ESC407 Lab 3

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1. **Elliptical orbit of a comet**

   (a) The function `fit_ellipse(x, y)` in `Lab3_Q1.py` fits a dataset of (x, y) coordinates to the general equation of an ellipse,

   $$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Fy + G = 0, \tag{1}$$

   by solving the corresponding eigenvalue problem, described as follows.
   Equation 1 can be written as

   $$f(a, X) = X \cdot a = 0,$$

   where $a = (A, 2B, C, 2D, 2F, G)$ and $X = (x^2, xy, y^2, x, y, 1)$. The parameters of the best fit ellipse are those that minimize the squared distance for each data point i,

   $$\delta(a, x) = \sum_{i=1}^{N} f(a, X)^2 = a^T(X^T X)a. \tag{2}$$

   To exclude the trivial solution, we impose the constraint $\gamma(a) = B^2 - 4AC < 0$ on $\delta(a, x)$. The constrained equation can be written as

   $$\begin{align} L(a) &= \delta(a, X) - \lambda(\gamma(a) - \phi) \tag{3} \\ &= a^T(X^T X)a - \lambda(a^T Y a - \phi), \tag{4} \end{align}$$

   where $\phi < 0$ and Y is a (6×6) constraint matrix such that

   $$a^T Y a = 4AC - B^2. \tag{5}$$

   To minimize L, we solve for where the derivative of L with respect to a is 0. Differentiating Equation 4 gives us the following eigenvalue problem, where $S = X^T X$,

   $$\frac{1}{\lambda} a = S^{-1} Y a, \tag{6}$$

   which can be solved with `np.linalg.eig`.
   Solving for the entries y_{ij} of Y from Equation 5, we find Y has the form of any skew-symmetric matrix Y', plus the sparse matrix Y_s, where the only non-zero entries are $y_{s,22} = 1/4$ and $y_{s,31} = -4$:

   $$Y = Y' + \begin{pmatrix} 0 & 0 & & \cdots & & 0 \\ 0 & \frac{1}{4} & \ddots & & & \\ -4 & 0 & 0 & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & \cdots & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

In the implementation of Equation 6, X can be written as an $N \times 6$ matrix where each row is the vector $X_i = (x_i^2, x_i y_i, y_i^2, x_i, y_i, 1)$ of the i^{th} data point. For simplicity and to decrease roundoff error, we will set Y' to be an empty matrix. We can find the parameters A, B, C, D, F, G from a, after using `np.linalg.eig` and selecting a to be the eigenvector with the largest corresponding eigenvalue.

(b) Figure 1 shows the best-fit ellipse from the given dataset. The ellipse appears to fit the data well, with small residuals; however, given that there are 5 independent fit parameters (since G serves as a constant factor) and only 5 data points, the ellipse is overfit to the data. The astronomer should take (at least 1) more datapoints to ensure a robust fit.

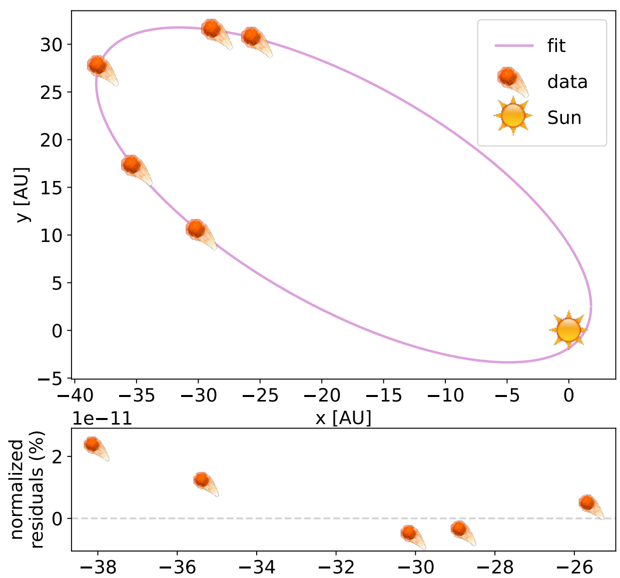

Figure 1: Best fit ellipse and data points, with $A = -0.0357$, $B = -0.0271$, $C = -0.0465$, $D = -0.2625$, $F = 0.1659$, and $G = 0.7752$.

2. **How many computational physicists does it take to screw in a lightbulb?**

(a) `get_etas(T, l1, l2, lN)` in `Lab3_Q2.py` calculates the efficiency of a perfect blackbody (lightbulb) at temperature `T`, for radiation between λ_1 =`l1` and λ_2 =`l2` using Gaussian Quadrature with `lN` points. The efficiency is defined as

$$\eta = \frac{E(\lambda_1, \lambda_2)}{E(0, \infty)},$$

where

$$E(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} \frac{2\pi A h c^2}{\lambda^5 (e^{hc/\lambda k_b T} - 1)} d\lambda \tag{8}$$

is the energy radiated between λ_1 and λ_2. The total energy emitted by the lightbulb, $E(0, \infty)$, can be evaluated analytically to be

$$E(0, \infty) = \frac{4 A k_b^4 \pi^5 T^4}{15 c^2 h^3}. \tag{9}$$

The analytical expression for $E(0,\infty)$ will be used to calculate η, since the undefined integrand at $\lambda = 0$ makes accurate numerical calculations impractical (Figure 2 shows an accurate calculation would require more than 5000 points). The area A is a constant factor for all E, which drops out of the calculation of η and is thus omitted from the code.

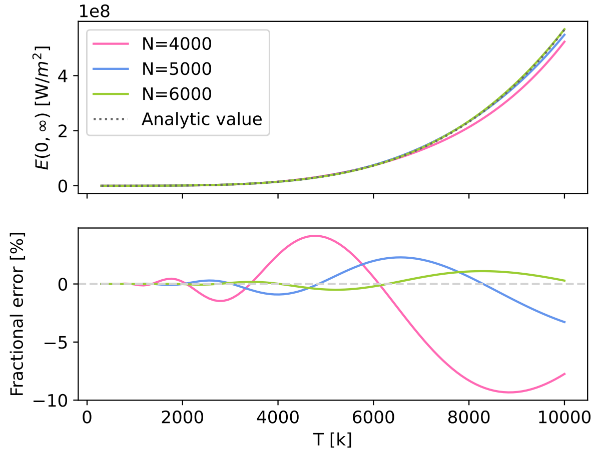

Figure 2: $E(0,\infty)$ obtained analytically and using Gaussian Quadrature with $A = 1\ \mathrm{m}^2$.

(b) Using `ln`=70 (roughly chosen to minimize error estimates in Figure 3 c)), we plot the efficiency η from 300 K to 10,000K in Figure 3. However, we would then also need to take the second derivative (which has a complicated expression and increases roundoff error due to added operations).

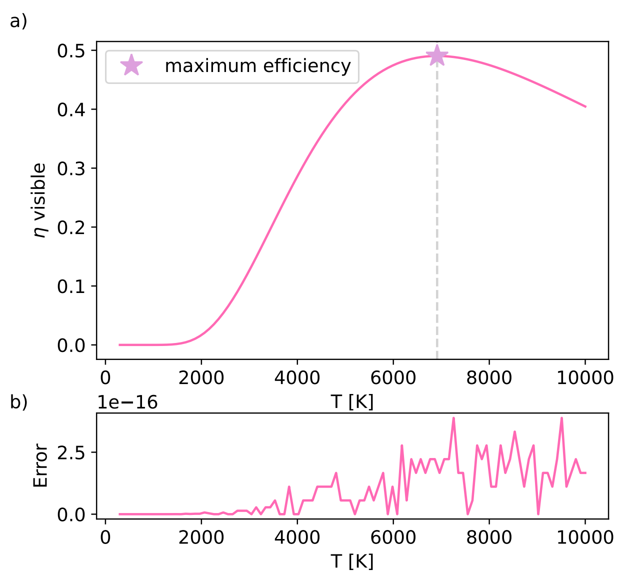

Figure 3: b) η from 300 K to 10,000K for visible (380-780 nm) radiation and c) error $\epsilon = (I_{2N} - I_N)/I_N$

(c) Using the golden ratio method to find the temperature of maximum efficiency, we find $T_{max} = 6913K$ and $\eta_{max} = 0.49$. Albeit slow, golden ratio suffices in this instance, since the majority of computational power in evaluating η arises from finding the weights and positions for Gaussian Quadrature, which only needs to be computed once across all iterations (since the integration bounds are constant). To speed up the calculation, we could also take the first derivative of η and search for its roots using Newton's method (Binary search is computationally equivalent to Golden Ratio, and it would be hard to isolate for T in $\partial\eta(T)_T$ with which to use the Relaxation method).

(d) For a lightbulb emitting in the infrared, using the same calculation procedure as in a) and c) (with `l1`=780 nm and `l2`=2250 nm), we find that the temperature of maximum efficiency is now $T_{max} = 2925K$ with $\eta_{max} = 0.67$. The IR efficiency from 300 to 10000 K is shown in Figure 4.

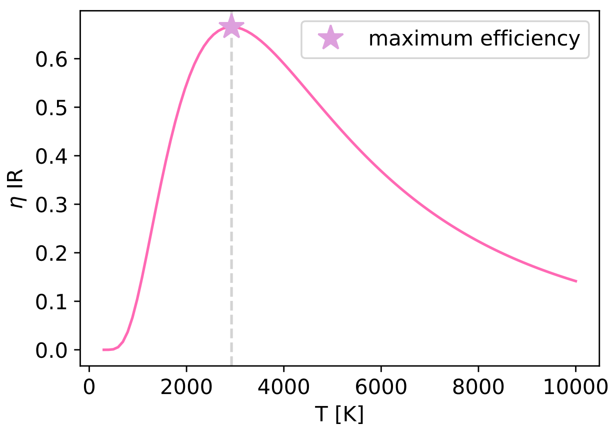

Figure 4: η from 300 K to 10,000K for IR (780-2250 nm) radiation.

The melting point of Tungsten is around 3700 K[1], which is above the optimal temperature for IR radiation, but below that of visible light. It is possible to operate a light bulb at maximum efficiency in the IR region, but not for visible wavelengths.

[1] *Tungsten.* Britannica, Sept. 2023. URL: `https://www.britannica.com/science/tungsten-chemical-element`