Discrete Signals & Systems
(Convolution & Correlation)
(DSP)
Part 3

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Related to Frequency Domain

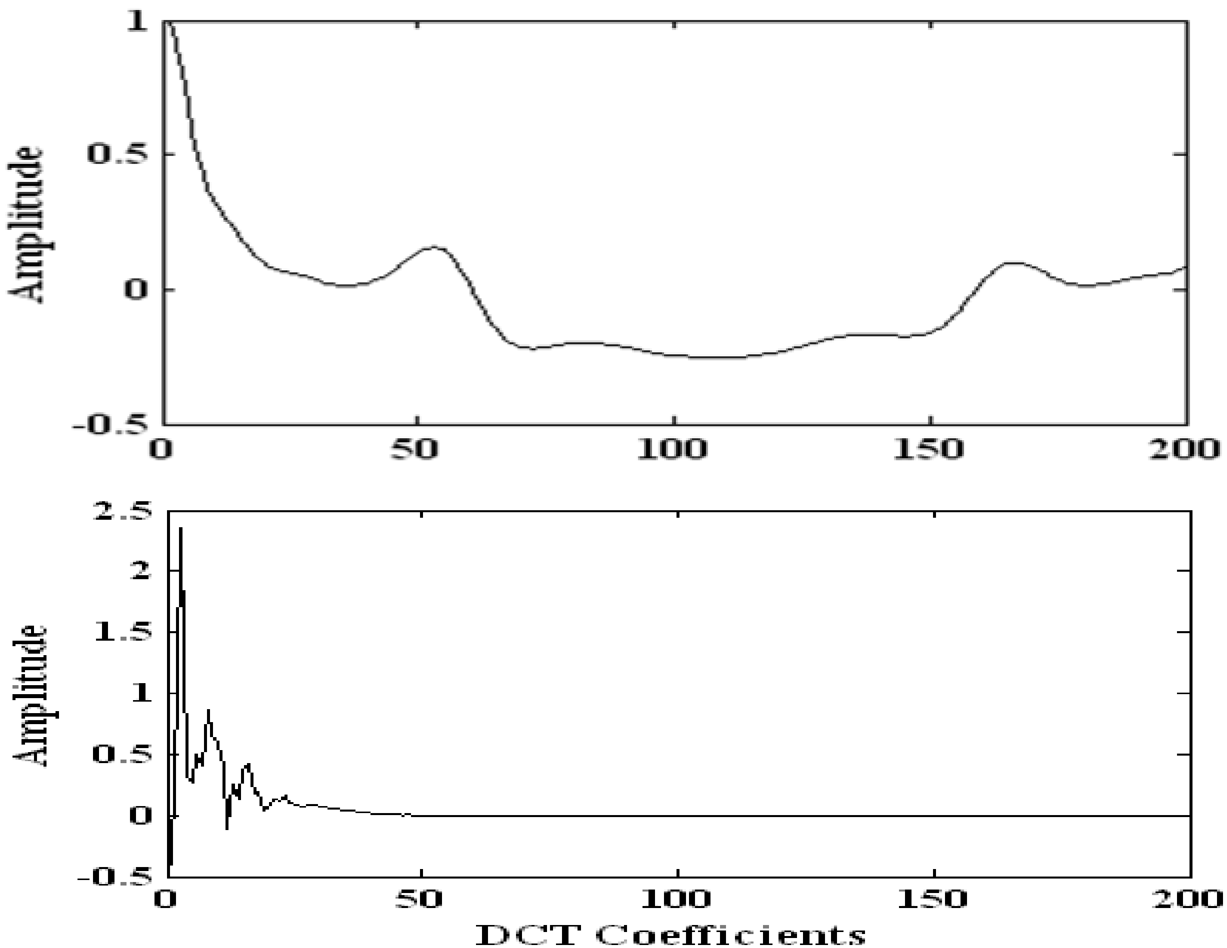
Discrete Cosine Transform (DCT)

❖ The DCT is frequently used in lossy data compression applications, such as the JPEG image format.

❖The property of the DCT that makes it quite suitable for compression is its high degree of "spectral compaction;" at a qualitative level, a signal's DCT representation tends to have more of its energy concentrated in a small number of coefficients when compared to other transforms like the DFT.

❖This is desirable for a compression algorithm; if you can approximately represent the original (time- or spatial-domain) signal using a relatively small set of DCT coefficients, then you can reduce your data storage requirement by only storing the DCT outputs that contain significant amounts of energy (ex: JPG images).

Discrete Cosine Transform (DCT)



Part-three Outlines
Discrete Systems

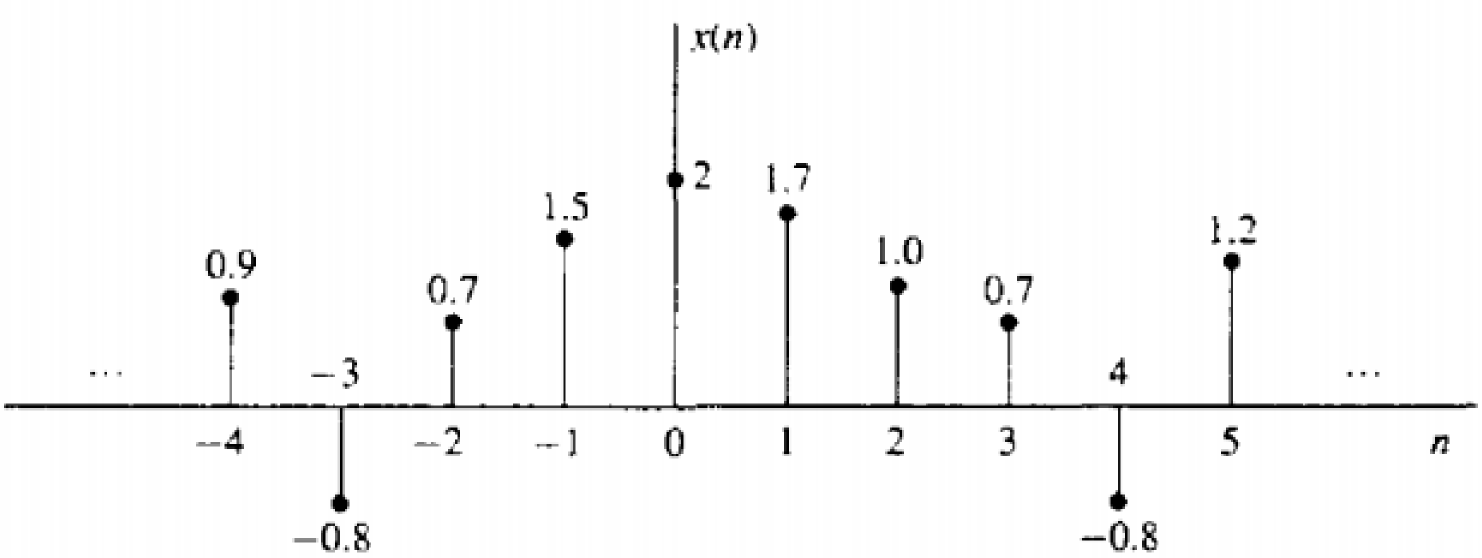
- *Discrete Signals*
- *Discrete Systems*
- *Block Diagram Representation of Discrete-Time Systems*
- *Classification of Discrete Systems*
- *Analyzing Linear Time Invariant (LTI) Systems*
- *Convolution*
- *Correlation*
- *Case Study*

Discrete Signals

- It is important to note that a discrete-time signal is not defined at instants between two successive samples. Also it is incorrect to think that x(n) is not defined for non-integer values of n. simply, the signal is not defined for non-integer values of n.
- In the sequel we will assume that a discrete-time signal is defined for every integer value n for $-\infty < n < \infty$. By tradition , we refer to x(n) as the "nth sample " of the signal even if the signal x(n) is inherently discrete time (i.e., not obtained by sampling an analog signal). If, indeed , x(n) was obtained from sampling an analog signal $X_a(t)$, then $x(n) = X_a(nT)$, where T is the sampling period (i.e., the time between successive samples).

Discrete Signals
Different Representations

Graphical Representation



Discrete Signals
Different Representations

1. Functional representation, such as

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases}$$

2. Tabular representation, such as

n	$\cdots$	-2	-1	0	1	2	3	4	5	$\ldots$
$x(n)$	$\cdots$	0	0	0	1	4	1	0	0	$\ldots$

3. Sequence representation

An infinite-duration signal or sequence with the time origin ($n = 0$) indicated by the symbol $\uparrow$ is represented as

$$x(n) = \{\ldots 0, \underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\}$$

A sequence $x(n)$, which is zero for $n < 0$, can be represented as

$$x(n) = \{\underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\}$$

The time origin for a sequence $x(n)$, which is zero for $n < 0$, is understood to be the first (leftmost) point in the sequence.

Discrete Signals
Different Representations

A finite-duration sequence can be represented as

$$x(n) = \{3, -1, \underset{\uparrow}{-2}, 5, 0, 4, -1\}$$

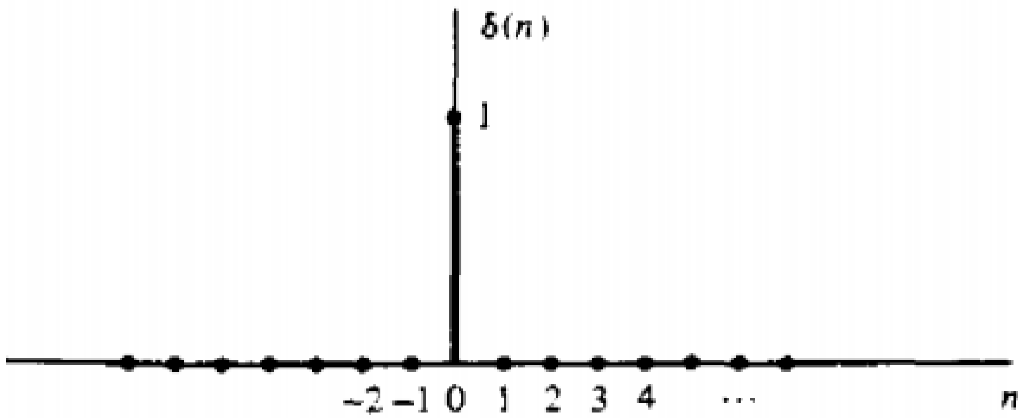
whereas a finite-duration sequence that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as

$$x(n) = \{\underset{\uparrow}{0}, 1, 4, 1\}$$

Discrete Signals
Elementary ones

Unit Sample (Impulse) Sequence

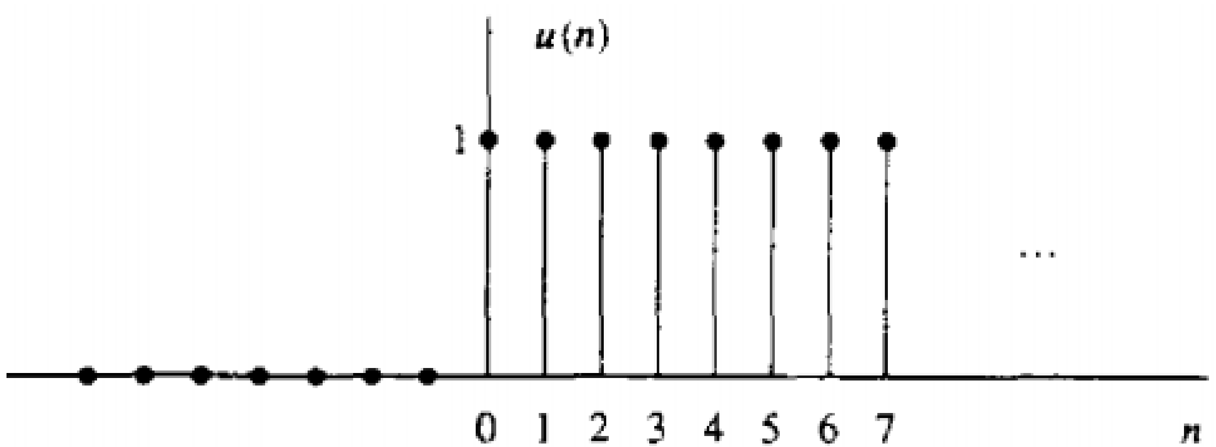
$$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases}$$



Discrete Signals
Elementary ones

Unit step Signal

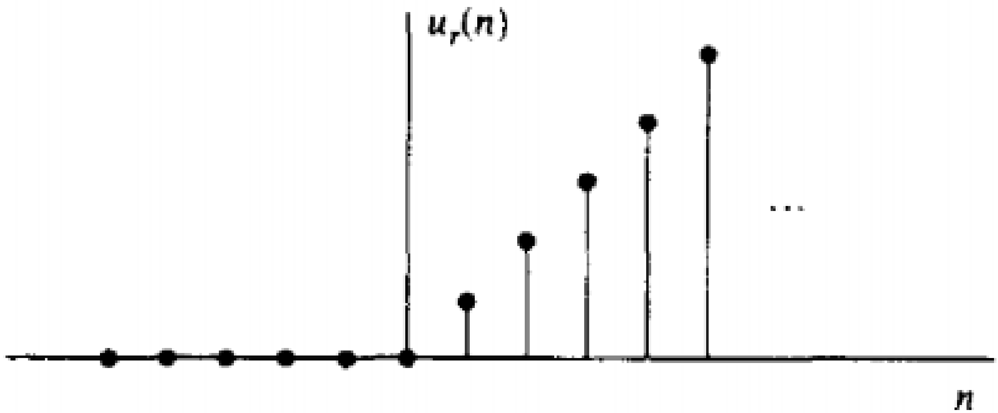
$$u(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$



Discrete Signals Elementary ones

Unit Ramp Signal

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$



Classification of Discrete Signals

❖Periodic & aPeriodic (non-periodic) Signals

A signal x(n) is periodic with period N (N > 0) if and only if

$$x(n+N) = x(n) \text{ for all } n$$

Else the signal is non-periodic and is called aperiodic signal

Classification of Discrete Signals

❖Symmetric & Anti-Symmetric Signals

A real valued signal x(n) is called symmetric (even) if

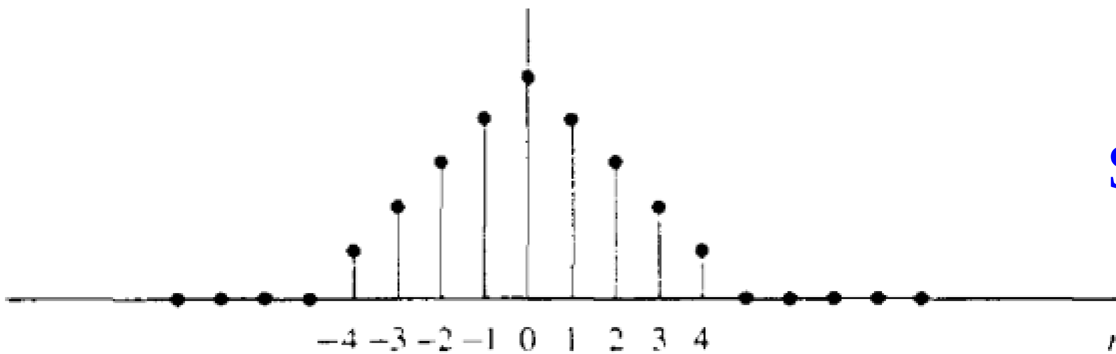
$$x(-n) = x(n)$$

O n the other hand , a signal x(n) is called anti-symmetric (odd) if

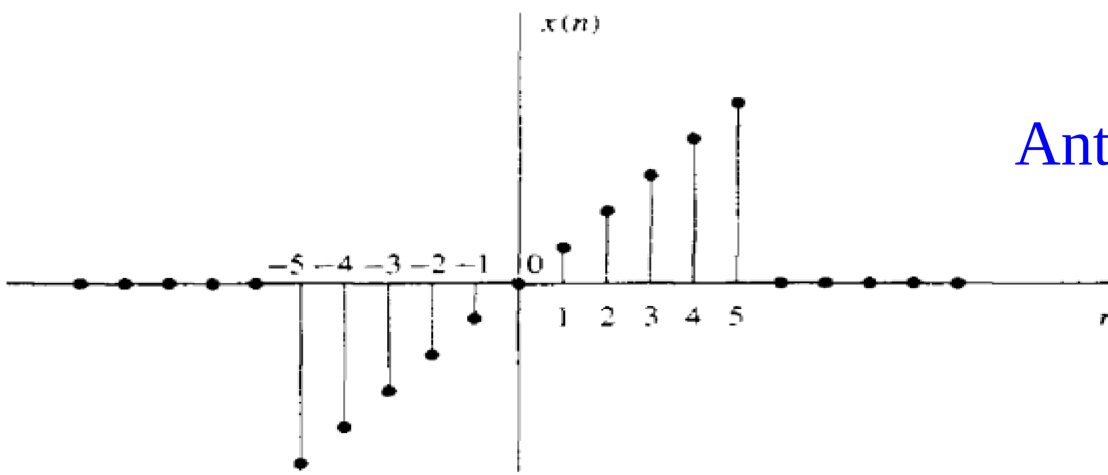
$$x(-n) = -x(n)$$

Classification of Discrete Signals

❖Symmetric & Anti-Symmetric Signals



Symmetric



Anti-Symmetric

Simple Manipulations of Discrete-Time Signals

❖**Transformation of the independent variable (time)**

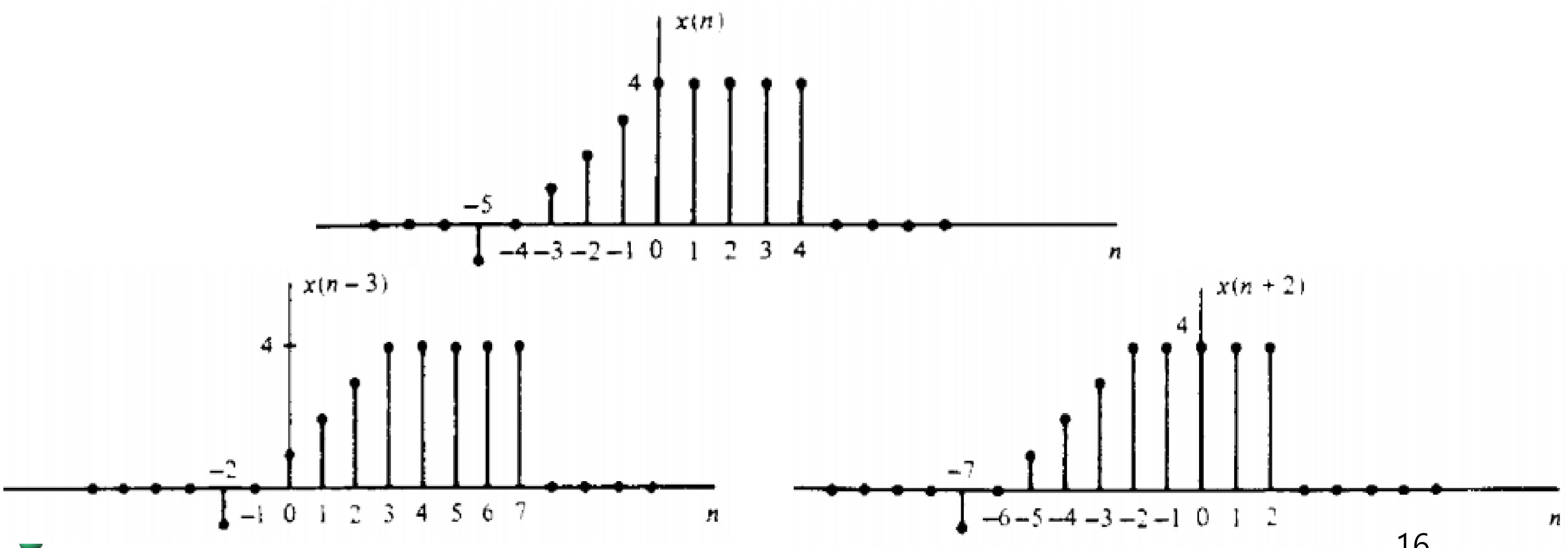
A signal x(n) may be shifted in time by replacing the independent variable n by n-k, where k is an integer. If k is a positive integer, the time shift results in a delay of the sig nal by k units of time. If k is a negative integer, the time s hift results in an advance of the signal by " k" units in time.

❖If the signal x(n) is stored, it is a relatively simple operation to modify the base by introducing a delay or an advance. On the other hand, if the signal is in real time, it is not possible to advance the signal in time.

Simple Manipulations of Discrete-Time Signals

❖**Transformation of the independent variable (time)**

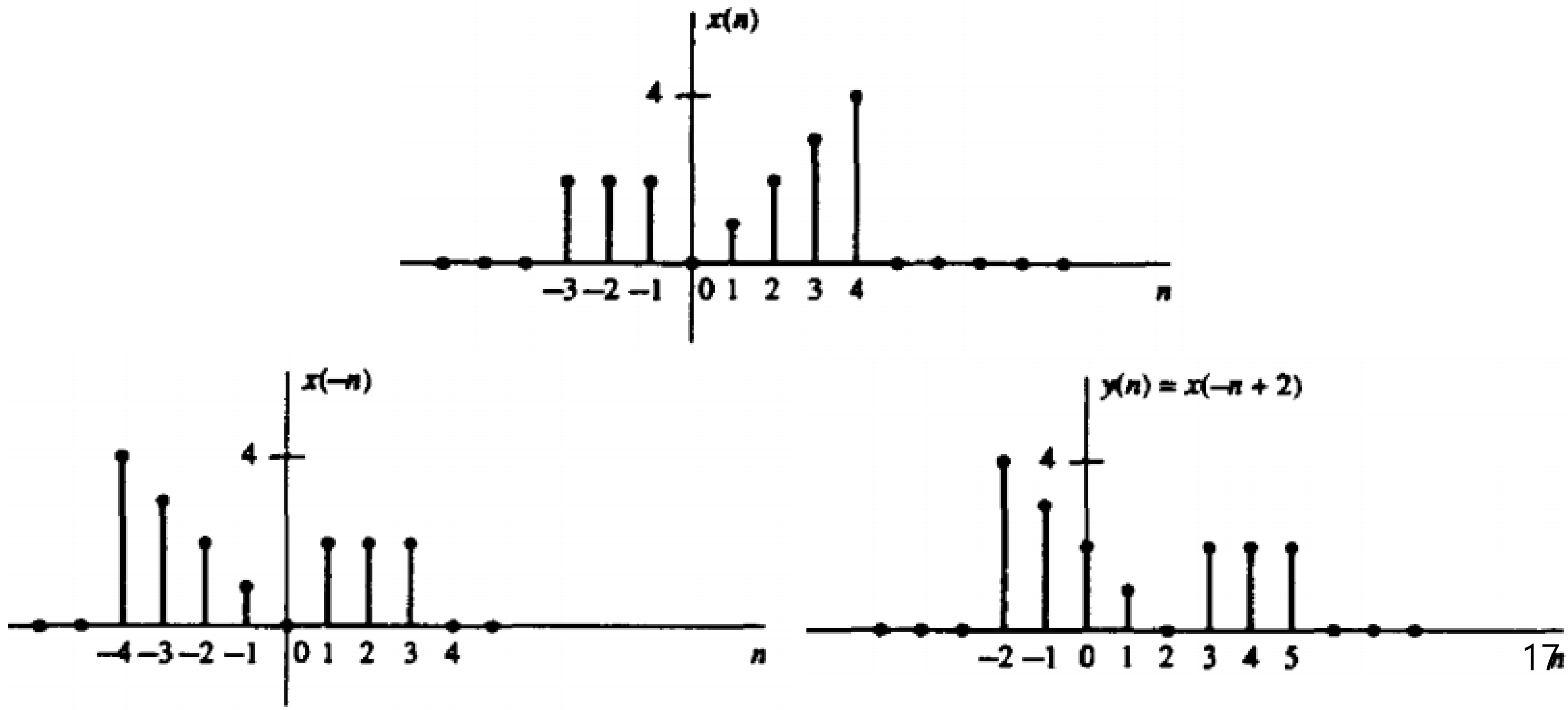
A signal x(n) is graphically represented. Show a graphical representation of the signals x(n-3) and x(n+2).



Simple Manipulations of Discrete-Time Signals

❖**Transformation of the independent variable (time) (folding)**

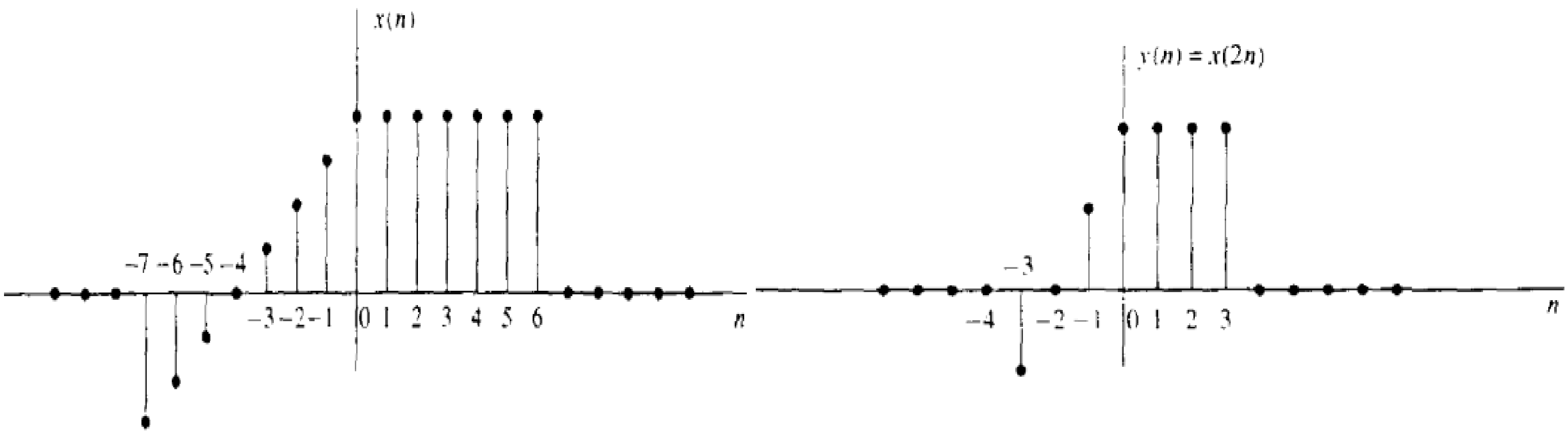
A signal x(n) is graphically represented. Show a graphical representation of the signals x(-n) and x(-n+2).



Simple Manipulations of Discrete-Time Signals

❖**Transformation of the independent variable (time) (sampling down)**

Show the graphical representation of the signal y(n) = x(2n)



Simple Manipulations of Discrete-Time Signals

❖**Addition, multiplication, and scaling of sequences**

Amplitude scaling o f a signal by a constant A is accomplished by multiplying the value o f every signal sample by A. Consequently , we obtain

$$y(n) = Ax(n) \qquad -\infty < n < \infty$$

The sum of two signals X_1 (n) and X_2 (n) is a signal y(n), whose value at any instant is equal to the sum of the values of these two signals at that instant, that is

$$y(n) = x_1(n) + x_2(n) \qquad -\infty < n < \infty$$

The product of two signals is similarly defined on a sample-to-sample basis as

$$y(n) = x_1(n)x_2(n) \qquad -\infty < n < \infty$$

Discrete time systems

- Device or algorithm that perform some prescribed operations on a discrete time signal.

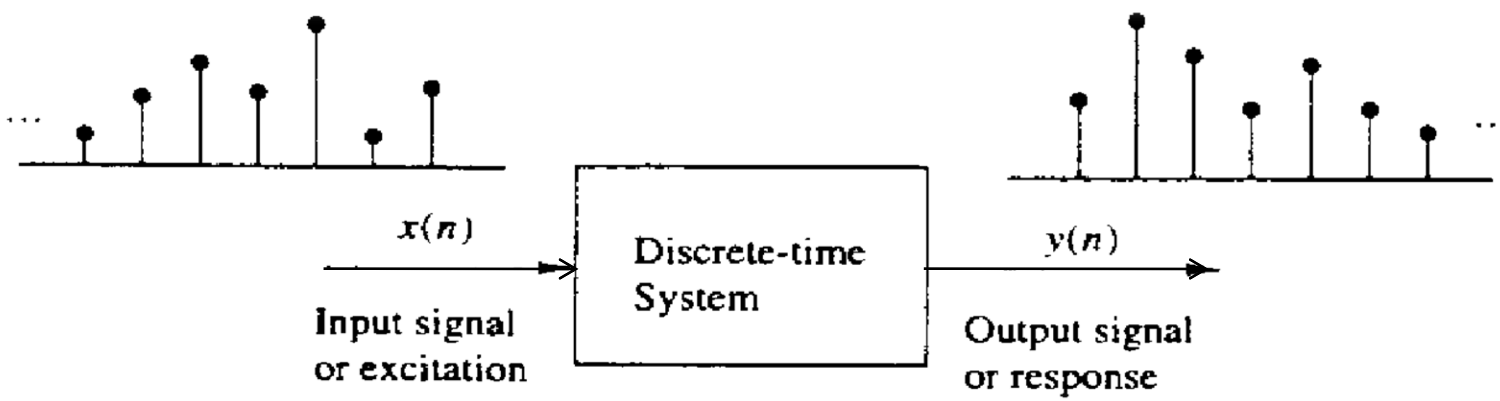
- Discrete time system is a device or algorithm that operate on discrete time signal called the **input or excitation** according to some well defined rules to produce another discrete time signal called **output or response** of the system.

Discrete time systems
Input/output description

- The input signal x(n) is transformed by the system into a signal y(n) expressed as:

$$y(n) \equiv \mathcal{T}[x(n)]$$

- г denotes the transformation(operator) or processing performed by the system on x(n) to produce y(n)



Discrete time systems
Input/output description

- Determine the response of the following systems to the input signal

$$x(n) = \begin{cases} |n|, & -3 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

(a) $y(n) = x(n)$

(b) $y(n) = x(n-1)$

(c) $y(n) = x(n+1)$

(d) $y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$

(e) $y(n) = max\{x(n+1), x(n), x(n-1)\}$

(f) $y(n) = \sum_{k=-\infty}^{n} x(k) = x(n) + x(n-1) + x(n-2) + \cdots$

Discrete time systems

Input/output description

$$x(n) = \{\ldots, 0, 3, 2, 1, \underset{\uparrow}{0}, 1, 2, 3, 0, \ldots\}$$

a) identity system.

b) system delays input by one sample.

Y(n) $= \{\ldots, 0, 3, 2, \underset{\uparrow}{1}, 0, 1, 2, 3, 0, \ldots\}$

c) Advances the input: y(n) $= \{\ldots, 0, 3, 2, 1, 0, \underset{\uparrow}{1}, 2, 3, 0, \ldots\}$

d) $y(n) = \{\ldots, 0, 1, \frac{5}{3}, 2, 1, \underset{\uparrow}{\frac{2}{3}}, 1, 2, \frac{5}{3}, 1, 0, \ldots\}$

e) $y(n) = \{0, 3, 3, 3, 2, \underset{\uparrow}{1}, 2, 3, 3, 3, 0, \ldots\}$

f) accumulator $y(n) = \{\ldots, 0, 3, 5, 6, \underset{\uparrow}{6}, 7, 9, 12, 12, \ldots\}$

Classification of Discrete Systems

❖ **Static vs. dynamic systems**

Static: or called memory less, if the output at any instant n depends at most on input sample at the same time, but no past or future samples of input. Any other case the system is said to be dynamic or have memory

❖ Examples

- Static $$y(n) = nx(n) + bx^3(n)$$
- Dynamic $$y(n) = x(n) + 3x(n-1)$$

$$y(n) = \sum_{k=0}^{n} x(n-k)$$

Time Invariant Vs. Time Variant systems

Definition

A relaxed system T is time invariant or shift invariant if and only if the input-output characteristics don't change with time

$$x(n) \xrightarrow{\mathcal{T}} y(n)$$

implies that $$x(n-k) \xrightarrow{\mathcal{T}} y(n-k)$$

for every input signal *x(n)* and every time shift k

Time Invariant Vs. Time Variant systems

If there exists a system with an arbitrary input sequence x(n), which produces an output denoted as y(n). Next we delay the input sequence by an amount k and compute the output. In general we can write the output as

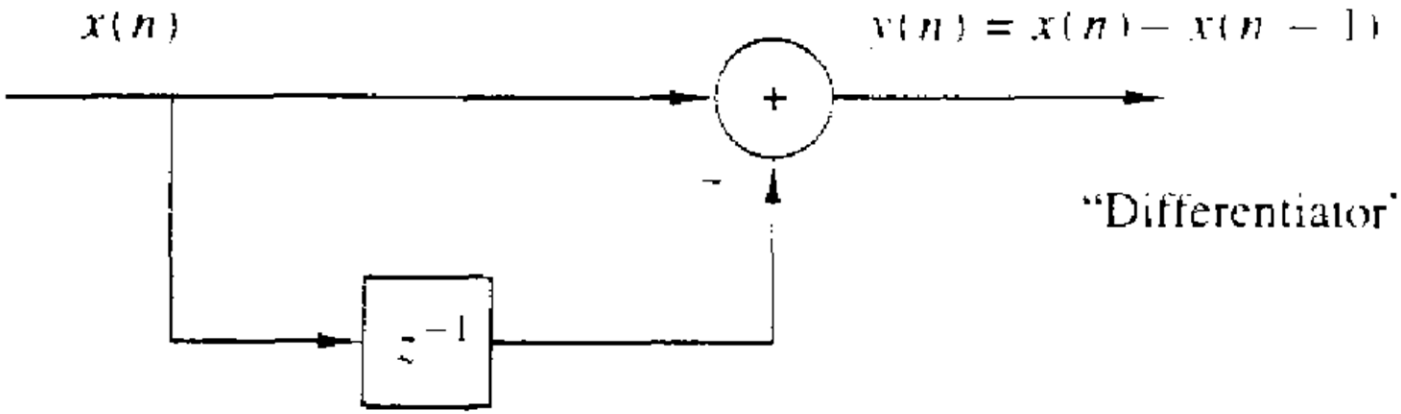
$$y(n,k) = T[x(n-k)]$$

Now if this output y (n, k) = y(n - k), for all possible values of k, the system is time invariant. On the other hand , if the output y(n, k) $\neq$ y(n - k), even for one value of k, the system is time variant.

Time Invariant Vs. Time Variant systems

given the system

$$y(n) = T[x(n)] = x(n) - x(n-1) \qquad (1)$$



(a)

Time Invariant Vs. Time Variant systems

delaying input with k units

$$y(n,k) = x(n-k) - x(n-k-1) \qquad (2)$$

delaying output with k units

$$y(n-k) = x(n-k) - x(n-k-1) \qquad (3)$$

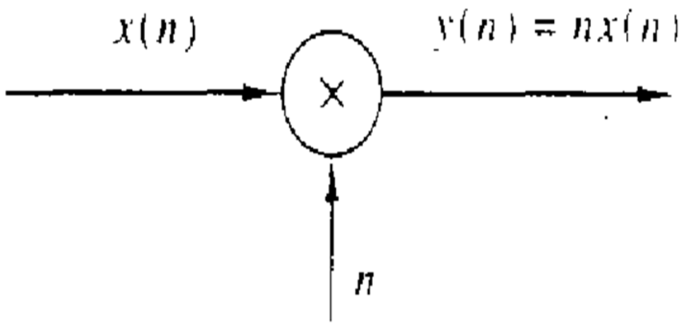
since the RHS of (2) & (3) are identical therefore y(n, k)=y(n-k)

system is time invariant

Time Invariant Vs. Time Variant systems

given the system

$$y(n) = T[x(n)] = nx(n)$$



"Time" multiplier

(b)

Time Invariant Vs. Time Variant systems

delaying input with k units

$$y(n,k) = nx(n-k) \qquad (2)$$

delaying output with k units

$$\begin{aligned} y(n-k) &= (n-k)x(n-k) \\ &= nx(n-k) - kx(n-k) \end{aligned} \qquad (3)$$

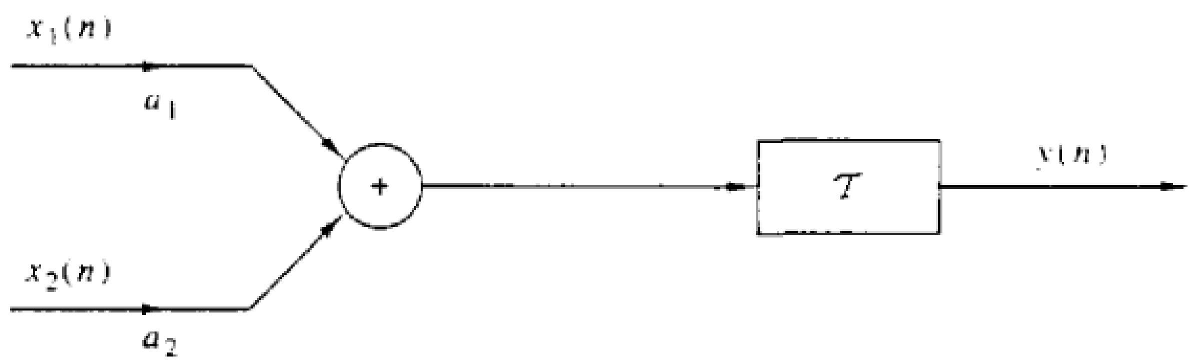
since the RHS of (2) & (3) are not identical , therefore $y(n, k) \neq y(n-k)$,system is time variant

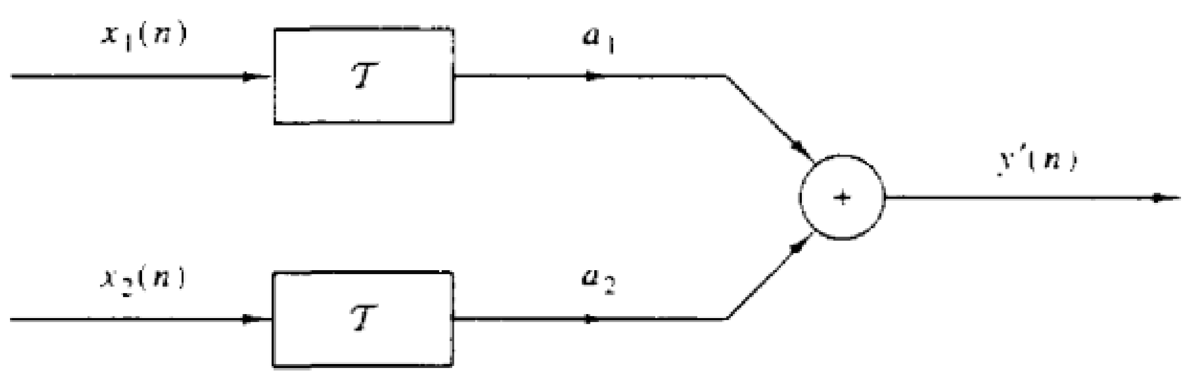
Linear Vs. Non Linear Systems

The general class of systems can also be subdivided into line ar systems and nonlinear systems. A linear system is one that satisfies the superposition principle. Simply stated, the principle of superposition requires that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum o f the responses (outputs) o f the system to each of the in dividual input signals. Hence, we have the following definition of linearity

$$\mathcal{T}[a_1x_1(n) + a_2x_2(n)] = a_1\mathcal{T}[x_1(n)] + a_2\mathcal{T}[x_2(n)]$$

Linear Vs. Non Linear Systems





Linear Vs. Non Linear Systems

Multiplicative or scaling property:

Where

$$y_1(n) = \mathcal{T}[x_1(n)]$$

$$\mathcal{T}[a_1 x_1(n)] = a_1 \mathcal{T}[x_1(n)] = a_1 y_1(n)$$

If the response of the system to the input $x_1(n)$ is $y_1(n)$ then the response to $a_1x_1(n)$ is $a_1y_1\,(n)$ any scaling of input results is an identical scaling of corresponding output

Linear Vs. Non Linear Systems

Addition property:

$$\mathcal{T}[x_1(n) + x_2(n)] = \mathcal{T}[x_1(n)] + \mathcal{T}[x_1(n)]$$
$$= y_1(n) + y_2(n)$$

The linearity condition can be extended arbitrarily to any weighted linear combination of signals by induction. In general, we have

$$x(n) = \sum_{k=1}^{M-1} a_k x_k(n) \xrightarrow{\mathcal{T}} y(n) = \sum_{k=1}^{M-1} a_k y_k(n)$$

A relaxed , linear system with zero input produces a zero output. If a system produces a non zero output with a zero input, the system may be either non relaxed or non linear. If a relaxed system does not satisfy the superposition principle as given by the definition above, it is called nonlinear.

Linear Vs. Non Linear Systems

Determine if the systems described by the following input–output equations are linear or nonlinear.

(a) $y(n) = nx(n)$ **(b)** $y(n) = x(n^2)$ **(c)** $y(n) = x^2(n)$

(d) $y(n) = Ax(n) + B$ **(e)** $y(n) = e^{x(n)}$

Linear Vs. Non Linear Systems

(a) For two input sequences $x_1(n)$ and $x_2(n)$, the corresponding outputs are

$$y_1(n) = nx_1(n)$$

$$y_2(n) = nx_2(n)$$

A linear combination of the two input sequences results in the output

$$y_3(n) = T[a_1x_1(n) + a_2x_2(n)] = n[a_1x_1(n) + a_2x_2(n)]$$

$$= a_1nx_1(n) + a_2nx_2(n) \quad \textbf{(1)}$$

On the other hand, a linear combination of the two outputs in results in the output

$$a_1y_1(n) + a_2y_2(n) = a_1nx_1(n) + a_2nx_2(n) \quad \textbf{(2)}$$

Since the right-hand sides of **(1)** and **(2)** are identical, the system is linear.

Linear Vs. Non Linear Systems

(b) As in part (a), we find the response of the system to two separate input signals $x_1(n)$ and $x_2(n)$. The result is

$$y_1(n) = x_1(n^2)$$

$$y_2(n) = x_2(n^2)$$

The output of the system to a linear combination of $x_1(n)$ and $x_2(n)$ is

$$y_3(n) = \mathcal{T}[a_1x_1(n) + a_2x_2(n)] = a_1x_1(n^2) + a_2x_2(n^2) \qquad \textbf{(1)}$$

Finally, a linear combination of the two outputs

$$a_1y_1(n) + a_2y_2(n) = a_1x_1(n^2) + a_2x_2(n^2) \qquad \textbf{(2)}$$

By comparing **(1)** with **(2)**, we conclude that the system is linear.

Linear Vs. Non Linear Systems

c) The responses of the system to two separate input signals are

$$y_1(n) = x_1^2(n)$$

$$y_2(n) = x_2^2(n)$$

The response of the system to a linear combination of these two input signals is

$$\begin{aligned} y_3(n) &= \mathcal{T}[a_1x_1(n) + a_2x_2(n)] \\ &= [a_1x_1(n) + a_2x_2(n)]^2 \\ &= a_1^2x_1^2(n) + 2a_1a_2x_1(n)x_2(n) + a_2^2x_2^2(n) \quad \textbf{(1)} \end{aligned}$$

On the other hand, if the system is linear, it would produce a linear combination of the two outputs in namely,

$$a_1y_1(n) + a_2y_2(n) = a_1x_1^2(n) + a_2x_2^2(n) \quad \textbf{(2)}$$

Since the actual output of the system, as given by **(1)**, is not equal to **(2)**, the system is nonlinear.

Linear Vs. Non Linear Systems

(d) Assuming that the system is excited by $x_1(n)$ and $x_2(n)$ separately, we obtain the corresponding outputs

$$y_1(n) = Ax_1(n) + B$$

$$y_2(n) = Ax_2(n) + B$$

A linear combination of $x_1(n)$ and $x_2(n)$ produces the output

$$\begin{aligned} y_3(n) &= \mathcal{T}[a_1x_1(n) + a_2x_2(n)] \\ &= A[a_1x_1(n) + a_2x_2(n)] + B \\ &= Aa_1x_1(n) + a_2Ax_2(n) + B \qquad \textbf{(1)} \end{aligned}$$

On the other hand, if the system were linear, its output to the linear combination of $x_1(n)$ and $x_2(n)$ would be a linear combination of $y_1(n)$ and $y_2(n)$, that is,

$$a_1y_1(n) + a_2y_2(n) = a_1Ax_1(n) + a_1B + a_2Ax_2(n) + a_2B \qquad \textbf{(2)}$$

- (1) = (2) if B=0 linear non relaxed system

Causal Vs. Non Causal Systems

- **Causal Systems:**

  A system is called causal if the output [y(n)] of the system at any time n depends only on the present and past inputs ex[x(n-1),x(n-2),….] But does not depend on future inputs ex[x(n+1), x(n+2),….]

$$y(n) = F[x(n), x(n-1), x(n-2), \ldots]$$

Where F[.] is any arbitrary function. Any system that don't satisfy the definition is non-causal system.

Causal Vs. Non Causal Systems

Determine if the systems described by the following input–output equations are causal or noncausal.

(a) $y(n) = x(n) - x(n-1)$ **(b)** $y(n) = \sum_{k=-\infty}^{n} x(k)$ **(c)** $y(n) = ax(n)$

(d) $y(n) = x(n) + 3x(n+4)$ **(e)** $y(n) = x(n^2)$ **(f)** $y(n) = x(2n)$

(g) $y(n) = x(-n)$

Stable Vs. non stable Systems

- **Stable Systems:**

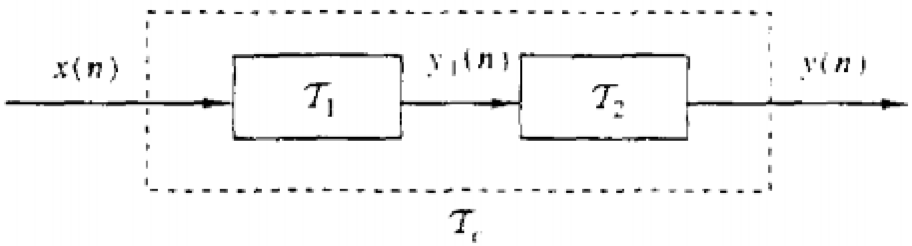
  An arbitrary relaxed system is said to be bounded input-bounded output (BIBO) stable if and only if every bounded input produces bounded output

$$|x(n)| \leq M_x < \infty \qquad |y(n)| \leq M_y < \infty$$

  Otherwise system is unstable

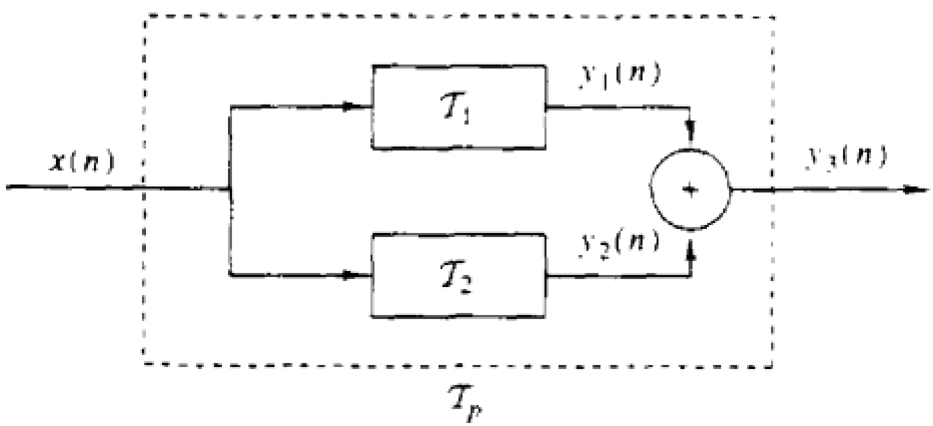
Interconnection of Discrete-Time Systems

Discrete-time systems can be interconnected to form larger systems. There are two basic ways in which systems can be in terconnected : in cascade (series) or in parallel. Conversely , we can take a larger system and break it down into smaller sub-s ystems for purposes of analysis and implementation.



series

$$\mathcal{T}_2\mathcal{T}_1 \neq \mathcal{T}_1\mathcal{T}_2$$



parallel

Analysis of Discrete Time Linear Time Invariant (LTI) Systems

❖ The method is based on the direct solution of the in put-output equation for the system. The input signal is resolved into sum of elementary signals Then the response of the system to the elementary signals are added to obtain total response to the system to the given input signal.

$$x(n) = \sum_{k} c_k x_k(n)$$

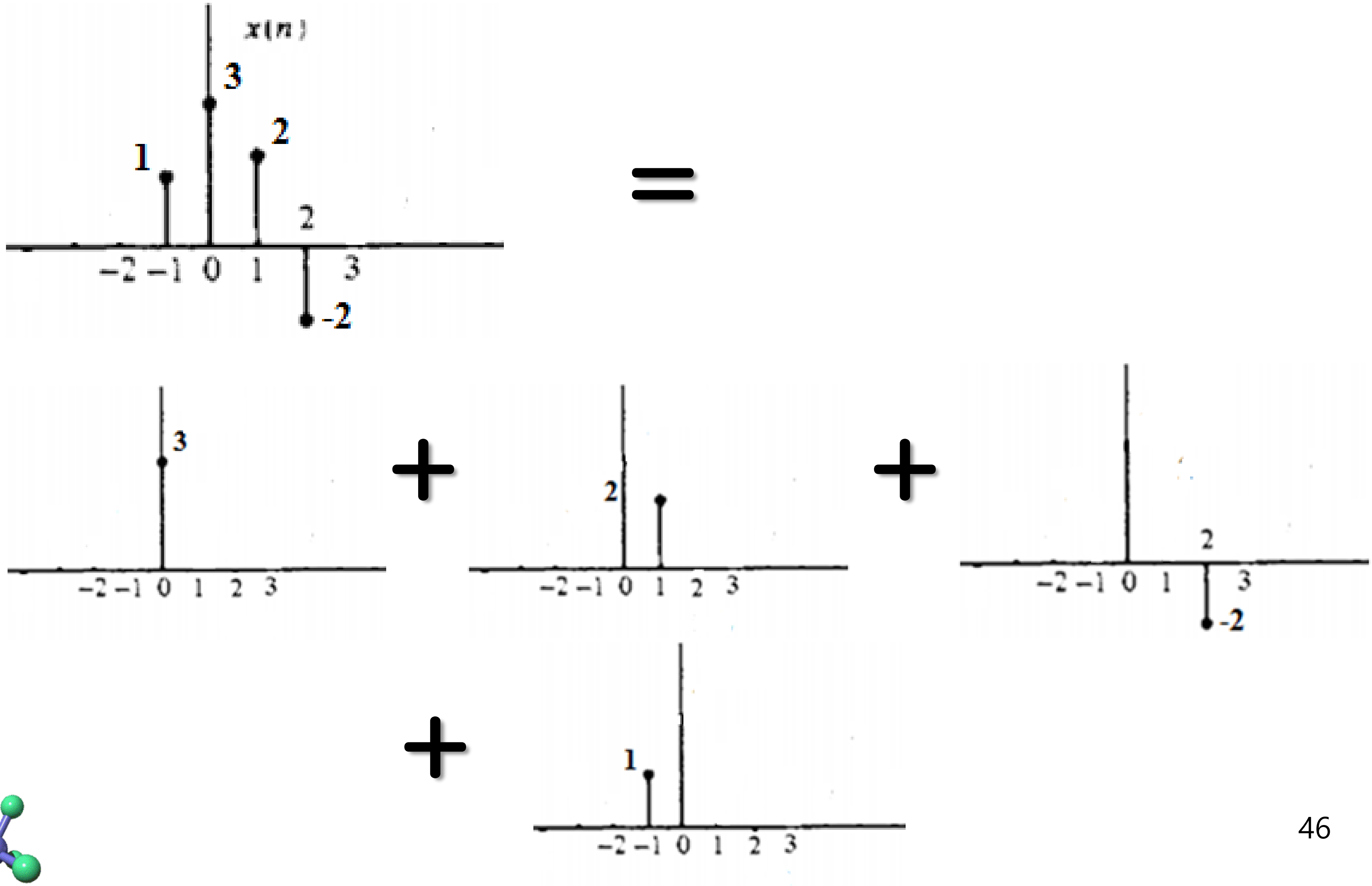
where the c_k are the set of amplitudes (weighting coefficients) in the decomposition of the signal x(n).

Resolution of Discrete Time Signals into Impulses

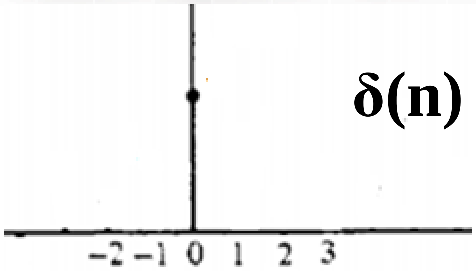
Suppose we have an arbitrary signal x(n) that we wish to resolve in a sum of unit sample sequence. Let the elementary signal be the unit impulse sequence. Multiply x(n) by δ (n-k) results another sequence that is zero everywhere except at n=k, where its value is x(k).

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

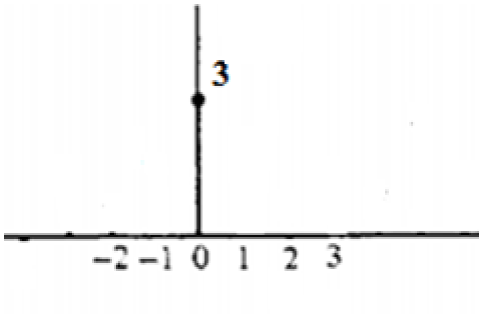
Resolution of Discrete Time Signals into Impulses



Resolution of Discrete Time Signals into Impulses

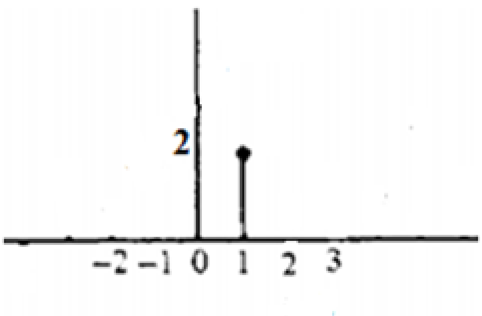


$\delta(n)$

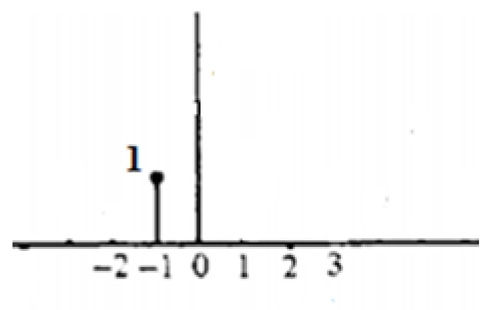


$= 3\delta(n)$

$= -2\delta(n-2)$



$= 2\delta(n-1)$



$= \delta(n+1)$

$$X(n) = -2\delta(n-2) + 2\delta(n-1) + 3\delta(n) + \delta(n+1)$$

Resolution of Discrete Time Signals into Impulses

Consider the special case of a finite-duration sequence given as

$$x(n) = \{2, \underset{\uparrow}{4}, 0, 3\}$$

Resolve the sequence $x(n)$ into a sum of weighted impulse sequences.

$$x(n) = 2\delta(n+1) + 4\delta(n) + 3\delta(n-2)$$

Response of LTI systems to Arbitrary inputs: the convolution sum

Having resolved an arbitrary input signal x(n) into a weighted sum of impulses, we are now ready to determine the response of any relaxed linear system to any input signal. First, we denote the response v(n, k) of the system to the input unit sample sequence at n = k it by the special symbol h(n, k), $-\infty < k < \infty$. That is

$$\mathbf{h(n) = \Gamma\,(\delta(n)\,)}$$

$$y(n, k) \equiv h(n, k) = \mathcal{T}[\delta(n-k)]$$

Response of LTI systems to Arbitrary inputs: the convolution sum

if the input is the arbitrary signal x (n) that is expressed as a sum of weighted impulses, that is

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

then the response of the system to x(n) is the corresponding sum of weighted outputs, that is

$$\begin{aligned} y(n) = \mathcal{T}[x(n)] &= \mathcal{T}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] \\ &= \sum_{k=-\infty}^{\infty} x(k)\mathcal{T}[\delta(n-k)] \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n,k) \end{aligned}$$

Response of LTI systems to Arbitrary inputs: the convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n,k)$$

the linearity property of the system is used in the previous equation but not its time-in variance property. Thus the expression in applies to any relaxed linear (time-variant) system.

If in addition, the system is time invariant, the response of the system to the delayed unit sample sequence δ(n-k) is

$$h(n-k) = \mathcal{T}[\delta(n-k)]$$

Consequently, the formula becomes

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

The Convolution Sum

This formula that gives the response y(n) o f the LTI system as a function of the input signal x(n) and the unit sample (impulse) response h(n) is called a convolution sum. We say that the input x(n) is convolved with the impulse.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

The Convolution Sum Analytically

$$
\begin{aligned}
x(n) &= \left\{\underset{\uparrow}{1}, 1, 1, 1\right\} \\
h(n) &= \left\{\underset{\uparrow}{6}, 5, 4, 3, 2, 1\right\} \\
y(n) &= \sum_{k=0}^{\infty} x(k)h(n-k) \\
y(0) &= x(0)h(0) = 6, \\
y(1) &= x(0)h(1) + x(1)h(0) = 11 \\
y(2) &= x(0)h(2) + x(1)h(1) + x(2)h(0) = 15 \\
y(3) &= x(0)h(3) + x(1)h(2) + x(2)h(1) + x(3)h(0) = 18 \\
y(4) &= x(0)h(4) + x(1)h(3) + x(2)h(2) + x(3)h(1) + x(4)h(0) = 14 \\
y(5) &= x(0)h(5) + x(1)h(4) + x(2)h(3) + x(3)h(2) + x(4)h(1) + x(5)h(0) = 10 \\
y(6) &= x(1)h(5) + x(2)h(4) + x(3)h(3) = 6 \\
y(7) &= x(2)h(5) + x(3)h(4) = 3 \\
y(8) &= x(3)h(5) = 1 \\
y(n) &= 0, n \geq 9 \\
y(n) &= \left\{\underset{\uparrow}{6}, 11, 15, 18, 14, 10, 6, 3, 1\right\}
\end{aligned}
$$

The Convolution Sum

Given the impulse response of LTI system is

$$h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$$

Determine the response of the system to the input signal

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$

Thanks for Your Attention