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Exercise 4

Solutions for exercises marked with a '$\star$' will be made available online, usually in the following week. Only if needed will these exercises be discussed during the tutorial. All other exercises are to be prepared at home and presented by the participants during the tutorial session.

4.1 Structure of a Singular Value Decomposition (SVD)

$$A_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$$

After importing `numpy` and `numpy.linalg` you can use the command `linalg.svd()` to get the singular value decomposition of A_1. Calculate the SVD of A_1 and familiarize yourself with the dimensions, content and structure of all matrices involved in the SVD, i.e. be able to state their properties.

4.2 Under-/Overdetermined Linear Systems*

Assume two linear systems: the first is underdetermined

$$A_1 \boldsymbol{x}_1 = \boldsymbol{b}_1, \quad \text{with} \quad A_1 = \begin{pmatrix} 5 & 0 & -1 \\ -2 & 2 & 3 \end{pmatrix}, \quad \boldsymbol{b}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

and the second is overdetermined

$$A_2 \boldsymbol{x}_2 = \boldsymbol{b}_2, \quad \text{with} \quad A_2 = \begin{pmatrix} 5 & 0 \\ -2 & 2 \\ 3 & -1 \end{pmatrix}, \quad \boldsymbol{b}_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}.$$

We will solve both systems using singular value decomposition, i.e.

$$A_1 = U_1 S_1 V_1^\top \quad \text{and} \quad A_2 = U_2 S_2 V_2^\top.$$

Write a Python script to solve the Under-/Overdetermined Linear Systems.

Hint: $U_\bullet$ and $V_\bullet$ are orthogonal, i.e. $U_\bullet^{-1} = U_\bullet^\top$ and $V_\bullet^{-1} = V_\bullet^\top$. Also, S_1 and S_2 are non-square matrices and can <u>not</u> be inverted directly. Instead use their *pseudoinverse* via the `numpy` function `pinv()` which is also part of `numpy.linalg`.

4.3 Zeroing

Let matrix A be of size 4×4 with elements

$$A_{ij} = \frac{1}{100 \cdot i + 0.01 \cdot j}.$$

Also, assume a 4×1 vector $\boldsymbol{b}$ with

$$b_i = \sum_{j=1}^{4} A_{ij}.$$

Write a Python script in order to do the following:

1. Compute A, its SVD and its condition number.

   Zero the smallest singular value and calculate the resulting $B = U\bar{S}V^\top$.
2. Compute the ranks of A and B.
3. Solve $A\boldsymbol{x}_1 = \boldsymbol{b}$ and $B\boldsymbol{x}_2 = \boldsymbol{b}$ <u>using the above SVD</u>.
4. Why is it necessary to set the smallest singular value to zero?