Introduction to the Finite Element Method

Lecture 18

P.S. Koutsourelakis

pk285@cornell.edu
369 Hollister Hall

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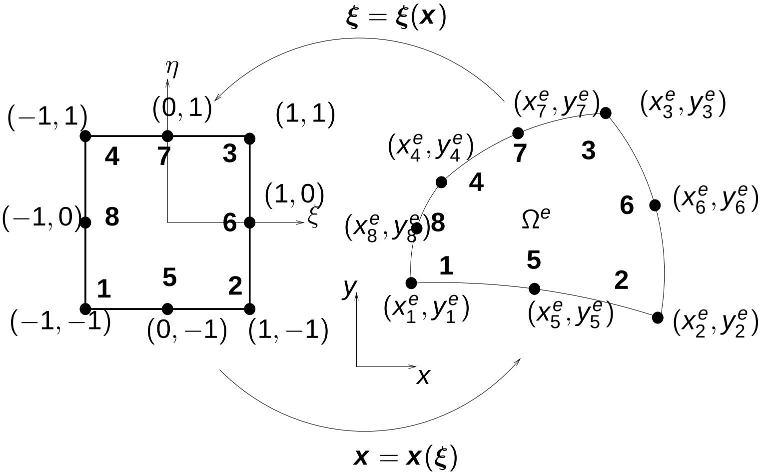
Higher-order isoparametric elements

- In many problems there might be a need to increase the number of shape functions (without increasing the number of elements) and/or represent more accurately the boundary of the domain.
- We discuss some higher order isoparametric elements that achieve this goal.
- There is no free lunch! Every time you increase the number of shape functions you will also increase the number of unknowns and the size of the system you need to solve.

Serendipity Elements



Serendipity Elements

- Serendipity elements consist of additional nodes *along the boundary*
- Observe the numbering!
- Let the mapping be defined as follows:

$$\left\{ \begin{array}{l} x(\xi,\eta) = \sum_{a=1}^{8} N_a(\xi,\eta) x_a^e \\ y(\xi,\eta) = \sum_{a=1}^{8} N_a(\xi,\eta) y_a^e \end{array} \right. \rightarrow \boldsymbol{x}(\boldsymbol{\xi}) = \sum_{a=1}^{8} N_a(\boldsymbol{\xi}) \boldsymbol{x}_a^e$$

- To obtain the functions N_a we first assume a form:

$$x(\xi,\eta) = b_0 + b_1\xi + b_2\eta + b_3\xi\eta + b_4\xi^2 + b_5\eta^2 + b_6\xi^2\eta + b_7\eta^2\xi$$
$$y(\xi,\eta) = c_0 + c_1\xi + c_2\eta + c_3\xi\eta + c_4\xi^2 + c_5\eta^2 + c_6\xi^2\eta + c_7\eta^2\xi$$

  where the parameters $\boldsymbol{b}$ and $\boldsymbol{c}$ should be determined by satisfying:

$$\begin{array}{l} x(\xi_a,\eta_a) = x_a^e \\ y(\xi_a,\eta_a) = y_a^e \end{array} \quad \text{for } a = 1,\ldots,8$$

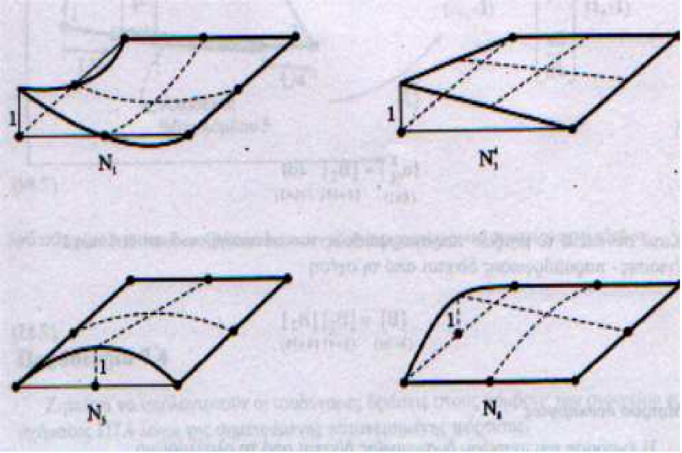
Serendipity Elements

- We obtain the following shape functions:

$$
\begin{aligned}
N_1 &= \tfrac{1}{4}(1-\xi)(1-\eta) - \tfrac{1}{2}(N_8+N_5)\\
N_2 &= \tfrac{1}{4}(1+\xi)(1-\eta) - \tfrac{1}{2}(N_5+N_6)\\
N_3 &= \tfrac{1}{4}(1+\xi)(1+\eta) - \tfrac{1}{2}(N_6+N_7)\\
N_4 &= \tfrac{1}{4}(1-\xi)(1+\eta) - \tfrac{1}{2}(N_7+N_8)\\
N_5 &= \tfrac{1}{2}(1-\xi^2)(1-\eta)\\
N_6 &= \tfrac{1}{2}(1+\xi)(1-\eta^2)\\
N_7 &= \tfrac{1}{2}(1-\xi^2)(1+\eta)\\
N_8 &= \tfrac{1}{2}(1-\xi)(1-\eta^2)
\end{aligned}
$$

- Note that N_1 arises from the *linear* shape function of the 4-node quad (which is equal to $1/2$ at nodes 5 and 8) by subtracting $1/2\ N_5$ and $1/2\ N_8$ in order to become zero at nodes 5 and 8.
- Similarly for N_2, N_3, N_4.

Serendipity Elements



- To find the local stiffness matrix $\boldsymbol{k}^e$:

$$
\begin{aligned}
\boldsymbol{k}^e &= \int_{\Omega^e} (\boldsymbol{B}^e)^T \, \boldsymbol{D} \, \boldsymbol{B}^e \, d\Omega \\
&= \int_{-1}^{1} \int_{-1}^{1} (\boldsymbol{B}^e(\xi,\eta))^T \, \boldsymbol{D} \, \boldsymbol{B}^e(\xi,\eta) \, j(\xi,\eta) \, d\xi \, d\eta
\end{aligned}
$$

- The strain displacement matrix $\boldsymbol{B}^e$:

$$
\underbrace{\boldsymbol{\epsilon}}_{3\times 1} = \underbrace{\boldsymbol{B}^e}_{3\times 16} \underbrace{\boldsymbol{d}^e}_{16\times 1} = \boldsymbol{B}^e \begin{bmatrix} d^e_{x,1} \\ d^e_{y,1} \\ \cdot \\ \cdot \\ d^e_{x,16} \\ d^e_{y,16} \end{bmatrix}
$$

Serendipity Elements

- Let's break $\boldsymbol{B}^e$ in two parts:

$$\epsilon = \begin{bmatrix} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_y}{\partial y} \\ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \end{bmatrix} = \frac{1}{j(\xi,\eta)} \begin{bmatrix} y_{,\eta} & -y_{,\xi} & 0 & 0 \\ 0 & 0 & -x_{,\eta} & x_{,\xi} \\ -x_{,\eta} & x_{,\xi} & y_{,\eta} & -y_{,\xi} \end{bmatrix} \begin{bmatrix} \frac{\partial u_x}{\partial \xi} \\ \frac{\partial u_x}{\partial \eta} \\ \frac{\partial u_y}{\partial \xi} \\ \frac{\partial u_y}{\partial \eta} \end{bmatrix}$$

or:

$$\epsilon = \boldsymbol{B}_1^e \begin{bmatrix} \frac{\partial u_x}{\partial \xi} \\ \frac{\partial u_x}{\partial \eta} \\ \frac{\partial u_y}{\partial \xi} \\ \frac{\partial u_y}{\partial \eta} \end{bmatrix}$$

- and the second part:

$$\begin{bmatrix} \frac{\partial u_x}{\partial \xi} \\ \frac{\partial u_x}{\partial \eta} \\ \frac{\partial u_y}{\partial \xi} \\ \frac{\partial u_y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} N_{1,\xi} & 0 & \dots & \dots & N_{9,\xi} & 0 \\ N_{1,\eta} & 0 & \dots & \dots & N_{9,\eta} & 0 \\ 0 & N_{1,\xi} & \dots & \dots & 0 & N_{9,\xi} \\ 0 & N_{1,\eta} & \dots & \dots & 0 & N_{9,\eta} \end{bmatrix} \boldsymbol{d}^e$$

or:

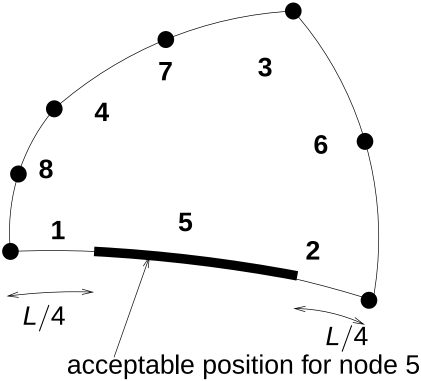
$$\begin{bmatrix} \frac{\partial u_x}{\partial \xi} \\ \frac{\partial u_x}{\partial \eta} \\ \frac{\partial u_y}{\partial \xi} \\ \frac{\partial u_y}{\partial \eta} \end{bmatrix} = \boldsymbol{B}_2^e \, \boldsymbol{d}^e$$

- Hence:

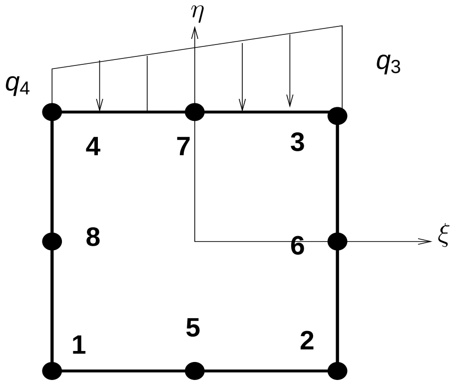
$$\underbrace{\boldsymbol{B}^e}_{3\times 16} = \underbrace{\boldsymbol{B}_1^e}_{3\times 4} \; \underbrace{\boldsymbol{B}_2^e}_{4\times 16}$$

Serendipity Elements

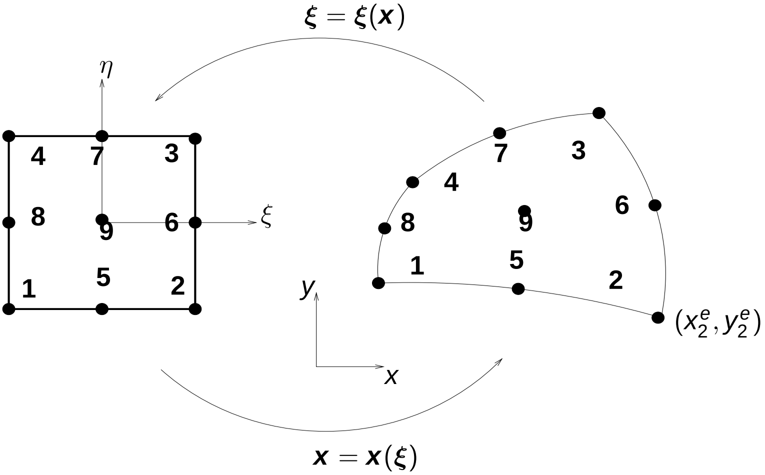
- When is the Jacobian determinant not zero?



- Example: Find local force vector



Lagrange Elements



Lagrange Elements

- *Lagrange elements* consist of additional nodes *in the interior of the element*
- Observe the numbering!
- *Lagrange polynomials*. Suppose one is given n pairs of values $(\xi_i, \phi_i = \phi(\xi_i))$. The the function ϕ can be approximated as follows:

$$\phi \approx L_1\phi_1 + L_2\phi_2 + \ldots + L_n\phi_n$$

where:

$$L_1 = \frac{(\xi_2-\xi)(\xi_3-\xi)\ldots(\xi_n-\xi)}{(\xi_2-\xi_1)(\xi_3-\xi_1)\ldots(\xi_n-\xi_1)}$$
$$L_2 = \frac{(\xi_1-\xi)(\xi_2-\xi)\ldots(\xi_n-\xi)}{(\xi_1-\xi_2)(\xi_3-\xi_3)\ldots(\xi_n-\xi_3)}$$
$$\ldots$$
$$L_n = \frac{(\xi_1-\xi)(\xi_2-\xi)\ldots(\xi_{n-1}-\xi)}{(\xi_1-\xi_n)(\xi_2-\xi_n)\ldots(\xi_{n-1}-\xi_n)}$$

- Note: $L_i(\xi_j) = \delta_{i,j}$

Lagrange Elements

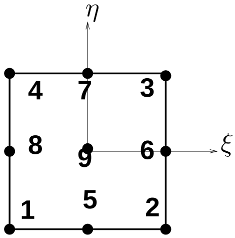
- For example, for $n = 2$, $\xi_1 = -1$, $\xi_2 = +1$:

$$
\begin{aligned}
L_1^{(2)} &= \frac{(\xi_2 - \xi)}{(\xi_2 - \xi_1)} = \frac{1}{2}(1 - \xi) \\
L_2^{(2)} &= \frac{(\xi_1 - \xi)}{(\xi_1 - \xi_2)} = \frac{1}{2}(1 + \xi)
\end{aligned}
$$

- For $n = 3$ and $\xi_3 = 0$:

$$
\begin{aligned}
L_1^{(3)} &= \frac{(\xi_2 - \xi)(\xi_3 - \xi)}{(\xi_2 - \xi_1)(\xi_3 - \xi_1)} = -\frac{1}{2}\xi(1 - \xi) \\
L_2^{(3)} &= \frac{(\xi_1 - \xi)(\xi_3 - \xi)}{(\xi_1 - \xi_2)(\xi_3 - \xi_2)} = \frac{1}{2}\xi(1 + \xi) \\
L_3^{(3)} &= \frac{(\xi_1 - \xi)(\xi_2 - \xi)}{(\xi_1 - \xi_3)(\xi_2 - \xi_3)} = -\frac{1}{2}(1 - \xi^2)
\end{aligned}
$$

Lagrange Elements



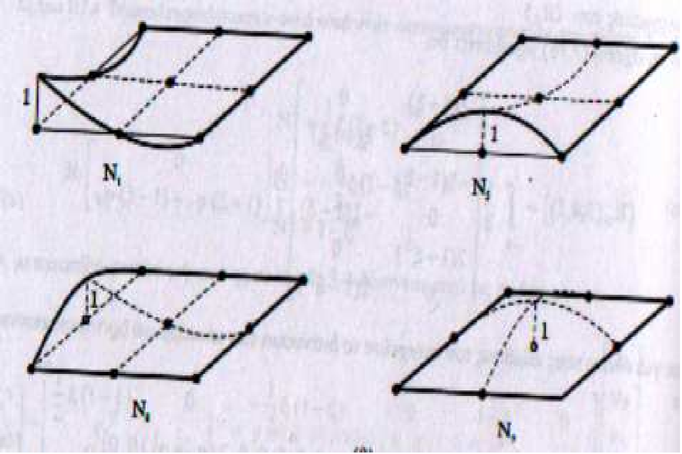
- To generate the shape functions for plane isoparametric elements, it suffices to multiply the Lagrange polynomials w.r.t. to ξ and η of appropriate order. For example, for the 9 node element:

$$N_1(\xi,\eta) = L_1^{(3)}(\xi)\,L_1^{(3)}(\eta) = \tfrac{1}{4}\xi\eta(1-\xi)(1-\eta)$$
$$N_6(\xi,\eta) = L_2^{(3)}(\xi)\,L_3^{(3)}(\eta) = \tfrac{1}{4}\xi(1+\xi)(1-\eta^2)$$

or :

$$N_9(\xi,\eta) = L_2^{(3)}(\xi)\,L_2^{(3)}(\eta) = (1-\xi^2)(1-\eta^2) \quad \text{bubble function}$$

Lagrange Elements



Lagrange Elements

- Naturally, we could derive the same shape functions if we follow the procedure discussed for serendipity elements and use interpolating functions of the form:

$$b_0 + b_1\xi + b_2\eta + b_3\xi\eta + b_4\xi^2 + b_5\eta^2 + b_6\xi^2\eta + b_7\eta^2\xi + b_8\xi^2\eta^2$$

- Conversely, we can use the Lagrange polynomials to re-derive the shape functions of the isoparametric elements we have seen before.
- For example, the 4-node quadrilateral:

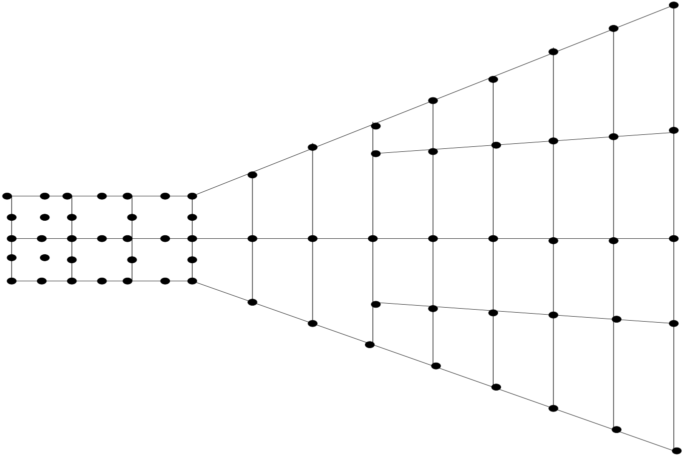
$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta) \quad = \quad L_1^{(2)}(\xi)L_1^{(2)}(\eta)$$

- Or the shape functions of *lateral* nodes of the 8-node serendipity element, e.g.:

$$N_5(\xi,\eta) = L_3^{(3)}(\xi)L_1^{(2)}(\eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$$

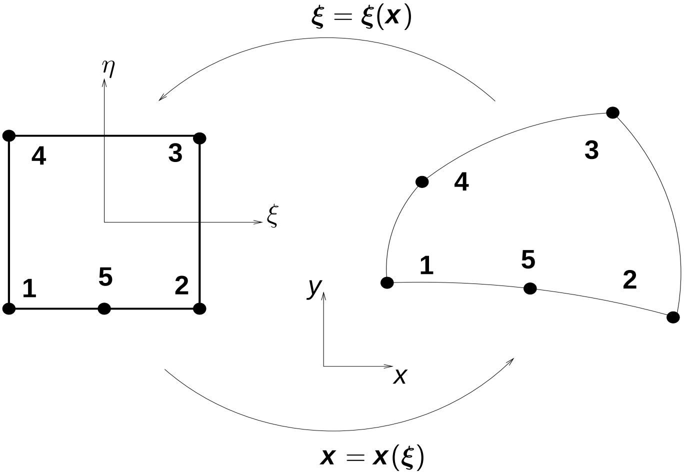
- One has to be more careful for the shape functions of the *corner nodes*!

Isoparametric Quadrilateral Elements with Variable Number of nodes



Isoparametric Quadrilateral Elements with Variable Number of nodes

- 5-node quadrilateral:



Isoparametric Quadrilateral Elements with Variable Number of nodes

- The shape function of *lateral node* can be found from Lagrange polynomials:

$$N_5 = L_3^{(3)}(\xi)\, L_1^{(2)}(\eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$$

- To find the shape functions of the *corner nodes* we use the ones from the 4-node quadrilateral and subtract $1/2N_5$ IF NEEDED so that they become 0 at all other nodes. Hence:

$$\begin{aligned} N_1 &= \tfrac{1}{4}(1-\xi)(1-\eta) - \tfrac{1}{2}N_5 \\ N_2 &= \tfrac{1}{4}(1+\xi)(1-\eta) - \tfrac{1}{2}N_5 \end{aligned}$$

and:

$$\begin{aligned} N_3 &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ N_4 &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned}$$

Summary of Shape functions for Isoparametric Quadrilateral Elements

	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$	$-\frac{1}{2}N_5$			$-\frac{1}{2}N_8$	$\frac{1}{4}N_9$
$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$	$-\frac{1}{2}N_5$	$-\frac{1}{2}N_6$			$\frac{1}{4}N_9$
$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$		$-\frac{1}{2}N_6$	$-\frac{1}{2}N_7$		$\frac{1}{4}N_9$
$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$			$-\frac{1}{2}N_7$	$-\frac{1}{2}N_8$	$\frac{1}{4}N_9$
$N_5 = \frac{1}{2}(1-\xi^2)(1-\eta)$					$-\frac{1}{2}N_9$
$N_5 = \frac{1}{2}(1+\xi)(1-\eta^2)$					$-\frac{1}{2}N_9$
$N_5 = \frac{1}{2}(1-\xi^2)(1+\eta)$					$-\frac{1}{2}N_9$
$N_5 = \frac{1}{2}(1-\xi)(1-\eta^2)$					$-\frac{1}{2}N_9$
$N_9 = (1-\xi^2)(1-\eta^2)$					

The shape functions in columns $i = 5 - 9$ are activated only if the respective node appears in the element and they are added to the appropriate shape function in the first column.