Generalized Barycentric Coordinates

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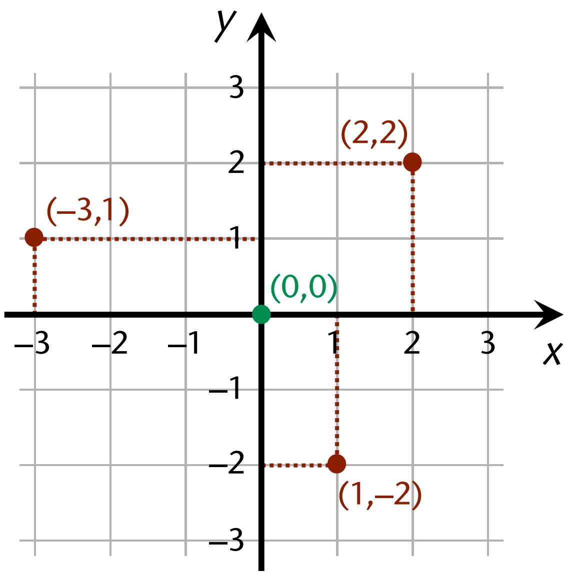
Cartesian coordinates

René Descartes (1596–1650)



point (2,2) with

- x-coordinate: 2
- y-coordinate: 2

mathematically:

$$(2,2) = 2 \cdot (1,0) + 2 \cdot (0,1)$$

in general:

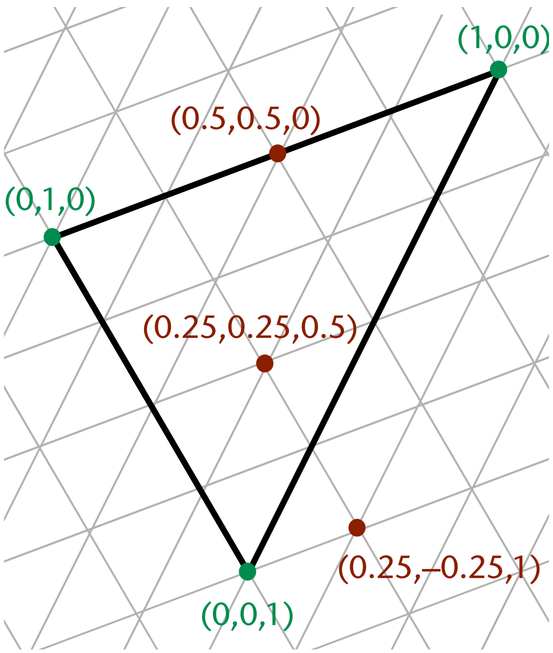
$$(x,y) = x \cdot (1,0) + y \cdot (0,1)$$

x- and y-coordinates w.r.t. ***base points***

(1,0) and (0,1)

Barycentric coordinates

August Ferdinand Möbius
(1790–1868)



point (a,b,c) with
3 coordinates w.r.t.
base points A, B, C

mathematically:

$$\begin{aligned}(a,b,c) = \quad & a \cdot A \\ + & b \cdot B \\ + & c \cdot C\end{aligned}$$

where

$$A = (1,0,0)$$
$$B = (0,1,0)$$
$$C = (0,0,1)$$

and

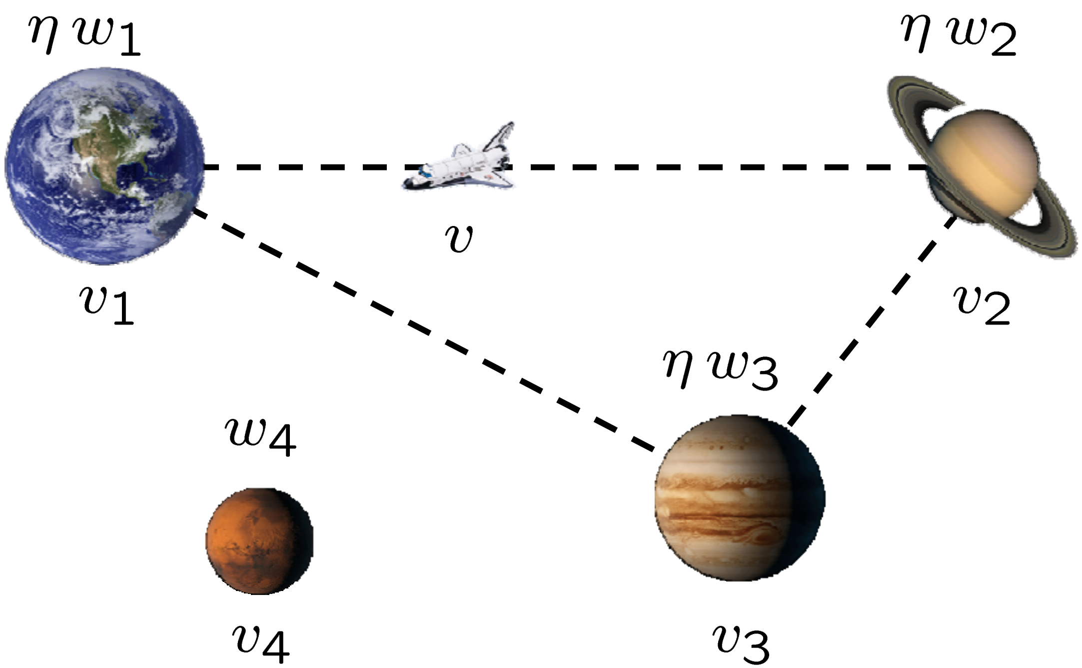
$$a + b + c = 1$$

Barycentric coordinates

- system of masses w_i at positions v_i
- position of the system's ***barycentre***:

$$\boxed{v = \frac{\sum_i w_i v_i}{\sum_i w_i}}$$

- w_i are the ***barycentric coordinates*** of v
- ***not unique***
- at least $d+1$ points needed to span $\mathbb{R}^d$



Barycentric coordinates

- **Theorem** [Möbius, 1827] :

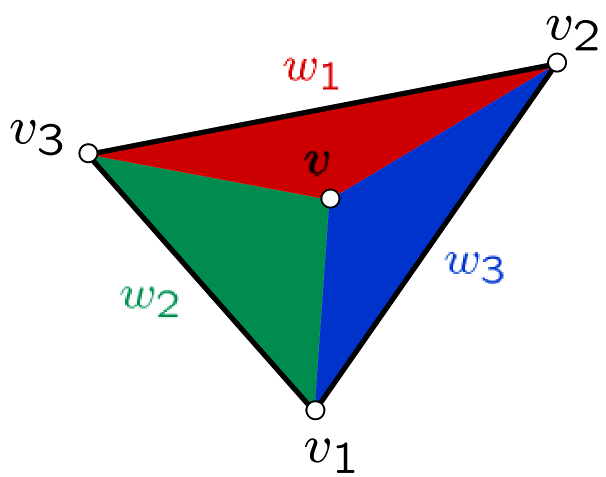
  The barycentric coordinates $w_1, \ldots, w_{d+1}$ *of* $v \in \mathbb{R}^d$ *with respect to* $v_1, \ldots, v_{d+1}$ *are* ***unique*** *up to a common factor*

- example: $d = 2$

$$v = \frac{w_1 v_1 + w_2 v_2 + w_3 v_3}{w_1 + w_2 + w_3}$$

$$\Longleftrightarrow$$

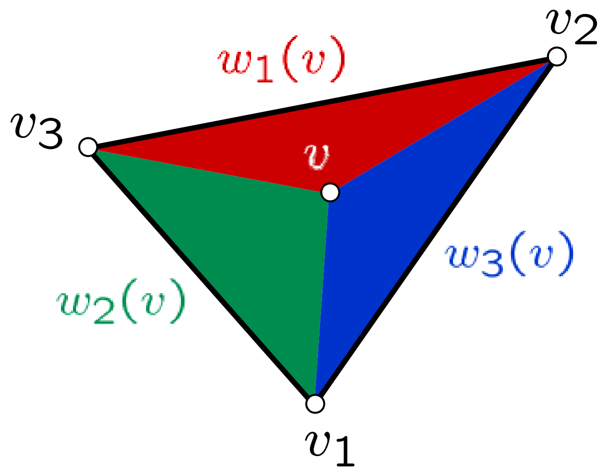
$$w_i = \eta A(v, v_{i+1}, v_{i+2})$$



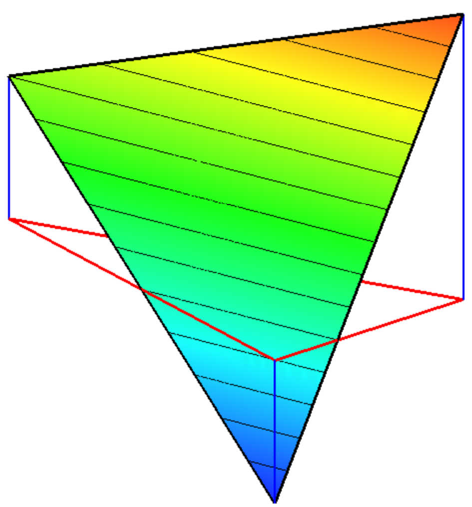
Barycentric coordinates for triangles

- normalized barycentric coordinates

$$b_i(v) = \frac{w_i(v)}{w_1(v) + w_2(v) + w_3(v)}$$



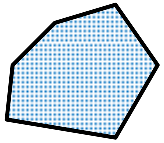
- properties
  - partition of unity $\sum_i b_i(v) = 1$
  - reproduction $\sum_i b_i(v) v_i = v$
  - positivity $b_i(v) > 0, \quad v \in \mathring{\triangle}$
  - Lagrange property $b_i(v_j) = \delta_{ij}$

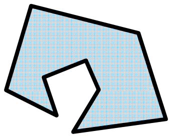


- application
  - linear interpolation of data $F(v) = \sum_{i=1}^{3} b_i(v) f_i$

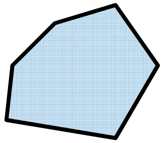
Generalized barycentric coordinates

- finite-element-method with polygonal elements
  - convex [Wachspress 1975]
  - weakly convex [Malsch & Dasgupta 2004]
  - arbitrary [Sukumar & Malsch 2006]



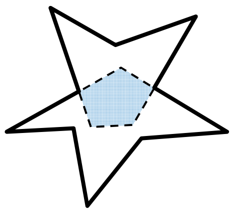


- interpolation of scattered data
  - *natural neighbour interpolants* [Sibson 1980]
  - – " – of higher order [Hiyoshi & Sugihara 2000]
  - Dirichlet tessellations [Farin 1990]

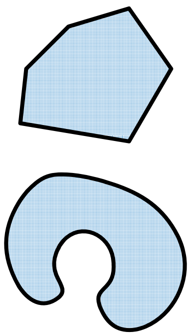


Generalized barycentric coordinates

- parameterization of piecewise linear surfaces
  - *shape preserving* coordinates [Floater 1997]
  - *discrete harmonic* (DH) coordinates [Eck et al. 1995]
  - *mean value* (MV) coordinates [Floater 2003]



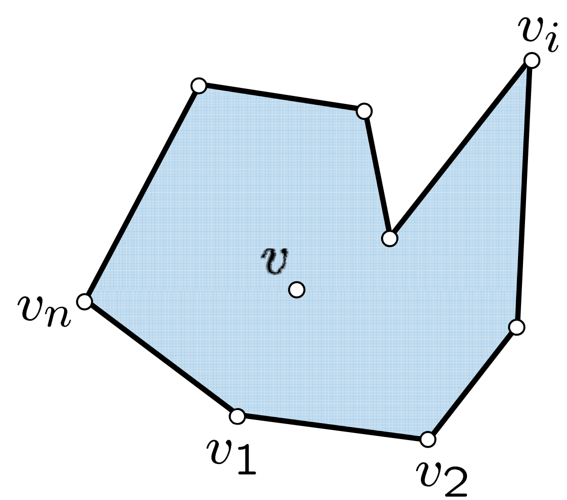
- other applications
  - discrete minimal surfaces [Pinkall & Polthier 1993]
  - colour interpolation [Meyer et al. 2002]
  - boundary value problems [Belyaev 2006]



Arbitrary polygons

- barycentric coordinates $w_1(v), \ldots, w_n(v)$

$$v = \frac{\sum_{i=1}^n w_i(v) v_i}{\sum_{j=1}^n w_j(v)}$$



- normalized coordinates

$$b_i(v) = \frac{w_i(v)}{\sum_{j=1}^n w_j(v)}$$

- properties
  - partition of unity $\sum_{i=1}^n b_i(v) = 1$
  - reproduction $\sum_{i=1}^n b_i(v) v_i = v$

linear precision

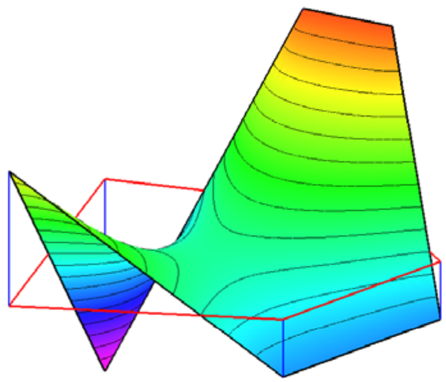
$$\left.\begin{array}{l} \sum_{i=1}^n b_i(v) = 1 \\ \sum_{i=1}^n b_i(v) v_i = v \end{array}\right\} \Rightarrow \sum_{i=1}^n b_i(v)\phi(v_i) = \phi(v)$$

for all $\phi \in \pi_1$

Convex polygons

[Floater, H. & Kós 2006]

- **Theorem:** If all $w_i(v) > 0$, then
  - positivity $b_i(v) > 0$
  - Lagrange property $b_i(v_j) = \delta_{ij}$
  - linear along boundary $b_i|_{[v_i,v_{i+1}]} \in \pi_1$

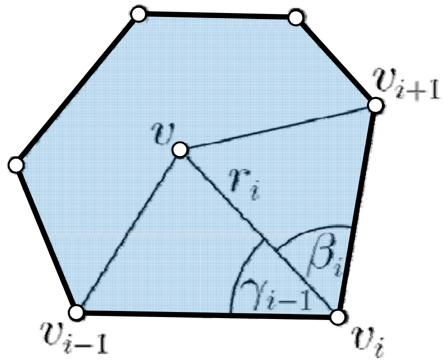


- application
  - interpolation of data given at the vertices $$F(v) = \sum_{i=1}^{n} b_i(v) f_i$$
  - $F(v)$ inside the convex hull of the f_i
  - direct and efficient evaluation

Examples

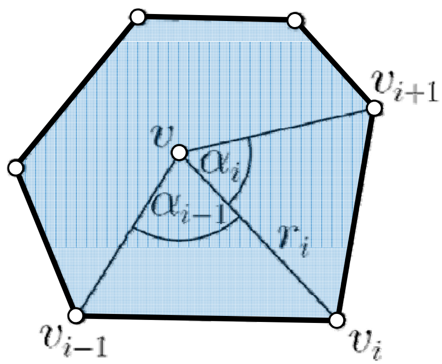
- ***Wachspress*** (WP) coordinates

$$w_i = \frac{\cot \gamma_{i-1} + \cot \beta_i}{{r_i}^2}$$



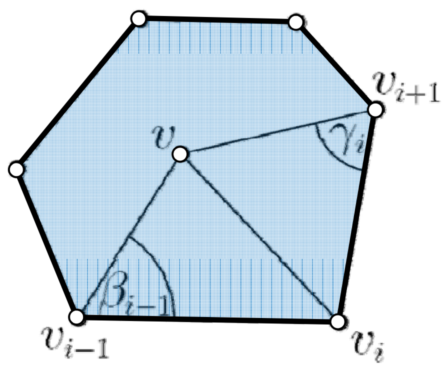
- ***mean value*** (MV) coordinates

$$w_i = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{r_i}$$



- ***discrete harmonic*** (DH) coordinates

$$w_i = \cot \beta_{i-1} + \cot \gamma_i$$



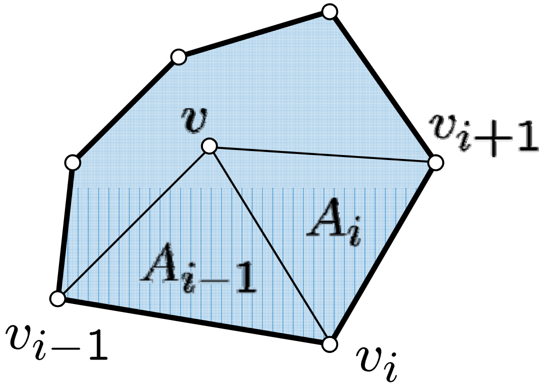
Normal form

[Floater, H. & Kós 2006]

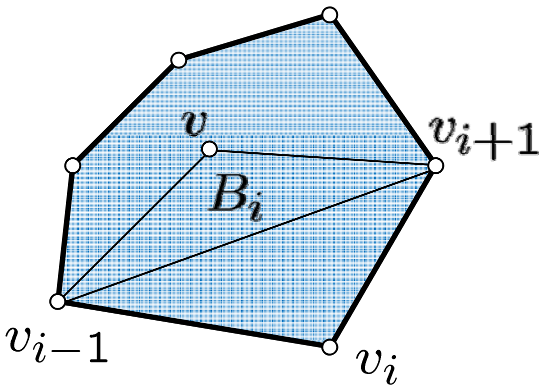
- **Theorem:** All barycentric coordinates can be written as

$$w_i = \frac{c_{i+1}A_{i-1} - c_i B_i + c_{i-1}A_i}{A_{i-1}A_i}$$

with certain real functions c_i



- *three-point coordinates*
  - $c_i = f(r_i)$ with $r_i = \|v - v_i\|$



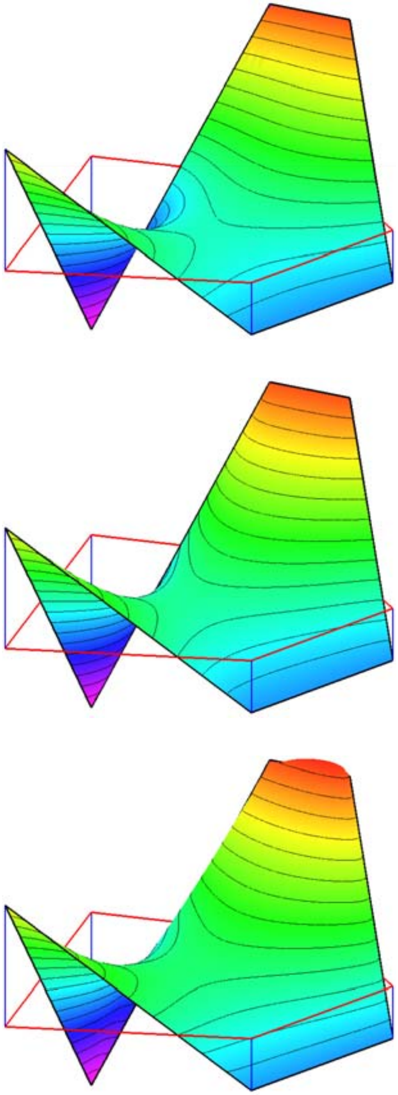
- **Theorem:** Such a generating function

$$f\colon \mathbb{R}^+ \to \mathbb{R}$$

exists for all three-point coordinates

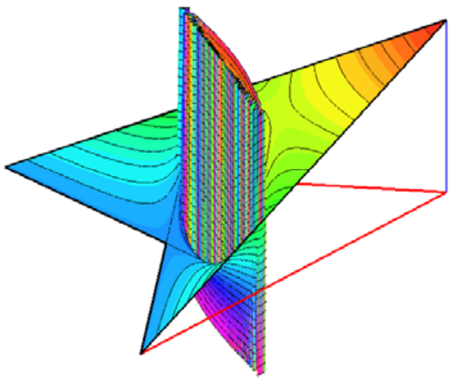
Three-point coordinates

- **Theorem:** $w_i(v) > 0$ if and only if f is
  - positive $f(r) > 0$
  - monotonic $f'(r) \geq 0$
  - convex $f''(r) \geq 0$
  - sub-linear $f'(r) \leq f(r)/r$
- examples
  - WP coordinates $f(r) = 1$
  - MV coordinates $f(r) = r$
  - DH coordinates $f(r) = r^2$



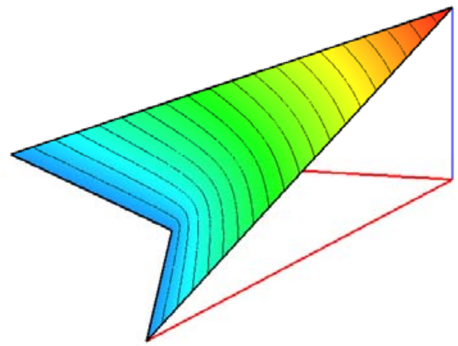
Non-convex polygons

Wachspress



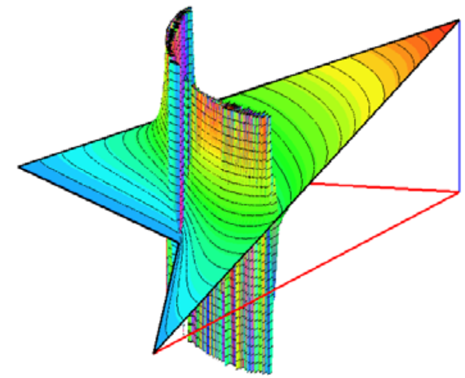
$f(r) = 1$

mean value



$f(r) = r$

discrete harmonic

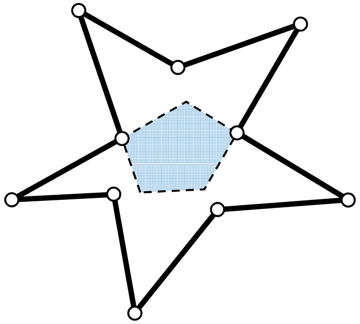


$f(r) = r^2$

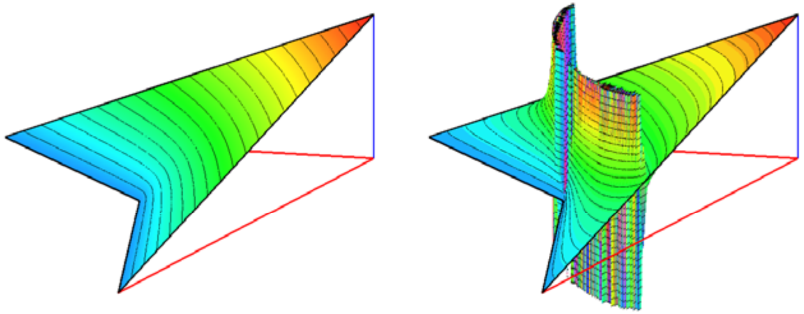
- poles, if $W(v) = \sum_{j=1}^{n} w_j(v) = 0$, because $b_i(v) = \dfrac{w_i(v)}{W(v)}$

Star-shaped polygons

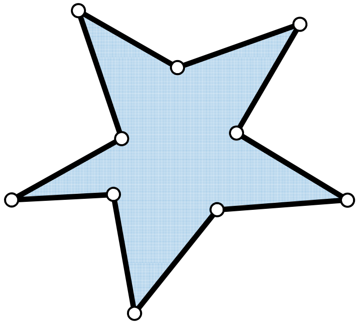
- **Theorem:** $W(v) \neq 0$ if and only if f is
  - positive $f(r) > 0$
  - super-linear $f'(r) \geq f(r)/r$



- examples
  - MV coordinates $f(r) = r$
  - DH coordinates $f(r) = r^2$



- **Theorem:** $W(v) = 0$ for some v if f is
  - strictly super-linear $f'(r) > f(r)/r$

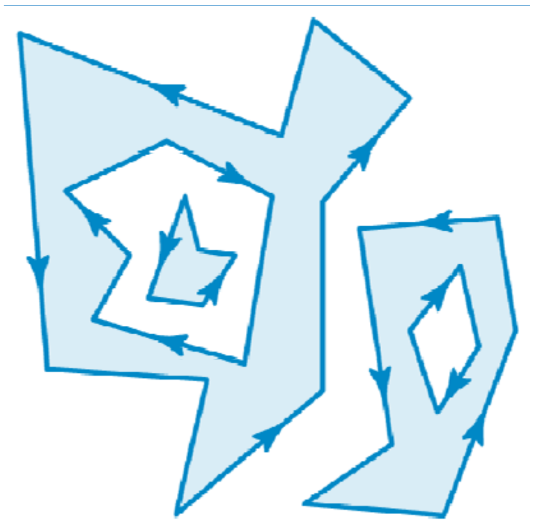


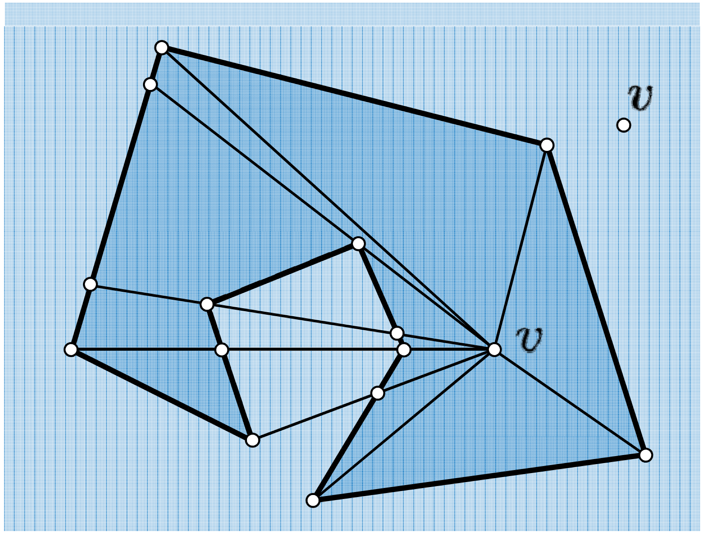
Mean value coordinates

[H. & Floater 2006]

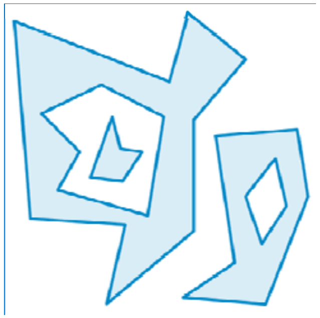
- **Theorem:** MV coordinates have no poles in $\mathbb{R}^2$

$$W(v) = \sum w_j(v) = \sum \kappa_i(v) \neq 0$$





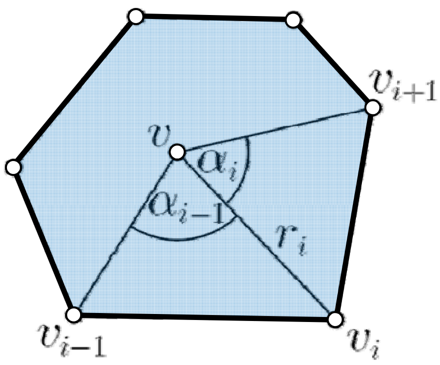
Mean value coordinates



- properties
  - well-defined everywhere in $\mathbb{R}^2$
  - Lagrange property $b_i(v_j) = \delta_{ij}$
  - linear along boundary $b_i|_{[v_i,v_{i+1}]} \in \pi_1$
  - linear precision $\sum_i b_i(v)\phi(v_i) = \phi(v)$ for $\phi \in \pi_1$
  - smoothness C^0 at v_i, otherwise C^∞
  - similarity invariance $b_i = \widehat{b}_i \circ \psi$ for $\widehat{\Omega} = \psi(\Omega)$
- application
  - direct interpolation of data $F(v) = \sum_{i=1}^{n} b_i(v) f_i$

Implementation

- Mean Value coordinates



$$w_i = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{r_i}$$

$$\tan(\alpha_i/2) = \frac{\sin \alpha_i}{1 + \cos \alpha_i} = \frac{r_i r_{i+1} \sin \alpha_i}{r_i r_{i+1} + r_i r_{i+1} \cos \alpha_i}$$

$$= \frac{\det(s_i, s_{i+1})}{r_i r_{i+1} + \langle s_i, s_{i+1} \rangle} = t_i$$

$$\boxed{s_i = v_i - v}$$

$$w_i = \frac{t_{i-1} + t_i}{r_i}$$

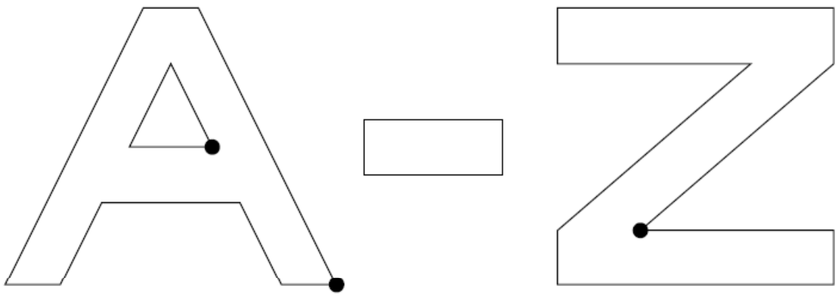
Implementation

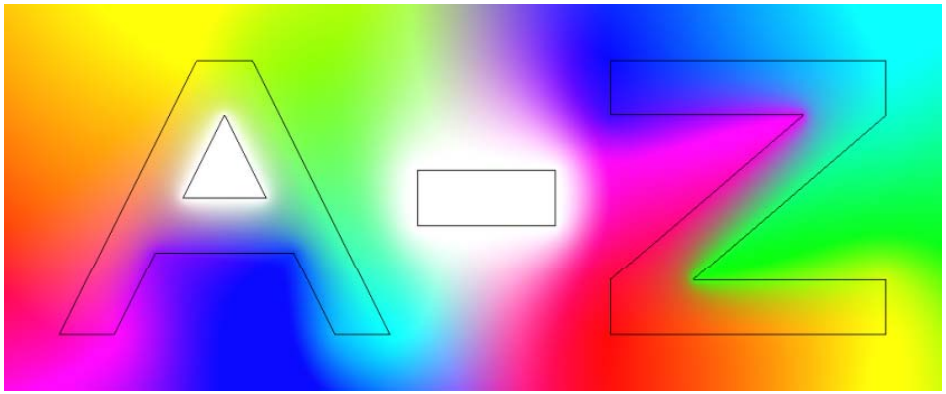
function $F(v)$

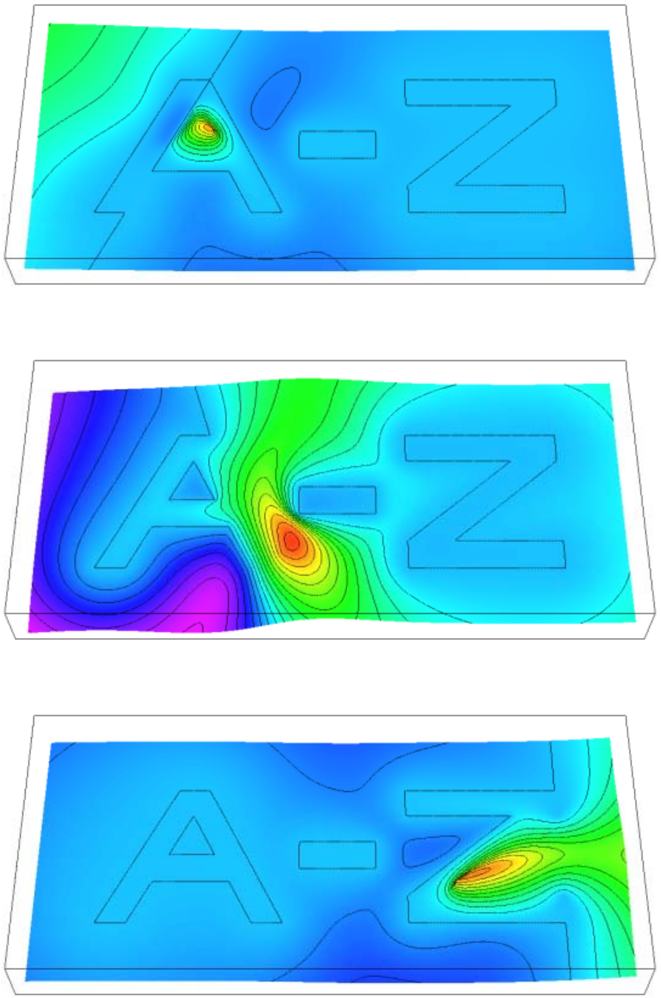
for $i = 1$ to n do
  $s_i := v_i - v$
  $r_i := \|s_i\|$
  if $r_i = 0$ then $// \; v = v_i$
    return f_i
for $i = 1$ to n do
  $A_i := \det(s_i, s_{i+1})$
  $D_i := \langle s_i, s_{i+1} \rangle$
  if $A_i = 0$ and $D_i < 0$ then $// \; v \in e_i$
    return $(r_{i+1} f_i + r_i f_{i+1}) / (r_i + r_{i+1})$
for $i = 1$ to n do
  $t_i := A_i / (r_i r_{i+1} + D_i)$
$f := 0$
$W := 0$
for $i = 1$ to n do
  $w := (t_{i-1} + t_i) / r_i$
  $f := f + w f_i$
  $W := W + w$
return f/W

- efficient and robust evaluation of the function $F(v)$

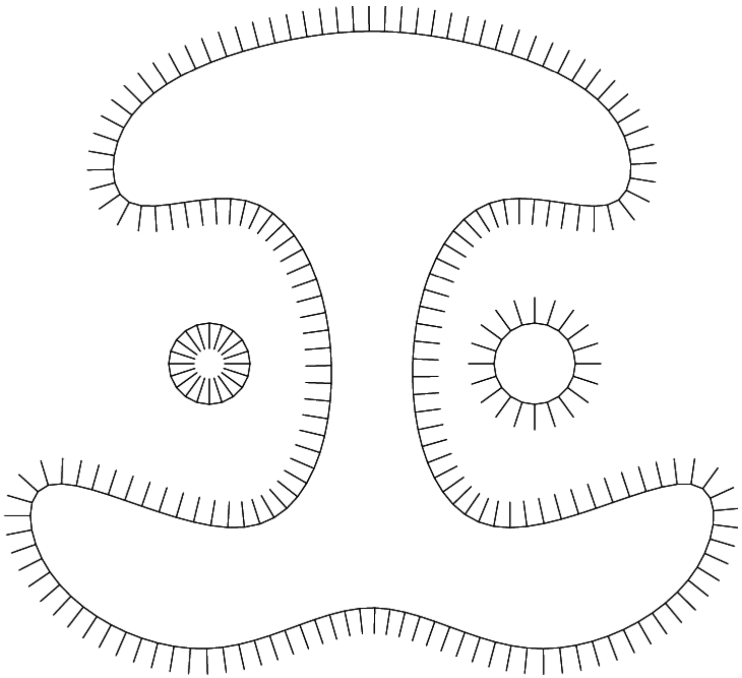
Colour interpolation

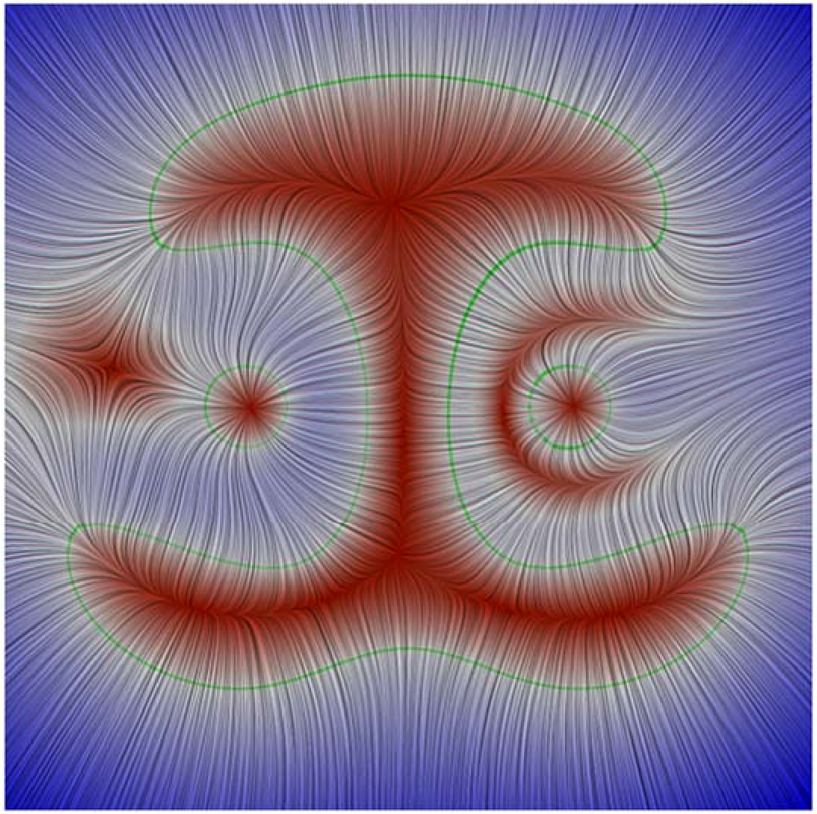




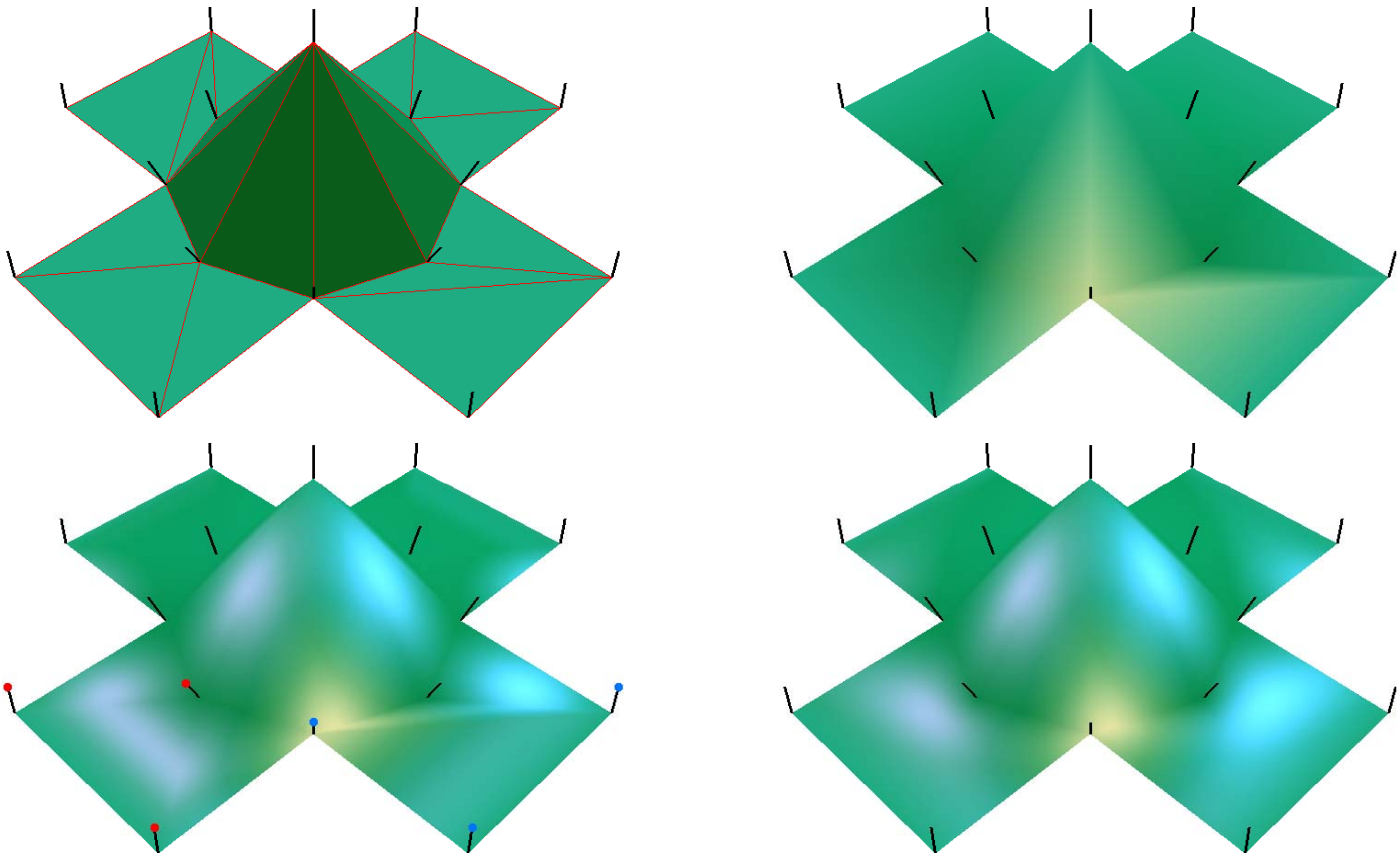


Vector fields

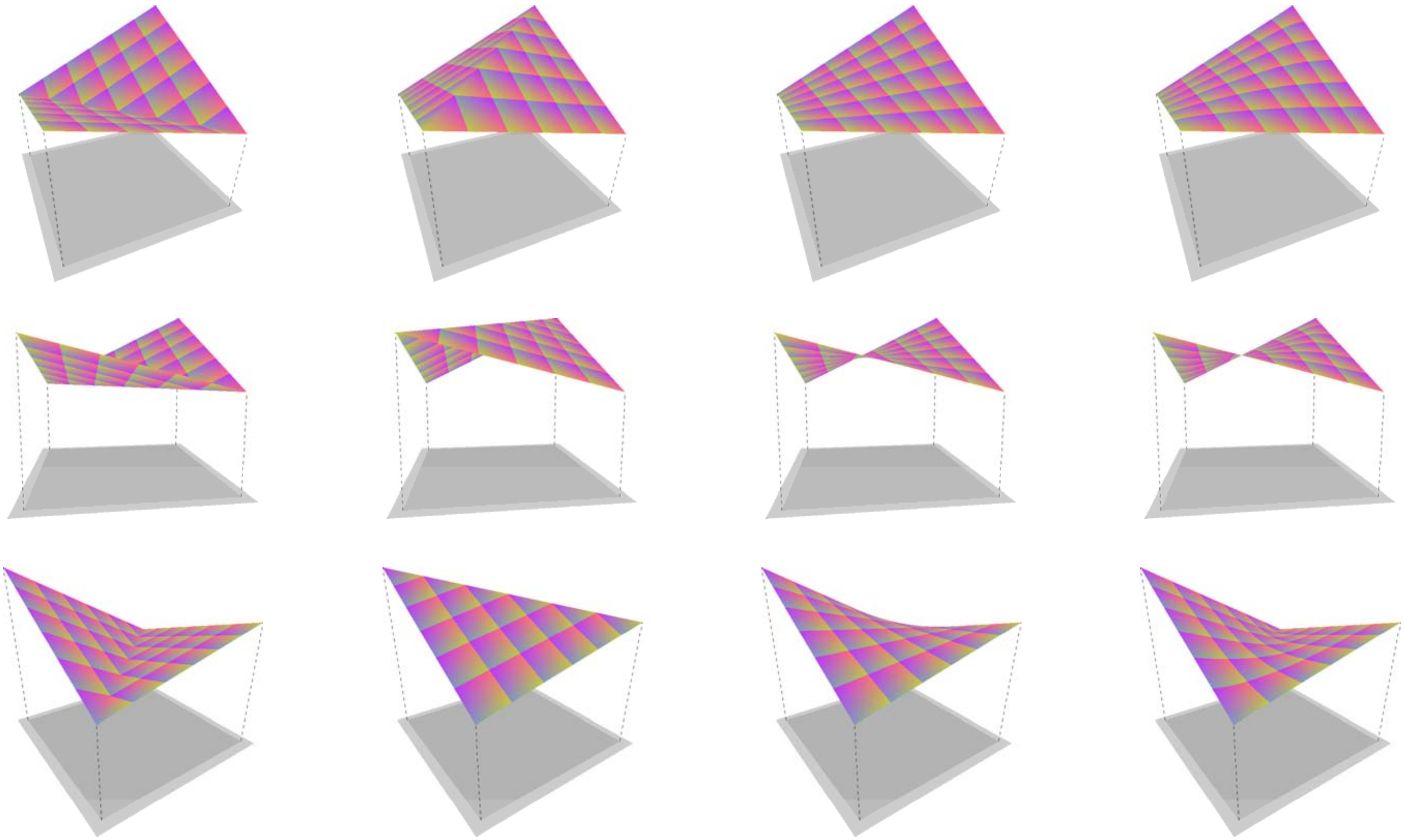




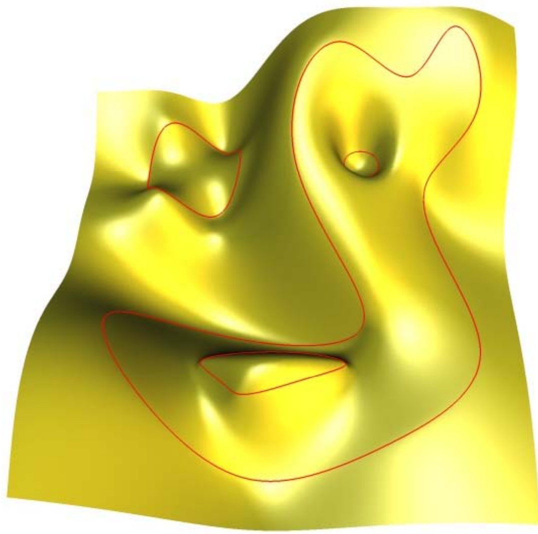
Smooth shading



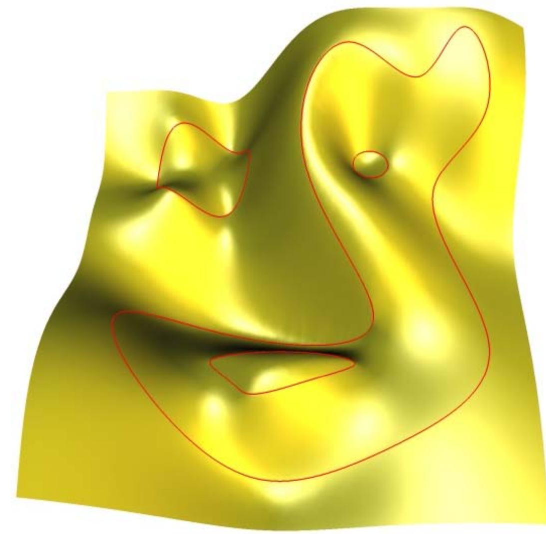
Rendering of quadrilateral elements



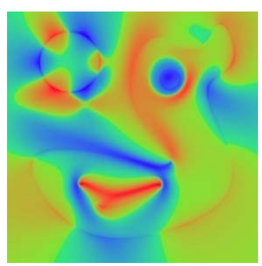
Transfinite interpolation



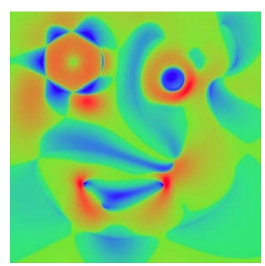
mean value coordinates



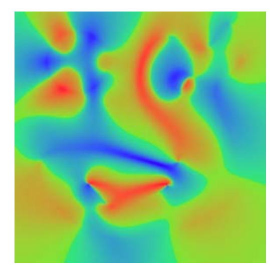
radial basis functions



H



K



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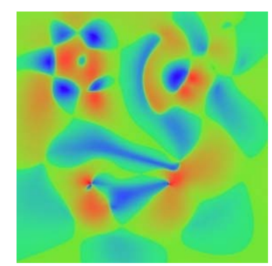
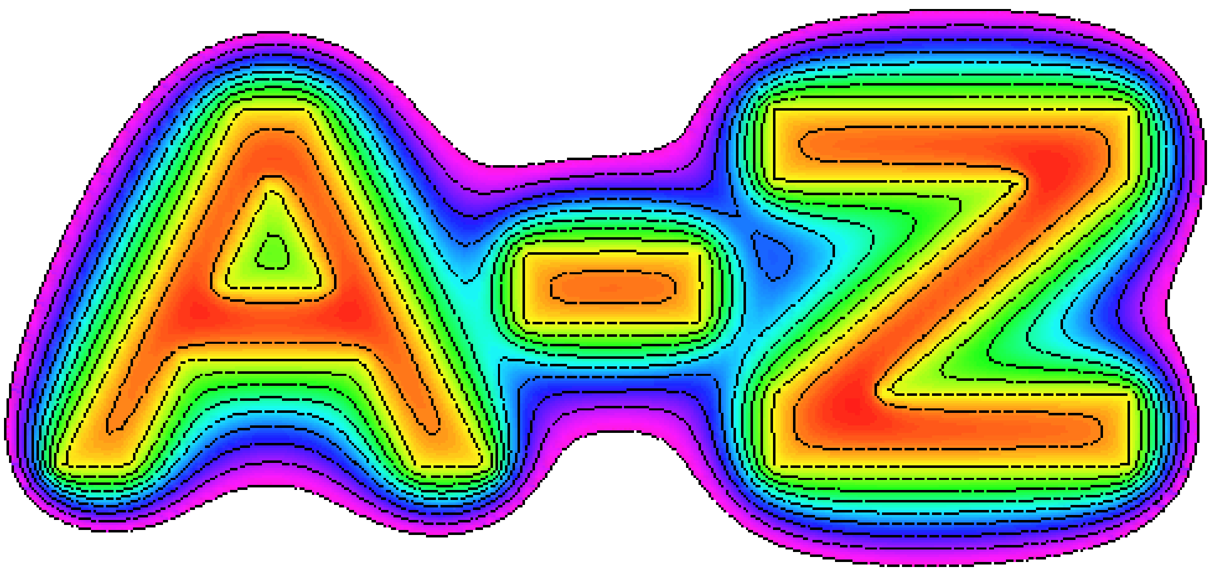


K

Smooth distance function

- Function $D = 1/W$ approximates the distance function
  - $D(v) = 0$ and $\|\nabla D(v)\| = 1/2$ along the boundary
  - smooth, except at the vertices



Mesh animation

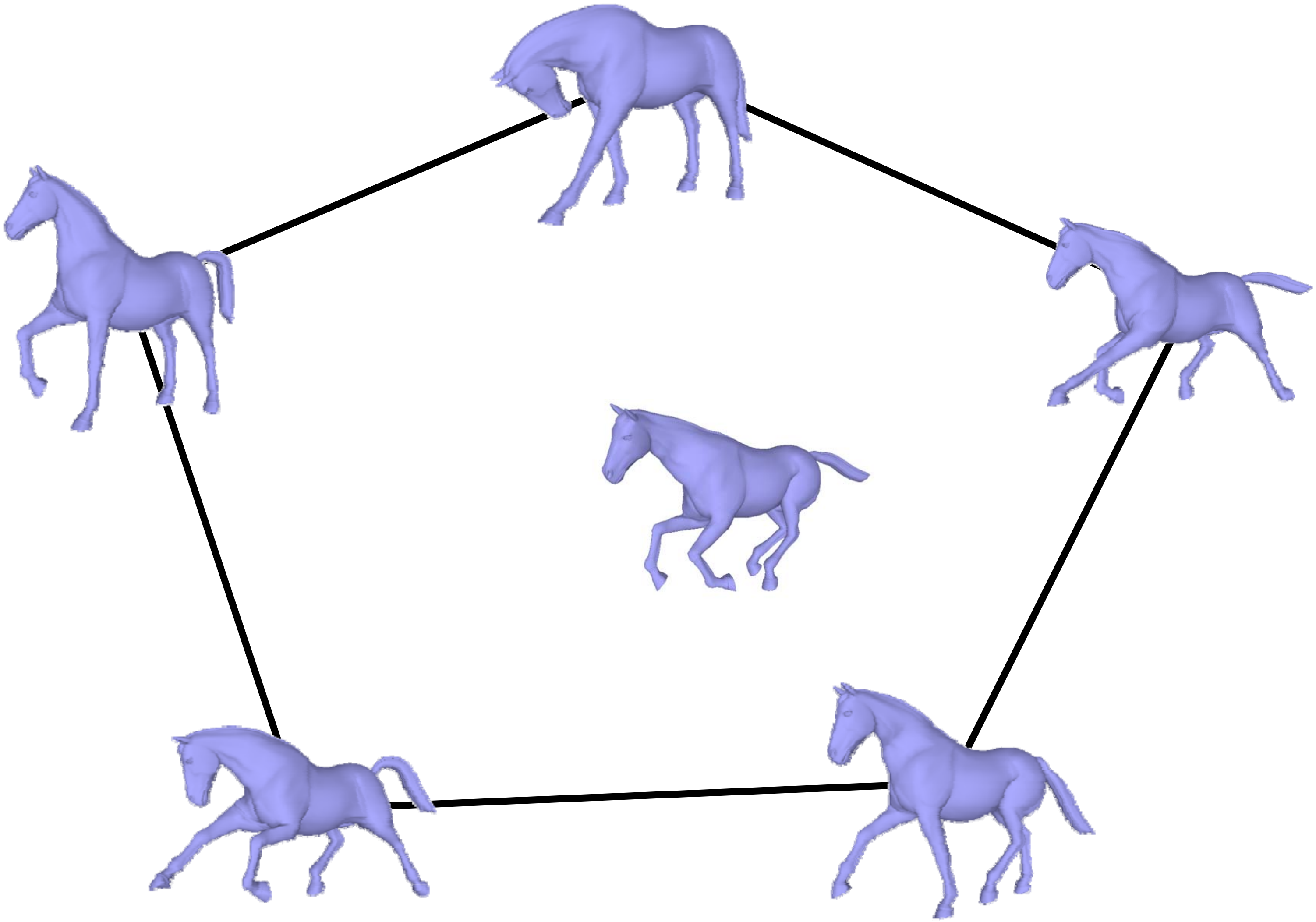
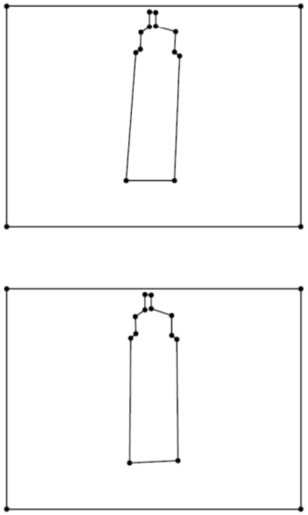


Image warping

original image



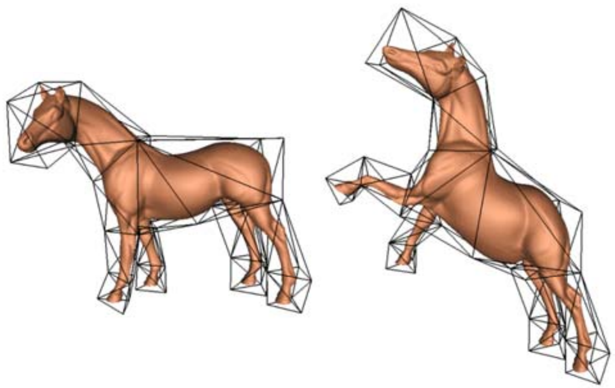
mask

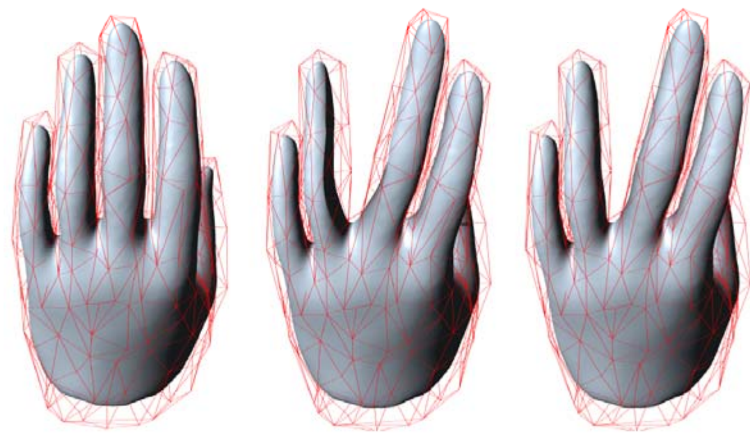
warped image

Mesh warping

- MV coordinates in 3D

[Ju et al. 2005]





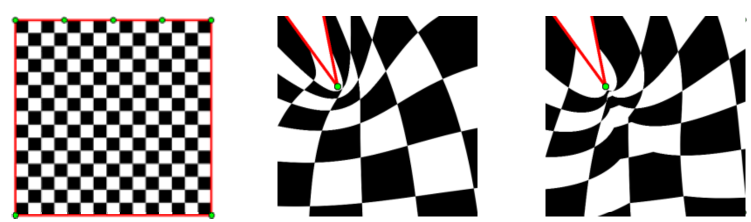
MVC PMVC

  - negative inside the domain

- positive MV coordinates

[Lipman et al. 2007]

  - only C^0-continuous
  - no closed form



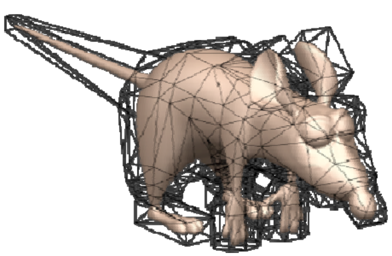
MVC PMVC

Harmonic coordinates

- define normalized coordinate b_i as solution of PDE

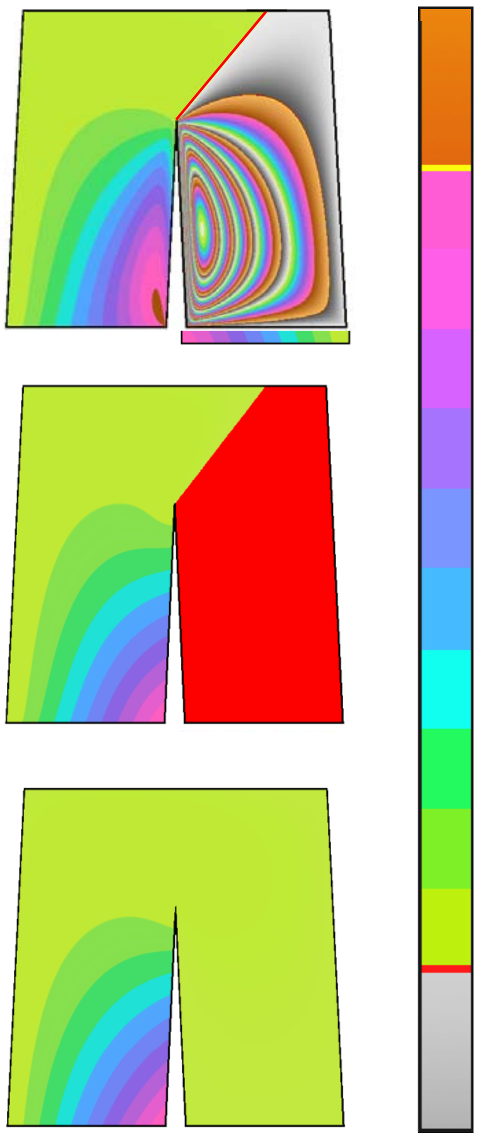
$$\Delta b_i = 0 \qquad \text{subject to} \qquad b_i(v_j) = \delta_{ij}, \quad b_i|_{[v_i,v_{i+1}]} \in \pi_1$$

- Lagrange property ✓ well-defined ✓ smooth ✓
- linear precision ✓ positivity ✓ efficient ✘
- animation for Ratatouille [Joshi et al. 2007]



Positive barycentric coordinates

- drawbacks so far …
- mean value coordinates
  - negative
- positive mean value coordinates
  - not smooth (only C^0)
- harmonic coordinates
  - rather expensive to compute
  - not smooth in practice



Maximum entropy coordinates

[H. & Sukumar 2008]

- based on maximizing the ***Shannon-Jaynes entropy***
- Lagrange property ✓ well-defined ✓ smooth (✓)
- linear precision ✓ positivity ✓ efficient (✓)