Engineering Mathematics 3

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Root-Finding Methods

In many engineering and scientific problems, we need to find the root x^* of a nonlinear equation:

$$f(x) = 0.$$

Three classical iterative methods for this purpose are the Newton-Raphson method, Bisection method, and Secant method.

1. Newton-Raphson Method

Idea: Use the tangent line at the current guess x_n to approximate the root.
Starting from an initial guess x_0, the iteration formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Derivation: At point x_n, approximate $f(x)$ by its first-order Taylor expansion:

$$f(x) \approx f(x_n) + f'(x_n)(x - x_n).$$

Setting this approximation to zero and solving for x:

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Algorithm steps:

1. Choose an initial guess x_0.
2. Repeat until convergence:
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$
3. Stop when $|x_{n+1} - x_n| < \varepsilon$ or $|f(x_{n+1})| < \varepsilon$.

Convergence:

- Quadratic convergence near the root if f is smooth and $f'(x^*) \neq 0$.
- Sensitive to initial guess; may fail if $f'(x_n) = 0$ or near a local extremum.

2. Bisection Method

Idea: Start with an interval $[a, b]$ where $f(a) \cdot f(b) < 0$. Halve the interval at each step to narrow down the root.
Algorithm steps:

1. Choose initial interval $[a, b]$ such that $f(a) \cdot f(b) < 0$.
2. Compute midpoint:
$$c = \frac{a+b}{2}.$$
3. If $|f(c)| < \varepsilon$ or interval width $(b-a)/2 < \varepsilon$, stop.
4. Otherwise, choose the subinterval $[a, c]$ or $[c, b]$ where f changes sign.
5. Repeat.

Convergence:

- Guaranteed to converge if f is continuous.
- Converges linearly: the error roughly halves each iteration.
- Slower than Newton-Raphson and Secant, but very robust.

Final error estimate:
$$\text{Error after } n \text{ iterations} \approx \frac{b-a}{2^n}.$$

3. Secant Method

Idea: Like Newton-Raphson but avoids computing the derivative by approximating it using two previous points.
Iteration formula:
$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}.$$

Algorithm steps:

1. Choose two initial guesses x_0, x_1.
2. Repeat until convergence:
$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}.$$
3. Update $x_{n-1} \leftarrow x_n$, $x_n \leftarrow x_{n+1}$.

Convergence:

- Superlinear convergence
- Usually faster than bisection; does not need the derivative.
- May fail if $f(x_n) - f(x_{n-1}) \approx 0$.

Method	Needs derivative?	Convergence rate	Robustness
Newton-Raphson	Yes	Quadratic	May fail if the initial guess is poor or if the derivative is zero
Bisection	No	Linear	Always converges if the function is continuous and sign change exists
Secant	No	Superlinear	Faster than bisection, but may fail if denominator becomes zero

Table 1: Comparison of root-finding methods

Example: Comparison of Root-Finding Methods

We analyze the convergence of three root-finding methods applied to the cubic polynomial:

$$f(x) = x^3 - 2x - 5 \tag{1}$$

with its derivative:

$$f'(x) = 3x^2 - 2 \tag{2}$$

Method	Root	Iterations	Final Error
Newton-Raphson	2.0945514815	5	0.0000000000
Bisection	2.0945520401	20	0.0000019073
Secant	2.0945514815	6	0.0000000003

Table 2: Comparison of root-finding methods

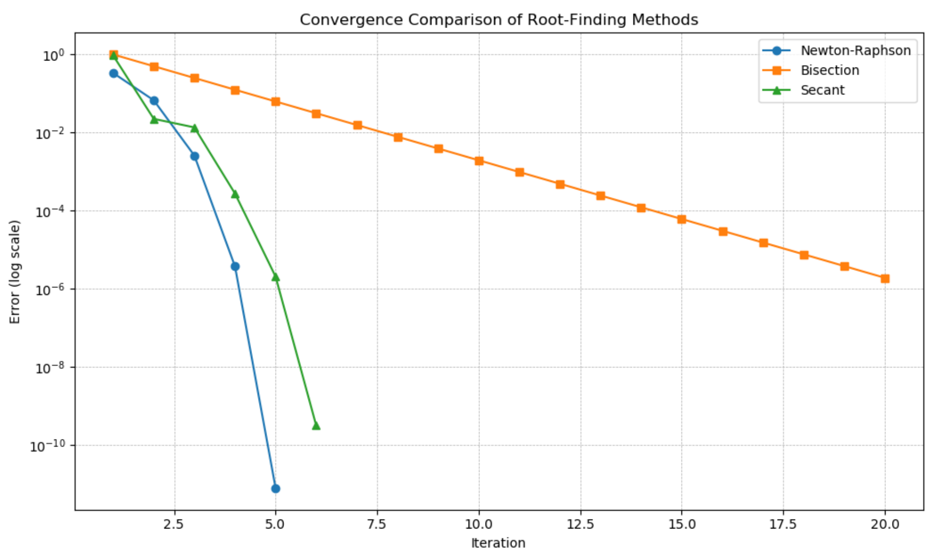

Figure 1: Illustration of Newton-Raphson, Bisection, and Secant methods

- Newton-Raphson is the fastest but requires derivative information and can be unstable
- Bisection is the most reliable but converges slowly
- Secant method offers a good compromise - faster than Bisection without requiring derivatives
- Newton-Raphson and Secant may fail to converge, while Bisection always converges when properly bracketed
- The choice depends on the specific problem requirements for speed, accuracy, and reliability

Example: Case Where Newton-Raphson Diverges

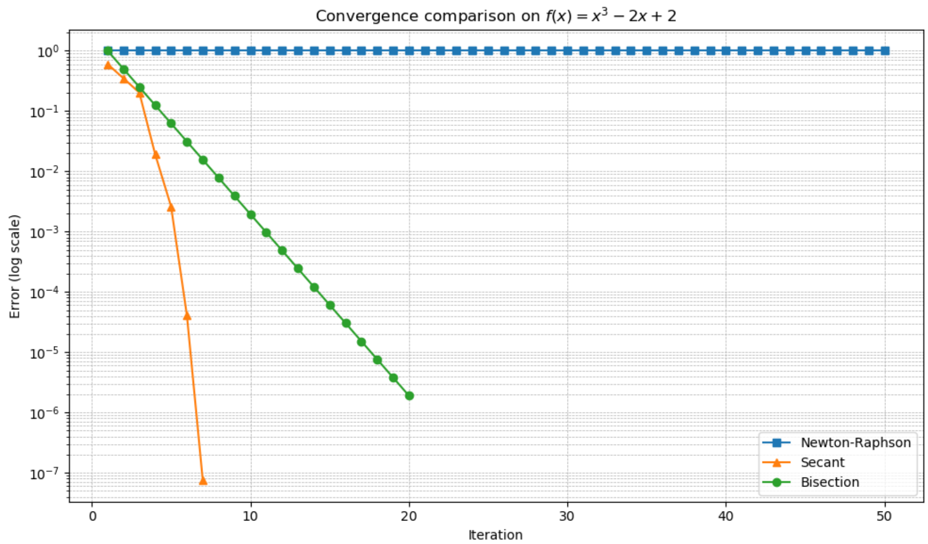

Figure 2: Illustration of Newton-Raphson, Bisection, and Secant methods

Consider the nonlinear function:

$$f(x) = x^3 - 2x + 2.$$

This function has at least one real root near $x \approx -1.7693$.

Newton-Raphson Method:
The iteration formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

where the derivative is:

$$f'(x) = 3x^2 - 2.$$

If we choose an initial guess $x_0 = 0$, then $f'(0) = -2$. The first iteration becomes:

$$x_1 = 0 - \frac{f(0)}{-2} = 0 - \frac{2}{-2} = 1.$$

At $x = 1$, the derivative $f'(1) = 1$, so the next step is:

$$x_2 = 1 - \frac{f(1)}{1} = 1 - (1 - 2 + 2) = 1 - 1 = 0.$$

Then, the iteration keeps oscillating between 0 and 1 without approaching the actual root at $x \approx -1.7693$. This happens because:

- The initial guess $x_0 = 0$ is close to a local extremum (near $x = \pm\sqrt{\frac{2}{3}} \approx \pm 0.8165$) where $f'(x) = 0$.
- Newton-Raphson relies heavily on the slope $f'(x)$. When the derivative is small or changes sign, the method can jump far from the root or enter a cycle.

Secant Method:
The secant method uses two initial guesses $x_0 = -2$ and $x_1 = -1$. Its iteration formula is:

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}.$$

Because the initial guesses bracket the actual root (i.e., $f(-2)$ and $f(-1)$ have opposite signs), the secant method converges toward the root near -1.7693, despite not knowing the derivative.

Bisection Method:
The bisection method starts with the interval $[-2, -1]$, where:

$$f(-2) = -8 + 4 + 2 = -2,$$

$$f(-1) = -1 + 2 + 2 = 3.$$

Since $f(-2) \cdot f(-1) < 0$, there is at least one root inside $[-2, -1]$. At each iteration, the interval is halved:

$$c = \frac{a+b}{2}.$$

Bisection always converges to the root because it only requires the function to be continuous and to change sign inside the interval. It does not rely on the derivative $f'(x)$.

Conclusion:

- Newton-Raphson fails in this case due to a poor initial guess near a local extremum where the derivative is small or changes sign.
- The Secant method and Bisection method succeed because they use information from two points (secant) or the interval sign change (bisection).
- This illustrates the sensitivity of Newton-Raphson to the choice of initial guess and its dependence on the derivative.

Tutorial Solutions

Question 7

We are given the volume of material as a function of the height h:

$$V(h) = 9.6h^2 + 5.76h + 0.768 + \frac{6000}{h} + \frac{600}{h^2}.$$

To find the value of h that minimizes $V(h)$, we need to solve:

$$V'(h) = 0.$$

Step 1: Compute derivatives

$$V'(h) = 19.2h + 5.76 - \frac{6000}{h^2} - \frac{1200}{h^3}.$$

$$V''(h) = 19.2 + \frac{12000}{h^3} + \frac{3600}{h^4}.$$

Step 2: Newton-Raphson method

The Newton-Raphson iteration for finding the critical point (minimum) is:

$$h_i = h_{i-1} - \frac{V'(h_{i-1})}{V''(h_{i-1})}.$$

Step 3: Iteration with initial guess $h_0 = 6$

Iteration 1:

$$h_0 = 6$$

$$V'(6) = 19.2 \cdot 6 + 5.76 - \frac{6000}{36} - \frac{1200}{216}$$
$$= 115.2 + 5.76 - 166.67 - 5.56 = -51.27$$

$$V''(6) = 19.2 + \frac{12000}{216} + \frac{3600}{1296}$$
$$= 19.2 + 55.56 + 2.78 = 77.54$$

$$h_1 = 6 - \frac{-51.27}{77.54} = 6 + 0.661 = 6.661$$

Iteration 2:

$$V'(6.661) = 19.2 \cdot 6.661 + 5.76 - \frac{6000}{(6.661)^2} - \frac{1200}{(6.661)^3}$$

Compute numerically (approximate):

$$19.2 \cdot 6.661 = 127.89, \quad (6.661)^2 = 44.38, \quad (6.661)^3 = 295.64$$

$$-\frac{6000}{44.38} = -135.19, \quad -\frac{1200}{295.64} = -4.06$$

$$V'(6.661) = 127.89 + 5.76 - 135.19 - 4.06 = -5.60$$

$$V''(6.661) = 19.2 + \frac{12000}{295.64} + \frac{3600}{1970.82}$$

$$= 19.2 + 40.60 + 1.83 = 61.63$$

$$h_2 = 6.661 - \frac{-5.60}{61.63} = 6.661 + 0.091 = 6.752$$

Iteration 3:
Continue similarly until $|h_i - h_{i-1}| < \varepsilon = 0.005$.

Step 4: Answer

After a few iterations, the method converges to:

$$\boxed{h \approx 6.75}$$

which minimizes the volume $V(h)$.

Question 8

Given:

$$x(t) = \left(\frac{1}{4} + t\right) e^{-4t} - \frac{1}{4}\cos(4t)$$

We need to solve $x(t) = 0$ for $t \in [4, 5]$.

Solution

Step 1: Compute the Derivative $x'(t)$

$$\begin{aligned} x'(t) &= \frac{d}{dt}\left[\left(\frac{1}{4}+t\right)e^{-4t}\right] - \frac{d}{dt}\left[\frac{1}{4}\cos(4t)\right] \\ &= e^{-4t} + \left(\frac{1}{4}+t\right)(-4)e^{-4t} + \sin(4t) \\ &= e^{-4t} - 4\left(\frac{1}{4}+t\right)e^{-4t} + \sin(4t) \\ &= -4te^{-4t} + \sin(4t) \end{aligned}$$

Step 2: Newton-Raphson Iteration Formula

$$t_{n+1} = t_n - \frac{x(t_n)}{x'(t_n)}$$

Step 3: Perform Iterations Starting at $t_0 = 4.5$

Iteration 1:

$$t_0 = 4.5$$

$$t_0 + \tfrac{1}{4} = 4.75, \quad e^{-4\cdot 4.5} = e^{-18} \approx 1.522 \times 10^{-8}$$

$$4.75 \cdot 1.522 \times 10^{-8} = 7.23 \times 10^{-8}$$

$\cos(18) \approx 0.6603$:

$$-\tfrac{1}{4} \cdot 0.6603 = -0.1651$$

$$x(4.5) = 7.23 \times 10^{-8} - 0.1651 \approx -0.1651$$

—
Compute $x'(4.5)$:

$$-4 \cdot 4.5 = -18, \quad -18 \cdot 1.522 \times 10^{-8} = -2.739 \times 10^{-7}$$

$\sin(18) \approx -0.7509$:

$$x'(4.5) = -2.739 \times 10^{-7} - 0.7509 \approx -0.7509$$

$$t_1 = 4.5 - \frac{-0.1651}{-0.7509} = 4.5 - 0.22 = 4.28$$

$$|4.28 - 4.5| = 0.22 > 0.05$$

—
Iteration 2:

$$t_2 = 4.28 - \frac{0.04}{-0.987} = 4.32$$

Iteration 3:
Continue similarly until $|t_i - t_{i-1}| < \varepsilon = 0.05$.

Step 4: Answer

After a few iterations, the method converges to:

$$\boxed{h \approx 4.32}$$

which is the root of the function $x(t)$.