# CEE570 / CSE 551 Class #31

## Previous Class

### Isoparametric elements (cont'd)

- Mapping of elements
- 4-node bilinear quadrilateral
- Jacobian matrix
- Understanding of [J] and J
- Examples
- Stiffness matrix
- Q8, T3, T6



# This Class

### Numerical integration: Gauss quadrature

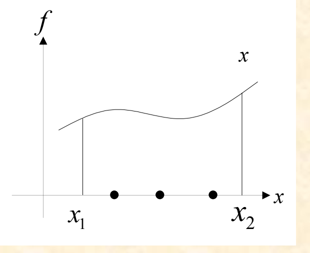
- Introduction to numerical integration and Gauss quadrature
- Gauss quadrature in one dimension (1D)
- Derivation of Gauss points and weights
  (one point, two-point, and n point quadrature)
- FEM example
- Gauss quadrature in 2D
- Exactness of Gauss quadrature in 2D
- Full, reduced and recommended integration in 2D
- Minimum integration order
- Gauss quadrature in 3D

## Introduction

### Numerical integration:

- Evaluating the integrand at specific points
- Multiplying each result by an appropriate weighting factor
- Summing up the results

$$\int_{x_1}^{x_2} f dx \approx W_1 f_1 + W_2 f_2 + \ldots + W_n f_n$$




**The choice of sampling points and weights defines different quadrature schemes.**



## Gauss quadrature (a numerical integration scheme)

- Formulated to compute exact integration for polynomials
- Use fewer sampling points compared to other integration schemes
- Is the most used numerical integration scheme to obtain element stiffness matrix in FEM

## One dimension Gauss quadrature

**Transform of integration limit: from $x_1$ and $x_2$ to -1 and 1 so that the formulas will be generalized**

$$x = \tfrac{1}{2}(1-\xi)x_1 + \tfrac{1}{2}(1+\xi)x_2$$

$$\boxed{\int_{x_1}^{x_2} f dx} \Rightarrow \boxed{\int_{-1}^{1} \phi d\xi}$$

$$\phi = f\left(x(\xi)\right) dx / d\xi$$

**Transformation Jacobian** $J = dx/d\xi = (x_2 - x_1)/2$

**From now on, we need to work only with -1 to 1 limits**



## One dimension Gauss quadrature

**Assume the integrand is a polynomial**

$$\phi(\xi) = a_0 + a_1\xi + a_2\xi^2 + \ldots + a_n\xi^n$$

**Exact integration for each terms:**

$$\int_{-1}^{1} 1.d\xi = 2 \qquad \int_{-1}^{1} \xi^2 d\xi = 2/3 \qquad \int_{-1}^{1} \xi^4 d\xi = 2/5$$

$$\int_{-1}^{1} \xi d\xi = 0 \qquad \int_{-1}^{1} \xi^3 d\xi = 0 \qquad \cdots$$

**Remarks: integration of odd order terms are always zero**

# Deriving sampling points and weights

**Based on the condition that all terms need to be integrated exactly**

**Example First order polynomial**

$$f(\xi) = a_0 + a_1\xi$$

$$\int_{-1}^{1} a_0 d\xi = 2a_0 = \sum_{i=1}^{n} W_i a_0 \xi_i^0$$

$$\int_{-1}^{1} a_1\xi d\xi = 0 = \sum_{i=1}^{n} W_i \xi_i$$

$W_i$ and $\xi_i$

**are weights and coordinates of Gauss points**

**There are two equations;**
**We need at least two unknowns for the solution to exist**
**At least 1 Gauss point needed (n=1)**



# One Gauss point rule (n=1)

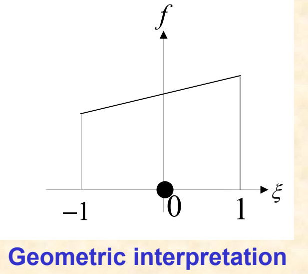
**Use 1 Gauss point: two unknown** $W_1$ and $\xi_1$

$$\int_{-1}^{1} a_0 d\xi = 2a_0 = W_1 a_0 \Rightarrow W_1 = 2$$

$$\int_{-1}^{1} a_1\xi d\xi = 0 = W_1\xi_1 \Rightarrow \xi_1 = 0$$

**Answer : one Gauss point**

$$\boxed{\xi_1 = 0 \quad W_1 = 2}$$




**Geometric interpretation**

## Two Gauss point rule (n=2)

**Example: Third order polynomial**

$$\phi(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$$

$$\int_{-1}^{1} a_0 d\xi = 2a_0 = \sum_{i=1}^{n} W_i a_0 \qquad \int_{-1}^{1} a_1\xi d\xi = 0 = \sum_{i=1}^{n} W_i a_1 \xi_i$$

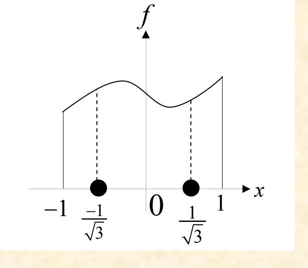
$$\int_{-1}^{1} a_2\xi^2 d\xi = 2/3a_2 = \sum_{i=1}^{n} W_i a_2 \xi_i^2 \qquad \int_{-1}^{1} a_3\xi^3 d\xi = 0 = \sum_{i=1}^{n} W_i a_3 \xi_i^3$$

**There are 4 equations;**
**Need at least 4 unknowns for the solution to exist**
**So, at least 2 Gauss points needed**



## Two Gauss point rule (n=2)

**Unknown:** $W_1, \xi_1, W_2, \xi_2$




$$W_1 + W_2 = 2$$

$$W_1\xi_1 + W_2\xi_2 = 0$$

$$W_1\xi_1^2 + W_2\xi_2^2 = 2/3$$

$$W_1\xi_1^3 + W_2\xi_2^3 = 0$$

**Solving the equations, we have:**

$$W_1 = 1$$

$$W_2 = 1$$

$$\xi_1 = \sqrt{1/3}$$

$$\xi_2 = -\sqrt{1/3}$$

## Three Gauss point rule (n=3)

**Fifth order polynomial**

$$\phi(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + a_5\xi^5$$

$$\int_{-1}^{1} a_0 d\xi = 2a_0 = \sum_{i=1}^{n} W_i a_0 \qquad \int_{-1}^{1} a_1\xi d\xi = 0 = \sum_{i=1}^{n} W_i a_1 \xi_i$$

$$\int_{-1}^{1} a_2\xi^2 d\xi = 2/3a_2 = \sum_{i=1}^{n} W_i a_2 \xi_i^2 \qquad \int_{-1}^{1} a_3\xi^3 d\xi = 0 = \sum_{i=1}^{n} W_i a_3 \xi_i^3$$

$$\int_{-1}^{1} a_4\xi^4 d\xi = 2/5a_4 = \sum_{i=1}^{n} W_i a_4 \xi_i^4 \qquad \int_{-1}^{1} a_5\xi^5 d\xi = 0 = \sum_{i=1}^{n} W_i a_5 \xi_i^5$$

**There are 6 equations;**
**Need at least 6 unknowns for the solution to exist**
**So, at least 3 Gauss points needed**



## Three Gauss point rule (n=3)

**Unknown:** $W_1, \xi_1, W_2, \xi_2, W_3, \xi_3$

$$W_1 + W_2 + W_3 = 2 \qquad W_1\xi_1^3 + W_2\xi_2^3 + W_3\xi_3^3 = 0$$

$$W_1\xi_1 + W_2\xi_2 + W_3\xi_3 = 0 \qquad W_1\xi_1^4 + W_2\xi_2^4 + W_3\xi_3^4 = 2/5$$

$$W_1\xi_1^2 + W_2\xi_2^2 + W_3\xi_3^2 = 2/3 \qquad W_1\xi_1^5 + W_2\xi_2^5 + W_3\xi_3^5 = 0$$

**Solving the equations, we have:**

$$W_1 = 5/9 \qquad \xi_1 = -\sqrt{3/5}$$

$$W_2 = 8/9 \qquad \xi_2 = 0$$

$$W_3 = 5/9 \qquad \xi_3 = \sqrt{3/5}$$

## Remarks on the derivation

**1) Note that the integration for odd order terms are always zero, e.g.**

$$W_1\xi_1 + W_2\xi_2 = 0$$
$$W_1\xi_1^3 + W_2\xi_2^3 = 0$$
$$\ldots$$

**This condition is equivalent to the symmetry of the Gauss points and weights**

**2) In general, polynomials of order *2n-1* is integrated exactly by *n* Gauss point rule**

**Bonus HW: derive the four point rule**

**Bonus HW: derive the five point rule**



## FEM example

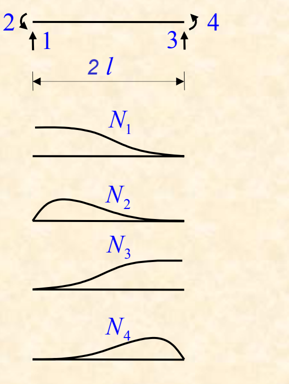
**Beam element** ( X = ξ l )

$$[K] = \int_{length} [B]^T [D][B] dx$$

$$[B] = \tfrac{1}{l^2}[N_{1,\xi\xi} \quad N_{2,\xi\xi} \quad N_{3,\xi\xi} \quad N_{4,\xi\xi}]$$

$[B]$ **Polynomial order 1**

$$[K] = \underbrace{\int_{-1}^{1} [B]^T [EI][B] l d\xi}_{\text{Polynomial order 2 (use two point Gauss quadrature)}}$$



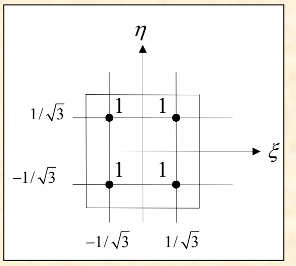

**Recall: cubic shape function**

$$[K] = (1)[B]^T [EI][B] l\Big|_{\sqrt{1/3}} + (1)[B]^T [EI][B] l\Big|_{-\sqrt{1/3}}$$

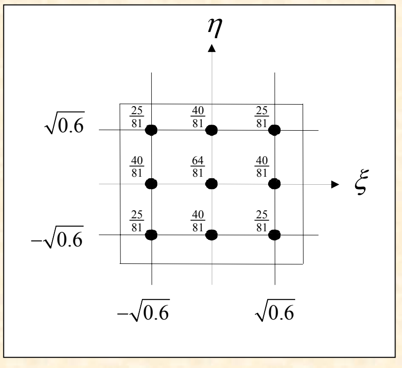
## Gauss quadrature in two dimensions

$$I=\int_{-1}^{1}\int_{-1}^{1}\phi(\xi,\eta)\,d\xi\, d\eta \approx \int_{-1}^{1}\left[\sum_i W_i \phi(\xi_i,\eta)\right]d\eta$$

$$=\sum_j W_j\left[\sum_i W_i \phi(\xi_i,\eta_j)\right]=\sum_j\sum_i W_j W_i \phi(\xi_i,\eta_j)$$




**2 point rule**




**3 point rule**

**m x n rule possible but not recommended**



## Exactness of Gauss quadrature in 2D

$$I=\int_{-1}^{1}\int_{-1}^{1}\xi^l\eta^m\,d\xi\, d\eta$$

**Gauss rule for exact Integration**

*Constant* $(l=m=0)$ $\quad 1$

*Linear* $(l+m=1)$ $\quad \xi \quad \eta$ — **One point**

*Quadratic* $(l+m=2)$ $\quad \xi^2 \quad \xi\eta \quad \eta^2$

*Cubic* $(l+m=3)$ $\quad \xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3$ — **2x2**

*Quartic* $(l+m=4)$ $\quad \xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4$

$\xi^3\eta^2 \quad \xi^2\eta^3$

$\xi^3\eta^3$

## Full, reduced and recommended integration in 2D

Table 6.8-1 Cook at al. 2nd Ed. pp226

| Element | Full | Reduced | Recommended |
|---|---|---|---|
| Q4 | 2x2 | 1 | 2x2 |
| Q8 | 3x3 | 2x2 | 2x2 |
| Q9 | 3x3 | 2x2 | 2x2 (\|J\|=const) 3x3 (\|J\| is not const) |
| Q12 | 4x4 | 3x3 | 3x3 |



## Minimum integration order

In order to pass the patch test, shape function should be able to represent constant strain.

$$\{\varepsilon\} = const$$

$$U_e = \tfrac{1}{2}\int_{V_e}\{\varepsilon\}^T\{\sigma\}\,dV_e = \tfrac{1}{2}\{\varepsilon\}\{\sigma\}\int_{V_e} dV_e = \tfrac{1}{2}U_o\int_{V_e} dV_e$$

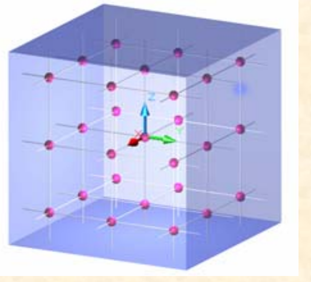
$$U_e = \tfrac{1}{2}U_o\int_{V_e}|J|\,d\xi\, d\eta\, d\varsigma$$

Minimum integration order should be able to compute exactly the volume of element.

## Three dimensions

$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \phi(\xi,\eta,\varsigma)\,d\xi\, d\eta\, d\varsigma$$

$$\approx \sum_i \sum_j \sum_k W_i W_j W_k \phi(\xi_i,\eta_j,\varsigma_k)$$



**3x3x3 Gauss points in 3D**

**Computational cost of 3x3x3 Gauss quadrature for the brick element:**

$\boxed{\text{27 (Gauss points) x 24x24 = 15,552 (function evaluations per element)}}$



## Next class

### Validity of isoparametric element

- Ability to represent rigid body motions
- Generalized Iso-P formulation: GIF
- Graded element and homogeneous element