g}(h(z))$$

$$h(z)-h(p) = \frac{Az+B}{Cz+D} - \frac{Ap+B}{Cp+D} = \frac{(AD-BC)}{(Cz+D)}\frac{(z-p)}{(Cp+D)} = (z-p)(\text{algo holomorfo})$$

Así, $Q(z) = (z-p)^n g(z)$, $Q\circ h^{-1}(z) = (z-p)^{\tilde{n}}\hat{g}(z) \Rightarrow n = \tilde{n}$.

· Hay necesariamente un cero y un polo (R no es nula, no es constante)

c) $p\in\mathbb{C} \rightarrow \mathrm{ord}(R,p)$ es finito.

$$z = \frac{x}{y} \;\Big|\; \begin{aligned} P(z) &= (z-p)^a A(z) \\ Q(z) &= (z-p)^{\tilde{a}}\tilde{A}(z) \end{aligned}\quad ,\ a \text{ o } \tilde{a} \text{ es } 0 \text{ (no coinciden)}$$

$$R(z) = \frac{P(z)}{Q(z)} = (z-p)^{a-\tilde{a}}\frac{A(z)}{\tilde{A}(z)} \Rightarrow \text{finito.}$$

d) ✓

Problema de la prueba. ¨

$\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(\theta) = \exp(i\theta) + i\exp(2i\theta)$

Calcular $\displaystyle\int_\gamma \frac{1}{z^{2015}-1}\, dz$.

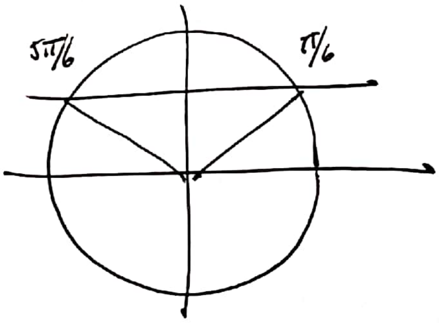
$\theta, \theta' \in [0, 2\pi]$, $\begin{cases} z = \exp(i\theta) \\ z' = i\exp(i\theta') \end{cases}$ | $\gamma(\theta) = \gamma(\theta') \Rightarrow (z - z')(1 + iz + iz') = 0$

Como $z \neq z' \Rightarrow$ $z + z' = i$

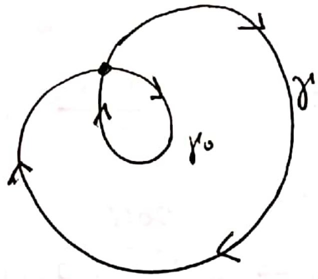
Así, $\begin{cases} \operatorname{sen}\theta + \operatorname{sen}\theta' = 1 \\ \cos\theta + \cos\theta' = 0 \end{cases}$ | Senos y cosenos son iguales módulo signo.

Caso I : $\operatorname{sen}\theta = \operatorname{sen}\theta' \Rightarrow \operatorname{sen}\theta = \operatorname{sen}\theta' = \frac{1}{2} \Rightarrow \theta, \theta' \in \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

(único caso)



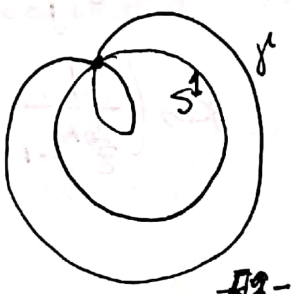
Curva γ



$\gamma_0 = \gamma\big|_{[\frac{\pi}{6}, \frac{5\pi}{6}]}$ curva de Jordan

$\gamma_1 = \gamma\big|_{[\frac{5\pi}{6}, \frac{13\pi}{6}]}$ curva de Jordan.

$|\gamma(\theta)| = 1 \Rightarrow (1 - \operatorname{sen}\theta)^2 + (\cos\theta)^2 = 1 \to \operatorname{sen}\theta = \frac{1}{2} \Rightarrow \theta \in \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$



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