Time Series Analysis Notes

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1 Overview of Time Series Characteristics

Definition 1. A **univariate time series** is a sequence of measurements of one variable at regular time steps.

Such data need not be IID.

1.1 Objectives of Analysis

1. Describe important features of any time series pattern
2. Explain how past affects future or how two time series "interact"
3. Forecasting
4. Use time series as a control standard, e.g., measurements of the quality of some manufacturing product

1.2 Model Types and Considerations

There are two basic types:

1. Relating time series values to past values and past prediction errors – called Autoregressive Integrated Moving Average or ARIMA models.
2. Regular regression models with time indices as x-variables.

Initial data exploration:

- Trend – overall progression of measurements
- Seasonality – periodic behavior based around calendar intervals such as seasons, quarters, months, weeks, etc.
- Outliers – data away from original data or away from some manipulation of it
- Long-run Cycle – periodic behavior unrelated to seasonality type
- Constant Variance – variance changing or not
- Abrupt Changes – significant disturbances to series or variance or other things

1.3 Autoregressive Models: Autocorrelation and Partial Autocorrelation

Definition 2. Let $\{y_t\}_{t=1}^{n}$ be a time series indexed by t. An **autoregressive model** is one where time series values are regressed on previous values. For example, a first-order autoregression AR(1) would look like:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \epsilon_t$$

and a second-order autoregression AR(2) would look like:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \epsilon_t.$$

In general, a k^{th}-order autoregression AR(k) is given by

$$y_t = (\beta_0, \beta_1, \ldots, \beta_n, 1)\,(1, y_{t-1}, \ldots, \beta_{t-n}, \epsilon_t)^T.$$

Usually the errors $\epsilon_t \overset{iid}{\sim} N(0, \sigma_\epsilon^2)$ and independent of y.

This begs the question: How do we choose the order for a given time series? There are two methods:

1. Autocorrelation function (ACF) and
2. Partial Autocorrelation function (PACF).

Definition 3. The coefficient of correlation between two values in a time series given by:

$$\text{Corr}(y_t, y_{t+k}) = r_k = \frac{c_k}{c_0} \quad \text{where} \quad c_k = \frac{1}{n}\sum_{t=1}^{n-k}(y_t - \bar{y})(y_{t+k} - \bar{y})$$

is called the **autocorrelation function (ACF)**. Recall what familiar term c_0 is. Notice the similarity to the covariance formula which is meant to exhibit the linear relationship between two variables. In this case, this is the covariance (or linear relationship) of lagged values. Here the **lag** is given by **k**.

This is one method. This method includes in it the influence of other lags on y_t. The second method removes this influence of the other lags in between...

Definition 4. Here the **Partial Autocorrelation Function (PACF)** is given by

$$f_k = \begin{cases} r_1 = \text{Corr}(y_t, y_0) & \text{if } k = 1; \\ \text{Corr}(y_t - y_t^{t-1}, y_0 - y_0^{k-1}) & \text{if } k \geq 2 \end{cases}.$$

Essentially, this has the effect of determining the linear correlation between y_t and y_{t+k} but removing the linear dependence of lags in between t and $t+k$, namely t_i such that $t < t_i < t+k$. One can also think of this as subtracting away the projection of y_t on the linear subspace spanned by $y_{t+1}, \ldots, y_{t+k-1}$.

We can additionally complicate the whole thing by considering certain trend behavior within our original time series via decomposition.

1.4 Complexifying ACF/PACF via Decomposition and/or Higher Order Trends

The typical decomposition of a time series involves:

- overall trend, m_t
- seasonality, s_t, and
- error, ϵ_t

so that

$$y_t = m_t + s_t + \epsilon_t.$$

Typical estimation involves first estimating the overall trend through linear filters. One example of this is a moving average given by some "window" size:

$$\hat{m_t} = \sum_{k=-a}^{a}\left(\frac{1}{1+2a}\right) y_{t+k}.$$

We can experiment with the window size to get a feel for a good overall trend. Then once this is done, we can estimate the seasonality by looking at what remains:

$$\hat{s_t} = y_t - \hat{m_t}.$$

Note that this $\hat{s}_t$ depends on the window size. So we can take the average of the window sized seasonality estimates to fix a single s_t. Then this gives us a way to calculate the error:

$$\epsilon_t = x_t - \hat{m_t} - \hat{s_t}.$$

Luckily there are packages for such decomposition.

Another way to complexify our model for our time series is to add quadratic trends to the model by considering not just linear time factors t, but higher order factors and interactions such as t^2, t^3, etc.