Yield Fluctuations: A study of Fiscal variables on its effect on Treasury Yields

Marvin Henry

Introduction:

In understanding the relationship between agents within the economy, emphasis must be placed in our government institutions and legislative bodies, especially regarding their spending habits. Many aspects of the present day topic of public finance were due to the evolution of the role of government post WW2 and leading to different perspectives on government spending (Buchanan, 1975). Considering that US fiscal debt has risen to a point of 121% of gross domestic product (Gaspar et al, 2025), and US debt was downgraded to double AA by three prominent rating agencies, one can infer that there is an issue (Bauer, 2025). Expectations on the role of government could be summarized as meeting the public wants and allocating resources to maximize their satisfaction (Buchanan, 1960). The methods for funding the government would include raising funding through taxes or deficit financing the government budget. To determine that decision, considerations must be made regarding the impact policy could have on consumer habits, firm profit maximization and fiscal debt accumulation (Barro, 1979; Barro, 1990; Zhang et al, 2008). To grasp expectations on fiscal stability, one can consider the fluctuations in the pricing and supply of sovereign debt. These instruments allow the government to facilitate deficit financing to meet its obligations, and can be used by the Federal Reserve to implement monetary policy through other means.

Methods that examine the pricing mechanics of government securities make use of vector-autoregressions and principal components analysis to determine the dynamics of the yield curve (Evans and Marshall, 2006; Ang and Piazzesi, 2003; Adrian et al, 2008). Establishing a vector of endogenous variables and a developing system depicting how the variables affect each other, the pricing dynamics can be examined. Furthermore, given the restrictions, the VAR model can be configured to respect market structures, knowledge about the institution or both (Killian et al, 2017). Then in consideration of the macroeconomic system, determining the factors that would impact the decisions to purchase or sell sovereign debt. On the question of portfolio selection, considerations would go into the expected value and variance of the product of interest (Markowitz, 1952). Then when the investor maximizes utility, the related attributes would have to be reviewed to determine what the true state or price could be(Fishburn, 1968). Therefore, when one determines the expected values and variances of the attributes that affect this product, then one can increase the likelihood of maximizing their utility.

To execute this analysis, this study makes use of macroeconomic variables to simulate an economy based on the Ramsey-Cass-Koopman model. In depicting a relationship between the level of capital and consumption, one can determine the dynamics of general behaviors in the economy. Simplifying assumptions allow for comparisons between agents within the system and examining the theoretical relationships between agents. With vector-autoregressions, using zero-restrictions to describe the interdependencies between variables allows one to reduce complexity. Additionally, defining these relationships depicts the exogeneity of the variables and provides better understanding on how the SVAR identification would look. Afterwards, the SVAR is constructed, utilizing an AB model to depict the contemporaneous innovations as functions of structural shocks. That determines the later impulse-response functions that help forecast relationships between tax revenue, bond pricing and supply.

Literature Review:

Consider a situation where there exists two actors, and both are attempting a transaction for their own benefit. Assuming that the participating actors behave rationally, can make unbiased comparisons between their desires, have equal bargaining skills and are aware of the other's preferences, then a solution can be determined (Nash, 1950). This bargaining situation alludes to a solution that induces the highest level of satisfaction and can be pointed out in the hypothetical perfect market. In relation to behaviors in a market, participating actors contribute to either forces of supply or demand, where services and goods meet consumption. Within financial markets, suppliers of financial products provide solutions of different risk complexity and structure that would meet the characteristics of that 'perfect product' sought out by demand. While the definition of a perfect product varies between investors or, if we are talking in-terms of the final goods market, between consumers, there exists characteristics that describe that product. That product would be the solution as described by John Nash, however there are preceding levels of uncertainty on the product's true characteristics or its true state.

The true state of that product exists in an event or a set of possible states, one can then find an associated probability distribution on the likelihood of a particular state occurring (Savage, 1972). From Savage's theory one can consider the states that have the highest likelihood of occurring within a set of gambles an individual can undertake (1972). Given that the true state is not known, one would order their preference of acts and their associated consequences given their prediction of the true state of the product (Savage, 1972). Additionally Savage's rationality assumptions contain an Archimedean rule postulating that an infinite number of states can exist, however this assumption can be relaxed (Gul, 1991). Insinuating that the number of acts and consequences are possibly finite and would only be indicative based on the information available on the product. That allows for a rational expectation(s) to be established which some would argue is not adequate enough to explain the decision-making process (Simon, 1966). However, consider Fisburn's explanation on multidimensional utility where the utility of attributes related to a good. Given the state of its attributes, the total satisfaction a decision related to that good, or its alternatives, can bring to an individual may vary (Fishburn, 1968). Then one can simulate a decomposition of factors that affect future satisfaction and determine an optimal solution that brings the two parties together and can lead to a satisfactory utility-maximizing agreement.

With advances made in utility theory, the decision-making process denoted as 'rational' can be instrumental in explaining the decisions of economic agents. As pointed out, attributes can influence decisions and, so therefore, expectations on these factors can affect the perceived utility the decision has. Muth's paper on rational expectations and price movements indicate that market expectations on future outcomes are conditional on a public set of information (1961). That public set of information would influence the expectations of supply and demand forces, ergo, prices movements would incorporate that information. When prices already account for information or expectations, that is synonymous to saying that a particular event or details have been "priced in". Therefore, only surprise information would affect the changes in prices and could, subsequently, affect the market as concluded from investigations on unanticipated monetary policy (Bernanke and Kuttner, 2005). Nerlove also postulates that expectations on future prices are specifically adaptive(1958). Those expectations are conditional on information that indicates a new standard quo, and is modeled through lags of those prices (Nerlove, 1958). One can see that information has become an important resource that influences the expectations on the perceived true state of some object. An individual can use the attributes of a decision, to order their decisions and that can be conditional on some sort of time preference (Fishburn, 1968). Therefore, decisions can be ordered sequentially, meaning that the preferences over time can shift and change in position given the attribute's total utility within that period amongst the alternatives. Supplemented with axioms established on preference ordering, decision modeling can be used to achieve the aim of understanding shifts in the perception of the economy (von Neumann et al, 1944).

Now the solution to be assessed is sovereign debt instruments, specifically the US treasury, which is used to meet obligations towards investors, government accounts and budgetary responsibilities. As a financial product it is notably the safest and most liquid asset available on the market as seen by Krishnamurthy and Vissing-Jorgenson's work in 2012. Results showed that changes in the supply of treasuries had a large effect on the yield spreads, and this conclusion was also reached through simulations derived in Roley's 1982 paper on federal debt management techniques. Both insights complements Gomes and Michaelides' 2008 findings which asserted that expanding the stock of riskless assets affected the risk premia and had effects on equity valuations. Furthermore, Reis denotes public debt as a 'safe haven' due to the lack of risk factors and magnitude of the risk factors as compared to private debt, which increases the attractiveness of sovereign debt (Reis, 2022). Then changes in the perceived return for both bonds and equities are due to uncertainty around the observable intrinsic risk factors (Baker et al, 2007; Connolly et al, 2005). What should follow is a consideration of what causes that uncertainty in the first place. Reflecting on Muth's outlook on rational expectations, public information that can be priced into the yield includes variables that impact tax-revenues and monetary policy decisions.

Within term structure literature macroeconomic variables have been used to predict the variation in treasury yields as seen in Evans and Marshall's 2006 paper. Discovering economic determinants of the nominal yield curve would then allow for analysis of the different economic arenas as they impact treasury yields. Found through a simple data vector of industrial production, personal consumption expenditures and sensitive materials price index account for at least 75% of the future variance in 5 year yields (Evans et al, 2006). Their study considered shocks from government deficits to proxy for the effect of fiscal shocks, and determined that these shocks represented a negligible amount of the variation in nominal yield movements. However, it was not made explicit how shifts in government revenues would potentially affect the valuation of treasuries. The tax stream would directly affect expected fiscal deficits, this isolates the focus to government revenue shocks and its effects on yields. Ang and Piazzesi(2003) extracted principal components to proxy factors of unemployment and price inflation and these determined 85% of forecasted variance in the yields for bills and notes. Progressing through this study requires understanding of macroeconomic theory that connects conceptualized behaviors between agents. So, this analysis will consider methods that are the underpinnings of dynamic stochastic general equilibrium models. The structure of DSGE models allows for the analysis of systematic change in policy on different parts of the economy (Christiano et al, 2018). By modeling different parts of the economy as agents, macroeconomists are able to capture the endogenous and exogenous forces that contribute to a general equilibrium condition. Finally, through defining those agents using functions steeped in microeconomic understanding on their behavior, combined with simplifying assumptions, identification is more feasible for the researcher.

To conduct a proper DSGE model a simulation of the interdependencies is used to provide a base case for comparisons between theory and real data. It is irrelevant whether one uses a Neoclassical Growth, Real Business Cycle, or another class of New Keynesian model. This study will implement a vector-autoregression model with a restriction matrix and a structural vector-autoregression that relies on economic theory established in the next section. The restriction matrix can be utilized when time series samples lack the size to ensure accurate parameter estimates(Lütkepohl, 2005). Furthermore, it allows for a stricter approach to exogeneity within the multivariate modelling strategy, allowing one to model relationships with respect to theory and literature. Later in the paper the identification of the restriction matrix will be determined through economic theory on growth within an economy that relies on capital accumulation. Alongside ideas about the steady state, a dynamic model that can depict its potential and determine the impact each variable has on each other proves effective in diagnosing a problem within complex systems(Bequette, 1998). Resources that provide the capabilities to analyze relationships within DSGE's can be found in the programming language Julia within the MacroModelling package (Kockerols, 2023).

Theory:

The growth model considers a closed economy with goods and factor markets that are perfectly competitive. The behaviors for the household and firms originate from the intuition established in the Ramsey-Cass-Koopman model, just the Ramsey model, or the Neoclassical Growth Model where accumulation of capital and consumption patterns are defined in laws of motion (Romer, 2019, pp 55-66). Furthermore, the Ramsey model makes use of the neoclassical production function to define the generation process of output in relation to factors of production. These are structured on ideas of the prolific Solow's Growth Model that introduces the relationship between output and capital with the intensive production function (Solow, 1956). Capital accumulation defines the behavior of the dynamic system, establishing capital as a state variable that illustrates the general behavior of the economic system. Utilizing the growth model helps isolate key variables that will be utilized in the model later in the study and defines the interdependent relationships that exist within the macroeconomic scenario.

Ramsey Model:

There is a set of homogenous households that attempt to maximize its lifetime utility function through optimal levels of consumption across multiple time periods as the form,

$$\boldsymbol{U} = \int_{t=0}^{\infty} \boldsymbol{e}^{-\rho t} \boldsymbol{u}(\boldsymbol{C}(\boldsymbol{t})) \frac{\boldsymbol{L}(\boldsymbol{t})}{\boldsymbol{H}} \boldsymbol{dt}, \tag{1}$$

illustrated by Romer (2019). U is the lifetime utility of the infinitely-lived household, integrating the function $u(C_t)$. The ρ represents the time preference to consume at time t. There exists a level of risk aversion θ that determines the household's desire to shift consumption levels between periods. Intuitively, that can take into account the expected value of responsibilities and obligations that the household should pay for within that duration,

$$\boldsymbol{u}(\boldsymbol{C}(\boldsymbol{t})) = \frac{\boldsymbol{C}(\boldsymbol{t})^{\mathbf{1}-\boldsymbol{\theta}}}{\mathbf{1}-\boldsymbol{\theta}}, \theta > \boldsymbol{0}. \tag{2}$$

The household would face the problem of constrained optimization, where the level of consumption would be affected by budgetary constraint. This budgetary constraint considers the level of capital and bonds held by the household, the level of disposable income given a tax rate and levels of consumption within an infinitesimal instance of each period of time. We allow taxes and bonds into the budgetary constraint and get,

$$\int_{t=0}^{\infty} \boldsymbol{e}^{(\mathrm{g}\,+\,\mathrm{n})\mathrm{t}} \boldsymbol{c}(\boldsymbol{t}) \boldsymbol{e}^{-\boldsymbol{Rt}} \boldsymbol{dt} = \boldsymbol{m}_0 + \int_{t=0}^{\infty} \boldsymbol{e}^{(\mathrm{g}\,+\,\mathrm{n})\mathrm{t}} [(\boldsymbol{1} - \boldsymbol{\tau}_w) \boldsymbol{w}(\boldsymbol{t})] \boldsymbol{e}^{-\boldsymbol{Rt}} \boldsymbol{dt}. \tag{3}$$

The identities included in the constraint are the household's initial sum of capital and bonds $\boldsymbol{m}_o$, level of household consumption, $\boldsymbol{c}_t$, the household's total wages, $\boldsymbol{w}_t$, tax rate on household's wages, τ_w, the interest rate, $\boldsymbol{R}$, the growth rate of technology, $\boldsymbol{g}$, the growth rate of population, $\boldsymbol{n}$ (See Appendix I to integrate budget constraint). To find create the Euler condition for per-capita consumption the utility function be adjusted to

$$\boldsymbol{U} = \boldsymbol{B} \int_{t=0}^{\infty} \boldsymbol{e}^{-\beta t} \frac{\boldsymbol{c}(\boldsymbol{t})^{\mathbf{1}-\boldsymbol{\theta}}}{\mathbf{1}-\boldsymbol{\theta}} \boldsymbol{dt}. \tag{4}$$

To understand how we got to this solution please review Appendix II and Romer's book(2019, pg 55). With this function, the Euler function can be determined by means of the Lagrangian and then the consumption law of motion can be derived. Romer's derivation process is similar to the way this paper went about it and produced this derivation of the Euler equation for consumption,

$$\frac{\dot{\boldsymbol{c}}}{\boldsymbol{c}_t} = \frac{[\boldsymbol{r} - \boldsymbol{\rho} - \boldsymbol{g\theta}]}{\boldsymbol{\theta}}, \tag{5}$$

$$\gamma_{\mathrm{C}} = \frac{\dot{\boldsymbol{C}}}{\boldsymbol{C}_t} = \frac{[\boldsymbol{f}'(\boldsymbol{k}_t) - \boldsymbol{\rho}]}{\boldsymbol{\theta}}. \tag{6}$$

The law of motion states that the "consumption per worker is rising if the real return exceeds the rate at which the household discounts future consumption"(Romer, 2019, pg 58). So, the household will choose an infinitesimally small amount to spend or save given the rate of return and the preference to consume at a particular time $\boldsymbol{t}$ (Romer, 2019). That would be in-line with the accepted effects of expansionary monetary policy where fluctuations in the real interest rate affect consumer spending(Mishkin, 1996).

The household would participate in the factor markets, and supply their services to firms through the renting of capital and labor. Given the assumption of perfectly competitive goods markets it means that marginal revenue equals marginal costs so the firm can not gain profit even through implementation of optimal levels of capital and labor,

$$\max\{\boldsymbol{F(K, AL)} - (\boldsymbol{wL} + \boldsymbol{RK})\} = 0. \tag{7}$$

Another assumption is that all firms abide by the conditions of a neoclassical production function. That consists of a multiplicative relationship between capital, technology and labor. So both inputs experience constant returns to scale,

$$\mathrm{Y} = \boldsymbol{F}(\boldsymbol{cK}, \boldsymbol{cAL}) = \boldsymbol{cF}(\boldsymbol{K}, \boldsymbol{AL}), \tag{8}$$

Next it must meet the Inada conditions which state that as total capital or labor units approach infinity the marginal product for an additional unit of either factor approaches 0 shown on line (). Also total capital and labor units approach 0 the marginal product for an additional unit of either factor approaches infinity, shown on line ().

$$\lim_{\boldsymbol{K}\to\infty} \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{K}} = \lim_{\boldsymbol{L}\to\infty} \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{L}} = \boldsymbol{0} \tag{9}$$

$$\lim_{\boldsymbol{K}\to \boldsymbol{0}} \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{K}} = \lim_{\boldsymbol{L}\to \boldsymbol{0}} \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{L}} = \infty$$

The general takeaway of the Inada conditions is that when a firm has no factor units its potential to produce output is infinite. As it accumulates capital and labor its potential to produce more output is smaller for an additional unit in either factor. Finally, the marginal products of either factor should be positive and marginally diminishing,

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{K}} > 0, \frac{\partial^2 \boldsymbol{F}}{\partial \boldsymbol{K}^2} < 0, \tag{10}$$

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{L}} > 0, \frac{\partial^2 \boldsymbol{F}}{\partial \boldsymbol{L}^2} < 0.$$

We also used a labor-augmenting production function that considers the process of output to be defined by the multiplicative relationship between capital and effective labor. Given the assumption for constant returns of scale, we can then create the intensive production function. By finding the product of F(K, AL) and c, which is equal to 1/AL, we get the intensive production function,

$$y = f(k) = K/AL\ ,\ f`(\mathrm{k}) > 0,\ f``(\mathrm{k}) < 0. \tag{11}$$

Firms would rent capital at the interest rate or the marginal product of capital and would pay wages to laborers and that would be the marginal product of labor. The marginal product of capital can be approximated as,

$$\boldsymbol{r}(\boldsymbol{t}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{k}} = \boldsymbol{f}\text{'}(\boldsymbol{k}(\boldsymbol{t})), \tag{12}$$

the marginal product of labor is approximated, with respect to the intensive production function,

$$\boldsymbol{w}_t = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{l}} = [\boldsymbol{f}(\boldsymbol{k}(\boldsymbol{t})) - \boldsymbol{k}(\boldsymbol{t})\boldsymbol{f}\text{'}(\boldsymbol{k}(\boldsymbol{t}))]. \tag{13}$$

In regards to labor and capital demand, we assume that the level of capital and labor at a time $\boldsymbol{t}$ meets the demand amounts in the respective factor markets. This indicates that the factor markets operate under perfect market mechanics, but this is not true, however, we utilize this simplifying assumption to promote ease in analytically solving the model. So under perfect market conditions the factor markets operate under like this,

$$\boldsymbol{K}_t = \boldsymbol{K}_d, \boldsymbol{L}_t = \boldsymbol{L}_d, \tag{14}$$

meaning the amount of the respective factor good at a particular time t is what the market demanded at that time.

The law of motion for capital can be seen as, essentially saying that per unit of effective labor investment into capital will be in the steady state when investment equals cost of capital.

Depreciation of capital is not included in the model as a simplifying assumption making aggregate investment equal to net increase in capital(See Appendix I for derivation). So the law of motion is shown as,

$$\dot{k} = f(k(t)) - c(t) - G(t) - k(t)(g+n). \tag{15}$$

The behavior of capital per capita can be explained through the interpretation of long term growth growth for aggregate capital in an economic system. For long term growth within Solow's model, change in aggregate capital is equates to the level of investment which is a function of the fraction amount of output spent on investing back into the firm (Solow, 1956). The generation of capital would affect the level of output in the economy, and this would be through its effect on the level of productivity within the economy with respect to the neoclassical production function. The Inada conditions are the evidence that support these claims on the effect capital can have on long term growth and production. Prior in this section, the Inada conditions were defined and it showed the increase of capital would impact output positively and infinitely, assuming there is no capital to begin with. That fact is complimented with the idea that as capital increased infinitely it would have a reduced marginal impact on output. However, if one considers the decades of technological advancements that led to new types of capital, each following technological revolution started at 0. Thus, there were multiple instances where the increases of capital started at a supposed level zero and integration of that new type of capital led to a large increase in productivity and output.

Investment is an important component for the development of capital in the economy, is necessary for growth and becomes a key influence in economic progress (Solow, 1962). Alongside the claims made by the Kaldor Facts on the progression of profit/capital ratio, which is "a steady rate of profit on capital, at least in the 'developed' capitalist societies" (Kaldor, 1961). One can define the aggregate growth of capital as,

$$\dot{K} = I(t), \tag{16}$$

where the change of aggregate capital between time periods equals aggregate investment. As seen in Appendix I, aggregate investment is a function of levels of output, consumer spending and government spending. Behavior of K' is incorporated into the per-capita function and allows fluctuations in investment function to be accounted for, which leads to the law of motion shown before. and the another assumption relates assets from the consumer budget constraint to capital so, $m = k$. That implies that assets progress in the same manner as capital; this approach is similar to Barro and Martin's explanation(2004). The process of exogenous government purchases, or G, per unit of effective labor is assumed to be funded with taxes on the consumer's income and nominal debt. One can rearrange the differential equation for nominal debt to make,

$$G(t) = \dot{B} - r(t)B(t) + T(t),\ T(t) = \tau_w wL. \tag{17}$$

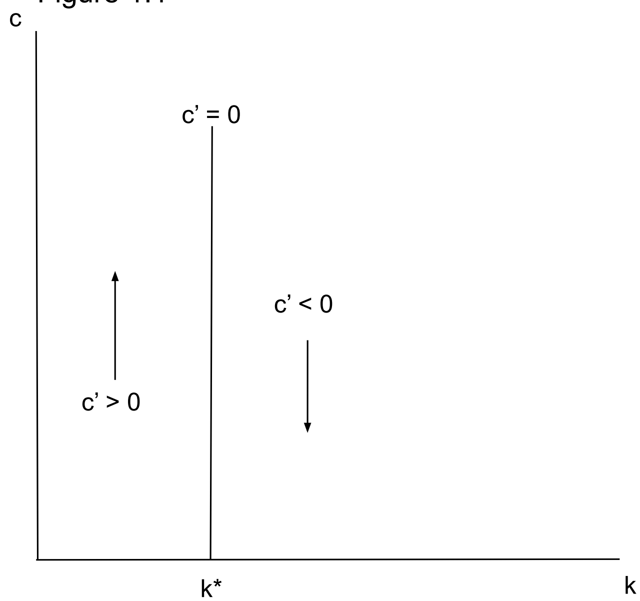
$B(t)$ represents the level of nominal debt; $T(t)$ is the level of taxes collected from taxes on wages. Based on this equation, the government is able to choose either taxes or bonds to fund its operations. In Appendix I, we see that the lines 45-47 show how the income tax does not directly affect the path of consumption. This would be in accordance with Ricardian Equivalence where the path of consumption is not affected whether the government chooses to finance itself with debt or tax revenue. That would also indicate that it would not impact the dynamics of the linear system, matching Barro and Martin's section on the effect of tax rates(2004, pg 146).

The dynamics of the model can be evaluated with both the law of motion of capital and consumption, components satisfying decentralized equilibrium conditions have been determined. Proving that there is a decentralized equilibrium satisfied ensures this model is Pareto Optimal, where no other agents can be . Within the decentralized equilibrium, there are endogenous variables and exogenous variables that need to be explicitly defined. Another assumption is that this model converges towards a balanced growth path or saddle path, where the endogenous variables are all growing at the same rate, this is also known as the steady state. The endogenous variables of concern are $\{\boldsymbol{k}, \boldsymbol{c}, \boldsymbol{y}\}$ and the exogenous variables of concern are $\{\boldsymbol{w}, \boldsymbol{r}, \boldsymbol{L}, \boldsymbol{A}\}$. With respect to the Euler equation, aggregate consumption would converge on the steady state when $f(k(t)) = \rho + \theta g$, thus leading to where the consumer's need to save meets their time discount factor to consume. Considering that $f(k(t)) = r(t)$ and r can be interpreted as a saving rate for the consumer, that idea seems more than plausible. Established conclusions for the Euler function are supported by the assumption that the initial value for consumption is given. When the rate of return . Now in regards to the steady state of capital we consider when k' is equal to 0, So algebraically we can equate steady state values for consumption per effective labor unit to the rest of the law of motion and that creates this equation,

$$c(t) = f(k(t)) - G(t) - k(t)(g + n). \tag{18}$$

The prior equation asserts the idea that consumption would meet the level of investment and costs of capital in the steady state. This characterises the balanced growth path(BGP) of the economy, and is similar to the BGP found for the Ramsey model in Campante and Velasco's textbook (2021). The balanced growth path is an interesting idea on its own, as it indicates the level for which all variables are growing at constant rates.

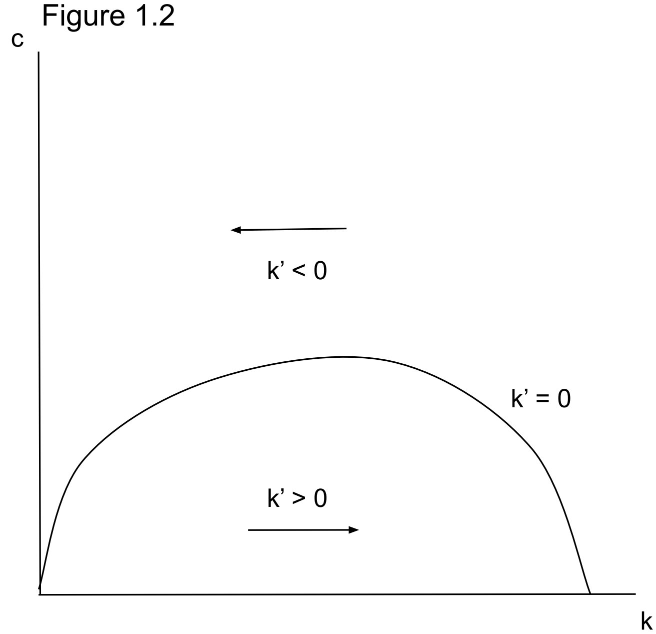
Figure 1.1



Dynamics of Consumption

As shown above the steady state for consumption would match with a value of capital that can maximize the wellness for both households and firms. The aforementioned condition that allows the change of consumption between time periods to be 0 is when the consumer’s saving rate equals the discounted time preference to consume. When the marginal product of capital increases the interest rate would follow, and cause an increase in consumption thus the optimal point of consumption would jump to another point on the steady state curve. Comparatively, when there is a change in the discount rate the curve would move along the x-axis. That defines changes in the consumer’s time preference to consume, a negative shock in that discount rate would shift the curve rightward to meet high levels of capital with higher levels of consumption. Furthermore these levels of consumption would still be on the steady state curve so the model would still transition to the BGP.

Figure 1.2



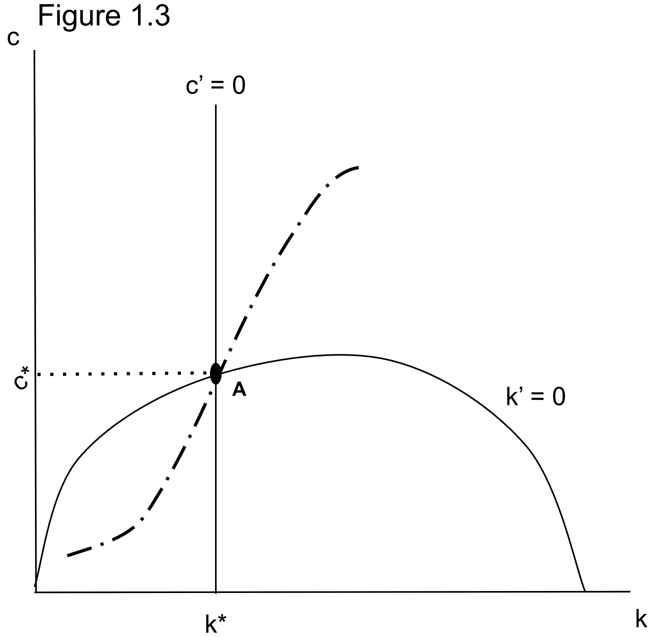
Dynamics of Capital

Figure 2.2 depicts the accumulation of capital in its steady state, and given the arrangement of the intensive production output would grow at the same rate as capital. Allows us to compare the behaviors of capital and consumption as the desired basket of consumption goods would require some level capital for production. As stated before, changes in the stock of capital would affect the marginal product of capital or the interest rate. In events where consumption exceeds corresponding capital levels, the change in capital would drop(Romer, 2019, pg. 60). That is a result of firms meeting consumers at their desired level of output requiring the level of output that converges onto the optimal level of capital. So, the changes of capital would be smaller to catch up to levels of consumption, that would be proportional to amounts of output delved out by the firm. Additionally, in regards to the steady state value for output that can be determined within the context of the intensive production function as being,

$$\boldsymbol{y* = f(k*) \Rightarrow \frac{Y*}{AL} = \frac{K*}{AL}}. \tag{19}$$

Given the levels of capital per unit of effective labor, one expects output to behave proportional to the amount of capital available for labor to use. Referring back to the Kaldor facts, “”. So, this where the steady state of output would be determined as a result of the values of capital. Given all of the established groundwork, one can consider the qualitative information or the implications the model can lead us to behaviors by the economy’s agents.

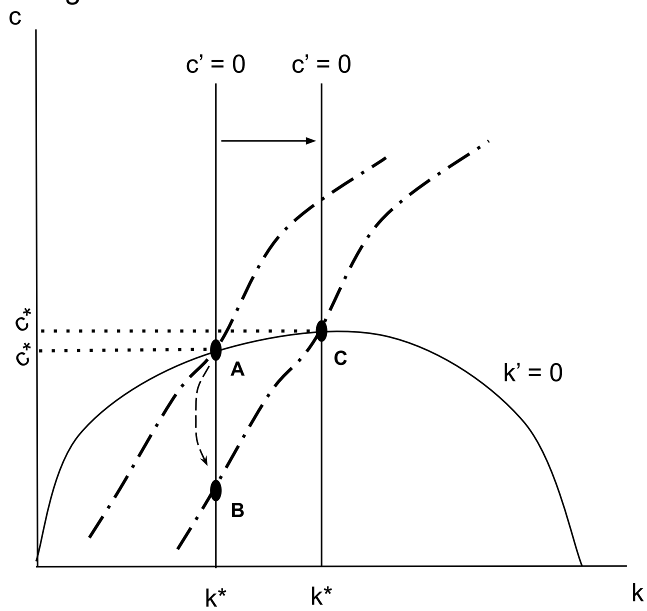
Figure 1.3



Saddle Path in the Ramsey Model

Figure 2.3 is a depiction of the dynamics of the optimal growth model, which are defined by the insights the law of motions provides us. Now, one can present scenarios to determine how the model can transition away and onto the balanced growth path. In comparison, the saddle paths refer to paths that converge on the saddle point, in this case that would be the dash-dotted line. Point A is the saddle-point that provides combinations of consumption and capital levels where the changes of variables between time periods would be 0. The saddle-path would converge from both sides of the saddle-point value. As capital moves towards its steady state value, there would be a value of consumption that would match that level of capital.

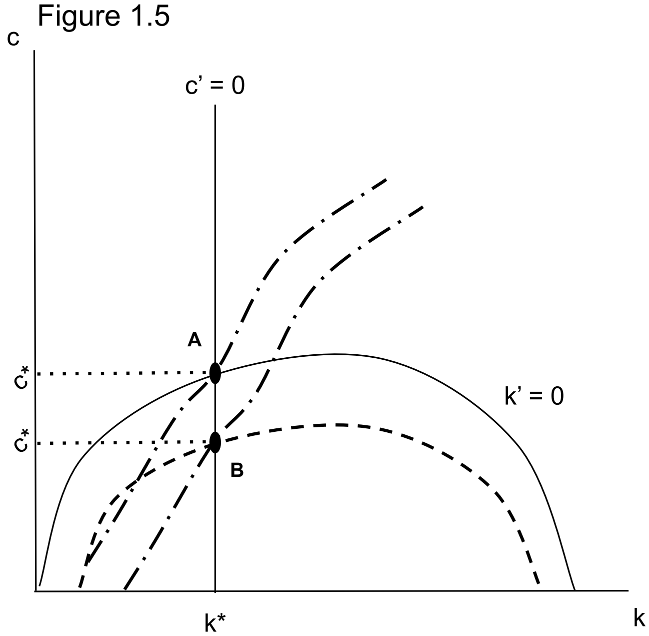
Figure 1.4



Scenario One - Time Discount Rate Drops

Figure 2.4 depicts the scenario where there are changes in consumer’s time preference, specifically this would be a drop in ρ. As stated that would illustrate the consumer’s preference to value today’s consumption compared to tomorrow’s consumption. So, there would be a higher level of consumption, which is seen at point C at the new intersection.

Figure 1.5



Scenario Two - Drop of Capital

The change in the linear system depicted in figure 2.5 shows a drop in the supply of private capital as a result of increases in government purchases. That would lead to consumption jumping down to match the new levels of capital from point A to B. As explained earlier consumption would match the levels of production, it could match the amount of capital within the economy to create that level of output. Figure 2.5 also demonstrates the impact of government spending shocks, as the government consumes more of the private capital it reduces the amount of available capital left for private firms.

Data and Methodology:

Model:

The agents considered will be modeled through a restricted vector autoregression(VAR) estimated with Ordinary Least Squares and tested with likelihood iterations using Maximum Likelihood estimation. As promoted as more effective than simultaneous equation models, macroeconomists are able to model interdependent and autoregressive relationships between time series variables with VAR models (Sims, 1980). Identification strategy is determined from the previous section where the relationships between agents and the choices they can make in maximizing their utility are considered. With an understanding of what forces are endogenous or exogenous, one is able to zero-restrict the VAR to encapsulate the interdependencies among the macro-variables. Structural VARs (SVAR) are known to be effective in modeling structural relationships between variables (Amisano and Giannini, 1997). Certain conditions can be implemented on SVARs, where instantaneous(short term) or long run restrictions can be imposed (Lütkepohl, 2005). Firstly the restricted VAR, will be used to test hypotheses on certain interdependencies and then an SVAR will be used to cement those findings.

The mentioned methodologies will be utilizing time series variables that have a unit root of I(0) to ensure stability in the estimation of the VAR processes, and we assume that all our variables meet the conditions for weak stationarity. The data processes are ensured for stationarity through the augmented Dickey-Fuller test and set the significance level to 5% to test the null hypothesis of a unit root in the time series. For the variables that do not meet the stationary conditions, we use first or second differencing to integrate our series to a unit root of 0. The model is depicted as so:

$$y_t = \boldsymbol{\phi}_1 y_{t-1} +,\dots,+ \boldsymbol{\phi}_p y_{t-p} + u_t, \tag{20}$$

where y_t represents a $K \times 1$ vector of endogenous variables; $\boldsymbol{\phi}_i$, where $i = 1,...,p$, are $K \times K$ matrix of coefficients; $\boldsymbol{\Theta}$ is a $K \times M$ matrix of coefficients; and u is a $K \times 1$ vector of unpredictable innovations that behaves as a zero-mean white-noise process with a covariance matrix where $u \sim \mathbf{(}0, \Sigma\mathbf{)}$**.** The assumption of the error term, emulates the classic linear regression model assumptions implying an average of 0 with a constant covariance (Killian et al, 2017). As previously stated, the restricted VAR will be identified with respect to the theory established in the previous section. Microeconomic behaviors of the agents will be implemented into the model through zero-restriction matrices, and that will include the endogenous shocks. So, the interdependencies can be seen through,

Dependent Variables

	Consumption	Capital	Interest Rates	Firm Profits	Govt Spending	Tax Receipts	Public Debt
Consumption	✓	✓		✓		✓	
Capital	✓	✓	✓	✓			
Interest Rates	✓	✓	✓	✓			✓
Firm Profits		✓		✓			
Govt Spending		✓			✓	✓	✓
Tax Receipts					✓	✓	✓
Public Debt					✓	✓	✓

With 1's and 0's that allows one to indicate that certain variables are exogenous to other processes that may occur within the system of variables. To determine the endogenous relationships within the SVAR, the Adjusted R-squared values will help in judging the extent for which the model depicts the variation of data.

Within the SVAR, the instantaneous relationships will be considered between variables and that can be captured using the Cholesky Decomposition method and the AB matrix model. Cholesky Decomposition allows one to detail unique causal chains, and allows one to make claims about the structure of the innovations(Killian et al, 2017). The analysis will not necessarily use this method but will be instrumental in explaining the AB model.

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ . & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ . & . & \mathbf{1} & \mathbf{0} \\ . & . & . & \mathbf{1} \end{bmatrix} \tag{21}$$

The above matrix called **A** denotes the instantaneous relationship between the observable variables, where instantaneous defines its impact as short term. Furthermore, contemporaneous can also depict these relationships as being impacting each other within the same period. The lower triangular of blank spaces is to structure the contemporaneous relationships between the variables, analogous to the changes within the short term. This structuring is known as the Cholesky Decomposition where the variable on the top impacts the variables below it. Important to note, that the restrictions would be structured with assumptions established with economic theory. Therefore, some variables would be denoted as null in its ability to impact the following dependent variable. The A matrix can be utilized alongside the B matrix where,

$$\mathrm{B} = \begin{bmatrix} . & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & . & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & . & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & . \end{bmatrix} \tag{22}$$

represents the structural shocks on each variable being modeled. The initial VAR model is defined as the reduced form where $\mathbf{u_t}$ equates to the weighted average of the structural shocks(Killian et al, 2017).

Through two matrices one is able to decompose the fluctuations of these variables into instantaneous relationships and structural shocks (Lütkepohl, 2005). An application of this model is shown in Chapter 8 from Killian and Lütkepohl's 2017 textbook. Assume a vector of macroeconomic variables $\boldsymbol{y}_t = (\boldsymbol{p}_t, \boldsymbol{gdp}_t, \boldsymbol{m}_t, \boldsymbol{i}_t)$, where $\boldsymbol{p}_t$ is log prices, $\boldsymbol{gdp}_t$ is logged real GDP, $\boldsymbol{m}_t$ is log of the M1 base and $\boldsymbol{i}_t$ is the federal funds rate (Killian et al, 2017). Aforementioned assumption about the innovation, is that it equals the weighted average

$$\begin{pmatrix} u_t^p \\ u_t^{gdp} \\ u_t^m \\ u_t^i \end{pmatrix} = \begin{bmatrix} b_0^{11} & 0 & 0 & 0 \\ b_0^{21} & b_0^{22} & 0 & 0 \\ b_0^{31} & b_0^{32} & b_0^{33} & 0 \\ b_0^{41} & b_0^{42} & b_0^{43} & b_0^{44} \end{bmatrix} \begin{pmatrix} w_{1t} \\ w_{2t} \\ w_{3t} \\ w_{4t} \end{pmatrix}.$$

of the structural shocks. So, one can depict the structure of each innovation. So the innovation for the money base would be $\boldsymbol{u}_t^m = \boldsymbol{w}_{1t}\boldsymbol{b}^{31}{}_0 + \boldsymbol{w}_{2t}\boldsymbol{b}^{32}{}_0 + \boldsymbol{w}_{3t}\boldsymbol{b}^{33}{}_0 + \boldsymbol{w}_{4t}0$(Killian et al, 2017). Then economic theory would be utilized to rationalize the structural shocks and its relations to the innovations. Another way to perceive this relationship is to view it as $\boldsymbol{A}u_t = \boldsymbol{B}\omega_t$ as interpreted by Giannini(1992). Finally, to interpret the results from the SVAR model, impulse-response graphs will be illustrated based on the estimation of the parameters. Impulse-response function graphs allow one to forecast a single variable's response to a standard deviation shock in another variable. In this study the vector of macroeconomic variables, where $\boldsymbol{y}_t = (\boldsymbol{i}_t, \boldsymbol{k}_t, \boldsymbol{c}_t, \boldsymbol{profits}_t, \boldsymbol{tax}_t, \boldsymbol{g}_t, \boldsymbol{debt}_t)$: $\boldsymbol{i}_t$ represents the nominal interest rate, $\boldsymbol{k}_t$ is firm capital, $\boldsymbol{c}_t$ personal consumption, $\boldsymbol{profits}_t$ domestic firm profits, $\boldsymbol{tax}_t$ tax levels, $\boldsymbol{g}_t$

government spending, and finally, $debt_t$ is public debt. Each variable, not including the interest rate, is transformed with logged first-differences depicting the percentage changes from one time period to the next.

Data:

From the Bureau of Economic Analysis, releases from the National Income and Product Accounts(NIPA) are implemented for the different agents represented in the above model. To proxy for the aggregate consumption and saving made by the consumer, personal consumption expenditures(PCE) and saving data are pulled. PCE is recognized as a reliable measure for types of goods and services consumed by the general household (Fox et al, 2024). The purchased goods, goods being either tangible or intangible, come from private sector business, not public sector. Also, within the Ramsey model prior government spending does not enter the household's utility function, this prevents one from formulating an assumption on direct relationship between household utility and transfers or consumed public goods. Domestic firm profits and firm capital values are also acquired from the BEA. Government spending and current tax receipts for levels of collected taxes are collected from NIPA. Current tax receipts would include taxes collected on income, capital, consumption, and any other taxable entity. This paper considers government spending which includes both federal and state government consumption expenditures and gross investment. This allows the focus to include any state issues that may rely on federally approved funds for local projects and its impact on the market yield for government bonds. The paper considers the impact of personal taxes on the impact on government spending which isn't the only source of tax revenue, so changes in current levels of tax receipts will be used instead. All of the macroeconomic variables are inflation adjusted for 2013 prices utilizing the consumer price index(CPI).

The Federal Reserve of Economic Data(FRED) provides daily observations of the federal funds rate and treasury yields for maturities of 24, 60, and 120 months notes. Also, the relationship between the term premium for the mentioned maturities will also be explored as well. Term premium estimates were a product of Adrian, Crump and Moench's findings on a linear regression methodology that used five principal components of treasury yields to derive the term structure (2013). Term premium estimates and yields will then be related to macroeconomic factors to relate these fluctuations to the change in bond price (Evans et al, 2006). This paper does not utilize the typical factor loading methodology as used in typical bond pricing models (Ang et al, 2003; Duffee, 2002; Evans et al, 2006). There are no transformations made for the yields and premiums, thus the models will regress onto nominal yields and premiums. With that in mind, the likelihood that the stability of the VAR is flawed is significantly higher since we are regressing on non-stationary series. However, we will just continue forward with the understanding that the causality tests and impulse response functions are biased due to spurious regression. Public debt data is utilized to control for the levels of funds the government has borrowed from the public and is the main variable of interest. Public debt will be seen in the SVAR when considering the increase in the deficit as impactful on the variation of nominal yields and premiums. The public debt data comes from the US Treasury, but was pulled from FRED through API requests. These are quarterly observations between 1966 and 2025, and the secondary SVAR model uses data that is between 2005 and 2025.

Results:

This section considers the data that will be utilized in the regression analysis, its behavior through time and the results of the multivariate models. The summary statistics section reviews the average growth and volatility of the macroeconomic variables that will be included into the models in the section after. These tables are constructed by looking at moments in time instead of exclusively summarizing the total dataset allowing for an analysis of patterns between the variables. Additionally, there is included a table that depicts the averages and volatility for treasury yields and estimated term premiums. That allows for a brief discussion on the impact of the short term rate on treasury yields, which is important in the realm of bond pricing and modeling. The next section revolves around the analysis and interpretation of results from the VAR(2) and SVAR(2) models that were developed in the data and methodology section. Impulse response function graphs will be included into the discussion to contribute to the analysis of the results from regressions.

Summary Statistics:

The variables that are used in the VAR model are depicted in the figure below, specifically showing the average growth of the macroeconomic variables. We can see that average growth of PCE and firm capital are about the same, indicating the Ramsey model's assumptions about the balanced growth path between capital and consumption are justified with real world data. For periods between and including 1986 and 2000, consumption and capital grew at rates at, or above, 2% on a quarterly basis. Similar insights can be drawn for 2006-2010, with firm capital levels and personal consumption growing around an average of 1.7% per quarter within that specific period. Furthermore, it was that period where domestic profits grew at about the same rate as both capital and consumption. Quarterly growth in government spending is shown to be averaging higher than tax receipts levels in multiple periods indicating large fiscal deficits developed during certain sections of time. Additionally, average growth of public debt exceeds levels of government spending and current tax receipts in every period but 2011-2015 and 1996-2000. That allows the market to consider the increase of public debt pertaining to the amount of bonds the treasury is willing to dispense to make up for the lack of tax revenue. Due to the obligation of paying back bondholders at maturity, the debt issuer needs to acknowledge the level of obligations it currently has before selling more debt instruments to meet their budgeted tasks. As Buchanon reviewed, Musgrave's perception on fiscal spending specifies that there is a particular branch that specializes in acquiring the funds from individuals and assets(1960). Those institutions must consider the means of acquiring those funds and the impact on economic potential (Buchanon, 1960), and that responsibility supports the requirement of efficient treasury auctions.

Figure 2.1: Average Growth of Macroeconomic Variables

Periods	PCE	Firm Capital	US Firm Profit	Govt Ex	Taxes	US Debt
1970-1975	3.87	3.61	3.28	3.54	2.89	3.48
1976-1980	4.77	5.74	3.55	4.30	5.08	4.54
1981-1985	3.52	3.26	4.34	3.51	2.57	4.96
1986-1990	2.65	2.06	1.11	2.48	2.69	3.76
1991-1995	2.02	2.02	4.00	1.51	2.12	2.69
1996-2000	2.21	2.90	0.84	1.78	2.84	1.26
2001-2005	1.96	1.07	3.89	2.22	0.99	2.51
2006-2010	1.70	1.80	1.75	2.01	1.84	3.26
2011-2015	1.33	2.21	1.36	0.57	2.65	1.92
2016-2020	1.30	1.07	1.79	1.49	0.80	2.37

Figure 2.2 describes the standard deviations of the macroeconomic variables within each of the periods. Standard deviation can describe the volatility of macroeconomic variables and can describe the behavior of the variables within each respective period. One can interpret a small deviation in a period as a dataset that is tighter around the average value of that sample. Therefore, with smaller deviations in a sample of time, we can assume that quarterly growth values varied closer to the average. Consumption levels in time periods including 1991 up to 2000 had growth values that were closer to the average indicating very little difference in

Figure 2.2: Average Volatility of Macroeconomic Variables

Periods	PCE	Firm Capital	US Firm Profit	Govt Ex	Taxes	US Debt
1970-1975	1.12	2.09	6.01	1.78	7.33	1.60
1976-1980	1.07	2.24	6.64	1.80	2.18	1.03
1981-1985	1.17	3.14	7.00	1.37	5.12	1.47
1986-1990	0.98	1.81	5.09	1.00	3.24	0.85
1991-1995	0.55	1.65	4.51	0.89	2.61	1.03
1996-2000	0.67	1.11	3.65	0.92	1.41	1.03
2001-2005	0.98	2.28	5.89	0.84	6.51	1.31
2006-2010	0.84	1.97	6.69	1.02	2.50	1.73
2011-2015	0.76	1.43	4.82	0.71	2.43	1.40
2016-2020	2.22	2.70	8.60	0.66	3.67	2.71

consumption from quarter to quarter in those time periods. In comparison, quarterly consumption growth between 2016 and 2020 varied greatly, indicating levels of consumption being largely different from period to period. Now as indicated in the theoretical model section public debt is impacted by the difference between government spending, taxes and real public debt. Larger deviations of taxes between quarters indicates wide growth patterns in the amount of taxes collected between time periods. Within the 2001-2005 period tax receipt growth grew at different levels between quarters, that would impact the size of the fiscal deficit between quarters. During that same period government spending between quarters was about 2.2%, that would reduce the likelihood of surpluses and increase public debt. Also, public debt growth averaged about 2.5% per quarter, growing at nearly the same rate as government spending.

In considering the patterns of the macroeconomic variables, it allows for an understanding into the bond pricing mechanisms that price its intrinsic risks into its yield. As expected paths of economic potential are illuminated, the market is able to accurately determine the worth of those debt instruments. Within the literature of bond pricing, that would fall in line with the expectations hypothesis where the long-term rate would be the average of future short-term rates (Lutz, 1940). However, in considering the time variation of these bond yields that allows one to reject the idea that short term interest rates exclusively impact long term yields (Duffee, 2002). From the VAR we will simulate shocks to display the impact of variables on future values of the 12, 60 and 120 month Treasuries. Impulse response functions will be used to show the change of the variables through time, by depicting the impact of positive standard deviation shock on the response variables. Furthermore, the same VAR(2) model is also applied for the term premiums of the aforementioned securities. Term premiums are the extra compensation the investor gets when they purchase a bond with a long maturity compared to a shorter maturity (Li et al, 2017). Term premiums comprise the term structure or the yield of the bond alongside the federal funds rate as the short term rate (Li et al, 2017). Placing the term premium in the model allows one to determine the value of longer maturities instruments compared to changes in certain macroeconomic factors. In tandem, previous literature indicates the changes in the supply of the treasuries impacted the size of the yield spread of treasuries and comparable products (Roley, 1982; Krishnamurthy et al, 2012). Therefore, when particular macroeconomic scenarios occur that impacts the likelihood of a large treasury auction, that would impact the price negatively leading to larger yields in the shorter term maturities. In comparison longer term yields would not be largely affected through time as general uncertainty is already priced into the bond.

Figure 2.3 illustrates quarterly averages yields and volatility for the federal funds rate, treasury yields and premiums within different time periods. Average yields of the fed rate have similar values to the shorter to medium term maturity bond yields. As recognized in yield-modeling and general macroeconomic literature, the main policy instrument of the central bank is impacting the level of short term rates (Diebold et al, 2005). Impacts of short term rates on bond rates are evident when bond yields are viewed as a term structure with a short rate and a term premium. Under that impression, bond yields would have average values that vary alongside the average of the short rate. Within 1991-1995, the Federal Funds rate averaged around 4.60% with 1-year Treasury note yields averaging at 4.94%, while 5-year note and 10-year note yields averaged above 6.30%. In this case bond yields would have the additional premium that gets higher as the maturity of the bond gets longer. The volatility of the fed rate behaves similarly to the one-year note while the 5-year and 10-year note have less volatile yields within the same time period. That would be an indication that there is lower deviation for longer term government yields, so the yield values vary lower around its true average. In comparison, shorter term yields have a similar standard deviation to the fed rate volatility, affirming that the values in shorter-term yields fluctuate alongside the fed rate.

Figure 2.3: Yields and Term Premiums

	Average Yields and Premiums						
		Term Premiums			Treasury Yields		
Periods	Fed Rate	ACMTP01	ACMTP05	ACMTP10	DGS1	DGS5	DGSTEN
1970-1975	6.68	0.26	0.89	1.37	6.64	6.96	7.01
1976-1980	8.64	0.47	1.42	2.22	8.41	8.54	8.72
1981-1985	12.27	0.96	2.85	4.02	11.35	12.24	12.36
1986-1990	8.43	0.59	1.75	2.59	7.49	8.13	8.38
1991-1995	4.60	0.59	1.91	2.70	4.94	6.40	6.91
1996-2000	5.78	0.31	0.78	1.28	5.46	5.82	5.92
2001-2005	2.34	0.30	1.18	1.79	2.46	3.73	4.42
2006-2010	2.77	0.16	0.75	1.32	2.76	3.48	4.08
2011-2015	0.10	-0.02	0.50	0.92	0.19	1.37	2.37
2016-2020	1.14	-0.31	-0.49	-0.51	1.31	1.70	2.02
	Average Volatility						
		Term Premiums			Treasury Yields		
Periods	Fed Rate	ACMTP01	ACMTP05	ACMTP10	DGS1	DGS5	DGSTEN
1970-1975	2.90	0.24	0.42	0.46	1.36	0.92	0.78
1976-1980	4.03	0.26	0.43	0.53	2.83	1.80	1.54
1981-1985	4.05	0.38	0.46	0.49	2.57	1.85	1.59
1986-1990	2.19	0.25	0.49	0.62	1.03	0.76	0.71
1991-1995	1.31	0.15	0.52	0.68	1.34	1.01	0.86
1996-2000	0.88	0.16	0.35	0.41	0.52	0.66	0.66
2001-2005	1.39	0.15	0.54	0.66	1.21	0.81	0.56
2006-2010	2.25	0.16	0.65	0.84	2.03	1.20	0.73
2011-2015	0.03	0.19	0.52	0.74	0.07	0.48	0.56
2016-2020	0.86	0.11	0.35	0.44	0.85	0.78	0.72

Regression Analysis:

To determine the structure of the SVAR, I examine the variations captured by the VAR models with first and second differenced data. In comparing the VAR models I determined how much variation is lost between the unrestricted and the restricted model using the adjusted R^2. The estimates for the VAR(2) models for both the sets of data and choice of model restriction are shown in figure 2.4. Second differenced data meeting the condition of being integrated with a unit root of 0 according to the augmented-Dickey Fuller test with 95% confidence. Logged first difference data shows the same results for all the relevant variables but government spending and public debt. Lack of stationarity in those variables will prove to impact the results of the VAR model, however the study shall continue using the data within the study. The difference between the R^2 negligible for the models with first differenced data, depicting pretty similar estimations of the data generation processes for the endogenous variables between the unrestricted and

Figure 2.4

	First Differenced Data VAR(2): Adj R^2	
Variables	Unrestricted	Restricted
PCE	0.8283803	0.8167013
Firm.Capital	0.7862580	0.7776023
Fed.Rate	0.9312148	0.9290199
US.Firm.Profit	0.2971854	0.2080915
Govt.Ex	0.8550439	0.8235362
Taxes	0.3966413	0.4036135
US.Debt	0.7865557	0.7825676

	Second Differenced Data VAR(2): Adj R^2	
Variables	Unrestricted	Restricted
PCE	0.3907707	0.3125934
Firm.Capital	0.5798143	0.5235469
Fed.Rate	0.9217187	0.9246485
US.Firm.Profit	0.7086089	0.6752815
Govt.Ex	0.4463114	0.4187025
Taxes	0.5824581	0.5852436
US.Debt	0.4356370	0.3755228

restricted models. Firm profits show a large difference between the restricted and unrestricted models of about 9% of unaccounted variation between the unrestricted and restricted models. Variation estimates for firm profits are low compared to the other endogenous variables, and is possibly due to the lack of consideration for fluctuation of workers. Not accounting for labor was a decision in response to the perception of wage in a Ramsey economy which does not consider labor turnover or growth. Within the context of a labor-leisure model, the market wage is related to the level of leisure undertaken by the worker and the level of consumption. However, this is a simplified analysis of the dynamics within an economic system that allows the researcher to understand how tax revenue impacts expected yields and premiums. The models are built with respect to the theory section, which assumes that firms maximize their profits by minimizing costs but will consider capital as a factor alone.

The unrestricted and restricted models for taxes, using first differenced data, shows limited capture of the variation as it solely accounted for any variation from consumption. As explained in the theory section, we assume a perfectly competitive goods market and that asserts that demanded quantity meets quantity supplied. So, firms supplying the demanded quantity can be interpreted as the level of income accrued by the firm being equal to the level of consumer spending. Fluctuations in the income generated would impact the level of taxable income for the firm, indicating the influence of demand shocks on tax levies. Due to increase in firm profits, there would be excess for investment allowing the firm to purchase more capital. An increased capital would prompt an increase in labor to impact levels of production, given that factors of production are considered to be complements for each other. More individuals with jobs would induce demand shocks that would lead to higher levels of consumer spending. Also, consumer spending would fuel wages and then general income levels increase leading personal income taxes to increase as a result. Then including consumer spending into the model accounts for the business cycle fluctuations that impact changes in tax levels quarterly. Thus, adding PCE into the tax receipt model is justified and additionally consumption taxes are also accounted for by including personal spending. Furthermore, as seen in figure 2.4, the restricted model for taxes depicts improvement from the unrestricted model in capturing fluctuations with the adjusted R^2 increasing by 0.4% or 0.004 affirming the inclusion of the consumption variable.

The SVAR structure that impacts the impulse response function graphs in both figures 2.5 and 2.7 structures the A matrix according to figure 2.4. Spaces represent the exogeneity of the variables in the columns when compared to the variables in the columns. There is always a contemporaneous relationship between the variable and its own lags, and that would be represented with a number one on that intersection of column and row. Example of that being, short term interest rate or the Fed Rate, this variable is unique on the account of it being the only exogenous variables in the system. Personal consumption expenditures are shown to be impacted by variations in firm capital, short term rates, and values of consumption expenditures. The

Figure 2.5

	Fed Rate	Firm Capital	PCE	US Firm Profits	Taxes	Govt Ex	US Public Debt	1-Year	5-year	10-year
Fed Rate	1	0	0	0	0	0	0	0	0	0
Firm Capital	.	1	0	0	0	0	0	0	0	0
PCE	.	.	1	0	0	0	0	0	0	0
US Firm Profits	.	.	.	1	0	0	0	0	0	0
Taxes	0	0	.	.	1	0	0	0	0	0
Govt Ex	.	.	0	0	.	1	0	0	0	0
US Public Debt	.	0	0	0	.	.	1	0	0	0
1-Year	.	.	.	.	.	.	.	1	0	0
5-year	.	.	.	.	.	.	.	.	1	0
10-year	.	.	.	.	.	.	.	.	.	1

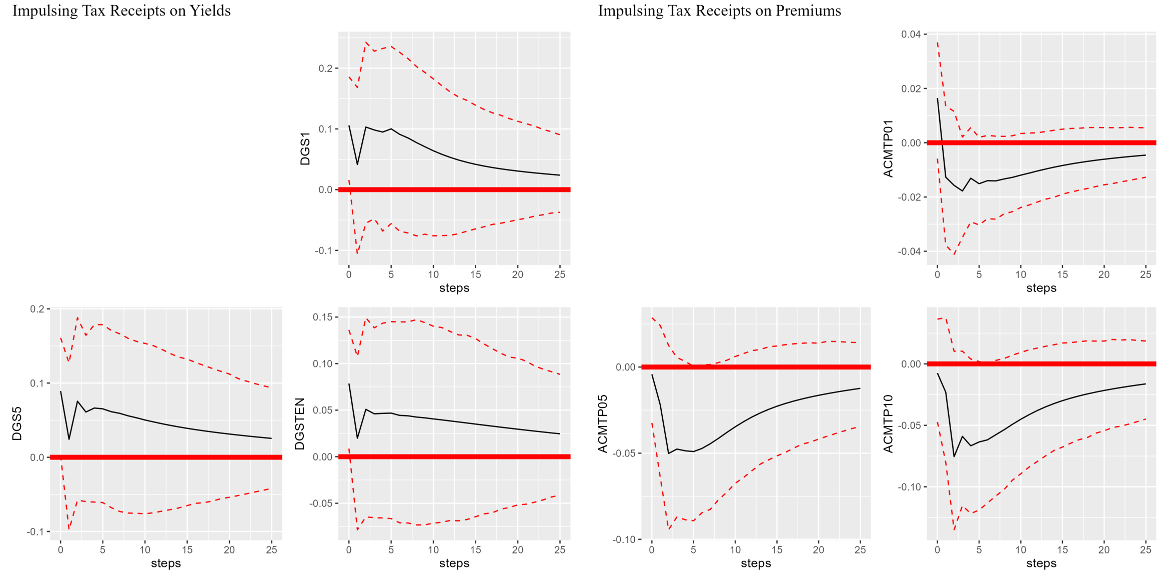
arrangement respects the theory established earlier, where the growth rate of aggregate consumption is impacted by the changes of capital per capita. The zeroes indicate that these variables would not impact the row variable that is being examined. Example being that taxes are exogenous to the short term rate, but is endogenous to consumption and firm profits so given this is a AB model innovations would be defined as $\boldsymbol{u}_t^{taxes} = \boldsymbol{w}_t^{pce}\boldsymbol{b}^{31}{}_0 + \boldsymbol{w}_t^{profits}\boldsymbol{b}^{32}{}_0$. This deviates from the theory established, but firm profits were added to the taxes function to compensate for a section of unaccounted variation in the restricted and unrestricted VAR model. Furthermore, real world data for current tax receipts includes corporate income taxes.

In interpreting the results of the SVAR impulse response functions are estimated and graphed to forecast dynamic relationships. Figure 2.6 shows the results of positive standard deviation shock of tax receipts on both yields and term premiums. For the term premiums, each depicts the same movement regardless of maturity, with a large spike at period zero which then decays sharply and then appreciates back towards 0 basis points. Comparably, the yields behave similarly however after decaying there is a secondary spike that decays at a flatter rate for longer

Figure 2.6

Impulsing Tax Receipts on Yields

Impulsing Tax Receipts on Premiums



term maturities, that being 60-month and 120-month maturities. The 12-month maturity's yield sticks at rate after spiking for more than one period and then begins its decay back towards 0 basis points. Increases in tax level can be an indication that the future quarterly deficits will have lower values, as lower deficits would indicate the Treasury does not need to sell debt to the market to meet obligations. Supply shocks in the market for treasuries would impact the basis point change, an instance where that would occur is a flight to safety and liquidity (Krishnamurty et al, 2012). This model did not consider the impacts of corresponding markets, however, in a previous paper, granger-causality tests depicted equity prices impacting treasury yields with 90% confidence (Henry, 2024). Plausibly, one can assert that there is intermarket dynamics that impacts market yields as well.

Figure 2.7

Impulsing Govt Expenditures on Yields

Impulsing Govt Expenditures on Premiums

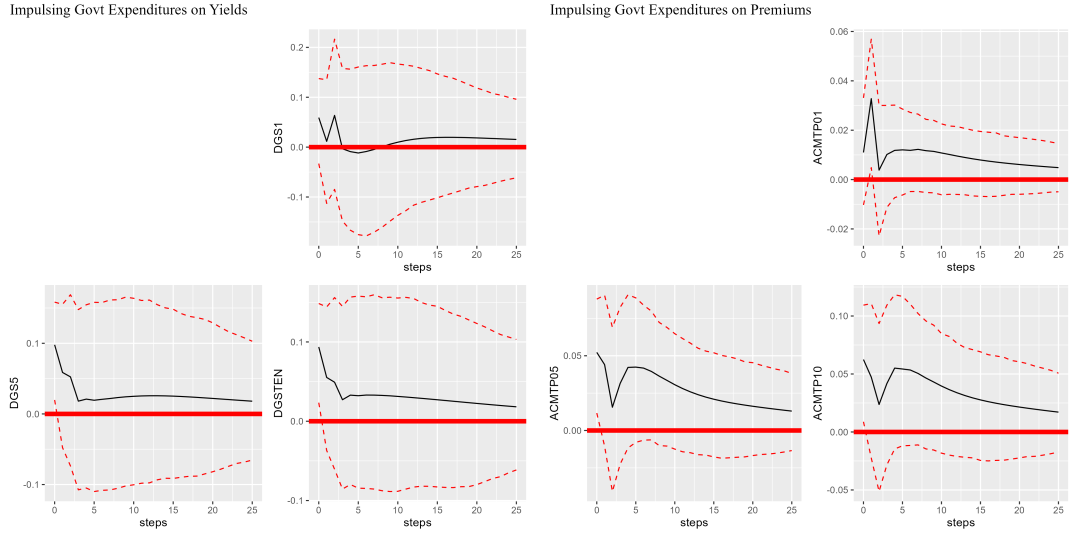


Figure 2.7 shows a large positive spike in term premiums for the 12-month security into period one that then decays sharply. This would also relate directly with the changes in supply, as the increase or decrease in marketable government debt could be predicted by the size of the deficit(Andritzky, 2012). Fig. 2.1 shows that the average growth of government spending was 2.0% quarterly between 2001-2005 and 2006-2010, and that outpaced average tax receipt growth within both those time periods. With such high spending amounts being approved without high levels of tax revenue, the premiums would spike due to expected fluctuations in the different maturities of the treasury maturities. Treasury yields spike at period zero and then decays quickly between period zero and five. That would be another expected supply shock at period zero, this utilizes ideas related to the impact of tax revenue on the supply of treasuries. This concerns the amount obligated the government would have to undertake, with levels of government spending exceeding taxes it would prompt a large debt/GDP. So, the government would finance their tasks with bond sales, and given the budget determined by Congress that would affect the size of the deficit. Explanations for figures 2.6 and 2.7 provide some indication that there is a relationship between the tax revenue, deficits and premiums but the 95% confidence bands are particularly

wide. Stronger evidence can be seen in figure 2.8, which makes use of different data for a smaller amount of time periods.

Figure 2.8

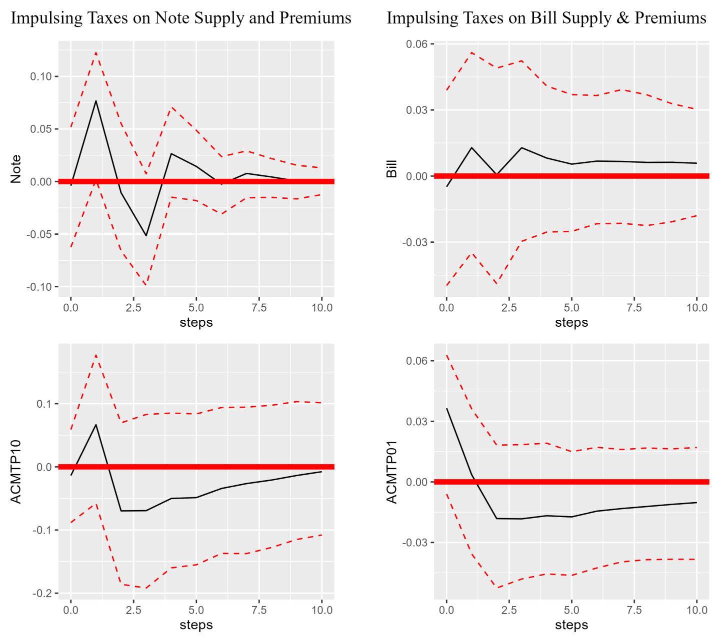


Figure 2.8 utilizes a different SVAR model than the previous figures, where this relates macroeconomic variables to the fluctuation in supply. Utilizing treasury supply data from the Fiscal Data site for the US Treasury, relating changes to the supply of government bonds to other macroeconomic variables. The complexity of the model was reduced and the Cholesky Decomposition method is utilized in this case. Let us say there is a vector, $\boldsymbol{y}_t$, that contains some endogenous variables, so $\boldsymbol{y}_t = (\boldsymbol{i}_t, \boldsymbol{k}_t, \boldsymbol{c}_t, \boldsymbol{tx}_t, \boldsymbol{g}_t, \boldsymbol{s}_t, \boldsymbol{p}_t)$. Establishing the relationship to supply and the fluctuations in tax revenue would cement the claim that the impact tax revenue can have on pricing for Treasuries. This SVAR tested bond supply and premiums for their respective maturity length, so one model had note supply and ten year treasury premiums and another had bills supply and one-year treasury premiums. Fig 2.8 shows tighter error bands for each response variable and the impulse variable in this group is taxes. The behavior of treasury notes depicts an immediate increase to the note supply after a shock in tax receipts. Then ten-year term premiums would follow that same behavior but then immediately drop and gradually return to near 0. If the investor is operating under the assumption that the economy has greater surpluses of taxes then investment into longer-term treasuries would be optimal. That assumes that the government has the money to meet its obligations and would reduce auctions on shorter-term maturities which can be seen during contractionary economic conditions (Goyenko et al, 2011). Furthermore, interest rate risk would be lower under conditions of a positive tax shock as that indicates optimal levels of employment and output without overbearing prices. Of course, there could be a situation where tax revenue can negatively impact the utility of consumers, however, very few

politicians would be willing to execute this form of tax policy. Finally the ten-year premiums depict a relationship where the premiums would spike into the first period, drop sharply and then gradually increase towards zero. In comparison, one year premiums would drop sharply and then make its way back towards zero.

Conclusion:

This study has established a relationship between the changes in government spending and tax revenues to changes in the term structure. With the neoclassical model, the impacts of macroeconomic variables and agents are depicted and allow for interpretation. Theory projects government spending as a factor that impacts the level of capital within the neoclassical system, specifically, shocks would reduce the supply of private capital. Levels of consumption would change due to output dropping as a result of changes in available capital and that would trickle into changes of tax levels. With the market clearing assumption applied onto the goods market, that allows for the level of consumption within the economy to be considered as the desired level of demand at that time. One future step is incorporating changes within the labor market to capture the movements within the business cycle that impact tax revenue levels for the government. With real business cycle models or new keynesian models to account for nominal rigidities, an economy can be simulated to better reflect the real world. Finally, in regards to growth modelling, ideas within investor preferred habitats will be for robust theory and utilization of software to depict simulations with calibrated parameters.

Through the use of vector-autoregression methods, a system of variables was depicted that imitates the true data-generation process of fiscal variables within the economy. The AB model provided the comparison between innovations of variables and their structural shocks. An improvement that can be made in the methodology is the use of advanced VAR-modeling techniques like time-varying parameter VARs or Bayesian VARs. Allowing for precise estimation for the parameters leading to robust interpretations from impulse response functions, Granger causality tests and forecast error variance decompositions. Additionally, considering how fiscal variables impact the term premium directly with Factor-augmented VAR models utilizing principal components analysis to simplify large amounts of highly correlated variables. Finally, through the empirical tests one can determine a relationship between changes of fiscal variables and the variation in sovereign bond yields. Indicating the impacts fiscal imbalance can have on the bond markets and encouraging healthy regulation, apt monetary policy and responsible federal spending.

Appendix I:

To compute the total differential we utilize the chain rule with aspects of the quotient rule to derive the per-capita law of motion equation for capital. The process that helps us compute that equation is shown with an example function ***z*** and the total differential for ***z*** is seen below as the ***dz*** / ***dt***. Keeping in mind that w, x, y, and z are all functions of time, we get this formula,

$$z = (w, x, y), \frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dw}\frac{dw}{dt} + \frac{dz}{dy}\frac{dy}{dt}. \qquad (1)$$

Next is when solving for a general solution for a differential equation, we can make the case with the function that describes the net increase of the labor force. The understanding is that the labor force's growth can be defined as a function of the growth rate of the population. Based on that idea we can get a general idea of what the labor force would be at time ***t***. So, what we see below is that relationship between the growth rate in population assuming a given level of population at time 0, or the initial population. $\dot{L}$ is the change of the labor force with respect to time, L is the level of population, ***n*** is the growth rate of the population.

$$\dot{L} = L_t n \qquad (2)$$

$$\frac{\dot{L}}{L} = n \qquad (3)$$

Integration on both sides allows us to get to the next step. Utilizing integration rules we are able to determine from (4) that we can get the natural log of L is equal to the growth rate multiplied by time to get line (5). We exponentiate both sides using Euler's number on line (6) and due to the natural log in the exponent spot we are able to determine the general solution in line (7).

$$\int \frac{1}{L}\dot{L}dt = \int ndt \qquad (4)$$

$$ln(L) = nt \qquad (5)$$

$$e^{ln(L)} = e^{nt} \qquad (6)$$

$$L(t) = L(0)e^{nt} \qquad (7)$$

One can utilize the same method for determining the general solution of labor for the process of A or technology. So, if the change of net technology is ***A'*** and it is equal to ***A***, which is the level of technology times the growth of technology $\boldsymbol{g}$ we get line (8). Then the general solution for technology in the economy is given as line (9) shows,

$$\dot{A} = A_t g, \tag{8}$$

$$A(t) = A(0)e^{gt}. \tag{9}$$

Now in certain cases, there may be a need for an integrating factor that helps to solve the solution. Furthermore, given the solution there is a transversality or solvency condition that helps us solve, refer to Campante and Velasco's textbook for clarity of its use (2021, pp 36, 367).

$$y' - P(x)y = Q(x) \tag{10}$$

$$I = e^{\int -P(x)dx} \tag{11}$$

Line (11) shows the integration factor, $\boldsymbol{I}$, that influences our ability to integrate the identities on both sides of the equal sign. One determines the integration factor by raising Euler's number to the integration of $\boldsymbol{P(x)dx}$ and that leads to line (11). $\boldsymbol{I}$ is then multiplied on both sides of line (10) which creates,

$$e^{\int -P(x)dx}y' - e^{-\int P(x)dx}P(x)y = e^{-\int P(x)dx}Q(x). \tag{12}$$

Given the results of line (12) we can integrate both sides with respect to $\boldsymbol{dx}$, leading us to,

$$\int e^{\int P(x)dx}y' - e^{\int P(x)dx}P(x)y\,dx = \int Q(x)e^{\int P(x)dx}dx. \tag{13}$$

Finally the integration of the left hand side is akin to a result determined with the product rule, as depicted with lines 14-15. Consider the product rule of a function $\boldsymbol{g(x)}$ that looks like:

$$g(x) = e^{\int -b(x)}a(x). \tag{14}$$

Then the derivative of $\boldsymbol{g(x)}$ that was a result of product rule which would look like this,

$$\frac{dg}{dx} = e^{\int -b(x)dx}a'(x) - b(x)e^{\int -b(x)dx}a(x). \tag{15}$$

So given the transition from lines (15) - (16), that leads to the conclusion of the integration part of the problem which is,

$$e^{\int P(x)dx}y = \int Q(x)e^{\int P(x)dx}dx. \tag{16}$$

Now to show the per-capita law of motion of capital utilizes the methods that got line (1) and (7). Note that an identity with an apostrophe is interpreted as an identity derived with respect to time and variables with subscript t are functions of time. The law of motion of aggregate capital is defined as the net increase aggregate of capital, ***K'***, is equal to aggregate investment, ***I***, minus depreciation of capital $\delta \boldsymbol{K}$. These identities are assumed to be functions of time. Line (10) is found in the beginning of most advanced macroeconomic or economic growth textbooks when talking about Solow's growth model(Romer(2019); Campante et al(2021); Barro et al(2004)).

$$\dot{\boldsymbol{K}} = \boldsymbol{I}_t - \boldsymbol{\delta K}_t \tag{17}$$

$$\dot{\boldsymbol{K}} = \boldsymbol{I}_t \tag{18}$$

Later derivations will utilize line (18) within the law of motion of capital. Then consider the resource constraint, as this will allow us to determine function for investment. So we assume the resource constraint is defined as total aggregate output is equal to the aggregate levels of consumption, investment and government spending. Then the function aggregate investment is equal to aggregate output minus consumption and government purchases.

$$\boldsymbol{Y}_t = \boldsymbol{C}_t + \boldsymbol{I}_t + \boldsymbol{G}_t \tag{19}$$

$$\boldsymbol{I}_t = \boldsymbol{Y}_t - \boldsymbol{C}_t - \boldsymbol{G}_t \tag{20}$$

To begin the total differential we consider the neoclassical production function that relates produced output to utilized units of factors of production: labor and capital. Campante and Velasco (2021) explains why line (13) satisfies the conditions of a neoclassical production function (pp 11-12). With this function we compute the intensive neoclassical production function that defines units of capital and output in terms of units of effective labor(Romer, 2019).

$$\boldsymbol{Y}_t = \boldsymbol{F(K}_t, \boldsymbol{A}_t\boldsymbol{L}_t\boldsymbol{)}, \boldsymbol{Y} \ / \ \boldsymbol{AL} = \boldsymbol{K} \ / \ \boldsymbol{AL} \tag{21}$$

Given the relationship between capital and effective labor, one is able to derive out the per-capita law of motion for capital. So, we call the per-capita increases of capital over time with ***k'***. Incorporated methods to find ***k'*** include the steps that determine solutions on lines (1) and (7). If one wants to review the derivation it is best to refer to Romer's derivation (2019, pp 10-17). Initial steps starts with considering the identity of ***k*** with lines(15-17) where,

$$\boldsymbol{k} = \boldsymbol{K}_t\boldsymbol{(A}_t\boldsymbol{L}_t\boldsymbol{)}^{-1}, \tag{22}$$

$$\dot{k} = \frac{1}{AL}\dot{K} - \frac{K}{[AL]^2}[\dot{A}L + A\dot{L}], \tag{23}$$

$$\dot{k} = \frac{\dot{K}}{AL} - \frac{K}{AL}[\frac{\dot{A}L}{AL} + \frac{A\dot{L}}{AL}]. \tag{24}$$

The quotient rule is present in the second term of ***k'*** on line (16) with the squared term of the multiplicative terms for effective labor in the divisor. That second term was multiplied by the sum of the partial derivatives of A and L with respect to time. Next, in line (17), the effective labor is divided into the partial derivatives to consider the growth rates of technology and population growth. Then, we get results that allow us to substitute identities into the derivations,

$$\dot{k} = \frac{\dot{K}}{AL} - \frac{K}{AL}[\frac{\dot{A}}{A} + \frac{\dot{L}}{L}], \tag{25}$$

$$\dot{k} = \frac{\dot{K}}{AL} - k[g + n], \tag{26}$$

$$\dot{k} = \frac{I(t)}{AL} - k[g + n], \tag{27}$$

The identities substituted into the function included: the ratio of capital and effective labor, the net increase of capital with respect to time and the aggregate investment in the economy. So after substituting and equation wrangling the results are,

$$\dot{k} = \frac{Y(t) - C(t) - G(t)}{AL} - k[g + n], \tag{28}$$

$$\dot{k} = y(t) - c(t) - G(t) - k(t)[g + n], \tag{29}$$

$$\dot{k} = f(k(t)) - c(t) - G(t) - k(t)[g + n]. \tag{30}$$

Line (22) shows the net increase in per-capita capital is defined by the output produced by capital units per unit of effective labor, deducted by personal consumption by the labor force, government purchases and costs of capital. This is one of the important identities that defines the Ramsey model since we consider changes in consumption into the law of motion. Furthermore, we include a government identity to proxy for the level of government purchases that crowds investment into new capital.

To find the optimal amount and dynamics of consumption we need to consider the budget constraint that influences spending habits. After defining the consumer budget constraint, derivations can be used to determine the Euler condition for consumption. Firstly, we are going to define the budget path that the household will undertake given the variables in the constraint. Assume the consumer chooses to consume today, $\boldsymbol{C}_t$, and owns capital tomorrow known as future assets, $\boldsymbol{M}_{t+1}$, and that equals the difference of income, ***W***, after taxes and assets today, $\boldsymbol{M}_t$, times the interest rate, ***R***,

$$\boldsymbol{M}_{t+1} + \boldsymbol{C}_t = (\boldsymbol{1} - \tau_w)\, \boldsymbol{W}_t + \boldsymbol{R}\boldsymbol{M}_t. \tag{31}$$

Then the function can be made in terms of assets equated to the level of saving which is the difference between income and consumption. After creating line (25), we set the equation to create the differential equation for assets, note that we replaced $\boldsymbol{M}_{t+1}$ with ***M'*** to denote the change of the variables as continuous time in line (32).

$$\boldsymbol{M}_{t+1} - \boldsymbol{R}\boldsymbol{M}_t = (\boldsymbol{1} - \tau_w)\, \boldsymbol{W}_t - \boldsymbol{C}_t \tag{32}$$

$$\dot{\boldsymbol{M}}(\boldsymbol{t}) - \boldsymbol{R}\boldsymbol{M}(\boldsymbol{t}) = (1 - \boldsymbol{\tau}_w)\boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{C}(\boldsymbol{t}) \tag{33}$$

Constructing the differential equation requires an integration factor, which I have chosen given the ***P(x)*** being the interest rate for which assets appreciated by. For understanding on how the integrating factor is utilized and determined, refer to lines (10-13). So, multiply the integrating factor, which is Euler's number raised to the integration of the interest rate, on both sides of the equal sign to make,

$$\boldsymbol{M'}\boldsymbol{e}^{\int -\boldsymbol{R}dt} - \boldsymbol{R}\boldsymbol{M}_t\boldsymbol{e}^{\int -\boldsymbol{R}dt} = [(\boldsymbol{1} - \tau_w)\, \boldsymbol{W}_t - \boldsymbol{C}_t]\boldsymbol{e}^{\int -\boldsymbol{R}dt}. \tag{34}$$

After finding integrating the exponents to make later steps legible we get,

$$\dot{\boldsymbol{M}}(\boldsymbol{t}) - \boldsymbol{R}\boldsymbol{M}(\boldsymbol{t})\boldsymbol{e}^{-\boldsymbol{R}(\boldsymbol{t})} = [(1 - \boldsymbol{\tau}_w)\boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{C}(\boldsymbol{t})]e^{-\boldsymbol{R}(\boldsymbol{t})} \tag{35}$$

and then integrate with respect to changes in time or ***dt***, to get,

$$\int \dot{\boldsymbol{M}}(\boldsymbol{t}) - \boldsymbol{R}\boldsymbol{M}(\boldsymbol{t})\boldsymbol{e}^{-\boldsymbol{R}(\boldsymbol{t})}\boldsymbol{dt} = \int [(1 - \boldsymbol{\tau}_w)\boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{C}(\boldsymbol{t})]e^{-\boldsymbol{R}(\boldsymbol{t})}\boldsymbol{dt}. \tag{36}$$

Based on the insights established on lines (14-16), we understand that the product rule can be utilized to get the derivative on the left side of the equal sign. So, since the result of the integration on the left hand side is known, the equation then equals,

$$\boldsymbol{M(t)e^{-Rt}} = \int_0^t [(\boldsymbol{1} - \tau_w)\boldsymbol{W(t)e^{-Rt}} - \boldsymbol{C(t)}]\boldsymbol{e^{-Rt}dt} + \boldsymbol{\kappa} \quad . \tag{37}$$

Then we implement a no-Ponzi game condition to constraint the behaviors of the consumer constraint to consumption and income. Specifically, the no-Ponzi game condition asserts that the consumer cannot accumulate debt for any reason (Campante et al, 2021, pg 36). Illustrating the no-Ponzi condition is done implementing the limit on line (37) from t to infinity,

$$\lim_{t\to\infty} [\boldsymbol{M(t)e^{-Rt}} = \int_0^t [(\boldsymbol{1} - \tau_w)\boldsymbol{W(t)} - \boldsymbol{C(t)}]\boldsymbol{e^{-Rt}dt} + \boldsymbol{\kappa}] \leq 0 \quad . \tag{38}$$

Since the limit of the left side of the equal sign is equal to 0 it leads to,

$$\lim_{\boldsymbol{t}\to\infty} \boldsymbol{Me^{-Rt}} = 0 \quad , \quad 0 = \int_0^t [(\boldsymbol{1} - \tau_w)\boldsymbol{W(t)} - \boldsymbol{C(t)}]\boldsymbol{e^{-Rt}dt} + \boldsymbol{M(0)} \tag{39}$$

Then equating changes of consumption to disposable income would lead to this version of the budget constraint,

$$\int_{t=0}^{\infty} \boldsymbol{C}_t\boldsymbol{e}^{\text{-}\boldsymbol{Rt}}\boldsymbol{dt} = \boldsymbol{M}_0 + \int_{t=0}^{\infty} [(\boldsymbol{1} - \tau_w)\,\boldsymbol{W}_t]\boldsymbol{e}^{\text{-}\boldsymbol{Rt}}\boldsymbol{dt} \quad . \tag{40}$$

The path of consumption is equal to assets in the initial period plus disposable income. Afterwards, to model the paths as per-capita figures for the household, we multiply both sides by the ratio of effective labor and the number of households. So, that gives the equation,

$$\int_{t=0}^{\infty} \frac{\boldsymbol{A}_t\boldsymbol{L}_t}{\boldsymbol{H}}\boldsymbol{c}_t\boldsymbol{e}^{\text{-}\boldsymbol{Rt}}\boldsymbol{dt} = \boldsymbol{m}_0\frac{\boldsymbol{A}_0\boldsymbol{L}_0}{\boldsymbol{H}} + \int_{t=0}^{\infty} \frac{\boldsymbol{A}_t\boldsymbol{L}_t}{\boldsymbol{H}}[(\boldsymbol{1} - \tau_w)\boldsymbol{w}_t]\boldsymbol{e}^{\text{-}\boldsymbol{Rt}}\boldsymbol{dt} \quad . \tag{41}$$

Since the processes for technology and labor have been established on lines (7) and (9), we substitute them in to create,

$$\int_{t=0}^{\infty} \frac{\boldsymbol{A}_0\boldsymbol{e}^{\boldsymbol{gt}}\boldsymbol{L}_0\boldsymbol{e}^{\boldsymbol{nt}}}{\boldsymbol{H}}\boldsymbol{c}_t\boldsymbol{e}^{\text{-}\boldsymbol{Rt}}\boldsymbol{dt} = \boldsymbol{m}_0\frac{\boldsymbol{A}_0\boldsymbol{L}_0}{\boldsymbol{H}} + \int_{t=0}^{\infty} \frac{\boldsymbol{A}_0\boldsymbol{e}^{\boldsymbol{gt}}\boldsymbol{L}_0\boldsymbol{e}^{\boldsymbol{nt}}}{\boldsymbol{H}}[(\boldsymbol{1} - \tau_w)\boldsymbol{w}_t]\boldsymbol{e}^{\text{-}\boldsymbol{Rt}}\boldsymbol{dt} \quad . \tag{42}$$

Then we multiply both sides by the inverse of the ratio of effective labor and total households to go to,

$$\frac{H}{A_0L_0}\left[\frac{A_0L_0}{H}\int_{t=0}^{\infty} e^{(g\,+\,n)t}c_t e^{-Rt}dt = m_0\frac{A_0L_0}{H} + \frac{A_0L_0}{H}\int_{t=0}^{\infty} e^{(g\,+\,n)t}[(1-\tau_w)w_t]e^{-Rt}dt\right], \quad (43)$$

$$\int_{t=0}^{\infty} e^{(g+n)t}c_t e^{-Rt}dt = m_0 + \int_{t=0}^{\infty} e^{(g+n)t}[(1-\tau_w)w_t]e^{-Rt}dt. \quad (44)$$

Line (44) will be the household budget constraint this paper exploits in the Ramsey model section of the paper.

$$\int_{t=0}^{\infty} e^{(g+n)t}c_t e^{-Rt}dt = m_0 + \int_{t=0}^{\infty} e^{(g+n)t}[w_t - w_t\tau_w]e^{-Rt}dt \quad (45)$$

$$\int_{t=0}^{\infty} e^{(g+n)t}c_t e^{-Rt}dt = m_0 + \int_{t=0}^{\infty} e^{(g+n)t}[w_t - \tau]e^{-Rt}dt \quad (46)$$

$$\int_{t=0}^{\infty} e^{(g+n)t}c_t e^{-Rt}dt = m_0 + \int_{t=0}^{\infty} e^{(g+n)t}w_t e^{-Rt}dt - \int_{t=0}^{\infty} e^{(g+n)t}\tau e^{-Rt}dt \quad (47)$$

Appendix II:

To determine a per-capita consumption utility function, this paper uses the derivations found by Romer(2019). Romer's details on the net change of capital, that it is defined by the aggregate output and consumption by effective labor units. Furthermore, since the flow of consumption equates to the flow of income, that income is predicated on the amount of effective labor units. Firstly, for per-capita form of consumption consider that aggregate consumption is a product of an assumed of the basket of goods and effective labor so,

$$C_t = c_t A_t L_t. \quad (1)$$

The utility function being used is of a class of Constant Relative Risk Aversion functions,

$$u(C_t) = \frac{C_t^{1-\theta}}{1-\theta}, \quad (2)$$

and the expectation is that we have consumption units that have a multiplicative relationship with consumption to get a per capita amount. Then we go ahead and multiply consumption by A_t to get,

$$u(c_t) = \frac{[A_t c_t]^{1-\theta}}{1-\theta}. \quad (3)$$

The multiplication of just technology is to evaluate the technology process within . So, the technology process determined in Appendix I can be substituted into for A_t and makes,

$$u(c_t) = \frac{[A_0 e^{gt} c_t]^{1-\theta}}{1-\theta}. \quad (4)$$

Then we distribute the exponent to make,

$$u(c_t) = \frac{A_0^{1-\theta} e^{gt(1-\theta)} c_t^{1-\theta}}{1-\theta}, \quad (5)$$

and now we can place this result into the integral to model the path of utility as a function of consumption per capita.,

$$U = \int_{t=0}^{\infty} A_0^{1-\theta} e^{gt(1-\theta)} e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} \frac{L_t}{H} dt. \quad (6)$$

Now the labor force process can be substituted in for L_t and that leads to,

$$U = \int_{t=0}^{\infty} A_0^{1-\theta} e^{gt(1-\theta)} e^{-\rho t} e^{nt} \frac{c_t^{1-\theta}}{1-\theta} \frac{L_0}{H} dt \quad (7)$$

To clean up the integral and to make it pretty, the initial values of technology and the labor force per household are assumed to be constants. Furthermore, given the rule of exponents, one can add the exponents together when the bases are the same and we denote that sum as $-\beta$, so

$$u(c_t) = A_0^{1-\theta} \frac{L_0}{H} \int_{t=0}^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} dt. \quad (8)$$

Finally we make the product of the constants equal to a variable B to make the final utility function,

$$u(c_t) = B \int_{t=0}^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} dt. \quad (9)$$

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