Interpolants within Convex Polygons: Wachspress' Shape Functions

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Abstract: During the 1970s, Wachspress developed shape functions for convex n-gons as polynomials of $n-2$ degree divided by the one of $n-3$. Originated from projective geometry these interpolants are linear on the sides and can exactly reproduce arbitrary linear fields. Here an alternative derivation is presented. The wide availability of computer algebra programs makes these high accuracy elements accessible. Associated C++ codes are particularly suited for large scale finite element analysis employed in the aerospace industry.

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Introduction

The power of the finite element method can be attributed to its capability to approximately solve equations of mathematical physics on regions of any shape. The domain, with an arbitrary boundary, is covered by an assembly of simple regions. Based on Ritz's formulation, Courant outlined a consistent solution procedure (Courant 1943) in which the approximate solution with supports on contiguous triangles, which cover the arbitrary geometry, are constructed. Tessellation with triangles does not pose any difficulty. In addition, spatially linear interpolants suffice as the finite element *test functions* on each triangular subdomain for potentials governed by the Laplace equation. These test functions, interpolants, or the finite element shape function are synonymous with the basis of the solution space. Hence their construction as closed form expressions is an important fundamental step in finite element technology.

In order to attain higher accuracy, quadrilateral elements with approximate quadratic shape functions were introduced (Taig 1961). The finite element literature seldom shows systematic ways to extend element formulation technology beyond quadrilaterals. Finite elements in the shape of pentagons, hexagons, and in general n-sided polygons are not available in commercial programs. In the literature for computational geometry an n-sided polygon is called an n-gon, which is considered here a single finite element.

Wachspress, around the year 1970 (Wachspress 1971), pioneered a mathematically rigorous formulation and completely solved the problems of generating shape functions on *convex* n-gons (Wachspress 1975). An identical method yields interpolants on triangles, quadrilaterals, and pentagons, and in all n-sided polygons, i.e., n-gons, according to his constructs of *projective geometry* (Samuel 1988). From the point of view of computer programs for finite elements, one procedure (also known as a module or subroutine in different programming languages) would then be enough to generate the entire class of convex elements.

According to the techniques of computational geometry (de Berg et al. 2000), optimal covering of arbitrary two-dimensional regions by *convex* n-gons can be achieved algorithmically. As a result, elements with a large number of sides, e.g., 200 sides to model an eye in biomedical engineering, become attractive in computational mechanics. The Wachspress shape functions are somewhat unconventional compared to the polynomials almost always used in commercial finite element computer codes. Wachspress proved that for an n-gon the shape functions are rational polynomials. Like rational numbers, an integer over another integer, rational polynomials are the ratios of two polynomials. Such interpolants are very popular with physicists and the applied mathematics literature is quite rich with rational polynomials (Werner and Bünger 1983), which are also known as the Padé series, named after the French mathematician.

Wachspress achieved the following essential features of interpolants:

- Each function is positive in the convex domain;
- Each function is linear on each side of the convex n-gon;
- A set of functions *exactly* interpolates an arbitrary linear field.

Here an alternative formulation is developed to enhance computations.

The Wachspress interpolants can be effectively used as finite element shape functions on n-sided polygons. This is a remarkable enhancement since the current Fortran and C-based computer codes allow only triangular and quadrilateral isoparametric finite elements. Applications that require high accuracy, in which secondary effects cannot be ignored, can be reliably carried out using elements with a large number of sides. Applications in media with random material properties and growth analysis in the healthcare industry can be cited as applications where Wachspress convex n-gons guarantee better accuracy than conventional triangles and isoparametric quadrilaterals.

The algebraic nature of the Wachspress formulation deterred Fortran or C users from launching well publicized computer programs to calculate shape functions on convex n-gons. No com-

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mercial developer has made such powerful and accurate element interpolants available to the application community during the last 30 years. The Appendix contains an illustration with a computer mathematics environment, *Mathematica* used by Wolfram (2002). With a number of recent advancements in C++ language to handle arrays and functions on arrays, fast procedural computation is now possible (Liu and Dasgupta 2002) for codes and examples. In UNIX-based operating systems, such shape function routines can be incorporated into existing finite element computer programs.

Demonstration Using A Pentagon

The isoparametric formulation given by Taig was an intuitive addition to the finite element repertoire limited by Courant triangles. Polynomial test functions in a computational domain, i.e., a square or a cube in transformed space parameterized by $(\eta-\zeta)$ or $(\eta-\zeta-\xi)$, respectively, sufficed to yield shape functions of arbitrary convex quadrilaterals in $x-y$ or hexahedra in $x-y-z$, respectively (Taig 1961).

Wachspress' construction is the only consistent procedure available to generate shape functions for the next level beyond isoparametric quadrilateral finite elements. The novelty of a shape function $\mathcal{N}_i^{(5)}(x,y)$ on a pentagon $(n=5)$ is demonstrated in the following rational polynomial form:

$$\frac{\alpha_1+\alpha_2x+\alpha_3x^2+\alpha_4xy+\alpha_5y^2+\alpha_5x^3+\alpha_6x^2y+\alpha_7xy^2+\alpha_8y^3}{\beta_1+\beta_2x+\beta_3x^2+\beta_4xy+\beta_5y^2} \tag{1}$$

An important characteristic of the i-th shape function $\mathcal{N}_i^{(n)}(x,y)$ on all convex n-gons is elaborated on below.

In general, for an n-sided convex polygon, a Wachspress shape function $\mathcal{N}_i^{(n)}(x,y)$ is a rational polynomial of the following form:

for a *convex* n-gon:

$$\mathcal{N}_i^{(n)}(x,y)=\frac{\mathcal{P}^{(n-2)}(x,y)}{\mathcal{P}^{(n-3)}(x,y)} \tag{2a}$$

$$\mathcal{P}^{(m)}(x,y){:}\, m\text{-degree (full) polynomial in } x,y \tag{2b}$$

Courant's triangles with $n=3$ matches the templet, Eq. (2*a*), where $\mathcal{P}^{(3-2)}(x,y)=\mathcal{P}^{(1)}(x,y)$ is a linear function in $x-y$ in the numerator; the denominator is $\mathcal{P}^{(3-3)}(x,y)=\mathcal{P}^{(0)}(x,y)$, which is a 0-th order polynomial, i.e., a constant.

In Wachspress' derivation of the shape functions, it is the denominator polynomial in Eq. (2*a*) that distinguishes his method from all others which is of primary interest. Its geometrical meaning is particularly significant from the viewpoint of projective geometry (Walker 1962; Samuel 1988). An illustration of a pentagon is presented below where the denominator polynomial and its geometrical meaning are explained.

Coordinates and Convention

Conventions for the coordinate direction and nodal point numbering are important because cyclical order is utilized. Consider the following pentagon:

$$(x_1,y_1)=(2,-1),\;(x_2,y_2)=(1,0.5),\;(x_3,y_3)=(-1,1)$$

$$(x_4,y_4)=(-3,-0.2),\;(x_5,y_5)=(-1.8,-0.8)$$

$$(x_iy_i){:}\,i\text{-th vertex} \tag{3}$$

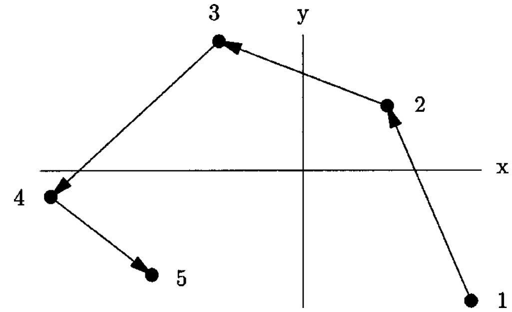

Fig. 1. Counterclockwise numbering of nodes

On a right-handed coordinate system the nodal points will traverse in a counterclockwise sense.

The positive conventions are illustrated in Fig. 1.

The numerator in Eq. (2*a*) can be obtained in terms of the equations of the sides s_i shown in Fig. 2.

The definitions are used in cyclical order in an n-gon, where the index i for nodes and sides can be assigned as follows:

$$i{:}n+1 \;\rightarrow\; i=1 \tag{4a}$$

$$i{:}n+2 \;\rightarrow\; i=2 \tag{4b}$$

$$i{:}0 \;\rightarrow\; i=n \tag{4c}$$

$$i{:}-1 \;\rightarrow\; i=n-1 \tag{4d}$$

$$s_i \text{ connects nodes } i \text{ and } i+1 \tag{4e}$$

In the $x-y$ framework the equation of s_i is $\ell_i(x,y)=0$. For straight lines, coefficients a_i, b_i, c_i in $\ell_i(x,y)=a_ix+b_iy+c_i$ can be obtained from the nodal coordinates: (x_i,y_i) and (x_{i+1},y_{i+1}).

The geometrical convexity property of the finite element region Ω allows us to select the origin as an internal point. The advantage is that no side will pass through the origin, hence the intercept form of a line equation $\ell_i(x,y)=0$ can be uniquely written with two coefficients, a_i and b_i, in the form:

$$\ell_i(x,y)=1-a_ix-b_iy=0;\quad (x,y)\in\Omega\rightarrow\ell_i(x,y)>0 \tag{5}$$

In terms of the nodal point coordinates (x_j,y_j) intercepts a_j and b_j are

$$a_i=\frac{y_i-y_{i-1}}{x_{i-1}y_i-x_iy_{i-1}};\quad b_i=\frac{x_{i-1}-x_i}{x_{i-1}y_i-x_iy_{i-1}} \tag{6}$$

This convenient intercept form for the line equation will be used here.

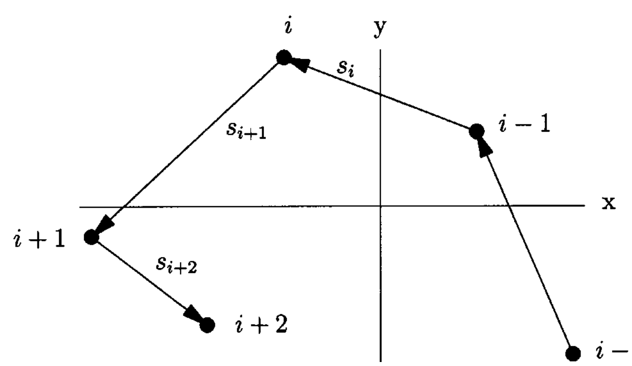

Fig. 2. Nodes and sides in an n-gon; i: node; s: side

For the nodes in Eq. (3) the line equations are

$$\ell_1 = 1 + 0.0588235x + 1.11765y, \quad \ell_2 = 1 - \frac{3x}{4} - \frac{y}{2}$$

$$\ell_3 = 1 - \frac{x}{3} - \frac{4y}{3} \tag{7a}$$

$$\ell_4 = 1 + 0.375x - 0.625y, \quad \ell_5 = 1 + 0.294118x + 0.588235y \tag{7b}$$

According to the scheme of Eqs. (4*c*) and (4*e*)

$$s_1 \text{ connects nodes 5 and 1} \tag{8a}$$

$$s_2 \text{ connects nodes 1 and 2} \tag{8b}$$

It is important to note that within the convex region Ω each $\ell_i(x,y)$ in Eqs. (7*a*) and (7*b*) is positive, which allows us to define the element region Ω:

from Eq. (5)

$$\Omega(x,y) = \bigcup_i \{\ell_i(x,y) > 0\} \tag{9}$$

Numerator Polynomial for Shape Functions

The numerator of $\mathcal{N}_i(x,y)$ is set to be proportional to the product of line equations, excluding the two sides through node *i*,

$$\mathcal{N}_i(x,y) \propto \prod_{\substack{j \neq i \\ j \neq i+1}}^{j=n} \ell_j(x,y) \tag{10}$$

This guarantees that the shape function $\mathcal{N}_i(x,y)$ vanishes along the boundary lines except those through node *i*.

Let us introduce weighting functions; $\varphi_i(x,y)$, such that

$$\mathcal{N}_i(x,y) = \varphi_i(x,y) \prod_{\substack{j \neq i \\ j \neq i-1}}^{j=n} \ell_j(x,y) \tag{11}$$

One of the necessary conditions for interpolants is that they should produce a uniform field exactly, which implies

$$\sum_i \mathcal{N}_i(x,y) = 1 \tag{12}$$

This requirement of the reproduction of a uniform field suggests the following structure for $\varphi_i(x,y)$. Let us start with scalar constants κ_i. Assign

$$\sum_i \left(\kappa_i \prod_{\substack{j \neq i \\ j \neq i-1}}^{j=n} \ell_j(x,y) \right) = \frac{1}{\varphi_0(x,y)} \tag{13a}$$

$$\varphi_i(x,y) = \kappa_i \varphi_0(x,y) \tag{13b}$$

This simplifies the problem of evaluating *n* number of functions $\varphi_i(x,y)$ in Eq. (11) to *n* number of scalar coefficients κ_i.

These unknown κ_i coefficients in Eq. (13*a*) are relative weights; see Eq. (13*b*). Without a loss of generality we shall assume the first coefficient $\kappa_1 = 1$. Now the unknown coefficients are κ_2, κ_3, κ_4, and κ_5.

According to Eqs. (11) and (13*b*) the numerators of a shape function $\mathcal{N}_i$, $\mathcal{N}um_i$ are listed below:

$$\mathcal{N}um_1 : 1 - 0.708333x - 0.15625x^2 + 0.09375x^3 - 2.45833y + 1.15625xy + 0.28125x^2y + 1.8125y^2 - 0.479167xy^2 - 0.416667y^3 \tag{14a}$$

$$\mathcal{N}um_2 : \kappa_2(1 + 0.335784x - 0.112745x^2 - 0.0367647x^3 - 1.3701y - 0.843137xy - 0.159314x^2y - 0.318627y^2 + 0.0735294xy^2 + 0.490196y^3) \tag{14b}$$

$$\mathcal{N}um_3 : \kappa_3(1 + 0.727941x + 0.149654x^2 + 0.00648789x^3 + 1.08088y + 0.782439xy + 0.125433x^2y - 0.408737y^2 + 0.0194637xy^2 - 0.4109y^3) \tag{14c}$$

$$\mathcal{N}um_4 : \kappa_4(1 - 0.397059x - 0.247405x^2 - 0.00129758x^3 + 1.20588y - 1.09256xy - 0.281142x^2y - 0.195502y^2 - 0.67474xy^2 - 0.32872y^3) \tag{14d}$$

$$\mathcal{N}um_5 : \kappa_5(1 - 1.02451x + 0.186275x^2 + 0.0147059x^3 - 0.715686y - 0.151961xy + 0.348039x^2y - 1.38235y^2 + 1.34314xy^2 + 0.745098y^3) \tag{14e}$$

Determination of the coefficients κ_i follows from equating the highest degree terms to zero in $\Sigma\mathcal{N}um_i$. The coefficients of x^3, x^2y, xy^2, and y^3 in $\Sigma\mathcal{N}um_i$ are, respectively,

$$\begin{gathered}
0.09375 - 0.0367647\kappa_2 + 0.00648789\kappa_3 - 0.0129758\kappa_4 + 0.0147059\kappa_5 \\
0.28125 - 0.159314\kappa_2 + 0.125433\kappa_3 - 0.281142\kappa_4 + 0.348039\kappa_5 \\
-0.479167 + 0.0735294\kappa_2 + 0.0194637\kappa_3 - 0.67474\kappa_4 + 1.34314\kappa_5 \\
-0.416667 + 0.490196\kappa_2 - 0.4109\kappa_3 - 0.32872\kappa_4 + 0.745098\kappa_5
\end{gathered} \tag{15}$$

These coefficients of x^3, x^2y, xy^2, and y^3 are equated to zero, leading to

$$\kappa_2 = 2.75, \quad \kappa_3 = 2.83333, \quad \kappa_4 = 2.40833, \quad \kappa_5 = 1.375 \tag{16}$$

These scalar constants reduce the sum of the terms in Eq. (13*a*),

$$\mathcal{N}um_1 + \mathcal{N}um_2 + \mathcal{N}um_3 + \mathcal{N}um_4 + \mathcal{N}um_5 \tag{17}$$

to

$$10.3667 - 0.0873775x - 0.381985x^2 - 8.32667 \ \ 10^{-17}x^3 - 1.2435y - 1.78566xy + 1.11022 \ \ 10^{-16}x^2y - 2.59338y^2 + 2.22045 \ \ 10^{-16}xy^2 + 2.22045 \ \ 10^{-16}y^3 \tag{18}$$

The scaled form, which sets the constant term to unity, of the expression $\Sigma\mathcal{N}um_i$ in Eq. (18) after ignoring the round-off error becomes

$$1 - 0.00842869x - 0.0368475x^2 - 0.119952y - 0.17225xy - 0.250166y^2 \tag{19}$$

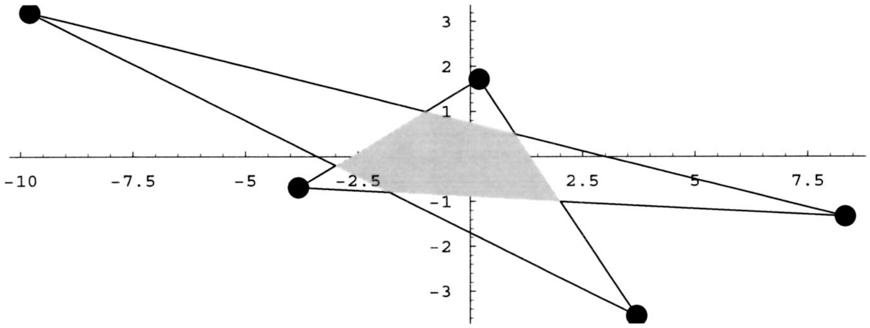

Fig. 3. External intersection points for a pentagon

It is remarkable that the Wachspress rational polynomial interpolants reproduce an arbitrary linear field exactly. The mathematical basis is his projective geometry formulation. The analytical treatment of Wachspress (1971) is not repeated here.

Wachspress' geometrical interpretation of the denominator polynomial in Eq. (19) is furnished next.

Denominator Polynomial for Shape Functions

The concept of an *external intersection point* (EIP) (Fig. 3), which was noted by Wachspress, plays an important role.

Consider two sides that do not share a common vertex. Their intersection is an external intersection point. Here, in the interest of clarity, we assume the sides to be nonparallel.

The maximum number of external intersection points in an n-gon is of interest in a geometrical interpretation. The total number of intersections of two sides is

$$ {}^{n}C_2=\binom{n}{2}=\frac{n!}{(n-2)!2!} \tag{20}$$

If we exclude n vertices, then the maximum number of external intersection points becomes

$$\frac{n!}{(n-2)!2!}-n=\frac{n(n-1)}{2}-n=\frac{n^2-3n}{2} \tag{21}$$

Now consider the total number of terms in the expansion of $(1+x+y)^n$, which is $(1+n)+[n(n+1)/2]$; excluding the constant there are $n+[n(n+1)/2]$ terms. For the denominator polynomial of degree $n-3$ the last expression is

$$(n-3)+\frac{(n-3)[(n-3)+1]}{2}=-\frac{3n}{2}+\frac{n^2}{2} \tag{22}$$

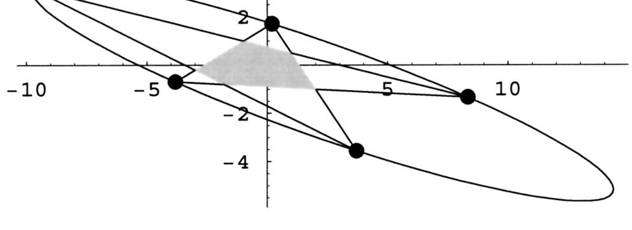

Fig. 4. Curve through external intersection points

This is the same as Eq. (21).

Wachspress proved that the denominator polynomial in Eq. (2*a*) is the equation of the algebraic curve through the external intersection points.

For a pentagon (Fig. 4), the maximum number of external intersection points from Eq. (22) is

$$\left.\frac{n^2-3n}{2}\right|_{n=5}=\frac{5^2-3\times 5}{2}=5 \tag{23}$$

The equation of a generalized ellipse,

$$\alpha_1 x^2+\alpha_2 xy+\alpha_3 y^2+\alpha_4 x+\alpha_5 y+1=0 \tag{24}$$

has five coefficients, $\alpha_1,\ldots,\alpha_5$, shown in Eq. (24). For the pentagon with nodes described in Eq. (3) the external intersection points are

$$(0.190476,1.71429),(-9.8,3.2),(-3.82258,-0.693548),$$
$$(3.7,-3.55),(8.33333,-1.33333) \tag{25}$$

and the elliptic curve through these points is

$$-0.0368475x^2-0.17225xy-0.250166y^2-0.0084287x$$
$$-0.119952y+1=0 \tag{26}$$

which is identical to Eq. (19). We can select the scalar constants κ_i in Eq. (13*a*) so that $\varphi_0(x,y)$ is the left-hand side of the equation of the algebraic curve through the external intersection points.

Based on the linearity of shape functions on the sides of the convex n-gon, an alternate determination of the scalar constants κ_i is developed in "Linear Distribution of the Perimeter."

Linear Distribution of the Perimeter

We start with Eqs. (11) and (13*b*) where a shape function $\mathcal{N}_j(x,y)$ can be written in terms of the equations of the boundary ℓ_i and unknown coefficients κ_i given in Eqs. (13*a*) and (13*b*). A comprehensive illustration using the pentagon with nodes described in Eq. (3) follows.

Let us consider the shape function associated with node 1. This node is common to sides s_1 and s_2. The corresponding ℓ_j, i.e., ℓ_1 and ℓ_2, do not appear in the numerator:

$$\mathcal{N}_1(x,y)$$
$$=\frac{\kappa_1\ell_3\ell_4\ell_5}{\kappa_1\ell_3\ell_4\ell_5+\kappa_2\ell_4\ell_5\ell_1+\kappa_3\ell_5\ell_1\ell_2+\kappa_4\ell_1\ell_2\ell_3+\kappa_5\ell_2\ell_3\ell_4} \tag{27a}$$

$$=\frac{\kappa_1\ell_3\ell_4\ell_5}{\varphi_0(x,y)};\ \text{cf. Eqs. } (13a) \text{ and } (13b) \tag{27b}$$

A compact definition

$$\mathcal{N}_i(x,y)=\frac{\kappa_i\ell_{(i+1)}(x,y)\ell_{(i+2)}(x,y)\ell_{(i+3)}(x,y)}{\Sigma_j\kappa_j\ell_{(j+1)}(x,y)\ell_{(j+2)}(x,y)\ell_{(j+3)}(x,y)} \tag{28a}$$

$$\varphi_0(x,y)=\sum_j \kappa_j\ell_{(j+1)}(x,y)\ell_{(j+2)}(x,y)\ell_{(j+3)}(x,y) \tag{28b}$$

encompasses the representation of Eqs. (27*a*), (13*a*), and (13*b*) where the cyclic order of Eqs. (4*a*)–(4*e*) prevails.

Refer to Fig. 2. Let us examine the behavior of $\mathcal{N}_1(x,y)$ along s_1 that connects nodes 5 and 1:

$$\mathcal{N}_1(x,y)|_{\ell_1=0}=\frac{\kappa_1\ell_3\ell_4\ell_5}{\kappa_1\ell_3\ell_4\ell_5+\kappa_5\ell_2\ell_3\ell_4}=\frac{\kappa_1\ell_5}{\kappa_1\ell_5+\kappa_5\ell_2} \tag{29a}$$

from Eq. (5):

$$=\left.\frac{\kappa_1(1-a_5x-b_5y)}{\kappa_1(1-a_5x-b_5y)+\kappa_5(1-a_2x-b_2y)}\right|_{\ell_1=0} \tag{29b}$$

In this kinematic development, a necessary requirement for a shape function is to be linear on the boundary. In general, Eq. (29*b*) is a nonlinear function in *x,y* since linear terms in *x,y* appear in the denominator. Now the question is whether it is possible to select κ_1 and κ_5 in such a way that the denominator

$$\kappa_1(1-a_5x-b_5y)+\kappa_5(1-a_2x-b_2y)|_{\ell_1=0} \tag{30}$$

becomes a constant. This possibility is investigated next.

Along side s_1, which connects nodes 5 and 1, a generic point can be represented in parametric form in terms of a single variable *t*:

$$x=x_5+t(x_1-x_5);\quad y=y_5+t(y_1-y_5) \tag{31}$$

The denominator of Eq. (29*b*) stated in Eq. (30) becomes

$$\begin{aligned}&(\kappa_1+\kappa_5-\kappa_1a_5x_5-\kappa_5a_2x_5-\kappa_1b_5y_5-\kappa_5b_2y_5)+t(-\kappa_1a_5x_1\\&-\kappa_5a_2x_1+\kappa_1a_5x_5+\kappa_5a_2x_5-\kappa_1b_5y_1-\kappa_5b_2y_1+\kappa_1b_5y_5\\&+\kappa_5b_2y_5)\end{aligned} \tag{32}$$

which can be a constant if the coefficient of $t=0$ when

$$\kappa_5=\kappa_1\left(\frac{a_5x_1-a_5x_5+b_5y_1-b_5y_5}{a_2x_5-b_2y_1+b_2y_5-a_2x_1}\right)$$

$$=\kappa_1\left(\frac{a_5(x_1-x_5)+b_5(y_1-y_5)}{a_2(x_5-x_1)+b_2(y_5-y_1)}\right) \tag{33}$$

As initiated in Eq. (3), if $\kappa_1=1$, then the numerical value of κ_5 from Eqs. (6) and (33) becomes

$$\kappa_5=1.375 \tag{34}$$

which is the same as in Eq. (16). This verifies the procedure.

Algebraically,

$$\kappa_1=\kappa_5\left(\frac{a_2(x_5-x_1)+b_2(y_5-y_1)}{a_5(x_1-x_5)+b_5(y_1-y_5)}\right) \tag{35}$$

This can be generalized to

$$\kappa_{i+1}=\kappa_i\left(\frac{a_{i+2}(x_i-x_{i+1})+b_{i+2}(y_i-y_{i+1})}{a_i(x_{i+1}-x_i)+b_i(y_{i+1}-y_i)}\right) \tag{36}$$

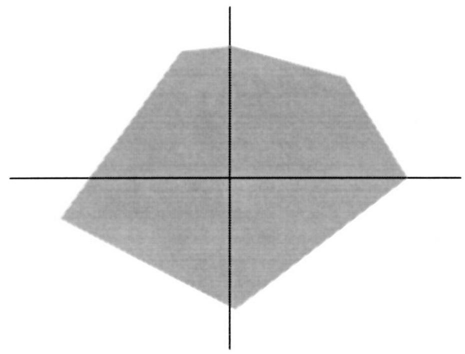

Fig. 5. Example of convex hexagon

Starting with $\kappa_1=1$ and coding Eqs. (6) and (36) all scalar weights can be computed quickly in a C++ program. This was demonstrated by Liu and Dasgupta (2002).

Generalization for a Convex *n*-Gon

We can now summarize the proposed derivation using forms similar to those already mentioned in the illustration of a pentagonal element.

In a convex *n*-gon an interior point is selected to be the origin of the right-hand coordinate system $x-y$. The nodal points are numbered counterclockwise as

$$(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n) \tag{37}$$

The subscript *i* cycles from 1 through *n*. A side s_i connects nodes $i-1$ and *i*. Its equation is $\ell_i=0$:

$$\ell_1=1-a_ix-b_iy$$

where

$$a_i=\frac{y_i-y_{i-1}}{x_{i-1}y_i-x_iy_{i-1}};\quad b_i=\frac{x_{i-1}-x_i}{x_{i-1}y_i-x_iy_{i-1}} \tag{38}$$

The numerator of a shape function $\mathcal{N}_i$ is $\mathcal{N}um_i$. The common denominator for all shape functions is $\Sigma_i\mathcal{N}um_i$. The rational polynomial expression is

$$\mathcal{N}_i(x,y)=\frac{\mathcal{N}um_i}{\Sigma_i\mathcal{N}um_i};\quad \mathcal{N}um_i=\kappa_i\prod_{\substack{j\neq i\\ j\neq i+1}}^{j=n}\ell_j(x,y) \tag{39}$$

and

$$\kappa_i=\kappa_{i-1}\left(\frac{a_{i+1}(x_{i-1}-x_i)+b_{i+1}(y_{i-1}-y_i)}{a_{i-1}(x_i-x_{i-1})+b_{i-1}(y_i-y_{i-1})}\right);\quad \kappa_1=1 \tag{40}$$

The equations are valid for all convex polygons of $n\geqslant 3$.

Numerical Example: Hexagon

We start with a convex hexagon randomly generated with a node at (1,0). The element is shown in Fig. 5. The nodes are

$$(0.648598,0.566481),(0.0183285,0.744137),$$
$$(-0.272096,0.716181),(-0.951595,-0.228752),$$
$$(0.0319588,-0.743882),(1,0) \tag{41}$$

Inadequacy of a Polynomial Shape Function

A full quadratic in *x,y*,

$$a_1+a_2x+a_3y+a_4x^2+a_5xy+a_6y^2 \tag{42}$$

has six coefficients, but this algebraic form does not yield a suitable shape function. The first shape function, which should be 1 at the first node and 0 at the remaining ones, can be solved from

$$[A]=\begin{bmatrix} 1 & 0.648598 & 0.566481 & 0.42068 & 0.367418 & 0.320901 \\ 1 & 0.0183285 & 0.744137 & 0.000335934 & 0.0136389 & 0.55374 \\ 1 & -0.272096 & 0.716181 & 0.0740363 & -0.19487 & 0.512915 \\ 1 & -0.951595 & -0.228752 & 0.905533 & 0.217679 & 0.0523275 \\ 1 & 0.0319588 & 0.743882 & 0.00102137 & -0.0237736 & 0.55336 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{43}$$

$$[A]=\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}; \quad \det[A]=1.774\ 10^{-16} \tag{44}$$

Observe the difficulties with numerical instability. The main objection is that the quadratic form does not guarantee positive values of each interpolant throughout the hexagonal domain. In addition, due to the presence of higher degree terms in the (Taylor form) polynomial, a shape function is invariably nonlinear along each side of hexagonal finite elements.

Step by Step Calculation

With the nodal point in Eq. (41), a specific example of Eq. (37), first the intercepts are evaluated according to Eq. (38). Note that the origin is an interior point, hence no side of the convex hexagon passes through the origin, and all coefficients, a_j and b_j, are finite. The line expressions ℓ_j are

$$\ell_1=1-1.x-0.620324y, \quad \ell_2=1-0.37618x-1.33457y$$
$$\ell_3=1+0.129664x-1.34703y$$
$$\ell_4=1+1.270149x-0.913604y$$
$$\ell_5=1+0.720275x+1.37524y, \quad \ell_6=1-1.x+1.30134y \tag{45}$$

The weights κ_j are computed from Eq. (40) using the values of (x_j,y_j) in Eq. (37) and (a_j,b_j) from Eq. (38). The value of κ_1 is assumed to be 1:

$$\kappa_1=1, \text{ assumed to start evaluation}$$
$$\kappa_2=0.617291, \ \kappa_3=1.44651, \ \kappa_4=2.18419 \tag{46}$$
$$\kappa_5=2.1, \ \kappa_6=1.74503$$

The numerators of the rational polynomial shape functions are calculated from Eq. (39):

$$\mathcal{N}um_1=(1+0.129664x-1.34703y)(1+1.27049x-0.913604y)$$
$$\times(1-1.x+1.30134y)(1+0.720275x+1.37524y)$$

$$\mathcal{N}um_2=0.617291(1+1.27049x-0.913604y)$$
$$\times(1-1.x-0.620324y)(1-1.x+1.30134y)$$
$$\times(1+0.720275x+1.37524y)$$

$$\mathcal{N}um_3=1.44651(1-0.37618x-1.33457y)(1-1.x-0.620324y)$$
$$\times(1-1.x+1.30134y)(1+0.720275x+1.37524y)$$

$$\mathcal{N}um_4=2.18419(1+0.129664x-1.34703y)(1-0.37618x$$
$$-1.33457y)(1-1.x-0.620324y)(1-1.x+1.30134y)$$

$$\mathcal{N}um_5=2.1(1+0.129664x-1.34703y)(1-0.37618x$$
$$-1.33457y)(1+1.27049x-0.913604y)(1-1.x$$
$$-0.620324y)$$

$$\mathcal{N}um_6=1.74503(1+0.129664x-1.34703y)(1-0.37618x$$
$$-1.33457y)(1+1.27049x-0.913604y)(1+0.720275x$$
$$+1.37524y) \tag{47}$$

It is important to observe that

$$\sum_i \mathcal{N}um_i=\mathcal{N}um_1+\mathcal{N}um_2+\mathcal{N}um_3+\mathcal{N}um_4+\mathcal{N}um_5+\mathcal{N}um_6$$
$$=9.09303-3.09329x-1.26426x^2+0.0676879x^3$$
$$+0.x^4-14.931y+3.52123xy+0.971354x^2y$$
$$+2.22045\ \ 10^{-16}x^3y+2.66957y^2+0.458584xy^2$$
$$+4.44089\ \ 10^{-16}x^2y^2+2.38195y^3$$
$$-8.88178\ \ 10^{-16}xy^3+4.44089\ \ 10^{-16}y^4 \tag{48}$$

the terms higher than cubics are due to round-off errors. The denominator polynomial [Eq. (2*a*)] is then

$$\mathcal{P}^{(3)}(x,y)=9.09303-3.09329x-1.26426x^2+0.0676879x^3$$
$$-14.931y+3.52123xy+0.971354x^2y+2.66957y^2$$
$$+0.458584xy^2+2.38195y^3 \tag{49}$$

Finally, the rational polynomial shape functions are calculated from Eqs. (47) and (49):

$$\mathcal{N}_i(x,y)=\frac{\mathcal{N}um_i(x,y)}{\mathcal{P}^{(3)}(x,y)} \tag{50}$$

Tchebycheff Condition

Shape functions $\mathcal{N}_i(x,y)$ are used to interpolate fields within element Ω from the nodal data in mathematical physics problems. Consider a thermomechanical computation of temperature distribution. The nodal temperatures are all positive numbers on the absolute scale. In order to guarantee that all interior points have physically acceptable positive temperatures, each interpolant should be positive in the interior of the element.

The form of Eq. (45) assures each $\ell_j(x,y)>0$, $(x,y)\in\Omega$. From Eq. (46) all weights κ_j are positive. Hence all numerators $\mathcal{N}um_i(x,y)$ in Eq. (47) are positive when $(x,y)\in\Omega$. The denominator polynomial is the sum of all these positive denominators, and hence is positive. Thus rational polynomials Eq. (50) will yield positive values inside the element.

Interpolation of Linear Fields

Linear behavior along the boundary is enforced according to Eqs. (32) and (33). Their ability to interpolate linear fields is examined now.

Consider a linear field:

$$v(x,y)=\alpha x+\beta y+\gamma;\quad \alpha,\beta,\gamma:\ \text{constants} \tag{51}$$

For nodes described in Eq. (41) the respective values are sampled to be

$$\begin{gathered}0.648598\alpha+0.566481\beta+\gamma,\quad 0.0183285\alpha+0.744137\beta+\gamma\\ -0.272096\alpha+0.716181\beta+\gamma,\quad -0.951595\alpha-0.228752\beta+\gamma\\ 0.0319588\alpha-0.743882\beta+\gamma,\quad 1.\alpha+\gamma\end{gathered} \tag{52}$$

These values are fed into the interpolating scheme:

$$\begin{gathered}(0.648598\alpha+0.566481\beta+\gamma)\mathcal{N}_1(x,y)+(0.0183285\alpha\\ +0.744137\beta+\gamma)\mathcal{N}_2(x,y)\\ (-0.272096\alpha+0.716181\beta+\gamma)\mathcal{N}_3(x,y)+(-0.951595\alpha\\ -0.228752\beta+\gamma)\mathcal{N}_4(x,y)\\ (0.0319588\alpha-0.743882\beta+\gamma)\mathcal{N}_5(x,y)+(1.\alpha+\gamma)\mathcal{N}_6(x,y)\end{gathered} \tag{53}$$

using the expressions for $\mathcal{N}_j$ from Eq. (50), and simplifying the sum $\alpha x+\beta y+\gamma$ is indeed recovered. This is a numerical demonstration that linear fields are interpolated exactly within the element. The analytical proof was given by Wachspress using projective geometry constructs.

Conclusions

The Wachspress interpolants for convex polygons are evaluated by enforcing linear distributions along the boundaries. In the object $x-y$ domain the shape functions are constructed as rational polynomials in (*x*,*y*). By setting the sum of all shape functions to unity, constant fields are exactly interpolated. Wachspress (1971), rigorously, i.e., analytically, proved exact interpolation of linear fields on the entire convex domain. This is numerically illustrated here on a hexagon.

An associated integration scheme on arbitrary polygons makes the rational polynomial interpolants attractive in finite element solutions. These shape functions exactly reproduce rigid body displacements and the constant strain cases. Irrespective of a large number of sides, *all* convex polygonal elements will pass the patch test (Irons and Razzaque 1972) for plane stress and plane strain situations.

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Appendix

Example of a Hexagon: Mathematica program

```
n=6; xy={{0.648598,0.0183285,-0.272096, -0.951595,
 0.0319588,1.}, {0.566481, 0.744137, 0.716181,
 -0.228752,-0.743882,0}}
```

```
nodes =Transpose[xy]
Evaluate[{Array[nx, n], Array[ny, n]}]=Transpose
 [nodes];
```

Line Equations

```
Intercept [{x1_,y1_},{x2_,y2_}]:=
 {y2-y1,x1-x2}/[{x1_,y1_},{x2_,y2_}]:={y2-y1,x1-x2}/
 Det[{{x1,y1},{x2,y2}}]
```

```
nodesToIntercepts [p_ ?MatrixQ]:=
Thread [Intercept [p, RotateRight [p]]]
```

```
nodesToLines [p_,{x_,y_}]:=
 1-Dot[nodesToIntercepts[p],{x,y}]
```

```
{ais, bis}=nodesToIntercepts[nodes]//Thread;
```

```
Evaluate[Array[a,6]]=ais; Evaluate[Array[b,6]]=bis;
```

Left-hand sides of line equations:

```
ls=nodesToLines [nodes, {x, y}]
```

```
{1-1.x-0.620324y,1-0.37618x-1.33457y,
 1+0.129664x-1.34703y,1+1.27049x
 -0.913604y,1+0.720275x+1.37524y,1-1.x+1.30134y}
```

Weights

```
k[1]=1; a[n+1]=a[1]; b[n+1]=b[1];
k[i]=k[i_]:=k[i-1]*(a[i+1]*(
       nx[i-1]-nx[i])
     +b[i+1]*
       (ny[i-1]-ny[i])
   )/(a[i-1]*(nx[i]-nx[i-1])
     +b[i-1]*(ny[i]-ny[i-1]))
```

```
ks=Array[k,n]
```

```
{1,0.617291,1.44651,2.18419,2.1,1.74503}
```

Shapes

```
num[1]=ks[[1]]*ls[[3]]*ls[[4]]*ls[[5]]*ls[[6]]
```

```
(1+0.129664x-1.34703y)(1+1.27049x-0.913604y)
 (1-1.x+1.30134y)(1+0.720275x+1.37524y)
```

```
num[2]=ks[[2]]*ls[[4]]*ls[[5]]*ls[[6]]*ls[[1]]
```

```
0.617291(1+1.27049x-0.913604y)(1-1.x-0.620324y)
 (1-1.x+1.30134y)(1+0.720275x+1.37524y)
```

```
num[3]=ks[[3]]*ls[[5]]*ls[[6]]*ls[[1]]*ls[[2]]
```

```
1.44651(1-0.37618x-1.33457y)(1-1.x-0.620354y)
 (1-1.x+1.30134y)(1+0.720275x+1.37524y)
```

```
num[4]=ks[[4]]*ls[[6]]*ls[[1]]*ls[[2]]*ls[[3]]
```

$2.18419(1+0.129664x-1.34703y)(1-0.37618x-1.33457y)(1-1.x-0.620324y)(1-1.x+1.30134y)$

```
num[5]=ks[[5]]*ls[[1]]*ls[[2]]*ls[[3]]*ls[[4]]
```

$2.1(1+0.129664x-1.34703y)(1-0.37618x-1.33457y)(1+1.27049x-0.913604y)(1-1.x-0.620324y)$

```
num[6]=ks[[6]]*ls[[2]]*ls[[3]]*ls[[4]]*ls[[5]]
```

$1.74503(1+0.129664x-1.34703y)(1-0.37618x-1.33457y)(1+1.27049x-0.913604y)(1+0.720275x+1.37524y)$

```
deno=Expand[
  num[1]+num[2]+num[3]+num[4]+num[5]+num[6]]
```

$$9.09303-3.09328x-1.26426x^2+0.0676876x^3+1.11022\times10^{-16}x^4-14.931y+3.52123xy+0.971354x^2y+6.66134\times10^{-16}x^3y+2.66957y^2+0.458582xy^2+4.44089\times10^{-16}x^2y^2+2.38195y^3-1.33227\times10^{-15}xy^3-4.44089\times10^{-16}y^4$$

```
Chop [deno]
```

$$9.09303-3.09328x-1.26426x^2+0.067687x^3-14.931y+3.52123xy+0.971354x^2y+2.66957y^2+0.458582xy^2+2.38195y^3$$

Verify Interpolation of an Arbitrary Linear Field

```
v={α,β,γ},{x,y,1}
```

$x\alpha+y\beta+\gamma$

Obtain the nodal point values:

```
val=(v/.Thread[{x,y}→#]) &/@ nodes
```

$\{0.648598\alpha+0.566481\beta+\gamma, 0.0183285\alpha+0.744137\beta+\gamma, -0.272096\alpha+0.716181\beta+\gamma, 0.951595\alpha-0.228752\beta+\gamma, 0.0319588\alpha-0.743882\beta+\gamma, 1.\alpha+\gamma\}$

Accumulate the contribution from each shape function:

```
terms=Chop[
   Plus@@Expand[Table[val[[i]]*num[i],{i,6}]]];
```

The interpolated field is

```
Expand [Apart[(terms)/Chop[deno], {y}] // Chop]
```

$\{1.x\alpha+1.y\beta+1.\gamma\}$

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