ESC407 Lab 3

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1. **Elliptical orbit of a comet**

   (a) The function `fit_ellipse(x, y)` in `Lab3_Q1.py` fits a dataset of (x, y) coordinates to the general equation of an ellipse,

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Fy + G = 0, \tag{1}$$

by solving the corresponding eigenvalue problem, described as follows.
Equation 1 can be written as

$$f(a, X) = X \cdot a = 0,$$

where $a = (A, 2B, C, 2D, 2F, G)$ and $X = (x^2, xy, y^2, x, y, 1)$. The parameters of the best fit ellipse are those that minimize the squared distance for each data point i,

$$\delta(a, x) = \sum_{i=1}^{N} f(a, X)^2 = a^T (X^T X) a. \tag{2}$$

To exclude the trivial solution, we impose the constraint $\gamma(a) = B^2 - 4AC < 0$ on $\delta(a, x)$. The constrained equation can be written as

$$\begin{align} L(a) &= \delta(a, X) - \lambda(\gamma(a) - \phi) \tag{3} \\ &= a^T (X^T X) a - \lambda(a^T Y a - \phi), \tag{4} \end{align}$$

where $\phi < 0$ and Y is a (6×6) constraint matrix such that

$$a^T Y a = 4AC - B^2. \tag{5}$$

To minimize L, we solve for where the derivative of L with respect to a is 0. Differentiating Equation 4 gives us the following eigenvalue problem, where $S = X^T X$,

$$\frac{1}{\lambda} a = S^{-1} Y a, \tag{6}$$

which can be solved with `np.linalg.eig`.
Solving for the entries y_{ij} of Y from Equation 5, we find Y has the form of any skew-symmetric matrix Y', plus the sparse matrix Y_s, where the only non-zero entries are $y_{s,22} = -1/4$ and $y_{s,31} = 4$:

$$Y = Y' + \begin{pmatrix} 0 & 0 & & \cdots & & 0 \\ 0 & -\frac{1}{4} & \ddots & & & \\ 4 & 0 & 0 & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & \cdots & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

In the implementation of Equation 6, X can be written as an $N \times 6$ matrix where each row is the vector $X_i = (x_i^2, x_i y_i, y_i^2, x_i, y_i, 1)$ of the i^{th} data point. For simplicity and to decrease roundoff error, we will set Y' to be an empty matrix. Thus, we can find the parameters A, B, C, D, F, G from a.

(b) Figure 1 shows the best-fit ellipse from the given dataset. The ellipse appears to fit the data well, with small residuals; however, the data covers a limited region of the orbit.

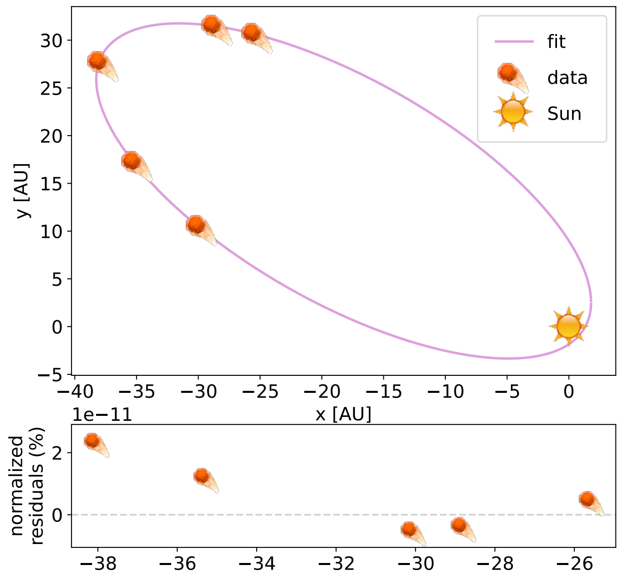

Figure 1: Best fit ellipse and data points, with $A = -0.0357$, $B = -0.0271$, $C = -0.0465$, $D = -0.2625$, $F = 0.1659$, and $G = 0.7752$.

2. **How many computational physicists does it take to screw in a lightbulb?**

(a) `get_etas(T, l1, l2, lN)` in `Lab3_Q2.py` calculates the efficiency of a perfect blackbody (lightbulb) at temperature `T`, for radiation between λ_1 =`l1` and λ_2 =`l2` using Gaussian Quadrature with `lN` points. The efficiency is defined as

$$\eta = \frac{E(\lambda_1, \lambda_2)}{E(0, \infty)},$$

where

$$E(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} \frac{2\pi A h c^2}{\lambda^5 (e^{hc/\lambda k_b T} - 1)} d\lambda \tag{8}$$

is the energy radiated between λ_1 and λ_2. The total energy emitted by the lightbulb, $E(0, \infty)$, can be evaluated analytically to be

$$E(0, \infty) = \frac{4 A k_b^4 \pi^5 T^4}{15 c^2 h^3}. \tag{9}$$

The analytical expression for $E(0, \infty)$ will be used to calculate η, since the undefined integrand at $\lambda = 0$ makes accurate numerical calculations impractical (Figure 2 shows an accurate calculation would require more than 5000 points). The area A is a constant factor for all E, which drops out of the calculation of η and is thus omitted from the code.

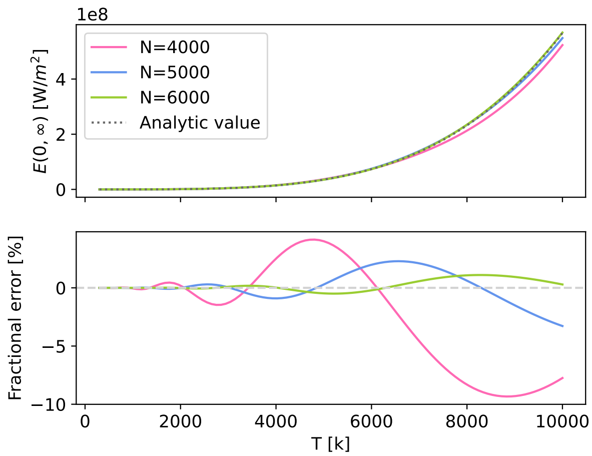

Figure 2: $E(0,\infty)$ obtained analytically and using Gaussian Quadrature with $A = 1$ m^2.

(b) Using `ln`=70 (roughly chosen to minimize error estimates in Figure 3 c)), we plot the efficiency η from 300 K to 10,000K in Figure 3. However, we would then also need to take the second derivative (which has a complicated expression and increases roundoff error due to added operations).

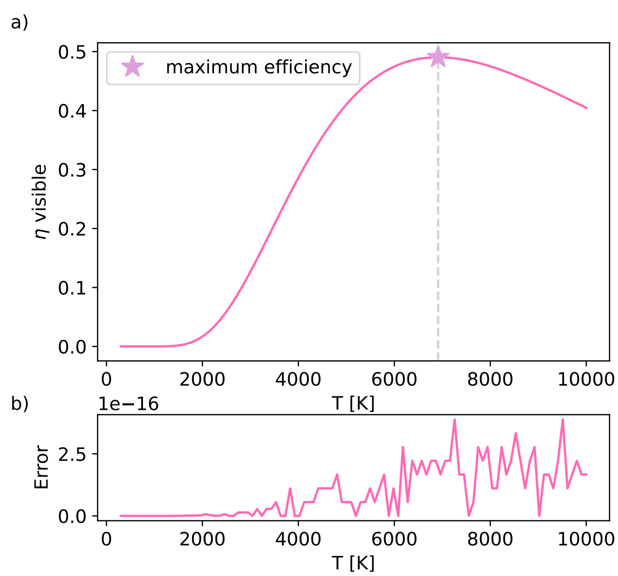

Figure 3: b) η from 300 K to 10,000K for visible (380-780 nm) radiation and c) error $\epsilon = (I_{2N} - I_N)/I_N$

(c) Using the golden ratio method to find the temperature of maximum efficiency, we find $T_{max} = 6913K$ and $\eta_{max} = 0.49$. Albeit slow, golden ratio suffices in this instance, since the majority of computational power in evaluating η arises from finding the weights and positions for Gaussian Quadrature, which only needs to be computed once across all iterations (since the integration bounds are constant)[1]

(d) For a lightbulb emitting in the infrared, using the same calculation procedure as in a) and c) (with `l1`=780 nm and `l2`=2250 nm), we find that the temperature of maximum efficiency is now $T_{max} = 2925K$ with $\eta_{max} = 0.67$. The IR efficiency from 300 to 10000 K is shown in Figure 4.

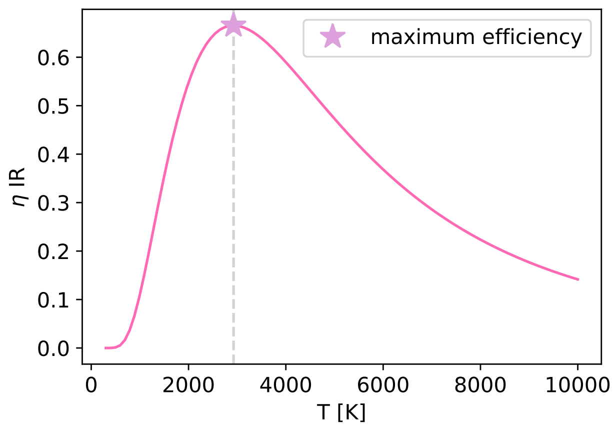

Figure 4: η from 300 K to 10,000K for IR (780-2250 nm) radiation.

The melting point of Tungsten is around 3700 K[2], which is above the optimal temperature for IR radiation, but below that of visible light. It is possible to operate a light bulb at maximum efficiency in the IR region, but not for visible wavelengths.

[1] To speed up the calculation, we could also take the first derivative of η and search for its roots using Newton's method (Binary search is computationally equivalent to Golden Ratio, and it would be hard to isolate for T in $\partial\eta(T)_T$ with which to use the Relaxation method).

[2] *Tungsten.* Britannica, Sept. 2023. URL: `https://www.britannica.com/science/tungsten-chemical-element`