Investigating soliton dynamics using numerical methods (python 3.6.4)

December 16, 2018

Abstract

The Korteweg de Vries partial differential equation is numerically solved using forward and central difference schemes. To step through solutions, the Runge-Kutta 4^{th} order scheme was employed. Soliton stability was found to depend on α. Larger α values restricted the range of dx and dt values. Soliton velocity was found to be 0.333 times the height with a strong linear relationship (r value = 0.999). During collisions, soliton area and shape were conserved. Wave breaking produced a trail of solitons and the number of solitons produced increased with a wider sine wave. Shock waves were found to be unstable without a diffusive term producing a trail of pulses. Adding a diffusive term lead to stable decay of the soliton.

1 Introduction

Solitons are single pulses which maintain their wave-form as they propagate. They were first observed in rivers where they would form at the front of ships or boats travelling at a certain speed and depth of water[1]. They are important to study because they also occur in other wave types such as electromagnetic waves. For example, they have been discovered to form in fibre optic cables when the dispersion cancels the non-linear term in a Gaussian pulse[2]. In this project the Korteweg de Vries equation will be solved numerically using forward and central difference schemes on the python 3.6.4 programming language. The stability of the solutions will be analysed, then once the solitons propagate their velocities and collisions will be investigated. Wave breaking and shock wave phenomena will also be looked into by changing the input wave form and by altering the descretisation.

2 Theory

The Korteweg de Vries (KdeV) partial differential equation (PDE) [3]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

where u is the solution, t is the time and x is the displacement (all scaled to dimensionless units), was solved numerically using a finite difference discretisation scheme. For this project, the soliton solution to the KdeV PDE will be investigated. A soliton is a single pulse which does not change shape during propagation. This is explained by the non-linear term and dispersive term from equation (1). The second term in equation (1) gives the wave speed as a function of the amplitude of disturbance, whereas the third term produces dispersive spreading of the pulse shape. These two effects cancel each other to produce a soliton.

Solving analytically gives:

$$u = 12\alpha^2 \operatorname{sech}^2(\alpha(x - 4^2 t)), \tag{2}$$

where α contributes to the amplitude of the soliton and has an effect on the wave velocity. Using finite difference approximations for the derivatives and u in equation (1) the KdeV was discritised:

$$u_i^j = \frac{u_{i+1}^j - u_{i-1}^j}{2} \tag{3}$$

$$u\frac{\partial u}{\partial x} = \frac{(u_{i+1}^j + u_{i-1}^j)(u_{i+1}^j - u_{i-1}^j)}{4h} \tag{4}$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{(u_{i+2}^j - 2u_{i+1}^j + u_{i-1}^j - u_{i-2}^j)}{2h^3}. \tag{5}$$

Using equations (3) and (4) the KdeV can be written as:

$$\frac{\partial u}{\partial t} = -\frac{(u^j_{i+1}+u^j_{i-1})(u^j_{i+1}-u^j_{i-1})}{4h} - \frac{(u^j_{i+2}-2u^j_{i+1}+2u^j_{i-1}-u^j_{i-2})}{2h^3}. \tag{6}$$

These are of second order accuracy because h^2 terms and higher order h are ignored when doin the Taylor expansions.

3 Method

In order to step through the solution in time, an Euler forward method is considered:

$$u^{j+1}_i = u^j_i - \frac{\Delta t(u^j_{i+1}+u^j_{i-1})(u^j_{i+1}-u^j_{i-1})}{4h} - \frac{\Delta t(u^j_{i+2}-2u^j_{i+1}+2u^j_{i-1}-u^j_{i-2})}{2h^3}. \tag{7}$$

To check this is consistent with the original PDE, Taylor expansions for time and space are substituted into equation (7) and letting Δt and h go to zero the KdeV is recovered. This method is unstable for the limit of small u. As there are periodic boundary conditions, von Neumann stability analysis can be used to check whether the Fourier modes amplitude increases after subsequent time steps. The amplitude factor for the Fourier mode has been shown to be larger than one, therefore the Euler forward difference is not stable.

Alternatively, a Runge-Kutta method of 4^{th} order accuracy is used. This uses four stages of weighted gradient calculation to compute an average one:

$$\begin{aligned} u^{j+1}_i &= u^j_i + \frac{k_1+2k_2+2k_3+k_4}{6} \\ k_1 &= \Delta t f(t_i, u_i) \\ k_2 &= \Delta t f(t_i+0.5h, u_i+0.5k_1) \\ k_3 &= \Delta t f(t_i+0.5h, u_i+0.5k_2) \\ k_4 &= \Delta t f(t_i+h, u_i+k_3), \end{aligned} \tag{8}$$

where $f = \frac{\partial u}{\partial t}$ from equation (6). When programming this, the analytic solution was used to produce the initial wave form and then equation (8) was used to propagate it. In order to implement boundary conditions, a function which shifted the initial solution was made. So for u_{i+1} the initial solutions were shifted by one position to the right.

The height of the soliton is not conserved during collisions due to the non-linear term, however the total area is because energy is conserved. Solitons have a condition for stability relating to the time and space steps, and the value of α which will be explored. The definition for stability will be that the height will not change more than one percent because the soliton is expected to maintain a constant height. To calculate the area of the soliton a Riemann integral was used. This method uses the numerical solutions and sums the product of step size and each solution to calculate the area:

$$A = \sum_{i=0}^{N} u_i h \tag{9}$$

where h is the step size which is constant and so the distance between each solution.

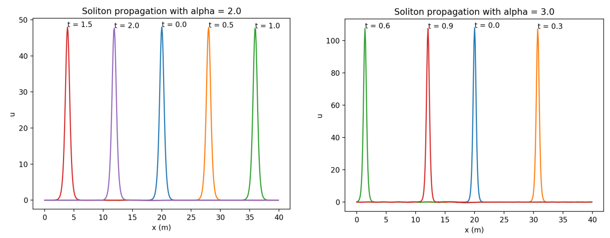
Wave breaking will be explored using a sine wave input because this is not a normal mode of the KdeV. Shock wave phenomena will be investigated by removing the dispersive term and introducing a diffusive term.

4 Results and analysis

4.1 Propagation

Using the 4^{th} order Runge-Kutta method detailed in equation (8) and the discretisation in equation (6), an animate function was produced which stepped through the numerical solutions with a set of dt, dx values. These dt, dx values were chosen so that the numerical solutions would be stable for a range of α values. Initially $dx = 0.05$ and $dt = 0.000125$ were chosen because they produced stable solitons.

Animations were produced for different soliton heights as shown in figures 1a and 1b. These animations show that the solitons propagate the entire spatial domain with a constant height and area. This means that the numerical solutions are stable.

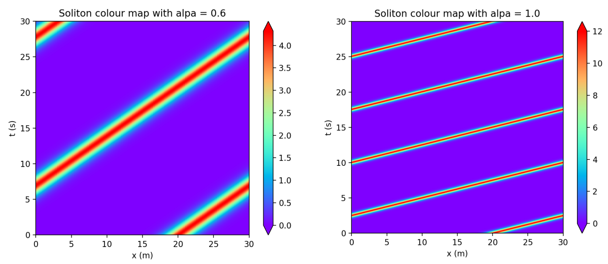


(a) Solitons plotted every 0.5 seconds.

(b) Solitons plotted every 0.3 seconds.

Figure 1: Solitons in 1a are much close together than those in 1b and this is due to the soliton velocity being dependent on the height. The height and area are constant as the solitons propagate.

Figures 2a and 2b show the solitons with colour mapping. The colour maps to the height of the soliton, so the widths can be deduced. As these plots are done with t and x the inverse of the gradient relates to the speed of the soliton. Comparing figures 2a and 2b, it can be seen that solitons with greater heights have smaller widths and that they travel faster.



(a) Soliton with $\alpha = 0.6$ simulated with colour mapping.

(b) Soliton with $\alpha = 1.0$ simulated with colour mapping.

Figure 2: The colour maps to the height of the soliton, also the spread can be seen because the purple means zero height. The speed of the soliton is the inverse of the gradient for these plots and can be used to compare the speeds of solitons with different heights. The gradient of soliton 2b is steeper than in 2a and so it is traveling faster. This soliton is also thinner.

4.2 Velocity

To observe how the soliton speed is related to its height, the position of the peak of each soliton was tracked in a given time interval and then this speed was plotted for different soliton heights. From the analytic solution shown in equation (2), the velocity of the pulse should be $4\alpha^2$. Comparing this to the soliton height, which is $12\alpha^2$, the height should be three times greater than the velocity. Figure 3 shows that there is a linear relationship between the soliton velocity and height with the gradient equal to 0.333. This means that the velocity is approximately a third of the height which is as expected from the analytic solution.

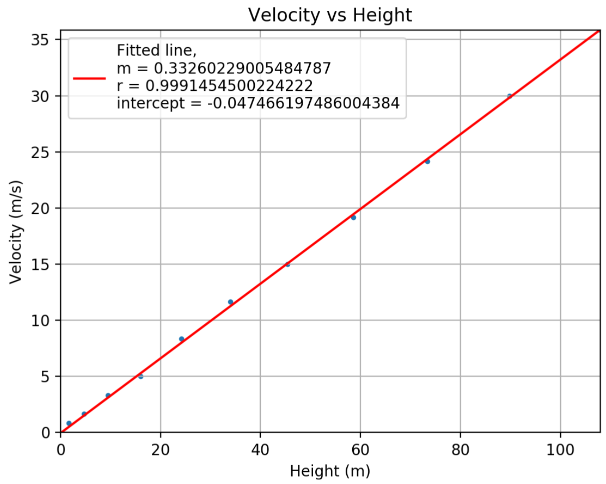
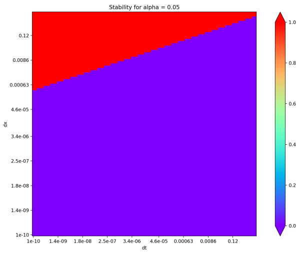


Figure 3: Velocity is plotted against height for solitons. The data points were fitted linearly with an r value equal to 0.999. This shows the linear relationship between the soliton velocity and height. The gradient is about 0.333 and the intercept is -0.047. For this plot, the value $dt = 0.000125$ and $dx = 0.05$ were used because they propagate the soliton in a stable manner.

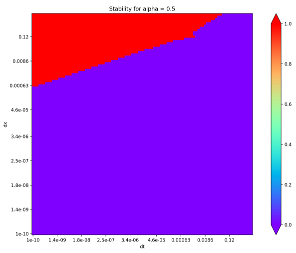
4.3 Stability

The stability of the soliton was defined as being stable if the soliton height remained within one percent of its initial value after each step. This is because the height of the soliton is expected to be constant so a one percent deviation will be considered as unstable. Logarithmic lists of dt and dx from 10^{-10} to $10^{0.2}$ were made and each combination was tested for solitons of different alpha. Each run was allowed to run for 1000 steps because the smaller time steps would take more steps to destabilize and this was adequate for my range of steps. At each step the height was compared to the initial height and then a colour map was plotted to show regions of stability.

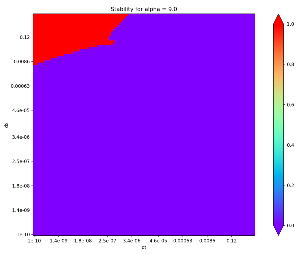
Comparing the plots in figure 4, it can be seen that as α increases the red region of stability decreases. This is because the value of α determines the height of the soliton. From figure 2 it can be seen that taller solitons have smaller widths, therefore the values of dx and dt needed to keep it stable will need to be much smaller than the width of the soliton. The minimum dx goes from 0.000063 to 0.00086. Also, a gradient change occurs for larger α. This is because dt is generally much smaller than dx for stability, and so for larger dx the dt will need to be much smaller, hence the increase in gradient.



(a) $\alpha = 0.05$.



(b) $\alpha = 0.5$.



(c) $\alpha = 9.0$.

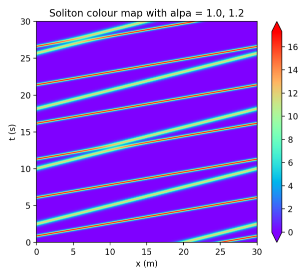
Figure 4: Stability plots show stable regions in red and unstable regions in purple. Plotted for three α values with the axes logarithmically scaled from 10^{-10} to $10^{0.2}$ on both axes. For small α linear relationship between dt and dx, however as α increases the gradient changes after a certian value. Also, as α increases the region of stability decreases.

4.4 Collisions

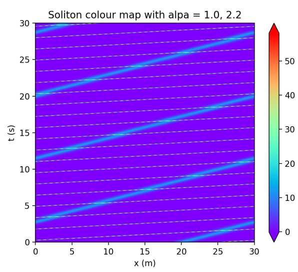
Two solitons were initiated with similar and very different α values to explore the dynamics of soliton collisions. They were separated in the spatial domain so that there was no initial overlap in order to observe and compare behaviour before, during and after collisions. As solitons with greater height travel faster, the taller one was positioned to start before the smaller one.

Comparing figure 5a with figure 5b, the phase shift for similar soliton heights is greater than those with very different heights. This is because the speeds of the solitons with similar height have speeds closer to each other. Therefore, they spend longer colliding compared to the solitons with greater height difference. Physically, this shows that solitons with similar height will interfere with each other more than those with very different heights during collisions. Also, it can be seen that there is an exchange of momentum between the two solitons of similar height, whereas the very different height solitons move relatively independently during collisions.

The shape of the pulses in both scenarios is the same before and after the collision. However during the collision this is not the case. During a collision, the taller soliton will have it's height reduced and it's width increased, and the shorter one will have it's height increased and it's width decreased. This is because the energy of the solitons is conserved and so the total area must be conserved. Comparing the areas before, during and after the collision using equation (9), it was observed that it remained constant. The non-linear term in equation (1) explains why there is no linear superposition of the waves.



(a) Soliton collision with very similar α.



(b) Soliton collision with very different α.

Figure 5: It can be seen that there is a phase shift during collisions and that the height and speed are the same before and after the collisions. In 5b each soliton moves less effected by the collision, as can be seen the lines for each soliton are much straighter during collisions than in figure 5a.

4.5 Wave breaking

To observe how wave forms other than solitons are propagated, a sine wave was used as the initial pulse. Due to the units of the PDE, the sine wave was setup to have a large period compared to the amplitude of the wave.

Figure 6 shows that the sine wave initially leans to the right and then breaks up into individual solitons of different heights. The height of the tallest soliton is greater than the amplitude of the sine wave. To explain this the fact that energy is conserved means the area is conserved, and so with many solitons, their combined areas must equal the area of the sine wave. When running a shorter period sine wave the number of solitons decreases however the height of the tallest one is always greater than the amplitude of the initial sine wave as can be seen in figure 7. This sine wave is not a normal mode of the KdeV as the soliton is and so the wave breaking into solitons is natural because these are the only normal modes which can propagate.

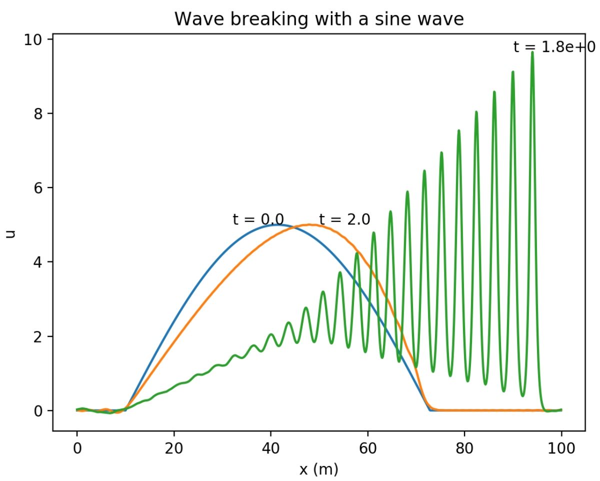


Figure 6: Initial positive part of sine wave is setup with period of 120 seconds and an amplitude of 5 so that the amplitude is much smaller than the period. This allows wavebreaking to occur. The wave initially leans to the right and starts to split into multiple solitons of different magnitude. The tallest soliton leads, and the other solitons move in order of height.

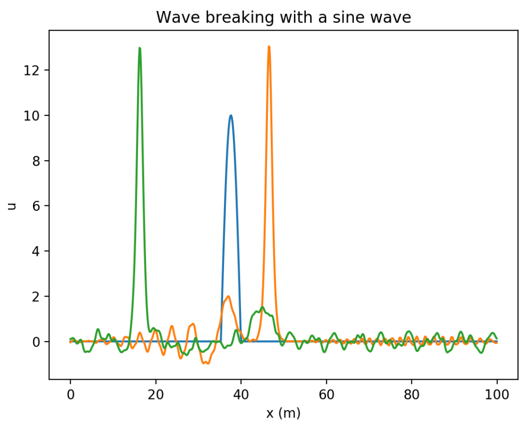


Figure 7: Initial positive part of sine wave is setup with period of 10 seconds and an amplitude of 10. The wave splits into multiple solitons with the tallest at around 12.5 and the next tallest at 2. The blue plot is the initial, then orange and green in increasing time.

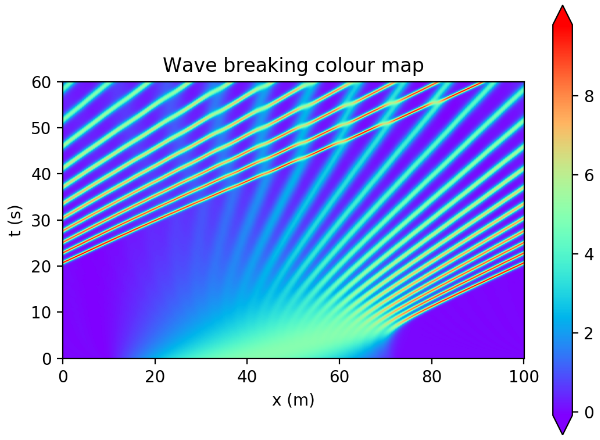
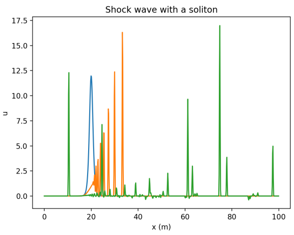


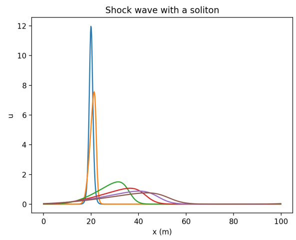
Figure 8: A sine wave with period 120 m and amplitude 5. Initially, wave shifts to the right before breaking into a train of solitons each of different amplitude.

4.6 Shock waves

Removing the final term in equation (1) will cause the non-linear term to be unbalanced and so will produce shock waves. Figure 9(a) illustrates that the soliton does not propagate as it did with the dispersive term, instead it separates into multiple pulses. If left to run for a few seconds the solution becomes unstable as the values blow up to infinity. In addition, the colour plot reveals that the waveforms are very narrow and independent of the wave height. The faint background ripples in figure 10 reveal the instability.



(a) Soliton propagation with non-dispersive term with $\alpha = 1$. Initial soliton in blue, then orange and then green in increasing time are shown. The soliton splits into pulses and then blows up (green plot is just before the explosion).



(b) Diffusion term added to stabilize shock wave. It can be seen that the soliton amplitude decreases and that the width increases. Both these changes occur non-linearly. This is a stable solution.

Figure 9: (a) shows no diffusion term and (b) shows soliton propagation with the diffusion term.

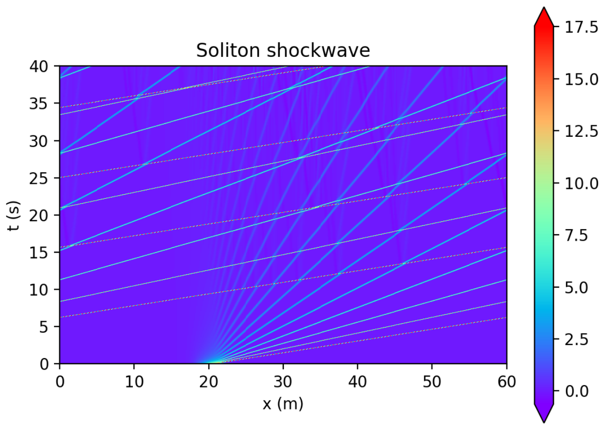


Figure 10: Colour plot for the same situation as in 9a. The widths of all pulses are the same despite the differences in height of the pulses.

To make this shock wave stable a diffusive term, $D\frac{\partial^2 u}{\partial x^2}$, was added and discretised as the other derivatives in section 2. Figure 9(b) shows the effect of this. A stable diffusion of the soliton is observed. This is expected because shock waves initially start with large amplitudes and as they propagate they lose energy to the surroundings which cause their amplitudes to decrease. The soliton causes the shock wave to go in one direction for x. Figure 11 shows this situation with a colour map and it can be seen that the wave spreads out and that the amplitude decays to zero at large distances from the initial soliton.

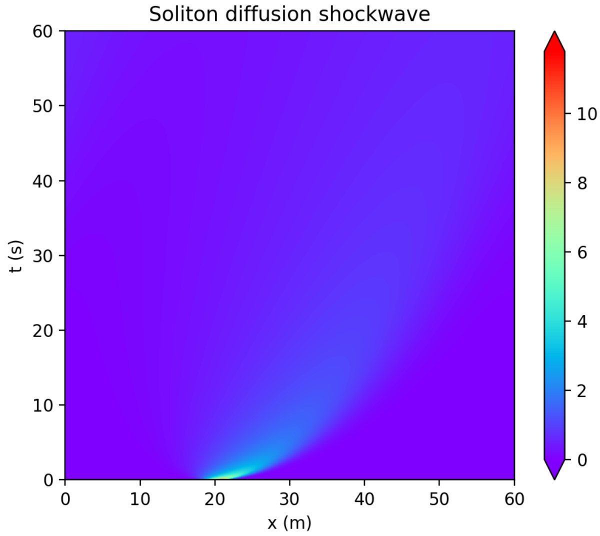


Figure 11: Soliton propagated with diffusion term D = 2.0. There is a smooth spreading of the initial waveform and eventually the amplitude drop to near zero.

5 Conclusion

The Kdev parabolic PDE was descritised using a forward difference scheme and central difference approximations for the derivatives. RK4 was used to step through the solutions numerically. This produced a stable soliton, where stability was defined in this report as the height remaining within one percent of the initial height. Solitons of different heights were propagated and traversed the entire spatial domain and the periodic boundary conditions worked. Taller solitons were observed to be thinner and to be faster than smaller ones, in fact the r value for the velocity vs height plot was 0.999 with a gradient of 0.333. Therefore, there is a strong linear relationship between velocity and height with the velocity being about a third of the height. Stability values for dt and dx decreased in range as α increased.

Collisions for similar solitons produced larger phase shifts and so interfere with each other more than those with very different α. Soliton area and shape was conserved before and after collisions. Wave breaking produced a trail of solitons and the number of solitons produced increased with a wider sine wave. Shock waves were found to be unstable without a diffusive term which was added, producing a trail of pulses. The diffusive term caused the soliton to decay in amplitude and spread out eventually reaching an amplitude of zero. To improve, the stability testing should be done with more checks for example the area also for the height the percentage off should be increased as dx becomes larger because the soliton peak will be less detailed. The relationship between soliton heights and when they are close enough to cause large enough phase shifts could be added.

References

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