Obs.: $\prod_{i=1}^{n}(T-x_i) = T^n - \sigma_1 T^{n-1} + \sigma_2 T^{n-2} - \cdots$

$\therefore$ $x_1, \ldots, x_n$ son enteros sobre R, luego $f \in K[x_1, \ldots, x_n]$ es entero sobre R, $\therefore$ $f \in R$ $\square$

Ejercicio: si $f \in \mathbb{Z}[x_1, \ldots, x_n]$ simétrico, ¿es cierto que $f \in \mathbb{Z}[\sigma_1, \ldots, \sigma_n]$?

Q2) Si $f \in \mathbb{Z}[x_1, \ldots, x_n]$ simmetrico è vero che

$$f \in \mathbb{Z}[\sigma_1, \ldots, \sigma_n] \ ?$$

[14/11/2013]

$1, \eta, \eta^2, \eta^3$

$$\eta + \eta^5 = 0$$

$$a + b\eta + c\eta^2 + d\eta^3 \longmapsto a - b\eta + c\eta^2 - d\eta^3$$

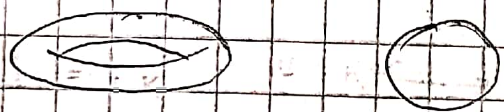
$$\varphi_5(a + c\eta^2) = a + c\eta^2$$

$$\eta^2 = i$$

$\mathbb{Q}(\sqrt{2})$ $\quad \sqrt{2} \cdot \sqrt{-2} = 2i \longrightarrow \mathbb{Q}(i)$

$\mathbb{Q}(\sqrt{-2})$

$$K = \mathbb{C}(x, \sqrt{x^3 - 1}) \neq \mathbb{C}(t)$$



$$y^2 = x^3 - 1$$

Mercoledì, Venerdì 14:30 – 16:00

ÁLGEBRAS

A espacio vectorial sobre un cuerpo K
con una estructura de anillo (con la misma suma)
tal que:

1) $\lambda(ab) = (\lambda a)b \quad \forall \lambda \in K \quad a,b \in A$

$= a(\lambda b)$

ley distributiva mixta

$\lambda(a+b)$

$(a+b)c = ac + bc \quad — \quad a,b,c \in A$
$c(a+b) = ca + cb$

$(a,b) \longmapsto ab$ bilineal

si A es un anillo unitario, podemos definir

$\varphi : K \longrightarrow A$

por $\varphi(\lambda) = \lambda 1_A$

$\varphi(\lambda+\mu) = (\lambda+\mu)1_A$

$= \lambda 1_A + \mu 1_A$

$= \varphi(\lambda) + \varphi(\mu)$

$$\varphi(\lambda\mu) = (\lambda\mu)1_A = \lambda(\mu 1_A)$$

$$= \lambda(1_A)(\mu 1_A) = (\lambda 1_A)(\mu 1_A) = \varphi(\lambda)\varphi(\mu)$$

PODEMOS IDENTIFICAR K CON SU IMAGEN EN A

ej) $A = M_n(K)$ $\quad 1_A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$$\lambda 1_A = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$$

ej) $A = K[X]$ $\quad 1_{K[X]} = 1$

$$\varphi : K \hookrightarrow K[X]$$

L/K EXTENSIÓN

L K-ALGEBRA

$1_L = 1_K \quad \varphi : K \hookrightarrow L$

$$\mathbb{H} = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R} \quad k = ij$$

$$\left\langle i, j \;\middle|\; \begin{matrix} i^2 = j^2 = -1 \\ ij = -ji \end{matrix} \right\rangle_{\mathbb{R}}$$

$\mathbb{H}$ algebra of division

$q \in \mathbb{H} \qquad q = a + bi + cj + dk$

$\bar{q} = a - bi - cj - dk$

$\overline{q+p} = \bar{q} + \bar{p}, \quad \overline{qp} = \bar{p} \cdot \bar{q}$

$N(q) = q\bar{q} = a^2 + b^2 + c^2 + d^2$

$q \neq 0 \quad N(q) \neq 0$

$$q^{-1} = \frac{\bar{q}}{N(q)} \qquad q\left(\frac{\bar{q}}{N(q)}\right) = \frac{N(q)}{N(q)} = 1$$

si $q \in \mathbb{H} \qquad q + \bar{q} = 2a$

$\quad = \mathrm{Tr}(q)$

$$q^{-1} = \frac{N(q)}{q} \Longrightarrow \bar{q}$$

$$\cancel{q^{-1} = N(q)} \qquad q^{-1} = \frac{\bar{q}}{N(q)} \Longrightarrow \bar{q} = N(q)\, q^{-1}$$

$$q + \bar{q} = TR(q)$$

$$q^2 + q\bar{q} = q\, TR(q)$$

$$q^2 - q\, TR(q) + N(q) = 0$$

$q \notin \mathbb{R}$

$$m_q(x) = x^2 - x\, TR(q) + N(q)$$

POL. MINIMAL

$$\mathbb{R}[q] \cong \frac{\mathbb{R}[x]}{(m_q(x))}$$

$m_q(x) = (x-b)(x-a) \qquad a,b \in \mathbb{R}$

$\Rightarrow (q-b)(q-a) = 0 \quad (\longrightarrow\!\leftarrow)$

$\therefore$ m_q NO TIENE RAICES REALES

$$\mathbb{R}[q] \cong \mathbb{C}$$

$\tilde{\mathbb{H}} \subseteq M_2(\mathbb{C})$

$$\tilde{\mathbb{H}} = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathbb{R}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

<u>OBS:</u> $\mathbb{H} \cong \tilde{\mathbb{H}}$

Anillo unitario y homomorfismo de anillos

$$\varphi : \mathbb{H} \longrightarrow \tilde{\mathbb{H}}$$

$$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{\tilde{\mathbb{H}}}$$

$$i \longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$j \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$k \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$\tilde{\mathbb{H}}$ $\mathbb{R}$ - álgebra

$\tilde{\mathbb{H}} \subseteq \tilde{\mathbb{H}}_{\mathbb{C}} \longleftarrow$ $\mathbb{C}$ - álgebra generada por $\tilde{\mathbb{H}}$

$$\tilde{\mathbb{H}}_{\mathbb{C}} = \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$\dim_{\mathbb{C}} \tilde{\mathbb{H}}_{\mathbb{C}} = 4 \qquad \dim_{\mathbb{R}} \tilde{\mathbb{H}}_{\mathbb{C}} = 8$

EXTENSIÓN DE UN $\mathbb{R}$-ÁLGEBRA A UNA $\mathbb{C}$-ÁLGEBRA

$$\tilde{\mathbb{H}}_{\mathbb{C}} \cong M_2(\mathbb{C})$$

EXTENSIÓN ESCALARES

PRODUCTO TENSORIAL

K cuerpo $\quad V, W$ E.V. SOBRE K

$V \otimes W$

$Z = K$-ESPACIO VECTORIAL CON BASE $\left\{ e_{v,w} \;\middle|\; \begin{matrix} v \in V \\ w \in W \end{matrix} \right\}$

X = SUBESPACIO GENERADO POR TODOS LOS ELEMENTOS DE LA FORMA

$$e_{v_1+v_2,w} - e_{v_1,w} - e_{v_2,w}$$

$$e_{\lambda v,w} - \lambda e_{v,w}$$

$$e_{v,w_1+w_2} - e_{v,w_1} - e_{v,w_2}$$

$$e_{v,\lambda w} - \lambda e_{v,w}$$

DEF: $V \otimes W = \mathbb{Z}/X$

$$\Updownarrow$$

$$v \otimes w = \overline{e_{v,w}}$$

$$(\lambda v) \otimes w = \lambda (v \otimes w)$$

$$\|$$

$$\overline{e_{\lambda v, w}} = \lambda \overline{e_{v,w}}$$

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$

$$v \otimes (\lambda w) = \lambda (v \otimes w)$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$

PROPIEDAD UNIVERSAL DEL PRODUCTO TENSORIAL

Si $b: V \times W \longrightarrow Y = (K\text{-E.V})$

es una función bilineal (lineal en cada variable)

entonces existe una función lineal única

$$\tilde{b}: V \otimes W \longrightarrow Y$$

con $b(v,w) = \tilde{b}(v \otimes w)$

$$v \otimes w = v' \otimes w'$$

Dem: $\hat{b}: Z \longrightarrow Y$

$$\hat{b}(e_{v,w}) = b(v,w)$$

PF: $\hat{b}(X) = \{0\}$

$$\hat{b}(e_{\lambda v,w} - \lambda e_{v,w}) = \hat{b}(e_{\lambda v,w}) - \lambda \hat{b}(e_{v,w})$$

$$= b(\lambda v, w) - \lambda b(v,w) = 0$$

$\exists!\ \tilde{b}: Z/X \longrightarrow Y$

$$\tilde{b}(\overline{e_{v,w}}) = b(v,w)$$

$$\tilde{b}(v\otimes w) = b(v,w)$$

$$\begin{tikzcd} Z \arrow[r, "\hat{b}"] \arrow[d, "\pi"'] & Y \\ Z/X \arrow[ur, "\exists!\, \tilde{b}"'] & \end{tikzcd}$$

$$\hat{b} = \tilde{b}\circ\tilde{\pi}$$

Prop: Si $v_1,\dots,v_n \in V$ son L.I. y $w_1,\dots,w_n \in W$ entonces

$$\sum_{i=1}^{n} v_i\otimes w_i = 0 \implies w_1 = \dots = w_n = 0 .$$

Dem: supongamos que no es así

sup que $w_1 \neq 0$

$$\exists\, \sigma \in W^* = \mathrm{Hom}_K(W,K)$$

con $\sigma(w_1)\neq 0$

$$b : V\times W \longrightarrow V$$

$$b(v,w) = \sigma(w)v$$

Ej: b es bilineal

$\exists! \ \tilde{b} : V \otimes W \longrightarrow V$

$\tilde{b}(v \otimes w) = \sigma(w) v$

$\tilde{b}\left(\sum_{i=1}^{n} v_i \otimes w_i\right) = \sum_{i=1}^{n} \sigma(w_i) v_i$

$\therefore \sum_{i=1}^{n} \sigma(w_i) v_i = 0 \qquad \therefore \sigma(w_i) = 0 \quad (\Leftarrow)$.

COR: SI B ES BASE DE V Y S ES BASE DE W

ENTONCES $B \bar{\otimes} S = \{ b \otimes s \mid b \in B,\ s \in S \}$

ES BASE DE $V \otimes W$.

1) $v \otimes w \in V \otimes W$

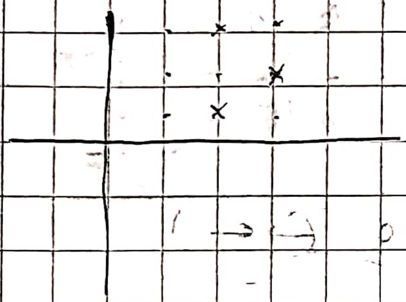
$\Rightarrow v \otimes w \in \langle B \bar{\otimes} S \rangle$

$v = \sum_{i=1}^{n} \lambda_i b_i$

$w = \sum_{j=1}^{m} \mu_j s_j$

$\Longrightarrow v \otimes w = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j (b_i \otimes s_j)$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_{ij}\left(b_i \otimes s_j\right)$$



$$\sum_{i=1}^{n} b_i \otimes \underbrace{\left(\sum_{j=1}^{m} \lambda_{ij}\, s_j\right)}_{0} = 0$$

$$\Longrightarrow \lambda_{ij} = 0.$$

PRODUCTO TENSORIAL DE ALGEBRAS
ES UN ALGEBRA

$$\dim_K (V \otimes W) = (\dim_K V)(\dim_K W)$$

26/11/2013)

$$\tilde{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad , \tilde{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$$

$$\mathbb{H}' = \mathbb{R} \oplus \mathbb{R}\tilde{i} \oplus \mathbb{R}\tilde{j} \oplus \mathbb{R}\tilde{i}\tilde{j} \cong \mathbb{H}$$

$$M_2(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C}\tilde{i} \oplus \mathbb{C}\tilde{j} \oplus \mathbb{C}\tilde{i}\tilde{j} \cong \mathbb{H}_{\mathbb{C}}$$

$$\therefore \quad \mathbb{H}_{\mathbb{C}} \cong M_2(\mathbb{C})$$

K cuerpo , $\alpha, \beta \in K^*$ $\quad$ car $K \neq 2$

$$D = \left(\frac{\alpha, \beta}{K}\right) = K \oplus Ki \oplus Kj \oplus Kij$$

$$i^2 = \alpha \ , \quad j^2 = \beta \quad ij = -ji$$

$$q = a + bi + cj + dij$$

$$\bar{q} = a - bi - cj - dij$$

Ej: $\quad N(q) = q\bar{q} \quad \in K$

$$TR(q) = q + \bar{q} \quad \in K$$

q es invertible ssi $N(q) \neq 0$, en ese caso

$$q^{-1} = \frac{1}{N(q)} \cdot \bar{q}$$

D ES ALGEBRA DE DIVISION SSI $N(q) \neq 0 \quad \forall q \in D$

HECHO: SI $\left(\frac{\alpha, \beta}{K}\right)$ NO ES UN ALGEBRA DE DIVISION,

ENTONCES $\left(\frac{\alpha, \beta}{K}\right) \cong \mathbb{M}_2(K)$

OBS: L/K EXT.

$$\left(\frac{\alpha, \beta}{K}\right) \otimes_K L \cong \left(\frac{\alpha, \beta}{L}\right)$$

SI $\left(\frac{\alpha, \beta}{L}\right) \cong \mathbb{M}_2(L)$ SE DICE QUE L ES UN

CUERPO DE DESCOMPOSICION DE D

$K(i) \cong K(\sqrt{\alpha}) \quad \alpha \neq \square$ EN K

$\alpha = 1 \qquad K(i) \cong \dfrac{K[x]}{(x^2-1)} \cong K \times K \qquad$ (TEO CHINO RESTOS)

EJ: $L = K(\sqrt{\alpha})$ SIEMPRE ES UN CUERPO DE DESCOMPOSICION DE D PUES

$$D_L = D \otimes_K L \supseteq K(i) \otimes_K L \cong L(i) \cong L \times L$$

$\Rightarrow D_L$ NO ES ALG. DE DIVISION

Ej: Álgebra cíclica L/K extensión cíclica

$Gal(L/K) = \langle \sigma \rangle$ $\quad \alpha \in K^*$ $\quad [L:K] = n$

$$\mathcal{A} = \left(\frac{L,\alpha}{K}\right) = L \oplus Li \oplus Li^2 \oplus \cdots \oplus Li^{n-1}$$

$$i^n = \alpha$$

$$i\lambda = \sigma(\lambda) i$$

$$\mathcal{A}_{\overline{K}} \cong ?$$

AF: $\mathcal{A}_{\overline{K}} \cong M_n(\overline{K})$

$L = K(\lambda)$ $\quad \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \ddots \; \lambda \end{pmatrix}$

$$\Lambda = \begin{pmatrix} \lambda & & & 0 \\ & \sigma(\lambda) & & \\ & & \sigma(\lambda) & \\ 0 & & & \ddots \; \sigma^{n-1}(\lambda) \end{pmatrix}$$

$\sigma(\lambda) = f(\lambda)$

$\sigma(\Lambda) = f(\Lambda)$

$$\sigma \Lambda = \begin{pmatrix} \sigma(\lambda) & & & 0 \\ & \sigma^2(\lambda) & & \\ & & \ddots \; \sigma^{n-1}(\lambda) & \\ 0 & & & \sigma^n(\lambda) = \lambda \end{pmatrix}$$

$\tilde{i} \Lambda = \sigma \Lambda \tilde{i}$

$$\Lambda e_1 = \lambda e_1$$

$$\Lambda e_2 = \sigma(\lambda) e_2$$

$$\tau \Lambda e_1 = \sigma(\Lambda) \tau e_1$$

$$\lambda \tau e_1 = \sigma(\Lambda) \tau e_1$$

$$\tau e_1 = \rho_n e_n$$

$$\tau = \begin{pmatrix} 0 & & \rho_1 \\ & \cdot^{\cdot^{\cdot}} & \\ & \rho_{n-1} & \\ \rho_n & & 0 \end{pmatrix}$$

$$\tau \Lambda e_2 = \sigma(\Lambda) \tau e_2$$

$$\sigma(\lambda) \tau e_2 = \sigma(\Lambda) \tau e_2$$

$$\tau e_2 = \rho_{n-1} e_1$$

$$\tau = \begin{pmatrix} 0 & \rho_1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \rho_{n-1} \\ \rho_n & & & 0 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ \alpha & & & 0 \end{pmatrix}$$

$K \subseteq L \qquad V, W$ E.V. L

$V \otimes_K W \qquad V \otimes_L W$

$4nm \qquad 2nm$

$$b : V \times W \longrightarrow V \otimes_L W$$

$$(v,w) \xmapsto{K\text{-LIN}} v \otimes w$$

$$\underset{K\text{-LIN}}{\varphi} : V \otimes_K W \longrightarrow\!\!\!\!\rightarrow V \otimes_L W$$

$V \otimes_L W$ ES NATURALMENTE UN COCIENTE DE $V \otimes_K W$

$l \in L \qquad v \otimes (lw) - (lv) \otimes w$ TIENE QUE ESTAR EN EL NUCLEO

$$\ker \varphi = \langle v \otimes (lw) - (lv) \otimes w \mid l \in L \;\; v \in V \;\; w \in W \rangle$$

$$V \hat{\otimes}_L W \cong \frac{V \otimes_K W}{Z}$$

$$\begin{tikzcd}
V \otimes_K W \arrow[r, "\varphi"] \arrow[d, "\pi"'] & V \otimes_L W \\
V \hat{\otimes}_L W \arrow[ur, dashed, "\exists ! \hat{\varphi}"'] &
\end{tikzcd}$$

POR OTRO LADO $\pi(v\otimes w) = v\hat{\otimes} w$

$$\bar{\pi} : V\times W \longrightarrow V\hat{\otimes}_L W$$

$$(v,w) \longmapsto v\hat{\otimes} w \quad L\text{-LINEAL.}$$

$$(lv,w) \longmapsto (lv)\hat{\otimes} w = l(v\hat{\otimes} w)$$

$$(v,lw) \longmapsto v\hat{\otimes}(lw) = (lv)\hat{\otimes} w = l(v\hat{\otimes} w)$$

$$\therefore \exists \psi$$

$$\hat{\varphi}(v\hat{\otimes} w) = v\otimes w$$

$$\psi(v\otimes w) = v\hat{\otimes} w$$

PROP: SI V ES K-ESPACIO VECTORIAL, W, X SON L-E.V. ENTONCES

$$V\otimes_K (W\otimes_L X) \cong (V\otimes_K W)\otimes_L X$$

□

KCLCF

V e.v $/K$

Prop: $(V_L)_F \cong V_F$

Dem: $(V_L)_F := V_L \otimes_L F = (V \otimes_K L) \otimes_L F$

$$\cong V \otimes_K (L \otimes_L F) \cong V \otimes_K F = V_F.$$

ej: A K-alg

$\mathbb{M}_n(A) \cong \mathbb{M}_n(K) \otimes A$

$\mathbb{M}_n(K) = \langle E_{ij} \rangle$ $\qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & & 0 \end{pmatrix}$

$\mathbb{M}_n(K) \otimes_K A = \langle E_{ij} \otimes a \rangle$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes a + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes b + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes c + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes d$$

Ej: $\mathbb{M}_n(K) \otimes \mathbb{M}_m(K)$

$$\mathbb{M}_n(K) \otimes \mathbb{M}_m(K) \;\cong\; \mathbb{M}_n(\mathbb{M}_m(K)) \cong \mathbb{M}_{nm}(K)$$

$$E_{ij} \otimes E_{lr} \sim E_{m(l-1)+i \;\; m(r-1)+j}$$

$$E_{ij} E_{st} = \delta_{js} E_{it}$$

Ej: $\mathbb{M}_n(\mathbb{C}\times\mathbb{C}) \cong \mathbb{M}_n(\mathbb{C}) \otimes_{\mathbb{C}} (\mathbb{C}\times\mathbb{C})$

$$\cong \left(\mathbb{M}_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}\right) \times \left(\mathbb{M}_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}\right)$$

$$\cong \mathbb{M}_n(\mathbb{C}) \times \mathbb{M}_n(\mathbb{C})$$

Ej: L/K EXT. GALO. SEPARABLE

$L \subseteq F \qquad L \otimes_K F \qquad L \otimes_K L \cong L \times L$

$$L \otimes_K F \cong L \otimes_K (L \otimes_L F) \cong (L \otimes_K L) \otimes_L F$$

$$\cong (L \times L) \otimes_L F$$

$$\cong (L \otimes_L F) \times (L \otimes_L F)$$

$$\cong F \times F.$$

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REPRESENTACIONES

A un K-álgebra, una ~~...~~ K-representación de A es un homomorfismo de K-álgebras

$$\varphi : A \longrightarrow \mathbb{M}_n(K) \quad (\text{de } \dim n)$$

$M = K^n$ es un A-módulo mediante

$$a.v = \varphi(a)(v)$$

$M_1 \leq M$ es submódulo ssi $\varphi(a) M_1 \subseteq M_1$

si $M_1 = \langle v_1, \dots, v_r \rangle$

$$M = \langle v_1, \dots, v_r, v_{r+1}, \dots, v_n \rangle$$

$$\varphi(a) = \left(\begin{array}{c|c} \Phi_1 & \Psi \\ \hline 0 & \Phi_2 \end{array} \right) \quad \forall a \in A$$

$$\varphi(a) = \begin{pmatrix} \varphi_1(a) & \Psi(a) \\ 0 & \varphi_2(a) \end{pmatrix}$$

$a \longmapsto \varphi_1(a)$, $a \longmapsto \varphi_2(a)$ son representaciones

$\varphi_1(a+b) = \varphi_1(a) + \varphi_1(b)$ $\qquad$ $\varphi_1(ab) = \varphi_1(a)\varphi_1(b)$

$\varphi_2(a+b) = \varphi_1(a) + \varphi_2(b)$ $\qquad$ $\varphi_2(ab) = \varphi_2(a)\varphi_2(b)$

φ_1, φ_2 se llaman componentes de la representación.

los módulos correspondientes

a φ_1 y φ_2 son M_1 y M/M_1

REPRESENTACIONES

A un K-ÁLGEBRA, una ~~...~~ K-REPRESENTACIÓN DE A ES UN HOMOMORFISMO DE K-ÁLGEBRAS

$$\varphi : A \longrightarrow \mathbb{M}_n(K) \quad (\text{de dim } n)$$

$M = K^n$ ES UN A-MÓDULO MEDIANTE

$$a.v = \underset{\substack{\cap \\ \mathbb{M}_n(K)}}{\varphi(a)}(v) \qquad \varphi(a)v = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$M_1 \leq M$ ES SUBMÓDULO SSI $\varphi(a) M_1 \subseteq M_1$

SI $M_1 = \langle v_1, \dots, v_r \rangle$

$$M = \langle v_1, \dots, v_r, v_{r+1}, \dots, v_n \rangle$$

$$\varphi(a) = \left(\begin{array}{c|c} \Phi_1 & \Psi \\ \hline 0 & \Phi_2 \end{array}\right) \qquad \forall a \in A$$

$$\varphi(a) = \begin{pmatrix} \varphi_1(a) & \Psi(a) \\ 0 & \varphi_2(a) \end{pmatrix}$$

$a \longmapsto \varphi_1(a)$, $a \longmapsto \varphi_2(a)$ SON REPRESENTACIONES

$\varphi_1(a+b) = \varphi_1(a) + \varphi_1(b)$. $\varphi_1(ab) = \varphi_1(a)\varphi_1(b)$

$\varphi_2(a+b) = \varphi_1(a) + \varphi_2(b)$ $\varphi_2(ab) = \varphi_2(a)\varphi_2(b)$

φ_1, φ_2 se llaman componentes de la representación:

los módulos correspondientes

a φ_1 y φ_2 son M_1 y M/M_1

28/11/2013)

A K-ÁLGEBRA

$$\rho : A \longrightarrow \mathbb{M}_n(K)$$

$M = K^n$ ES UN A-MOD

CON $a \cdot v = \rho(a)(v)$

SI $M_1 \subseteq M$ ES SUBMÓDULO

PODEMOS ESCOGER UNA BASE DE MODO QUE

$$\rho(a) = \begin{pmatrix} \rho_1(a) & * \\ 0 & \rho_2(a) \end{pmatrix}$$

ρ_1 ρ_2

$\rho_1(a)$, $\rho_2(a)$ SON LAS REP CORRESPONDIENTES A M_1 Y M/M_1

SE DICEN COMPONENTES DE ρ.

SI NO HAY SUBMÓDULOS NO TRIVIALES DIREMOS QUE ρ (O M) ES IRREDUCIBLE

SI A ES K-ÁLGEBRA DE DIMENSIÓN n PODEMOS IDENTIFICARLA CON K^n COMO E.V.

$$\rho : A \longrightarrow \mathbb{M}_n(K)$$

$$a \longmapsto m_a \longleftarrow \text{MULTIPLICACIÓN POR } a$$

$m_a : A \longrightarrow A$

$[m_a] : K^n \quad K^n$

$A = \mathbb{C}$ como $\mathbb{R}$-álgebra

$$m_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\{1, i\}$ matriz $= \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$$\rho(a+bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Prop: si M es irreducible, $Am = M$ $\forall m \in M$ $m \neq 0$

Dem: Am es un submódulo de M, y $m = 1 \cdot m \in Am$ luego, si $m \neq 0$ es no trivial.

Prop: si M es irreducible, entonces existe un ideal izquierdo J en A tq $M \cong A/J$ (ideal izq $\equiv$ submódulo)

Dem.

$\varphi : A \longrightarrow M$ $\quad \varphi(a) = am$

homomorfismo de módulos

$$M = \varphi(A) \cong A/\ker\varphi \checkmark$$

ideal izquierdo (submódulo)

PROP: SEAN M_1, M_2 DOS MODULOS INDESCOMPONIBLES, Y SEA

$\varphi : M_1 \longrightarrow M_2$ UN HOMOMORFISMO DE MODULOS.

ENTONCES $\varphi = 0$ Ó φ ES ISOMORFISMO.

DEM: $\ker \varphi \subseteq M_1$ SUBMODULO

SI $\varphi \neq 0$, $\ker \varphi \neq M_1$, LUEGO $\ker \varphi = \{0\}$

SI $\varphi \neq 0$, $\operatorname{Im} \varphi \neq \{0\}$, LUEGO $\operatorname{Im} \varphi = M_2$

Ej: SEA D UN ÁLGEBRA DE DIVISIÓN $\overbrace{}^{\text{SOBRE } K}$ $A = M_n(D)$ (ES K-ÁLGEBRA)

$$\begin{array}{c} \Psi \\ T = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \end{array}$$

$v \in D^n$ $\quad v = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$

$$Tv = \begin{pmatrix} a_{11}d_1 + \cdots + a_{1n}d_n \\ \vdots \\ a_{n1}d_1 + \cdots + a_{nn}d_n \end{pmatrix}$$

$M = D^n$ ES UN A-MODULO.

$\dim_K (D) = s \qquad \dim_K (D^n) = sn$

$$\rho : \mathbb{M}_n(D) \longrightarrow \mathbb{M}_{sn}(K)$$

M es irreducible $\quad m \in M$ cualquiera

$$m = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \neq 0 \qquad d_i \neq 0$$

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{fila } j \qquad m = \sum_{j=1}^{n} d_j e_j$$

$$E_{jl} = \begin{array}{cc} & l \\ \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} & \leftarrow j \end{array} \qquad E_{jl}\, e_k = \begin{cases} e_j & l = k \\ 0 & \text{si no} \end{cases}$$

$$E_{ji}\, m = E_{ji}\left(\sum_{l=1}^{n} d_l e_l \right) = d_i\, e_j$$

$$d^{-1} E_{ji}\, m = e_j$$

$$\sum_{j=1}^{n} a_j e_j = \left(\sum_{j=1}^{n} a_j d^{-1} E_{ji} \right) m$$

$$Am = M \qquad M \text{ irred.}$$

Consideremos ahora A como A-módulo.

Sea A

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (c_1\ c_2\ \cdots\ c_n)$$

$$s\,(c_1|c_2|\cdots|c_n) = (sc_1|sc_2|\cdots|sc_n)$$

$$\therefore\ A \cong \underbrace{M \times M \times \cdots \times M}_{n\text{-veces}} \quad \text{como } A\text{-módulo}$$

$$A = \bigoplus_{i=1}^{n} M_i \qquad M_i \cong M$$

i-ésima columna

Sea M' un A-módulo irreducible $M' \cong A/J$ J-ideal izq.

Supongamos que $M \not\cong M'$

Sea $\varphi : A \twoheadrightarrow A$ la proyección

$\varphi(M_i) = 0$ (pues $M_i \not\cong A/J$)

luego $\varphi(A) = 0$ ($\times$)

M es el único módulo irreducible sobre A.

Si $\rho : A \longrightarrow \mathbb{M}_n(K)$ es una representación

podemos escoger la base de K^n de modo que

$$\rho(a) = \begin{matrix} \rho_1(a) & * & * \\ 0 & \rho_2(a) & \\ & \ddots & * \\ 0 & 0 & \rho_n(a) \end{matrix}$$

$\{0\} \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_t = M$

$\rho_i(a)$ rep que corresponde a M_i / M_{i-1}

siempre puede suceder que los ρ_i son irreducibles

Supongamos ahora que $M \cong M_1 \oplus M_2$

con M_1, M_2 submódulos

$M/M_1 \cong M_2$

$\rho : A \longrightarrow \mathbb{M}_n(K)$ corresp. a M

$\{v_1, \dots, v_r\}$ base de M_1

$\{v_{r+1}, \dots, v_n\}$ base de M_2

$$\rho(a) = \begin{pmatrix} \rho_1(a) & 0 \\ 0 & \rho_2(a) \end{pmatrix}$$

ρ (o M) se dice descomponible

y ρ_1, ρ_2 se dicen constituyentes

UN MÓDULO M SE DICE TOTALMENTE DESCOMPONIBLE SI ES SUMA DIRECTA DE MÓDULOS IRREDUCIBLES ES:

$$\rho(a) = \begin{pmatrix} \rho_1(a) & & 0 \\ & \ddots & \\ 0 & & \rho_n(a) \end{pmatrix}$$ CON ρ_i IRRED

$$M \cong \bigoplus_{i=1}^{n} M_i \qquad M_i \text{ IRRED.}$$

$$A = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a, b \in K \right\} \qquad A \subseteq M_2(K)$$

$$M = K^2 \qquad M_1 = \langle e_1 \rangle$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} = y \begin{pmatrix} 1 \\ 0 \end{pmatrix} \supseteq M_1$$

$$M_2 \subseteq M_1 \;,\; M \neq \{0\} \;,\; M \supseteq M_1$$

$$M_2 = M_1 \text{ ó } M_2 = M :$$

$$\{0\} \subseteq M_1 \subseteq M$$ NO HAY MÁS SUBMÓDULOS

$$\vec{a} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$ LA REP. IRRED ES $\rho(\vec{a}) = a$

$$\rho \not\sim \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$ ESTA REPRESENTACIÓN NO ES TOT. DESCOMPONIBLE

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$$\varphi : A \longrightarrow \mathbb{M}_3(K)$$

$$\varphi\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \left(\begin{array}{cc|c} a & b & 0 \\ 0 & a & 0 \\ \hline 0 & 0 & a \end{array}\right)$$

ρ_1, ρ_2

$$\varphi = \rho_1 \oplus \rho_2$$

DEF: UN IDEAL IZQUIERDO $\underbrace{J}_{\text{de } A}$ SE DICE PROYECTIVO SI EXISTE UN IDEAL IZQUIERDO I DE A CON $A = J \oplus I$

SUPONDREMOS EN LO QUE SIGUE QUE A ES SUMA DIRECTA DE IDEALES IZQUIERDOS IRREDUCIBLES (PROYECTIVOS)

$$A = J_1 \oplus \cdots \oplus J_s$$

$\mathbb{M}_n(D) \cong M_1 \oplus \cdots \oplus M_n$ $\quad M_i \longleftarrow$ i-ESIMA COLUMNA.

PROP: BAJO LAS HIPOTESIS PRECEDENTES, TODO A-MODULO ES TOTALMENTE DESCOMPONIBLE

DEM: SEA M UN A-MODULO Y M_1 UN SUBMODULO IRREDUCIBLE CON M/M_1

$$\begin{pmatrix} \rho_1 & & * \\ & \rho_2 & \\ 0 & & \ddots \; \rho_n \end{pmatrix}$$

TODOS LOS MODULOS SE ASUMEN DE DIM FINITA

PD: $M = M_1 \oplus M_1'$

$A = J_1 \oplus \cdots \oplus J_s \qquad M_1 \cong A/J$

$\varphi : A \twoheadrightarrow A/J$

$\varphi(J_i) = 0$ ó $\varphi(J_i) = A/J$ φ iso

$\therefore M_1 \cong J_i$ algún i $\qquad M/M_1 \cong J_i$ algún i

supongamos $M/M_1 \cong J_1 \qquad J_1' = J_2 \oplus \cdots \oplus J_s$

$$1 = 1_A = \underset{\underset{J_1}{\Uparrow}}{a} + b \in J_1'$$

AFIRMACIÓN: $\exists\ \psi : M/M_1 \longrightarrow M$ homomorfismo con

$\pi : M \longrightarrow M/M_1$

$\pi \circ \psi = id_{M/M_1}$

podemos suponer $M/M_1 = J_1$

$M_1' = \operatorname{im} \psi$

sea $m \in M$

$m = \psi(\pi(m)) + (m - \psi(\pi(m)))$

$\psi(\pi(m)) \in \operatorname{Im}\psi = M_1'$

$\pi(m - \psi(\pi(m))) = \pi(m) - \pi\psi\pi(m) = 0$

$\therefore m - \psi(\pi(m)) \in M_1$

$\cancel{M = M_1 \oplus M_1'}$: $M = M_1 + M_1'$

$m \in M_1 \cap M_1'$

$\pi(m) = 0 \quad (m \in M_1)$

$m = \psi(m_1)$

$0 = \pi(m) = \pi(\psi(m_1)) = m_1$

$m = \psi(0) = 0$

$M = M_1 \oplus M_1' \qquad M_1' \cong M/M_1 \cong J_1$

se termina por inducción //

DF: M modulo $M_1 \subseteq M$ submódulo

$$M/M_1 \cong J_1 \qquad A = J_1 \oplus J_1' \cong J_1 \times J_1' \quad (\text{como módulo})$$

$$M \times J_1' \big/ M_1 \times J_1' \cong M/M_1 \cong J_1$$

$$M \times J_1' \big/ M_1 \times \{0\} \underbrace{\cong M/M_1 \times J' \cong J_1 \times J_1' \cong A}_{\tilde{\pi}}$$

$$\tilde{\varphi} : A \longrightarrow M \times J_1'$$

$$\tilde{\pi} \circ \tilde{\varphi} = id_A$$

$$\tilde{\varphi}(a) = a\,\tilde{\varphi}(1) = a m' \quad \text{con} \quad \tilde{\pi}(m') = 1$$

$$\tilde{\pi} \circ \tilde{\varphi}(a) = \tilde{\pi}(a\,\tilde{\varphi}(1)) = a\,\tilde{\pi}(\tilde{\varphi}(1)) = a\,\tilde{\pi}(m') = a$$

$$\tilde{\varphi} : A = J_1 \times J_1' \longrightarrow M \times J_1'$$

$\tilde{\varphi}(a,b)$ $\quad a \in J_1$

$\quad b \in J_1'$

$$\tilde{\varphi}(a,b) = (\varphi_1(a,b), \varphi_2(a,b))$$

$$\tilde{\varphi}(0,b) = (\varphi_1(a,b), \varphi_2(0,b))$$

$$\tilde{\pi}(\tilde{\Psi}(a,b)) = (a,b)$$

$$\left(\pi(\psi_1(a,b)), b\right)$$

$$\left(\pi(\psi_1(a,b)), \psi_2(a,b)\right)$$

$$\psi_2(a,b) = b$$

$$\pi(\psi_1(a,b)) = a$$

$$\psi : M/M_1 \longrightarrow M$$

$$\psi(a) = \psi_1(a,0),$$

$$\pi(\psi(a)) = a$$

$$M \subseteq A^n$$

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M/M_1 \longrightarrow 0$$

$$\begin{array}{c} M/M_1 \\ \cong \\ A^n \end{array}$$

$$M \cong M_1 \oplus M/M_1$$

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PRODUCTOS

$A = A_1 \times A_2$ PRODUCTO DE ANILLOS

$(1,0), \quad (0,1) \in A$

$\overset{\shortparallel}{p} \qquad p^c = 1 - p$

$p p^c = p(1-p) = p - p^2 = 0$

$p p^c = 0$

$p + p^c = 1$

$p^2 = p \qquad (p^c)^2 = p^c$

$pa = ap \quad \forall a \in A \qquad a = (a_1, a_2)$

M A-MÓDULO

pM SUBMÓDULO

$a \in A$

$a(pM) = p(aM) \subseteq pM$

$p^c M$ SUBMÓDULO

$m \in pM \cap p^c M$

$m = p m_1, \quad m = p^c m_2$

$pm = p(p m_1) = p^2 m_1 = p m_1$

$pm = p(p^c m_2) = 0$

$\therefore m = 0$

$\therefore \quad PM \cap P^cM = \{0\}$

$m \in M \Rightarrow \quad m = 1 \cdot m$

$\qquad\qquad\qquad = (P + P^c)m$

$\qquad\qquad\qquad = Pm + P^c m$

$\therefore \quad PM + P^cM = M$

$\therefore \quad M = PM \oplus P^cM$

$$\begin{array}{ccccc} A & = & PA & \oplus & P^cA \\ & & \wr\!\wr & & \wr\!\wr \\ & & A_1 & & A_2 \end{array}$$

$M = M_1 \times M_2$

M_1 es A_1- módulo

M_2 es A_2- módulo

en particular toda representación de A es de la forma

$$\rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}$$

donde $\rho_1(a_1, a_2) =$

DONDE $\rho_1(a_1, a_2) = \rho'_1(a_1)$

CON ρ'_1 REP DE A_1

Y $\rho_2(a_1, a_2) = \rho'_2(a_2)$

CON ρ'_2 REP. DE A_2

EJ: K CUERPO L, F EXTENSIONES FINITAS DE K

SUPONGAMOS PRIMERO QUE SON EXTENSIONES GALOISIANAS DE K

$$\begin{tikzcd} & FL \arrow[dl, dash] \arrow[dr, dash] & \\ F \arrow[dr, dash] & & L \arrow[dl, dash] \\ & F\cap L = E \arrow[d, dash] & \\ & K & \end{tikzcd}$$

$$F \otimes_K L \cong F \otimes_E \left(E \otimes_K L \right) \cong F \otimes_E \left(E \otimes_K E \right) \otimes_E L$$

$E = K(\alpha)$ α RAIZ DE $f(x) = 0$

f TIENE TODAS SUS RAICES EN E

$$E \cong \frac{K[x]}{(f)}$$