Virtual Element Methods

F. Brezzi

IMATI-C.N.R., Pavia, Italy

IUSS-Istituto Universitario di Studi Superiori, Pavia, Italy

KAU-King Abdulaziz University, Jeddah, Saudi Arabia

From joint works with: L. Beiraõ da Veiga, A. Cangiani G. Manzini, L.D. Marini and A. Russo

Milano, 17-19 September, 2012

Workshop on Polygonal and Polyhedral Meshes

Outline

In FEM the degrees of freedom are used to reconstruct polynomials (or isoparametric images of polynomials) in each element.

Ingredients:

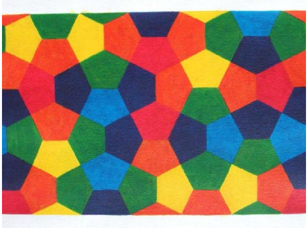
- the geometry of the element (e.g.: triangles)
- the degrees of freedom; say, N d.o.f. per element
- in each element, a space of polynomials of dim. N.

The ingredients must match

- *Unisolvence* N numbers $\leftrightarrow$ one and only one polynomial
- *Continuity*

Virtual Elements on pentagons

Assume now that we want to decompose the domain Ω, for instance, in pentagons, obviously not necessarily regular.



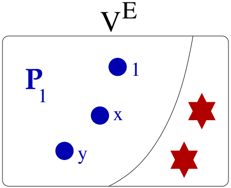
How to take a polynomial space of dimension 5 (e.g., to be associated to the nodal values)?

VEM: We take, as unknowns, the values at the vertices. Then, for every pentagon E and for every fixed set of 5 vertex-values we define the corresponding function (say, φ_h) **first on** ∂E, by: φ_h is linear on each edge.

VEM: We take, as unknowns, the values at the vertices. Then, for every pentagon E and for every fixed set of 5 vertex-values we define the corresponding function (say, φ_h) **first on** ∂E, by: φ_h is linear on each edge. Hence, from the vertex values we have φ_h on the whole ∂E. At this point we decide that all our functions, inside E, should be *harmonic*. Hence, from its 5 nodal values φ_h is first defined on ∂E (by linear interpolation), and from its value on ∂E it is defined inside (by harmonic extension). In summary, the value of φ_h on E is uniquely determined by its nodal values.

Virtual Elements for Laplace Equation

In E we have therefore a local space V^E, of dimension 5, that contains as a **subspace** the space $\mathbb{P}_1$ of polynomials of degree ≤ 1, plus two additional functions that are not polynomials (and that, unfortunately, we cannot compute, or, at least, not in a cheap-enough way).



Classical option: use some *numerical integration formula*.
In VEM, however, we proceed *differently*.

Reminder: implementation of Classic FEM

Find $u \in V \equiv H^1_0(\Omega)$ *s. t.* $-\Delta u = f$. That is:

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f \, v \, d\Omega \quad \forall \, v \in V.$$

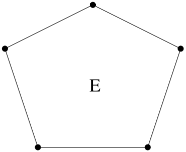
Setting V_h = continuous piecewise linear functions vanishing at the boundary, we look for u_h in V_h such that

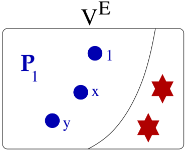
$$\int_\Omega \nabla u_h \cdot \nabla v_h \, d\Omega = \int_\Omega f \, v_h \, d\Omega \quad \forall \, v_h \in V_h.$$

The final matrix is then computed as the sum of the contributions of the single elements:

$$A_{i,j} \equiv \int_\Omega \nabla v^j \cdot \nabla v^i \, d\Omega = \sum_E \int_E \nabla v^j \cdot \nabla v^i \, dE.$$

How to use VEM





$V_h := \{v \in V : \; v \text{ linear on each edge}, -\Delta v = 0 \; \text{ in } E \,\forall E\}$

$$a^E(p_1, v) = \int_E \nabla p_1 \cdot \nabla v \, dE = \int_{\partial E} \frac{\partial p_1}{\partial n} v \, d\ell =: a_h^E(p_1, v)$$

If u is in $\mathbb{P}_1(E)$, then $a^E(u, v)$ can be computed *exactly*. We have then to decide *how to chose* ${a_h}^E(u, v)$ *when both* u *and* v *are* **not** *in* $\mathbb{P}_1(E)$.

Continuous and discretized problem

We consider the continuous problem *Find* $u \in V \equiv H^1_0(\Omega)$ *such that*

$$a(u, v) \equiv \int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f \, v \, d\Omega \quad \forall \, v \in V,$$

and its discretized version: *Find* $u_h \in V_h$ *such that*

$$a_h(u_h, v_h) = (f_h, \ v_h) \quad \forall \, v_h \in V_h,$$

and we look for *sufficient conditions* on a_h that ensure *all the good properties that you would have with standard Finite Elements*

The two basic properties

H1 (Consistency) $a_h^E(p_1, v) = a^E(p_1, v)$

$$\forall E,\ \forall v \in V^E,\ \forall p_1 \in \mathbb{P}_1(E).$$

H2 (Stability) $\exists\, \alpha^*,\ \alpha_* > 0$ such that:

$$\alpha_*\, a^E(v,v) \le a_h^E(v,v) \le \alpha^*\, a^E(v,v) \quad \forall E,\ \forall\, v \in V^E.$$

Under Assumptions **H1** and **H2** the discrete problem has a unique solution. Moreover the **Patch Test** of order 1 is **satisfied**:*on any patch of elements, if the exact solution is a global polynomial of degree* 1*, then the exact solution and the approximate solution* **coincide**.

Incidentally: $\|u - u_h\|_1 = O(h)$.

Convergence Theorem

Theorem

Under the above assumptions **H1** and **H2**, *for every approximation* u_I *of* u *in* V_h *and for every approximation* u_p *of* u *that is piecewise in* $\mathbb{P}_1$, *we have*

$$\|u - u_h\|_V \leq C\Big(\|u - u_I\|_V + \|u - u_p\|_{h,V} + \|f - f_h\|_{(V_h)'}\Big)$$

where

$$\|f - f_h\|_{(V_h)'} := \sup_{v_h \in V_h} \frac{(f, v_h) - (f_h, v_h)}{\|v_h\|_V}$$

Proof of convergence

Set $\delta_h := u_h - u_I$

$$
\begin{aligned}
\alpha_* \, \alpha \|\delta_h\|_V^2 \le \alpha_* \, a(\delta_h, \delta_h) \le a_h(\delta_h, \delta_h) &= a_h(u_h, \delta_h) - a_h(u_I, \delta_h) \\
&= (f_h, \delta_h) - \sum_E a_h^E(u_I, \delta_h) \\
&= (f_h, \delta_h) - \sum_E \Big(a_h^E(u_I - u_p, \delta_h) + a_h^E(u_p, \delta_h) \Big) \\
&= (f_h, \delta_h) - \sum_E \Big(a_h^E(u_I - u_p, \delta_h) + a^E(u_p, \delta_h) \Big) \\
&= (f_h, \delta_h) - \sum_E \Big(a_h^E(u_I - u_p, \delta_h) + a^E(u_p - u, \delta_h) \Big) - a(u, \delta_h) \\
&= (f_h, \delta_h) - \sum_E \Big(a_h^E(u_I - u_p, \delta_h) + a^E(u_p - u, \delta_h) \Big) - (f, \delta_h) \\
&= \; (f_h - f, \delta_h) - \sum_E \Big(a_h^E(u_I - u_p, \delta_h) + a^E(u_p - u, \delta_h) \Big).
\end{aligned}
$$

We construct first on each E an operator Π_1^∇ from V^E into $\mathbb{P}_1(E)$ defined by

$$\sum_{V_i = \text{vertex of } E} (v - \Pi_1^\nabla v)(V_i) = 0 \qquad a^E(v - \Pi_1^\nabla v, p_1) = 0 \; \forall \, p_1$$

Note that $\Pi_1^\nabla p_1 = p_1$ for all p_1 in $\mathbb{P}_1(E)$.

Then we set, for all u and v in V^E

$$a_h^E(u,v) := a^E(\Pi_1^\nabla u, \Pi_1^\nabla v) + S(u - \Pi_1^\nabla u, v - \Pi_1^\nabla v)$$

where the *stabilizing* bilinear form S is (for instance) the Euclidean inner product in $\mathbb{R}^5$.

Proof of **H1** and **H2**

Consistency:

$$a_h^E(p_k, v_h) = a^E(p_k, \Pi^k v_h) = a^E(\Pi^k v_h, p_k) = a^E(p_k, v_h)$$

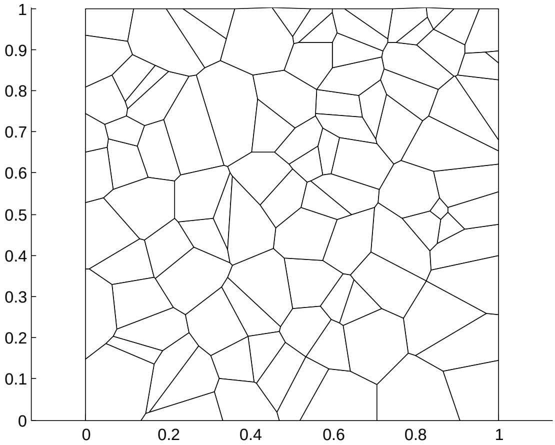
Stability (upper bound):

$$a_h^E(v_h, v_h) \le a^E(\Pi^k v_h, \Pi^k v_h) + c_1 a^E(v_h - \Pi^k v_h, v_h - \Pi^k v_h)$$
$$= a^E(v_h, \Pi^k v_h) + c_1 a^E(v_h - \Pi^k v_h, v_h) \le \alpha^* a^E(v_h, v_h)$$

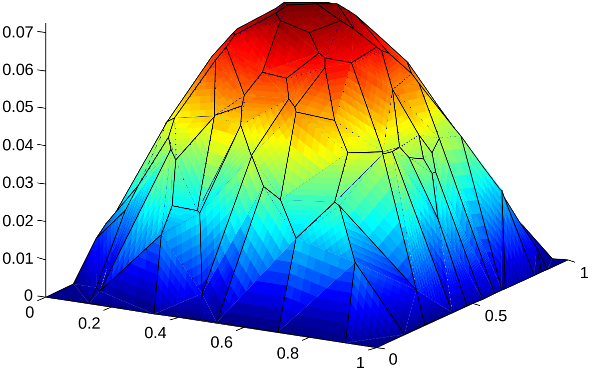
Stability (lower bound):

$$a_h^E(v_h, v_h) \ge a^E(\Pi^k v_h, \Pi^k v_h) + c_0 a^E(v_h - \Pi^k v_h, v_h - \Pi^k v_h)$$
$$\ge \alpha_*(a^E(v_h, \Pi^k v_h) + a^E(v_h - \Pi^k v_h, v_h)) = \alpha_* a^E(v_h, v_h)$$

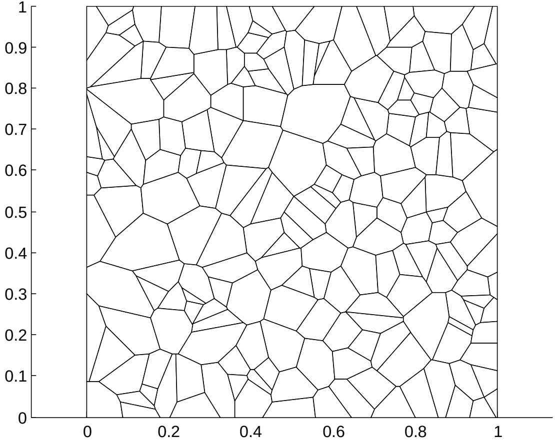
Robustness of the method (by A. Russo)



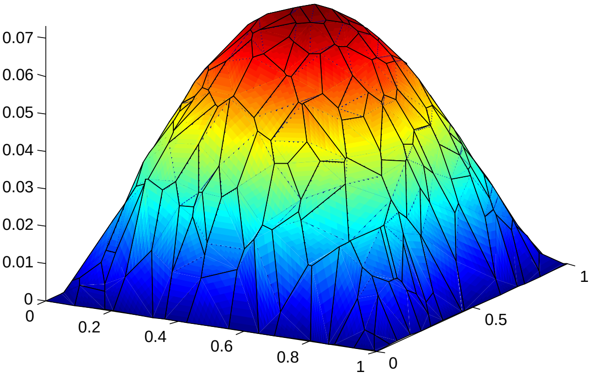
Robustness of the method (by A. Russo)



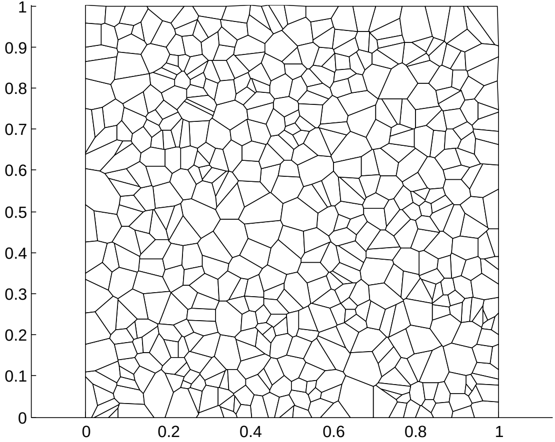
Robustness of the method (by A. Russo)



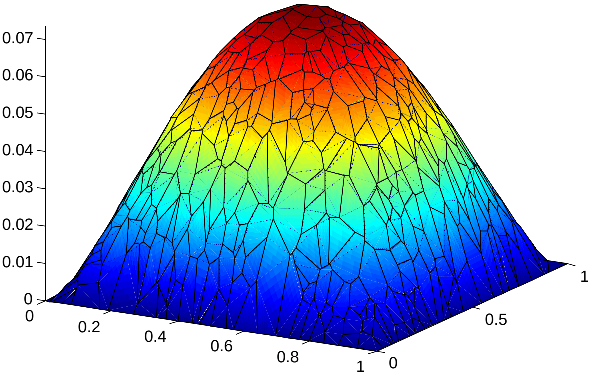
Robustness of the method (by A. Russo)



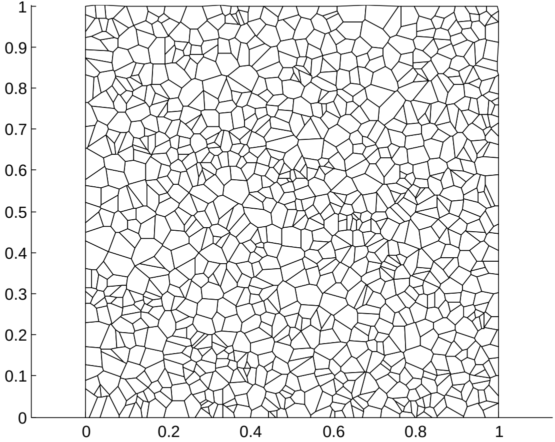
Robustness of the method (by A. Russo)



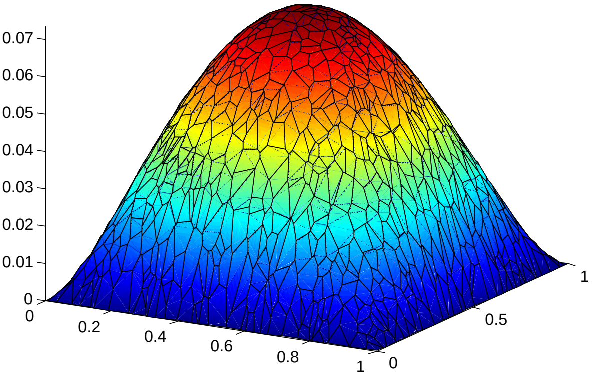
Robustness of the method (by A. Russo)



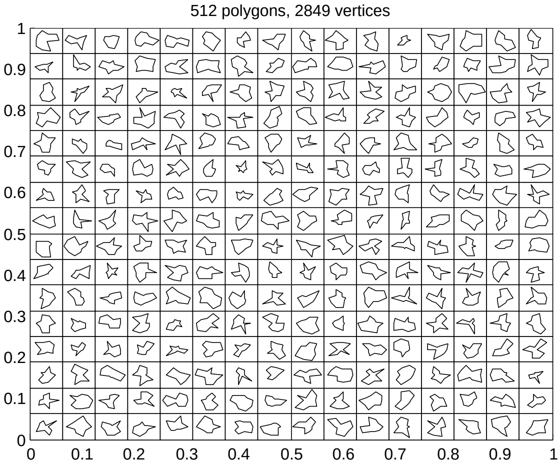
Robustness of the method (by A. Russo)

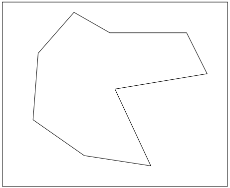


Robustness of the method (by A. Russo)

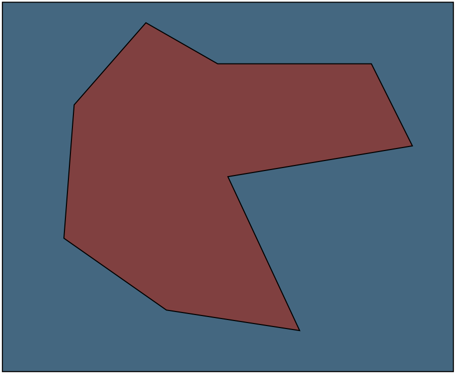


Robustness of the method (by A. Russo)



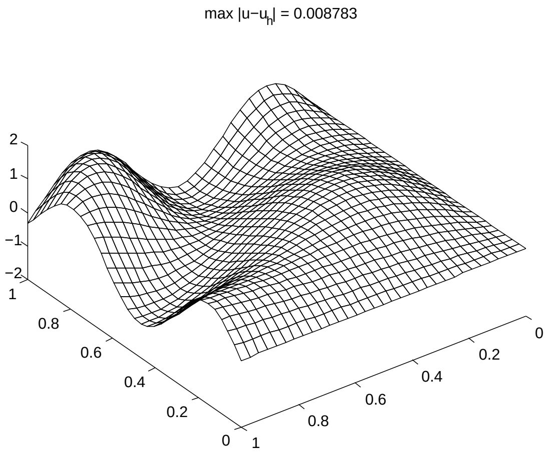


Robustness of the method



Note that the pink element is a polygon with 9 edges, while the blue element is a polygon (not simply connected) with 13 edges. We are exact on linears...

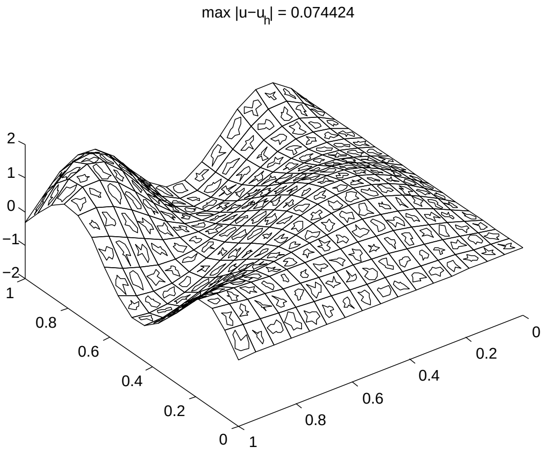
Robustness of the method (by A. Russo)



For reasons of "glastnost", we take as exact solution

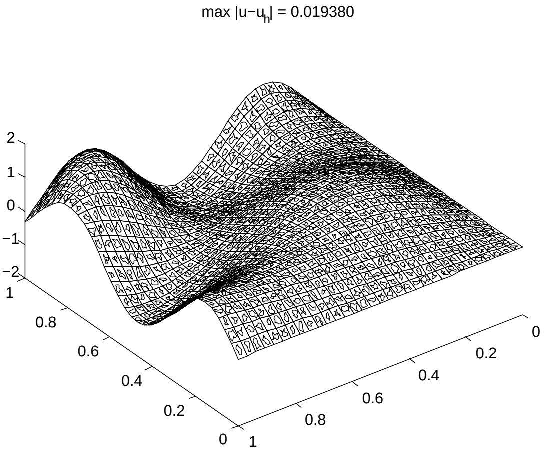
$$w = x(x-0.3)^3(2-y)^2\sin(2\pi x)\sin(2\pi y) + \sin(10xy)$$

Robustness of the method (by A. Russo)



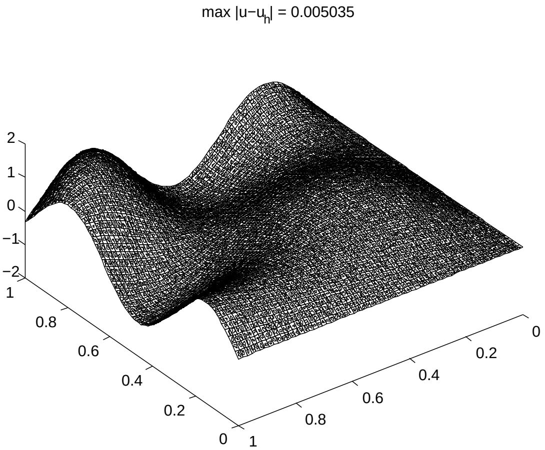
This is on a mesh of 512 (16 $\times$ 16 little squares) elements.

Robustness of the method (by A. Russo)



This is on a mesh of 2048 (32×32 little squares) elements.

Robustness of the method (by A. Russo)

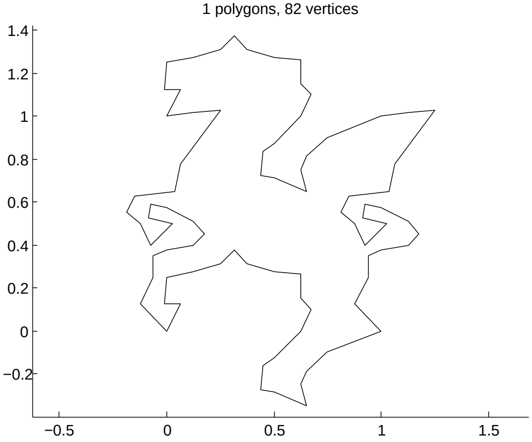


And this is on a mesh of 8192 (64×64 little squares) elements.

The next steps? (by M.C. Escher)

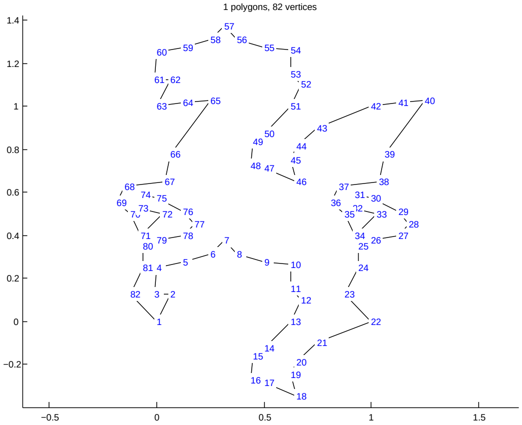
The next steps? (by M.C. Escher)

Going berserk (by A. Russo)



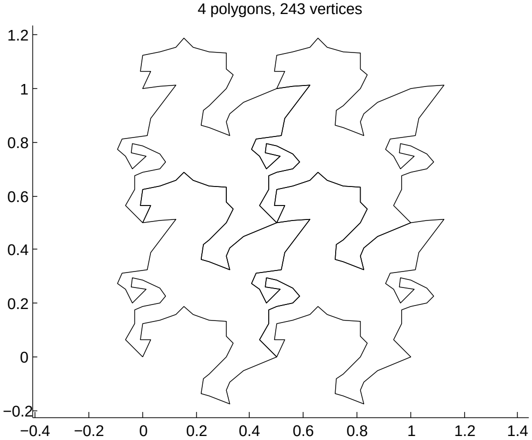
The first step: a pegasus-shaped polygon

Going berserk (by A. Russo)



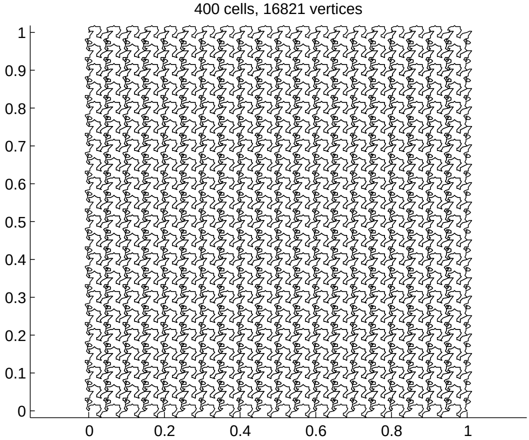
The second step: local numbering of nodes.

Going berserk (by A. Russo)



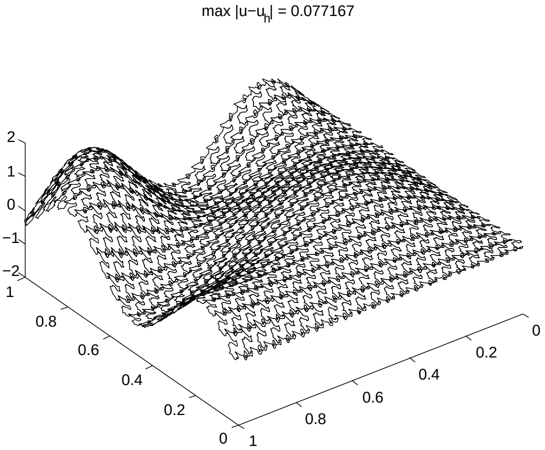
The third step: a mesh of 2×2 pegasus.

Going totally berserk (by A. Russo)



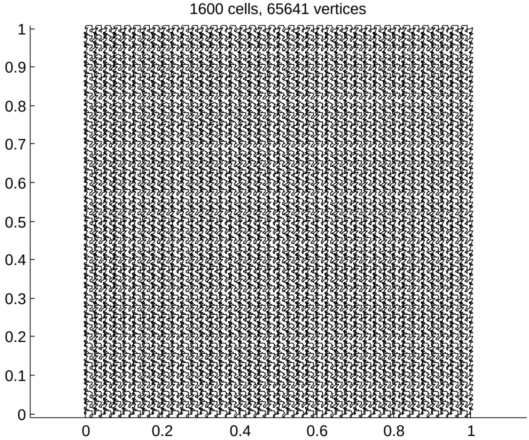
A mesh of 20×20 pegasus.

Going totally berserk (by A. Russo)

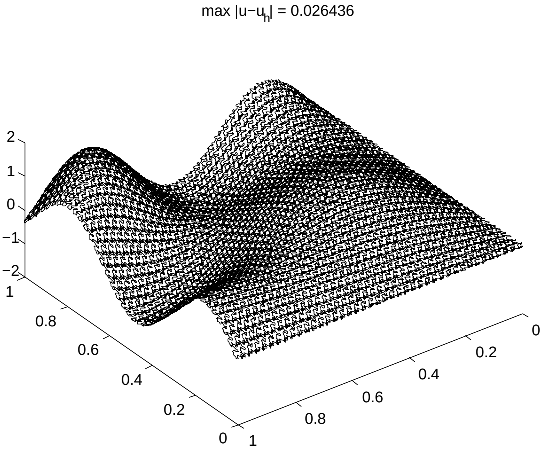


Solution on a 20×20-pegasus mesh.

Going **totally** berserk (by A. Russo)



A mesh of 40×40 pegasus.



Solution on a 40×40-pegasus mesh.

Higher order VEM

$$V_{h,k} := \{v \in V : \; v \text{ of degree } k \text{ on each edge, with } \Delta v \in \mathbb{P}_{k-2} \text{ in } E \;\forall E\}$$

In each E the functions in V^E are identified by

- their value at ∂E,
- (for $k > 1$)the moments up to the order $k-2$ in E

One can prove that these d.o.f. are *unisolvent*.

$$a^E(p_k, v) = \int_E \nabla p_k \cdot \nabla v \, dE$$

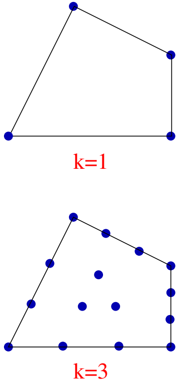
$$= -\int_E \Delta p_k \, v \, dE + \int_{\partial E} \frac{\partial p_k}{\partial n} v \, d\ell =: a_h^E(p_k, v)$$

Higher order VEM

On an element with n edges, the local V^E space of degree k will have:
nk d.o.f. at the boundary and $k(k-1)/2$ internal d.o.f.

Higher order VEM

On an element with n edges, the local V^E space of degree k will have:
nk d.o.f. at the boundary and $k(k-1)/2$ internal d.o.f.



How to satisfy **H1** and **H2**

We construct first on each E an operator Π_k^∇ from V^E into $\mathbb{P}_k(E)$ defined by

$$\int_E (v - \Pi_k^\nabla v)\, dE = 0 \qquad a^E(v - \Pi_k^\nabla v, p_k) = 0\, \forall\, p_k \in \mathbb{P}_1$$

Then we set, for all u and v in V^E

$$a_h^E(u,v) := a^E(\Pi_k^\nabla u, \Pi_k^\nabla v) + S(u - \Pi_k^\nabla u, v - \Pi_k^\nabla v)$$

where, here too, the *stabilizing* bilinear form S is (for instance) the Euclidean inner product in $\mathbb{R}^N$.
The resulting scheme will satisfy a **Patch Test of order** k.

Consider now a (toy) 2d linear elasticity problem, with (unrealistic) homogeneous kinematic boundary conditions all over $\partial\Omega$.

The internal energy, in terms of the displacements $\mathbf{u}$ is

$$\text{Internal Energy} = \mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \mathrm{d}x + \frac{\lambda}{2} \int_\Omega \mathit{div}\mathbf{u}\, \mathit{div}\mathbf{v} \mathrm{d}x$$

where $\boldsymbol{\varepsilon}(\mathbf{u})$ (classical symmetric gradient) is the *strain* tensor and μ and λ are the classical *Lamé coefficients*.

The bilinear form associated with the internal energy is

$$a(\mathbf{u}, \mathbf{v}) = 2\mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \mathrm{d}x + \lambda \int_\Omega \mathit{div}\mathbf{u}\, \mathit{div}\mathbf{v} \mathrm{d}x.$$

The corresponding Green formula is

$$a(\mathbf{u},\mathbf{v}) = -2\mu\int_\Omega (\mathbf{div}(\varepsilon(\mathbf{u}))\cdot\mathbf{v}\mathrm{d}x - \lambda\int_\Omega(\nabla div\mathbf{u})\cdot\mathbf{v}\mathrm{d}x$$
$$+\,2\mu\int_{\partial\Omega}\mathbf{M_n}(\mathbf{u})\cdot\mathbf{v}\mathrm{d}s + \lambda\int_{\partial\Omega} div\mathbf{u}\,(\mathbf{v}\cdot\mathbf{n})\mathrm{d}s$$

where $\mathbf{M_n}$ is a first order trace operator. We also set $A_\mu := -\mathbf{div}\varepsilon$, $A_\lambda := -\nabla div$ and

$$A_{\lambda,\mu} := 2\mu\,A_\mu + \lambda\,A_\lambda.$$

Towards VEM for linear elasticity

$$a^E(\mathbf{u},\mathbf{v}) = -2\mu\int_E \mathbf{div}(\varepsilon(\mathbf{u}))\cdot\mathbf{v}\mathrm{d}x - \lambda\int_E(\nabla\, div\mathbf{u})\cdot\mathbf{v}\mathrm{d}x + 2\mu\int_{\partial E}\mathbf{M}_{\mathbf{n}}(\mathbf{u})\cdot\mathbf{v}\mathrm{d}s + \lambda\int_{\partial E} div\mathbf{u}\,(\mathbf{v}\cdot\mathbf{n})\mathrm{d}s.$$

Assume that our degrees of freedom allow the reconstruction, at the boundary, of vector valued polynomials of degree k. Assume further that we have, as internal degrees of freedom, the moments up to the order $k-2$. Then for every element E, for every $\mathbf{u}\in\mathbf{P}_k$, and for every $\mathbf{v}$ in the discretized subspace the local matrix $a^E(\mathbf{u},\mathbf{v})$ is uniquely computable.

$$V_{h,k} := \{\mathbf{v} \in (H^1_0(\Omega))^2 : \text{ such that } \mathbf{v} \in (\mathbb{P}_k(e))^2 \,\forall\, e,\ A_{\lambda,\mu}\mathbf{v} \in (\mathbb{P}_{k-2}(E))^2\,\forall E\}$$

In each E the vectors $\mathbf{v}$ in V^E are identified by

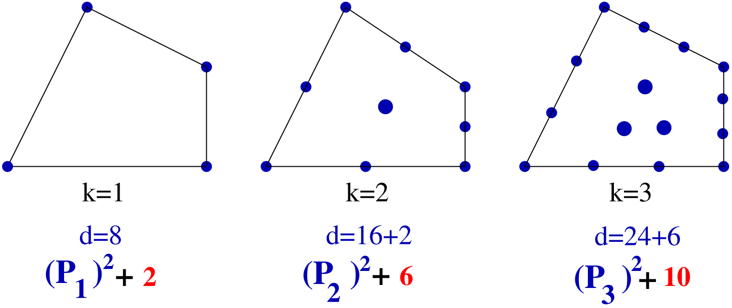
- their values at ∂E,
- (for $k > 1$) the moments up to the order $k-2$ in E

One can prove that these d.o.f. are *unisolvent*. Moreover

$$a^E(\mathbf{p}_k, \mathbf{v}) = -2\mu \int_E \mathbf{div}(\varepsilon(\mathbf{p}_k))\cdot\mathbf{v}\,\mathrm{d}x - \lambda \int_E (\nabla\, div\mathbf{p}_k)\cdot\mathbf{v}\,\mathrm{d}x$$

$$+ 2\mu \int_{\partial E} \mathbf{M}_{\mathbf{n}}(\mathbf{p}_k)\cdot\mathbf{v}\,\mathrm{d}s + \lambda \int_{\partial E} div\mathbf{p}_k\,(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}s$$

$$=: 2\mu\, a^E_{h,\mu}(\mathbf{p}_k, \mathbf{v}) + \lambda\, a^E_{h,\lambda}(\mathbf{p}_k, \mathbf{v}).$$


k=1
d=8
$(P_1)^2 + 2$
k=2
d=16+2
$(P_2)^2 + 6$
k=3
d=24+6
$(P_3)^2 + 10$

Again, we construct on each E a projection operator $\Pi_k^{el}: \ V^E \rightarrow (\mathbb{P}_k(E))^2$ defined by

$$\sum_{vertices} (\mathbf{v} - \Pi_k^{el}\mathbf{v}) = 0, \text{ and } a^E(\mathbf{v} - \Pi_k^{el}\mathbf{v}, \mathbf{p}_k) = 0 \, \forall \, \mathbf{p}_k,$$

and then set, for all $\mathbf{u}$ and $\mathbf{v}$ in V^E

$$a_h^E(\mathbf{u}, \mathbf{v}) := a^E(\Pi_k^{el}\mathbf{u}, \Pi_k^{el}\mathbf{v}) + S(\mathbf{u} - \Pi_k^{el}\mathbf{u}, \mathbf{v} - \Pi_k^{el}\mathbf{v})$$

where the *stabilizing* bilinear form S, once more, is for instance the Euclidean inner product in $\mathbb{R}^N$. The resulting scheme will again satisfy a **Patch Test of order** k

To understand *what to do* for the nearly incompressible case ($\lambda >> \mu$) it is convenient (as usual) to consider the $(\mathbf{u}, p)$ formulation, introducing $p := \lambda\mathbf{u}$ as an additional unknown.
The simplest approach consists in introducing the space Q_h of discontinuous local polynomials of degree $k-1$ (with zero global mean value), and define an operator *Div* from V_h to Q_h by

$$\int_E Div\mathbf{v}_h q_h\,dx = -\int_E \mathbf{v}_h \cdot \nabla q_h\,dx + \int_{\partial E} \mathbf{v}_h \cdot \mathbf{n} q_h\,ds$$

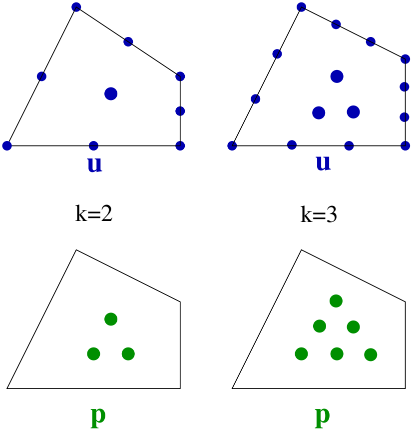
Note that $\nabla q_h \in (\mathbb{P}_{k-2})^2$ and hence $Div\mathbf{v}_h$ is *computable*.

The discrete problem will then be

$$\begin{cases} \textit{Find } (\mathbf{u}_h, p_h) \in V_h \times Q_h \textit{ such that} \\ 2\mu\, a_{h,\mu}(\mathbf{u}_h, \mathbf{v}_h) + (p_h, Div\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in V_h \\ (q_h, Div\mathbf{u}_h) - \dfrac{1}{\lambda}(p_h, q_h) = 0 \qquad \forall\, q_h \in Q_h \end{cases}$$

It is not difficult to check that, for $k \geq 2$, the *inf-sup* condition is satisfied for the operator *Div*, and that this implies optimal order convergence. Then we learn that, in the original formulation, we can write

$$2\mu\, a_{h,\mu}(\mathbf{u}_h, \mathbf{v}_h) + \lambda(Div\mathbf{u}_h, Div\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in V_h$$



These elements could be seen as a generalization of the $(\mathbb{P}_k + \mathbb{B}_{k+1})^2$-(discontinuous $\mathbb{P}_{k-1}$) elements for Stokes.

Plate bending - Kirchhoff-Love

We consider now the Kirchhoff-Love model for the bending of thin plates.

At rest the midsection of the plate occupies the region Ω.

After deformation the (scaled) total energy of the plate could be written as

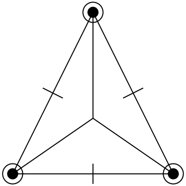
$$\frac{1}{2}\int_\Omega D\Big((1-\nu)\, w_{/ij}w_{/ij} + \nu w_{/ii}w_{/jj}\Big)\, dx - \int_\Omega f\, w\, dx$$

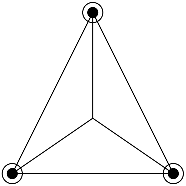
where w=transversal displacement, f=external load density, ν=Poisson ratio, D=elastic coefficient.

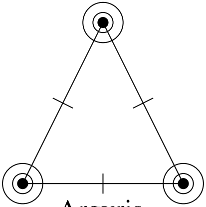
It is well known that the approximation of plate problems (in the K-L formulation) requires the use of C^1 elements.

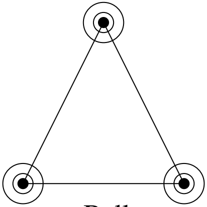
C^1 Finite Elements

There are relatively few C^1 Finite Elements on the market. Here are some:


HCT


reduced HCT


Argyris


Bell

Programming C^1 elements

Cod liver oil (Dorschleberöl)

General idea of VEM for plates

Step 1. *We first choose the degree, r, of our shape functions, and the degree, s, of their normal derivative on each edge.*

Step 2. *We choose d.o.f. such that: i) they determine uniquely v and v_n on ∂E, ii) their restriction on each edge determine uniquely the value of v and v_n on that edge.*

Step 3. *Finally we fix a polynomial degree, k such that $k \leq r$ and $k-1 \leq s$. This will be our order of accuracy.*

Step 4. *If $k \geq 4$ we then add to the boundary degrees of freedom the moments*

$$\int_E v\, q\, dx, \qquad q \in \mathbb{P}_{k-4}(E).$$

Constructing VEM for plates

For r, s, k, with $r \geq k$ and $s \geq k-1$ we set

$$V_h := \{v \in V : \; v \in \mathbb{P}_r(e),\; v_n \in \mathbb{P}_s(e)$$
$$\forall \text{ edge } e \text{ and } \Delta^2 v \in \mathbb{P}_{k-4}(E) \;\forall \text{ element } E\}$$

In each E the functions in V^E are identified by

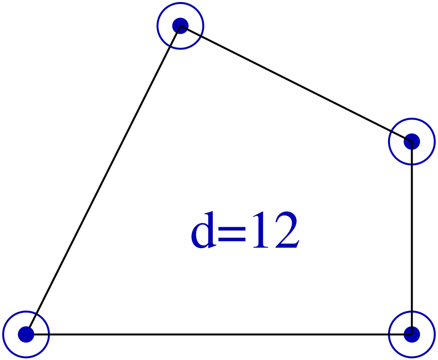
- their value and the value of their derivatives on ∂E,
- (for $k > 3$) the moments up to the order $k-4$ in E

These d.o.f. are *unisolvent*. Moreover, for $p \in \mathbb{P}_k$

$$a^E(v,p) := \int_E D\Big((1-\nu)\, v_{/ij} p_{/ij} + \nu v_{/ii} p_{/jj}\Big)\, dx$$

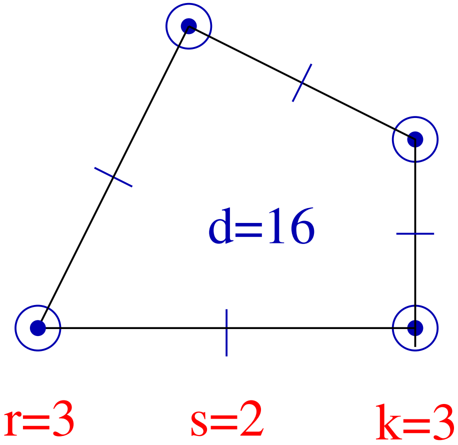
$$= D \int_E v\, \Delta^2 p\, dx + \int_{\partial E} \nabla v \cdot \mathbf{M_n}(p) - v\, Q_n(p)\, d\ell =: a_h^E(v,p)$$

Example 1

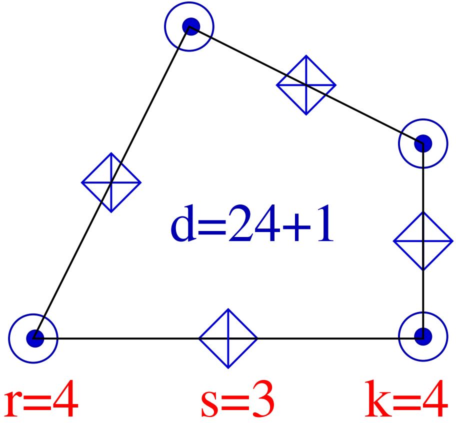


r=3 s=1 k=2

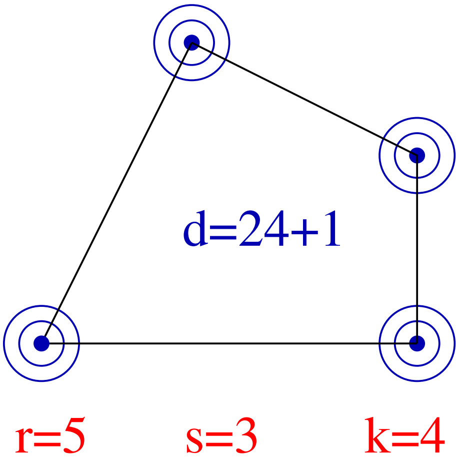
Example 2



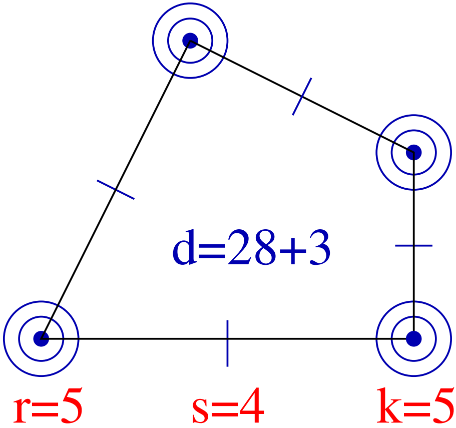
Example 3



Example 4

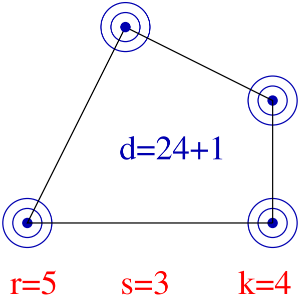


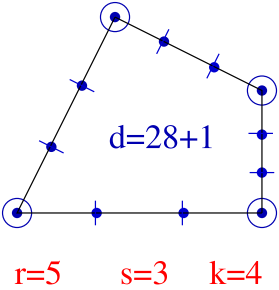
Example 5



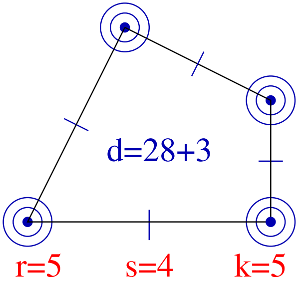
Example 6

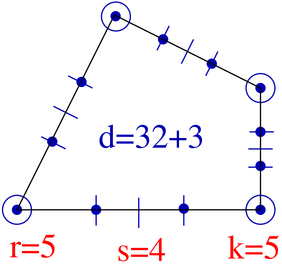
For the same r, s, and k we might use different degrees of freedom.





For the same r, s, and k we might use different degrees of freedom.





We construct on each E a projection operator Π_k^M from V^E into $\mathbb{P}_k(E)$ defined by

$$\sum_{vertices}(v-\Pi_k^M v)=0,\ \sum_{vertices}\nabla(v-\Pi_k^M v)=0 \text{ and}$$

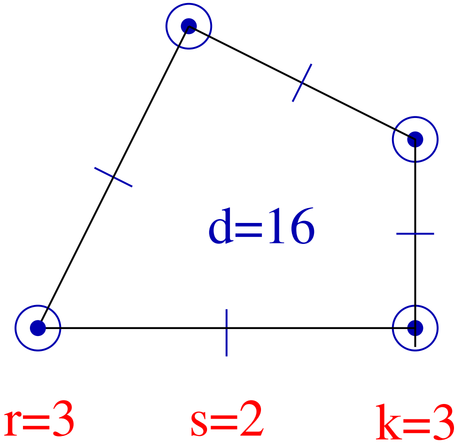
$$a^E(v-\Pi_k^M v, p_k)=0\,\forall\, p_k,$$

and then set, for all u and v in V^E

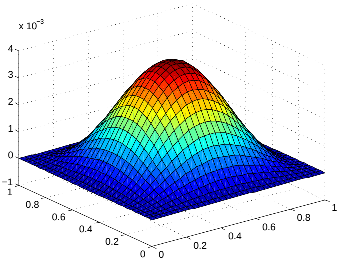
$$a_h^E(u,v):=a^E(\Pi_k^M u,\Pi_k^M v)+S(u-\Pi_k^M u, v-\Pi_k^M v)$$

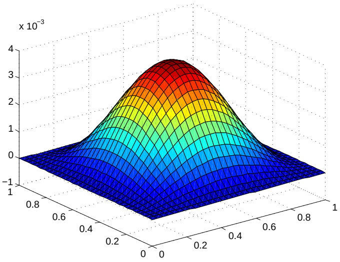
where the *stabilizing* bilinear form is again (for instance) the Euclidean inner product in $\mathbb{R}^N$. The resulting scheme will again satisfy a **Patch Test of order k**

Numerical experiments on the 3-2 element



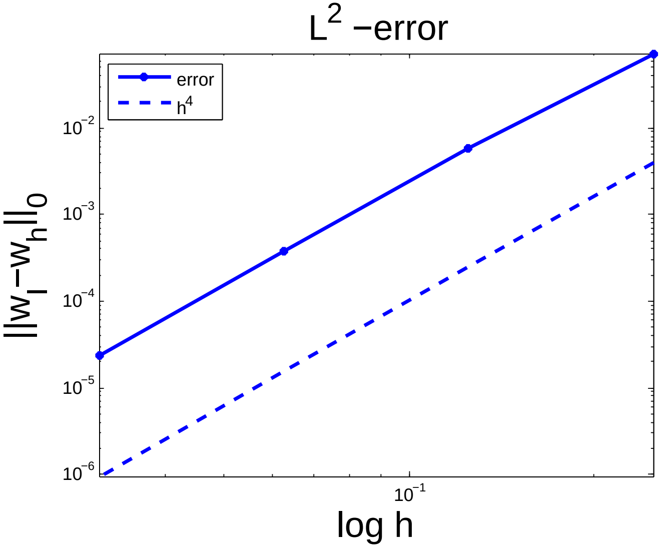
Exact and approximate solution



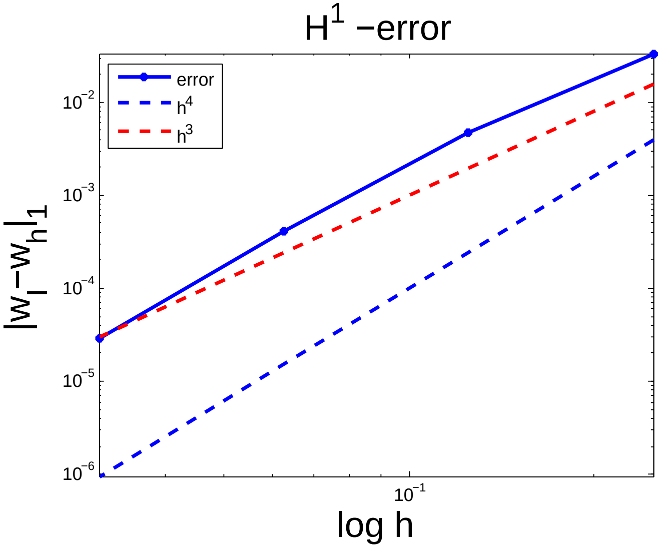


Exact solution (left): $w = x^2(1-x)^2y^2(1-y)^2$ on the unit square $]0,1[\times]0,1[$. The approximate solution is computed with the $r = 3$, $s = 2$, $k = 3$ element on a grid of uniform 32×32 square (BLUSH!) elements.

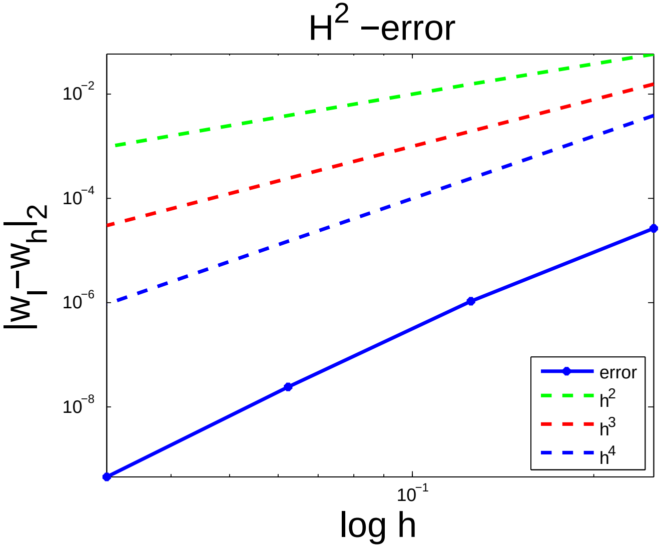
Behaviour of the L^2 error



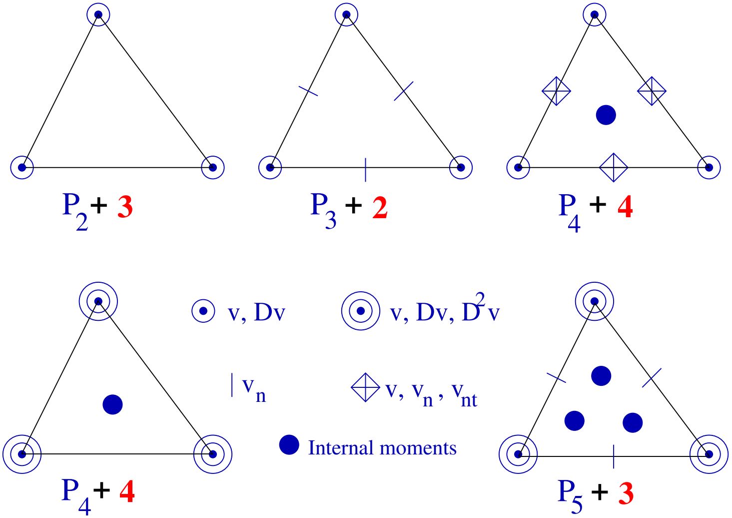
Behaviour of the H^1 error



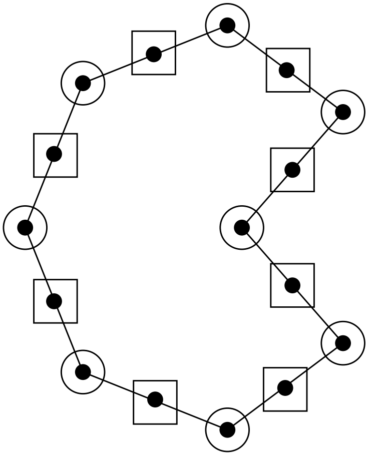
Behaviour of the H^2 error



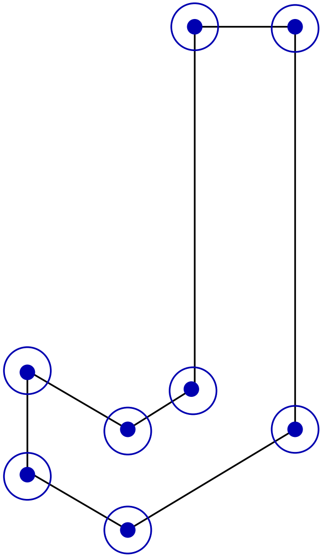
VEM on Triangles



VEM on general shapes



VEM on general shapes



Conclusions

• VEM preserve all the *good features* of MFD, but are much more simple and more elegant.

• In some cases, they could be seen "just" as a generalization to more general geometries of classical FEM

• In other cases, they offer additional possibilities. This is for instance the case for general quadrilaterals, hanging nodes, and C^k elements with $k \geq 1$.

• They are almost *newborn*. To assess their true interest for engineering computations still requires a *long long way* to go.