PHY407 Formal Lab Report 3

Fall 2023

Background

Ellipses: The standard equation for an ellipse is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{1}$$

where a and b are the semi-major and semi-minor axes, respectively, and (h, k) is the centre of the ellipse.

The general equation of an ellipse is given by

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Fy + G = 0 \tag{2}$$

where A, B, C, D, F, and G are constants.

Fitting with constraints: Given enough coordinate values for (x, y), one can solve Equation 2 for the constants of the best-fit ellipse, using linear algebra. This equation can be written as

$$f(a, X) = X \cdot a = 0,$$

for the parameters $a = (A, 2B, C, 2D, 2F, G)$ and $X = (x^2, xy, y^2, x, y, 1)$. To fit this function we could minimize the squared distance

$$\delta(a, x) = \sum_{i=1}^{N} f(a, X)^2$$

for each measurement point i, using a package such as `numpy.linalg` . However, the result would be the trivial solution $A = B = C = D = F = G = 0$, which is not what we want.

To make progress, we need to impose a constraint on the solution to require an ellipse, which can be written as $\gamma(a) = B^2 - 4AC < 0$, and instead of minimizing the squared distance we minimize the constrained function

$$L(a) = \delta(a, X) - \lambda(\gamma(a) - \phi), \tag{3}$$

where $\phi < 0$.

Re–writing equation 3 in terms of a and X we get

$$\begin{aligned} L(a) &= \delta(a, X) - \lambda(\gamma(a) - \phi), \\ &= (Xa)^T(Xa) - \lambda(a^T Y a - \phi), \\ &= a^T(X^T X)a - \lambda(a^T Y a - \phi), \end{aligned} \tag{4}$$

where Y is the constraint matrix (i.e. $a^T Y a = 4AC - B^2$).

Minimizing function L means finding where the derivative with respect to a is zero:

$$\begin{aligned} \partial L(a)_a &= (X^T X)a - \lambda\, Y a = 0 \\ (X^T X)a &= \lambda Y a. \end{aligned} \tag{5}$$

Replacing $S = X^T X$ for convenience:

$$Sa = \lambda Y a. \tag{6}$$

The last equation should be recognizable as an eigenvalue equation. Re–writing as

$$\frac{1}{\lambda} a = S^{-1} Y a \tag{7}$$

it can be solved with `numpy.eig`, using the eigenvector with the largest eigenvalue as the likely solution to the constrained problem:

```
S_inv=...
Y=...
E, V = np.eig(np.dot(S_inv,Y)) # E contains the eigenvalues, V the eigenvectors.
```

The eigenvector decomposition gives you the parameters a which can then be converted to the ellipse parameters A, B, C, D, F, G

Blackbody radiation: If I was your prof for PHY254 you'll remember this topic...

An incandescent light bulb usually contains a metal filament that is thin enough so that an electrical current will heat it sufficiently to radiate thermally and visibly. Almost all of the power consumed by the bulb is radiated as blackbody radiation, with only a fraction of the radiation in the visible part of the spectrum. If the light bulb is assumed (somewhat inaccurately) to be a perfect blackbody, then the power radiated per unit wavelength λ obeys

$$I(\lambda) = \frac{2\pi\, Ahc^2}{\lambda^5(e^{hc/\lambda k_b T} - 1)}, \tag{8}$$

where A is the surface area of the filament, h is Planck's constant, c is the speed of light, and k_B is Boltzmann's constant. The total energy radiated between two wavelengths λ_1 and λ_2 is then

$$E(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} I(\lambda) d\lambda, \tag{9}$$

and a lightbulb efficiency can be defined as the ratio of desirable light to the total energy emitted by the lightbulb,

$$\eta = \frac{E(\lambda_1, \lambda_2)}{E(0, \infty)} \tag{10}$$

X	Y
-38.04	27.71
-35.28	17.27
-25.58	30.68
-28.80	31.50
-30.06	10.53

Table 1: An astronomically suspicious dataset of observations of a comet, in astronomical units (1AU = 1.496×10^{11} metres).

Tungsten current–temperature function: For Tungsten filaments the resistance of the wire varies with temperature as

$$R = R_0 \left(1 + \alpha(T - T_0)\right), \tag{11}$$

where $R_0 = 1.1\Omega$ and $\alpha = 4.5 \times 10^{-3} K^{-1}$. This equation can be inverted to give the temperature (in Kelvins) at a given input voltage and current:

$$T = 300 + \frac{\frac{V/I}{R_0} - 1}{\alpha}, \tag{12}$$

Questions

1. **Elliptical orbit of a comet**

   An astronomer wants to determine the orbit of a comet about the Sun. To do so, the astronomer sets up a Cartesian coordinate system in the plane of the orbit, setting the Sun's location as the origin, and then makes observations of the comet's position in this system five times. The measured coordinates are given in Table 1.

   (a) Write the code to fit the dataset for the constants of the best-fit ellipse (see the relevant Background sections for some details of the method.) **Hand in your code. In the report, make sure to document your method for fitting the ellipse, referring to equations as needed.**

   (b) Plot the ellipse you have found, along with the measured points from the dataset. Does the ellipse appear to be a good fit to the data? **Hand in the plot, and brief written answer.**

2. **How many computational physicists does it take to screw in a lightbulb?** To answer this old riddle, we first need to do a lot of calculations to see what's going on with the lightbulb.

   (a) Write a Python function to take the temperature T and calculate the efficiency η of a lightbulb, for radiation between 380nm and 780nm. **Submit your code.**

(b) Use the function to make a graph of the efficiency from 300K to 10,000K (note, you should see an extremum in the efficiency curve.) **Submit your plot.**

(c) Calculate the temperature of maximum efficiency of the light bulb to within 1K, using the search method of your choice. For this level of accuracy you will need accurate physical constants from 'scipy', and accurate integration and search methods. **Submit your code, and temperature of maximum efficiency. In the report, make sure to specify which search method you used, and why.**

(d) How would your answer change if you wanted to make an light bulb that emits in the infrared (780nm-2250nm) instead of the visible? Is it possible to run the Tungsten bulb at maximum efficiency for either wavelength region? **Submit the new temperature of maximum efficiency, and written answers (including explanations of how you got them.)**