Digital Signal Processing
(DSP)
Part 2

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Part-two Outlines
Discrete Transforms

- *Fourier Series*
- *Fourier Transform*
- *Discrete Fourier Transform (DFT)*
- *DFT inverse*
- *Properties of DFT*
- *Complexity of DFT*

Fourier Series

❖ Any periodic continuous waveform f(t) can be represented as the sum of an infinite number of sinusoidal and cosinusoidal terms, together with a constant term, this representation being the Fourier series given by

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nwt) + \sum_{n=1}^{\infty} \mathrm{b}_n \sin(nwt)$$

$$a_0 = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} f(t)dt \qquad b_n = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} f(t)\sin(nwt)dt$$

$$a_n = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} f(t)\cos(nwt)dt$$

The frequencies $n\omega$ are known as the nth harmonics of ω.

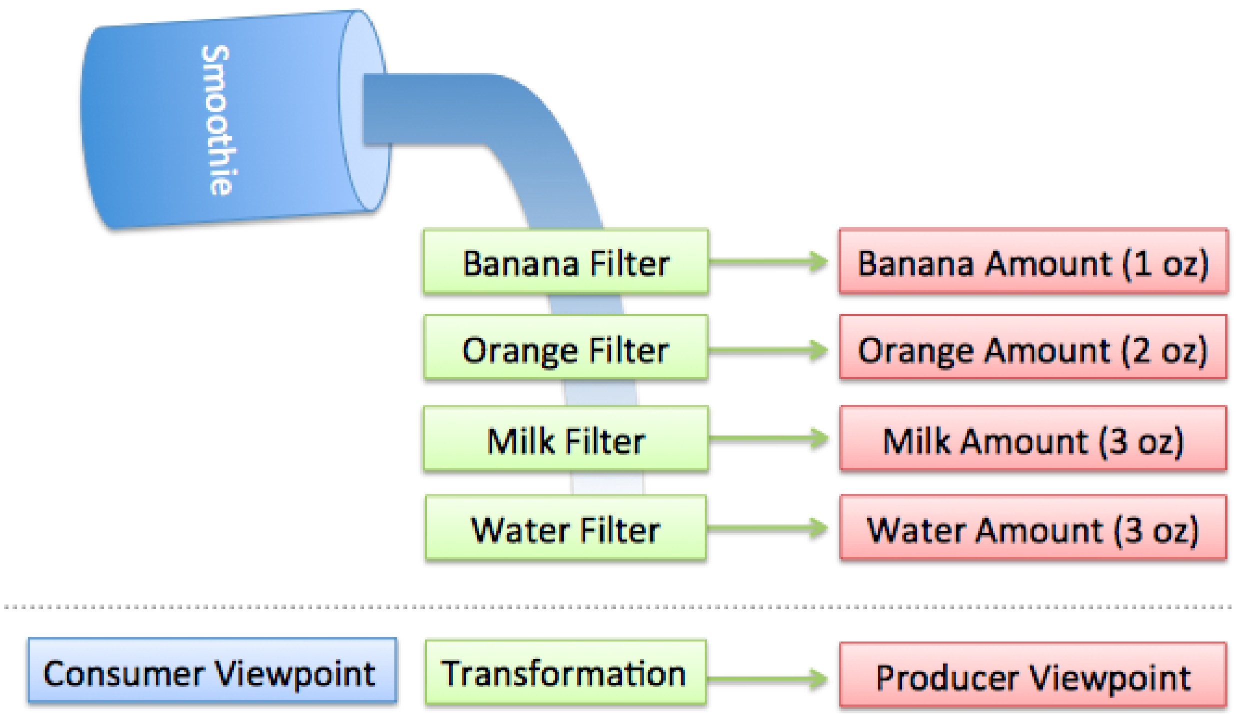
Fourier Series

Fourier Series

❖The Fourier Transform changes our perspective from consumer to producer, turning What I see ?, into How was it made ?

- **What does the Fourier Transform do?** Given a smoothie, it finds the recipe.
- **How?** Run the smoothie through filters to extract each ingredient.
- **Why?** Recipes are easier to analyze, compare, and modify than the smoothie itself.
- **How do we get the smoothie back?** Blend the ingredients.
  - **Filters must be independent**. The banana filter needs to capture bananas, and nothing else. Adding more oranges should never affect the banana reading.
  - **Filters must be complete**. We won't get the real recipe if we leave out a filter ("There were mangoes too!"). Our collection of filters must catch every last ingredient.
  - **Ingredients must be combine-able**. Smoothies can be separated and re-combined without issue (A cookie? Not so much. Who wants crumbs?). The ingredients, when separated and combined in any order, must make the same result.

Smoothie to Recipe



Fourier Series

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The frequencies $n\omega$ are known as the nth harmonics of ω.

Fourier Series

where t is an independent variable which often represents time but could, for example, represent distance or any other quantity, $f(t)$ is often a varying voltage versus time waveform, but could be any other waveform, $\omega = 2\pi/T_p$ is known as the first harmonic, or fundamental, angular frequency, related to the fundamental frequency, f, by $\omega = 2\pi f$, T_p is the repetition period of the waveform,

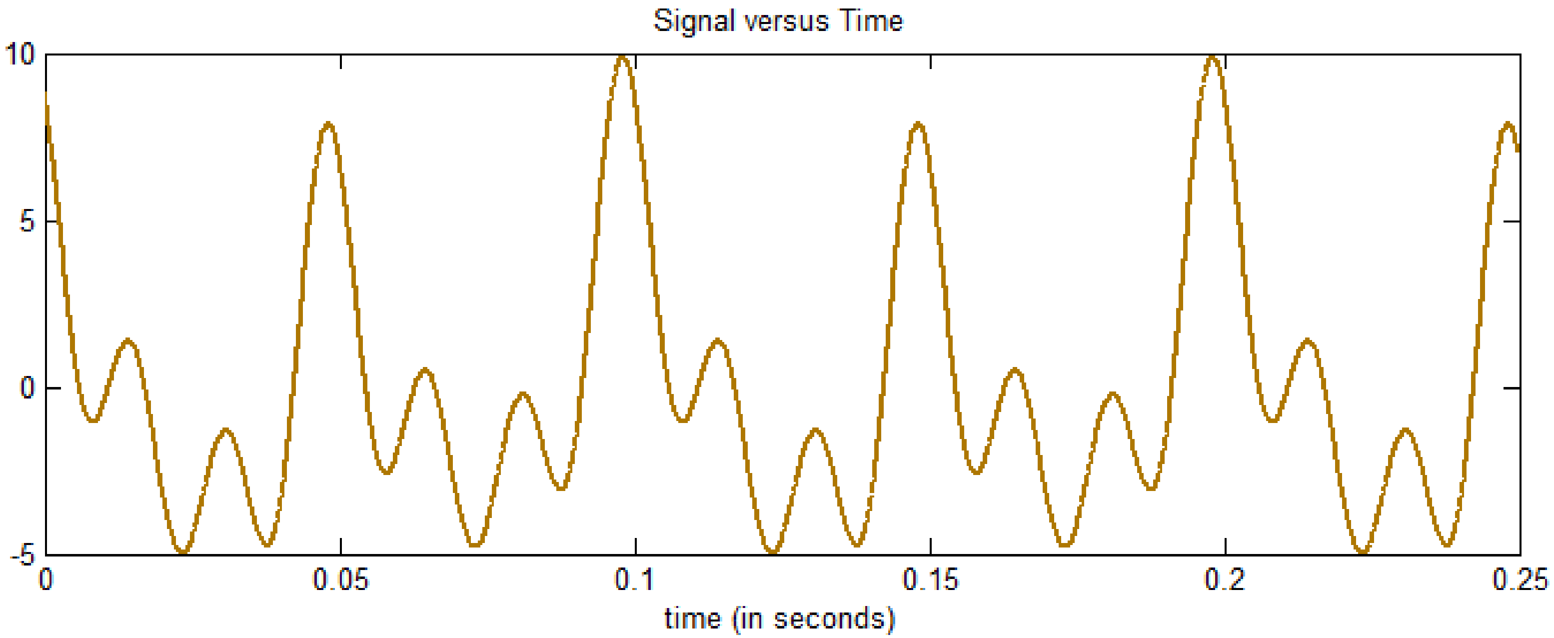
❖This series can be written more compactly using exponential notation and the advantage in that form is that it is more easily manipulated mathematically the series that becomes

$$f(t) = \sum_{n=-\infty}^{\infty} d_n e^{jn\omega t}$$

$$d_n = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} f(t) e^{-jn\omega t} \, dt$$

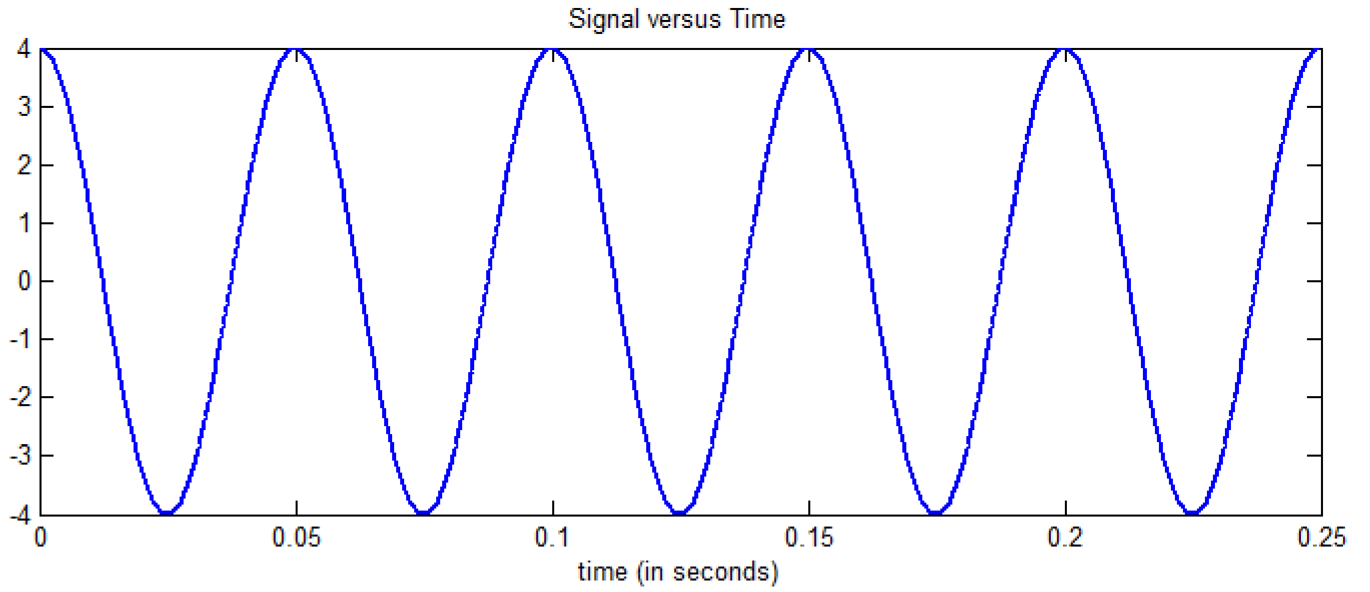
❖is complex and $|d_n|$ has the unit of volts.

Formation of signals



Formation of signals

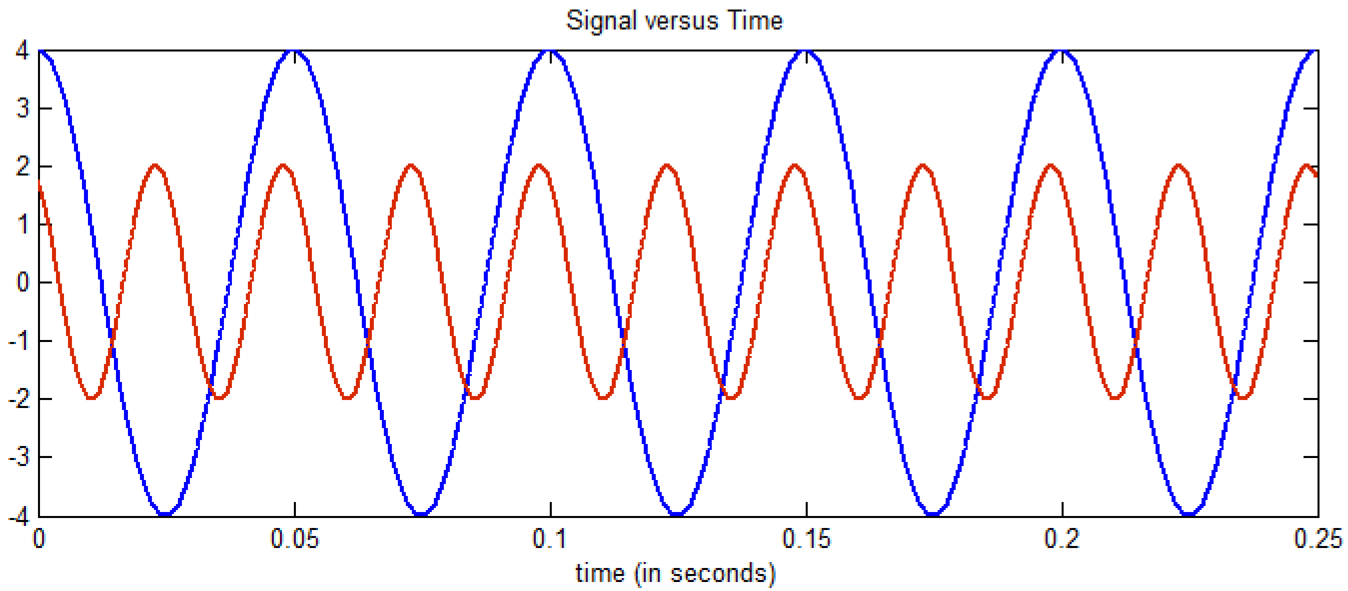
- F= 20 HZ
- A=4
- Θ=0



Formation of signals

- F= 20 HZ
- A=4
- Θ=0 rad

- F= 40 HZ
- A=2
- Θ=0.5 rad



Formation of signals

- F= 20 HZ
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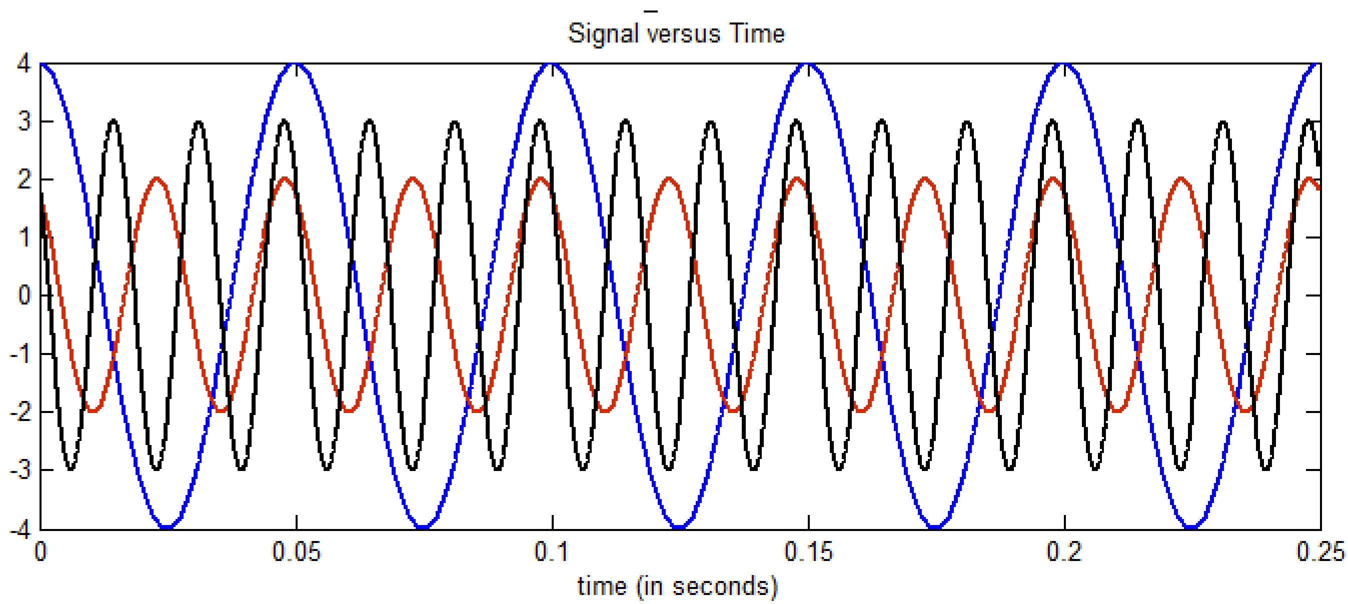
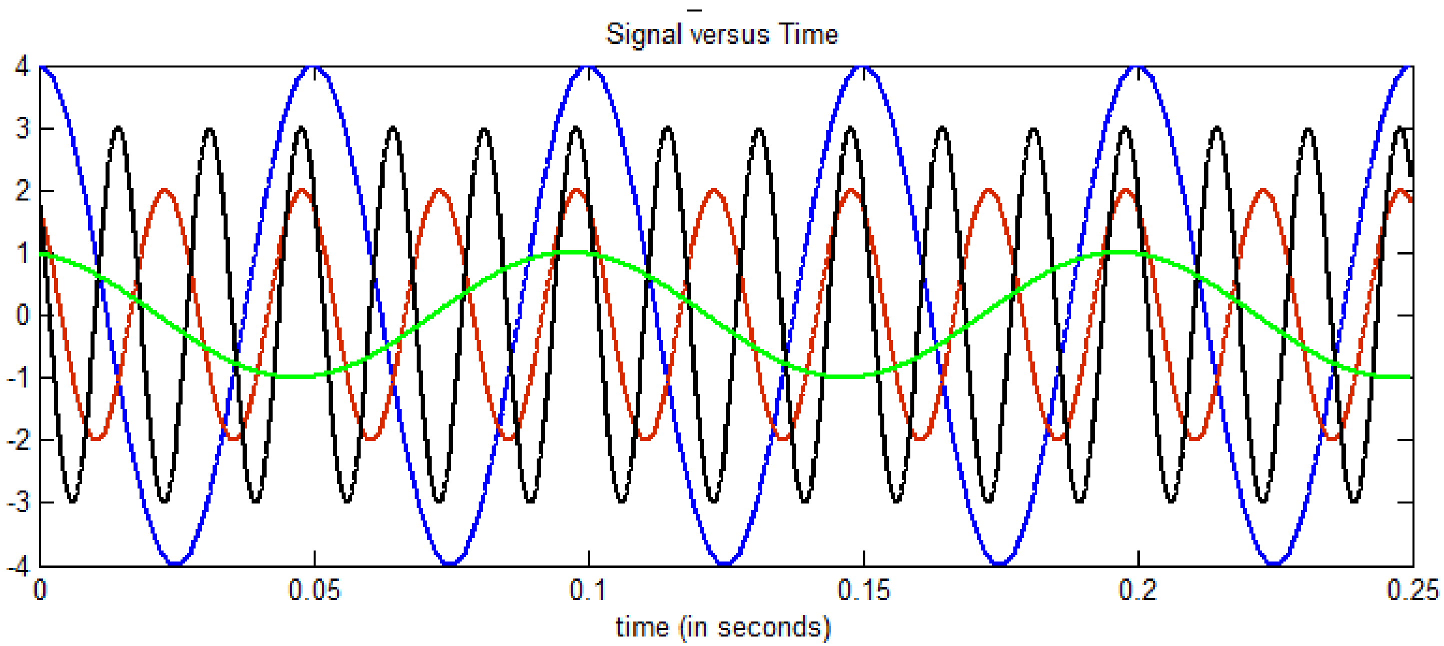


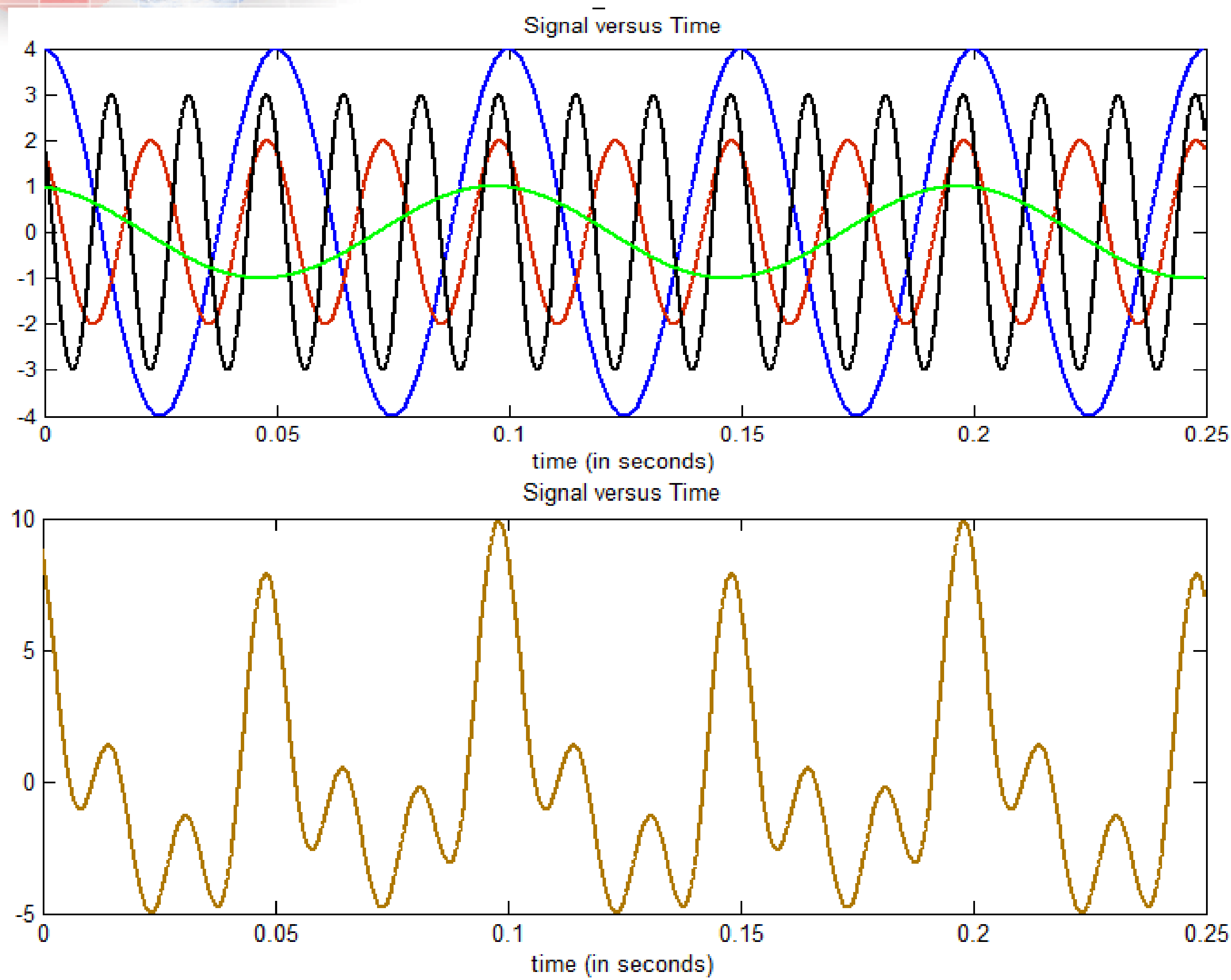
- F= 60 HZ
- A=3
- Θ=0.8 rad

Formation of signals

- F= 20 HZ
- A=4
- Θ=0
- F= 40 HZ
- A=2
- Θ=0.5 rad
- F= 60 HZ
- A=3
- Θ=0.8 rad
- F= 10 HZ
- A=1
- Θ=0.2 rad



Formation of signals



Fourier Transform

Non-periodic Signals

❖By increasing the period T_p to be infinite. As T_p is increased the spacing between the harmonic components, $1/T_p = w/2\pi$, decreases to $dw/2\pi$ eventually becoming zero. This corresponds to a change from the discrete frequency variable nw to the continuous variable w, and the amplitude and phase spectra become continuous. Thus $d_n \to d(w)$ as $T_p \to \infty$. With this modifications,

$$d(\omega) = \frac{d\omega}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \quad\Longrightarrow\quad \frac{d(\omega)}{d\omega/2\pi} = F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$F(j\omega) = \mathrm{Re}(j\omega) + j\,\mathrm{Im}(j\omega) = |F(j\omega)|e^{j\phi(\omega)}$$

$$|F(j\omega)| = [\mathrm{Re}^2(j\omega) + \mathrm{Im}^2(j\omega)]^{1/2}$$

$$\phi(\omega) = \tan^{-1}[\mathrm{Im}(j\omega)/\mathrm{Re}(j\omega)]$$

Discrete Fourier Transform (DFT)

❖Assume that a waveform has been sampled at regular time intervals T to produce the sample sequence {x(nT)} = x(0), x(T),…x[(N-1)T] Of N sample values, where n is the sample number from n=0 to n=N-1.The DFT of x(nT) is then defined as the sequence of complex values {X(kΩ)} = X(0), X(Ω),…… X[(N-1)Ω] in the frequency Domain where Ω is the first harmonic frequency given by Ω= 2π/NT. Thus, the X(kΩ) have real and imagery components in general so that for the kth harmonic

$$X(k) = R(k) + jI(k)$$

$$|X(k)| = [R^2(k) + I^2(k)]^{1/2}$$

$$\phi(k) = \tan^{-1}[I(k)/R(k)]$$

Discrete Fourier Transform (DFT)

❖ N real data values (in time domain) transform to N complex DFT values in (frequency domain). The DFT values, X(k) are given by

$$X(k) = F_{\mathrm{D}}[x(nT)] = \sum_{n=0}^{N-1} x(nT)\mathrm{e}^{-\mathrm{j}k\Omega nT}, \; k = 0, 1, \ldots, N-1$$

Discrete Fourier Transform DFT

Sometimes we don't know the value of T but may be eliminated using $\Omega = 2\pi/NT$ giving

$$X(k) = \sum_{n=0}^{N-1} x(nT)e^{-jk\Omega nT}$$

$$= \sum_{n=0}^{N-1} x(nT)e^{-jk2\pi n/N}$$

DFT Problems

1) Compute the DFT for the following sequence x(n) = { 1, 0, 0, 1}.

$$X(k) = \sum_{n=0}^{N-1} x(nT)e^{-jk2\pi n/N}$$

$$X(0) = 1 + 0 + 0 + 1 = 2$$

$$X(1) = 1 + 0 + 0 + 1e^{-j2\pi 3/4} = 1 + e^{-j3\pi/2}$$

$$= 1 + \cos\left(\frac{3\pi}{2}\right) - j\sin\left(\frac{3\pi}{2}\right) = 1 + j$$

$$X(2) = 1 + 0 + 0 + 1e^{-j4\pi 3/4} = 1 + 0 + 0 + e^{-j3\pi} = 1 - 1 = 0$$

$$X(3) = 1 + 0 + 0 + e^{-j9\pi/2} = 1 - j$$

It has therefore been shown that the time series $\{1, 0, 0, 1\}$ has the DFT given by the complex sequence $\{2, 1 + j, 0, 1 - j\}$.

DFT Problems

2) Compute the DFT for the following sequence $x(n) = \{ 0, 1 , 2 , 3\}$.

Ans: $X(k) = \{ 6, -2+2j, -2, -2-2j\}$

Relation between FT & DFT

❖ by putting x(nT)=f(t), kΩ=w and nT= t, so the two transforms may be expected to have similar properties. The transforms are not however equal. Thus, making these substitutions and putting dt= T and replacing the integral by a summation gives

$$\sum_{n=0}^{N-1} x(nT)e^{-jk\Omega nT}\,T = F(j\omega) \qquad 0 \leq t \leq (N-1)T.$$

This reveals

$$F(j\omega) = TX(k)$$

Showing that the Fourier transform components are related to the DFT components by sampling interval, and may be obtained by multiplying the DFT components by the sampling interval.

Important Properties of DFT

❖An important property of the DFT may be deduced if the kth component of DFT, X(k) is compared with (K+N)th component, X(k+N) thus,

$$X(k) = \sum_{n=0} x(nT)e^{-jk\Omega nT}$$

$$= \sum_{n=0}^{N-1} x(nT)e^{-jk2\pi n/N}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(nT)e^{-jk2\pi n/N}e^{-jN2\pi n/N}$$

$$= \sum_{n=0}^{N-1} x(nT)e^{-jk2\pi n/N}e^{-j2\pi n}$$

$$= \sum_{n=0}^{N-1} x(nT)e^{-jk2\pi n/N} = X(k)$$

Important Properties of DFT

❖The fact that X(k+N)=X(k) shows that the DFT is periodic with period N. this is the cyclical property of the DFT. The values of DFT are repetitive.

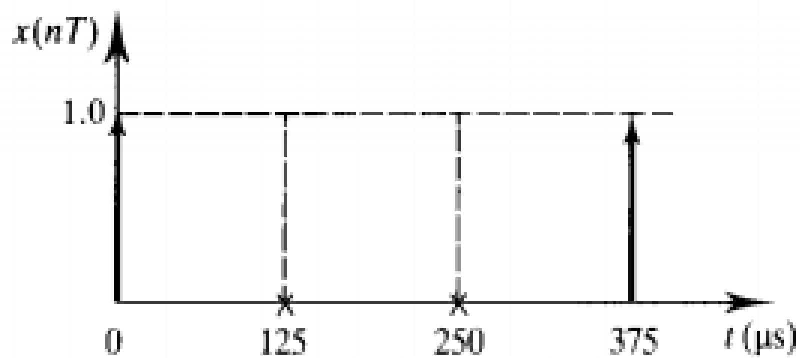
❖ the amplitude spectrum distribution of N-point DFT is symmetrical about harmonic N/2. similarly the phase function being odd exhibits anti-symmetry about harmonic N/2

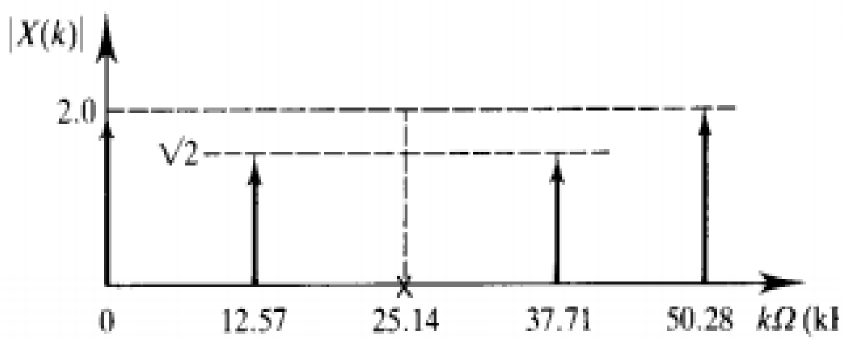
❖The harmonic N/2 represents F_{max} the maximum frequency present in the signal. Thus all the signal components are fully represented in an amplitude spectrum plotted up to F_{max} (harmonic N/2) and it is unnecessary to plot further points.

❖Thus, N real points data transform to N/2 complex DFT values of practical significance.

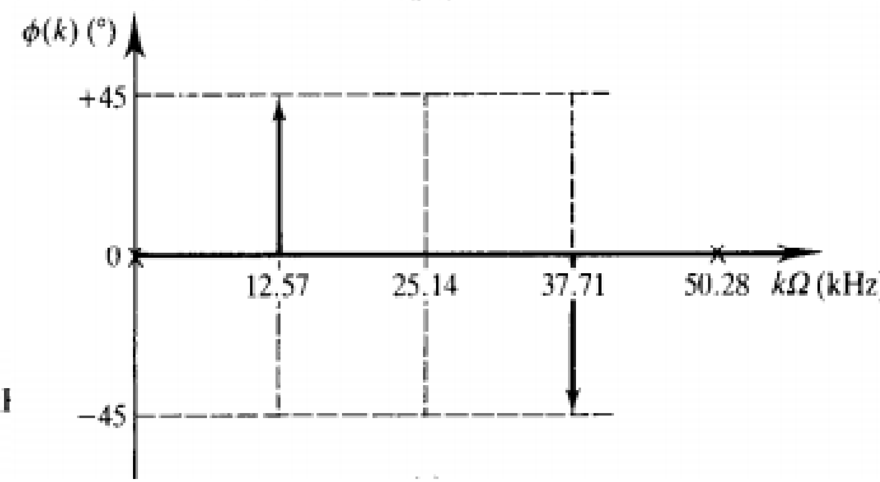
DFT Problems

If T for the sequence {1, 0, 0, 1} equals 125µs then the sampling frequency equals 8kHz and we can have the following representations



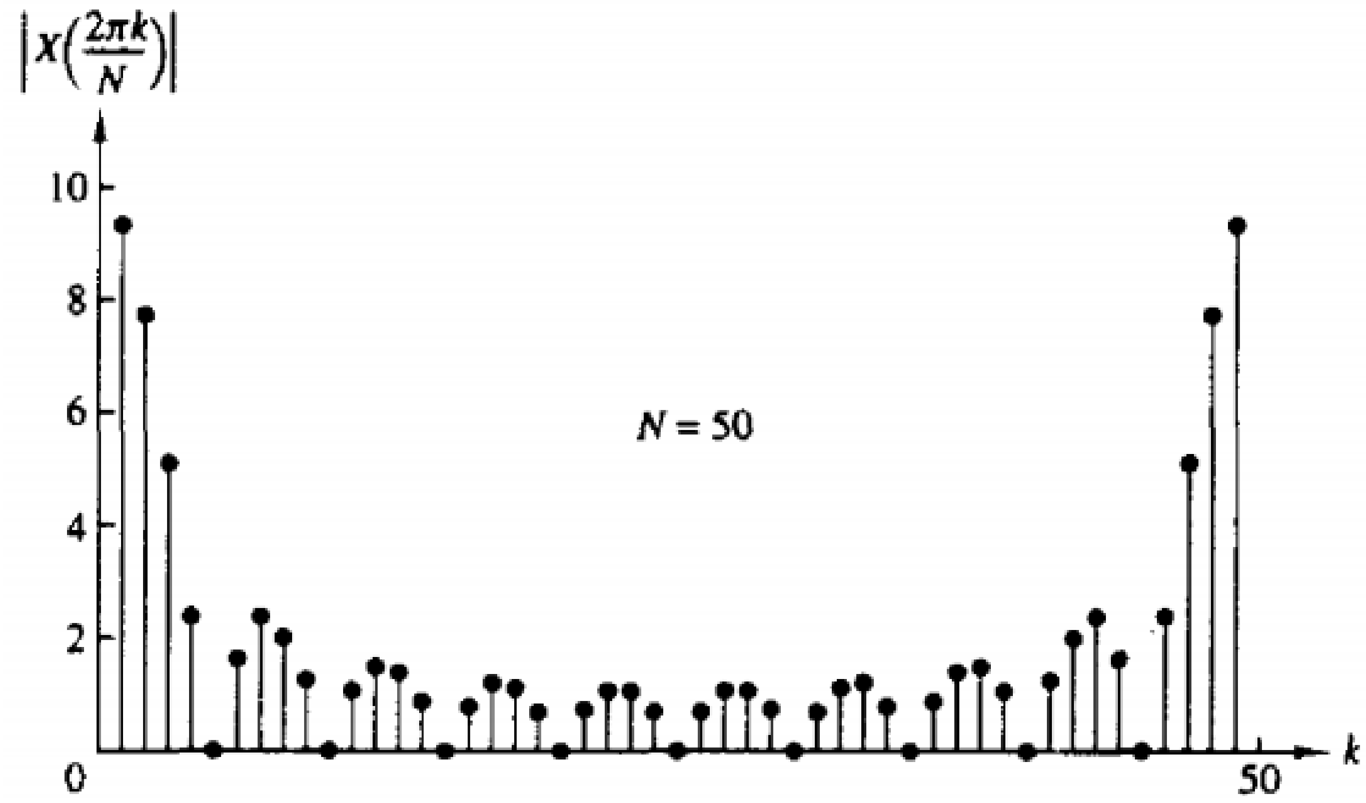


(b)



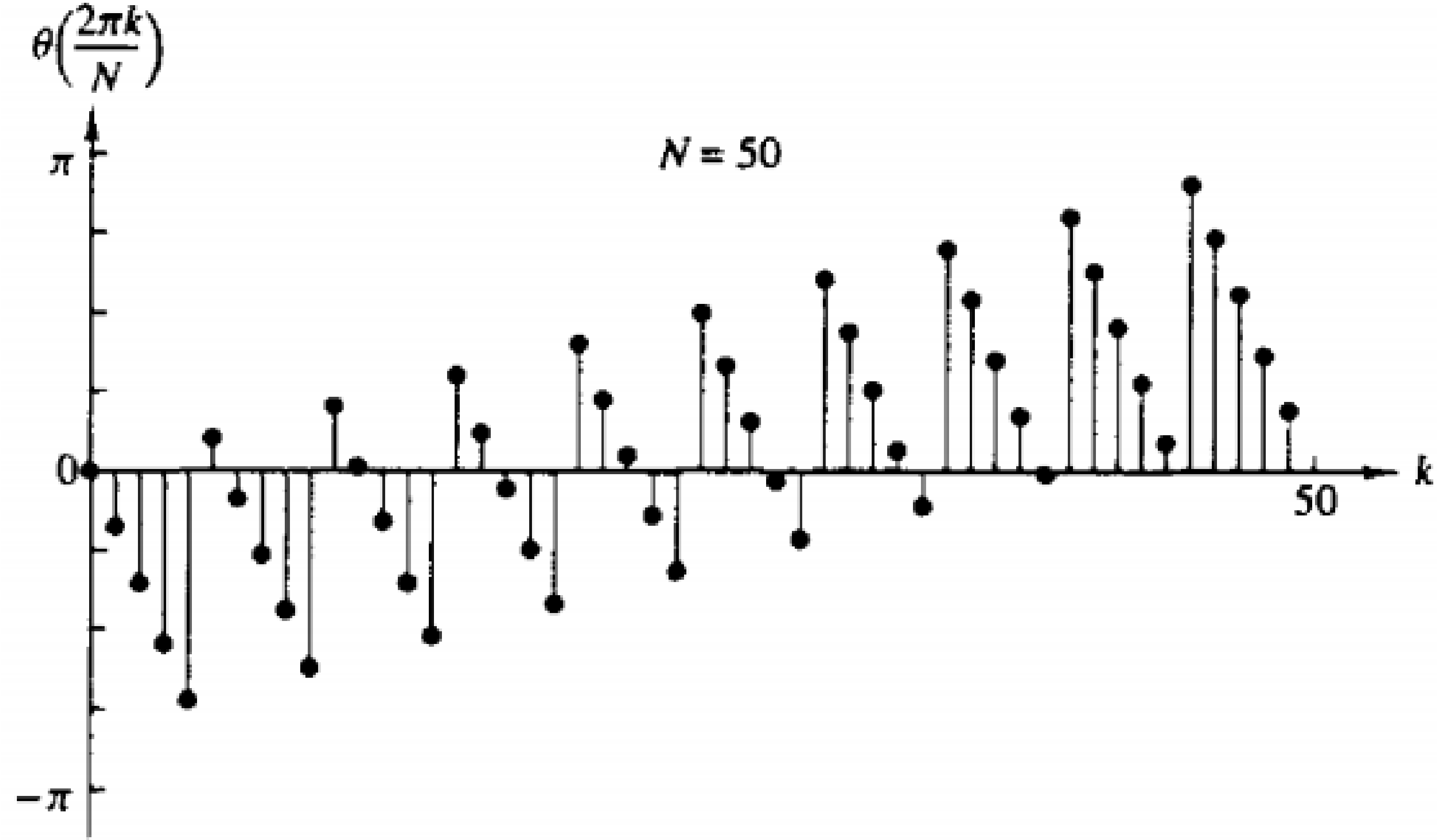
DFT Examples

❖**Amplitude**



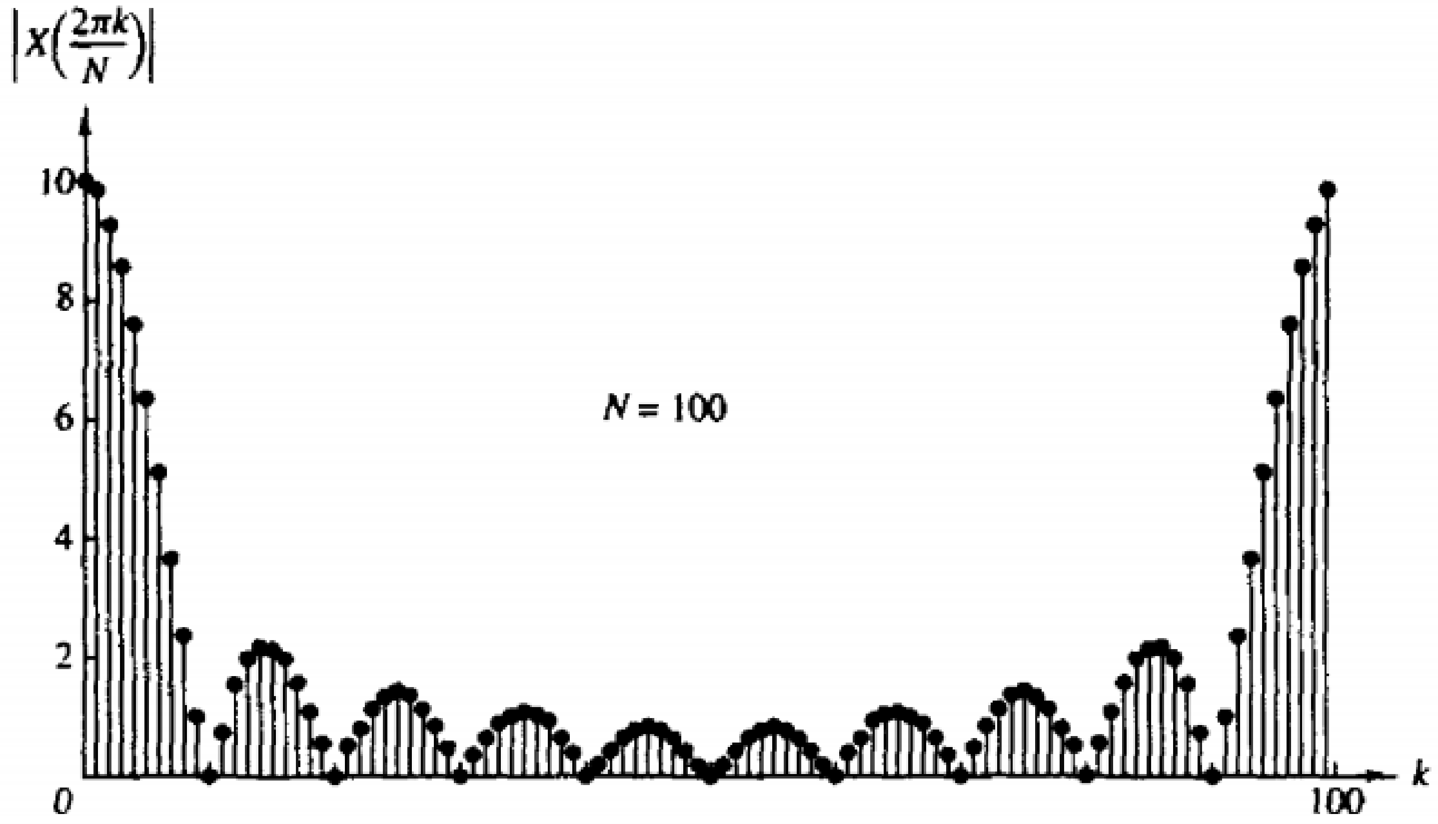
DFT Examples

❖**Phase**



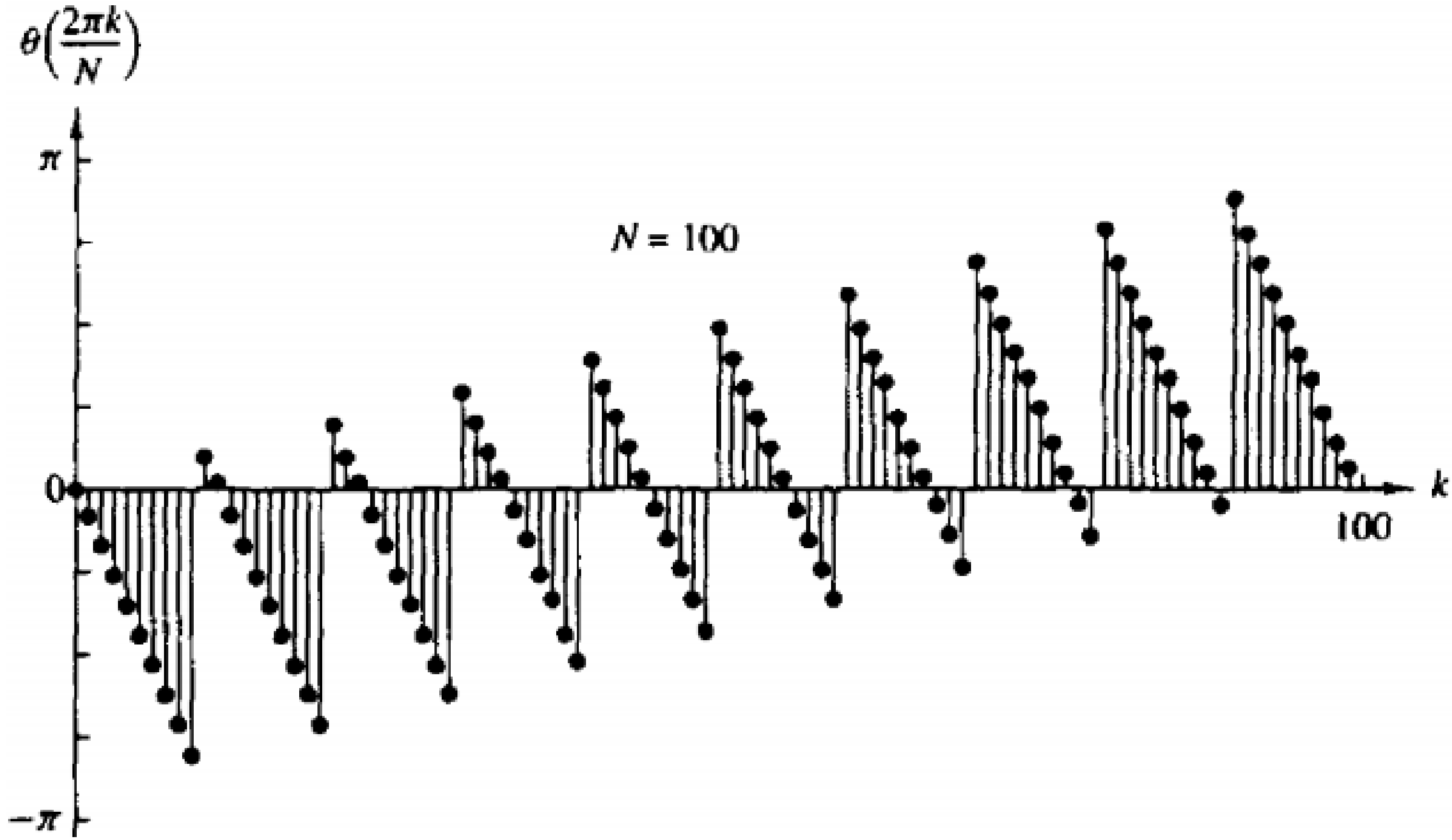
DFT Examples

❖**Amplitude**



DFT Examples

❖ **Phase**



Inverse DFT (IDFT)

- It is necessary to be able to carry out discrete transformation from the frequency to the time domain. This may be achieved using the inverse discrete Fourier transform (IDFT), defined by

$$x(nT) = F_{\mathrm{D}}^{-1}[X(k)] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\mathrm{e}^{\mathrm{j}k\Omega nT}, \; n = 0, 1, \ldots, N-1$$

- The analogy for the inverse Fourier transform is obvious, obtained by dividing IDFT by N.

IDFT Problems

1) Compute the data sequence back from its DFT components $\{2, 1+j, 0, 1-j\}$

IDFT Problems

$$= \tfrac{1}{4}[X(0) + X(1) + X(2) + X(3)]$$
$$= \tfrac{1}{4}[2 + (1 + \mathrm{j}) + 0 + (1 - \mathrm{j})] = 1$$

as expected. With $n = 1$,

$$x(nT) = x(T) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\mathrm{e}^{\mathrm{j}k\Omega T}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\mathrm{e}^{\mathrm{j}k2\pi/N} = \frac{1}{4}\sum_{k=0}^{N-1} X(k)\mathrm{e}^{\mathrm{j}k\pi/2}$$
$$= \tfrac{1}{4}[2 + (1 + \mathrm{j})\mathrm{e}^{\mathrm{j}\pi/2} + 0 + (1 - \mathrm{j})\mathrm{e}^{\mathrm{j}3\pi/2}]$$
$$= \tfrac{1}{4}[2 + (1 + \mathrm{j})\mathrm{j} + (1 - \mathrm{j})(-\mathrm{j})]$$
$$= \tfrac{1}{4}(2 + \mathrm{j} - 1 - \mathrm{j} - 1) = 0$$

as expected. With $n = 2$,

$$x(nT) = x(2T) = \frac{1}{N}\sum_{k=0}^{N-1} x(k)\mathrm{e}^{\mathrm{j}k\pi}$$
$$= \tfrac{1}{4}[2 + (1 + \mathrm{j})\mathrm{e}^{\mathrm{j}\pi} + (1 - \mathrm{j})\mathrm{e}^{\mathrm{j}3\pi}] = \tfrac{1}{4}[2 - (1 + \mathrm{j}) - (1 - \mathrm{j})]$$
$$= 0$$

again, as expected. Finally, with $n = 3$,

$$x(nT) = x(3T) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\mathrm{e}^{\mathrm{j}k3\pi/2}$$
$$= \tfrac{1}{4}[2 + (1 + \mathrm{j})\mathrm{e}^{\mathrm{j}3\pi/2} + (1 - \mathrm{j})\mathrm{e}^{\mathrm{j}9\pi/2}]$$
$$= \tfrac{1}{4}[2 + (1 + \mathrm{j})(-\mathrm{j}) + (1 - \mathrm{j})\mathrm{j}] = \tfrac{1}{4}(2 - \mathrm{j} + 1 + \mathrm{j} + 1) = 1$$

IDFT Problems

2) Compute the data sequence back from its DFT components $\{6, -2+2j, -2, -2-2j\}$

IDFT Problems

2) Compute the data sequence back from its DFT components { 6, -2+2j, -2, -2-2j}

Time & Frequency Domains

- Time domain: All time information (samples) is available but no information about frequency components.

- Frequency domain : Frequency components are known but no information about time.

- Do we need sometimes information about both time & frequency ??

  YES, when we need to when a specific frequency occurs since it doesn’t exist all the time

Low & High Frequencies

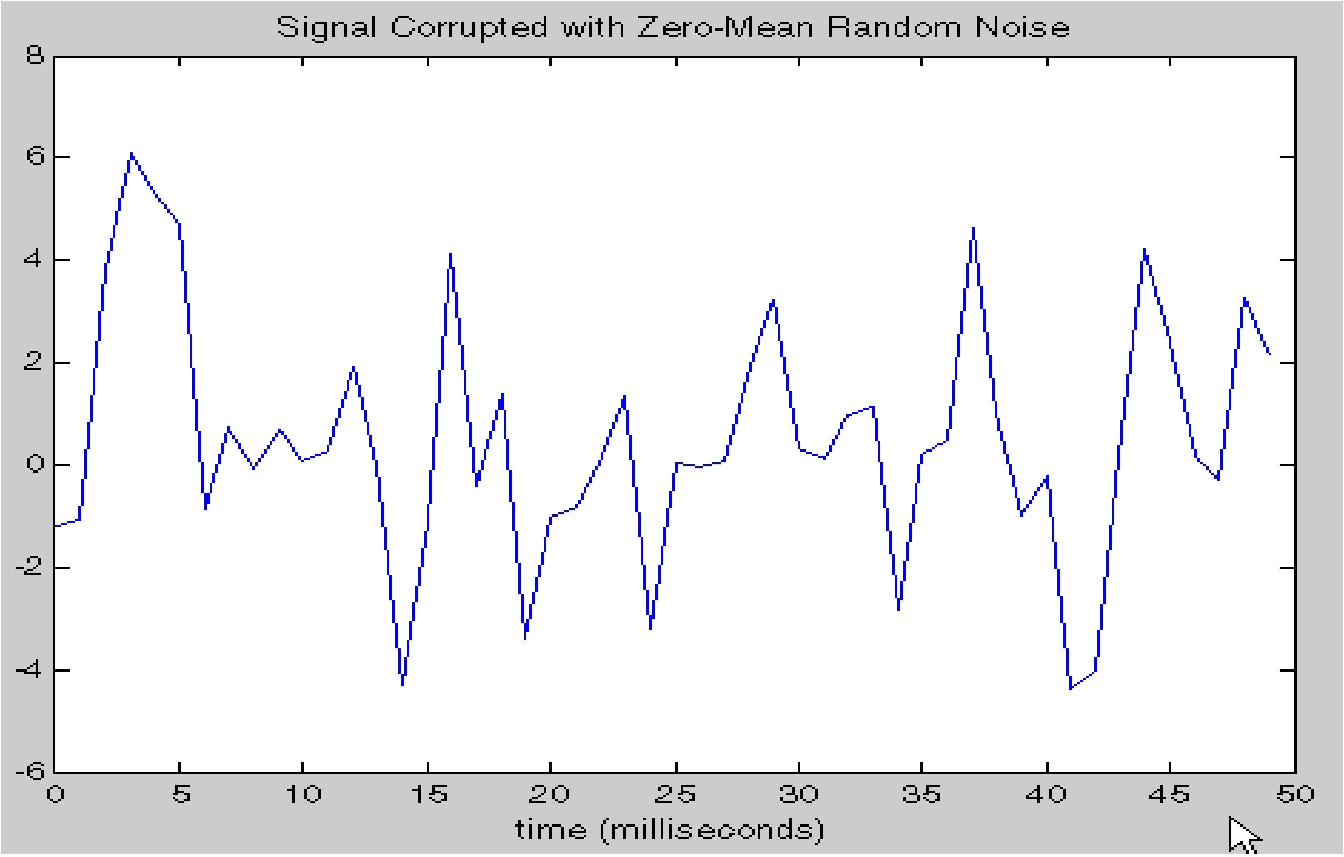
- Low Frequencies: Slow small change in values
  signal → smooth   image -> blurring

- High Frequencies: Sudden noticeable change in values
  signal → includes sharp peaks   image → includes sharp edges

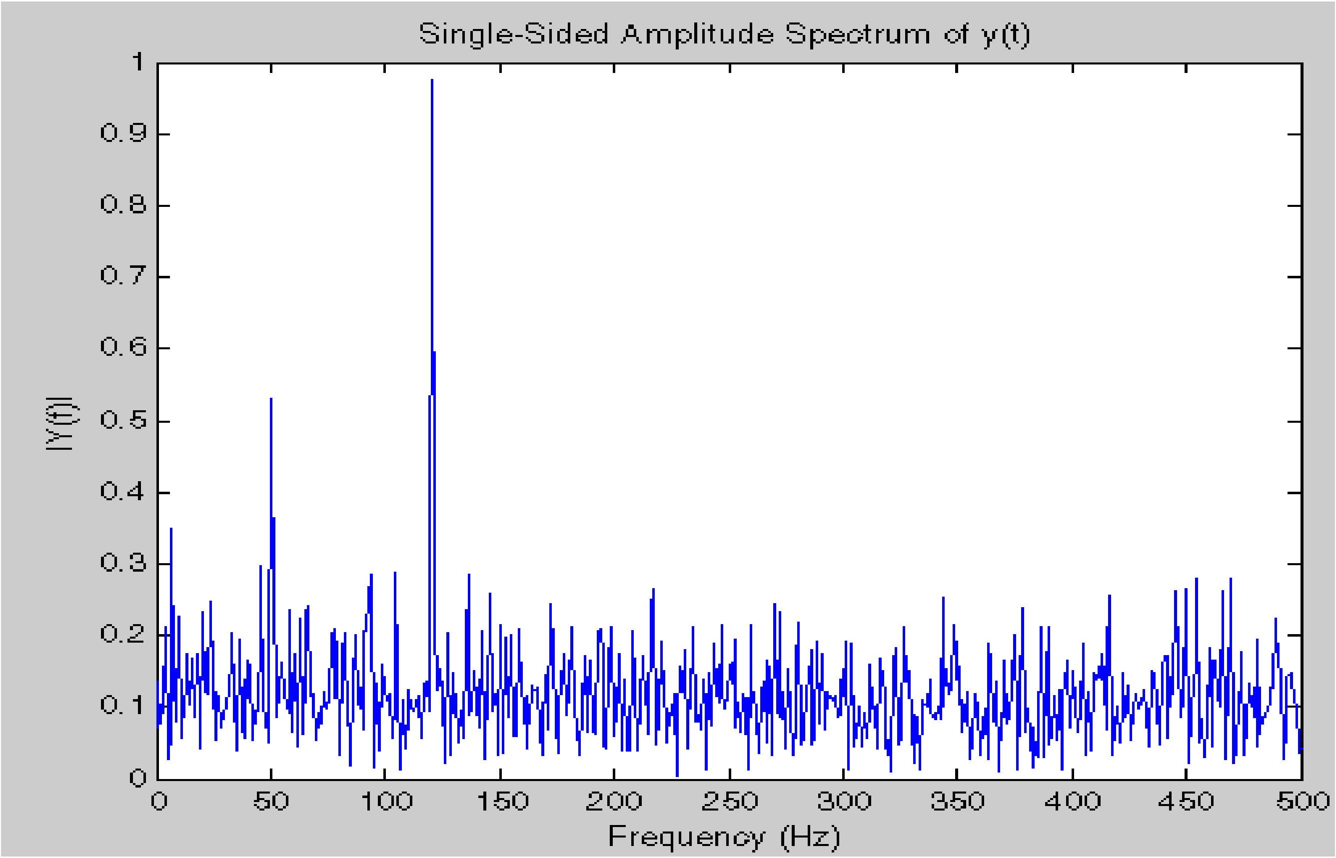
- Smoothing of signals and images preserves low frequencies
  Smoothing → Moving average or low pass filters

- Sharpening of signals and images preserves high frequencies
  Sharpening → Derivative or high pass filters

Frequency Selection (noise removal)



Frequency Selection (noise removal)

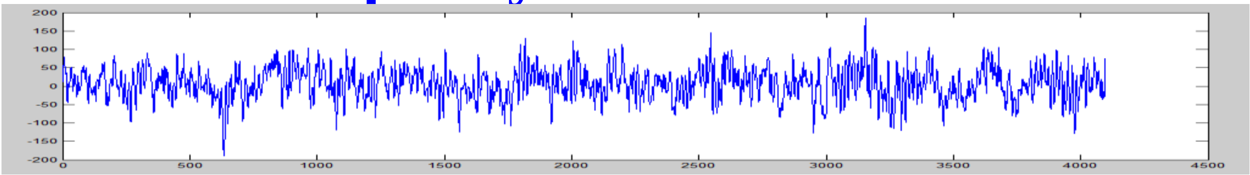


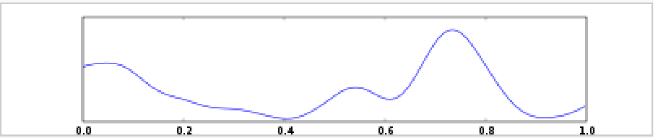
Frequency Selection (Band Selection)

- Sometimes we need to concentrate on a specific frequency band and neglect the others. Band pass filters can be used.

Example: EEG Frequency Bands

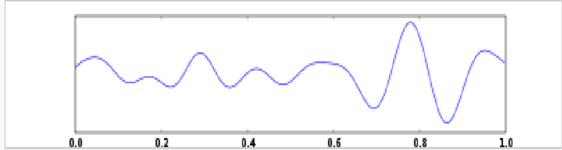
EEG





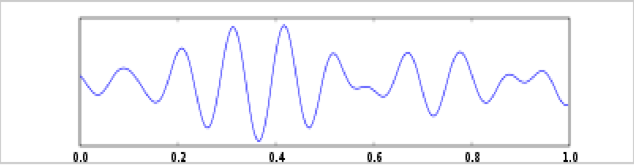
delta waves.

< 4 HZ



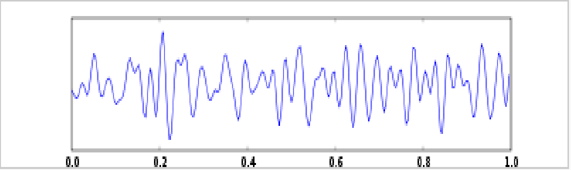
theta waves.

4 to 7 HZ



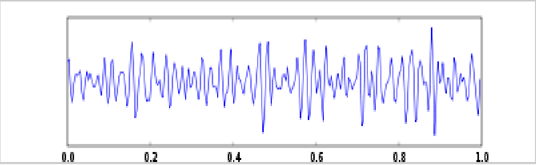
alpha waves.

8 to 14 HZ



beta waves.

15 to 30 HZ



gamma waves.

>30 HZ

DC Component Removal (mean removal)

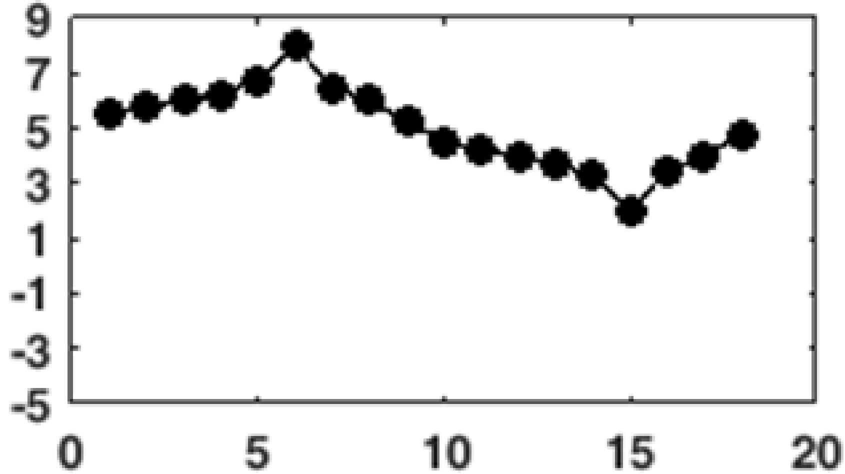
- The DC component if it is not equal zero, it makes the signal oscillates on a horizontal line rather than the X axis. Thus, the signal is shifted upwards or downwards.

- We are usually interested in removing the shift and allow the signal to oscillate back on the X axis.

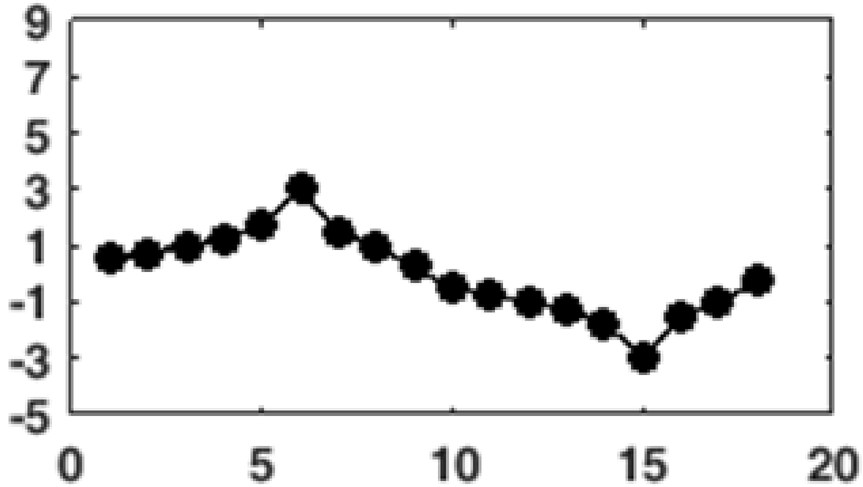
- For time Domain → Subtract the mean of the signal from it.

- For Frequency domain → remove the first harmonic X(0).

DC Component Removal (mean removal)



DC equals 5



After removing DC

Computational Complexity of DFT

- Each term consists of a multiplication of an exponential term which is always complex by another term which is real or complex. Each of the product terms is added together. Therefore, for N points DFT there will be N complex multiplication and N-1 complex additions to be calculated. There are also N points to be evaluated. Thus, N^2 complex multiplications and N(N-1) complex additions.

- What happens if we have N=1000 or N = 10000 ?? **Huge computation**

- Can we reduce the computation ?? **Yes**

Thanks for Your Attention