Chapter 2: Sets, Functions, and Matrices

Math 275 | Summer 2021

Sets

- A set is an unordered collection of distinct objects.
- A set contains elements.
- $a \in A$ means a is an element of A.
- There are multiple methods for describing set:

Roster Method

- This method lists all the elements of a set using curly brackets.
- $\{a, b, c, d\}$ means a set that has four elements: a, b, c, and d.
- Set that represents vowels of English Vowels: $V = \{a, e, i, o, u\}$
- You can also use ellipses, so that $\{1, 2, \ldots, 99\}$ is equal to the set of positive integers less than 100.

Set Builder Notation

- Has all the properties of that set in the brackets
- General Form: $\{x \mid x \text{ has property } P\}$, means the set of x, so that x has Property P
- Pretty easy to read
- $O = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$, means all positive integers that are and are less 10.

Common Sets

- $\mathbf{N}$, natural numbers, which are positive integers and 0.
- $\mathbf{Z}$, all integers, plus or minus, makes it set of positive or negative integers
- $\mathbf{Q}$, set of all rational numbers
- $\mathbf{R}$, set of all real numbers
- $\mathbf{C}$, set of all complex numbers

Interval Notation

- Describing sets with () and [], and their combinations for interval.
- Self explanatory

Set Equality

- A set is equal to another set iff they have the same elements.
- **Order doesn't matter in a set**, so that $\{3, 4\}$ is the same as $\{4, 3\}$.
- Duplication does not matter in a set, so that $\{3, 3, 4, 4, 4, 5\}$ equals $\{3, 4, 5\}$.

Universal Set and Empty Set

- The universal set U is the set containing everything currently under consideration.
- The empty set is a set with no elements, denoted by $\emptyset$, or $\{\}$.
- Note that $\emptyset$ **IS NOT THE SAME** as $\{\emptyset\}$.
- One is the empty set, another is a set that has the empty set.
- From the book:
  > The empty set can be thought of as an empty folder and the set consisting of just the empty set can be thought of as a folder with exactly one folder inside, namely, the empty folder.

Subsets

- Kinda like the name implies, they are sets within sets.
- Set A is a **subset** of set B, iff every element of A is also an element of B.
  - This means B must be at least the same size of A, or bigger.
- The notation $A \subseteq B$ is used to indicate that A is a subset of B
- {1, 2} is a subset of {1, 2, 3}.
  - {1} and {2} are also subsets of both.
- $\emptyset$ is **always** a subset of ANY set.
- A set is always a subset of *itself*.
  - Basically, $A \subseteq A$ always.

Proper Subsets

- If we have two sets A and B, and $A \subseteq B$,but $A \neq B$,then we say A is a proper subset of B.
- Is denoted with $A \subset B$

Power Sets

- The power set is the set of all subsets of a set.
- Denoted by 2^A or $P(A)$.
- If $A = \{a, b\}$ then $P(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$
- Notice that the power set is NOT the set of each individual element, rather it is the set of all subsets.
  - Basically make sure to have brackets around everything in the power set.

Set Cardinality and Size of Sets

- If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is finite. Otherwise it is infinite.
- The cardinality of a finite set A, denoted by |A|, is the number of (distinct) elements of A.

1. $|\varnothing| = 0$ Empty set has no elements.
2. $|\{1, 2, 3\}| = 3$
3. $|\{\varnothing\}| = 1$ Set of empty set has 1 element.
4. $|\{1, 2, 3\}| = 1$

Notice how that subsets count as one set.

Cartesian Products

The Cartesian Product of sets A and B (denoted with $A \times B$), represents the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

For example:

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

Note that the Cartesian product is NOT commutative, i.e. A x B does not equal B x A.

Set Operations

Basically, operations you can do on sets.

Union

Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$.

Basically adding all elements in both sets to one set.

Intersection

Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 3\}$; that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$.

Two sets are called disjoint if their intersection is the empty set.

Difference

The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

Basically, subtract elements that are in B from A.

Complement

Let U be the universal set. The complement of the set A, denoted by $\bar{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

Set Identities

Identity	Name
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws

Identity	Name
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\bar{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \bar{A} \cup \bar{B}$ $\overline{A \cup B} = \bar{A} \cap \bar{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \bar{A} = U$ $A \cap \bar{A} = \emptyset$	Complement laws

Notice how DeMorgan's law is very similar.

Functions

- Also called mappings and transformations
- For x, there can only be one y.

Injective

- Also called one to one.
- Each x points to a different y.

Surjective

- Means every y, has a corresponding x value.
- $y = x^2$ is NOT surjective, because when $y = -1$, there is no x.
- Also called onto.

Bijective

- Both injective and surjective
- Also called one-to-one correspondence

Inverse of Functions

- A function has an inverse, if when you switch x and y, the new equation is also a function.
- Basically, **iff** the original function is a bijection.

Floor and Ceil

The floor function, denoted $f(x) = \lfloor x \rfloor$ is the largest integer less than or equal to x. The ceiling function, denoted $f(x) = \lceil x \rceil$ is the smallest integer greater than or equal to x.

- Basically for floor, look at the ones decimal place for the answer.
  - As it lowers it to that integer.
- For ceil, add 1 to the ones decimal place for answer.
  - As it increases it to that integer.

Example:

- floor(1/2) = floor(0.5), since ones place is 0, answer is 0.
- ceil(1/2) = ceil(.5), since ones place is 0, we add 1 to get 1, for the answer.
- It's pretty self explanatory from the names, themselves. Floor lowers, ceils increases.
- Floor or ceil of that integer is itself.

Sequences & Summations

- Sequences are ordered lists of elements (usually numbers).
- Summations are when you add all the elements in the sequence.

Empty String

From the book:

> Sequences of the form a1, a2, ... , an are often used in computer science. These finite se- quences are also called strings. This string is also denoted by a1a2 ... an... The length of a string is the number of terms in this string. The empty string, denoted by λ, is the string that has no terms `[think null character in C-strings]`. The empty string has length zero. The string abcd is a string of length four.

Arithmetic Regression

Arithmetic regression has form:

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

Geometric Progression

Geometric progression is a sequence in the form of:

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

Summing these terms gives us a geomettic series: $\sum_{k=0}^{n} ar^k = \dfrac{ar^{n+1} - a}{r - 1}, r \neq 0$

Recurrence Relation

- A recurrence relation for the sequence a_n is an equation that expresses an in terms of one or more of the previous terms of the sequence.
- Think of the Fibonacci Sequence: $f_n = f_{n-1} + f_{n-2}$.
    - Current element defined by sum of two last elements.
- The initial conditions of a recurrence relation define it. That is a recurrence relation together with its initial conditions determines a unique solution.
- A **closed formula** for a sequence, defines in terms of its initial conditions only, and can give you the term at any n, regardless of whether you know its previous terms.

Product Notation

- Just like sigma, but you multiply instead of add, and you use big pi, rather than big sigma: $\prod_{i=a}^{b} f(i)$

On Cardinality and Countability of Infinite Sets

- The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B.
- The set of positive even integers is countably infinite
- The set of (positive and negative) integers is countably infinite
- The set of positive rational numbers is countably infinite
- The set of real numbers is uncountably infinite.

A set is countably infinite if its elements can be put in one-to-one correspondence with the set of natural numbers, otherwise uncountbaly infinite.

Matrices

- Rectangular of numbers

Zero-One Matrix

- A matrix all of whose entries are either 0 or 1 is called a zero–one matrix:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- These 1's and 0's can represent computer bits, and we can treat them as truth values.
    - **1 IS TRUE**
    - **0 IS FALSE**
- Because we are treating them as truth values, we can use the Boolean operations ∧ (and) and ∨ (or).
    - $1 \wedge 1 \equiv 1$
    - $1 \wedge 0 \equiv 0$
    - $1 \vee 1 \equiv 1$
    - etc, basically just T and F, but with 1 and 0
- Using these properties, we can define new operations for zero-one matrices

Join

- The join of two matrices is defined for two zero-one matrices of equal dimensions.

- The join of two matrices A, B, results in a new matrix with equal dimensions, with the *(i, j)*th entry corresponding to $A_{ij} \vee B_{ij}$.
- Basically you look at the entry of the same location for two matrices, and you do OR on them.

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Meet

- The meet is similar to join except it uses ∧ (and) instead of ∨.
- The meet of two matrices is defined for two zero-one matrices of equal dimensions.
- The join of two matrices A, B, results in a new matrix with equal dimensions, with the *(i, j)*th entry corresponding to $A_{ij} \wedge B_{ij}$.
- Basically you look at the entry of the same location for two matrices, and you do AND on them.

Example, same matrices as above, but meet instead of join:

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Boolean Product

- Denoted with: ⊙
- The boolean product of two zero-one matrices is defined for matrices that you can do regular matrix multiplication with.
  - This means if you take the boolean product of $\mathbf{A} \odot \mathbf{B}$, you should be able to do regular matrix multiplication with them.
  - That is number of columns of A should equal number rows of B. (Inner dimensions match)
- You basically do matrix multiplication, but instead of addition you use ∨, and multiplication you use ∧.
- Also like matrix multiplication, the boolean product is NOT COMMUTATIVE.

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ 1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix} = \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$