Universidad de las Américas
Algebra II, MA141
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Propuesta Cátedra III

Problema 1. $T(x,y,z) = (x+y+2z,\ 2x+5y-2z,\ 5x+2y+16z)$

a). $(x,y,z) \in \ker(T)$ siempre y cuando $T(x,y,z) = (0,0,0)$

$$(x+y+2z,\ 2x+5y-2z,\ 5x+2y+16z) = (0,0,0)$$

$$\Leftrightarrow \quad \begin{cases} x+y+2z=0 \\ 2x+5y-2z=0 \\ 5x+2y+16z=0 \end{cases} \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 1 & 2 \\ 2 & 5 & -2 \\ 5 & 2 & 16 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Aplicamos operaciones elementales:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 5 & -2 & 0 \\ 5 & 2 & 16 & 0 \end{array}\right) \xrightarrow[f_3-5f_1 \to f_3]{f_2-2f_1 \to f_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & -3 & 6 & 0 \end{array}\right) \xrightarrow{f_2+f_3 \to f_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{f_2 \to \frac{1}{3}f_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{f_1-f_2 \to f_1} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Así: $\begin{cases} x+4z=0 \\ y+2z=0 \end{cases} \longrightarrow \begin{array}{l} x=-4z \\ y=-2z \end{array}$

$$(x,y,z) = (-4z,-2z,z) = z(-4,-2,1)$$

una base de $\ker(T)$ es $\{(-4,-2,1)\}$.

b. $T(x,y,z) = (x+y+2z,\ 2x+5y-2z,\ 5x+2y+16z)$

$= (x, 2x, 5x) + (y, 5y, 2y) + (2z, -2z, 16z)$

$= x(1,2,5) + y(1,5,2) + z(2,-2,16)$

Luego $Im(T) = \langle (1,2,5), (1,5,2), (2,-2,16) \rangle$

Buscamos una base aplicando operaciones elementales a la matriz $\begin{pmatrix} 1 & 2 & 5 \\ 1 & 5 & 2 \\ 2 & -2 & 16 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 2 & 5 \\ 1 & 5 & 2 \\ 2 & -2 & 16 \end{pmatrix} \xrightarrow[f_3 - 2f_1 \to f_3]{f_2 - f_1 \to f_2} \begin{pmatrix} 1 & 2 & 5 \\ 0 & 3 & -3 \\ 0 & -6 & 6 \end{pmatrix} \xrightarrow{f_3 + 2f_2 \to f_3} \begin{pmatrix} 1 & 2 & 5 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{f_2 \to \frac{1}{3} f_2} \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{f_1 - 2f_2 \to f_1} \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Así, una base de $Im(T)$ es $\{(1,0,7), (0,1,-1)\}$.

c. $\text{nulidad}(T) = \dim(\ker(T)) = 1$

$\text{rango}(T) = \dim(Im(T)) = 2$

Observación: Se cumple el teorema de la dimensión:

$$\dim(\mathbb{R}^3) = \text{rango}(T) + \text{nulidad}(T)$$

$$3 = 2 + 1 .$$

Problema 2.

$$T(1,0,0) = (5,-3,8)$$
$$T(0,1,0) = (0,-8,6)$$
$$T(0,0,1) = (-4,10,-1)$$

$$
\begin{aligned}
T(x,y,z) &= T(\,x(1,0,0)+y(0,1,0)+z(0,0,1))\\
&= x\,T(1,0,0)+y\,T(0,1,0)+z\,T(0,0,1)\\
&= x(5,-3,8)+y(0,-8,6)+z(-4,10,-1)\\
&= (5x,-3x,8x)+(0,-8y,6y)+(-4z,10z,-z)\\
&= (5x-4z,\,-3x-8y+10z,\,8x+6y-z)
\end{aligned}
$$

Por lo tanto: $T(x,y,z) = (5x-4z,\,-3x-8y+10z,\,8x+6y-z)$

c. $v+w = (1,5,10)+(-2,7,-9) = (-1,12,1)$

$$[v+w]_{\mathcal{C}} = [-1,12,1]$$

$$[v+w]_{\mathcal{B}} = [I]_{\mathcal{C}}^{\mathcal{B}}[v+w]_{\mathcal{C}} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} -1 \\ 12 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ 12 \\ 1/2 \end{pmatrix}$$

$$\therefore [v+w]_{\mathcal{B}} = [13,12,1/2]$$

Ahora:

$$[v]_{\mathcal{B}}+[w]_{\mathcal{B}} = [4,5,5]+[9,7,-9/2] = [13,12,5-9/2]$$
$$= [13,12,1/2]$$

Se cumple la igualdad $[v]_{\mathcal{B}}+[w]_{\mathcal{B}} = [v+w]_{\mathcal{B}}$

Problema 4.

a. Polinomio característico $p(\lambda) = \det\begin{pmatrix} 2-\lambda & 3 \\ 6 & -4-\lambda \end{pmatrix}$

$$\det\begin{pmatrix} 2-\lambda & 3 \\ 6 & -4-\lambda \end{pmatrix} = (2-\lambda)(-4-\lambda)-18 = -8-2\lambda+4\lambda+\lambda^2-18$$
$$= \lambda^2+2\lambda-26$$

$$p(\lambda)=0 \iff \lambda^2+2\lambda-26=0$$

$$\lambda = \frac{-2 \pm \sqrt{4+104}}{2} = \frac{-2\pm\sqrt{108}}{2} = \frac{-2\pm 6\sqrt{3}}{2} = -1 \pm 3\sqrt{3}$$

Los valores propios de A son $-1+3\sqrt{3}$ y $-1-3\sqrt{3}$.

b. Sea $v=(x,y)$ vector propio asociado al valor propio $-1+3\sqrt{3}$

$$\begin{pmatrix} 2-(-1+3\sqrt{3}) & 3 \\ 6 & -4-(-1+3\sqrt{3}) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-3\sqrt{3} & 3 \\ 6 & -3-3\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 3-3\sqrt{3} & 3 & 0 \\ 6 & -3-3\sqrt{3} & 0 \end{array}\right) \xrightarrow[f_2 \to \frac{1}{3}f_2]{f_1 \to \frac{1}{3}f_1} \left(\begin{array}{cc|c} 1-\sqrt{3} & 1 & 0 \\ 2 & -1-\sqrt{3} & 0 \end{array}\right)$$

$$\xrightarrow{f_2 \leftrightarrow f_1} \left(\begin{array}{cc|c} 2 & -1-\sqrt{3} & 0 \\ 1-\sqrt{3} & 1 & 0 \end{array}\right) \xrightarrow{f_1 \to \frac{1}{2}f_1} \left(\begin{array}{cc|c} 1 & -\frac{1}{2}-\frac{\sqrt{3}}{2} & 0 \\ 1-\sqrt{3} & 1 & 0 \end{array}\right)$$

$$\xrightarrow{f_2-(1-\sqrt{3})f_1 \to f_2} \left(\begin{array}{cc|c} 1 & -\frac{1}{2}-\frac{\sqrt{3}}{2} & 0 \\ 0 & 1-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)(1-\sqrt{3}) & 0 \end{array}\right)$$

$$\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)(1-\sqrt{3}) = -\frac{1}{2}+\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}+\frac{3}{2} = \frac{2}{2} = 1$$

Finalmente : $\left(\begin{array}{cc|c} 1 & -\frac{1}{2}-\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$

Así: $v = \left(\frac{1}{2} + \frac{\sqrt{3}}{2} , 1 \right)$

es un vector propio asociado a $-1 + 3\sqrt{3}$

Sea $w = (x, y)$ un vector propio asociado al valor propio $-1 - 3\sqrt{3}$:

$$\begin{pmatrix} 2 - (-1 - 3\sqrt{3}) & 3 \\ 6 & -4 - (-1 - 3\sqrt{3}) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 + 3\sqrt{3} & 3 \\ 6 & -3 + 3\sqrt{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 3 + 3\sqrt{3} & 3 & 0 \\ 6 & -3 + 3\sqrt{3} & 0 \end{array} \right) \xrightarrow[f_2 \to \frac{1}{6} f_2]{f_1 \to \frac{1}{3} f_1} \left(\begin{array}{cc|c} 1 + \sqrt{3} & 1 & 0 \\ 2 & -\frac{1}{2} + \frac{\sqrt{3}}{2} & 0 \end{array} \right)$$

$$\xrightarrow{f_2 \leftrightarrow f_1} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2} & 0 \\ 1 + \sqrt{3} & 1 & 0 \end{array} \right) \xrightarrow{f_2 - (1+\sqrt{3}) f_1 \to f_2} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2} & 0 \\ 0 & 1 - (1+\sqrt{3})(-\frac{1}{2} + \frac{\sqrt{3}}{2}) & 0 \end{array} \right)$$

$$(1 + \sqrt{3})\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + \frac{3}{2} = \frac{2}{2} = 1$$

Luego: $\left(\begin{array}{cc|c} 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 \end{array} \right)$

Así: $w = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} , -1 \right)$

es un vector propio asociado al valor propio $-1 - 3\sqrt{3}$.