ESC407 Lab 3

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1. **Elliptical orbit of a comet**

   (a) The function `fit_ellipse(x, y)` in `Lab3_Q1.py` fits a dataset of (x, y) coordinates to the general equation of an ellipse,

   $$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Fy + G = 0, \tag{1}$$

   by solving the corresponding eigenvalue problem, described as follows.
   Equation 1 can be written as

   $$f(a, X) = X \cdot a = 0,$$

   where $a = (A, 2B, C, 2D, 2F, G)$ and $X = (x^2, xy, y^2, x, y, 1)$. The parameters of the best fit ellipse are those that minimize the squared distance for each data point i,

   $$\delta(a, x) = \sum_{i=1}^{N} f(a, X)^2 = a^T(X^TX)a. \tag{2}$$

   To exclude the trivial solution, we impose the constraint $\gamma(a) = B^2 - 4AC < 0$ on $\delta(a, x)$. The constrained equation can be written as

   $$\begin{aligned} L(a) &= \delta(a, X) - \lambda(\gamma(a) - \phi) && (3)\\ &= a^T(X^TX)a - \lambda(a^TYa - \phi), && (4) \end{aligned}$$

   where $\phi < 0$ and Y is a (6×6) constraint matrix such that

   $$a^TYa = 4AC - B^2. \tag{5}$$

   To minimize L, we solve for where the derivative of L with respect to a is 0. Differentiating Equation 4 gives us the following eigenvalue problem, where $S = X^TX$,

   $$\frac{1}{\lambda}a = S^{-1}Ya, \tag{6}$$

   which can be solved with `np.linalg.eig`.
   Solving for the entries y_{ij} of Y from Equation 5, we find Y has the form of any skew-symmetric matrix Y' plus the sparse matrix Y_s where the only non-zero entries are $y_{s,22} = -1/4$ and $y_{s,31} = 4$:

   $$Y = Y' + \begin{pmatrix} 0 & 0 & & \cdots & & 0 \\ 0 & -\frac{1}{4} & \ddots & & & \\ 4 & 0 & 0 & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & \cdots & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

In the implementation of Equation 6, X can be written as an $N \times 6$ matrix where each row is the vector $X_i = (x_i^2, x_i y_i, y_i^2, x_i, y_i, 1)$ of the i^{th} data point. For simplicity, we will set all upper diagonal entries of Y' to 1. Thus, we can find the parameters A, B, C, D, F, G from a.

(b) Figure 1 shows the best-fit ellipse from the given dataset.

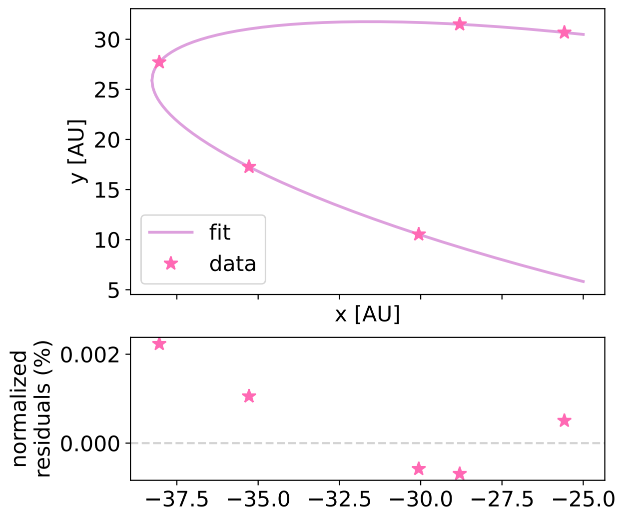

Figure 1: Best fit ellipse and data points, with $A = -0.0353$, $B = -0.0268$, $C = -0.0460$, $D = -0.2625$, $F = 0.1640$, and $G = 0.7813$.

2. **How many computational physicists does it take to screw in a lightbulb?**

(a) `get_etas(T, l1, l2, lN)` in `Lab3_Q2.py` calculates the efficiency of a perfect blackbody (lightbulb) at temperature T, for radiation between $\lambda_1 =$`l1` and $\lambda_2 =$`l2`. The energy radiated between λ_1 and λ_2,

$$E(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} I(\lambda) d\lambda, \tag{8}$$

is calculated using Gaussian Quadrature with `lN` points. The total energy emitted by the lightbulb, $E(0, \infty)$, can be derived analytically when $\lambda_1 = 0$ and $\lambda_2 = \infty$ as

$$E(0, \infty) = \frac{4Ak_b^4 \pi^5 T^4}{15c^2 h^3}. \tag{9}$$

The area A is a constant factor for all E, which drops out of the calculation of η. For simplicity, we set $A = 1$ m^2 (and omit it from the code). To determine an appropriate `lN`, we compare $E(0, \infty)$ obtained analytically and using Gaussian Quadrature (after applying a change of variables $\lambda = \frac{z}{1-z}$ to the improper integral).

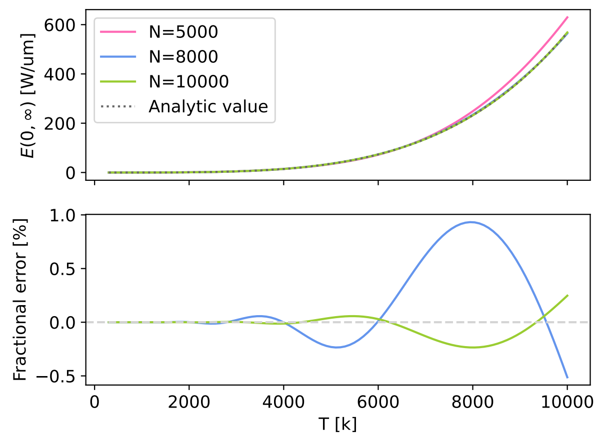

Figure 2: $E(0, \infty)$ obtained analytically and using Gaussian Quadrature for $A = 1$ m^2.

(b) Using `ln`=10,000, we plot the efficiency η from 300 K to 10,000K in Figure 3 a).

(c) Using the golden ratio method to find the temperature of maximum efficiency, we find $T_{max} = 6913K$ and $\eta_{max} = 0.49$. Albeit slow, golden ratio suffices in this instance, since the majority of computational power in evaluating η arises from finding the weights and positions for Gaussian Quadrature, which only needs to be computed once across all iterations (since the integration bounds are constant)[1].

(d) For a lightbulb emitting in the infrared, using the same calculation procedure as in a) and c) (with `l1`=780 nm and `l2`=2250 nm), we find that the temperature of maximum efficiency is now $T_{max} = 2925K$ with $\eta_{max} = 0.67$. The IR efficiency from 300 to 10000 K is shown in Figure 3 b). The temperature of maximum efficiency is impractical for both wavelength ranges, requiring temperatures in the 1000°Cs. However, the effective temperature of the sun, at 5772 K, is just below the maximum efficiency for emission in the visible wavelength range [2].

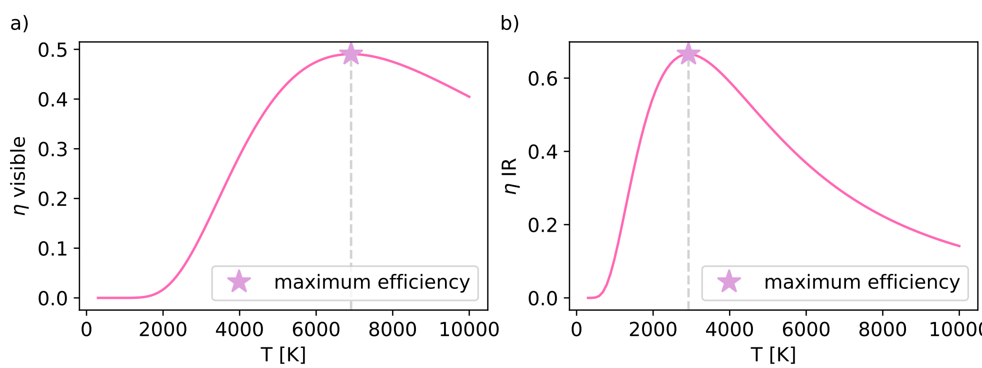

Figure 3: η from 300 K to 10,000K for a) visible (380-780 nm) and b) IR (780-2250 nm) radiation.

[1] To speed up the calculation, we could also take the first derivative of η and search for its roots using Newton's method (Binary search is computationally equivalent to Golden Ratio, and it would be hard to isolate for T in $\partial\eta(T)_T$ with which to use the Relaxation method). However, we would then also need to take the second derivative (which has a complicated expression and increases roundoff error due to added operations).

[2] David Williams. *Sun Fact Sheet*. NASA, Nov. 1973. URL: `https://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html`